# On the effectivity problem of the universality theorem and the denseness problem for zeta and $L$-functions

(ゼータ関数及び$L$-関数の普遍性定理の定量的評価と稠密性定理について)

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Abstract. This thesis is the summary of author's two studies on the value-distribution of zeta and $L$-functions.

The first study is on the denseness problem for the iterated integrals of the logarithm of the Riemann zeta-function $\zeta(s)$, which is a joint work with Shōta Inoue [**9**]. We give a result for the denseness of the values of the iterated integrals on horizontal lines. By using this result under the Riemann Hypothesis, we obtain the denseness of the values $\int_0^t \log\zeta(1/2+it')dt'$. Moreover, we show that, for any $m \geq 2$, the denseness of the values of an $m$ times iterated integral on the critical line is equivalent to the Riemann Hypothesis.

The second study is on the effectivity problem of the universality theorem for zeta and $L$-functions. Recently, Garunkštis, Laurinčikas, Matsumoto, J. & R. Steuding showed an effective universality-type theorem for the Riemann zeta-function by using an effective multi-dimensional denseness result of Voronin. We will generalize Voronin's effective result and their theorem to the elements of the Selberg class satisfying some conditions.

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# Contents

## 1. Introduction

Let $s = \sigma + it$ be a complex variable with $\mathrm{Re}(s) = \sigma$ and $\mathrm{Im}(s) = t$. The Riemann zeta-function $\zeta(s)$ is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\sigma > 1$. This series can be analytically continued to the whole complex plane and has a pole at $s = 1$ with residue 1. The Riemann zeta-function $\zeta(s)$ plays a great role in analytic number theory and is known to be connected with the distribution of prime numbers. In particular, the distribution of zeros of $\zeta(s)$ is closely related to that of the prime numbers.

Let us recall the basic properties of the Riemann zeta-function.

- The Euler product representation

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}$$

  holds for $\mathrm{Re}(s) > 1$, where the product is taken over all prime numbers $p$.
- The functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

  holds.

The Euler product representation implies that the Riemann zeta-function is zero free for $\mathrm{Re}(s) > 1$. By the functional equation, we can find that the Riemann zeta-function has zeros at $s = -2, -4, -6, \ldots$ and no zeros for $\mathrm{Re}(s) < 0$ except for these zeros. These zeros of $\zeta(s)$ located in $\mathrm{Re}(s) < 0$ are called trivial zeros. Thus, our main interest is to reveal the distribution of zeros of $\zeta(s)$ in $0 \le \mathrm{Re}(s) \le 1$. These zeros of $\zeta(s)$ located in $0 \le \mathrm{Re}(s) \le 1$ are called non-trivial zeros, and the strip $\{s \in \mathbb{C}; 0 \le \mathrm{Re}(s) \le 1\}$ is called the critical strip. The famous Riemann Hypothesis states that the real part of all non-trivial zeros of $\zeta(s)$ equals $1/2$.

### 1.1. The denseness results for the Riemann zeta-function.

1.1.1. *Known facts and unsolved problem for the denseness theorems.* As we have seen in the above, it is important to study the value-distribution of the Riemann zeta-function in the critical strip to understand the distribution of prime numbers. However, the behavior of the values of the Riemann zeta-function in the critical strip is extremely complicated. Each of the following famous results is one of the results which express the complexity of the values of the Riemann zeta-function.

Theorem 1.1 (Bohr and Courant in 1914 [**4**]). *For any $1/2 < \sigma \le 1$, the set $\{\zeta(\sigma + it)\ ;\ t \in \mathbb{R}\}$ is dense in the complex plane.*

Theorem 1.2 (Bohr in 1916 [**3**]). *For any $1/2 < \sigma \le 1$, the set $\{\log\zeta(\sigma + it)\ ;\ t \in \mathbb{R}\}$ is dense in the complex plane.*

Note that the former theorem immediately follows from the latter one. Here, we define the branch of $\log \zeta(s)$. Let $G$ denote

$$G = \mathbb{C} \setminus \left\{ \bigcup_{\rho=\beta+i\gamma} \{s = \sigma + it \ ; \ \sigma \leq \beta\} \cup (-\infty, 1] \right\}$$

and define $\log \zeta(s)$ by

$$\log \zeta(s) = \int_{\infty}^{\sigma} \frac{\zeta'}{\zeta}(\alpha + it) d\alpha$$

for $s = \sigma + it \in G$. Here $\rho$ denotes the non-trivial zeros of $\zeta(s)$.

Furthermore, the probabilistic improvements of these two theorems have been known as the Bohr-Jessen limit theorem [**5**]. To state this theorem, we recall the notion of weak convergence from probability theory. A family of probability measure $(\mu_T)_{T>0}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ is said to converge weakly to a probability measure $\mu$ if $\int_{\mathbb{C}} f d\mu_T \to \int_{\mathbb{C}} f d\mu$ holds as $T \to \infty$ for any bounded continuous function $f$ on $\mathbb{C}$, where $\mathcal{B}(S)$ stands for the Borel $\sigma$-field of the topological space $S$. Here and in what follows, let $\mathrm{meas}(\cdot)$ denote the one-dimensional Lebesgue measure.

THEOREM 1.3. *Let $1/2 < \sigma \leq 1$. For $T > 0$, define the probability measure $\mu_{\sigma,T}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ by*

$$\mu_{\sigma,T}(A) = \frac{1}{T} \mathrm{meas} \{t \in [0, T] \ ; \ \log \zeta(\sigma + it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

*Then there exists the unique probability measure $\mu_\sigma$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that the family $(\mu_{\sigma,T})_{T>0}$ converges weakly to $\mu_\sigma$ as $T \to \infty$. Moreover, $\mu_\sigma$ has the probability density function which is continuous and takes everywhere positive values.*

The statement of Theorem 1.3 is a little different from that in [**5**] and written in terms of modern probability theory. Later, Jessen-Wintner [**20**] gave an alternative proof by a probabilistic argument. For further developments of the Bohr-Jessen limit theorem, see e.g. [**16**], [**17**], [**24**] and [**25**].

Here we mention some known facts about the denseness of the set of the values $\zeta(\sigma + it)$ for the other values of $\sigma$. For $\sigma > 1$, it is classically known that the set $\{\zeta(\sigma + it) \ ; \ t \in \mathbb{R}\}$ is bounded. Hence the set $\{\zeta(\sigma + it) \ ; \ t \in \mathbb{R}\}$ is not dense in $\mathbb{C}$ in this case. For $\sigma < 1/2$, Garunkštis and Steuding [**14**] proved that the set $\{\zeta(\sigma + it) \ ; \ t \in \mathbb{R}\}$ is not dense in $\mathbb{C}$ under the Riemann Hypothesis. For $\sigma = 1/2$, Kowalski and Nikeghbali [**23**, Corollary 9] gave a sufficient condition for the denseness of the set $\{\zeta(1/2 + it) \ ; \ t \in \mathbb{R}\}$. However it is so strong that no one has proved even that the Riemann Hypothesis implies their condition. At present, the denseness of the set $\{\zeta(1/2 + it) \ ; \ t \in \mathbb{R}\}$ is still open.

PROBLEM 1. *Is the set $\{\zeta(1/2 + it) \ ; \ t \in \mathbb{R}\}$ dense in the complex plane?*

Garunkštis and Steuding [**14**] showed that the set $\{(\zeta(1/2 + it), \zeta'(1/2 + it)) \ ; \ t \in \mathbb{R}\}$ is not dense in $\mathbb{C}^2$, and we may guess that the answer of Problem 1 is negative from this result. One of the important results in the attempt to understand the distribution of the values for the Riemann zeta function in the critical line is Selberg's work [**39, 40**]. He studied the moments of $\log \zeta(1/2 + it)$ to obtain the following limit theorem.

Theorem 1.4. *If $A$ is a Jordan measurable set in the complex plane, then*

$$\lim_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\left\{t\in[T,2T]\ ;\ \frac{\log\zeta(1/2+it)}{\sqrt{\frac{1}{2}\log\log T}}\in A\right\}=\frac{1}{2\pi}\iint_A e^{-(x^2+y^2)/2}dxdy.$$

1.1.2. *Statement of the main results.* The contents of this subsubsection are based on the paper [**9**]. This is a joint work with Shōta Inoue.

In order to mention the main results, we give some definitions. Define the function $\eta_m(s)$ for $m\in\mathbb{N}$ by

$$\eta_m(\sigma+it)=\int_0^t \eta_{m-1}(\sigma+it')dt'+c_m(\sigma),$$

where

$$\eta_0(\sigma+it)=\log\zeta(\sigma+it)\quad\text{and}\quad c_m(\sigma)=\frac{i^m}{(m-1)!}\int_0^\infty(\alpha-\sigma)^{m-1}\log\zeta(\alpha)d\alpha.$$

The function $\eta_m(s)$ is the $m$ times iterated integral of $\log\zeta(s)$ on the vertical line, which was introduced by Inoue [**18**]. In this thesis, we discuss the denseness problem of $\eta_m(s)$ for $\sigma\geq 1/2$ and $m\geq 1$.

Theorem 1.5. *Let $1/2\leq\sigma<1$. If the number of zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ with $\beta>\sigma$ is finite, then the set*

$$\left\{\int_0^t\log\zeta(\sigma+it')dt'\ ;\ t\in[0,\infty)\right\}$$

*is dense in the complex plane. Moreover, for each integer $m\geq 2$, the following statements are equivalent.*

(I) *The Riemann zeta-function does not have any zeros whose real parts are greater than $\sigma$.*
(II) *The set $\{\eta_m(\sigma+it)\ ;\ t\in[0,\infty)\}$ is dense in the complex plane.*

From this theorem, we see that the Riemann Hypothesis implies that the set

$$\left\{\int_0^t\log\zeta(1/2+it')dt'\ ;\ t\in[0,\infty)\right\}$$

is dense in the complex plane. Moreover, the equivalence in Theorem 1.5 would be a new type of statement which gives the relation between the denseness of values of the Riemann zeta-function and the Riemann Hypothesis.

Now we introduce the function $\tilde{\eta}_m(s)$ which is closely related to $\eta_m(s)$. Define the function $\tilde{\eta}_m(\sigma+it)$ for $m\in\mathbb{N}$ by

$$\tilde{\eta}_m(\sigma+it)=\int_\sigma^\infty\tilde{\eta}_{m-1}(\alpha+it)d\alpha,$$

where $\tilde{\eta}_0(\sigma+it)=\log\zeta(\sigma+it)$. This function is the $m$ times iterated integral of $\log\zeta(\sigma+it)$ on the horizontal line. By Littlewood's lemma, we can obtain the following connection between $\eta_m(s)$ and $\tilde{\eta}_m(s)$.

Lemma 1.6 (Lemma 1 in [**18**]). *Let $m$ be a positive integer, and let $t > 0$. Then, for any $\sigma \geq 1/2$, we have*

$$\eta_m(\sigma + it) = i^m \tilde{\eta}_m(\sigma + it) + 2\pi \sum_{k=0}^{m-1} \frac{i^{m-1-k}}{(m-k)!k!} \sum_{\substack{0<\gamma<t \\ \beta>\sigma}} (\beta - \sigma)^{m-k}(t-\gamma)^k.$$

From this connection, it is important to analyze the function $\tilde{\eta}_m(s)$ to know the property of $\eta_m(s)$. The function $\tilde{\eta}_m(s)$ is holomorphic in the same region as in the case of $\log \zeta(s)$ and has some properties similar to that of $\log \zeta(s)$. From this observation, we obtain the following theorem unconditionally

Theorem 1.7. *Let $1/2 \leq \sigma < 1$, and $m$ be a positive integer. Let $T_0$ be any positive number. Then the set*

$$\{\tilde{\eta}_m(\sigma + it) \ ; \ t \in [T_0, \infty)\}$$

*is dense in the complex plane.*

As mentioned above, Bohr developed his denseness results with Jessen from the viewpoint of probability theory in [**5**]. Following their method, Inoue and the author will continue their study with Mine in a subsequent paper [**10**]. They will give deeper results such as an analog of Lamzouri's study [**24**] and of the study of Lamzouri, Lester and Radziwiłł [**25**].

### 1.2. The multi-dimensional denseness theorem and the universality theorem.

1.2.1. *Voronin's work and related results.* In 1972, Voronin [**46**] generalized Bohr's results to obtain the following multi-dimensional denseness theorems.

Theorem 1.8. *Let $n \in \mathbb{N}$ and $h > 0$. Let $s_1, \ldots, s_n$ satisfy $1/2 < \mathrm{Re}(s_k) \leq 1$ for $k = 1, \ldots, n$ and $s_k \neq s_j$ for $k \neq j$. Then the set*

$$\{(\zeta(s_1 + imh), \ldots, \zeta(s_n + imh)) \ ; \ m \in \mathbb{N}\}$$

*is dense in $\mathbb{C}^n$.*

Theorem 1.9. *Let $n \in \mathbb{N}$ and $h > 0$ and $1/2 < \mathrm{Re}(s) \leq 1$. Then the set*

$$\{(\zeta(s + imh), \zeta'(s + imh), \ldots, \zeta^{(n-1)}(s + imh)) \ ; \ m \in \mathbb{N}\}$$

*is dense in $\mathbb{C}^n$.*

In particular, the following theorem immediately follows from Theorem 1.9.

Theorem 1.10. *Let $n \in \mathbb{N}$ and $1/2 < \sigma \leq 1$. Then the set*

$$\{(\zeta(\sigma + it), \zeta'(\sigma + it), \ldots, \zeta^{(n-1)}(\sigma + it)) \ ; \ t \in \mathbb{R}\}$$

*is dense in $\mathbb{C}^n$.*

In 1975, Voronin [**47**] discovered the universality theorem for the Riemann zeta-function, which states as follows;

Theorem 1.11. *Let $\mathcal{K}$ be a compact subset of $\mathcal{D} = \{s \in \mathbb{C} \ ; \ 1/2 < \sigma < 1\}$ with connected complement, and let $f$ be a non-vanishing continuous on $\mathcal{K}$ that is holomorphic in the interior of $\mathcal{K}$. Then we have, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] \ ; \ \max_{s\in\mathcal{K}} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0. \tag{1}$$

Roughly speaking, Theorem 1.11 says that the Riemann zeta-function can approximate any non-vanishing holomorphic functions. Note that Theorem 1.11 is not the original one by Voronin but is the form by Reich [**37**]. Voronin discussed the case when the above $\mathcal{K}$ is replaced by a closed disk $\{s \in \mathbb{C} \ ; \ |s + 3/4| \le r\}$ with $0 < r < 1/4$. Note that we can also see Voronin's proof of the universality theorem in the textbook [**21**].

The theory of the universality theorem has been developed in various directions. One of the remarkable results is Bagchi's work [**1**]. He gave a probabilistic proof of the universality theorem for the Riemann zeta-function, which was a different approach from Voronin's original proof. To state this result, we will give some notations. Let $\gamma$ denote the unit circle on the complex plane and define $\Omega = \prod_p \gamma_p$, $\gamma_p = \gamma$. Since $\Omega$ is compact, there exists the probability Haar measure $\mathbf{m}$ on $(\Omega, \mathcal{B}(\Omega))$. For any $\omega = (\omega(p))_p \in \Omega$ and $n \in \mathbb{N}$ with $n \ge 2$, let $\omega(1) = 1$ and let

$$\omega(n) = \prod_{j=1}^{k} \omega(p_j)^{r_j},$$

where $n = p_1^{r_1} \cdots p_k^{r_k}$ is the prime factorization of $n$. Let $\mathcal{D} = \{s \ ; \ 1/2 < \sigma < 1\}$ and let $\mathcal{H}(\mathcal{D})$ denote the set of holomorphic function on $\mathcal{D}$ equipped with the topology of uniform convergence on compact subsets. Define the probability measures $\nu_T$ and $\nu$ on $(\mathcal{H}(\mathcal{D}), \mathcal{B}(\mathcal{H}(\mathcal{D})))$ by

$$\nu_T(A) = \frac{1}{T} \operatorname{meas}\{\tau \in [0,T] \ ; \ \zeta(s+i\tau) \in A\}$$

and

$$\nu(A) = \mathbf{m}\{\omega \in \Omega \ ; \ \zeta(s,\omega) \in A\}$$

for $A \in \mathcal{B}(\mathcal{H}(\mathcal{D}))$, where $\mathcal{H}(\mathcal{D})$-valued random variable $\zeta(s,\omega)$ is defined by

$$\zeta(s,\omega) = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s}.$$

Bagchi proved the following.

Theorem 1.12. *We have the followings;*

(i) *The probability measures $\nu_T$ converges weakly to $\nu$ as $T \to \infty$.*
(ii) *The support of the probability measure $\nu$ coincides with the set*

$$\{\varphi \in \mathcal{H}(\mathcal{D}) \ ; \ \varphi(s) \neq 0 \quad \textit{for} \quad s \in \mathcal{D} \quad \textit{or} \quad \varphi(s) \equiv 0\}.$$

Combining Merglyan's theorem with this theorem we obtain the universality theorem. Merglyan's theorem asserts the following;

Theorem 1.13 (see e.g. [**38**]). *Let $\mathcal{K}$ be a compact subset of $\mathbb{C}$ with connected complement. If $f$ is continuous function on $\mathcal{K}$ which is holomorphic in the interior of $\mathcal{K}$, and if $\varepsilon > 0$, then there exists a polynomial $P$ such that $|f(s) - P(s)| < \varepsilon$ for any $s \in \mathcal{K}$.*

At present, Bagch's approach is a standard method of proving universality theorems for a wide class of zeta and $L$-functions. Refer to [**27**], [**43**], [**22**] etc. for the details of Bagchi's theory. For other developments of the universality, see e.g. a survey paper [**31**].

As another development of the universality theorem, there are some studies of the refinement to an effective form of Voronin's universality theorem. There arose the questions on how large the value (1) is or how small the shift $\tau$ we can take to make $\zeta(s+i\tau)$ approximate a given non-vanishing holomorphic function. In Voronin's proof of the universality theorem, Pečerskiĭ's rearrangement theorem [**35**] in Hilbert space and Kronecker's approximation theorem [**21**, Appendix 8, Theorem 1] are used. These theorems assert the followings;

THEOREM 1.14 (Pečerskiĭ). *Let $\langle x, y\rangle_{\mathcal{H}}$ denote the inner product of $x$ and $y$ belonging to a real Hilbert space $\mathcal{H}$. The norm $\|x\|_{\mathcal{H}}$ of $x \in \mathcal{H}$ is canonically defined by $\|x\|_{\mathcal{H}} = \sqrt{\langle x, x\rangle_{\mathcal{H}}}$. Suppose that a sequence $\{u_n\}_{n=1}^{\infty}$ on $\mathcal{H}$ satisfies the the following conditions;*

- $\sum_{n=1}^{\infty} \|u_n\|_{\mathcal{H}} < \infty$,
- *For any $e \in \mathcal{H}$ with $\|e\|_{\mathcal{H}} = 1$, there exists a bijective mapping $l = l(e) : \mathbb{N} \ni k \mapsto l_k \in \mathbb{N}$ such that the series $\sum_{k=1}^{\infty} \langle u_{l_k}, e\rangle_{\mathcal{H}}$ converges conditionally.*

*Then, for any $v \in \mathcal{H}$, there exists a bijective mapping $j = j(v) : \mathbb{N} \ni k \mapsto j_k \in \mathbb{N}$ such that*

$$\sum_{k=1}^{\infty} u_{j_k} = v \quad \text{in } \mathcal{H}.$$

THEOREM 1.15 (Kronecker' approximation theorem). *Let $A$ be a Jordan measurable subregion of $[0,1)^N$, and $a_1, \dots, a_N$ be real numbers linearly independent over $\mathbb{Q}$. Set, for any $T > 0$,*

$$I(T, A) = \{t \in [0, T] \ ; \ (\{a_1 t\}, \dots, \{a_N t\}) \in A\}.$$

*Here $\{x\}$ means the fractional part of $x$. Then we have*

$$\lim_{T \to +\infty} \frac{\mathrm{meas}(I(T, A))}{T} = \mathrm{meas}(A).$$

Since these two theorems are ineffective, it is difficult in general to obtain effective results of the universality theorem. Good [**15**] was the first to make progress on this effectivization problem. He combined Montgomery's results [**32**] about the extreme values of $\log \zeta(s)$ with Voronin's results to get some effective results of the universality theorem. After that, Garunkštis [**11**] extended Good's idea to obtain some explicit results in a small region for the above effecitivization problems.

Another approach by using the Taylor series expansion is taken by Garunkštis, Laurinčikas, Matsumoto, J. & R. Steuding [**13**]. They refined Matsumoto's weak version of the universality theorem into the effective form. Matsumoto's theorem is written in the survey paper [**30**, Section 3]. To see what is the refinement, we recall how to prove the Matsumoto's theorem. Let $g(s)$ be a function holomorphic at $s = s_0$. By the Taylor series expansions of $\zeta(s+i\tau)$ and $g(s)$, the equations

$$\zeta(s+i\tau) = \sum_{k=0}^{\infty} \frac{\zeta^{(k)}(s_0+i\tau)}{k!}(s-s_0)^k \quad \text{and} \quad g(s) = \sum_{k=0}^{\infty} \frac{g^{(k)}(s_0)}{k!}(s-s_0)^k$$

hold around $s = s_0$. Letting the coefficients $\zeta^{(k)}(s_0 + i\tau)$ approximate $g^{(k)}(s_0)$ simultaneously by using Theorem 1.10, we can deduce a weak version of the universality theorem which is valid only around $s = s_0$. However, since Theorem 1.10 is ineffective, we can not obtain the effective result of this weak approximation theorem. To obtain the effective version of this result, Garunkštis, Laurinčikas, Matsumoto, J. & R. Steuding directed their attention to the following theorem by Voronin [**48**], which is an effective version of Theorem 1.10.

THEOREM 1.16. *Let $N \in \mathbb{N}$ and $\sigma_0 \in (1/2, 1)$ and $\underline{b} = (b_0, b_1, \ldots, b_{N-1}) \in \mathbb{C}^N$ with $|b_0| > \varepsilon > 0$. Then a sufficient condition for the system of inequalities*

$$\left|\zeta^{(k)}(\sigma_0 + it) - b_k\right| < \varepsilon, \quad k = 0, \ldots, N-1$$

*to have a solution $t \in [T, 2T]$ is that*

$$T > c_0(N, \sigma_0) \exp\exp\left(c_1(N, \sigma_0) A(N, \underline{b}, \varepsilon)^{\frac{8}{1-\sigma_0} + \frac{8}{\sigma_0 - 1/2}}\right),$$

*where $c_0(N, \sigma_0)$ and $c_1(N, \sigma_0)$ are a positive* [1]*, effectively computable constant, depending only on $N$ and $\sigma_0$, and*

$$A(N, \underline{b}, \varepsilon) = |\log b_0| + \left(\frac{\|\underline{b}\|}{\varepsilon}\right)^{N^2}$$

*with $\|\underline{b}\| = \sum_{0 \le k < N} |b_k|$. Here the above branch of $\log b_0$ can be taken arbitrarily.*

We remark that this result is also written in the textbook [**21**], and the above statement is the form described in the textbook. This result is also regarded as a kind of $\Omega$-results, which Voronin called it. He proved this effective result cleverly without using ineffective results as mentioned above. In his proof, Pečerskiĭ's theorem is replaced by a geometrical argument and an argument in which the system of the linear equation and the prime number theorem for short interval are used. Kronecker's approximation theorem is replaced by a kind of amplification technique in which we estimate a certain weighted mean value. Garunkštis, Laurinčikas, Matsumoto, J. & R. Steuding [**13**] used this effective result to refine Matsumoto's theorem. Since the author found a slight mistake in their statement [**13**], we shall mention the modified version of this statement as follows;

THEOREM 1.17 (Modified version of the result [**13**]). *Let $s_0 = \sigma_0 + it_0$, $1/2 < \sigma_0 < 1$, $r > 0$, $\mathcal{K} = \{s \in \mathbb{C} \ ; \ |s - s_0| \le r\}$, and suppose that $g : \mathcal{K} \to \mathbb{C}$ is an analytic function with $g(s_0) \neq 0$. Put $M(g) = \max_{|s-s_0|=r} |g(s)|$. Fix $\varepsilon \in (0, 1)$ and $0 < \delta_0 < 1$. If $N = N(\delta_0, \varepsilon, g)$ and $T = T(g, \varepsilon, \sigma_0, \delta_0, N)$ satisfy*

$$M(g) \frac{\delta_0^N}{1 - \delta_0} < \frac{\varepsilon}{3}$$

*and*

$$T \ge \max\left\{c_0(\sigma_0, N) \exp\exp\left(c_1(\sigma_0, N) A\left(N, \mathbf{g}, (\varepsilon/3)\exp(-\delta_0 r)\right)^{\frac{8}{1-\sigma_0} + \frac{8}{\sigma_0 - 1/2}}\right), r\right\},$$

*respectively, then there exists $\tau \in [T - t_0, 2T - t_0]$ such that*

$$\max_{|s-s_0| \le \delta r} |\zeta(s + i\tau) - g(s)| < \varepsilon$$

[1]It is written in [**21**] that the constant $c_1(N, \sigma_0)$ can be taken 5. However one cannot prove this fact by the method in [**21**]. This is probably a mistake.

*for any* $0 \leq \delta \leq \delta_0$ *satisfying*
$$M(\tau)\frac{\delta^N}{1-\delta} < \frac{\varepsilon}{3}.$$
*Here* $c_0(\sigma_0, N)$, $c_1(\sigma_0, N)$ *and* $A\left(N, \mathbf{g}, (\varepsilon/3)\exp(-\delta_0 r)\right)$ *are the same constants as in Theorem 1.16 with* $\mathbf{g} = \left(g(s_0), g'(s_0), \ldots, g^{(N-1)}(s_0)\right)$ *, and* $M(\tau)$ *is defined by* $M(\tau) = \max_{|s-s_0|=r} |\zeta(s+i\tau)|$.

This correction is inspired by Matsumoto's paper [**30**]. We will mention this correction in the proof of Corollary 1.21.

Remark 1.18. We will explain the mistake in [**13**]. Let the settings be the same as in Theorem 1.17. In [**13**], they showed that the inequality
$$|\zeta(s+i\tau) - g(s)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\exp(\delta r) + M(\tau)\frac{\delta^n}{1-\delta} =: G(\delta)$$
holds for $0 \leq \delta \leq \delta_0$, which is the second line from the top of the page 214 in [**13**]. Here $n$ and $\tau$ are some positive numbers depending on $\delta_0$ and other parameters, and $M(\tau)$ is given by $M(\tau) = \max_{|s-s_0|=r} |\zeta(s+i\tau)|$. After that, they stated that one can choose $\delta$ so that $G(\delta) = \varepsilon$ since $G(0) = (2/3)\varepsilon$ and $\lim_{\delta \nearrow 1} G(\delta) = \infty$. However this argument can not always realized since the above inequality is valid only for $0 \leq \delta \leq \delta_0 < 1$.

Our goal in this thesis is to generalize these results in [**48**] and [**13**] to the Selberg class $\mathcal{S}$ with some conditions.

As other recent effective results, we give examples like [**15**], [**11**], [**12**], [**41**], [**42**], [**13**], [**26**] and refer to [**28**] for a good survey of effectivization problem of the universality theorem.

1.2.2. *Statement of the main results.* The contents of this subsubsection are based on the paper [**8**]. To state the main theorem, we start to recall the definition of the Selberg class $\mathcal{S}$. The Dirichlet series
$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$
is said to belong to the Selberg class $\mathcal{S}$ if $\mathcal{L}(s)$ satisfies the following axioms;

(i) Ramanujan hypothesis: $a(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$.

(ii) Analytic continuation: there exists a nonnegative integer $m$ such that $(s-1)^m \mathcal{L}(s)$ is an entire function of finite order.

(iii) Functional equation: $\mathcal{L}(s)$ satisfies a functional equation of the type
$$\mathcal{H}_{\mathcal{L}}(s) = \omega \overline{\mathcal{H}_{\mathcal{L}}(1-\overline{s})}$$
where
$$\mathcal{H}_{\mathcal{L}}(s) = \mathcal{L}(s) R^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j) = \gamma(s)\,\mathcal{L}(s)$$
with positive real numbers $R$, $\lambda_j$ and complex numbers $\mu_j$ and $\omega$ with $\operatorname{Re}\mu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product: $\log \mathcal{L}(s) = \sum_{n=1}^{\infty} b(n) n^{-s}$, where $b(n) = 0$ unless $n = p^m$ with a prime number $p$ and $m \geq 1$, and $b(n) \ll n^\vartheta$ for some $\vartheta < 1/2$.

Now we give some definitions. The zeros of $\mathcal{L}(s)$ which are not derived from the poles of the $\gamma$-factor $\gamma(s)$ and are not equal to possible zeros of $\mathcal{L}(s)$ at $s = 0, 1$ are called non-trivial zeros, and denote by $\rho = \beta + i\gamma$ such zeros throughout this thesis. Let $N_{\mathcal{L}}(T)$ denote the number of non-trivial zeros with multiplicity satisfying $0 \le \beta \le 1$ and $|\gamma| \le T$. We remark that it is known that

$$N_{\mathcal{L}}(T) = \frac{d_{\mathcal{L}}}{\pi} T \log T + c_{\mathcal{L}} T + O(\log T) \tag{2}$$

holds, where $d_{\mathcal{L}} = 2\sum_{j=1}^{f} \lambda_j$ and $c_{\mathcal{L}}$ is some constant depending on $\mathcal{L}$. This $d_{\mathcal{L}}$ is called the degree of $\mathcal{L}(s)$ and known to be invariant in the Selberg class $\mathcal{S}$. For other properties of the Selberg class $\mathcal{S}$, we refer to a survey paper [**36**] for example.

In this thesis, we further assume the following three conditions.

(C1) There exists a $\kappa = \kappa(\mathcal{L}) > 0$ such that

$$\frac{1}{\pi(X)} \sum_{p \le X} |a(p)|^2 \sim \kappa \quad \text{as } X \to \infty.$$

(C2) There exists a $\sigma_{\mathcal{L}} \ge 1/2$ such that for any fixed $\sigma > \sigma_{\mathcal{L}}$

$$N_{\mathcal{L}}(\sigma, T) \ll T^{1-\Delta_{\mathcal{L}}(\sigma)}$$

as $T \to \infty$ with some positive real number $\Delta_{\mathcal{L}}(\sigma) > 0$, where $N_{\mathcal{L}}(\sigma, T)$ denote the number of non-trivial zeros of $\mathcal{L}(s)$ with multiplicity satisfying $\beta \ge \sigma$ and $|\gamma| \le T$. The implicit constant may depend on $\sigma$.

(C3) There exists an $E_{\mathcal{L}} > 0$ such that

$$\sum_{X < p \le X+H} |a(p)|^2 \sim \kappa \frac{H}{\log X} \quad \text{and} \quad \pi(X+H) - \pi(X) \sim \frac{H}{\log X}$$

hold for $X \ge H \ge X^{1-E_{\mathcal{L}}}(\log X)^D$ with some $D \ge 1$ as $X \to \infty$.

The above implicit constants appearing in the symbol $O(\cdot)$ and $\ll$ may depend on $\mathcal{L}(s)$. Remark that the universality theorem for the element of the Selberg class $\mathcal{S}$ satisfying the condition (C1) is proved in a certain strip by Nagoshi and Steuding [**34**].

Here, we define a branch of $\log \mathcal{L}(s)$. Let $G(\mathcal{L})$ denote

$$G(\mathcal{L}) = \{s \ ; \ \sigma > 1/2\} \setminus \left\{ \left( \bigcup_{\rho=\beta+i\gamma} \{s = \sigma + i\gamma \ ; \ \sigma \le \beta\} \right) \cup (-\infty, 1] \right\}$$

and we define $\log \mathcal{L}(s)$ by

$$\log \mathcal{L}(s) = \int_{\infty}^{\sigma} \frac{\mathcal{L}'}{\mathcal{L}}(\alpha + it) d\alpha$$

for $s = \sigma + it \in G(\mathcal{L})$.

In this thesis, we first show the following two results which are generalizations of the results in [**48**].

THEOREM 1.19. *Let $\mathcal{L}(s)=\sum_{n=1}^{\infty} a(n)n^{-s}$ be an element of the Selberg class $\mathcal{S}$ satisfying the conditions* (C1), (C2) *and* (C3). *Let $\max\{\sigma_{\mathcal{L}}, 1-2E_{\mathcal{L}}\}<\sigma_0<1$, $N\in\mathbb{N}$, $\varepsilon\in(0,1)$, $\underline{c}=(c_0,c_1,\dots,c_{N-1})\in\mathbb{C}^N$. Then a sufficient condition for the system of inequalities*

$$\left|\frac{d^k}{ds^k}\log\mathcal{L}(\sigma_0+it)-c_k\right|<\varepsilon \quad \text{for } k=0,1,\dots,N-1$$

*to have a solution $t\in[T,2T]$ is that*

$$T\geq\exp\exp\left(C_1(\mathcal{L},\sigma_0,N)\left(\|\underline{c}\|+\frac{1}{\varepsilon}\right)^{d(\sigma_0,E_{\mathcal{L}})}\right),$$

*where $C_1(\mathcal{L},\sigma_0,N)$ and $d(\sigma_0,E_{\mathcal{L}})$ are effectively computable positive constants depending on $\mathcal{L},\sigma_0,N$ and on $\sigma_0,E_{\mathcal{L}}$ respectively.*

COROLLARY 1.20. *Under the same hypothesis of Theorem 1.19 with $c_0\neq 0$, a sufficient condition for the system of inequalities*

$$\left|\frac{d^k}{ds^k}\mathcal{L}(\sigma_0+it)-c_k\right|<\varepsilon \quad \text{for } k=0,1,\dots,N-1$$

*to have a solution $t\in[T,2T]$ is that*

$$T\geq\exp\exp\left(C_2(\mathcal{L},\sigma_0,N)B(N,\underline{c},\varepsilon)^{d(\sigma_0,E_{\mathcal{L}})}\right),$$

*where $C_2(\mathcal{L},\sigma_0,N)$ is effectively computable positive constant depending on $\mathcal{L},\sigma_0,N$, and $d(\sigma_0,E_{\mathcal{L}})$ is the same constant as in Theorem 1.19, and*

$$B(N,\underline{c},\varepsilon)=|\log c_0|+\left(\frac{\|\underline{c}\|}{|c_0|}\right)^{(N-1)^2}\frac{1+|c_0|}{\varepsilon}.$$

*Here the above branch of $\log c_0$ can be taken arbitrarily.*

By combining Corollary 1.20 with the method as in [**13**], we have the following corollary.

COROLLARY 1.21. *Let $\mathcal{L}(s)=\sum_{n=1}^{\infty} a(n)n^{-s}$ be an element of the Selberg class $\mathcal{S}$ satisfying the conditions* (C1), (C2) *and* (C3). *Let $s_0=\sigma_0+it_0$, $\max\{\sigma_{\mathcal{L}},1-2E_{\mathcal{L}}\}<\sigma_0<1$, $r>0$, $\mathcal{K}=\{s\in\mathbb{C}\ ;\ |s-s_0|\leq r\}$, and suppose that $g:\mathcal{K}\to\mathbb{C}$ is an analytic function with $g(s_0)\neq 0$. Put $M(g)=\max_{|s-s_0|=r}|g(s)|$. Fix $\varepsilon\in(0,1)$ and $0<\delta_0<1$. If $N=N(\delta_0,\varepsilon,M(g))$ and $T=T(\mathcal{L},g,\varepsilon,\sigma_0,\delta_0,N)$ satisfy*

$$M(g)\frac{\delta_0^N}{1-\delta_0}<\frac{\varepsilon}{3}$$

*and*

$$T\geq\max\left\{\exp\exp\left(C_2(\mathcal{L},\sigma_0,N)B\left(N,\mathbf{g},(\varepsilon/3)\exp(-\delta_0 r)\right)^{d(\sigma_0,E_{\mathcal{L}})}\right),r\right\},$$

*respectively, then there exists $\tau\in[T-t_0,2T-t_0]$ such that*

$$\max_{|s-s_0|\leq\delta r}|\mathcal{L}(s+i\tau)-g(s)|<\varepsilon$$

*for any $0\leq\delta\leq\delta_0$ satisfying*

$$M(\tau;\mathcal{L})\frac{\delta^N}{1-\delta}<\frac{\varepsilon}{3}.$$

*Here* $C_2(\mathcal{L},\sigma_0,N)$ *and* $B\left(N,\mathbf{g},(\varepsilon/3)\exp(-\delta_0 r)\right)$ *are the same constants as in Corollary 1.20 with* $\mathbf{g}=\left(g(s_0),g'(s_0),\ldots,g^{(N-1)}(s_0)\right)$ *, and* $M(\tau;\mathcal{L})$ *is defined by* $M(\tau;\mathcal{L})=\max_{|s-s_0|=r}|\mathcal{L}(s+i\tau)|$.

At the end of this section, we give some examples. First, we will see that one can apply these results to the Riemann zeta-function in the range $1/2<\sigma_0<1$. In this case, the condition (C1) is well-known as the prime number theorem. For the condition (C2), we can take $\sigma_\zeta=1/2$ by the zero-density theorem (see e.g. [**45**, Theorem 9.19]). For the condition (C3), it is known that

$$\pi(X+H)-\pi(X)\sim\frac{H}{\log X}\quad\text{for}\quad X^{7/12}(\log X)^{22}\leq H\leq X$$

holds (see e.g. [**19**, Theorem 12.8]). Hence we can take $E_\zeta=5/12$ and we obtain $\max\{\sigma_\zeta,1-2E_\zeta\}=1/2$ in this case. For other examples, we can confirm that the Dirichlet $L$-function $L(s,\chi)$ of primitive characters $\chi$ and the Dedekind zeta-function $\zeta_K(s)$ belong to the Selberg class $\mathcal{S}$ and satisfy the conditions (C1), (C2) and (C3).

## 2. Proof of main results in 1.1.2

In this subsection, we will show the main results in 1.1.2.

**2.1. Key propositions of the proof.** Our first purpose is to show Theorem 1.7. In the proof of Theorem 1.7, the following two propositions play an important role.

In the following, the symbol meas$(\cdot)$ stands for the one-dimensional Lebesgue measure, and $\mathrm{Li}_m(z)$ means the polylogarithmic function defined as $\sum_{n=1}^{\infty} \frac{z^n}{n^m}$ for $|z| < 1$.

PROPOSITION 2.1. *Let $m$ be a positive integer. Then for any $\sigma \geq 1/2$, $T \geq X^{135}$, $\varepsilon > 0$, we have*

$$\lim_{X \to +\infty} \frac{1}{T} \operatorname{meas} \left\{ t \in [0, T] \ ; \ \left| \tilde{\eta}_m(\sigma + it) - \sum_{p \leq X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma - it})}{(\log p)^m} \right| < \varepsilon \right\} = 1.$$

The important point of this proposition is that $\tilde{\eta}_m(s)$ can be approximated by the Dirichlet polynomial even on the critical line. To prove this proposition, we must control exactly the contribution of nontrivial zeros of $\zeta(s)$, and we therefore need a strong zero density estimate of the Riemann zeta-function like Selberg's result [**39**, Theorem 1]. More precisely, we require that there exist numbers $c > 0$, $A < 2m + 1$ such that

$$N(\sigma, T) \ll T^{1 - c(\sigma - 1/2)} (\log T)^A$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$. Here, $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ with multiplicity satisfying $\beta > \sigma$ and $0 < \gamma \leq T$. Therefore, to prove Proposition 2.1, we need a strong zero density estimate comparable to the assumption by Bombieri and Hejhal [**6**]. On the other hand, when we discuss the denseness of $\tilde{\eta}_m(s)$ for fixed $\frac{1}{2} < \sigma < 1$, it suffices to use the weaker estimate

$$N(\sigma, T) \ll T^{1 - c(\sigma - 1/2) + \varepsilon}$$

for every $\varepsilon > 0$. Hence, there is an essential difference of the depth between the discussion in the case $\frac{1}{2} < \sigma < 1$ and that in the case $\sigma = \frac{1}{2}$ in Proposition 2.1. We will explain this point more closely later.

In contrast, we can prove the following proposition by almost the same method as in [**3**], [**4**].

PROPOSITION 2.2. *Let $m$ be a positive integer, $1/2 \leq \sigma < 1$. Let $a$ be any complex number, and $\varepsilon$ be any positive number. If we take a sufficiently large number $N_0 = N_0(m, \sigma, a, \varepsilon)$, then, for any integer $N \geq N_0$, there exists some Jordan measurable set $\Theta_0 = \Theta_0(m, \sigma, a, \varepsilon, N) \subset [0, 1)^{\pi(N)}$ with* meas$(\Theta_0) > 0$ *such that*

$$\left| \sum_{p \leq N} \frac{\mathrm{Li}_{m+1}(p^{-\sigma} \exp(-2\pi i \theta_p))}{(\log p)^m} - a \right| < \varepsilon.$$

*for any $\underline{\theta} = (\theta_{p_n})_{n=1}^{\pi(N)} \in \Theta_0$.*

Roughly speaking, Proposition 2.1 means that $\tilde{\eta}_m(\sigma + it)$ "almost" equals the finite sum of polylogarithmic functions when the number of the terms of the sum is sufficiently large, and Proposition 2.2 that any complex number can be approximated by the finite sum of polylogarithmic functions when the number of the terms of the sum is sufficiently large.

**2.2. Proof of Proposition 2.1.** In this section, we prove Proposition 2.1. In order to prove it, we prepare two lemmas.

LEMMA 2.3. *Let $m$ be a positive integer, and $\sigma \geq 1/2$. Let $T$ be large. Then, for $3 \leq X \leq T^{\frac{1}{135}}$, we have*

$$\frac{1}{T}\int_{14}^{T}\left|\tilde{\eta}_m(\sigma+it) - \sum_{2\leq n\leq X}\frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}}\right|^2 dt \ll \frac{X^{1-2\sigma}}{(\log X)^{2m}}.$$

Here, we refer the following theorem to prove this lemma.

LEMMA 2.4. *Let $m$, $k$ be positive integers. Let $T$ be large, and $X \geq 3$ with $X \leq T^{\frac{1}{135k}}$. Then, for $\sigma \geq 1/2$, we have*

$$\int_{14}^{T}\left|\eta_m(\sigma+it) - i^m\sum_{2\leq n\leq X}\frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} - Y_m(\sigma+it)\right|^{2k} dt \tag{3}$$

$$\ll 2^k k!\left(\frac{2m+1}{2m} + \frac{C}{\log X}\right)^k \frac{X^{k(1-2\sigma)}}{(\log X)^{2km}} + C^k k^{2k(m+1)}\frac{T^{\frac{1-2\sigma}{135}}}{(\log T)^{2km}}.$$

This lemma is Theorem 5 in [**18**]. As we mentioned in the previous section, the proof of this lemma requires a strong zero density estimate like Selberg's result. In fact, if we only knew the estimate

$$N(\sigma, T) \ll T^{1-c(\sigma-1/2)}(\log T)^A$$

for some $c > 0$, $A \geq 1$, then the right hand side of (3) in the case $k = 1$ becomes

$$O\left(\frac{X^{1-2\sigma}}{(\log X)^{2m}} + \frac{T^{\frac{1-2\sigma}{135}}}{(\log T)^{2m+1-A}}\right).$$

Hence, the power of the logarithmic factor of the zero density estimate plays an important role in the case $\sigma = 1/2$.

PROOF. By Theorem 5 in [**18**], we have

$$\frac{1}{T}\int_{14}^{T}\left|\eta_m(\sigma+it) - i^m\sum_{2\leq n\leq X}\frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} - Y_m(\sigma+it)\right|^2 dt \ll \frac{X^{1-2\sigma}}{(\log X)^{2m}},$$

where

$$Y_m(\sigma+it) = 2\pi\sum_{k=0}^{m-1}\frac{i^{m-1-k}}{(m-k)!k!}\sum_{\substack{0<\gamma<t \\ \beta>\sigma}}(\beta-\sigma)^{m-k}(t-\gamma)^k. \tag{4}$$

Further, by Lemma 1.6, we see that

$$\eta_m(\sigma+it) - Y_m(\sigma+it) = i^m\tilde{\eta}_m(\sigma+it).$$

Hence we obtain this lemma. □

Lemma 2.5. *Let $m$ be an integer, $\sigma \geq 1/2$. Let $T$ be large. Then for $3 \leq X \leq T^{1/4}$, we have*

$$\frac{1}{T}\int_0^T \left| \sum_{p \leq X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 dt \ll \frac{X^{1-2\sigma}}{(\log X)^{2m+1}},$$

*where the function $\Lambda(n)$ is the von Mangoldt function.*

Proof. By definitions of the polylogarithmic function and the von Mangoldt function, we find that

$$\sum_{p \leq X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} = \sum_{p \leq X} \sum_{k > \frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m}$$
$$= \sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m} + O\left(\frac{X^{1-3\sigma}}{(\log X)^m}\right).$$

Here, we can write

$$\left| \sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m} \right|^2$$
$$= \sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{p^{-2k\sigma}}{k^{2(m+1)}(\log p)^{2m}} +$$
$$+ \sum_{p_1 \leq X} \sum_{p_2 \leq X} \sum_{\substack{\frac{\log X}{\log p_1} < k_1 \leq 3\frac{\log X}{\log p_1} \\ (p_1,k_1) \neq (p_2,k_2)}} \sum_{\frac{\log X}{\log p_2} < k_2 \leq 3\frac{\log X}{\log p_2}} \frac{(p_1^{k_1}p_2^{k_2})^{-\sigma}(p_1^{k_1}/p_2^{k_2})^{-it}}{(k_1k_2)^{m+1}(\log p_1 \log p_2)^m}.$$

Therefore, it holds that

$$\int_0^T \left| \sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m} \right|^2 dt$$
$$= T \sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{p^{-2k\sigma}}{k^{2(m+1)}(\log p)^{2m}} +$$
$$+ O\left( X^3 \left( \sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{1}{p^{k\sigma}k^{m+1}(\log p)^m} \right)^2 \right)$$
$$\ll T\frac{X^{1-2\sigma}}{(\log X)^{2m+1}} + \frac{X^{5-2\sigma}}{(\log X)^{2(m+1)}} \ll T\frac{X^{1-2\sigma}}{(\log X)^{2m+1}}.$$

Hence we have

$$\int_0^T \left| \sum_{p\le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2\le n\le X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 dt$$
$$\ll \int_0^T \left| \sum_{p\le X} \sum_{\frac{\log X}{\log p}<k\le 3\frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m} \right|^2 dt + T\frac{X^{2-6\sigma}}{(\log X)^{2m}} \ll T\frac{X^{1-2\sigma}}{(\log X)^{2m+1}},$$

which completes the proof of this lemma. □

Proof of Proposition 2.1. By Lemma 2.3 and Lemma 2.5, for $X \le T^{1/135}$, we find that

$$\frac{1}{T}\int_{14}^T \left| \tilde{\eta}_m(\sigma+it) - \sum_{p\le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} \right|^2 dt$$
$$\ll \frac{1}{T}\int_{14}^T \left| \tilde{\eta}_m(\sigma+it) - \sum_{2\le n\le X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 dt$$
$$+ \frac{1}{T}\int_{14}^T \left| \sum_{p\le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2\le n\le X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 dt$$
$$\ll \frac{X^{1-2\sigma}}{(\log X)^{2m}}.$$

By using this estimate, for any fixed $\varepsilon > 0$, we have

$$\frac{1}{T}\,\mathrm{meas}\left\{ t\in[0,T]\ ;\ \left| \tilde{\eta}_m(\sigma+it) - \sum_{p\le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} \right| \ge \varepsilon \right\} \ll \frac{X^{1-2\sigma}}{\varepsilon^2(\log X)^{2m}} + \frac{1}{T}.$$

Hence, for any $T \ge X^{135}$, it holds that

$$\frac{1}{T}\,\mathrm{meas}\left\{ t\in[0,T]\ ;\ \left| \tilde{\eta}_m(\sigma+it) - \sum_{p\le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} \right| \ge \varepsilon \right\} \to 0$$

as $X \to +\infty$. Thus, we obtain Proposition 2.1. □

### 2.3. Proof of Proposition 2.2.

In this section, we prove Proposition 2.2 by the method described in [**21**, VIII.3], [**48**]. First of all, we will show the following elementary geometric lemma.

Lemma 2.6. *Let $N$ be a positive integer larger than two. Suppose that the positive numbers $r_1, r_2, \ldots, r_N$ satisfy the condition*

$$r_{n_0} \le \sum_{\substack{n=1 \\ n\ne n_0}}^{N} r_n, \qquad (5)$$

*where* $r_{n_0} = \max\{r_n \ ; \ n=1,2,\ldots,N\}$. *Then we have*

$$\left\{ \sum_{n=1}^{N} r_n \exp(-2\pi i\theta_n) \in \mathbb{C} \ ; \ \theta_n \in [0,1) \right\} = \left\{ z \in \mathbb{C} \ ; \ |z| \le \sum_{n=1}^{N} r_n \right\}. \tag{6}$$

Proof. By Proposition 3.3 in [**7**], it immediately follows that

$$\left\{ \sum_{n=1}^{N} r_n \exp(-2\pi i\theta_n) \in \mathbb{C} \ ; \ \theta_n \in [0,1) \right\}$$

is the closed circle with center origin and radius $\sum_{n=1}^{N} r_n$. Note that their $T_n$ becomes zero under assumption (5). □

Next, we introduce the following definitions.

Definition 2.7. Let $m$ be a positive integer and $\mathcal{M}$ a finite subset of the set of prime numbers. For $\sigma \ge 1/2$ and $\underline{\theta} = (\theta_p)_{p\in\mathcal{M}} \in [0,1)^{\mathcal{M}}$, we define the functions

$$\varphi_{m,\mathcal{M}}(\sigma,\underline{\theta}) := \sum_{p\in\mathcal{M}} \frac{\exp(-2\pi i\theta_p)}{p^{\sigma}(\log p)^m},$$

$$\tilde{\eta}_{m,\mathcal{M}}(\sigma,\underline{\theta}) := \sum_{p\in\mathcal{M}} \frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p))}{(\log p)^m} = \sum_{p\in\mathcal{M}} \sum_{k=1}^{\infty} \frac{\exp(-2\pi ik\theta_p)}{k^{m+1}p^{k\sigma}(\log p)^m}.$$

Definition 2.8. Let $p_n$ be the $n$-th prime number. Put

$$\underline{\theta}^{(0)} = \left(\theta_{p_n}^{(0)}\right)_{n\in\mathbb{N}} = (0,1/2,0,1/2,\ldots) \in [0,1)^{\mathbb{N}},$$

and

$$\gamma_{m,\sigma} = \sum_{p} \sum_{k=1}^{\infty} \frac{\exp(-2\pi ik\theta_p^{(0)})}{k^{m+1}p^{k\sigma}(\log p)^m}.$$

Note that the series for $\gamma_{m,\sigma}$ is convergent for $\sigma \ge 1/2$.

Proof of Proposition 2.2. Fix a complex number $a$ and $1/2 \le \sigma < 1$. Let $U$ be a positive real parameter. We take a sufficiently large number $N = N(U,m,\sigma,a)$ for which the estimates

$$|a-\gamma_{m,\sigma}| \le \sum_{p\in\mathcal{M}} \frac{1}{p^{\sigma}(\log p)^m},$$

$$\frac{1}{p_{\min}^{\sigma}(\log p_{\min})^m} \le \sum_{p\in\mathcal{M}\setminus\{p_{\min}\}} \frac{1}{p^{\sigma}(\log p)^m}$$

are satisfied, where $\mathcal{M} = \mathcal{M}(U,N)$ is defined as $\{p \ ; \ p \text{ prime}, U < p \le N\}$, and $p_{\min}$ is the minimum of $\mathcal{M}$. Note that the existence of such $N$ is guaranteed by $\sum_p \frac{1}{p^{\sigma}(\log p)^m} = \infty$. Then, by Lemma 2.6, the function

$$\varphi_{m,\mathcal{M}}(\sigma,\cdot) \ : \ [0,1)^{\mathcal{M}} \ni \underline{\theta} \longmapsto \varphi_{m,\mathcal{M}}(\sigma,\underline{\theta}) \in \left\{ z \in \mathbb{C} \ ; \ |z| \le \sum_{p\in\mathcal{M}} \frac{1}{p^{\sigma}(\log p)^m} \right\}$$

is surjective. Hence, there exists some $\underline{\theta}^{(1)} = \underline{\theta}(m,\sigma,U,N)^{(1)} = (\theta_p^{(1)})_{p\in\mathcal{M}} \in [0,1)^{\mathcal{M}}$ such that

$$\varphi_{m,\mathcal{M}}(\sigma,\underline{\theta}^{(1)}) = a - \gamma_{m,\sigma}.$$

By using the prime number theorem, we find that

$$\begin{aligned}\tilde{\eta}_{m,\mathcal{M}}(\sigma,\underline{\theta}^{(1)}) &= \varphi_{m,\mathcal{M}}(\sigma,\underline{\theta}^{(1)}) + \sum_{p\in\mathcal{M}}\sum_{k=2}^{\infty}\frac{\exp(-2\pi ik\theta_p^{(1)})}{k^{m+1}p^{k\sigma}(\log p)^m}\\ &= a - \gamma_{m,\sigma} + O\left(\frac{1}{(\log U)^m}\right).\end{aligned}$$

For any prime number $p$, we put

$$\theta_p^{(2)} = \begin{cases}\theta_p^{(0)} & \text{if} \quad p\notin\mathcal{M},\\ \theta_p^{(1)} & \text{if} \quad p\in\mathcal{M}.\end{cases}$$

Then it holds that

$$\begin{aligned}&\sum_{p\le N}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(2)}))}{(\log p)^m}\\ =&\sum_{p\in\mathcal{M}}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(1)}))}{(\log p)^m} + \sum_{p\le U}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(0)}))}{(\log p)^m}\\ =&\tilde{\eta}_{m,\mathcal{M}}(\sigma,\underline{\theta}^{(1)}) + \gamma_{m,\sigma} + \sum_{p>U}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(0)}))}{(\log p)^m},\end{aligned}$$

and additionally, by using the prime number theorem and simple calculations of alternating series,

$$\begin{aligned}\sum_{p>U}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(0)}))}{(\log p)^m} &= \sum_{p>U}\frac{\exp(-2\pi i\theta_p^{(0)}))}{p^{\sigma}(\log p)^m} + O\left(\sum_{p>U}\frac{1}{p^{2\sigma}(\log p)^m}\right)\\ &\ll \frac{1}{(\log U)^m}.\end{aligned}$$

Hence, by taking a sufficiently large $U = U(\varepsilon)$ and noting the continuity of the function $\sum_{p\le N}\frac{\mathrm{Li}_{m+1}(p^{\sigma}\exp(-2\pi i\theta_p))}{(\log p)^m}$ with respect to $(\theta_p)_{p\le N}\in[0,1)^{\pi(N)}$, we obtain this proposition. □

**2.4. Proof of Theorem 1.7.** In this section, we prove Theorem 1.7. Here, we use the following lemma related with Kronecker's approximation theorem.

LEMMA 2.9. *Let $A$ be a Jordan measurable subregion of $[0,1)^N$, and $a_1,\dots,a_N$ be real numbers linearly independent over $\mathbb{Q}$. Set, for any $T>0$,*

$$I(T,A) = \{t\in[0,T]\ ;\ (\{a_1t\},\dots,\{a_Nt\})\in A\}.$$

*Then we have*

$$\lim_{T\to+\infty}\frac{\mathrm{meas}(I(T,A))}{T} = \mathrm{meas}(A).$$

PROOF. This lemma is Theorem 1 of Appendix 8 in [**21**] □

Let us start the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $\varepsilon > 0$ be any small number, $a$ any fixed complex number, $\frac{1}{2} \le \sigma < 1$, and let $T_0$ be any positive number. Define $S_M(\theta_1, \dots, \theta_M; \sigma, m)$ and $S_{M,N}(\theta_{M+1}, \dots, \theta_N; \sigma, m)$ by

$$S_M(\theta_1, \dots, \theta_M; \sigma, m) = \sum_{n \le M} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma} e^{-2\pi i \theta_n})}{(\log p_n)^m},$$

$$S_{M,N}(\theta_{M+1}, \dots, \theta_N; \sigma, m) = \sum_{M < n \le N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma} e^{-2\pi i \theta_n})}{(\log p_n)^m}.$$

Then, by Proposition 2.2, we can take a sufficiently large $M_0 = M_0(m, \sigma, a, \varepsilon)$ so that for any $M \ge M_0$, there exists some Jordan measurable subset $\Theta_1^{(M)} = \Theta_1^{(M)}(m, \sigma, a, \varepsilon, M)$ of $[0,1)^M$ such that $\delta_M := \mathrm{meas}(\Theta_1^{(M)}) > 0$ and

$$|S_M(\theta_1, \dots, \theta_M; \sigma, m) - a| < \varepsilon$$

for any $(\theta_1, \dots, \theta_M) \in \Theta_1^{(M)}$. We also find that

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 |S_{M,N}(\theta_{M+1}, \dots, \theta_N; \sigma, m)|^2 d\theta_{M+1} \cdots d\theta_N \\
&= \int_0^1 \cdots \int_0^1 \left| \sum_{M < n \le N} \sum_{k=1}^{\infty} \frac{p_n^{-\sigma k} e^{-2\pi i k \theta_n}}{k^{m+1} (\log p_n)^m} \right|^2 d\theta_{M+1} \cdots d\theta_N \\
&= \sum_{M < n_1 \le N} \sum_{M < n_2 \le N} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left\{ \frac{(p_{n_1} p_{n_2})^{-\sigma k}}{(k_1 k_2)^{m+1} (\log p_{n_1} \log p_{n_2})^m} \times \right. \\
&\qquad\qquad \left. \times \int_0^1 \cdots \int_0^1 e^{-2\pi i (k_1 \theta_{n_1} - k_2 \theta_{n_2})} d\theta_{M+1} \cdots d\theta_N \right\} \\
&= \sum_{M < n \le N} \sum_{k=1}^{\infty} \frac{1}{k^{2(m+1)} p_n^{2\sigma k} (\log p_n)^{2m}} \ll \sum_{M < n \le N} \frac{1}{p_n (\log p_n)^{2m}}.
\end{aligned}$$

Note that the last sum tends to zero as $M \to +\infty$. Therefore, there exists some large number $M_1 = M_1(m, \varepsilon)$ such that, for any $N > M \ge M_1$, it holds that

$$\mathrm{meas}\left( \{ (\theta_{M+1}, \dots, \theta_N) \in [0,1)^{N-M} \ ; \ |S_{M,N}(\theta_{M+1}, \dots, \theta_N; \sigma, m)| < \varepsilon \} \right) > \frac{1}{2}.$$

Here we denote the set of the content of $\mathrm{meas}(\cdot)$ in the above inequality by $\Theta_2^{(M,N)} = \Theta_2^{(M,N)}(M, N, \varepsilon)$.

We put $M_2 = \max\{M_0, M_1\}$ and $\Theta_3 = \Theta_1^{(M_2)} \times \Theta_2^{(M_2,N)}$ for any $N > M_2$. Then $\Theta_3$ is a subset of $[0,1)^N$ satisfying $\mathrm{meas}(\Theta_3) > \delta_{M_2}/2$. Hence, putting

$$\mathcal{I}(T) = \left\{ t \in [T_0, T] \ ; \ \left( \left\{ \frac{t}{2\pi} \log p_1 \right\}, \dots, \left\{ \frac{t}{2\pi} \log p_N \right\} \right) \in \Theta_3 \right\}$$

and applying Lemma 2.9, for any positive integer $N > M_2$, there exists some large number $T_N > T_0$ such that $\mathrm{meas}(\mathcal{I}(T)) > \delta_{M_2} T/2$ holds for any $T \geq T_N$. On the other hand, by Proposition 2.1, there exists a large number $N_0 = N_0(\varepsilon, \delta_{M_2})$ such that

$$\mathrm{meas}\left\{ t \in [T_0, T] \ ; \ \left| \tilde{\eta}_m(\sigma + it) - \sum_{n \leq N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma - it})}{(\log p_n)^m} \right| < \varepsilon \right\} > (1 - \delta_{M_2}/4)T$$

for any $N \geq N_0$, $T \geq p_N^{135}$.

Therefore, for any $N \geq \max\{N_0, M_2 + 1\}$, $T \geq \max\{T_N, p_N^{135}\}$, there exists some $t_0 \in [T_0, T]$ such that

$$\left( \left\{ \frac{t_0}{2\pi} \log p_1 \right\}, \ldots, \left\{ \frac{t_0}{2\pi} \log p_N \right\} \right) \in \Theta_3,$$

and

$$\left| \tilde{\eta}_m(\sigma + it_0) - \sum_{n \leq N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma - it_0})}{(\log p_n)^m} \right| < \varepsilon.$$

Then we have

$$\begin{aligned} &|\tilde{\eta}_m(\sigma + it_0) - a| \\ &\leq \left| \tilde{\eta}_m(\sigma + it_0) - \sum_{n \leq N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma} e^{-it_0 \log p_n})}{(\log p_n)^m} \right| + \left| \sum_{n \leq M_2} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma} e^{-it_0 \log p_n})}{(\log p_n)^m} - a \right| \\ &\quad + \left| \sum_{M_2 < n \leq N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma} e^{-it_0 \log p_n})}{(\log p_n)^m} \right| < 3\varepsilon. \end{aligned}$$

This completes the proof of Theorem 1.7. □

**2.5. Proof of Theorem 1.5.** In this section, we prove Theorem 1.5. Here, we prepare the following lemma.

LEMMA 2.10. *Let $\sigma \geq 1/2$ and $m$ be a positive integer. Then we have*

$$\eta_m(s) = Y_m(s) + O_m(\log t),$$

*where $Y_m$ is defined by* (4).

PROOF. This lemma is equation (2.2) in [**18**]. □

PROOF OF THEOREM 1.5. First, we show Theorem 1.5 in the case $m = 1$. If the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta > \sigma$ is finite, then there exists a sufficiently large $T_0$ such that $Y_1(\sigma + it) \equiv b$ for $t \geq T_0$, where $b$ is a positive real number. Therefore, by Lemma 1.6, we have

$$\int_0^t \log \zeta(\sigma + it') dt' = i\tilde{\eta}_1(\sigma + it) + b$$

for any $t \geq T_0$. By this formula, we obtain

$$\left\{ \int_0^t \log \zeta(\sigma + it')dt' \ ; \ t \in [0, \infty) \right\} \supset \left\{ \int_0^t \log \zeta(\sigma + it')dt' \ ; \ t \in [T_0, \infty) \right\}$$
$$= \{ i\tilde{\eta}_1(\sigma + it) + b \ ; \ t \in [T_0, \infty) \} \, .$$

If a set $A \subset \mathbb{C}$ is dense in $\mathbb{C}$, then for any $c_1 \in \mathbb{C} \setminus \{0\}$ and $c_2 \in \mathbb{C}$, the set $\{c_1 a + c_2 \ ; \ a \in A\}$ is also dense in $\mathbb{C}$. By this fact and Theorem 1.7, the set $\{i\tilde{\eta}_1(\sigma + it) + b \ ; \ t \in [T_0, \infty)\}$ is dense in $\mathbb{C}$. Thus, the set $\left\{ \int_0^t \log \zeta(\sigma + it')dt' \ ; \ t \in [0, \infty) \right\}$ is dense in $\mathbb{C}$ under this assumption.

Next, for $m \in \mathbb{Z}_{\geq 2}$, we show the equivalence of (I) and (II). The implication (I) $\Rightarrow$ (II) is clear since the equation $\eta_m(\sigma + it) = i^m \tilde{\eta}_m(\sigma + it)$ holds by assuming (I).

In the following, we show the inverse implication (II) $\Rightarrow$ (I). By Lemma 2.10, if (I) is false, then the estimate $|\eta_m(\sigma + it)| \gg_m t^{m-1}$ holds. Therefore, for some $T_2 > 0$, we have

$$\overline{\{\eta_m(\sigma + it) \ ; \ t \in [T_2, \infty)\}} \subset \mathbb{C} \setminus \{z \mid |z| \leq 1\} \, .$$

Here, $\overline{A}$ means the closure of the set $A$. Since $\{\eta_m(\sigma + it) \ ; \ t \in [0, T_2]\}$ is a piecewise smooth curve of finite length, $\mu\left(\overline{\{\eta_m(\sigma + it) \ ; \ t \in [0, T_2]\}}\right) = 0$. Here $\mu$ is the Lebesgue measure in $\mathbb{C}$. Therefore, we obtain

$$\{z \ ; \ |z| \leq 1\} \not\subset \overline{\{\eta_m(\sigma + it) \ ; \ t \in [0, T_2]\}}.$$

Hence, if (I) is false, then the set $\{\eta_m(\sigma + it) \ ; \ t \in [0, \infty)\}$ is not dense in $\mathbb{C}$. Thus, we obtain the implication (II) $\Rightarrow$ (I). $\square$

## 3. Proof of main results in 1.2.2

**3.1. Preliminaries.** In this section, let $\mathcal{L}(s)$ be an element of the Selberg class $\mathcal{S}$ which is represented by

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right) \quad \text{for} \quad \sigma > 1.$$

3.1.1. *Definitions and Notations.* We use the following definitions and notations.

- Let $\mathbb{N}_0$ denote the set of nonnegative integers.
- Let $\mathcal{P}$ denote the set of all prime numbers.
- For a set $A$, let $\mathbb{R}^A$ denote the family of elements in $\mathbb{R}$ indexed by $A$.
- For any $Q > 0$, let $\mathcal{P}(Q)$ denote the set of prime numbers smaller than or equal to $Q$.
- We define the generalized von Mangoldt function $\Lambda_{\mathcal{L}}(n)$ by

$$-\frac{\mathcal{L}'}{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \Lambda_{\mathcal{L}}(n) n^{-s}$$

  for $\sigma > 1$, that is, $\Lambda_{\mathcal{L}}(n) = b(n) \log n$.
- Let $\mathcal{M}$ be a finite subset of $\mathcal{P}$ and $s = \sigma + it$ be a complex number with $\sigma > 1/2$. For any $\mathcal{M} \subset \mathcal{N} \subset \mathcal{P}$ and $\underline{\theta} = (\theta_p)_{p\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}$, we denote $\varphi_{\mathcal{M}}(s, \underline{\theta})$ and $\log \mathcal{L}_{\mathcal{M}}(s, \underline{\theta})$ by

$$\varphi_{\mathcal{M}}(s, \underline{\theta}) = \sum_{p\in\mathcal{M}} \frac{b(p)\exp(-2\pi i\theta_p)}{p^s}$$

  and

$$\log \mathcal{L}_{\mathcal{M}}(s, \underline{\theta}) = \sum_{p\in\mathcal{M}} \sum_{l=1}^{\infty} \frac{b(p^l)\exp(-2\pi i l\theta_p)}{p^{ls}}.$$

  Note that the series $\log \mathcal{L}_{\mathcal{M}}(s, \underline{\theta})$ converges absolutely by the estimate $b(p^l) \ll p^{l\vartheta}$ with some $\vartheta < 1/2$ coming from the axiom (iv) of the Selberg class. If $\mathcal{M} = \{p\}$, we abbreviate $\varphi_{\{p\}}(s, \underline{\theta})$ and $\log \mathcal{L}_{\{p\}}(s, \underline{\theta})$ to $\varphi_p(s, \underline{\theta})$ and $\log \mathcal{L}_p(s, \underline{\theta})$ respectively, and if $\underline{\theta} = (0)_{p\in\mathcal{N}}$, we do $\log \mathcal{L}_{\mathcal{M}}(s, (0)_{p\in\mathcal{N}})$ to $\log \mathcal{L}_{\mathcal{M}}(s)$.

**3.2. Some known results.** In this subsection, we summarize the results not coming from analytic number theory. The followings are used in [**21**] and [**48**].

3.2.1. *A certain estimate coming from the Vandermonde matrix.*

Lemma 3.1. *Let $X > e$, $N \in \mathbb{N}$ and $\underline{a} = (a_0, a_1, \dots, a_{N-1}) \in \mathbb{C}^N$. Put $X_j = 2^j X$ for $j = 0, 1, \dots, N-1$. Then the system of linear equations in the unknown $\underline{z} = (z_0, z_1, \dots, z_{N-1}) \in \mathbb{C}^N$;*

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ -\log X_0 & -\log X_1 & \cdots & -\log X_{N-1} \\ (-\log X_0)^2 & (-\log X_1)^2 & \cdots & (-\log X_{N-1})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (-\log X_0)^{N-1} & (-\log X_1)^{N-1} & \cdots & (-\log X_{N-1})^{N-1} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{N-1} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \end{pmatrix}$$

*has a unique solution* $\underline{z}_0 = \underline{z}_0(X, \underline{a}) \in \mathbb{C}^N$ *and the estimate*

$$\|\underline{z}_0\| \ll_N (\log X)^{N-1} \|\underline{a}\|$$

*holds.*

This lemma is used in [**21**] and [**48**], however the proof was written very roughly in these papers. For this reason, we will give a proof in this thesis.

Proof. Fix $\underline{a} = (a_0, a_1, \ldots, a_{N-1}) \in \mathbb{C}^N$. Let $\mathbf{U} = (U_0, U_1, \ldots, U_{N-1})$ be indeterminate and $\underline{z} = (z_0, z_1, \ldots, z_{N-1})$ variable in $\mathbb{C}(\mathbf{U})^N$, where $\mathbb{C}(\mathbf{U})$ denotes the field of rational functions. We first consider the following system of linear equations;

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ U_0 & U_1 & \ldots & U_{N-1} \\ U_0^2 & U_1^2 & \ldots & U_{N-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ U_0^{N-1} & U_1^{N-1} & \cdots & U_{N-1}^{N-1} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{N-1} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \end{pmatrix}. \tag{7}$$

Let $B(\mathbf{U})$ denote the first matrix in the left hand side of (7). Since $B(\mathbf{U})$ is the Vandermonde matrix, we have

$$\det(B(\mathbf{U})) = \prod_{0 \le l < k \le N-1} (U_k - U_l) \neq 0 \quad \text{in } \mathbb{C}[\mathbf{U}]. \tag{8}$$

Hence the system of linear equations (7) has a solution $\underline{z} = \underline{z}(\mathbf{U}, \underline{a}) \in \mathbb{C}(\mathbf{U})^N$ such that

$${}^t\underline{z} = \frac{1}{\det(B(\mathbf{U}))} \widetilde{B}(\mathbf{U}) \cdot {}^t\underline{a},$$

where $\widetilde{B}(\mathbf{U}) = {}^t(\widetilde{b}_{i,j}(\mathbf{U}))_{0 \le i,j \le N-1}$ denotes the adjugate matrix of $B(\mathbf{U})$. Here the symbol ${}^tD$ stands for the transposed matrix of a matrix $D$. We fix $0 \le i, j \le N-1$ and put

$$\Delta_j(\mathbf{U}) = \prod_{\substack{0 \le l < k \le N-1; \\ l,k \neq j}} (U_k - U_l).$$

Then we find that $\Delta_j(\mathbf{U})$ divides $\widetilde{b}_{i,j}(\mathbf{U})$, and hence there exists $f_{i,j}(\mathbf{U}) \in \mathbb{C}[\mathbf{U}]$ such that $\widetilde{b}_{i,j}(\mathbf{U}) = f_{i,j}(\mathbf{U})\Delta_j(\mathbf{U})$. By the definition of the determinant, we have

$$\deg\left(\widetilde{b}_{i,j}(\mathbf{U})\right) = \sum_{\beta=0}^{N-1} \beta - i.$$

Here the statement $\deg(g(\mathbf{U})) = n$ means that

$$\max\left\{i_1 + \cdots + i_{N-1} \ ; \ c_{i_1,\ldots,i_{N-1}} \neq 0\right\} = n$$

for $g(\mathbf{U}) = \sum_{i_0 \ldots, i_{N-1}} c_{i_0,\ldots,i_{N-1}} U_0^{i_0} \cdots U_{N-1}^{i_{N-1}} \in \mathbb{C}[\mathbf{U}]$. On the other hand, we find that

$$\begin{aligned} \deg(\Delta_j(\mathbf{U})) &= \#\left\{(l,k) \in \{0,1,\ldots,N-1\} \ ; \ l < k, l, k \neq j\right\} \\ &= \sum_{\beta=0}^{N-1} \beta - (N-1). \end{aligned}$$

Therefore we have

$$\begin{aligned}\deg\left(f_{i,j}(\mathbf{U})\right) &= \deg\left(\widetilde{b}_{i,j}(\mathbf{U})\right) - \deg\left(\Delta_j(\mathbf{U})\right)\\ &= \left(\sum_{\beta=0}^{N-1}\beta - i\right) - \left(\sum_{\beta=0}^{N-1}\beta - (N-1)\right)\\ &= N-1-i \leq N-1.\end{aligned}$$

By substituting $\mathbf{U}$ into $\mathbf{X} = (-\log X_0, -\log X_1, \ldots, -\log X_{N-1})$ and letting $\underline{z}_0 = \underline{z}_0(X, \underline{a})$ be a solution of the system of linear equations (7) in this case, the estimates

$$\left|\widetilde{b}_{i,j}(\mathbf{X})\right| = |\Delta_j(\mathbf{X})|\,|f_{i,j}(\mathbf{X})| \ll_N (\log X)^{N-1} \quad \text{and} \quad \det\left(B(\mathbf{X})\right) \asymp_N 1$$

hold by the equation (8) and the definition of $\Delta_j(\mathbf{X})$. Consequently, we obtain

$$\|\underline{z}_0\| \ll_N \left(\sum_{0\leq i,j\leq N-1}\left|\widetilde{b}_{i,j}(\mathbf{X})\right|\right)\|\underline{a}\| \ll_N (\log X)^{N-1}\|\underline{a}\|.$$

This completes the proof. □

3.2.2. *Elementary geometric lemma.*

LEMMA 3.2. *Let $N$ be a positive integer larger than two. Suppose that the positive numbers $r_1 \leq r_2 \leq \cdots \leq r_N$ satisfy $r_N \leq \sum_{n=1}^{N-1} r_n$. Then we have*

$$\left\{\sum_{n=1}^{N} r_n \exp(-2\pi i\theta_n) \in \mathbb{C}\ ;\ \theta_n \in [0,1)\right\} = \left\{z \in \mathbb{C}\ ;\ |z| \leq \sum_{n=1}^{N} r_n\right\}.$$

PROOF. The proof is written in [**7**] roughly and in [**44**] precisely. □

3.2.3. *The mollifier and the estimate for its Fourier series expansion.* Let $Q, M > 2$ and put $\delta = \delta_Q = Q^{-1} \in (0,1)$. We will prepare a mollifier on $\mathbb{R}^{\mathcal{P}(Q)}$ and its truncated formula as follows; We take $\varphi \in C^\infty(\mathbb{R})$ satisfying

$$\varphi(x) \geq 0, \quad \operatorname{supp}(\varphi) \subset [-1,1], \quad \int_{-\infty}^{\infty}\varphi(x)dx = 1.$$

Throughout this subsection, let the implicit constants depend on $\varphi$. We define the function $\varphi_\delta : [-1/2, 1/2] \to \mathbb{R}$ by

$$\varphi_\delta(\theta) = \frac{1}{\delta}\varphi\left(\frac{\theta}{\delta}\right), \quad \theta \in [-1/2, 1/2]$$

and extend $\varphi_\delta$ onto $\mathbb{R}$ by periodicity with period 1. We also define $\Phi_Q(\underline{\theta}) : \mathbb{R}^{\mathcal{P}(Q)} \to \mathbb{R}$ by

$$\Phi_Q(\underline{\theta}) = \prod_{p\leq Q}\varphi_\delta(\theta_p), \quad \underline{\theta} = (\theta_p)_{p\in\mathcal{P}(Q)} \in \mathbb{R}^{\mathcal{P}(Q)}.$$

Note that

$$\Phi_Q(\underline{\theta}) \neq 0 \quad \text{implies} \quad \underline{\theta} \in [-\delta,\delta]^{\mathcal{P}(Q)} + \mathbb{Z}^{\mathcal{P}(Q)}. \tag{9}$$

For any $\theta_0 \in \mathbb{R}$, the function $\varphi_\delta(\theta-\theta_0)$, $\theta \in \mathbb{R}$ can be represented as a Fourier series

$$\varphi_\delta(\theta-\theta_0) = \sum_{n\in\mathbb{Z}} \alpha_n(\theta_0)\exp(2\pi i n\theta)$$

where the Fourier coefficients are given by

$$\alpha_n(\theta_0) = \int_{-1/2}^{1/2} \varphi_\delta(\theta-\theta_0)\exp(-2\pi i n\theta)d\theta.$$

By integration by parts, we have

$$\alpha_0(\theta_0) = 1 \quad \text{and} \quad |\alpha_n(\theta_0)| \le \min\{1, C_\varphi/\delta^2 n^2\} \tag{10}$$

with some positive constant $C_\varphi$ depending on $\varphi$. Then we have

$$\begin{aligned}\Phi_Q(\underline{\theta}-\underline{\theta}^{(0)}) &= \sum_{\underline{n}\in\mathbb{Z}^{\mathcal{P}(Q)}} \beta_{\underline{n}}(\underline{\theta}^{(0)})\exp\left(2\pi i\langle \underline{n},\underline{\theta}\rangle\right) \\ &= \sum_{\substack{\underline{n}=(n_p)_{p\in\mathcal{P}(Q)}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{p\le Q}|n_p|\le M}} \beta_{\underline{n}}(\underline{\theta}^{(0)})\exp\left(2\pi i\langle \underline{n},\underline{\theta}\rangle\right) + O\left(\sum_{\substack{\underline{n}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{p\le Q}|n_p|>M}} \left|\beta_{\underline{n}}(\underline{\theta}^{(0)})\right|\right)\end{aligned}$$

for $\underline{\theta} = (\theta_p)_{p\in\mathcal{P}(Q)} \in \mathbb{R}^{\mathcal{P}(Q)}$, where

$$\langle \underline{n},\underline{\theta}\rangle = \sum_{p\le Q} n_p\theta_p \quad \text{and} \quad \beta_{\underline{n}}(\underline{\theta}^{(0)}) = \prod_{p\le Q}\alpha_{n_p}(\theta_p^{(0)}).$$

Note that the estimates

$$\beta_{\mathbf{0}}(\underline{\theta}^{(0)}) = 1 \quad \text{and} \quad |\beta_{\underline{n}}(\underline{\theta}^{(0)})| \le \prod_{p\le Q}\min\{1, C_\varphi/\delta^2 n_p^2\}$$

hold by the estimate (10). By using the prime number theorem and by noting

$$\left\{\underline{n} = (n_p)_{p\in\mathcal{P}(Q)} \in \mathbb{Z}^{\mathcal{P}(Q)}\ ;\ \max_{p\le Q}|n_p| > M\right\} = \bigcup_{q\in\mathcal{P}(Q)}\left\{\underline{n}\in\mathbb{Z}^{\mathcal{P}(Q)}\ ;\ |n_q| > M\right\},$$

we have

$$\sum_{\underline{n}\in\mathbb{Z}^{\mathcal{P}(Q)}} |\beta_{\underline{n}}(\underline{\theta}^{(0)})| \le \left(\sum_{n\in\mathbb{Z}}\min\left\{1, \frac{C_\varphi}{\delta^2 n^2}\right\}\right)^{\pi(Q)} \ll \exp(C_0 Q) \tag{11}$$

and

$$\begin{aligned}&\sum_{\substack{\underline{n}=(n_p)_{p\in\mathcal{P}(Q)}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{p\le Q}|n_p|>M}} \left|\beta_{\underline{n}}(\underline{\theta}^{(0)})\right| \\ \le&\pi(Q)\left(\sum_{\substack{n\in\mathbb{Z};\\ |n|>M}}\frac{1}{\delta^2 n^2}\right)\left(\sum_{n\in\mathbb{Z}}\min\left\{1, \frac{1}{\delta^2 n^2}\right\}\right)^{\pi(Q)-1} \ll \frac{1}{M}\exp(C_1 Q)\end{aligned}$$

for some positive constant $C_0, C_1$ depending on $\varphi$. Hence we have

$$\Phi_Q(\underline{\theta} - \underline{\theta}^{(0)}) = \sum_{\substack{\underline{n}=(n_p)_{p\in\mathcal{P}(Q)}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{p\le Q}|n_p|\le M}} \beta_{\underline{n}}(\underline{\theta}^{(0)}) \exp\left(2\pi i\langle \underline{n}, \underline{\theta}\rangle\right) + O\left(\frac{1}{M}\exp(C_1 Q)\right). \tag{12}$$

3.2.4. *Known results for the Selberg class $\mathcal{S}$.*

LEMMA 3.3. *The following statements hold.*

(i) *We have $a(p) = b(p)$ for all primes $p$.*

(ii) *For any $\varepsilon > 0$, we have the inequality*

$$|b(p^l)| \ll_\varepsilon (2^l - 1)p^{l\varepsilon}/l$$

*for all primes $p$ and all $l \in \mathbb{N}$, where the implicit constant may depend on $\mathcal{L}(s)$.*

PROOF. The proof can be found in [**33**, Exercise 8.2.9]. □

LEMMA 3.4. *Let $q_j(r) = (\mu_j + r)/\lambda_j$ for $j = 1, \dots f$ and $r \in \mathbb{N}_0$. If $x > 1$ and $s \neq 0, 1, -q_j(r), \rho$ for $j = 1, \dots f$ and $r \in \mathbb{N}_0$, then we have*

$$\begin{aligned}\frac{\mathcal{L}'}{\mathcal{L}}(s) = &-\sum_{n\le x^2}\frac{\Lambda_{\mathcal{L},x}(n)}{n^s} + \frac{1}{\log x}\sum_{j=1}^{f}\sum_{r=0}^{\infty}\frac{x^{-q_j(r)-s} - x^{-2(q_j(r)+s)}}{(q_j(r)+s)^2}\\ &+ m_{\mathcal{L}}\frac{x^{-2s} - x^{-s}}{s^2\log x} + m_{\mathcal{L}}\frac{x^{2(1-s)} - x^{1-s}}{(1-s)^2\log x} + \frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s} - x^{2(\rho-s)}}{(\rho-s)^2},\end{aligned} \tag{13}$$

*where $\Lambda_{\mathcal{L},x}(n)$ is defined by*

$$\Lambda_{\mathcal{L},x}(n) = \Lambda_{\mathcal{L}}(n) \quad \text{for } 1 \le n \le x, \quad \Lambda_{\mathcal{L}}(n)\frac{\log(x^2/n)}{\log x} \quad \text{for } x \le n \le x^2,$$

*and $m_{\mathcal{L}}$ is defined by*

$$m_{\mathcal{L}} = \begin{cases}\text{the order of pole of } \mathcal{L}(s) \text{ at } s = 1 & \text{if } \mathcal{L}(s) \text{ has a pole at } s = 1,\\ 0 & \text{if } \mathcal{L}(s) \text{ has no zeros or poles at } s = 1,\\ (-1)\times(\text{the order of zero of } \mathcal{L}(s) \text{ at } s = 1) & \text{if } \mathcal{L}(1) = 0.\end{cases}$$

PROOF. This follows by the same argument as in [**45**, Theorem 14.20]. □

### 3.3. Proofs.

3.3.1. *Proof of Theorem 1.19.* We fix $\max\{\sigma_{\mathcal{L}}, 1 - 2E_{\mathcal{L}}\} < \sigma_0 < 1$ and $N \in \mathbb{N}$, $\underline{c} = (c_k)_{k=0}^{N-1} \in \mathbb{C}^N$ and take $\varepsilon \in (0,1)$. We begin with the following lemma.

LEMMA 3.5. *There exist $d_1(\sigma_0, E_{\mathcal{L}}) > 0$ and $C(\mathcal{L}, \sigma_0, N) > 0$ such that if*

$$Q > C(\mathcal{L}, \sigma_0, N)\left(\|\underline{c}\| + 1/\varepsilon\right)^{d_1(\sigma_0, E_{\mathcal{L}})},$$

*then there exist $\underline{\theta}^{(\star)} = \underline{\theta}^{(\star)}(Q) = (\theta_p^{(\star)})_{p\in\mathcal{P}(Q)} \in \mathbb{R}^{\mathcal{P}(Q)}$ such that*

$$\left|\frac{d^k}{ds^k}\log\mathcal{L}_{\mathcal{P}(Q)}(\sigma_0, \underline{\theta}^{(\star)}) - c_k\right| < \frac{\varepsilon}{3} \quad \text{for } k = 0, 1, \dots, N-1.$$

*Here $(d^k/ds^k)F(\sigma)$ means $(d^k/ds^k)F(s)|_{s=\sigma}$ for holomorphic $F(s)$.*

PROOF OF LEMMA 3.5. We divide the proof into several steps.

STEP 1. We will show the following claim and give a certain convergent series.

CLAIM 3.6. *There exists $\underline{\theta}^{(0)} = (\theta^{(0)}_{p_n})_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ such that the estimate*

$$\left| \sum_{p \le \xi} b(p) \exp(-2\pi i \theta^{(0)}_p) \right| \le C_{\mathcal{L},\eta} \xi^{\eta} \tag{14}$$

*holds for any $\xi > 0$ when the estimate $|b(p)| = |a(p)| \le C_{\mathcal{L},\eta} p^{\eta}$ holds for any prime number $p$ with some positive number $\eta$.*

PROOF OF CLAIM 3.6. We put $\mathcal{P}_0 = \{p \in \mathcal{P} \ ; \ b(p) \neq 0\}$ and $\{p_n\}_{n=1}^{\infty}$ which satisfy $\{p_n\}_{n=1}^{\infty} = \mathcal{P}_0$ and $p_n < p_{n+1}$ for any $n \in \mathbb{N}$. For any $n \ge 1$, put

$$b(p_n) = |b(p_n)| \exp(2\pi i \theta^{(\mathcal{L})}_{p_n}),$$

and we take $\underline{\theta}^{(0)} = (\theta^{(0)}_{p_n})_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ so that

$$\theta^{(0)}_{p_1} = \theta^{(\mathcal{L})}_{p_1}, \quad \theta^{(0)}_{p_2} = 1/2 + \theta^{(\mathcal{L})}_{p_2},$$

$$\theta^{(0)}_{p_3} = \begin{cases} \theta^{(\mathcal{L})}_{p_3} & \text{if} \quad \displaystyle\sum_{j=1}^{2} b(p_j) \exp(-2\pi i \theta^{(0)}_{p_j}) \le 0, \\ 1/2 + \theta^{(\mathcal{L})}_{p_3} & \text{if} \quad \displaystyle\sum_{j=1}^{2} b(p_j) \exp(-2\pi i \theta^{(0)}_{p_j}) > 0, \end{cases}$$

$$\vdots$$

$$\theta^{(0)}_{p_l} = \begin{cases} \theta^{(\mathcal{L})}_{p_l} & \text{if} \quad \displaystyle\sum_{j=1}^{l-1} b(p_j) \exp(-2\pi i \theta^{(0)}_{p_j}) \le 0, \\ 1/2 + \theta^{(\mathcal{L})}_{p_l} & \text{if} \quad \displaystyle\sum_{j=1}^{l-1} b(p_j) \exp(-2\pi i \theta^{(0)}_{p_j}) > 0. \end{cases}$$

By the construction of $\underline{\theta}^{(0)} \in \mathbb{R}^{\mathbb{N}}$, we have the estimate (14) for any $\xi > 0$ when the estimate $|b(p)| = |a(p)| \le C_{\mathcal{L},\eta} p^{\eta}$ holds for any prime number $p$ with some positive number $\eta$. Taking $\theta_p = 0$ for $p \in \mathcal{P} \setminus \mathcal{P}_0$, we have the conclusion. □

We put

$$\gamma_k = \sum_{p} \sum_{l=1}^{\infty} \frac{(-\log p^l)^k b(p^l) \exp(-2\pi i l \theta^{(0)}_p)}{p^{l\sigma_0}}$$

for any $k = 0, \ldots, N-1$ and $\underline{\gamma} = (\gamma_k)_{k=0}^{N-1}$. Since it holds that

$$\sum_{p} \sum_{l=2}^{\infty} \frac{|b(p^l)| (\log p^l)^k}{p^{l\sigma_0}} < \infty$$

by an argument similar to (2.13) in [**34**], we find that the series $\gamma_k$ is convergent by partial summation.

Step 2. We will use the positive density method introduced by Laurinčikas and Matsumoto [**29**]. The following idea is due to [**34**]. Put $\mu = \sqrt{\kappa/8}$ and $\rho = \kappa/4$. Then we have

$$\kappa - \rho > 0 \quad \text{and} \quad \kappa - 2\mu^2 - \rho > 0.$$

We first take a positive number $\eta$ so that

$$0 < \eta < 1/2(1 - E_{\mathcal{L}}), \tag{15}$$

which is chosen more precisely later (see (32)). Then there exists $C_{\mathcal{L},\eta} > 0$ such that $|a(p)| \leq C_{\mathcal{L},\eta} p^{\eta}$ for any prime number $p$ by the axiom (i) of the Selberg class $\mathcal{S}$. Let $U$ be a large positive parameter depending on $\mathcal{L}, \sigma_0, N, \eta$, and let $U^{1-E_{\mathcal{L}}}(\log U)^{D+1} \leq H \leq U$. We put

$$\mathcal{M}_{\mu,j}^{(U,H)} = \{p \in \mathcal{P} \ ; \ 2^j U < p \leq 2^j U + H, |a(p)| > \mu\}$$

for $j = 0, 1, \dots N-1$.

Claim 3.7. *We have*

$$\#(\mathcal{M}_{\mu,j}^{(U,H)}) \gg_{\mathcal{L},N,\eta} \frac{H}{U^{2\eta} \log U}, \tag{16}$$

*where $\#(A)$ denotes the cardinality of the set $A$.*

Proof of claim 3.7. Put $\pi_\mu(x) = \#\{p \in \mathcal{P} \ ; \ p \leq x, |a(p)| > \mu\}$. When $\alpha > \beta \geq 1$, it holds that

$$\sum_{\alpha < p \leq \beta} |a(p)|^2 \leq (C_{\mathcal{L},\eta}^2 \beta^{2\eta} - \mu^2)(\pi_\mu(\beta) - \pi_\mu(\alpha)) + \mu^2(\pi(\beta) - \pi(\alpha)) \tag{17}$$

by (2.26) in [**34**]. By the condition (C3), it holds that

$$\pi(2^j U + H) - \pi(2^j U) \leq 2\frac{H}{\log U} \tag{18}$$

and

$$\sum_{2^j U < p \leq 2^j U + H} |a(p)|^2 \geq (\kappa - \rho)\frac{H}{\log U}. \tag{19}$$

Substituting $\alpha = 2^j U$ , $\beta = 2^j U + H$ for $j = 0, 1, \dots, N-1$ into (17) and using the estimate (18) and (19), we have

$$\begin{aligned}\#(\mathcal{M}_{\mu,j}^{(U,H)}) =& \pi_\mu(2^j U + H) - \pi_\mu(2^j U) \\ \geq& (\kappa - 2\mu^2 - \rho)\frac{H}{(4^{N\eta} C_{\mathcal{L},\eta}^2 U^{2\eta} - \mu^2)\log U} \\ \gg_{\mathcal{L},N,\eta}& \frac{H}{U^{2\eta}\log U}.\end{aligned}$$

This completes the proof. □

Step 3. We will use Lemma 3.2 in this step. Let $X$ be a sufficiently large positive number depending on $\mathcal{L}, \sigma_0, N, \eta$. We may assume that $\#(\mathcal{M}_{\mu,0}^{(X,X)}) \geq N$ by (15) and (16). Fix the distinct primes $p_{k_0}, p_{k_2}, \dots, p_{k_{N-1}} \in \mathcal{M}_{\mu,0}^{(X,X)}$. Let $Y \geq 2X + 1$ be a positive parameter which is determined later (see (20)), and let $Y^{1-E_{\mathcal{L}}}(\log Y)^{D+1} \le H \le Y$ which is determined later (see (31)). Let

$$\mathcal{M}_j^{(Y,H)} = \left\{p \in \mathcal{P}\ ;\ 2^jY < p \le 2^jY + H\right\}$$

and put

$$\mathcal{M}_\mu^{(Y,H)} = \bigsqcup_{j=0}^{N-1} \mathcal{M}_{\mu,j}^{(Y,H)} \quad \text{and} \quad \mathcal{M}^{(Y,H)} = \bigsqcup_{j=0}^{N-1} \mathcal{M}_j^{(Y,H)}.$$

Claim 3.8. *We consider the following two conditions;*

(Y1) *For any $p_X \in \mathcal{M}_{\mu,0}^{(X,X)}$ and $p_Y \in \mathcal{M}^{(Y,H)}$, it holds that*

$$\frac{|a(p_X)|}{p_X^{\sigma_0}} \ge \frac{|a(p_Y)|}{p_Y^{\sigma_0}},$$

(Y2) *For any $p_X \in \mathcal{M}_{\mu,0}^{(X,X)}$ and $j = 0, 1, \ldots, N-1$, it holds that*

$$\frac{|a(p_X)|}{p_X^{\sigma_0}} \le \sum_{p \in \mathcal{M}_j^{(Y,H)}} \frac{|a(p)|}{p^{\sigma_0}}.$$

*Then the choice*

$$Y = \left(\frac{C_{\mathcal{L},\eta}}{\mu}\right)^{\frac{1}{\sigma_0-\eta}} (2X)^{\frac{\sigma_0}{\sigma_0-\eta}}, \tag{20}$$

*yields the condition* (Y1)*, and the estimate*

$$\frac{1}{X^{\sigma_0-\eta}} \ll_{\mathcal{L},\sigma_0,N,\eta} \frac{H}{Y^{\sigma_0+2\eta}(\log Y)^2}. \tag{21}$$

*yields the condition* (Y2)*.* (*We take $H$ suitably which satisfies the bound* (21) *later* (*see* (32)).)

Proof of Claim 3.8. We first consider the condition (Y1). Since the estimates

$$\frac{|a(p_X)|}{p_X^{\sigma_0}} \ge \frac{\mu}{p_X^{\sigma_0}} \ge \frac{\mu}{(2X)^{\sigma_0}} \quad \text{and} \quad \frac{|a(p_Y)|}{p_Y^{\sigma_0}} \le \frac{C_{\mathcal{L},\eta}}{p_Y^{\sigma_0-\eta}} \le \frac{C_{\mathcal{L},\eta}}{Y^{\sigma_0-\eta}}$$

hold for $p_X \in \mathcal{M}_{\mu,0}^{(X,X)}$ and $p_Y \in \mathcal{M}^{(Y,H)}$, the choice $Y$ in (20) yields the condition (Y1).

Next we consider the condition (Y2). Now it holds that

$$\sum_{p \in \mathcal{M}_j^{(Y,H)}} \frac{|a(p)|}{p^{\sigma_0}} \ge \sum_{p \in \mathcal{M}_{\mu,j}^{(Y,H)}} \frac{|a(p)|}{p^{\sigma_0}} \ge \frac{\mu}{(2^NY)^{\sigma_0}} \# \left(\mathcal{M}_{\mu,j}^{(Y,H)}\right) \gg_{\mathcal{L},\sigma_0,N,\eta} \frac{H}{Y^{\sigma_0+2\eta}\log Y} \tag{22}$$

by (16). By using the above estimate and the estimate $|a(p_X)|p_X^{-\sigma_0} \le C_{\mathcal{L},\eta}X^{-(\sigma_0-\eta)}$ for $p_X \in \mathcal{M}_{\mu,0}^{(X,X)}$, the condition (Y2) holds when the estimate (21) holds. □

For any $j = 0, 1, \ldots, N-1$, put

$$\mathcal{M}_j = \{p_{k_j}\} \sqcup \mathcal{M}_j^{(Y,H)} \quad \text{and} \quad \mathcal{M} = \bigsqcup_{j=0}^{N-1} \mathcal{M}_j.$$

In what follows, we take

$$Y = \left( \frac{C_{\mathcal{L},\eta}}{\mu} \right)^{\frac{1}{\sigma_0 - \eta}} (2X)^{\frac{\sigma_0}{\sigma_0-\eta}}.$$

Then we have

$$\left\{ \sum_{p \in \mathcal{M}_j} \frac{b(p)\exp(-2\pi i \theta_p)}{p^{\sigma_0}} \ ; \ (\theta_p)_{p\in\mathcal{M}_j} \in [0,1)^{\mathcal{M}_j} \right\} = \left\{ z \in \mathbb{C} \ ; \ |z| \le \sum_{p\in\mathcal{M}_j} \frac{|b(p)|}{p^{\sigma_0}} \right\} \tag{23}$$

by Lemma 3.2 and by (i) of Lemma 3.3 when the estimate (21) holds.

Step 4. Let $\underline{\theta} = (\theta_p)_{p\in\mathcal{M}} \in \mathbb{R}^{\mathcal{M}}$ and write $\underline{\theta}_j = (\theta_p)_{p\in\mathcal{M}_j}$. We will prove the following claim.

Claim 3.9. *For $j, k = 0, 1, \ldots N-1$ and for $\underline{\theta} = (\theta_p)_{p\in\mathcal{M}} \in \mathbb{R}^{\mathcal{M}}$, we have*

$$\frac{\partial^k}{\partial s^k} \varphi_{\mathcal{M}_j}(\sigma_0, \underline{\theta}_j) = (-\log Y_j)^k \varphi_{\mathcal{M}_j}(\sigma_0, \underline{\theta}_j) + R_{j,k}, \tag{24}$$

*where $Y_j = 2^j Y$ and*

$$R_{j,k} \ll_{\mathcal{L},\sigma_0,N,\eta} (\log Y)^{N-2} H^2 Y^{-(1+\sigma_0-\eta)} + \frac{(\log X)^{N-1}}{X^{\sigma_0-\eta}}. \tag{25}$$

Proof of Claim 3.9. By the equation (24), we have

$$\begin{aligned} R_{j,k} = &\sum_{p\in\mathcal{M}_j^{(Y,H)}} \left\{ (-\log p)^k - (-\log Y_j)^k \right\} b(p) p^{-\sigma_0} \exp(-2\pi i \theta_p) \\ &+ \left\{ (-\log p_{k_j})^k - (-\log Y_j)^k \right\} b(p_{k_j}) p_{k_j}^{-\sigma_0} \exp(-2\pi i \theta_{p_{k_j}}). \end{aligned}$$

Using the mean value theorem for $(-\log p)^k - (-\log Y_j)^k$ in the first term, we have

$$R_{j,k} \ll_N \sum_{p\in\mathcal{M}_j^{(Y,H)}} (\log Y)^{k-1} \frac{H}{Y} |b(p)| p^{-\sigma_0} \quad + |(\log p_{k_j})^k - (\log Y_j)^k| |b(p_{k_j})| p_{k_j}^{-\sigma_0}.$$

By using the estimate $|b(p)| = |a(p)| \ll_{\mathcal{L},\eta} p^\eta$, it holds that

$$\begin{aligned} \sum_{p\in\mathcal{M}_j^{(Y,H)}} (\log Y)^{k-1} \frac{H}{Y} |b(p)| p^{-\sigma_0} &\ll_{\mathcal{L},\eta} (\log Y)^{k-1} \frac{H}{Y} \sum_{p\in\mathcal{M}_j^{(Y,H)}} p^{\eta-\sigma_0} \\ &\le (\log Y)^{k-1} \frac{H}{Y} Y^{\eta-\sigma_0} \sum_{p\in\mathcal{M}_j^{(Y,H)}} 1 \\ &\ll (\log Y)^{N-2} H^2 Y^{-(1+\sigma_0-\eta)}, \end{aligned}$$

and

$$\begin{aligned} &|(\log p_{k_j})^k - (\log Y_j)^k| |b(p_{k_j})| p_{k_j}^{-\sigma_0} \\ &\quad \ll_{\mathcal{L},N,\eta} (\log Y)^{N-1} p_{k_j}^{\eta-\sigma_0} \ll \frac{(\log Y)^{N-1}}{X^{\sigma_0-\eta}} \ll_{\mathcal{L},\sigma_0,N,\eta} \frac{(\log X)^{N-1}}{X^{\sigma_0-\eta}}. \end{aligned}$$

This completes the proof. $\square$

STEP 5. We now consider the following system of linear equations in the unknown $z_j$:

$$\sum_{j=0}^{N-1}(-\log Y_j)^k z_j = c_k - \gamma_k \quad \text{for } k = 0, 1, \ldots, N-1. \tag{26}$$

Since the coefficient matrix of this system is the Vandermonde matrix, Lemma 3.1 implies that it has a unique solution $\underline{z} = \underline{z}(Y, \underline{c}, \underline{\gamma}) = (z_0, z_1, \ldots, z_{N-1})$ which satisfies the bound

$$\|\underline{z}\| \ll_N (\log Y)^{N-1}\|\underline{c} - \underline{\gamma}\|. \tag{27}$$

CLAIM 3.10. *A sufficient condition that the system of equations*

$$\varphi_{\mathcal{M}_j}(\sigma_0, \underline{\theta}_j) = z_j, \quad \textit{for} \quad j = 0, 1, \ldots, N-1 \tag{28}$$

*has a solution $\underline{\theta} \in \mathbb{R}^{\mathcal{M}}$ is that the estimate* (21) *and the estimate*

$$\frac{H}{Y^{\sigma_0+2\eta}(\log Y)^{N+1}} \gg_{\mathcal{L},\sigma_0,N,\eta} \|\underline{c} - \underline{\gamma}\| + 1. \tag{29}$$

*hold.*

PROOF OF CLAIM 3.10. It is enough to take $H$ to establish

$$\|\underline{z}\| \leq \sum_{p\in\mathcal{M}_j} \frac{|b(p)|}{p^{\sigma_0}} \tag{30}$$

by (23). The bound (22) and (27) give the proof. $\square$

STEP 6. Note that, by $\sigma_0 \in (\max\{\sigma_{\mathcal{L}}, 1-2E_{\mathcal{L}}\}, 1)$, it holds that $\left(\sigma_0, \frac{1+\sigma_0}{2}\right)\cap(1-E_{\mathcal{L}}, 1) \neq \emptyset$. We will show the following claim.

CLAIM 3.11. *Let $H = Y^A$ and choose $A = A(\sigma_0, E_{\mathcal{L}})$ and $\eta = \eta(\sigma_0, E_{\mathcal{L}})$ such that*

$$A = A(\sigma_0, E_{\mathcal{L}}) = \frac{1}{2}\left(\max\{\sigma_0, 1-E_{\mathcal{L}}\} + \frac{1+\sigma_0}{2}\right) \in \left(\sigma_0, \frac{1+\sigma_0}{2}\right) \cap (1-E_{\mathcal{L}}, 1) \tag{31}$$

*and*

$$\eta = \eta(\sigma_0, E_{\mathcal{L}}) = \frac{1}{2}\min\left\{\frac{1-E_{\mathcal{L}}}{2}, \frac{A(\sigma_0, E_{\mathcal{L}})-\sigma_0}{2}, 1+\sigma_0-2A(\sigma_0, E_{\mathcal{L}})\right\} > 0. \tag{32}$$

*Put*

$$d_1^{(1)}(\sigma_0, E_{\mathcal{L}}) = \frac{\sigma_0}{\sigma_0-\eta}\left(A - \sigma_0 - 2\eta\right) > 0$$

*and*

$$B(\sigma_0, E_{\mathcal{L}}) = \min\left\{\frac{\sigma_0}{\sigma_0-\eta}\left(1+\sigma_0-2A-\eta\right), \sigma_0-\eta\right\} > 0.$$

*If the estimate*

$$X \geq C^{(1)}(\mathcal{L}, \sigma_0, N)\left(\|\underline{c}-\underline{\gamma}\|_N + 1\right)^{2(d_1^{(1)}(\sigma_0,E_{\mathcal{L}}))^{-1}} \tag{33}$$

*holds with sufficiently large $C^{(1)}(\mathcal{L},\sigma_0,N)$ depending on $\mathcal{L},\sigma_0,N$, then there exists $\underline{\theta}^{(1)} = \underline{\theta}^{(1)}(\mathcal{L},\sigma_0,N,X) = (\theta_p^{(1)})_{p\in\mathcal{M}} \in \mathbb{R}^{\mathcal{M}}$ such that*

$$\left|\sum_{j=0}^{N-1} \frac{\partial^k}{\partial s^k}\varphi_{\mathcal{M}_j}(\sigma_0,\underline{\theta}_j^{(1)}) - (c_k-\gamma_k)\right| \ll_{\mathcal{L},\sigma_0,N} X^{-B(\sigma_0,E_{\mathcal{L}})}(\log X)^{N-1} \tag{34}$$

*holds for any $k=0,1,\dots,N-1$.*

Proof of Claim 3.11. Let X satisfy the bound (33). Then we have (21). By substituting (20), the left hand side of (29) equals

$$\frac{H}{Y^{\sigma_0+2\eta}(\log Y)^{N+1}} = \frac{Y^{A-\sigma_0-2\eta}}{(\log Y)^{N+1}} \asymp_{\mathcal{L},\sigma_0,N} \frac{X^{d_1^{(1)}(\sigma_0,E_{\mathcal{L}})}}{(\log X)^{N+1}}, \tag{35}$$

and the estimate

$$R_{j,k} \ll_{\mathcal{L},\sigma_0,N} X^{-B(\sigma_0,E_{\mathcal{L}})}(\log X)^{N-1} \tag{36}$$

holds by (25). By the estimate (35), we have (29). Hence Claim 3.10 with the estimates (21) and (29) implies that there exists $\underline{\theta}^{(1)} = \underline{\theta}^{(1)}(\mathcal{L},\sigma_0,N,X) = (\theta_p^{(1)})_{p\in\mathcal{M}} \in \mathbb{R}^{\mathcal{M}}$ such that the system of equations (28) holds. Therefore, by (24), (26), (28) and (36), we have the conclusion. □

Step 7. To finish the proof, we give some estimates. Put

$$\delta_0 = \frac{1}{2}(\sigma_0-1/2), \quad l_0 = \frac{2}{\sigma_0-1/2}.$$

Then we have

$$\left|\frac{\partial^k}{\partial s^k}\log\mathcal{L}_p(\sigma_0,\theta_p) - \frac{\partial^k}{\partial s^k}\varphi_p(\sigma_0,\theta_p)\right| \le \sum_{l=2}^{\infty} \frac{l^{N-1}|b(p^l)|(\log p)^{N-1}}{p^{l\sigma_0}}.$$

for $k=0,1,\dots,N-1$ and $\theta_p\in\mathbb{R}$. By a calculation similar to (2.12) in [**34**], we have

$$\sum_{l=2}^{\infty} \frac{l^{N-1}|b(p^l)|(\log p)^{N-1}}{p^{l\sigma_0}} \ll_{\mathcal{L},\sigma_0,N} \frac{(\log p)^{N-1}}{p^{2(\sigma_0-\delta_0)}} + \frac{(\log p)^{N-1}}{p^{l_0(\sigma_0-1/2)}} \ll \frac{(\log p)^{N-1}}{p^{\sigma_0+1/2}}.$$

Hence it holds that

$$\left|\frac{\partial^k}{\partial s^k}\log\mathcal{L}_p(\sigma_0,\theta_p) - \frac{\partial^k}{\partial s^k}\varphi_p(\sigma_0,\theta_p)\right| \ll_{\mathcal{L},\sigma_0,N} \frac{(\log p)^{N-1}}{p^{\sigma_0+1/2}} \tag{37}$$

for $k=0,1,\dots,N-1$ and $\theta_p\in\mathbb{R}$.

From now, let $Q>2^N Y$ and let $X$ satisfy (33). Put

$$\theta_p^{(\star)} = \begin{cases} \theta_p^{(0)} & \text{if } p\in\mathcal{P}\setminus\mathcal{M}, \\ \theta_p^{(1)} & \text{if } p\in\mathcal{M}. \end{cases}$$

Then we have the following estimates.

Claim 3.12. *We have*

$$\frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)\setminus\mathcal{M}}(\sigma_0, \underline{\theta}^{(0)}) = \gamma_k + O_{\mathcal{L},\sigma_0,N}\left(X^{1/2-\sigma_0}(\log X)^{N-1}\right), \tag{38}$$

$$\frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{M}}(\sigma_0, \underline{\theta}^{(1)}) - \frac{\partial^k}{\partial s^k} \varphi_{\mathcal{M}}(\sigma_0, \underline{\theta}^{(1)}) \ll_{\mathcal{L},\sigma_0,N} X^{1/2-\sigma_0}(\log X)^{N-1} \tag{39}$$

*hold for any* $k = 0, 1, \ldots, N-1$.

Proof of Claim 3.12. The estimate (39) follows from the estimate (37). Next, we will show the estimate (38). We can write

$$\frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)\setminus\mathcal{M}}(\sigma_0, \underline{\theta}^{(0)}) = \gamma_k - \left(\sum_{p>Q} + \sum_{p\in\mathcal{M}}\right) \frac{\partial^k}{\partial s^k} \log \mathcal{L}_p(\sigma_0, \underline{\theta}^{(0)}).$$

By the estimate (37), we have

$$\begin{aligned}
&\left(\sum_{p>Q} + \sum_{p\in\mathcal{M}}\right) \left(\frac{\partial^k}{\partial s^k} \log \mathcal{L}_p(\sigma_0, \underline{\theta}^{(0)}) - \frac{\partial^k}{\partial s^k} \varphi_p(\sigma_0, \underline{\theta}^{(0)})\right) \\
&\ll_{\mathcal{L},\sigma_0,N} \sum_{p>X} \frac{(\log p)^{N-1}}{p^{\sigma_0+1/2}} \ll_N X^{1/2}(\log X)^{N-1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(-1)^k \sum_{p>Q} \frac{\partial^k}{\partial s^k} \varphi_p(\sigma_0, \underline{\theta}^{(0)}) &= \sum_{p>Q} \frac{b(p)(\log p)^k \exp(-2\pi i \theta_p^{(0)})}{p^{\sigma_0}} \\
&= \left[\left(\sum_{p\le\xi} b(p) \exp(-2\pi i \theta_p^{(0)})\right) \frac{(\log \xi)^k}{\xi^{\sigma_0}}\right]_{\xi=Q}^{\xi=\infty} \\
&\quad - \int_Q^\infty \left(\sum_{p\le\xi} b(p) \exp(-2\pi i \theta_p^{(0)})\right) d\left(\frac{(\log \xi)^k}{\xi^{\sigma_0}}\right) \\
&\ll_{\mathcal{L},\sigma_0,N} Q^{1/2-\sigma_0}(\log Q)^{N-1} \le X^{1/2-\sigma_0}(\log X)^{N-1}
\end{aligned}$$

by partial summation and the estimate (14). By a calculation similar to the above, we have

$$\begin{aligned}
&(-1)^k \sum_{p\in\mathcal{M}_j} \frac{\partial^k}{\partial s^k} \varphi_p(\sigma_0, \underline{\theta}^{(0)}) \\
&= \frac{b(p_{k_j})(\log p_{k_j})^k \exp(-2\pi i \theta_{p_{k_j}}^{(0)})}{p^{\sigma_0}} + \sum_{Y_j<p\le Y_j+H} \frac{b(p)(\log p)^k \exp(-2\pi i \theta_p^{(0)})}{p^{\sigma_0}} \\
&\ll_{\mathcal{L},\sigma_0,N} X^{1/2-\sigma_0}(\log X)^{N-1}
\end{aligned}$$

for any $j = 0, 1, \ldots, N-1$. This completes the proof. □

Step 8. We will finish the proof in this step. By Claim 3.12, we have

$$\begin{aligned}
&\frac{\partial^k}{\partial s^k}\log \mathcal{L}_{\mathcal{P}(Q)}(\sigma_0,\underline{\theta}^{(\star)})\\
&=\frac{\partial^k}{\partial s^k}\varphi_{\mathcal{M}}(\sigma_0,\underline{\theta}_j^{(1)})+\left(\frac{\partial^k}{\partial s^k}\log\mathcal{L}_{\mathcal{M}}(\sigma_0,\underline{\theta}^{(1)})-\frac{\partial^k}{\partial s^k}\varphi_{\mathcal{M}}(\sigma_0,\underline{\theta}^{(1)})\right)\\
&\qquad+\frac{\partial^k}{\partial s^k}\log\mathcal{L}_{\mathcal{P}(Q)\setminus\mathcal{M}}(\sigma_0,\underline{\theta}^{(0)})\\
&=c_k-\gamma_k+O_{\mathcal{L},\sigma_0,N}\left(X^{-B(\sigma_0,E_{\mathcal{L}})}(\log X)^{N-1}\right)\\
&\qquad+\gamma_k+O_{\mathcal{L},\sigma_0,N}\left(X^{1/2-\sigma_0}(\log X)^{N-1}\right)\\
&=c_k+O_{\mathcal{L},\sigma_0,N}\left(X^{-\min\{B(\sigma_0,E_{\mathcal{L}}),\sigma_0-1/2\}}(\log X)^{N-1}\right)
\end{aligned}$$

for any $k=0,1,\dots,N-1$. Hence we have

$$\left|\frac{\partial^k}{\partial s^k}\log\mathcal{L}_{\mathcal{P}(Q)}(\sigma_0,\underline{\theta}^{(\star)})-c_k\right|\ll_{\mathcal{L},\sigma_0,N}X^{-\min\{B(\sigma_0,E_{\mathcal{L}}),\sigma_0-1/2\}}(\log X)^{N-1}$$

for any $k=0,1,\dots,N-1$. Putting

$$d_1(\sigma_0,E_{\mathcal{L}})=\frac{2\sigma_0}{\sigma_0-\eta}\max\left\{(d_1^{(1)}(\sigma_0,E_{\mathcal{L}}))^{-1},(\min\{B(\sigma_0,E_{\mathcal{L}}),\sigma_0-1/2\})^{-1}\right\},$$

letting $C(\mathcal{L},\sigma_0,N)$ be sufficiently large depending on $\mathcal{L},\sigma_0,N$, and using

$$\|\underline{c}-\underline{\gamma}\|+1/\varepsilon\ll_{\mathcal{L}}\|\underline{c}\|+1/\varepsilon,$$

we have the conclusion. □

Proof of Theorem 1.19. We divide the proof into several steps.

Step 1. First we will give settings and mention the strategy of the proof. Let $Q$ satisfy

$$Q>C_1^{(1)}(\mathcal{L},\sigma_0,N)\left(\|\underline{c}\|+1/\varepsilon\right)^{d_1(\sigma_0,E_{\mathcal{L}})},$$

where $C_1^{(1)}(\mathcal{L},\sigma_0,N)$ is a sufficiently large constant depending on $\mathcal{L},\sigma_0,N$ with $C_1^{(1)}(\mathcal{L},\sigma_0,N)\geq 2^{8/(\sigma_0-1/2)}$. Let $\underline{\theta}^{(\star)}=(\theta_p^{(\star)})_{p\in\mathcal{P}(Q)}\in\mathbb{R}^{\mathcal{P}(Q)}$ be as in Lemma 3.5 and put $\delta=Q^{-1}$. We put

$$\mathcal{I}=\int_{D_T}\sum_{k=0}^{N-1}\Phi_Q\left(\underline{\gamma}(t)-\underline{\theta}^{(\star)}\right)\left|(\log\mathcal{L}(\sigma_0+it))^{(k)}-\left(\log\mathcal{L}_{\mathcal{P}(Q)}(\sigma_0+it)\right)^{(k)}\right|^2dt,$$

where $\Phi_Q(\underline{\theta})$ is the mollifier defined in subsection 3.2.3,

$$\underline{\gamma}(t)=\left(\frac{\log p}{2\pi}t\right)_{p\in\mathcal{P}(Q)}\in\mathbb{R}^{\mathcal{P}(Q)},$$

and $D_T$ is the subset of $[T,2T]$ which is defined as follows. For each nontrivial zeros $\rho=\beta+i\gamma$ of $\mathcal{L}(s)$, we define

$$P_\rho^{(h)}=\{s=\sigma+it\ ;\ (1/2)(\sigma_{\mathcal{L}}+\sigma_0)\leq\sigma\leq 15,\quad |t-\gamma|\leq h\}$$

with the positive parameter $10 \le h < T$, and we put

$$D_T = D_T(h) = \left\{ t \in [T, 2T] \ ; \ \sigma_0 + it \notin \bigcup_{\rho \ ; \ \beta > (1/2)(\sigma_{\mathcal{L}} + \sigma_0)} P_\rho^{(h)} \right\}.$$

Now, we mention the strategy of the proof. We want to choose $T$ depending on $\mathcal{L}, \sigma_0, N, \underline{c}, \varepsilon$ and choose $Q$ and $h$ depending on $T$ so that

$$\mathcal{I} \le \left(\frac{\varepsilon}{3}\right)^2 \int_{D_T} \Phi_Q \left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt, \tag{40}$$

$$\int_{D_T} \Phi_Q \left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt \ge \frac{T}{2}, \tag{41}$$

and

$$\left| \frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)} (\sigma_0, \underline{\theta}) - \frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)} \left(\sigma_0, \underline{\theta}^{(\star)}\right) \right| < \frac{\varepsilon}{3} \tag{42}$$

for $|\theta_p - \theta_p^{(\star)}| < \delta$, $p \le Q$ and $k = 0, 1, \ldots, N-1$. Once we have such choices, there exists $t_0 \in [T, 2T]$ such that

$$\left| \frac{\partial^k}{\partial s^k} \log \mathcal{L}(\sigma_0 + it_0) - \frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)}(\sigma_0 + it_0) \right| \le \frac{\varepsilon}{3}$$

for any $k = 0, 1, \ldots, N-1$ and $\Phi_Q \left(\underline{\gamma}(t_0) - \underline{\theta}^{(\star)}\right) > 0$. By (42) and (9) and by noting the equation $\log \mathcal{L}_{\mathcal{P}(Q)}(\sigma_0 + it_0) = \log \mathcal{L}_{\mathcal{P}(Q)} \left(\sigma_0, \underline{\gamma}(t_0)\right)$, we have

$$\left| \frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)} (\sigma_0 + it_0) - \frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)} \left(\sigma_0, \underline{\theta}^{(\star)}\right) \right| < \frac{\varepsilon}{3}$$

for any $k = 0, \ldots, N-1$ by substituting $\underline{\theta} = \underline{\gamma}(t_0)$. These estimates and Lemma 3.5 give the inequalities

$$\left| \frac{d^k}{ds^k} \log \mathcal{L}(\sigma_0 + it_0) - c_k \right| < \varepsilon \quad \text{for } k = 0, 1, \ldots, N-1.$$

Step 2. We give a certain estimate toward the estimate (42). By using the estimates $|e^{i\alpha} - 1| \le |\alpha|$ for $\alpha \in \mathbb{R}$ and $b(p^l) \ll_{\mathcal{L}} p^{l/2}$, we have

$$\begin{aligned} \left| \frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)} (\sigma_0, \underline{\theta}) - \frac{\partial^k}{\partial s^k} \log \mathcal{L}_{\mathcal{P}(Q)} \left(\sigma_0, \underline{\theta}^{(\star)}\right) \right| &\ll \sum_{p \le Q} \sum_{l=1}^{\infty} \frac{l^{k+1} (\log p)^k |b(p^l)|}{p^{l\sigma_0}} \delta \\ &\ll_{\mathcal{L}, \sigma_0, N} Q^{-1} \sum_{p \le Q} \frac{(\log p)^{N-1}}{p^{\sigma_0 - 1/2}} \ll_{\sigma_0, N} Q^{1/2 - \sigma_0} (\log Q)^{N-1} \end{aligned} \tag{43}$$

for $|\theta_p - \theta_p^{(\star)}| < \delta$, $p \le Q$ and $k = 0, 1, \ldots, N-1$.

Step 3. We will estimate $\mathcal{I}$. To estimate $\mathcal{I}$, we use the formula (13) in Lemma 3.4. First we will give the formula similar to (13) for $\log \mathcal{L}(s)$ . Put

$$F(s,z)=\int_{s+10}^{s}\frac{x^{z-w}-x^{2(z-w)}}{(w-z)^2}dw.$$

Integrating (13), we obtain

$$\begin{aligned}\log \mathcal{L}(s) &= \log \mathcal{L}(s+10)+\sum_{n\le x^2}\frac{\Lambda_{\mathcal{L},x}(n)}{n^s\log n}-\sum_{n\le x^2}\frac{\Lambda_{\mathcal{L},x}(n)}{n^{s+10}\log n}\\ &\quad -\frac{m_{\mathcal{L}}}{\log x}F(s,1)-\frac{m_{\mathcal{L}}}{\log x}F(s,0)+\frac{1}{\log x}\sum_{\rho}F(s,\rho)+\frac{1}{\log x}\sum_{j=1}^{f}\sum_{r=0}^{\infty}F(s,-q_j(r))\\ &=\sum_{n\le x^2}\frac{\Lambda_{\mathcal{L},x}(n)}{n^s\log n}+\sum_{n>x}\left(\Lambda_{\mathcal{L}}(n)-\Lambda_{\mathcal{L},x}(n)\right)\frac{1}{n^{s+10}\log n}\\ &\quad -\frac{m_{\mathcal{L}}}{\log x}F(s,1)-\frac{m_{\mathcal{L}}}{\log x}F(s,0)+\frac{1}{\log x}\sum_{\rho}F(s,\rho)+\frac{1}{\log x}\sum_{j=1}^{f}\sum_{r=0}^{\infty}F(s,-q_j(r))\end{aligned} \tag{44}$$

if $t$ is not equal to 0 and the imaginary part of any zero of $\mathcal{L}(s)$. Let $Q<x\le T$ and put

$$\mathcal{I}_k=\int_{D_T}\Phi_Q\left(\underline{\gamma}(t)-\underline{\theta}^{(\star)}\right)\left|\left(\log \mathcal{L}(\sigma_0+it)\right)^{(k)}-\left(\log \mathcal{L}_{\mathcal{P}(Q)}(\sigma_0+it)\right)^{(k)}\right|^2dt.$$

We will estimate $\mathcal{I}_k$ for $k=1,\dots,N-1$. For $k=0$, we have the same upper bound by using the formula (44). For any $k=1,2,\dots,N-1$, the estimate

$$\mathcal{I}_k\ll\mathcal{A}_k+\mathcal{B}_k+\mathcal{C}_k+\mathcal{D}_k+\mathcal{E}_k+\mathcal{F}_k,$$

holds, where

$$\mathcal{A}_k = \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right)\left|\left(\sum_{n\le x^2}\frac{\Lambda_{\mathcal{L},x}(n)}{n^{s_0}}\right)^{(k-1)} - \left(\sum_{n\le Q}\frac{\Lambda_{\mathcal{L}}(n)}{n^{s_0}}\right)^{(k-1)}\right|^2 dt,$$

$$\mathcal{B}_k = \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right)\left|\left(\sum_{p\le Q}\sum_{l>\frac{\log Q}{\log p}}\frac{\Lambda_{\mathcal{L}}(p^l)}{p^{ls_0}}\right)^{(k-1)}\right|^2 dt,$$

$$\mathcal{C}_k = \frac{1}{(\log x)^2}\int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right)\times$$
$$\times\left|\left(\sum_{j=1}^{f}\sum_{r=0}^{\infty}\frac{x^{-q_j(r)-s_0} - x^{-2(q_j(r)+s_0)}}{(q_j(r)+s_0)^2}\right)^{(k-1)}\right|^2 dt,$$

$$\mathcal{D}_k = \frac{m_{\mathcal{L}}^2}{(\log x)^2}\int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right)\left|\left(\frac{x^{2(1-s_0)} - x^{1-s_0}}{(1-s_0)^2}\right)^{(k-1)}\right|^2 dt,$$

$$\mathcal{E}_k = \frac{m_{\mathcal{L}}^2}{(\log x)^2}\int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right)\left|\left(\frac{x^{-2s_0} - x^{-s_0}}{s_0^2}\right)^{(k-1)}\right|^2 dt,$$

$$\mathcal{F}_k = \frac{1}{(\log x)^2}\int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right)\left|\left(\sum_{\rho}\frac{x^{\rho-s_0} - x^{2(\rho-s_0)}}{(\rho-s_0)^2}\right)^{(k-1)}\right|^2 dt,$$

and $s_0 = \sigma_0 + it$.

*Bound for $\mathcal{A}_k$.* We can write

$$\sum_{Q<n\le x^2}\frac{\Lambda_{\mathcal{L},x}(n)(\log n)^{k-1}}{n^{s_0}}$$
$$= \sum_{Q<p\le x^2}\frac{\Lambda_{\mathcal{L},x}(p)(\log p)^{k-1}}{p^{s_0}} + \sum_{\substack{Q<n\le x^2;\\ n=p^l, l\ge 2}}\frac{\Lambda_{\mathcal{L},x}(p^l)(\log p^l)^{k-1}}{p^{ls_0}},$$

and

$$\sum_{\substack{Q<n\le x^2;\\ n=p^l, l\ge 2}}\frac{\Lambda_{\mathcal{L},x}(p^l)(\log p^l)^{k-1}}{p^{ls_0}}$$

(45)
$$\ll \sum_{p\le\sqrt{Q}}\sum_{l>\frac{\log Q}{\log p}}\frac{|\Lambda_{\mathcal{L},x}(p^l)|(\log p^l)^{k-1}}{p^{l\sigma_0}} + \sum_{p>\sqrt{Q}}\sum_{l=2}^{\infty}\frac{|\Lambda_{\mathcal{L},x}(p^l)|(\log p^l)^{k-1}}{p^{l\sigma_0}}.$$

Note that

$$\sum_{l>X} \frac{l^k}{p^{l\sigma}} \ll_{\sigma,N} \frac{X^k}{p^{X\sigma}}. \tag{46}$$

holds for $X \geq 1$, $\sigma > 0$ and for $k = 0, 1, \ldots N-1$. We will estimate the first term of (45). By using the estimate $b(p^l) \ll_{\mathcal{L}} p^{l/2}$ and (46), we have

$$\sum_{l>\frac{\log Q}{\log p}} \frac{|\Lambda_{\mathcal{L},x}(p^l)|(\log p^l)^{k-1}}{p^{l\sigma_0}} = \sum_{l>\frac{\log Q}{\log p}} \frac{|b(p^l)|(\log p^l)^k}{p^{l\sigma_0}} \ll_{\mathcal{L},\sigma_0,N} Q^{1/2-\sigma_0}(\log Q)^{N-1} \tag{47}$$

for any $p \leq \sqrt{Q}$ and $k = 0, 1, \ldots, N-1$. Put

$$\eta_0 = \frac{\sigma_0 - 1/2}{4}.$$

Since Lemma 3.3 (ii) yields the estimate

$$b(p^l) \ll_{\mathcal{L},\sigma_0} (2^l - 1)\frac{p^{\eta_0 l}}{l} \leq \frac{p^{\left(\eta_0 + \frac{\log 2}{\log p}\right)l}}{l},$$

we have, by the estimate (46),

$$\begin{aligned}\sum_{l>\frac{\log Q}{\log p}} \frac{|b(p^l)|(\log p^l)^k}{p^{l\sigma_0}} &\ll_{\mathcal{L},\sigma_0} (\log p)^k \sum_{l>\frac{\log Q}{\log p}} \frac{l^{k-1}}{p^{(\sigma_0-\eta_0-\frac{\log 2}{\log p})l}} \\ &\leq (\log p)^k \sum_{l>\frac{\log Q}{\log p}} \frac{l^{k-1}}{p^{(\sigma_0-1/2(\sigma_0-1/2))l}} \ll_{\sigma_0,N} Q^{-\sigma_0-1/2(1/2-\sigma_0)}(\log Q)^{N-1}\end{aligned} \tag{48}$$

for $p \geq 2^{4/(\sigma_0-1/2)}$. By the estimate (47) and (48), we have

$$\begin{aligned}&\sum_{p\leq\sqrt{Q}} \sum_{l>\frac{\log Q}{\log p}} \frac{|\Lambda_{\mathcal{L},x}(p^l)|(\log p^l)^{k-1}}{p^{l\sigma_0}} \\ &= \left(\sum_{2\leq p<2^{\frac{4}{\sigma_0-1/2}}} + \sum_{2^{\frac{4}{\sigma_0-1/2}}\leq p\leq\sqrt{Q}}\right) \sum_{l>\frac{\log Q}{\log p}} \frac{|\Lambda_{\mathcal{L},x}(p^l)|(\log p^l)^{k-1}}{p^{l\sigma_0}} \\ &\ll_{\mathcal{L},\sigma_0,N} Q^{1/2-\sigma_0}(\log Q)^{N-1} + \sum_{p\leq\sqrt{Q}} Q^{-\sigma_0-1/2(1/2-\sigma_0)}(\log Q)^{N-1} \\ &\ll Q^{1/2(1/2-\sigma_0)}(\log Q)^{N-1}.\end{aligned}$$

As for the second term of (45), by an argument similar to (48), we have

$$\sum_{p>\sqrt{Q}} \sum_{l=2}^{\infty} \frac{|\Lambda_{\mathcal{L},x}(p^l)|(\log p^l)^{k-1}}{p^{l\sigma_0}} \ll_{\mathcal{L},\sigma_0,N} Q^{(3/4)(1/2-\sigma_0)}(\log Q)^{N-1}.$$

Hence we have

$$\sum_{\substack{Q<n\le x^2;\\ n=p^l, l\ge 2}} \frac{\Lambda_{\mathcal{L},x}(p^l)(\log p^l)^{k-1}}{p^{ls_0}} \ll_{\mathcal{L},\sigma_0,N} Q^{1/2(1/2-\sigma_0)}(\log Q)^{N-1}. \tag{49}$$

Therefore, we have

$$\begin{aligned}\mathcal{A}_k &\ll_{\mathcal{L},\sigma_0,N} \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) \left|\sum_{Q<p\le x^2} \frac{\Lambda_{\mathcal{L},x}(p)(\log p)^{k-1}}{p^{s_0}}\right|^2 dt \\ &\quad + Q^{1/2-\sigma_0}(\log Q)^{2N-2} \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt =: \mathcal{A}_k^{(1)} + \mathcal{A}_k^{(2)}.\end{aligned}$$

By the formula (12) with $\beta_{\underline{n}} := \beta_{\underline{n}}(\underline{\theta}^{(\star)})$, we have

$$\begin{aligned}&\mathcal{A}_k^{(1)} \\ &\ll \sum_{\substack{\underline{n}=(n_q)_{q\in\mathcal{P}(Q)}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{q\in\mathcal{P}(Q)}|n_q|\le M}} |\beta_{\underline{n}}| \left|\int_T^{2T} \exp\left(it\sum_{q\in\mathcal{P}(Q)} n_q \log q\right) \left|\sum_{Q<p\le x^2} \frac{\Lambda_{\mathcal{L},x}(p)(\log p)^{k-1}}{p^{s_0}}\right|^2 dt\right| \\ &\quad + \frac{1}{M}\exp(C_1 Q) \int_T^{2T} \left|\sum_{Q<p\le x^2} \frac{\Lambda_{\mathcal{L},x}(p)(\log p)^{k-1}}{p^{s_0}}\right|^2 dt \\ &\ll \int_T^{2T} \left|\sum_{Q<p\le x^2} \frac{\Lambda_{\mathcal{L},x}(p)(\log p)^{k-1}}{p^{s_0}}\right|^2 dt \\ &\quad + \sum_{\substack{\mathbf{0}\ne\underline{n}=(n_q)_{q\in\mathcal{P}(Q)}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{q\in\mathcal{P}(Q)}|n_q|\le M}} |\beta_{\underline{n}}| \left|\int_T^{2T} \exp\left(it\sum_{q\in\mathcal{P}(Q)} n_q \log q\right) \left|\sum_{Q<p\le x^2} \frac{\Lambda_{\mathcal{L},x}(p)(\log p)^{k-1}}{p^{s_0}}\right|^2 dt\right| \\ &\quad + \frac{1}{M}\exp(C_1 Q) \int_T^{2T} \left|\sum_{Q<p\le x^2} \frac{\Lambda_{\mathcal{L},x}(p)(\log p)^{k-1}}{p^{s_0}}\right|^2 dt =: \mathcal{A}_k^{(1,1)} + \mathcal{A}_k^{(1,2)} + \mathcal{A}_k^{(1,3)}.\end{aligned}$$

By using the estimates $|b(p)| = |a(p)| \ll_{\mathcal{L},\sigma_0} p^{\eta_0}$ and

$$\sum_{1\le n_1\le n_2\le T} \frac{1}{n_1^\alpha n_2^\alpha \log(n_2/n_1)} \ll_\alpha T^{2-2\alpha}\log T \quad \text{for} \quad 1/2 \le \alpha < 1,$$

we have

$$\begin{aligned}
&\mathcal{A}_k^{(1,1)}\\
&\leq \sum_{Q<p\leq x^2} \frac{|b(p)|^2(\log p)^{2k}}{p^{2\sigma_0}} T + O\left( \sum_{Q<p_1<p_2\leq x^2} \frac{|b(p_1)||b(p_2)|(\log p_1)^k(\log p_2)^k}{p_1^{\sigma_0}p_2^{\sigma_0}\log(p_2/p_1)} \right)\\
&\ll_{\mathcal{L},\sigma_0,N} Q^{(3/2)(1/2-\sigma_0)}(\log Q)^{2N-2}T + x^2(\log x)^{2N-1}.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathcal{A}_k^{(1,3)} &= \frac{1}{M}\exp(C_1 Q)\mathcal{A}_k^{(1,1)}\\
&\ll_{\mathcal{L},\sigma_0,N} \frac{1}{M}\exp(C_1 Q)\left(Q^{(3/2)(1/2-\sigma_0)}(\log Q)^{2N-2}T + x^2(\log x)^{2N-1}\right).
\end{aligned}$$

We will estimate $\mathcal{A}_k^{(1,2)}$. Fix $\mathbf{0} \neq \underline{n} = (n_q)_{q\in\mathcal{P}(Q)} \in \mathbb{Z}^{\mathcal{P}(Q)}$ with $\max_{q\in\mathcal{P}(Q)}|n_q| \leq M$. Then we have

$$\begin{aligned}
&\left| \int_T^{2T} \exp\left( it \sum_{q\in\mathcal{P}(Q)} n_q \log q \right) \left| \sum_{Q<p\leq x^2} \frac{\Lambda_{\mathcal{L},x}(p)(\log p)^{k-1}}{p^{s_0}} \right|^2 dt \right|\\
&\leq \sum_{Q<p\leq x^2} \frac{|b(p)|^2(\log p)^{2k}}{p^{2\sigma_0}} \left| \int_T^{2T} \exp\left( it \sum_{q\in\mathcal{P}(Q)} n_q \log q \right) dt \right|\\
&+O\Bigg( \sum_{Q<p_1<p_2\leq x^2} \frac{|b(p_1)||b(p_2)|(\log p_1)^k(\log p_2)^k}{p_1^{\sigma_0}p_2^{\sigma_0}} \times\\
&\qquad \times \left| \int_T^{2T} \exp\left( it \sum_{q\in\mathcal{P}(Q)} n_q \log q \right) \left(\frac{p_2}{p_1}\right)^{it} dt \right| \Bigg) =: \widetilde{\mathcal{A}}_k^{(1,2,1)}(\underline{n}) + \widetilde{\mathcal{A}}_k^{(1,2,2)}(\underline{n})
\end{aligned}$$

holds. We will estimate

$$\int_T^{2T} \exp\left( it \sum_{q\in\mathcal{P}(Q)} n_q \log q \right) dt.$$

Note that $\sum_{q\in\mathcal{P}(Q)} n_q \log q \neq 0$. Put

$$\mathcal{Q}^+(\underline{n}) = \{q \in \mathcal{P}(Q)\ ;\ n_q \geq 0\}, \quad \mathcal{Q}^-(\underline{n}) = \{q \in \mathcal{P}(Q)\ ;\ n_q < 0\},$$

and

$$Q^+(\underline{n}) = \prod_{q\in\mathcal{Q}^+(\underline{n})} q^{n_q}, \quad Q^-(\underline{n}) = \prod_{q\in\mathcal{Q}^-(\underline{n})} q^{-n_q}.$$

Then it holds that

$$\sum_{q\in\mathcal{P}(Q)} n_q \log q = \log\left(\frac{Q^+(\underline{n})}{Q^-(\underline{n})}\right).$$

By the asymptotic formula $\sum_{p\le X}\log p \sim X$ as $X\to\infty$, we have

$$\max\{Q^{+}(\underline{n}), Q^{-}(\underline{n})\} \le \exp\left(M\sum_{q\in\mathcal{P}(Q)}\log q\right) \le \exp\left(C_2QM\right) \tag{50}$$

for some absolute positive constant $C_2$. Hence, since it holds that

$$|\log(n/m)| > \frac{1}{\max\{m,n\}}$$

for any distinct positive integers $m$ and $n$, we have

$$\int_T^{2T}\exp\left(it\sum_{q\in\mathcal{P}(Q)}n_q\log q\right)dt \ll \exp\left(C_2QM\right). \tag{51}$$

Therefore we have

$$\widetilde{\mathcal{A}}_k^{(1,2,1)}(\underline{n}) \ll_{\sigma_0,N} Q^{(3/2)(1/2-\sigma_0)}(\log Q)^{2N-2}\exp\left(C_2QM\right). \tag{52}$$

By calculations similar to (52), we have

$$\begin{aligned}
&\widetilde{\mathcal{A}}_k^{(1,2,2)}(\underline{n})\\
&\ll \sum_{Q<p_1<p_2\le x^2}\frac{|b(p_1)||b(p_2)|(\log p_1)^k(\log p_2)^k}{p_1^{\sigma_0}p_2^{\sigma_0}}\left|\int_T^{2T}\left(\frac{p_2Q^{+}(\underline{n})}{p_1Q^{-}(\underline{n})}\right)^{it}dt\right|\\
&\ll \sum_{Q<p_1<p_2\le x^2}\frac{|b(p_1)||b(p_2)|(\log p_1)^k(\log p_2)^k}{p_1^{\sigma_0}p_2^{\sigma_0}}\max\{p_2Q^{+}(\underline{n}), p_1Q^{-}(\underline{n})\}\\
&\ll_{\mathcal{L},\sigma_0,N} x^4(\log x)^{2N-2}\exp\left(C_2QM\right).
\end{aligned}$$

Hence we have, by the estimate (11),

$$\begin{aligned}
\mathcal{A}_k^{(1,2)} &\le \sum_{\substack{\mathbf{0}\ne\underline{n}=(n_q)_{q\in\mathcal{P}(Q)}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{q\in\mathcal{P}(Q)}|n_q|\le M}}|\beta_{\underline{n}}|\left(\tilde{\mathcal{A}}_k^{(1,2,1)}(\underline{n})+\tilde{\mathcal{A}}_k^{(1,2,2)}(\underline{n})\right)\\
&\ll_{\mathcal{L},\sigma_0,N}\left(\sum_{\substack{\mathbf{0}\ne\underline{n}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{q\in\mathcal{P}(Q)}|n_q|\le M}}|\beta_{\underline{n}}|\right)x^4(\log x)^{2N-2}\exp\left(C_2QM\right)\\
&\ll x^4(\log x)^{2N-2}\exp(C_0Q)\exp\left(C_2QM\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\mathcal{A}_k^{(1)} \\
&\ll_{\mathcal{L},\sigma_0,N} \left(1+\frac{1}{M}\exp(C_1Q)\right)\left(Q^{(3/2)(1/2-\sigma_0)}(\log Q)^{2N-2}T + x^2(\log x)^{2N-1}\right) \\
&\quad + x^4(\log x)^{2N-2}\exp(C_0Q)\exp\left(C_2QM\right) \\
&\ll Q^{(3/2)(1/2-\sigma_0)}(\log Q)^{2N-2}T + x^4(\log x)^{2N-2}\exp\exp\left(C_3Q\right),
\end{aligned}$$

where we take $M=\exp(2C_1Q)$, and $C_3>2C_1$ is a positive absolute constant. Therefore we have

$$\begin{aligned}
\mathcal{A}_k \ll_{\mathcal{L},\sigma_0,N} & Q^{(3/2)(1/2-\sigma_0)}(\log Q)^{2N-2}T \\
& + x^4(\log x)^{2N-2}\exp\exp\left(C_3Q\right) \\
& + Q^{1/2-\sigma_0}(\log Q)^{2N-2}\int_{D_T}\Phi_Q\left(\underline{\gamma}(t)-\underline{\theta}^{(\star)}\right)dt.
\end{aligned}$$

*Bound for $\mathcal{B}_k$.* By calculations similar to (49), we have

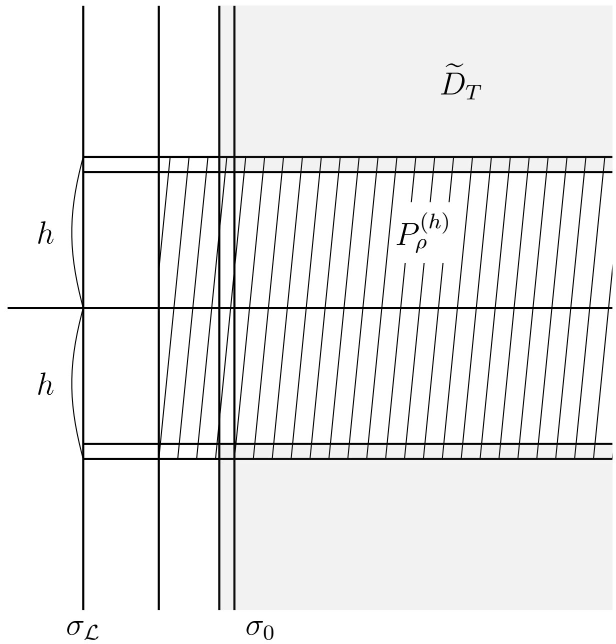
$$\sum_{p\le Q}\sum_{l>\frac{\log Q}{\log p}}\frac{b(p^l)(\log p^l)^k}{p^{ls_0}} \ll_{\mathcal{L},N,\sigma_0} Q^{1/2(1/2-\sigma_0)}(\log Q)^{N-1}.$$

Hence we have

$$\mathcal{B}_k \ll_{\mathcal{L},N,\sigma_0} Q^{1/2-\sigma_0}(\log Q)^{2N-2}\int_{D_T}\Phi_Q\left(\underline{\gamma}(t)-\underline{\theta}^{(\star)}\right)dt.$$

We will estimate $\mathcal{C}_k$, $\mathcal{D}_k$, $\mathcal{E}_k$, $\mathcal{F}_k$ using the following bound. We define

$$\begin{aligned}
&\widetilde{D}_T=\widetilde{D}_T(h) \\
=&\left\{s\ ;\ \sigma\ge\sigma_0-1/10(\sigma_0-\sigma_{\mathcal{L}}), T-1\le t\le 2T+1, s\notin\bigcup_{\rho\ ;\ \beta>1/2(\sigma_{\mathcal{L}}+\sigma_0)}P_\rho^{(h-1)}\right\}.
\end{aligned}$$




Then, by using the Cauchy integral formula, the estimate

$$|g^{(k-1)}(s_0)| = \left| \frac{(k-1)!}{2\pi i} \int_{|z-s_0|=1/10(\sigma_0-\sigma_{\mathcal{L}})} \frac{g(z)}{(z-s_0)^k} dz \right| \ll_{\mathcal{L},\sigma_0,N} \sup_{z\in \widetilde{D}_T} |g(z)| \tag{53}$$

holds for any holomorphic function $g(s)$ on $\widetilde{D}_T$ and $t \in D_T$ and for any $k = 1, \ldots, N-1$.

*Bound for $\mathcal{C}_k$, $\mathcal{D}_k$, $\mathcal{E}_k$.* By using the estimate (53), we can easily check that

$$\begin{aligned}
\mathcal{C}_k &\ll_{\mathcal{L},\sigma_0,N} \frac{1}{T^4 x^{2A(\sigma_0)} (\log x)^2} \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt,\\
\mathcal{D}_k &\ll_{\mathcal{L},\sigma_0,N} \frac{x^{4(1-A(\sigma_0))}}{T^4 (\log x)^2} \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt,\\
\mathcal{E}_k &\ll_{\mathcal{L},\sigma_0,N} \frac{1}{T^4 x^{2A(\sigma_0)} (\log x)^2} \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt,
\end{aligned}$$

where $A(\sigma_0) = \sigma_0 - 1/10(\sigma_0 - \sigma_{\mathcal{L}})$.

*Bound for $\mathcal{F}_k$.* We will estimate

$$\sup_{s\in\widetilde{D}_T} \left| \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(\rho-s)^2} \right|^2.$$

Fix $s = \sigma + it \in \widetilde{D}_T$. We divide the sum into two sums;

$$\begin{aligned}
&\sum_{\rho} \frac{x^{\rho - s} - x^{2(\rho - s)}}{(\rho - s)^2} \\
&= \sum_{\substack{\rho; \\ 0 < \beta \le (1/2)(\sigma_{\mathcal{L}} + \sigma_0)}} \frac{x^{\rho - s} - x^{2(\rho - s)}}{(\rho - s)^2} + \sum_{\substack{\rho; \\ (1/2)(\sigma_{\mathcal{L}} + \sigma_0) < \beta \le 1}} \frac{x^{\rho - s} - x^{2(\rho - s)}}{(\rho - s)^2} \\
&=: \Sigma_L + \Sigma_R.
\end{aligned}$$

By using the estimate $N_{\mathcal{L}}(T+1) - N_{\mathcal{L}}(T) \ll_{\mathcal{L}} \log T$ for $T \ge 2$ which is deduced by (2), we have

$$\begin{aligned}
\Sigma_R &\ll x^{2(1-\sigma)} \sum_{m \ge h-2} \sum_{m \le |\gamma - t| < m+1} \frac{1}{|\gamma - t|^2} \\
&\le x^{2(1-\sigma)} \sum_{m \ge h-2} \frac{1}{m^2} \sum_{m \le |\gamma - t| < m+1} 1 \ll_{\mathcal{L}} x^{2(1-\sigma)} \sum_{m \ge h-2} \frac{\log(t+m)}{m^2} \\
&\ll x^{2(1-\sigma)} \sum_{m \ge h-2} \frac{\max\{\log t, \log m\}}{m^2} \ll \frac{x \log T}{h}.
\end{aligned}$$

On the other hand, we have

$$\Sigma_L \ll_{\sigma_0} x^{-2/5(\sigma_0 - \sigma_{\mathcal{L}})} \sum_{\rho} \frac{1}{1 + (t - \gamma)^2} \ll_{\mathcal{L}} x^{-2/5(\sigma_0 - \sigma_{\mathcal{L}})} \log T.$$

Therefore we obtain

$$\begin{aligned}
&\mathcal{F}_k \\
&\ll_{\mathcal{L}, \sigma_0, N} \frac{1}{(\log x)^2} \left( \frac{x^2 (\log T)^2}{h^2} + x^{-4/5(\sigma_0 - \sigma_{\mathcal{L}})} (\log T)^2 \right) \int_{D_T} \Phi_Q \left( \underline{\gamma}(t) - \underline{\theta}^{(\star)} \right) dt.
\end{aligned}$$

Step 4. Next we will give the lower bound for $\int_{D_T} \Phi_Q \left( \underline{\gamma}(t) - \underline{\theta}^{(\star)} \right) dt$. By the estimate (11), it holds that

$$\begin{aligned}
&\int_{D_T} \Phi_Q \left( \underline{\gamma}(t) - \underline{\theta}^{(\star)} \right) dt \\
&= \int_T^{2T} \Phi_Q \left( \underline{\gamma}(t) - \underline{\theta}^{(\star)} \right) dt - \int_{[T, 2T] \setminus D_T} \Phi_Q \left( \underline{\gamma}(t) - \underline{\theta}^{(\star)} \right) dt \\
&\ge \int_T^{2T} \Phi_Q \left( \underline{\gamma}(t) - \underline{\theta}^{(\star)} \right) dt - 2h \sum_{\underline{n} \in \mathbb{Z}^{\mathcal{P}(Q)}} |\beta_{\underline{n}}| N_{\mathcal{L}} \left( 1/2(\sigma_{\mathcal{L}} + \sigma_0), 2T \right) \\
&= \int_T^{2T} \Phi_Q \left( \underline{\gamma}(t) - \underline{\theta}^{(\star)} \right) dt + O \left( h \exp(C_0 Q) N_{\mathcal{L}} \left( 1/2(\sigma_{\mathcal{L}} + \sigma_0), 2T \right) \right).
\end{aligned}$$

By the estimates (11), (12) and (51) and by substituting $M = \exp(2C_1Q)$, we have

$$\begin{aligned}&\int_T^{2T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt \\ \geq& T - \sum_{\substack{\mathbf{0}\neq \underline{n}\in\mathbb{Z}^{\mathcal{P}(Q)};\\ \max_{p\leq Q}|n_p|\leq M}} |\beta_{\underline{n}}| \left| \int_T^{2T} \exp\left(it\sum_{p\leq Q} n_p \log p\right) dt\right| + O\left(\frac{T}{M}\exp(C_1Q)\right) \\ =& T + O\left(\exp\exp(C_3Q)\right) + O\left(\frac{T}{M}\exp(C_1Q)\right) \\ =& \left(1 + O\left(\frac{\exp\exp(C_3Q)}{T}\right) + O\left(\frac{1}{\exp(C_1Q)}\right)\right) T.\end{aligned}$$

Taking $h = T^{\Delta_{\mathcal{L}}(1/2(1/2+\sigma_0))/2}$, we have

$$h\exp(C_0Q)N_{\mathcal{L}}\left(1/2(\sigma_{\mathcal{L}}+\sigma_0), 2T\right) \ll_{\mathcal{L},\sigma_0} \exp(C_0Q)T^{1-\frac{\Delta_{\mathcal{L}}(1/2(\sigma_{\mathcal{L}}+\sigma_0))}{2}}$$

by the condition (C2). Hence we have

$$\begin{aligned}\int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt \geq \Biggl(1 + O_{\mathcal{L},\sigma_0}\left(\exp(C_0Q)T^{-\frac{\Delta(1/2(\sigma_{\mathcal{L}}+\sigma_0))}{2}}\right) + \\ + O\left(\frac{\exp\exp(C_3Q)}{T}\right) + O\left(\frac{1}{\exp(C_1Q)}\right)\Biggr) T.\end{aligned}$$

Taking $T = \exp\exp(CQ)$, $C = C_0 + C_3$, we have the inequality (41).

Step 5. We will finish the proof in this step. By the estimate (41) and by the estimates $\mathcal{A}_k$-$\mathcal{F}_k$, we obtain

$$\begin{aligned}\mathcal{I} \ll_{\mathcal{L},\sigma_0,N} \Biggl(Q^{1/2-\sigma_0}(\log Q)^{2N-2} + \frac{x^4(\log x)^{2N-2}}{T^{1/2}} + \frac{x^2(\log T)^2}{T^{\Delta_{\mathcal{L}}(1/2(1/2+\sigma_0))}} + \\ + x^{-4/5(\sigma_0-\sigma_{\mathcal{L}})}(\log T)^2\Biggr) \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt.\end{aligned}$$

Taking

$$x = T^{\mu} \quad \text{with} \quad \mu = \min\left\{1/200, \Delta(1/2(\sigma_{\mathcal{L}}+\sigma_0))/10\right\},$$

we have

$$\mathcal{I} \ll_{\mathcal{L},\sigma_0,N} Q^{1/2(1/2-\sigma_0)} \int_{D_T} \Phi_Q\left(\underline{\gamma}(t) - \underline{\theta}^{(\star)}\right) dt. \tag{54}$$

Put

$$d(\sigma_0, E_{\mathcal{L}}) = \max\left\{d_1(\sigma_0, E_{\mathcal{L}}), \frac{8}{\sigma_0 - 1/2}\right\}.$$

By the estimates (43) and (54) and step 4, we have (40), (41) and (42) if

$$Q \geq C_1^{(1)}(\mathcal{L}, \sigma_0, N)\left(\|\underline{c}\|_N + 1/\varepsilon\right)^{d(\sigma_0, E_{\mathcal{L}})}$$

holds. Taking $C_1(\mathcal{L}, \sigma_0, N) = CC_1^{(1)}(\mathcal{L}, \sigma_0, N)$, we have the conclusion. □

3.3.2. *Proof of Corollary 1.20.* We will prove by using the same method as in [**48**] and [**21**]. Before stating the proof, we give some notations and lemmas.

Let $\mathbb{C}[[X]]$ be the formal power series ring with coefficients in $\mathbb{C}$ and indeterminate $X$. We refer the reader to [**2**] for the details of the general theory of the formal power series ring for example. In what follows, we will use the following notations;

- Let $\mathbb{R}_+$ denote the set of nonnegative real numbers.
- For any $N \in \mathbb{N}$, we write $[N] = [1, N] \cap \mathbb{N}$.
- For empty index $\emptyset \in \mathbb{C}^0$, we adopt the convention that $\|\emptyset\|_0 = 0$.
- Let $N \in \mathbb{N}$ and $[N] \subset A \subset \mathbb{N}$. For $\boldsymbol{z} = (z_j)_{j\in A} \in \mathbb{C}^A$, we write $\boldsymbol{z}_{[N]} = (z_j)_{j=1}^N \in \mathbb{C}^N$.
- Let $\alpha(X) = \sum_{n=0}^\infty a_n X^n \in \mathbb{C}[[X]]$ and $\beta(X) = \sum_{n=0}^\infty b_n X^n \in \mathbb{C}[[X]]$ with $b_n \in \mathbb{R}_+$ for any $n \in \mathbb{N}_0$. The statement $\alpha \trianglelefteq \beta$ means that $|a_n| \le b_n$ holds for any $n \in \mathbb{N}$. We also define $\alpha^{\mathrm{abs}}(X) \in \mathbb{C}[[X]]$ by $\alpha^{\mathrm{abs}}(X) = \sum_{n=0}^\infty |a_n| X^n$.
- Let $\underline{Z} = (Z_j)_{j=1}^\infty \in \mathbb{C}^{\mathbb{N}}$ be the indeteminate and let $N$ be a positive integer. For a multi-index $\underline{i} = (i_1, \dots, i_N) \in \mathbb{N}_0^N$ and for $f(\underline{Z}) \in \mathbb{C}[\underline{Z}]$, define the symbol $\partial^{\underline{i}}$ as a differential operator given by

$$\partial^{\underline{i}}(f(\underline{Z})) = \frac{\partial^{|\underline{i}|} f}{\partial Z_1^{i_1} \cdots \partial Z_N^{i_N}}(\underline{Z}), \quad |\underline{i}| = i_1 + \cdots + i_N.$$

  In addition, we write $\underline{i}! = i_1! \cdots i_N!$ and $(\underline{Z}_{[N]})^{\underline{i}} = Z_1^{i_1} \cdots Z_N^{i_N}$ .

Let $\underline{Z} = (Z_j)_{j=1}^\infty$ and $\underline{W} = (W_j)_{j=1}^\infty$ be the indeterminates and write $\underline{Z}_{[n]} = (Z_j)_{j=1}^n$ and $\underline{W}_{[n]} = (W_j)_{j=1}^n$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$, define the polynomials $F_n(\underline{Z}_{[n]}) \in \mathbb{R}[\underline{Z}_{[n]}]$ and $G_n(\underline{W}_{[n]}) \in \mathbb{R}[\underline{Z}_{[n]}]$ by

$$\exp\left(\sum_{n=1}^\infty Z_n X^n\right) = 1 + \sum_{n=1}^\infty F_n(\underline{Z}_{[n]}) X^n \tag{55}$$

and

$$\log\left(1 + \sum_{n=1}^\infty W_n X^n\right) = \sum_{n=1}^\infty G_n(\underline{W}_{[n]}) X^n.$$

Then we have $\deg(F_n) = n$. We also define the maps

$$\underline{F} : \mathbb{C}^{\mathbb{N}} \ni \underline{z} = (z_j)_{j=1}^\infty \mapsto \underline{F}(\underline{z}) = (F_j(\underline{z}_j))_{j=1}^\infty \in \mathbb{C}^{\mathbb{N}},$$

$$\underline{F}_{[N]} : \mathbb{C}^N \ni \boldsymbol{z} = (z_j)_{j=1}^N \mapsto \underline{F}_{[N]}(\boldsymbol{z}) = (F_j(\boldsymbol{z}_{[j]}))_{j=1}^N \in \mathbb{C}^{\mathbb{N}},$$

and

$$\underline{G} : \mathbb{C}^{\mathbb{N}} \ni \underline{w} = (w_j)_{j=1}^\infty \mapsto \underline{G}(\underline{w}) = (G_j(\underline{w}_{[j]}))_{j=1}^\infty \in \mathbb{C}^{\mathbb{N}}.$$

We can easily check that $\underline{G}$ is the inverse mapping of $\underline{F}$. For complex variables $\underline{z} = (z_j)_{j=1}^\infty$, define

$$f(X; \underline{z}) = \sum_{n=1}^\infty z_n X^n,$$

$$g(X, \underline{z}) = \log\left(1 + \sum_{n=1}^\infty z_n X^n\right)$$

and

$$h(X;\underline{z}) = -\log\left(1-\sum_{n=1}^{\infty}|z_n|X^n\right).$$

Note that the relations

$$f(X;\underline{z}) = g(X,\underline{F}(\underline{z})) \quad \text{and} \quad g(X,\underline{z}) \trianglelefteq h(X;\underline{z}) \tag{56}$$

hold.

LEMMA 3.13. *Let $\underline{z}=(z_j)_{j=1}^{\infty}$ be complex variables.*

(i) *We have $f^{\mathrm{abs}}(X;\underline{z}) \trianglelefteq h(X;\underline{F}(\underline{z}))$.*
(ii) *Let $N\in\mathbb{N}$ and $\underline{i}=(i_1,\dots,i_{N-1})\in\mathbb{N}_0^{N-1}$ with $|\underline{i}|\geq 1$. Then we have $f^{\mathrm{abs}}(X;\partial^{\underline{i}}\underline{F}(\underline{z}))\trianglelefteq X^{S(\underline{i})}\exp\left(f^{\mathrm{abs}}(X;\underline{z})\right)$, where $\partial^{\underline{i}}\underline{F}(\underline{z})=(\partial^{\underline{i}}F_j(\underline{z}_{[j]}))_{j=1}^{\infty}$ and $S(\underline{i})=i_1+2i_2+\cdots(N-1)i_{N-1}$.*

PROOF. Note that $\alpha(X)\trianglelefteq\beta(X)$ implies $\alpha^{\mathrm{abs}}(X)\trianglelefteq\beta(X)$ for $\alpha(X),\beta(X)\in\mathbb{C}[[X]]$. This and the relation (56) deduce the first assertion. We fix $N\in\mathbb{N}$ and $\underline{i}=(i_1,\dots,i_{N-1})\in\mathbb{N}_0^{N-1}$ with $|\underline{i}|\geq 1$. By differentiating $\exp\left(f(X;\underline{Z})\right)$, we have

$$\frac{\partial}{\partial Z_j}\exp\left(f(X;\underline{Z})\right) = X^j\exp\left(f(X;\underline{Z})\right).$$

Hence, applying $\partial^{\underline{i}}$ on the both sides of the equation (55) with $\underline{z}$ in place of $\underline{Z}$, we obtain

$$X^{S(\underline{i})}\exp\left(f(X;\underline{z})\right) = f(X;\partial^{\underline{i}}\underline{F}(\underline{z})).$$

We can confirm that $\exp\left(f(X;\underline{z})\right)\trianglelefteq\exp\left(f^{\mathrm{abs}}(X;\underline{z})\right)$, and this completes the proof. □

LEMMA 3.14. *Fix $\varepsilon>0$ and $N\in\mathbb{N}$. Let $z_0,a_0\in\mathbb{C}$ and $\boldsymbol{z}=(z_1,z_2,\dots,z_{N-1})$, $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_{N-1})\in\mathbb{C}^{N-1}$.*

(i) *We have*

$$\|\boldsymbol{\alpha}\| \ll_N \left(1+\|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\|\right)^{N-1}$$

(ii) *If $\|(z_0,\boldsymbol{z})-(\alpha_0,\boldsymbol{\alpha})\|<\delta<1$, then we have*

$$|e^{z_0}-e^{\alpha_0}|<2|e^{\alpha_0}|\delta$$

*and*

$$\left\|e^{z_0}\underline{F}_{[N-1]}(\boldsymbol{z})-e^{\alpha_0}\underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\| \ll_N |e^{\alpha_0}|\left(1+\left\|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\|\right)^{(N-1)^2}\delta.$$

PROOF. Let

$$\underline{\boldsymbol{\alpha}^{(0)}} = (\alpha_1,\dots,\alpha_{N-1},0,0,\dots)\in\mathbb{C}^{\mathbb{N}}$$

and

$$\underline{w^{(0)}} = (F_1(\boldsymbol{\alpha}_{[1]}),F_2(\boldsymbol{\alpha}_{[2]}),\dots,F_{N-1}(\boldsymbol{\alpha}),0,0,\dots)\in\mathbb{C}^{\mathbb{N}}.$$

Note that

$$\underline{G}\left(\underline{w^{(0)}}\right)_{[N-1]} = \boldsymbol{\alpha},$$

holds. Then we have

$$\sum_{n=1}^{N-1}|\alpha_n|X^n = f^{\mathrm{abs}}\left(X;\underline{\alpha^{(0)}}\right) \trianglelefteq f^{\mathrm{abs}}\left(X;\underline{G}(\underline{w^{(0)}})\right) \tag{57}$$

$$\trianglelefteq h\left(X;\underline{F}\left(\underline{G}(\underline{w^{(0)}})\right)\right) = h\left(X;\underline{w^{(0)}}\right) = -\log\left(1-\sum_{n=1}^{N-1}|F_n(\boldsymbol{\alpha}_{[n]})|X^n\right)$$

by (i) of Lemma 3.13. Evaluating (57) at $X = \left(3\left(1+\left\|F_{[N-1]}(\boldsymbol{\alpha})\right\|\right)\right)^{-1}$, we have

$$\left(\frac{1}{3\left(1+\left\|F_{[N-1]}(\boldsymbol{\alpha})\right\|\right)}\right)^{N-1}\sum_{n=1}^{N-1}|\alpha_n| \le \sum_{n=1}^{N-1}|\alpha_n|\left(\frac{1}{3\left(1+\left\|F_{[N-1]}(\boldsymbol{\alpha})\right\|\right)}\right)^n$$

$$\le -\log\left(1-\sum_{n=1}^{N-1}\left(\frac{1}{3}\right)^n\right) \ll 1,$$

which deduces $\|\boldsymbol{\alpha}\| \ll_N \left(1+\left\|F_{[N-1]}(\boldsymbol{\alpha})\right\|\right)^{N-1}$. Hence, we have the first assertion of this lemma.

To prove the second assertion of this lemma, we will show

$$\sum_{n=1}^{N-1}\left|\partial^{\underline{i}}F_n(\boldsymbol{\alpha}_{[n]})\right| \ll_N \left(1+\|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\|\right)^{(N-1)^2} \tag{58}$$

for $\underline{i}\in\mathbb{N}_0^n$ and $|\underline{i}|\ge 1$. By (ii) of Lemma 3.13, we have

$$\sum_{n=1}^{N-1}\left|\partial^{\underline{i}}F_n(\boldsymbol{\alpha}_{[n]})\right|X^n \trianglelefteq f^{\mathrm{abs}}(X;\partial^{\underline{i}}\underline{F}(\underline{\alpha^{(0)}})) \tag{59}$$

$$\trianglelefteq X^{S(\underline{i})}\exp\left(f^{\mathrm{abs}}(X;\underline{\alpha^{(0)}})\right) = X^{S(\underline{i})}\exp\left(\sum_{n=1}^{N-1}|\alpha_n|X^n\right).$$

By evaluating (59) at $X = (3(1+\|\boldsymbol{\alpha}\|))^{-1}$, we have

$$\left(\frac{1}{3(1+\|\boldsymbol{\alpha}\|)}\right)^{N-1}\sum_{n=1}^{N-1}\left|\partial^{\underline{i}}F_n(\boldsymbol{\alpha}_{[n]})\right| \le \sum_{n=1}^{N-1}\left|\partial^{\underline{i}}F_n(\boldsymbol{\alpha}_{[n]})\right|\left(\frac{1}{3(1+\|\boldsymbol{\alpha}\|)}\right)^n$$

$$\le \left(\frac{1}{3}\right)^{S(\underline{i})}\exp\left(\sum_{n=1}^{N-1}\left(\frac{1}{3}\right)^n\right) \ll 1,$$

which implies $\sum_{n=1}^{N-1}\left|\partial^{\underline{i}}F_n(\boldsymbol{\alpha}_{[n]})\right| \ll_N (1+\|\boldsymbol{\alpha}\|)^{N-1}$. By (i) of Lemma 3.14, we obtain

$$\sum_{n=1}^{N-1}\left|\partial^{\underline{i}}F_n(\boldsymbol{\alpha}_{[n]})\right| \ll_N \left(1+\|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\|\right)^{(N-1)^2},$$

which gives the estimate (58).

We will show the second assertion. Let $\|(z_0, \boldsymbol{z}) - (\alpha_0, \boldsymbol{\alpha})\| < \delta < 1$. By the Taylor series expansion, we have

$$|e^{z_0} - e^{\alpha_0}| = |e^{\alpha_0}| \left|e^{z_0-\alpha_0} - 1\right| \leq |e^{\alpha_0}| \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \leq 2\,|e^{\alpha_0}|\,\delta. \tag{60}$$

By the Taylor series expansion, by the equation (58) and by using $\deg F_n = n$ , we have

$$\begin{aligned} &F_n(\boldsymbol{z}_{[n]}) - F_n(\boldsymbol{\alpha}_{[n]}) \\ &= \sum_{\substack{\underline{i}=(i_1,\ldots,i_n)\in\mathbb{N}_0^n;\\ 1\leq|\underline{i}|\leq n}} \frac{\partial^{\underline{i}} F_n(\boldsymbol{\alpha}_{[n]})}{\underline{i}!} \left(\boldsymbol{z}_{[n]} - \boldsymbol{\alpha}_{[n]}\right)^{\underline{i}} \ll_N \left(1 + \|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\|\right)^{(N-1)^2} \delta \end{aligned} \tag{61}$$

for $1 \leq n \leq N-1$. Hence we deduce

$$\left\|\underline{F}_{[N-1]}(\boldsymbol{z}) - \underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\| \ll_N \left(1 + \|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\|\right)^{(N-1)^2} \delta.$$

Therefore we obtain

$$\begin{aligned} &\left\|e^{z_0}\underline{F}_{[N-1]}(\boldsymbol{z}) - e^{\alpha_0}\underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\| \\ \leq &|e^{z_0}| \left\|\underline{F}_{[N-1]}(\boldsymbol{z}) - \underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\| + \left\|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\| |e^{z_0} - e^{\alpha_0}| \\ \leq &|e^{z_0} - e^{\alpha_0}| \left\|\underline{F}_{[N-1]}(\boldsymbol{z}) - \underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\| \\ &\quad + |e^{\alpha_0}| \left\|\underline{F}_{[N-1]}(\boldsymbol{z}) - \underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\| + \left\|\underline{F}_{N-1}(\boldsymbol{\alpha})\right\| |e^{z_0} - e^{\alpha_0}| \\ \ll_N &|e^{\alpha_0}| \left(1 + \left\|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\|\right)^{(N-1)^2} \delta \end{aligned}$$

by the estimates (60) and (61). This completes the proof. □

Proof of Corollary 1.20. Let $\varepsilon \in (0,1)$, $\underline{c} = (c_k)_{k=0}^{N-1} \in \mathbb{C}^N$ with $|c_0| \neq 0$. Put

$$z_0(t) = \log \mathcal{L}(\sigma_0 + it),$$

$$\boldsymbol{z}(t) = \left(\frac{1}{1!}\frac{d}{ds}\log\mathcal{L}(\sigma_0+it), \ldots, \frac{1}{(N-1)!}\frac{d^{N-1}}{ds^{N-1}}\log\mathcal{L}(\sigma_0+it)\right),$$

$$\beta_1 = \frac{c_1}{c_0 \cdot 1!},\ \beta_2 = \frac{c_2}{c_0 \cdot 2!} \ldots,\ \beta_{N-1} = \frac{c_{N-1}}{c_0(N-1)!}, \boldsymbol{\beta} = (\beta_k)_{k=1}^{N-1}$$

and

$$\alpha_0 = \log c_0,\ \alpha_1 = G_1(\boldsymbol{\beta}_{[1]}), \ldots, \alpha_n = G_{N-1}(\boldsymbol{\beta}_{[N-1]}), \boldsymbol{\alpha} = (\alpha_j)_{j=1}^{N-1}.$$

Note that $\boldsymbol{\beta} = \underline{F}_{[N-1]}(\boldsymbol{\alpha})$ holds. Let

$$\delta = \delta(\varepsilon, \underline{c}, \Delta_N) = \frac{\Delta_N \cdot \varepsilon}{(1 + |e^{\alpha_0}|)\left(1 + \left\|\underline{F}_{[N-1]}(\boldsymbol{\alpha})\right\|\right)^{(N-1)^2}} \in (0,1),$$

where $\Delta_N$ is sufficiently small depending on $N$. Note that, by the relation (55), we have

$$e^{z_0(t)}\underline{F}_{[N-1]}(\boldsymbol{z}(t)) = \left(\frac{1}{1!}\frac{d}{ds}\mathcal{L}(\sigma_0+it), \ldots, \frac{1}{(N-1)!}\frac{d^{N-1}}{ds^{N-1}}\mathcal{L}(\sigma_0+it)\right)$$

for $\mathcal{L}(\sigma_0 + it) \neq 0$. Hence if

$$\|(z_0(t), \boldsymbol{z}(t)) - (\alpha_0, \boldsymbol{\alpha})\| < \delta \quad \text{and} \quad \mathcal{L}(\sigma_0 + it) \neq 0 \tag{62}$$

hold, then we have

$$\left|\frac{d^k}{ds^k}\,\mathcal{L}(\sigma_0+it)-c_k\right|<\varepsilon \quad \text{for } k=0,1,\dots,N-1$$

by Lemma 3.14 (ii). To get the inequality (62), it is enough to notice

$$\left|\frac{d^k}{ds^k}\log\mathcal{L}(\sigma_0+it)-k!\alpha_k\right|<\frac{\delta}{N} \quad \text{for } k=0,1,\dots,N-1.$$

By Lemma 3.14 (i), we have

$$\|(\alpha_0,1!\alpha_1,\dots,(N-1)!\alpha_{N-1})\|+\frac{N}{\delta}\ll_N |\log c_0|+\left(\frac{\|\underline{c}\|}{|c_0|}\right)^{(N-1)^2}\frac{1+|c_0|}{\varepsilon}.$$

Combining with Theorem 1.19, we have the conclusion. □

3.3.3. *Proof of Corollary 1.21.* Although the proof is almost the same as in [**13**], we shall give the full details.

Proof of Corollary 1.21. Let the setting be as in Corollary 1.21. We will use the Taylor expansion series to prove the corollary. Recall the Cauchy integral formula

$$g^{(k)}(s_0)=\frac{k!}{2\pi i}\int_{|z-s_0|=r}\frac{g(z)}{(z-s_0)^{k+1}}dz.$$

Hence we have

$$|g^{(k)}(s_0)(s-s_0)^k|\le k!M(g)\delta_0^k$$

for $|s-s_0|\le\delta_0 r$. By using the Taylor expansion series, we have

$$\Sigma_1:=\left|g(s)-\sum_{0\le k<N}\frac{g^{(k)}(s_0)}{k!}(s-s_0)^k\right|\le M(g)\sum_{k=N}^{\infty}\delta_0^k=M(g)\frac{\delta_0^N}{1-\delta_0}$$

for $|s-s_0|\le\delta_0 r$. We chose $N=N(\delta_0,\varepsilon,M(g))$ such that

$$M(g)\frac{\delta_0^N}{1-\delta_0}<\frac{\varepsilon}{3}.$$

We apply Corollary 1.20 with $c_k=g^{(k)}(s_0)$ and $(\varepsilon/3)\exp(-\delta_0 r)$ in place of $\varepsilon$. We chose $T=T(\mathcal{L},\sigma_0,g,\varepsilon,\delta_0,N)$ such that

$$T\ge\max\left\{\exp\exp\left(C_2(\mathcal{L},\sigma_0,N)B\left(N,\mathbf{g},(\varepsilon/3)\exp(-\delta_0 r)\right)^{d(\sigma_0,E_{\mathcal{L}})}\right),r\right\}.$$

Then there exists $t_1\in[T,2T]$ such that

$$\left|\mathcal{L}^{(k)}(\sigma_0+it_1)-g^{(k)}(\sigma_0+it_0)\right|<\frac{\varepsilon}{3}\exp(-\delta_0 r)$$

for $0\le k<N$. Put $\tau:=t_1-t_0$ and note that $\sigma_0+it_1=s_0+i\tau$ holds. Remark that our choice of $T\ge r$ make the disc $\{s\ ;\ |s-s_0|\le\delta_0 r\}+i\tau$ avoid from including the pole of $\mathcal{L}(s)$.

Hence we have

$$\Sigma_2 := \left| \sum_{0 \le k < N} \frac{\mathcal{L}^{(k)}(s_0 + i\tau)}{k!}(s - s_0)^k - \sum_{0 \le k < N} \frac{g^{(k)}(s_0)}{k!}(s - s_0)^k \right|$$
$$< \frac{\varepsilon}{3} \exp(-\delta_0 r) \sum_{0 \le k < N} \frac{(\delta_0 r)^k}{k!} < \frac{\varepsilon}{3}$$

for $|s - s_0| \le \delta_0 r$. On the other hand, we have

$$\Sigma_3 := \left| \mathcal{L}(s + i\tau) - \sum_{0 \le k < N} \frac{\mathcal{L}^{(k)}(s_0 + i\tau)}{k!}(s - s_0)^k \right| < M(\tau) \frac{\delta^N}{1 - \delta}$$

for $|s - s_0| \le \delta r$ and $0 < \delta \le \delta_0$. Therefore we have

$$|\mathcal{L}(s + i\tau) - g(s)| \le \Sigma_1 + \Sigma_2 + \Sigma_3 < \frac{2}{3}\varepsilon + M(\tau) \frac{\delta^N}{1 - \delta}$$

for $|s - s_0| \le \delta r$ and $0 < \delta \le \delta_0$. Choosing $\delta$ which satisfies

$$M(\tau; \mathcal{L}) \frac{\delta^N}{1 - \delta} < \frac{\varepsilon}{3},$$

we have the conclusion. □

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