Higher dimensional case of sharper estimates of Ohsawa–Takegoshi L^2-extension theorem

(大沢–竹腰 L^2 拡張定理の精密な評価の高次元化)

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Introduction

The *Ohsawa–Takegoshi L^2-extension theorem* [**21**] states the following: Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, V be a closed complex submanifold of Ω and φ be a plurisubharmonic function on Ω. Suppose that a holomorphic function $f \in \mathcal{O}(V)$ is given. Then there exists an $F \in \mathcal{O}(\Omega)$ satisfying the L^2-estimate

$$\int_{\Omega} |F|^2 e^{-\varphi} \leq C \int_V |f|^2 e^{-\varphi},$$

where C is the positive constant that is independent of the weight φ and a given $f \in \mathcal{O}(V)$. The Ohsawa–Takegoshi L^2-extension theorem and its generalizations are applied widely in the studies of complex analysis as well as complex geometry and algebraic geometry. For example, Demailly's approximation of plurisubharmonic functions [**9**], Siu's invariance of plurigenera [**23**] and so forth. The *Green function* on a domain in $\mathbb{C}$ is a solution of the Laplace equation. More precisely, this function is the upper envelope of negative subharmonic functions with a logarithmic pole at the given point. Since the Green function have many information of the domain on which this is defined, many studies were conducted in complex analysis. The Suita conjecture was a long-standing conjecture about a relationship between Bergman kernels and logarithmic capacities. The study of the interplay of the Ohsawa–Takegoshi L^2-extension theorem and the Green function was essential in its resolution.

In [**5**], [**6**] and [**11**], the *optimal L^2-extension theorem* was proved. This means that we can determine the positive constant C in the best possible way. Many problems including Suita conjecture were solved by using the optimal L^2-estimate. Here we state the Berndtsson–Lempert type optimal L^2-extension theorem [**5**].

Theorem A (optimal L^2-extension theorem, [**5**], [**6**], [**11**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, V a closed complex submanifold of Ω with codimension k and φ a plurisubharmonic function on Ω. Let G be a negative plurisubharmonic function on Ω such that*

$$\log d_V^2(z) - B(z) \leq G(z) \leq \log d_V^2(z) + A(z), \tag{0.0.1}$$

where $d_V(z)$ is the distance between $z \in \Omega$ and V, $A(z)$ and $B(z)$ are continuous functions on Ω. Then for any holomorphic function f on V with $\int_V |f|^2 e^{-\varphi + kB} < \infty$, there exists a holomorphic function F such that $F|_V = f$ and

$$\int_{\Omega} |F|^2 e^{-\varphi} \leq \sigma_k \int_V |f|^2 e^{-\varphi + kB},$$

where σ_k is the volume of the unit ball in $\mathbb{C}^k$.

A negative plurisubharmonic function G satisfying (0.0.1) is called a Green-type function on Ω with poles along V [**13**]. Characterizing domain $\Omega \subset \mathbb{C}^n$ and closed subvarieties $V \subset \Omega$ admitting a Green-type function with poles along V in terms of their geometry is an open problem.

Hosono proposed in [**13**] an idea of getting an L^2-estimate sharper than the one of Theorem A by allowing constants depending on weight functions. In other words, we can find a positive constant C' depending on weight functions sharper than the optimal constant C. Here we state this result in [**13**].

Theorem B (Sharper estimates of L^2-extension theorem, [**13**]). *Let Ω be a bounded domain in $\mathbb{C}$ containing 0 and φ a subharmonic function on Ω with $\varphi(0) = 0$. Let $\tilde{\Omega}$ be a domain in $\mathbb{C}^2$ defined by*

$$\tilde{\Omega} := \{(z, w) \in \mathbb{C}^2 : z \in \Omega, |w|^2 < e^{-\varphi(z)}\}$$

and $\tilde{G}$ a Green-type function on $\tilde{\Omega}$ with poles $\{z = 0\}$, i.e., $\tilde{G}$ is a negative plurisubharmonic function on $\tilde{\Omega}$ such that there exists continuous functions $\tilde{A}, \tilde{B}$ on $\tilde{\Omega}$ such that

$$\log |z|^2 - \tilde{B}(z, w) \leq \tilde{G}(z, w) \leq \log |z|^2 + \tilde{A}(z, w).$$

(1) *Then there exists a holomorphic function f on Ω such that $f(0) = 1$ and*

$$\int_{\Omega} |f(z)|^2 e^{-\varphi(z)} \leq \int_{|w|<1} e^{\tilde{B}(0,w)}.$$

(2) *Suppose that $\tilde{\Omega}$ is a strictly pseudoconvex domain. Then one can make the estimate in* (1) *strictly sharper than the one in Theorem A, i.e., there exist functions $\tilde{G}$ and $\tilde{B}$ satisfying the above conditions such that*

$$\int_{|w|<1} e^{\tilde{B}(0,w)} < \pi e^{B(0)}$$

where $B(z) = G(z) - \log |z|^2$ is a harmonic function on Ω and $G(z)$ is the Green function on Ω with a pole at $\{0\}$.

It is an idea used in the proof of Theorem B to use solutions of the Dirichlet problem for complex Monge-Ampère equation to construct the Green-type function with poles along $\{z = 0\}$. As an application of Theorem B, in the case where Ω is the unit disc $\{|z| < 1\}$ in $\mathbb{C}$ and φ is a radial subharmonic function, Hosono was able to determine the L^2-minimum extension of the function 1 on the subvariety $\{0\}$.

In this thesis, for a bounded pseudoconvex domain in $\mathbb{C}^n$ and a closed complex submanifold of it with some conditions, we generalize Theorem B. In general, for a closed complex submanifold, it is difficult to construct the Green-type function because we do not know whether the logarithmic distance from a closed complex submanifold is plurisubharmonic. To generalize this result, we establish an analogue of Theorem A by using the theory of the pluricomplex Green function with poles along subvarieties [**22**]. Our first main theorem is as follows.

Theorem C (Theorem 4.1.1, [**16**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, V a closed complex submanifold of Ω with codimension k such that V has bounded global generators $\psi = (\psi_1, \dots, \psi_k)$ and there exists a positive constant C such that $\frac{1}{C} \leq |J_\psi|$ near V, where J_ψ is a Jacobian of ψ for suitable coordinates. Let φ be a plurisubharmonic function on Ω and $G_{\Omega,V}$ the pluricomplex Green function on Ω with poles along V. Assume that there exists some continuous function B on Ω such that*

$$\log|\psi(z)| - B(z) \leq G_{\Omega,V}(z).$$

Then for any holomorphic function f on V with $\int_V |f|^2 e^{-\varphi+2kB} < \infty$, there exists a holomorphic function F on Ω such that $F|_V = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} \leq C\sigma_k \int_V |f|^2 e^{-\varphi+2kB},$$

where σ_k is the volume of the unit ball in $\mathbb{C}^k$.

In Chapter 4, we give a geometric interpretation for the assumptions of Theorem C.

Under the conditions in Theorem C, we consider a pseudoconvex domain $\tilde{\Omega}$ in $\mathbb{C}^{n+k}$ defined by

$$\tilde{\Omega} = \{(z, w) \in \mathbb{C}^{n+k} : z \in \Omega, |w|^2 < e^{-\frac{\varphi(z)}{k}}\}$$

and a closed complex submanifold $\tilde{V}$ of $\tilde{\Omega}$ such that

$$\tilde{V} = \{\tilde{\psi}_1 = \cdots = \tilde{\psi}_k = 0\},$$

where $\tilde{\psi}_i(z, w) := \psi_i(z)$ are holomorphic functions on $\tilde{\Omega}$. Let $\tilde{G}$ be the pluricomplex Green function on $\tilde{\Omega}$ with poles along $\tilde{V}$ or a subsolution of it such that there exists a continuous function $\tilde{B}(z, w)$ on $\tilde{\Omega}$ such that

$$\log|\tilde{\psi}(z, w)| - \tilde{B}(z, w) \leq \tilde{G}(z, w). \tag{0.0.2}$$

Then, by using Theorem C, we can obtain the following higher dimensional case of Theorem B.

Theorem D (Theorem 4.2.1, [**16**]). *Under the above setting, the following statements hold.*

(1) *For any holomorphic function f on V with $\int_V |f|^2 e^{-\varphi+2kB} < \infty$, there exists a holomorphic function F on Ω such that $F|_V = f$ and*

$$\int_\Omega |F|^2 e^{-\varphi} \leq C \int_{\tilde{V}} |\tilde{f}|^2 e^{2k\tilde{B}},$$

where C is a positive constant determined from Theorem C and the holomorphic function $\tilde{f}$ on $\tilde{V}$ is defined by $\tilde{f}(z, w) := f(z)$.

(2) *Suppose that $\tilde{\Omega}$ is a strictly pseudoconvex domain and $-B(z)$ is a plurisubharmonic function. Then one can make the estimate in* (1) *strictly sharper than the one in Theorem C, i.e., there exist functions $\tilde{G}$ and $\tilde{B}$ satisfying the above conditions such that*

$$\int_{\tilde{V}} |\tilde{f}|^2 e^{2k\tilde{B}} < \sigma_k \int_V |f|^2 e^{-\varphi+2kB}.$$

As an application of Theorem D, we can show that for a unit ball $\Omega = \mathbb{B}^n$ in $\mathbb{C}^n$, a closed complex submanifold $V = \{z_1 = \cdots = z_k = 0\} = \{z' = 0\}$ and a radial plurisubharmonic function φ with respect to V, i.e., $\varphi(z) = \varphi(|z'|)$, one can obtain the L^2-minimum extension of holomorphic functions f on V.

In addition, we consider the sharper estimates in terms of the Azukawa pseudometric. Specifically, we aim at the comparison with the result which was obtained in [**14**]. To consider it, we prove the following result.

Theorem E (Theorem 4.3.3, [**16**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, V be a closed complex submanifold defined by $V = \{z_1 = \cdots = z_k = 0\}$ and φ be a plurisubharmonic function on Ω. Let $G_{\Omega,V}$ be the pluricomplex Green function on Ω with poles along V. We assume that there exists the limit*

$$A_{\Omega,V,w}(X) := \lim_{\lambda\to 0} \left(G_{\Omega,V}(\lambda X, w) - \log|\lambda|\right),$$

where $(0,\ldots,0,w) \in V$ and $0 \neq X \in \mathbb{C}^k$. We define $I_{\Omega,V,w} := \{X \in \mathbb{C}^n : A_{\Omega,V,w}(X) < 0\}$. Then for any holomorphic function f on V with $\int_V \mathrm{vol}(I_{\Omega,V,w})|f^2|e^{-\varphi} < \infty$, there exists a holomorphic function F on Ω such that $F|_V = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} \leq \int_V \mathrm{vol}(I_{\Omega,V,w})|f^2|e^{-\varphi}.$$

Using the above result, we study a sharper estimate in a specific example.

This thesis is organized as follows. From Chapter 1 to Chapter 3 are preliminary for Chapter 4. We will give some definitions, basic properties and known results. More precisely, in Chapter 1, we introduce basics of the theory of the complex Monge–Ampère equations. In Chapter 2, we introduce the theory of the pluricomplex Green functions. In Chapter 3, we introduce the Ohsawa–Takegoshi L^2-extension theorem. Chapter 4 is the main part of this thesis. we prove Theorem C and Theorem D. In Section 4.1, we prove Theorem C following the argument of the proof of Berndtsson–Lempert type L^2-extension theorem [**15**]. In Section 4.2, we prove Theorem D by using Theorem C. In section 4.3, as a special case, we consider the sharper estimates in terms of the Azukawa pseudometric. In particular, we prove Theorem E and study in a specific example. In Section 4.4, as an application of Theorem D, in the case where $\Omega = \mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$, $V = \{z_1 = \cdots = z_k = 0\}$ is a closed complex submanifold and φ is a radial plurisubharmonic function with respect to V, we prove that one can obtain the L^2-minimum extension of holomorphic functions f on V.

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CHAPTER 1

Complex Monge–Ampère equation

In this chapter, we introduce the theory of the complex Monge–Ampère equations. Reference are [**3**], [**12**] and [**19**]. In particular, we treat the definition and some properties of the complex Monge-Ampère operator (measure) and the Dirichlet problem for the complex Monge–Ampère equation.

1.1. Basic properties of complex Monge–Ampère operator

The complex Monge-Ampère operator $\varphi \to (dd^c\varphi)^n$ gives a non-negative Radon measure where φ is a plurisubharmonic function. In this section, we explain the definition and some properties of the complex Monge-Ampère operator (measure), following the works of Bedford–Taylor [**3**].

Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. In other words, there exists a smooth strictly plurisubharmonic function ρ such that $\Omega = \{\rho < 0\}$. Let $PSH(\Omega)$ be the set of all plurisubharmonic functions on Ω.

If a plurisubharmonic function u belongs to $C^2(\Omega)$, we can calculate $(dd^c u)^n$ pointwise. As a general case, we consider the case where a function u belongs to $PSH(\Omega) \cap L^\infty_{loc}(\Omega)$.

Let T be a closed positive (p,p)-current in a domain Ω, $0 \leq p \leq n-1$. Then T decomposed as

$$T = i^{p^2} \sum_{|I|=p,|J|=p} T_{I,J} dz_I \wedge d\bar{z}_J,$$

where the coefficients $T_{I,J}$ are complex Borel measures. A locally bounded Borel function u is locally integrable with respect to the coefficients of T. Therefore uT is well-defined current by

$$\langle uT, \Psi \rangle := \langle T, u\Psi \rangle$$

where Ψ is the test form of bidegree $(n-p, n-p)$. From the argument of the case where u is smooth, we can define the $(p+1, p+1)$-current $dd^c u \wedge T$.

Definition 1.1.1 ([**3**]). Let T be a closed positive current of bidegree (p,p) on a domain $\Omega \subset \mathbb{C}^n$ and $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. The current $dd^c u \wedge T$ is the $(p+1, p+1)$-current defined by

$$dd^c u \wedge T := dd^c(uT).$$

In other words, for any test form Ψ of bidegree (q,q), $q = n-p-1$,

$$\langle dd^c u \wedge T, \Psi \rangle = \langle uT, dd^c\Psi \rangle .$$

Let $\{u_j\}$ be locally bounded plurisubharmonic functions on Ω which decrease to $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. Then, $dd^c u_j \wedge T \to dd^c u \wedge T$ in the weak sense of currents on Ω. Therefore $dd^c u \wedge T$ is a closed positive current. By repeating this process inductively, we can define the intersection of currents in the following way:

DEFINITION 1.1.2 ([**3**]). If $u_1, \ldots, u_p \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ and T is a closed positive current of bidegree (q, q) where $p + q \leq n$, we define the current $dd^c u_1 \wedge \ldots \wedge dd^c u_p \wedge T$ by

$$dd^c u_1 \wedge \ldots \wedge dd^c u_p \wedge T := dd^c (u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_p \wedge T).$$

Then we can define the complex Monge-Ampère measure.

DEFINITION 1.1.3 ([**3**]). The complex Monge-Ampère measure of locally bounded plurisubharmonic function u is defined by

$$(dd^c u)^n := dd^c u \wedge \ldots \wedge dd^c u,$$

where the right-hand side is n-times wedge product.

Here we introduce some properties of the complex Monge–Ampère operator(measure).

THEOREM 1.1.4 (Continuity of complex Monge–Ampère operator, [**3**]). *Let T be a closed positive current of bidegree (p, p) on a domain $\Omega \subset \mathbb{C}^n$ and $\{u_1^j\}, \ldots, \{u_q^j\}$ be decreasing(resp. increasing) sequences of locally bounded plurisubharmonic functions converging respectively to $u_1, \ldots, u_q \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$, where $p + q \leq n$. Then*

$$dd^c u_1^j \wedge \ldots \wedge dd^c u_q^j \wedge T \to dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$$

in the weak sense of currents.

The following theorem is one of the most important tools in pluripotential theory. This result is used in the proof of the uniqueness of solutions of the Dirichlet problem for the complex Monge–Ampère equations. Therefore it is known as a non-linear version of the maximum principle.

THEOREM 1.1.5 (Comparison principle, [**3**]). *Assume that functions $u, v \in PSH(\Omega) \cap L^\infty(\Omega)$ satisfy the condition $\liminf_{z \to \partial\Omega} (u(z) - v(z)) \geq 0$. Then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

COROLLARY 1.1.6 ([**3**]). *Assume that functions $u, v \in PSH(\Omega) \cap L^\infty(\Omega)$ satisfy the condition $\liminf_{z \to \partial\Omega} (u(z) - v(z)) \geq 0$. If $(dd^c u)^n \leq (dd^c v)^n$, then $v \leq u$ in Ω.*

1.2. Dirichlet problem for complex Monge–Ampère equation

In this section, we treat the Dirichlet problem for the complex Monge–Ampère equation. More precisely, it states the following : Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Suppose that the functions $\varphi \in C(\partial\Omega)$ and $0 \le f \in C(\bar{\Omega})$ are given. Then we consider the following type of the Dirichlet problem:

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega}), & \\ (dd^c u)^n = f\beta^n & \text{on } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.2.1}$$

where we set $\beta := \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$. To construct the solution of the Dirichlet problem (1.2.1) and show it is unique, we use the method of upper envelopes due to Perron and the comparison principle.

Let H_n^+ be the set of all semi-positive Hermitian $n \times n$ matrices. We set $\dot{H}_n^+ := \{H \in H_n^+ : \det H = n^{-n}\}$. For any $H = (h_{i\bar{j}}) \in \dot{H}_n^+$, we consider

$$\Delta_H := \sum_{i,j=1}^{n} h^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j},$$

where $(h^{i\bar{j}})$ is the matrix $\bar{H}^{-1}$. Then the following proposition holds.

PROPOSITION 1.2.1. *Let u be a locally bounded plurisubharmonic function on Ω and $0 \le f$ be a continuous function on Ω. The following conditions are equivalent:*
(1)$\Delta_H u \ge f^{\frac{1}{n}}$ for all $H \in \dot{H}_n^+$,
(2)$(dd^c u)^n \ge f\beta^n$.

Using these equivalent conditions, we define the class $\mathcal{V}$ of the subsolutions of the Dirichlet problem (1.2.1).

DEFINITION 1.2.2 ([**3**]). Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, φ be a continuous function near $\partial\Omega$ and $0 \le f$ be a continuous function on $\bar{\Omega}$. Then we define the class $\mathcal{V}$ by

$$\mathcal{V} = \mathcal{V}_{\Omega,\varphi,f} := \{v \in PSH(\Omega) \cap L^\infty(\Omega) : v|_{\partial\Omega} \le \varphi, \Delta_H v \ge f^{\frac{1}{n}}, \forall H \in \dot{H}_n^+\}.$$

Then we define the Perron–Bremermann envelope.

PROPOSITION 1.2.3 (Perron–Bremermann envelope, [**3**]). *The class $\mathcal{V}$ is non-empty, stable under maxima and bounded above in Ω. Then the Perron–Bremermann envelope*

$$U_{\Omega,\varphi,f}(z) := \sup\{v(z) : v \in \mathcal{V}\}$$

is plurisubharmonic on Ω.

The Perron–Bremermann envelope U is a continuous plurisubharmonic function which belongs to $\mathcal{V}$ and satisfies $U = \varphi$ on $\partial\Omega$. To show $dd^c U = f\beta^n$, we consider the case of

$\Omega = \mathbb{B}^n$. When the functions φ and $0 \leq f$ satisfies $\varphi \in C^{1,1}(\partial\mathbb{B}^n)$ and $f^{\frac{1}{n}} \in C^{1,1}(\bar{\mathbb{B}}^n)$, the Perron–Bremermann envelope U satisfies $U \in C^{1,1}_{loc}(\mathbb{B}^n)$, i.e., U admits second-order partial derivatives almost everywhere in $\mathbb{B}^n$ which are locally bounded in $\mathbb{B}^n$. Then we can show the following:

THEOREM 1.2.4 ([**3**]). *Let $\Omega = \mathbb{B}^n$ be a unit ball in $\mathbb{C}^n$. Assume that the function φ and $0 \leq f$ satisfies $\varphi \in C^{1,1}(\partial\mathbb{B}^n)$ and $f^{\frac{1}{n}} \in C^{1,1}(\bar{\mathbb{B}}^n)$. Then $U = U_{\Omega,\varphi,f}$ is the unique solution to the Dirichlet problem (1.2.1).*

Using an approximation, we can show the case that the function φ and $0 \leq f$ satisfies $\varphi \in C(\partial\mathbb{B}^n)$ and $f \in C(\bar{\mathbb{B}}^n)$. This result yields the following:

THEOREM 1.2.5 ([**3**]). *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$, φ be a continuous function near $\partial\Omega$ and $0 \leq f$ be a continuous function on $\bar{\Omega}$. Then $U = U_{\Omega,\varphi,f}$ is the unique solution to the Dirichlet problem (1.2.1).*

REMARK 1.2.6. After that, Theorem 1.2.5 generalizes to the setting where the function $0 \leq f$ satisfies $f \in L^p(\Omega)$, $p > 1$ (more generally, f belongs to the Orlicz space). Kołodziej [**18**] has shown that the Dirichlet problem (1.2.1) has a unique solution.

CHAPTER 2

Pluricomplex Green function

In this chapter, we introduce the concept of pluricomplex Green functions and state some properties which we need in Chapter 3 and 4. At first, we treat the case that the pole is a point. Next, we treat the case that the poles are in subvarieties.

2.1. Pluricomplex Green function with a pole at a point

The concept of pluricomplex Green functions are pluripotential theoretic generalizations of the concept of Green functions which are the solutions of the Laplace equation. It plays important roles in complex and algebraic geometry and complex dynamics in several variables. Klimek first introduced the concept of pluricomplex Green functions in [**17**]. After that, many studies of it, for example, Demailly [**8**] and Azukawa [**1**] [**2**], was conducted in complex analysis. In this section, we introduce the definition and some properties of it.

Definition 2.1.1 ([**17**]). Let Ω be a domain in $\mathbb{C}^n$ and w be a point in Ω. Then we define the class $\mathcal{F}_w$ by

$$\mathcal{F}_w := \{u \in PSH(\Omega) : u < 0, u(z) \leq \log|z-w| + C \text{ near } w\}.$$

Definition 2.1.2 ([**17**]). The pluricomplex Green function $g_{\Omega,w}$ with a pole at w is the upper envelope of all functions in $\mathcal{F}_w$, i.e., for any $z \in \Omega$,

$$g_{\Omega,w}(z) := \sup\{u(z) : u \in \mathcal{F}_w\}.$$

When $\mathcal{F}_w = \varnothing$, we define $g_{\Omega,w} = -\infty$. The pluricomplex Green function $g_{\Omega,w}$ has the following important property.

Theorem 2.1.3 ([**17**]). *For any $w \in \Omega$, $g_{\Omega,w}$ is plurisubharmonic, i.e., $g_{\Omega,w} \in \mathcal{F}_w$.*

In other words, $g_{\Omega,w}$ is plurisubharmonic without taking the upper semi-continuous regularization.

The Azukawa pseudometric is a generalization of the logarithmic capacitiy which is defined by the Green function in the complex variables. It is known that the Azukawa pseudometric has relations to the Carathéodory pseudometric and the Kobayashi pseudomeric. Further, it also has a relation to the higher dimensional case of the Suita conjecture as it appears in the lower bound of the Bergman kernel [**7**]. It is defined as follows:

Definition 2.1.4 ([**1**]). The (logarithmic) Azukawa pseudometric is defined as follows: for any $X \in \mathbb{C}^n$,

$$A_{\Omega,w}(X) := \limsup_{\lambda \to 0}(g_{\Omega,w}(w+\lambda X) - \log|\lambda|).$$

It has a following property: for any $w \in \Omega$, $X \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, $A_{\Omega,w}(\lambda X) = A_{\Omega,w}(X) + \log|\lambda|$.

Here we introduce some properties of the pluricomplex Green function and the Azukawa pseudometric.

THEOREM 2.1.5 (Decreasing property). *Let $F : \Omega_1 \to \Omega_2$ be a holomorphic map. Then for any $z, w \in \Omega_1$ and $X \in \mathbb{C}^n$,*

$$g_{\Omega_2,F(w)}(F(z)) \leq g_{\Omega_1,w}(z),$$

$$A_{\Omega_2,F(w)}(F'(w)X) \leq A_{\Omega_1,w}(X).$$

If the holomorphic map F is a biholomorphic, then the above inequalities become equalities.

The domain $\Omega \subset \mathbb{C}^n$ is a bounded hyperconvex domain if there exists a negative continuous plurisubharmonic function ρ on Ω such that for any $c > 0$, the sublevel set $\{\rho < -c\}$ is relatively compact in Ω. The following theorem describes the boundary behavior of $g_{\Omega,w}$.

THEOREM 2.1.6 (Boundary behavior of $g_{\Omega,V}$, [**8**]). *Let Ω be a bounded hyperconvex domain. Then $g_{\Omega,w}$ is continuous on $\Omega \setminus \{w\}$ and $g_{\Omega,w}(z) \to 0$ as $z \to \partial\Omega$.*

Related to the above theorem, the pluricomplex Green functions are characterized as the solutions of the complex Monge-Ampère equation with a logarithmic pole.

THEOREM 2.1.7 ([**8**]). *If Ω be a bounded hyperconvex domain, the pluricomplex Green function $g_{\Omega,w}$ is the unique solution of the following Dirichlet problem: for $w \in \Omega$,*

$$\begin{cases} g_{\Omega,w} \in PSH(\Omega) \cap C(\bar{\Omega} \setminus \{w\}), & \\ (dd^c g_{\Omega,w})^n = \delta_w & \text{on } \Omega, \\ g_{\Omega,w} = \log|\cdot - w| + O(1) & \text{near } w, \\ g_{\Omega,w} = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_w is the Dirac measure concentrated at w.

At last, we state the relations between the Azukawa pseudo metric and the Carathéodory pseudometric and the Kobayashi pseudometric.

DEFINITION 2.1.8. (1) The Carathéodory pseudometric is definded for $w \in \Omega$ and $X \in \mathbb{C}^n$ by

$$C_{\Omega,w}(X) := \sup\{\rho(f(w); f_*X)) : f \in \mathcal{O}(\Omega, \mathbb{D})\}$$

where ρ is the Poicaré metric on the unit disc $\mathbb{D}$.

(2) The Kobayashi pseudometric is defined by

$$K_{\Omega,w}(X) := \inf\{\rho(w'; Y) : f \in \mathcal{O}(\mathbb{D}, \Omega), f(w') = w, f_*Y = X\}.$$

THEOREM 2.1.9 ([**2**]). *For any $w \in \Omega$ and $X \in \mathbb{C}^n$,*

$$C_{\Omega,w}(X) \leq \exp(A_{\Omega,w}(X)) \leq K_{\Omega,w}(X).$$

2.2. Pluricomplex Green function with poles along subvarieties

The pluricomplex Green function with poles along subvarieties [**22**] is a kind of the generalization of the pluricomplex Green function with a pole at a point. From the various recent studies, for example, Berndtsson–Lempert [**5**] and Hosono [**13**], it turns out that these play important roles. In this section, we explain the definition and some properties of it, following the work of Rashkovskii–Sigurdsson [**22**].

Let Ω be a domain in $\mathbb{C}^n$ and V be a anlytic subvariety of Ω. In other words, for any $z \in V$, there exist a neighborhood U of z and holomorphic functions $\psi_1, \dots, \psi_k$ on U such that $V \cap U = \{\psi_1 = \cdots = \psi_k = 0\}$. Let $\mathcal{O}_\Omega$ be the sheaf of germs of locally defined holomorphic functions on Ω and $\mathcal{I}_V$ be the coherent ideal sheaf of V in $\mathcal{O}_\Omega$.

Definition 2.2.1 ([**22**]). Let Ω be a domain in $\mathbb{C}^n$ and V be a anlytic subvariety of Ω. The class $\mathcal{F}_V$ consists of all negative plurisubharmonic functions u on Ω such that for any $z \in \Omega$, there exist local generators $\psi_1, \dots, \psi_k$ of $\mathcal{I}_V$ near z and a constant C depending on u and generators such that $u \leq \log|\psi| + C$ near z where we denote $\psi = (\psi_1, \dots, \psi_k)$.

Definition 2.2.2 ([**22**]). The pluricomplex Green function $G_{\Omega,V}$ with poles along V is the upper envelope of all functions in $\mathcal{F}_V$, i.e., for any $z \in \Omega$,

$$G_{\Omega,V}(z) := \sup\{u(z) : u \in \mathcal{F}_V\}.$$

When $\mathcal{F}_V = \varnothing$, we define $G_{\Omega,V} = -\infty$. The pluricomplex Green function $G_{\Omega,V}$ has the following important property.

Theorem 2.2.3 (Plurisubharmonicity of $G_{\Omega,V}$, [**22**]). *If V is closed, then $G_{\Omega,V}$ is plurisubharmonic, i.e., $G_{\Omega,V} \in \mathcal{F}_V$.*

In other words, if V is closed, $G_{\Omega,V}$ is plurisubharmonic without taking the upper semi-continuous regularization.

From now on, we consider V has bounded global generators ψ. This means that there exist bounded holomorphic functions $\psi_1, \dots, \psi_k$ on Ω such that

$$V = \{\psi_1 = \cdots = \psi_k = 0\}.$$

Then, by boundedness of generators, we can take a positive constant M such that $\frac{|\psi|}{M} < 1$, i.e., $\log\frac{|\psi|}{M} \in \mathcal{F}_V$. Since $\mathcal{F}_V \neq \varnothing$, there exists the pluricomplex Green function $G_{\Omega,V}$ with poles along V which is not equal to $-\infty$ identically.

Theorem 2.2.4 ([**22**]). *Let V has bounded global generators ψ. Then near V,*

$$G_{\Omega,V} = \log|\psi| + O(1).$$

A plurisubharmonic function v on Ω is a strong plurisubharmonic barrier at $p \in \partial\Omega$ if $\sup_{\Omega\setminus U} v < 0$ for every neighbourhood U of p and $v(z) \to 0$ as $z \to p$. The following theorem describes the boundary behavior of $G_{\Omega,V}$.

THEOREM 2.2.5 (Boundary behavior, [**22**]). *Let V has bounded global generators and Ω has a strong plurisubharmonic barrier at $p \in \partial\Omega \setminus V$. Then $G_{\Omega,V}(z) \to 0$ as $z \to p$.*

In the case where V has bounded global generators, there exist conditions about the uniqueness of $G_{\Omega,V}$ as follows:

THEOREM 2.2.6 (Uniqueness of $G_{\Omega,V}$, [**22**]). *Let V has bounded global generators ψ and u a negative plurisubharmonic function on Ω satisfying the following properties:*

(1) u is locally bounded and maximal on $\Omega \setminus V$.

(2) For any $\epsilon > 0$, there exists a compact set $K \subset \Omega$ such that $u \geq G_{\Omega,V} - \epsilon$ on $\Omega \setminus K$.

(3) u has logarithmic poles along V. i.e., $u = \log|\psi| + O(1)$ near V.

Then $u = G_{\Omega,V}$.

As Theorem 2.1.7, when V has bounded global generators, we can calculate the Monge–Ampère measure of $G_{\Omega,V}$.

THEOREM 2.2.7. *(Monge–Ampère measure of $G_{\Omega,V}$, [**22**]) Let V has bounded global generators $\psi = (\psi_1, \dots, \psi_k)$. Then*

$$(dd^c G_{\Omega,V})^k = [V] + Q,$$

where $[V]$ is the integral current with respect to V and Q is a closed positive current of bidegree (k,k) such that $\chi_V Q = 0$ where χ_V is the characteristic function of V.

CHAPTER 3

Ohsawa–Takegoshi L^2-extension theorem

In this chapter, we explain the Ohsawa–Takegoshi L^2-extension theorem. In particular, we treat the optimal L^2-extension theorem and the sharper estimate of it.

3.1. Berndtsson–Lempert type optimal L^2-extension theorem

The Ohsawa–Takegoshi L^2-extension theorem was obtained for the first time in [**21**]. The Ohsawa–Takegoshi L^2-extension theorem and its generalizations are applied widely in the studies of complex analysis as well as complex geometry and algebraic geometry. The statement of it is as follows:

THEOREM 3.1.1 (Ohsawa–Takegoshi L^2-extension theorem, [**21**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\sup_\Omega |z_n| \leq 1$, $H = \Omega \cap \{z_n = 0\}$ a hyperplane of Ω and φ a plurisubharmonic function on Ω. Then for any holomorphic function f on H with $\int_H |f|^2 e^{-\varphi} < \infty$, there exists a holomorphic function F on Ω such that $F|_H = f$ and*

$$\int_\Omega |F|^2 e^{-\varphi} \leq C \int_H |f|^2 e^{-\varphi},$$

where $C < 1620\pi$ is the constant that is independent of f and φ.

Recently, Błocki[**6**] and Guan-Zhou[**11**] proved an optimal L^2-extension theorem, i.e., we can determine the positive constant C in the best possible way. In their proofs, variants of Hörmander's L^2-estimate for $\bar{\partial}$-equation improved by a careful choice of auxiliary functions as solutions of a certain ODE play an important role. After that Berndtsson–Lempert obtained a new proof of the optimal L^2-extension theorem in [**5**]. Here we state the Berndtsson–Lempert type optimal L^2-extension theorem.

THEOREM 3.1.2 (Optimal L^2-extension theorem, [**5**], [**6**], [**11**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, V a closed complex submanifold of Ω with codimension k and φ a plurisubharmonic function on Ω. Let G be a negative plurisubharmonic function on Ω such that*

$$\log d_V^2(z) - B(z) \leq G(z) \leq \log d_V^2(z) + A(z),$$

where $d_V(z)$ is distance between $z \in \Omega$ and V, $A(z)$ and $B(z)$ are continuous functions on Ω. Then for any holomorphic function f on V with $\int_V |f|^2 e^{-\varphi + kB} < \infty$, there exists a *holomorphic function F such that $F|_V = f$ and*

$$\int_\Omega |F|^2 e^{-\varphi} \leq \sigma_k \int_V |f|^2 e^{-\varphi + kB},$$

where σ_k is the volume of the unit ball in $\mathbb{C}^k$.

As we need the argument of the proof of Theorem 3.1.2, we explain the details of it in Chapter 4. The key result used in the proof of Theorem 3.1.2 is the variational result of [**4**] which is generalization of the result in Maitani–Yamaguchi [**20**].

Theorem 3.1.3 (Berndtsson's convexity theorem, [**4**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\Phi \in PSH \cap C^\infty(\overline{\Omega \times \Delta})$ where $\Delta = \{t \in \mathbb{C} : |t| < 1\}$ is the unit disc. For each $t \in \Delta$, we define a plurisubharmonic function $\phi_t(z) := \Phi(z,t)$ on Ω. Let $A^2(\Omega, \phi_t) = A^2(\Omega)$ be a Hilbert space of holomorphic functions F on Ω with $\int_\Omega |F|^2 e^{-\phi_t} < \infty$ and $\xi \in (A^2(\Omega))^*$ be a bounded linear functional on $A^2(\Omega)$. Then the function*

$$t \mapsto \log \|\xi\|_{(A^2(\Omega))^*}$$

is subharmonic.

Remark 3.1.4. Conversely, Theorem 3.1.3 is obtained from the optimal L^2-extension theorem [**11**]. Further, Theorem 3.1.3 is obtained from the non-optimal L^2-extension theorem [**10**]. From these results, Theorem 3.1.3 holds for singular weights.

On the other hand, Theorem 3.1.3 and its generalization are used in the studies of the existence of Kähler–Einstein metric and its generalization.

3.2. Sharper estimates of Ohsawa–Takegoshi L^2-extension theorem

Hosono proposed in [**13**] an idea of getting an L^2-estimate sharper than the one of Theorem 3.1.2 by allowing constants depending on weight functions, i.e., we can determine the positive constant C' depending on weight functions sharper than the optimal constant C. Here we state this result in [**13**].

Let Ω be a bounded domain in $\mathbb{C}$ with $0 \in \Omega$ and φ a subharmonic function on Ω. Here, we define the domain $\tilde{\Omega}$ in $\mathbb{C}^2$ by

$$\tilde{\Omega} := \{(z,w) \in \mathbb{C}^2 : z \in \Omega, |w|^2 < e^{-\varphi(z)}\}.$$

Since φ is subharmonic, the domain $\tilde{\Omega}$ is pseudoconvex domain in $\mathbb{C}^2$. Let $\tilde{G}$ be a Green-type function on $\tilde{\Omega}$ with poles $\{z = 0\}$. In other words, $\tilde{G}$ is a negative plurisubharmonic function on $\tilde{\Omega}$ such that there exists continuous functions $\tilde{A}, \tilde{B}$ on $\tilde{\Omega}$ such that

$$\log |z|^2 - \tilde{B}(z,w) \leq \tilde{G}(z,w) \leq \log |z|^2 + \tilde{A}(z,w).$$

Then the following theorem holds.

Theorem 3.2.1 (Sharper estimates of L^2-extension theorem, [**13**]). *Let Ω be a bounded domain in $\mathbb{C}$, φ be a subharmonic function on Ω with $\varphi(0) = 0$. Let $\tilde{\Omega}$, $\tilde{G}$ and $\tilde{B}$ be defined as above.*

(1) *There exists a holomorphic function f on Ω such that $f(0) = 1$ and*

$$\int_\Omega |f(z)|^2 e^{-\varphi(z)} \leq \int_{|w|<1} e^{\tilde{B}(0,w)}.$$

(2) *Suppose that $\tilde{\Omega}$ is a strictly pseudoconvex domain. Then one can make the estimate in* (1) *strictly sharper than the one in Theorem A, i.e., there exist functions $\tilde{G}$ and $\tilde{B}$ satisfying the above conditions such that*

$$\int_{|w|<1} e^{\tilde{B}(0,w)} < \pi e^{B(0)}$$

where $B(z) = G(z) - \log |z|^2$ is a harmonic function on Ω and $G(z)$ is a Green function on Ω.

As the proof of our main theorems is based on the argument of the proof of Theorem 3.1.2 and 3.2.1, we will explain the argument in details in Chapter 4. The key result used in the proof of Theorem 3.2.1 is the solutions of the Dirichlet problem for the complex Monge–Ampère equations and the comparison principle.

As an application of Theorem 3.2.1, in the case where $\Omega = \{|z| < 1\}$ is a unit disc in $\mathbb{C}$ and φ is a radial subharmonic function, i.e., $\varphi(z) = \varphi(|z|)$, Hosono [**13**] proved that we can determine the L^2-minimum extension of the function 1 on the subvariety $\{0\}$.

CHAPTER 4

Higher dimesional case of sharper estimates of Ohsawa–Takegoshi L^2-extension theorem

In this chapter, we generalize Theorem 3.2.1 to the setting where Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and V is a closed complex submanifold of it with some conditions. In order to do so, we establish an analogue of Theorem 3.1.2 by using the theory of the pluricomplex Green function with poles along a subvariety. The content of this chapter is based on [**16**].

4.1. Establishment of an analogue of Berndtsson–Lempert type L^2-extension theorem

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, V be a closed complex submanifold of Ω with codimension $k \le n$. We assume that V has bounded global generators. This means that there exists bounded holomorphic functions $\psi_1, \dots, \psi_k$ on Ω such that

$$V = \{\psi_1 = \cdots = \psi_k = 0\}.$$

Then by boundedness of generators, there exists the pluricomplex Green function $G_{\Omega,V}$ with poles along V which is not equal to $-\infty$ identically.

Since V is a submanifold, for any $z^\circ \in V$, we can take the coordinate $z = (z_1, \dots, z_n)$ near z° such that

$$J_\psi := \det \frac{\partial(\psi_1, \dots, \psi_k)}{\partial(z_1, \dots, z_k)} \neq 0.$$

In particular, in the case where V has bounded global generator, for any $z \in \Omega$, we can find a re-ordering of linear coordinates $z' := (z_1, \dots, z_k)$ and $z'' := (z_{k+1}, \dots, z_n)$ such that $J_\psi = \det \frac{\partial(\psi_1,\dots,\psi_k)}{\partial(z_1,\dots,z_k)} \neq 0$.

In this setting, we establish an analogue of Theorem 3.1.2.

Theorem 4.1.1 ([**16**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, V a closed complex submanifold of Ω with codimension k such that V has bounded global generators $\psi = (\psi_1, \dots, \psi_k)$ and there exists a positive constant C such that $\frac{1}{C} \le |J_\psi|$ near V. Let φ be a plurisubharmonic function on Ω and $G_{\Omega,V}$ the pluricomplex Green function on Ω with poles along V. Assume that there exists some continuous function B on Ω such that*

$$\log|\psi(z)| - B(z) \le G_{\Omega,V}(z). \tag{4.1.1}$$

Then for any holomorphic function f on V with $\int_V |f|^2 e^{-\varphi+2kB} < \infty$*, there exists a holomorphic function F on Ω such that $F|_V = f$ and*

$$\int_\Omega |F|^2 e^{-\varphi} \leq C\sigma_k \int_V |f|^2 e^{-\varphi+2kB},$$

where σ_k is the volume of the unit ball in $\mathbb{C}^k$.

Remark 4.1.2. Theorem 4.1.1 is obtained even if we use an another function in $\mathcal{F}_V$ satisfying (4.1.1) in substitution for $G_{\Omega,V}$. The L^2-estimate obtained from the conclusion will be sharper if $B(z)$ is smaller. Therefore, we need to take the pluricomplex Green function to obtain a better L^2-estimate.

proof. To prove Theorem 4.1.1, we follow the argument in the proof of [**15**, Theorem 2.1].

Let $A^2(\Omega,\varphi) = A^2(\Omega)$ be a Hilbert space of holomorphic functions F on Ω with $\int_\Omega |F|^2 e^{-\varphi} < \infty$. We may assume that φ is continuous.

Let $F^\circ \in A^2(\Omega,\varphi)$ be an arbitrary L^2-extension of f to Ω. Here, we consider a sequences of domains Ω_j such that Ω_j approximates Ω from inside. From now on, we will discuss on Ω_j. For simplicity, we omit the subscript j.

Let F_0 be the L^2-minimum extension of f. The L^2-norm $||F_0||_{A^2(\Omega)}$ of F_0 is equal to the L^2-norm $\|F^\circ\|_{A^2(\Omega)/A^2(\Omega)\cap\mathcal{I}_V}$ of F°, where we denote $\mathcal{I}_V := \{F \in \mathcal{O}(\Omega) : F|_V = 0\}$. Actually,

$$\begin{aligned}
\|F_0\|_{A^2(\Omega)} &= \sup_{0\neq e\in A^2(\Omega)} \frac{|(e,F_0)_{A^2(\Omega)}|}{\|e\|_{A^2(\Omega)}} \\
&= \sup_{0\neq e\in \mathcal{I}_V^\perp} \frac{|(e,F_0)_{A^2(\Omega)}|}{\|e\|_{A^2(\Omega)}} \\
&= \sup_{0\neq \xi\in A^2(\Omega)^*\cap \mathrm{Ann}\mathcal{I}_V} \frac{|\langle \xi, F_0\rangle|}{||\xi||_{A^2(\Omega)^*}} \\
&= \sup_{0\neq \xi\in A^2(\Omega)^*\cap \mathrm{Ann}\mathcal{I}_V} \frac{|\langle \xi, F^\circ\rangle|}{||\xi||_{A^2(\Omega)^*}} \\
&= ||F^\circ||_{A^2(\Omega)/A^2(\Omega)\cap\mathcal{I}_V},
\end{aligned}$$

where we denote $\mathrm{Ann}\mathcal{I}_V := \{\xi \in A^2(\Omega)^* : \xi|_{A^2(\Omega)\cap\mathcal{I}_V} = 0\}$. Then we deal with the linear form ξ in the following way. For a fixed smooth function g on V with compact support, we define a linear functional ξ_g on $A^2(\Omega)$ by

$$\langle \xi_g, h\rangle := \sigma_k \int_V h\bar{g} e^{-\varphi+2kB}, \quad h \in A^2(\Omega,\varphi).$$

The set of such functionals ξ_g is a dense subspace of $(A^2(\Omega)/A^2(\Omega)\cap\mathcal{I}_V)^*$. Therefore, the L^2-norm $\|F^\circ\|_{A^2(\Omega)/A^2(\Omega)\cap\mathcal{I}_V}$ can be written as

$$\sup_g \frac{|\langle \xi_g, F^\circ\rangle|}{||\xi_g||_{A^2(\Omega)^*}}.$$

For $p > 0$, $t \in \mathbb{C}$ with $\mathrm{Re}\, t \leq 0$, we define

$$\varphi_{t,p}(z) := \varphi(z) + p \max \left\{ G_{\Omega,V}(z) - \frac{\mathrm{Re} t}{2}, 0 \right\}.$$

For any fixed p, by Theorem 3.1.3,

$$t \longmapsto \log ||\xi_g||_{A^2(\Omega,\varphi_{t,p})^*}$$

is subharmonic. In particular, $\varphi_{t,p}$ is convex in $\mathrm{Re}\, t$. Therefore, we can assume that $t \in \mathbb{R}_{\leq 0}$. The following lemma describes the asymptotic behavior of $||\xi_g||_{A^2(\Omega,\varphi_{t,p})^*}$ when $t \to -\infty$. We will write $||\xi_g||_{A^2(\Omega,\varphi_{t,p})^*} = ||\xi_g||_{t,p}$ for simplicity.

LEMMA 4.1.3 ([**5**]Lemma 3.2). *For fixed $p > 0$, it follows that*

$$||\xi_g||_{t,p} e^{\frac{kt}{2}} = O(1).$$

when $t \to -\infty$. In particular, $||\xi_g||_{t,p} e^{\frac{kt}{2}}$ is increasing in t.

Let $F_{t,p}$ be the L^2-minimum extension of f in $A^2(\Omega, \varphi_{t,p})$. By Lemma 4.1.3,

$$e^{-\frac{kt}{2}} ||F_{t,p}||_{A^2(\Omega,\varphi_{t,p})} = \sup_g \frac{\langle \xi_g, F^\circ \rangle}{e^{\frac{kt}{2}} ||\xi_g||_{t,p}}$$

is decreasing in t. Therefore, it follows that

$$\begin{aligned} ||F_0||_{A^2(\Omega,\varphi)} &\leq e^{-\frac{kt}{2}} ||F_{t,p}||_{A^2(\Omega,\varphi_{t,p})} \\ &\leq e^{-\frac{kt}{2}} ||F^\circ||_{A^2(\Omega,\varphi_{t,p})}. \end{aligned}$$

For fixed $t < 0$, the right hand side of the last inequalities converges to

$$e^{-\frac{kt}{2}} \left(\int_{\{G_{\Omega,V} < \frac{t}{2}\}} |F^\circ|^2 e^{-\varphi} \right)^{\frac{1}{2}}$$

when $p \to \infty$.

The subscript j will be specified from here. We prepare the following lemma.

LEMMA 4.1.4 ([**16**]). *Let $\chi \geq 0$ be a continuous function on $\bar{\Omega}$ and integrable on V. Then it follows that*

$$\limsup_{t \to -\infty} e^{-kt} \int_{\Omega_j \cap \Omega_t} \chi \leq C\sigma_k \int_{\Omega_j \cap V} \chi e^{2kB},$$

where we denote $\Omega_t = \left\{ G_{\Omega,V} < \dfrac{t}{2} \right\}$.

proof. By using the assumption on $G_{\Omega,V}$, we have

$$\Omega_t = \{|\psi| < e^{\frac{t}{2}+B}\}.$$

Since the submanifold $\overline{\Omega_j \cap V}$ is compact in $\mathbb{C}^n$, there exists a finite open covering $\{U_i\}_{i=1}^N$ such that there exists a change of numbering of linear coordinates depending on i so that we have

$$J_\psi = \det \frac{\partial(\psi_1, \dots, \psi_k)}{\partial(z_1, \dots, z_k)} \neq 0 \quad \text{on each } U_i.$$

Let $t < 0$ be a sufficiently negative number. Since every point in V belongs to some U_i, we have

$$\begin{aligned} e^{-kt} \int_{\Omega_j \cap \Omega_t} \chi &= e^{-kt} \int_{\Omega_j \cap \{|\psi| < e^{\frac{t}{2}+B}\}} \chi \\ &= e^{-kt} \int_{\Omega_j \cap V} dz'' \int_{\{|\psi| < e^{\frac{t}{2}+B}\}} \chi(\psi^{-1}, z'') \frac{1}{|J_\psi|} d\psi \\ &\leq C e^{-kt} \int_{\Omega_j \cap V} dz'' \int_{\{|\psi| < e^{\frac{t}{2}+B}\}} \chi(\psi^{-1}, z'') d\psi. \end{aligned} \tag{4.1.2}$$

By the continuity of χ, we can calculate the right hand side of (4.1.2) as follow:

$$\begin{aligned} Ce^{-kt} \int_{\Omega_j \cap V} dz'' \int_{\{|\psi| < e^{\frac{t}{2}+B}\}} \chi(\psi^{-1}, z'') d\psi &\leq Ce^{-kt} \int_{\Omega_j \cap V} (\chi(0, z'') + \epsilon) dz'' \int_{\{|\psi| < e^{\frac{t}{2}+B}\}} d\psi \\ &= Ce^{-kt} \int_{\Omega_j \cap V} (\chi(0, z'') + \epsilon) \sigma_k e^{2k(\frac{t}{2}+B)} dz'' \\ &= C\sigma_k \int_{\Omega_j \cap V} (\chi(0, z'') + \epsilon) e^{2kB} dz''. \end{aligned}$$

Take the upper limits with respect to t of both sides of the above inequalities and $t \to -\infty$, we get the conclusion. □

We conclude from Lemma 4.1.4 that

$$\begin{aligned} ||F_0^{(j)}||^2_{A^2(\Omega_j, \varphi)} &\leq \limsup_{t \to -\infty} e^{-kt} ||F^\circ||^2_{A^2(\Omega_j \cap \{G < \frac{t}{2}\}, \varphi)} \\ &\leq C\sigma_k \int_V |f|^2 e^{-\varphi + 2kB}. \end{aligned}$$

Therefore, the L^2-norms $||F_0^{(j)}||_{A^2(\Omega_j,\varphi)}$ are uniformly bounded with respect to j. After taking a subsequence of $\{F_0^{(j)}\}$, it converges in $A^2(\Omega_{j_0}, \varphi)$ for every fixed j_0, thus the limit $F_0^{(\infty)} \in A^2(\Omega, \varphi)$ of it satisfies

$$||F_0^{(\infty)}||_{A^2(\Omega,\varphi)} \leq C\sigma_k \int_V |f|^2 e^{-\varphi + 2kB}.$$

□

4.2. Proof of higher dimensional case of sharper estimates

In this section, we prove the higher dimensional case of Theorem 3.2.1 by using Theorem 4.1.1. Our setting is as follows.

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and V a closed complex submanifold of Ω with codimension k such that V has bounded global generators $\psi = (\psi_1, \ldots, \psi_k)$. Suppose that there exists a positive constant C such that $\frac{1}{C} \leq |J_\psi|$ near V. Let φ be a plurisubharmonic function on Ω and $G_{\Omega,V}$ the pluricomplex Green function on Ω with poles along V. Assume that there exists some continuous function B on Ω such that

$$\log |\psi(z)| - B(z) \leq G_{\Omega,V}(z).$$

Let $\tilde{\Omega}$ be a pseudoconvex domain in $\mathbb{C}^{n+k}$ defined by

$$\tilde{\Omega} = \{(z, w) \in \mathbb{C}^{n+k} : z \in \Omega, |w|^2 < e^{-\frac{\varphi(z)}{k}}\}$$

and $\tilde{V}$ a closed complex submanifold of $\tilde{\Omega}$ such that

$$\tilde{V} = \{\tilde{\psi}_1 = \cdots = \tilde{\psi}_k = 0\},$$

where $\tilde{\psi}_i(z, w) := \psi_i(z)$ are holomorphic functions on $\tilde{\Omega}$. Let $\tilde{G}$ be the pluricomplex Green function on $\tilde{\Omega}$ with poles along $\tilde{V}$ or an another function in $\mathcal{F}_V$ such that there exists continuous function $\tilde{B}(z, w)$ on $\tilde{\Omega}$ such that

$$\log |\tilde{\psi}(z, w)| - \tilde{B}(z, w) \leq \tilde{G}(z, w). \tag{4.2.1}$$

Then, the following theorem holds.

THEOREM 4.2.1 ([**16**]). *Under the above setting, the following statements hold.*

(1) *For any holomorphic function f on V with $\int_V |f|^2 e^{-\varphi+2kB} < \infty$, there exists a holomorphic function F on Ω such that $F|_V = f$ and*

$$\int_\Omega |F|^2 e^{-\varphi} \leq C \int_{\tilde{V}} |\tilde{f}|^2 e^{2k\tilde{B}},$$

where C is a positive constant determined from Theorem 4.1.1 and the holomorphic function $\tilde{f}$ on $\tilde{V}$ is defined by $\tilde{f}(z, w) := f(z)$.

(2) *Suppose that $\tilde{\Omega}$ is a strictly pseudoconvex domain and $-B(z)$ is a plurisubharmonic function. Then one can make the estimate in* (1) *strictly sharper than one in Theorem 4.1.1, i.e., there exist functions $\tilde{G}$ and $\tilde{B}$ satisfying the above conditions such that*

$$\int_{\tilde{V}} |\tilde{f}|^2 e^{2k\tilde{B}} < \sigma_k \int_V |f|^2 e^{-\varphi+2kB}.$$

proof. (1) By applying Theorem 4.1.1 with the trivial metric $e^{-\tilde{\varphi}} \equiv 1$ to the holomorphic function $\tilde{f}(z, w) := f(z)$ on $\tilde{V}$, we get a holomorphic function $\tilde{F}$ on $\tilde{\Omega}$ satisfying the properties that $\tilde{F}|_{\tilde{V}} = \tilde{f}$ and

$$\int_{\tilde{\Omega}} |\tilde{F}|^2 \leq C\sigma_k \int_{\tilde{V}} |\tilde{f}|^2 e^{2k\tilde{B}} = C\sigma_k \int_{\tilde{V}} |f|^2 e^{2k\tilde{B}}.$$

We consider a holomorphic function $F(z) := \tilde{F}(z, 0)$ on Ω. For any $z \in V$, we have

$$F(z) = \tilde{F}(z, 0) = \tilde{f}(z, 0) = f(z),$$

i.e., $F|_V = f$ holds. For fixed $z \in \Omega$, by the mean value inequality, we have

$$|F(z)|^2 = |\tilde{F}(z,0)|^2 \leq \frac{1}{\sigma_k e^{-\varphi(z)}} \int_{|w|^2 < e^{-\frac{\varphi(z)}{k}}} |\tilde{F}|^2.$$

Therefore, it follows that

$$\sigma_k \int_\Omega |F|^2 e^{-\varphi} \leq \int_{\tilde{\Omega}} |\tilde{F}|^2.$$

Then the following inequality holds.

$$\int_\Omega |F|^2 e^{-\varphi} \leq C \int_{\tilde{V}} |f|^2 e^{2k\tilde{B}}.$$

(2) We assume that $\tilde{\Omega}$ is a strictly pseudoconvex domain. Then by Theorem 1.2.5, there exists $\tilde{u} \in PSH(\tilde{\Omega}) \cap \mathcal{C}(\bar{\tilde{\Omega}})$ such that

$$(dd^c\tilde{u})^{n+k} = 0 \quad \text{on } \tilde{\Omega}, \text{ and}$$
$$\tilde{u} = -\max(\log|\tilde{\psi}|, D) \quad \text{on } \partial\tilde{\Omega},$$

where D is a sufficiently negative constant such that

$$\max(\log|\psi|, D) - B(z) < 0$$

on Ω. Here, we define $\tilde{G} := \log|\tilde{\psi}| + \tilde{u}$. On $\partial\tilde{\Omega}$, we have

$$\tilde{G} = \log|\tilde{\psi}| - \max(\log|\tilde{\psi}|, D) \leq \log|\tilde{\psi}| - \log|\tilde{\psi}| = 0.$$

By the maximum principle, it follows that $\tilde{G} < 0$ on $\tilde{\Omega}$. By continuity of $\tilde{u}$, we can choose $\tilde{B} = -\tilde{u}$.

Then it is sufficient to prove the following inequality

$$\int_{\tilde{V}} |f|^2 e^{2k\tilde{B}} < \sigma_k \int_V |f|^2 e^{-\varphi + 2kB}.$$

First, we calculate each side of the above inequality separately:

$$\int_{\tilde{V}} |f|^2 e^{2k\tilde{B}} = \int_V |f(z)|^2 \left(\int_{|w|^2 < e^{-\frac{\varphi}{k}}} e^{2k\tilde{B}(z,w)} \right),$$

$$\begin{aligned} \sigma_k \int_V |f|^2 e^{-\varphi+2kB} &= \int_V |f(z)|^2 e^{-\varphi+2kB(z)} \left(\int_{|w|^2 < e^{-\frac{\varphi}{k}}} e^{\varphi(z)} \right) \\ &= \int_V |f(z)|^2 \left(\int_{|w|^2 < e^{-\frac{\varphi}{k}}} e^{2kB(z)} \right). \end{aligned}$$

Therefore, we need to compare the value of $\int_{|w|^2 < e^{-\frac{\varphi}{k}}} e^{2k\tilde{B}(z,w)}$ and $\int_{|w|^2 < e^{-\frac{\varphi}{k}}} e^{2kB(z)}$.

We define a plurisubharmonic function $\tilde{u}'(z,w) := -B(z)$ on $\tilde{\Omega}$. Then the function $\tilde{u}'$ satisfies the following conditions.

$$(dd^c\tilde{u}')^{n+k} \geq 0 \quad \text{on } \tilde{\Omega},$$
$$\tilde{u}' = -B \quad \text{on } \partial\tilde{\Omega}.$$

Since $\tilde{u} = -\max(\log|\tilde{\psi}|, D) \geq \tilde{u}' = -B$ on $\partial\tilde{\Omega}$ and $(dd^c\tilde{u})^{n+k} \leq (dd^c\tilde{u}')^{n+k}$ on $\tilde{\Omega}$, Corollary 1.1.6 yields $\tilde{u} \geq \tilde{u}'$ on $\tilde{\Omega}$. In particular, for any $z \in V$, it follows that

$$\tilde{u}(z,w) \geq \tilde{u}'(z,w) = -B(z)$$

on $\{|w|^2 < e^{-\frac{\varphi(z)}{k}}\}$. Since $\tilde{u} = -D > -B$ on $\tilde{V} \cap \partial\tilde{\Omega}$ and $\tilde{u}$ is continuous near $\partial\tilde{\Omega}$, for any $z \in V$, $\{w : |w| < e^{-\frac{\varphi(z)}{k}}, \tilde{u}(z,w) > -B(z)\}$ has positive measure. Therefore, we have

$$\int_{|w|^2<e^{-\frac{\varphi}{k}}} e^{2k\tilde{B}(z,w)} < \int_{|w|^2<e^{-\frac{\varphi}{k}}} e^{2kB(z)}.$$

□

Remark 4.2.2. When $B(z)$ is a continuous function on Ω with $\log|\psi(z)| - B(z) \leq G_{\Omega,V}(z)$ on Ω, we can obtain a continuous function $B'(z)$ on Ω such that $-B'(z)$ is plurisubharmonic and $\log|\psi(z)| - B'(z) \leq G_{\Omega,V}(z)$ on Ω. In fact, by Theorem 1.2.5, there exists $-B' \in PSH(\tilde{\Omega}) \cap \mathcal{C}(\bar{\tilde{\Omega}})$ such that

$$(dd^c(-B'))^{n+k} = 0 \quad \text{on } \tilde{\Omega} \text{ and}$$
$$-B' = -B \quad \text{on } \partial\tilde{\Omega}.$$

Then, since

$$\log|\tilde{\psi}| - B' = \log|\tilde{\psi}| - B(z) \leq G_{\Omega,V}(z) \leq 0$$

on $\partial\tilde{\Omega}$, from the maximum principle, it follows that $\log|\tilde{\psi}| - B' \leq 0$ on $\tilde{\Omega}$. In particular, we have $\log|\psi| - B'(z,0) \leq G_{\Omega,V}(z)$ on Ω.

But, we do not know whether we can obtain the sharper estimates after replacing the function B with B'.

4.3. Toward sharper estimates of the Ohsawa–Takegoshi L^2-extension theorem in terms of the Azukawa pseudometric

The L^2-estimate obtained from the conclusion will be sharper if $B(z)$ is smaller. Therefore, we need to take the pluricomplex Green function to obtain a better L^2-estimate. As a special case, the following result was obtained in [**14**].

Theorem 4.3.1 ([**14**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, w a point in Ω and φ a plurisubharmonic function on Ω. Let $g_{\Omega,w}$ be the pluricomplex Green function on Ω with a pole at w and $A_{\Omega,w}$ the Azukawa pseudometric. We assume that there exists the limit*

$$A_{\Omega,w}(X) = \lim_{\lambda\to 0} (g_{\Omega,w}(w+\lambda X) - \log|\lambda|) \quad X \in \mathbb{C}^n.$$

Then there exists a holomorphic function f on Ω such that $f(w) = 1$ and

$$\int_\Omega |f|^2 e^{-\varphi} \leq \mathrm{vol}(I_{\Omega,w}) e^{-\varphi(w)},$$

where $I_{\Omega,w}$ is the Azukawa indicatrix defined by $I_{\Omega,w} := \{X \in \mathbb{C}^n : A_{\Omega,w}(X) < 0\}$ and $\mathrm{vol}(I_{\Omega,w})$ *is the euclidean volume of $I_{\Omega,w}$.*

REMARK 4.3.2. In [**24**], it is shown that the assumptions of Theorem 4.3.1 holds on a bounded hyperconvex domain.

When the submanifold V is $\{z_1 = \cdots = z_k = 0\}$, we can generalize Theorem 4.3.1 by using the pluricomplex Green function with poles along V.

THEOREM 4.3.3 ([**16**]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, V a closed complex submanifold defined by $V = \{z_1 = \cdots = z_k = 0\}$ and φ a plurisubharmonic function on Ω. Let $G_{\Omega,V}$ be the pluricomplex Green function on Ω with poles along V. We assume that there exists the limit*

$$A_{\Omega,V,w}(X) := \lim_{\lambda\to 0}\left(G_{\Omega,V}(\lambda X, w) - \log|\lambda|\right),$$

where $(0,\ldots,0,w) \in V$ and $0 \neq X \in \mathbb{C}^k$. We define $I_{\Omega,V,w} := \{X \in \mathbb{C}^n : A_{\Omega,V,w}(X) < 0\}$. Then for any holomorphic function f on V with $\int_V \mathrm{vol}(I_{\Omega,V,w})|f|^2 e^{-\varphi} < \infty$, there exists a holomorphic function F on Ω such that $F|_V = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} \leq \int_V \mathrm{vol}(I_{\Omega,V,w})|f|^2 e^{-\varphi}.$$

proof. It is sufficient to prove the following lemma.

LEMMA 4.3.4 ([**16**]). *Let $\chi \geq 0$ be a continuous function on $\bar{\Omega}$. Then we have*

$$\limsup_{t\to-\infty} e^{-kt}\int_{\Omega_t}\chi \leq \int_V \mathrm{vol}(I_{\Omega,V,z''})\chi,$$

where $\Omega_t := \left\{G_{\Omega,V} < \dfrac{t}{2}\right\}$ and $(0,\ldots,0,z'') \in V$.

proof. For any $\delta > 0$ and sufficiently negative t, by continuity of χ, we have

$$e^{-kt}\int_{\Omega_{t-2\delta}}\chi \leq e^{-kt}\int_V (\chi(0,z'') + \epsilon)dz'' \int_{\{G_{\Omega,V}(z',z'')<\frac{t}{2}-\delta\}} dz'.$$

Here, for any $(0,\ldots,0,z'') \in V$, we consider the value of $e^{-kt}\int_{\{G_{\Omega,V}(z',z'')<\frac{t}{2}-\delta\}} dz'$. Using the substitution $z' = e^{\frac{t}{2}}\tilde{z}'$, we have

$$e^{-kt}\int_{\{G_{\Omega,V}(z',z'')<\frac{t}{2}-\delta\}} dz' = \int_{\{G_{\Omega,V}(e^{\frac{t}{2}}\tilde{z}',z'')-\log e^{\frac{t}{2}}<-\delta\}} d\tilde{z}'. \tag{4.3.1}$$

By the assumptions of $G_{\Omega,V}$, take the upper limits with respect to t of both sides of (4.3.1), then the right-hand side of (4.3.1) converges to something whose magnitude is at most

$$\int_{\{A_{\Omega,V,z''}(\tilde{z}')\leq-\delta\}} d\tilde{z}'.$$

This value can be estimated as follow:

$$\int_{\{A_{\Omega,V,z''}(\tilde{z}')\leq-\delta\}} d\tilde{z}' \leq \int_{\{A_{\Omega,V,z''}(\tilde{z}')<0\}} d\tilde{z}' = \mathrm{vol}(I_{\Omega,V,z''}).$$

Therefore, by $\delta \to 0$, we can get the following inequality

$$\limsup_{t\to-\infty} e^{-kt}\int_{\Omega_t}\chi \leq \int_V \mathrm{vol}(I_{\Omega,V,z''})\chi.$$

□

By replacing Lemma 4.1.4 in the proof of Theorem 4.1.1 with Lemma 4.3.4, we can prove Theorem 4.3.3. □

EXAMPLE 4.3.5 ([**16**]). Here, for $n \geq 2$, we consider a unit ball $\Omega = \mathbb{B}^n$ in $\mathbb{C}^n$ and $\varphi(z) = -n\log(1-|z|^2)$. In this situation, the pluricomplex Green function $g_{\mathbb{B}^n,0}(z)$ with a pole at 0 is equal to $\log|z|$ and the Azukawa pseudometric $A_{\mathbb{B}^n,0}(X)$ is equal to $\log|X|$. Since the Azukawa indicatrix is $I_{\mathbb{B}^n,0} = \{|X|<1\}$, therefore we have $\mathrm{vol}(I_{\mathbb{B}^n,0}) = \sigma_n$. On the other hand, since $\tilde{\Omega} = \{|w|^2+|z|^2<1\}$ in $\mathbb{C}^{2n}$ and $\tilde{V} = \{|w|<1\}$, the pluricomplex Green function $G_{\tilde{\Omega},\tilde{V}}$ with poles along $\tilde{V}$ is equal to $\log\dfrac{|z|}{\sqrt{1-|w|^2}}$ and $A_{\tilde{\Omega},\tilde{V},w}(X)$ is equal to $\log|X| - \frac{1}{2}\log(1-|w|^2)$. Since $I_{\tilde{\Omega},\tilde{V},w} = \{|X| < (1-|w|^2)^{\frac{1}{2}}\}$, we have $\mathrm{vol}(I_{\tilde{\Omega},\tilde{V},w}) = \sigma_n(1-|w|^2)^n$. Then

$$\begin{aligned}\int_{\tilde{V}} \mathrm{vol}(I_{\tilde{\Omega},\tilde{V},w}) &= \int_{|w|<1}\sigma_n(1-|w|^2)^n \\ &= \sigma_n\int_{S^{2n-1}}\int_0^1(1-r^2)^n r^{2n-1}drdS \\ &= \sigma_n\mu_n\int_0^1(1-r^2)^n r^{2n-1}dr \\ &= \frac{\sigma_n\mu_n}{2}B(n,n+1) \\ &= \frac{\sigma_n\mu_n}{2}\frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)} \\ &= \frac{\sigma_n\mu_n}{2}\frac{(n-1)!n!}{(2n)!},\end{aligned}$$

where μ_n is the volume of S^{2n-1}, B is the Beta function and Γ is the Gamma function. For any $n \geq 2$, since $\mu_n = \dfrac{2\pi^n}{\Gamma(n)} = \dfrac{2\pi^n}{(n-1)!}$, it follows that

$$\frac{\mu_n}{2}\frac{(n-1)!n!}{(2n)!} < 1.$$

Therefore, in this situation, it follows that $\int_{\tilde{V}} \mathrm{vol}(I_{\tilde{\Omega},\tilde{V},w}) < \mathrm{vol}(I_{\mathbb{B}^n,0})$. From this observation, we can expect that the sharper estimates of Ohsawa–Takegoshi L^2-extension theorem in terms of the Azukawa pseudometric holds.

4.4. Radial case in $\mathbb{C}^n$

In [**13**], Hosono obtained the L^2-minimum extension of the function 1 on the subvariety $\{0\}$ by applying Theorem 3.2.1 to the case where $\Omega = \{|z| < 1\}$ is a unit disc in $\mathbb{C}$ and φ is a radial subharmonic function, i.e., $\varphi(z) = \varphi(|z|)$. Similarly, in this subsection, we obtain the L^2-minimum extension of holomorphic functions f on V in the setting where $\Omega = \mathbb{B}^n$ is a unit ball in $\mathbb{C}^n$, V is a closed submanifold defined by $V = \{z_1 = \cdots = z_k = 0\} = \{z' = 0\}$ and φ is a radial plurisubharmonic function with respect to V, i.e., $\varphi(z) = \varphi(|z'|)$, by applying Theorem 4.2.1 in this setting.

For $z \in \Omega$, we denote $z = (z', z'')$ where $z' = (z_1, \dots, z_k), z'' = (z_{k+1}, \dots, z_n)$. We may assume that $\varphi(0) = 0$. Under the above setting, we can write $\varphi(z) = ku(\log |z'|^2)$ where u is a convex increasing function on $\mathbb{R}_{<0}$. Assume that u is strictly increasing and for fixed z'', $\lim_{t \to \log(1-|z''|^2)-0} u(t) = \infty$. Define the plurisubharmonic function ψ by

$$\psi(w) := -\frac{1}{2} u^{-1}(-\log |w|^2).$$

Then

Proposition 4.4.1 ([**16**]). *For fixed z'', we have*

$$\int_{|z'|^2+|z''|^2<1} e^{-\varphi} = \int_{|w|<1} e^{-2k\psi(w)}. \tag{4.4.1}$$

proof. Let μ_k be the volume of S^{2k-1}. First, we calculate the left hand side of (4.4.1). Using the substitution $2\log r = t$, we have

$$\begin{aligned} \int_{|z'|^2+|z''|^2<1} e^{-\varphi} &= \int_{|z'|^2<1-|z''|^2} e^{-ku(\log |z|^2)} \\ &= \mu_k \int_0^1 e^{-ku(\log r^2)} r^{2k-1} dr \\ &= \frac{\mu_k}{2} \int_{-\infty}^{\log(1-|z''|^2)} e^{-ku(t)} e^{kt} dt. \end{aligned} \tag{4.4.2}$$

Next, we calculate the right hand side of (4.4.1). At first, using the substitution $2k\log r = t$, we have

$$\begin{aligned}\int_{|w|<1} e^{-2k\psi} &= \int_{|w|<1} e^{ku^{-1}(-\log|w|^2)}\\ &= \mu_k\int_0^1 e^{ku^{-1}(-\log r^2)}r^{2k-1}dr\\ &= \frac{\mu_k}{2k}\int_{-\infty}^0 e^{ku^{-1}(-\frac{t}{k})}e^t dt. \qquad (4.4.3)\end{aligned}$$

Then letting $u^{-1}(-\frac{t}{k}) = s$, we see that the right hand side of (4.4.3) is equal to

$$\frac{\mu_k}{2}\int_{-\infty}^{\log(1-|z''|^2)} e^{ks}e^{-ku(s)}u'(s)ds. \qquad (4.4.4)$$

And finally, using the substitution $ks - ku(s) = q$, we calculate the difference between (4.4.2) and (4.4.4) as follows:

$$\begin{aligned}&\frac{\mu_k}{2}\int_{-\infty}^{\log(1-|z''|^2)} e^{-ku(t)}e^{kt}dt - \frac{\mu_k}{2}\int_{-\infty}^{\log(1-|z''|^2)} e^{ks}e^{-ku(s)}u'(s)ds\\ &= \frac{\mu_k}{2k}\int_{-\infty}^{\log(1-|z''|^2)} k(1-u'(s))e^{ks-ku(s)}ds\\ &= \frac{\mu_k}{2k}\int_{-\infty}^{-\infty} e^q dq = 0.\end{aligned}$$

□

We define $\tilde{G}(z,w) := \log|z| + \psi(w)$. From Proposition 4.4.1 and Theorem 4.1.1, we infer that for any holomorphic function f on V, there exists a holomorphic function F on $\mathbb{B}^n$ such that $F|_V = f$ and

$$\int_{\mathbb{B}^n} |F|^2e^{-\varphi} \le \int_{|z'|^2+|z''|^2<1} |f(0,z'')|^2e^{-\varphi}.$$

Therefore, in this case, by the above inequality and the mean value inequality, we can obtain that the L^2-minimum extension with respect to φ is $F(z) = f(0,z'')$ in holomorphic functions on $\mathbb{B}^n$ with $F|_V = f$.

In the general case, for any $\epsilon > 0$, we define $ku_\epsilon(\log|z'|^2) := \varphi(z) - \epsilon\log(1-|z''|^2-|z'|^2)$. Then u_ϵ is a strictly increasing and satisfies that for fixed z'', $\lim_{t\to\log(1-|z''|^2)-0} u_\epsilon(t) = \infty$. Therefore, for any $\epsilon > 0$, we can obtain that the L^2-minimum extension with respect to $ku_\epsilon(\log|z|^2)$ is $F(z) = f(0,z'')$ in holomorphic functions on $\mathbb{B}^n$ with $F|_V = f$. Finally, by $\epsilon \to 0$, we get the conclusion.

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