

Computation of the index of some meromorphic
functions of degree 3 on tori

(トーラス上のある3次有理型関数の指数の計算)

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Contents

1	Introduction	3
2	Preliminaries	8
2.1	Preliminaries on a Riemann surface and meromorphic function	8
2.2	Preliminaries on the index and the nullity of a meromorphic function . . .	9
3	Setting	17
3.1	Torus	17
3.2	Vector spaces $H(w)$ and $\widehat{H}(w)$	17
4	Computation of $\text{Nul}(tw)$	20
5	Computation of $\text{Ind}(tw)$	30
6	Appendix	32

1 Introduction

The index of a nonconstant meromorphic function g on a compact Riemann surface is an invariant of g , which is defined as the number of negative eigenvalues of the differential operator $L := -\Delta - |dG|^2$, where Δ is the Laplacian with respect to a conformal metric $ds^2 = \lambda d\zeta d\bar{\zeta}$ on the Riemann surface, defined by $\Delta := \frac{4}{\lambda} \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}$ using a local coordinate ζ , $G: M \rightarrow S^2$ is the holomorphic map corresponding to the meromorphic function g and $|dG|$ is the norm of the differential dG of G . The multiplicity of the eigenvalue 0 of L is called the nullity of g and denoted by $\text{Nul}(g)$. The operator L depends on how to choose a conformal metric, but the index and the nullity do not depend on how to choose a conformal metric.

The index of a meromorphic function is closely related to the index (Morse index) of a complete minimal surface with finite total curvature. Huber [6] and Osserman [11] proved if the total curvature of a complete oriented minimal surface in \mathbb{R}^3 is finite, this minimal surface is identified with a Riemann surface given by excluding finite points from a compact Riemann surface, and the Gauss map on this minimal surface is extended to a meromorphic function on the compact Riemann surface. Fischer-Colbrie[3] and Gulliver-Lawson [4], [5] proved that for a complete oriented minimal surface in \mathbb{R}^3 , the index is finite if and only if the total curvature is finite. This is a qualitative study of the index. Fischer-Colbrie proved when the total curvature a complete oriented minimal surface in \mathbb{R}^3 is finite, the index coincides with the index of the extended Gauss map of this minimal surface. Tysk [12] proved the index of a complete oriented minimal surface in \mathbb{R}^3 is bounded from above by some scalar multiple of the total curvature. This is the first quantitative study of the relationship between the index and the total curvature. Study of lower bound of index was done by Choe [1] and Nayatani [9]. Nayatani [10] studied for the index and the nullity of the operator L_g associated to any meromorphic function g on a compact Riemann surface M , how they change under a certain deformation g_t of g (t is a positive real number). He considered the derivative \wp' of the Weierstrass \wp -function corresponding to the square lattice $\mathbb{Z} \oplus i\mathbb{Z}$ as a meromorphic function g , and computed the index of g_t when t is sufficiently small and the nullity of g_t for all t . In particular, he showed that there are two values $t_1, t_2 (t_1 < t_2)$ of t such that the nullity is 4. Furthermore, he investigated the change of index when t becomes large. He showed that the indices of t_1g, t_2g are 5, and since t_2g is the Gauss map of the Costa surface, he could conclude that the index of the Costa surface is 5.

In this paper, we study the index $\text{Ind}(g)$ and the nullity $\text{Nul}(g)$ of certain nonconstant meromorphic functions g from a compact Riemann surface M to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In order to compute $\text{Nul}(g)$, we recall the real vector space $H(g)$ (see (2.5)) which was introduced by Ejiri-Kotani [2] and Montiel-Ros [7]. By the formula $\text{Nul}(g) = 3 + \dim_{\mathbb{R}} H(g)$, we can

compute $\text{Nul}(g)$. If the genus of the Riemann surface M is 1, that is, M is homeomorphic to the torus, $H(g)$ can be defined as follows.

$$H(g) = \left\{ f\omega \mid f : M \rightarrow \widehat{\mathbb{C}} \text{ is a meromorphic function,} \right. \\ \left. D(f) + \widetilde{B}(g) \geq 0, \text{Res}_{p_i}(f\omega) = 0, i = 1, \dots, \mu, \right. \\ \left. \text{Re} \int_{\alpha} (1 - g^2, i(1 + g^2), 2g)(f\omega) = 0, \forall \alpha \text{ closed curve} \right\},$$

where ω is a fixed nonzero holomorphic one-form on M , p_1, \dots, p_{μ} are the ramification points of g with ramification indices e_1, \dots, e_{μ} , and $D(f)$ is the divisor of f , and when $P(g)$ is the polar divisor of g , $\widetilde{B}(g)$ is the divisor defined by $\widetilde{B}(g) = \sum_{i=1}^{\mu} e_i p_i - 2P(g)$. Since the definition of $H(g)$ is complicated as it includes the period condition, we introduce the complex vector space

$$\widehat{H}(g) = \left\{ f\omega \mid f : M \rightarrow \widehat{\mathbb{C}} \text{ is a meromorphic function,} \right. \\ \left. D(f) + \widetilde{B}(g) \geq 0, \text{Res}_{p_i}(f\omega) = 0, i = 1, \dots, \mu \right\}$$

that is easier to handle, excluding the period condition. $H(g)$ is a real subspace of $\widehat{H}(g)$. For $t \in \mathbb{C} \setminus \{0\}$, $H(tg) \neq H(g)$ in general, but $\widehat{H}(tg) = \widehat{H}(g)$.

As already mentioned, Nayatani[10] computed the index of Costa surface. The compact Riemann surface of the Costa surface is \mathbb{C} divided by the square lattice $\mathbb{Z} \oplus i\mathbb{Z}$, which is homeomorphic to a torus, and the Gauss map of Costa surface is a scalar multiple of the derivative \wp' of the Weierstrass \wp -function. $\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$ is isomorphic to

$$M_1 = \left\{ (z, w) \in \widehat{\mathbb{C}}^2 \mid w^2 = z(z^2 - 1) \right\}$$

as a Riemann surface, and \wp' coincides with the meromorphic function w except for a scalar multiple. As a generalization of Nayatani's setting, we consider a one-parameter family

$$M_a = \left\{ (z, w) \in \widehat{\mathbb{C}}^2 \mid w^2 = z(z - a) \left(z + \frac{1}{a} \right) \right\}, a \geq 1$$

of Riemann surfaces homeomorphic to a torus and the meromorphic function w , and we tackled the problem of computing the index and nullity of tw , $t > 0$. We note that M_a is isomorphic, as a Riemann surface, to \mathbb{C} divided by the rectangular lattice $\mathbb{Z} \oplus ic\mathbb{Z}$, $c > 0$. As a result, we are able to determine $\text{Ind}(tw)$ for all a in the range $1 \leq a \leq a_0$ (where a_0 can be numerically evaluated) and all $t > 0$. This is the main theorem of this dissertation.

First we determine $\widehat{H}(w)$.

Lemma 1.1. (Lemma 4.1) $\widehat{H}(w)$ is a three dimensional complex vector space spanned by

$$\begin{aligned}\eta_1 &= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z - A_1)^2(z - A_2)^2} \frac{dz}{w}, \\ \eta_2 &= \frac{z^2 - A_1A_2}{(z - A_1)^2(z - A_2)^2} \frac{dz}{w}, \\ \eta_3 &= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z - A_1)^2(z - A_2)^2} dz.\end{aligned}$$

Then we determine $t > 0$ so that the dimension of $H(tw)$ is 1 or more, and find exactly two values $t = t_1(a), t_2(a)$ ($t_1(a) < t_2(a)$). We then determine $\text{Nul}(tw)$.

Lemma 1.2. (Lemma 4.4) For each a in the range $1 \leq a \leq a_0$ (where a_0 can be numerically evaluated) there are positive real numbers $t_1(a), t_2(a)$ ($t_1(a) < t_2(a)$) such that

$$\dim_{\mathbb{R}} H(tw) = \begin{cases} 1, & t = t_1(a), t_2(a), \\ 0, & t > 0, t \neq t_1(a), t_2(a). \end{cases}$$

Therefore,

$$\text{Nul}(tw) = \begin{cases} 4, & t = t_1(a), t_2(a), \\ 3, & t \neq t_1(a), t_2(a). \end{cases}$$

By using the fact that $\text{Nul}(tw)$ also changes if $\text{Ind}(tw)$ changes when a, t move, we determine $\text{Ind}(tw)$ for all a in the range $1 \leq a \leq a_0$ (where a_0 can be numerically evaluated) and all $t > 0$.

Theorem 1.3. (Theorem 5.2) If $t_1(a)$ and $t_2(a)$ are as described in Lemma 1.2, then

$$\text{Ind}(tw) = \begin{cases} 5, & 0 < t \leq t_1(a), t_2(a) \leq t, \\ 6, & t_1(a) < t < t_2(a) \end{cases}$$

for any a in the range $1 \leq a \leq a_0$ (where a_0 can be numerically evaluated).

Organization of this thesis

This paper is organized as follows. In Section 2, first we prepare basic concepts and materials on the Riemann surface and the meromorphic function and differential. Then we define the index and nullity of a meromorphic function on a compact Riemann surface. We also recall the vector spaces which are used in the computation of nullity and were introduced by Ejiri-Kotani [2] and Montiel-Ros [7]. In Section 3 we consider a certain family of meromorphic functions g_a of degree three defined on Riemann surfaces M_a , $a \geq 1$, homeomorphic to the torus, and describe the above vector spaces in these special cases. In Section 4 we compute the nullity of tg_a for all $t > 0$ and a in the range $1 \leq a \leq a_0$, where a_0 is a constant which can be numerically evaluated. In Section 5 we compute the index of tg_a for all t and a in the same range.

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2 Preliminaries

In this section, we prepare basic concepts and materials which will be used in this thesis.

2.1 Preliminaries on a Riemann surface and meromorphic function

As this thesis is concerned with a meromorphic function on a Riemann surface, we recall their definitions. We begin with the Riemann surface.

Definition 2.1. Let M be a connected Hausdorff space. When there is a set $A = \{(U_1, \phi_1), (U_2, \phi_2), \dots\}$ of the pairs (U_i, ϕ_i) of open subsets U_i in M and homeomorphisms ϕ_i from U_i to an open subsets in \mathbb{C} , and A satisfies the following conditions, (M, A) is called Riemann surface.

- (i) U_1, U_2, \dots cover M .
- (ii) For any $i, j = 1, 2, \dots$, when $U_i \cap U_j \neq \emptyset$, the coordinate change

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$$

is holomorphic.

Next we define the meromorphic function and related functions/differentials.

Definition 2.2. A complex-valued function $f : M \rightarrow \mathbb{C}$ is called *holomorphic* if $f \circ \phi_i^{-1}$ is a holomorphic function on $\phi_i(U_i)$ for all i .

Definition 2.3. If a differential 1-form ω can be locally written as $\omega = h(\zeta)d\zeta$ with a local holomorphic coordinate ζ and a holomorphic function $h(\zeta)$, ω is called a *holomorphic differential* on M .

Definition 2.4. A function $f : M \rightarrow \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called *meromorphic* if there exists an open set $M' \subset M$ which satisfies the following conditions,

- (i) $M \setminus M'$ is empty or discrete in M .
- (ii) $f|_{M'}$ is holomorphic.
- (iii) For any $p \in M \setminus M'$, $\lim_{z \rightarrow p} |f(z)| = \infty$.

Definition 2.5. When ω satisfies the following conditions, ω is called a *meromorphic differential* on M : There is an open set $M' \subset M$ such that $M \setminus M'$ is discrete and

(i) $\omega|_{M'}$ is a holomorphic differential on M' .

(ii) For any $p \in M \setminus M'$, there is a local holomorphic coordinate ζ with $\zeta(p) = 0$ such that $\omega = m(\zeta)d\zeta$ near p , where $m(\zeta)$ is a meromorphic function and has $\zeta = 0$ as a pole.

Finally, we collect basic notions related to the meromorphic function and differential which will be used later.

Definition 2.6. Let ω be a meromorphic differential on M , let $p \in M$, and we express ω as $\omega = m(\zeta)d\zeta$ for a local holomorphic coordinate ζ with $\zeta(p) = 0$. The residue of the meromorphic function $m(\zeta)$ at $\zeta = 0$ is called the *residue* of ω at p .

In the following two definitions, let g be a nonconstant meromorphic function from a compact Riemann surface M to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 2.7. Let $p \in M$. If there exists a local holomorphic coordinate ζ with $\zeta(p) = 0$ such that

$$g(\zeta) = \begin{cases} g(0) + \zeta^e, & \text{if } p \text{ is not a pole,} \\ \frac{1}{\zeta^e}, & \text{if } p \text{ is a pole,} \end{cases}$$

then $e(\geq 1)$ is called the *ramification index* of g at p . If $e \geq 2$, then p is called a *ramification point* of g .

Definition 2.8. Let p_1, \dots, p_μ be the ramification points of g and e_1, \dots, e_μ be their ramification indices. Then

$$B(g) = \sum_{i=1}^{\mu} (e_i - 1)p_i$$

is called the *ramification divisor* of g .

2.2 Preliminaries on the index and the nullity of a meromorphic function

In this subsection, we define the index and nullity of a meromorphic function on a compact Riemann surface. We also recall the vector spaces which are used in the computation of nullity.

Let M be a Riemann surface, and g be a nonconstant meromorphic function from a compact Riemann surface M to $\widehat{\mathbb{C}}$. We fix a conformal metric $ds^2 = \lambda dz d\bar{z}$, where λ is a positive function on M , and consider the differential operator $L = -\Delta - |dG|^2$. Here, $\Delta := \frac{1}{\lambda} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{4}{\lambda} \frac{\partial^2}{\partial z \partial \bar{z}}$ is the Laplace-Beltrami operator of ds^2 .

$$G = \left(\frac{2\operatorname{Re}(g)}{|g|^2 + 1}, \frac{2\operatorname{Im}(g)}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right) : M \rightarrow S^2$$

is the holomorphic map corresponding to the meromorphic function g , where S^2 is the unit sphere of \mathbb{R}^3 . $|dG|^2$ is the square of the norm of dG with respect to the metric ds^2 , that is, $|dG|^2 = \frac{1}{\lambda} \left(\left| \frac{\partial G}{\partial x} \right|^2 + \left| \frac{\partial G}{\partial y} \right|^2 \right) = \frac{4}{\lambda} \left| \frac{\partial G}{\partial z} \right|^2$.

Lemma 2.9. For functions u and v on M ,

$$\int_M (\Delta u) v dA = - \int_M \langle du, dv \rangle dA, \quad (2.1)$$

$$\int_M (\Delta u) v dA = \int_M u (\Delta v) dA, \quad (2.2)$$

$$\int_M (Lu) v dA = \int_M u (Lv) dA \quad (2.3)$$

hold. Here, $dA = \lambda dx \wedge dy$ is the area element with respect to ds^2 .

Proof.

$$\begin{aligned} \lambda dz \wedge d\bar{z} &= \lambda(dx + idy) \wedge (dx - idy) \\ &= -i\lambda dx \wedge dy + i\lambda dy \wedge dx \\ &= -2i\lambda dx \wedge dy. \end{aligned}$$

Therefore,

$$\begin{aligned} dA &= \lambda dx \wedge dy \\ &= \frac{1}{-2i} \lambda dz \wedge d\bar{z} \\ &= \frac{i}{2} \lambda dz \wedge d\bar{z}. \end{aligned}$$

Thus, the left side of (2.1) becomes

$$\begin{aligned} \int_M (\Delta u) v dA &= \int_M \frac{4}{\lambda} \frac{\partial^2 u}{\partial z \partial \bar{z}} v \cdot \frac{i}{2} \lambda dz \wedge d\bar{z} \\ &= \int_M 2i \frac{\partial^2 u}{\partial z \partial \bar{z}} v dz \wedge d\bar{z}, \end{aligned}$$

and the right side (2.1) becomes

$$\begin{aligned}
& - \int_M \langle du, dv \rangle dA \\
& = - \left\langle \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}, \frac{\partial v}{\partial z} dz + \frac{\partial v}{\partial \bar{z}} d\bar{z} \right\rangle \frac{i}{2} \lambda dz \wedge d\bar{z} \\
& = -i \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial \bar{z}} + \frac{\partial u}{\partial \bar{z}} \frac{\partial v}{\partial z} \right) dz \wedge d\bar{z}.
\end{aligned}$$

On the other hand, let $\beta = -v \left(-\frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z} \right)$. We obtain

$$\begin{aligned}
d\beta & = d \left(-v \frac{\partial u}{\partial z} \right) \wedge dz + d \left(v \frac{\partial u}{\partial \bar{z}} \right) \wedge d\bar{z} \\
& = \left[\frac{\partial}{\partial z} \left(-v \frac{\partial u}{\partial z} \right) dz + \frac{\partial}{\partial \bar{z}} \left(-v \frac{\partial u}{\partial z} \right) d\bar{z} \right] \wedge dz + \left[\frac{\partial}{\partial z} \left(v \frac{\partial u}{\partial \bar{z}} \right) dz + \frac{\partial}{\partial \bar{z}} \left(v \frac{\partial u}{\partial \bar{z}} \right) d\bar{z} \right] \wedge d\bar{z} \\
& = \frac{\partial}{\partial \bar{z}} \left(-v \frac{\partial u}{\partial z} \right) d\bar{z} \wedge dz + \frac{\partial}{\partial z} \left(v \frac{\partial u}{\partial \bar{z}} \right) dz \wedge d\bar{z} \\
& = - \left(\frac{\partial v}{\partial \bar{z}} \frac{\partial u}{\partial z} + v \frac{\partial^2 u}{\partial z \partial \bar{z}} \right) d\bar{z} \wedge dz + \left(\frac{\partial v}{\partial z} \frac{\partial u}{\partial \bar{z}} + v \frac{\partial^2 u}{\partial z \partial \bar{z}} \right) dz \wedge d\bar{z} \\
& = \left(\frac{\partial v}{\partial \bar{z}} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial u}{\partial \bar{z}} \right) dz \wedge d\bar{z} + 2v \frac{\partial^2 u}{\partial z \partial \bar{z}} dz \wedge d\bar{z} \\
& = \frac{1}{i} ((\Delta u) v dA + \langle du, dv \rangle dA).
\end{aligned}$$

Since $\int_M d(i\beta) = 0$ by the theorem of Stokes,

$$\int_M ((\Delta u) v dA + \langle du, dv \rangle dA) = 0.$$

Therefore, (2.1) is proved. (2.2), (2.3) follow from (2.1). This completes the proof of the lemma. \square

Remark 1. For two different eigenvalues λ, μ of L , an eigenfunction u of L corresponding to λ , and an eigenfunction v of L corresponding to μ , we have

$$\int_M uv dA = 0.$$

Actually,

$$\begin{aligned}
\int_M (Lu) v dA & = \lambda \int_M uv dA, \\
\int_M u (Lv) dA & = \mu \int_M uv dA
\end{aligned}$$

by $Lu = \lambda u$, $Lv = \mu v$. By (2.3), we obtain

$$(\lambda - \mu) \int_M uv dA = 0.$$

Thus, since $\lambda \neq \mu$,

$$\int_M uv dA = 0.$$

Claim 1. $N(g) = \{u \in C^\infty(M) \mid Lu = 0\}$ does not depend on how the conformal metric ds^2 is taken.

Actually,

$$\tilde{L} = \phi L, \quad \phi = \frac{\lambda}{\tilde{\lambda}} > 0$$

for two conformal metrics $ds^2 = \lambda dz d\bar{z}$, $\tilde{ds}^2 = \tilde{\lambda} dz d\bar{z}$. (Note that ϕ is globally defined on M .) Therefore, $\tilde{L}u = 0$ if and only if $Lu = 0$. Thus, $N(g)$ does not depend on the choice of conformal metric ds^2 . \square

We now define the index and the nullity of g .

Definition 2.10. We define the index $\text{Ind}(g)$ of the meromorphic function g as the number of negative eigenvalues (counted with multiplicities) of the operator L , and define the nullity $\text{Nul}(g)$ of g as

$$\text{Nul}(g) = \dim N(g).$$

The nullity $\text{Nul}(g)$ does not depend on the choice of conformal metric ds^2 by Claim 1. The index $\text{Ind}(g)$ also does not depend on the choice of ds^2 by the following discussion.

The bilinear form associated with L is represented by $Q: C^\infty(M) \times C^\infty(M) \rightarrow \mathbb{R}$. That is, for $u, v \in C^\infty(M)$, we define $Q(u, v)$ as

$$\begin{aligned} Q(u, v) &= \int_M (Lu)v dA \\ &= \int_M (-\Delta u - |dG|^2 u)v dA. \end{aligned}$$

By (2.3), $Q(u, v) = Q(v, u)$, that is, Q is symmetric. By (2.1),

$$Q(u, u) = \int_M (|du|^2 - |dG|^2 u^2) dA$$

for a function u on M .

Remark 2. Q does not depend on the choice of conformal metric ds^2 on M .

Actually, if we let $ds^2 = \lambda(dx^2 + dy^2)$, we obtain

$$\begin{aligned}
Q(u, u) &= \int_M (|du|^2 - |dG|^2 u^2) dA \\
&= \int_M \left(\frac{1}{\lambda} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) - \frac{1}{\lambda} \left(\left| \frac{\partial G}{\partial x} \right|^2 + \left| \frac{\partial G}{\partial y} \right|^2 \right) u^2 \right) \lambda dx dy \\
&= \int_M \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 - \left(\left| \frac{\partial G}{\partial x} \right|^2 + \left| \frac{\partial G}{\partial y} \right|^2 \right) u^2 \right) dx dy. \tag{2.4}
\end{aligned}$$

Since λ is not included in the rightmost side of (2.4), Q does not depend on the choice of conformal metric ds^2 on M .

Let the eigenspace V_λ corresponding to the eigenvalue λ of L be $V_\lambda = \{u \in C^\infty(M) \mid Lu = \lambda u\}$, $\mu_1 < \mu_2 < \dots < \mu_k < 0$ be the set of all negative eigenvalues of L , and $V = V_{\mu_1} \oplus V_{\mu_2} \oplus \dots \oplus V_{\mu_k} \subset C^\infty(M)$.

Claim 2. Q is negative definite on V , that is, $Q(u, u) < 0$ for any $u \in V \setminus \{0\}$.

Actually, for any $u \in V$, u can be written as

$$u = \sum_{i=1}^k u_i, \quad u_i \in V_{\mu_i}.$$

Then

$$\begin{aligned}
Q(u, v) &= \int_M (Lu)u dA \\
&= \int_M \left(L \sum_{i=1}^k u_i \right) \left(\sum_{j=1}^k u_j \right) dA \\
&= \int_M \left(\sum_{i=1}^k Lu_i \right) \left(\sum_{j=1}^k u_j \right) dA \\
&= \int_M \left(\sum_{i=1}^k \mu_i u_i \right) \left(\sum_{j=1}^k u_j \right) dA \\
&= \int_M \left(\sum_{i,j=1}^k \mu_i u_i u_j \right) dA \\
&= \sum_{i,j=1}^k \mu_i \int_M u_i u_j dA \\
&= \sum_{i=1}^k \mu_i \int_M u_i^2 dA,
\end{aligned}$$

where we have used Remark 1. Therefore, when $u \neq 0$, $Q(u, u) < 0$. □

In fact, one can show that V is a maximal subspace of $C^\infty(M)$ on which Q is negative definite. Thus, the index $\text{Ind}(g)$ coincides with the dimension of a maximal subspace of $C^\infty(M)$ on which Q is negative definite. Hence, by Remark 2, $\text{Ind}(g)$ does not depend on the choice of conformal metric ds^2 .

Let $ds_{S^2}^2 = \frac{4}{(1+|z|^2)^2} dzd\bar{z}$ be the standard Riemannian metric on $\widehat{\mathbb{C}}$. Let ζ be a local holomorphic coordinate on M . The pull-back ds_g^2 of $ds_{S^2}^2$ by g can be written as $ds_g^2 = g^* ds_{S^2}^2 = \lambda d\zeta d\bar{\zeta}$, where $\lambda = \frac{4|g'|^2}{(1+|g|^2)^2}$. Let $\Delta_g = \frac{4}{\lambda} \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}$ be the Laplacian of ds_g^2 . If L corresponding to this ds_g^2 is represented by L_g , $L_g = -\Delta_g - 2$.

Since $\lambda = 0$ at the ramification points of g , $ds_g^2 = 0$ at the ramification points. Although ds^2 is not strictly a conformal metric, one can show that $\text{Ind}(g)$ can be computed as the number of negative eigenvalues (counted with multiplicities) of L_g and $\text{Nul}(g)$ can be computed as the multiplicity of the eigenvalue 0 of L_g . In other words, we have the following.

Lemma 2.11. $\text{Ind}(g)$ can be computed as the number of eigenvalues (counted with multiplicities) of $-\Delta_g$ which are smaller than 2. $\text{Nul}(g)$ can be computed as the multiplicity of the eigenvalue 2 of $-\Delta_g$.

Remark 3. Let M' be a complete oriented minimal surface in \mathbb{R}^3 and ds^2 be the first fundamental form on M' . The operator L corresponding to ds^2 becomes the Jacobi operator $L = -\Delta + 2K$. Here, Δ is the Laplacian corresponding to ds^2 and K is the Gaussian curvature of ds^2 . Then the (*Morse*) *index* of M' is defined as

$$\text{Ind}(M') = \sup \{ \dim V \mid V \subset C_0^\infty(M'), Q \text{ is negative definite on } V \}.$$

Here, $Q(u, v) = \int_{M'} (Lu)v dA$, $u, v \in C_0^\infty(M')$. Furthermore, Fischer-Colbrie [3] proved that when the total curvature of M' is finite, the index of M' coincides with the index of the extended Gauss map of M' .

Example 1. The index of the catenoid is 1. In fact, the catenoid is identified with $\mathbb{C} - \{0\}$ as a Riemann surface, and its extended Gauss map is the meromorphic function $g(z) = z$ on $\widehat{\mathbb{C}}$. Therefore, $ds_g^2 = ds_{S^2}^2$, the standard metric of the unit sphere. Thus, $\text{Ind}(g)$ coincides with the number of eigenvalues (counted with multiplicities) of $-\Delta_{S^2}$ which are smaller than 2. Since 0 is the only such eigenvalue and has multiplicity 1, we conclude that $\text{Ind}(g) = 1$. Thus, the index of the catenoid is 1 by Remark 3.

Proposition 2.12. $\text{Nul}(g) \geq 3$.

Proof. We define $L(g)$ as $L(g) = \{a \cdot G \mid a \in \mathbb{R}^3\}$. Since $G = (G_1, G_2, G_3): M \rightarrow S^2 \subset$

\mathbb{R}^3 is a holomorphic map, G satisfies the harmonic map equation $\Delta G + |dG|^2 G = 0$.

$$\begin{aligned}
L(a \cdot G) &= L\left(\sum_{i=1}^3 a_i G_i\right) \\
&= \sum_{i=1}^3 a_i L G_i \\
&= \sum_{i=1}^3 a_i (-\Delta G_i - |dG|^2 G_i) \\
&= 0.
\end{aligned}$$

Therefore, $a \cdot G \in N(g)$. Thus, $L(g) \subset N(g)$. The dimension of $L(g)$ is three. In fact, if this is not true, we have a linear relation $a_1 G_1 + a_2 G_2 + a_3 G_3 = 0$, and this means that the image of G lies in a great circle of S^2 . But this implies G is a constant map (and g is a constant function) as it is holomorphic. This contradicts the assumption that g is nonconstant. Therefore, $\dim L(g) = 3$ and $\dim N(g) \geq 3$. This completes the proof of the proposition. \square

As mentioned in the above proof, $L(g) \subset N(g)$, and $\text{Nul}(g) > 3$ if and only if $N(g) \setminus L(g) \neq \emptyset$. In order to compute $\text{Nul}(g)$, we recall the work of Ejiri-Kotani [2] and Montiel-Ros [7]. They observed that an element of $N(g) \setminus L(g)$ appears in the following way.

Definition 2.13. Let $X: M' = M \setminus \{p_1, \dots, p_\mu\} \rightarrow \mathbb{R}^3$ be a complete branched minimal immersion of finite total curvature, where M is a compact Riemann surface. An end p_i is said to be *planer* if there exists a unit vector $a_i \in S^2$ such that $\langle X, a_i \rangle$ is bounded in a neighborhood of p_i .

Proposition 2.14 (Ejiri-Kotani [2] and Montiel-Ros [7]). Let $g: M \rightarrow \widehat{\mathbb{C}}$ be a nonconstant meromorphic function on a compact Riemann surface M , and $G: M \rightarrow S^2$ be the holomorphic map corresponding to g . Let $X: M' = M \setminus \{p_1, \dots, p_\mu\} \rightarrow \mathbb{R}^3$ be a complete branched minimal immersion of finite total curvature whose extended Gauss map is g and whose ends are all planer. Then $u = \langle X, G \rangle: M' \rightarrow \mathbb{R}$, the support function of X , extends to M smoothly and gives an element of $N(g) \setminus L(g)$.

On the contrary, Ejiri-Kotani [2] and Montiel-Ros [7] proved that any element of $N(g) \setminus L(g)$ appeared as the support function of a complete branched minimal surface with planer ends.

Theorem 2.15 (Ejiri-Kotani [2] and Montiel-Ros [7]). For any $u \in N(g) \setminus L(g)$, there exists a complete branched minimal immersion $X: M' = M \setminus \{p_1, \dots, p_\mu\} \rightarrow \mathbb{R}^3$ whose

ends are all planer and whose extended Gauss map coincides with g such that $u = \langle X, G \rangle$ on M , where $G: M \rightarrow S^2$ is the holomorphic map corresponding to g . In terms of the Weierstrass representation formula, this assertion is stated as follows. Let $H(g)$ be the real vector space defined by

$$H(g) = \left\{ \omega \in H^0(K(M) + \tilde{B}(g)) \mid \text{Res}_{p_i} \omega = 0, \forall i = 1, \dots, \mu, \right. \\ \left. \text{Re} \int_{\alpha} ((1 - g^2), i(1 + g^2), 2g)\omega = 0, \forall \alpha \in H_1(M, \mathbb{Z}) \right\}, \quad (2.5)$$

where $K(M)$ is the canonical divisor of M , p_i are the ramification points of g , $P(g)$ is the polar divisor of g and $\tilde{B}(g)$ is the divisor defined by $\tilde{B}(g) = \sum_{i=1}^{\mu} e_i p_i - 2P(g)$. Then for any $u \in N(g) \setminus L(g)$, there exists $\omega \in H(g) \setminus \{0\}$ such that $u = \langle X_{\omega}, G \rangle$ on M , where $X_{\omega}: M \setminus \{p_1, \dots, p_{\mu}\} \rightarrow \mathbb{R}^3$ is the branched minimal immersion defined by

$$X_{\omega}(p) = \text{Re} \int_{p_0}^p ((1 - g^2), i(1 + g^2), 2g)\omega. \quad (2.6)$$

In particular, we have the linear isomorphism $N(G)/L(G) \cong H(g)$.

Corollary 2.16 (Ejiri-Kotani [2] and Montiel-Ros [7]). $\text{Nul}(g)$ can be computed from the dimension of $H(g)$ by the following formula:

$$\text{Nul}(g) - 3 = \dim_{\mathbb{R}} H(g). \quad (2.7)$$

The complex vector space $\hat{H}(g)$ defined as follows plays an auxiliary role in the computation of $H(g)$.

Definition 2.17. Define a complex vector space $\hat{H}(g)$ as

$$\hat{H}(g) = \left\{ \omega \in H^0(K(M) + \tilde{B}(g)) \mid \text{Res}_{p_i} \omega = 0, i = 1, \dots, \mu \right\}.$$

For $t \in \mathbb{C} \setminus \{0\}$, $H(tg) \neq H(g)$ in general, but $\hat{H}(tg) = \hat{H}(g)$.

3 Setting

In this section, we consider a certain family of meromorphic functions of degree three defined on Riemann surfaces homeomorphic to the torus, and describe the above vector spaces in these special cases.

3.1 Torus

We first define the Riemann surfaces. Let

$$M_a = \left\{ (z, w) \in \widehat{\mathbb{C}}^2 \mid w^2 = z(z-a) \left(z + \frac{1}{a} \right) \right\}, \quad a \geq 1.$$

If $(r_1, \theta_1), (r_2, \theta_2), (r_3, \theta_3)$ are the polar coordinates centered at $0, a, -\frac{1}{a}$, then $z \in \widehat{\mathbb{C}}$ is represented in three ways as

$$z = r_1 e^{i\theta_1}, z = a + r_2 e^{i\theta_2}, z = -\frac{1}{a} + r_3 e^{i\theta_3}, \quad r_1, r_2, r_3 \geq 0, 0 \leq \theta_1, \theta_2, \theta_3 < 2\pi.$$

Define the two branches w_1, w_2 of w by

$$w_1 = \sqrt{r_1 r_2 r_3} e^{\frac{i(\theta_1 + \theta_2 + \theta_3)}{2}}, \quad w_2 = \sqrt{r_1 r_2 r_3} e^{\frac{i(\theta_1 + \theta_2 + \theta_3)}{2} + \pi} = -w_1.$$

Prepare two copies of the Riemann spheres $\widehat{\mathbb{C}}$, let them be $\widehat{\mathbb{C}}_1$ and $\widehat{\mathbb{C}}_2$, respectively, and consider w_1 as a function on $\widehat{\mathbb{C}}_1$ and w_2 as a function on $\widehat{\mathbb{C}}_2$. Put in a slit in the half line connecting $z = a$ and $z = \infty$ on $\widehat{\mathbb{C}}_1$, and let the upper part (of the slit) be l_1 and the lower part be \tilde{l}_1 . Put in a slit in the half line connecting $z = a$ and $z = \infty$ on $\widehat{\mathbb{C}}_2$, and let the upper part be l_2 and the lower part be \tilde{l}_2 . Put in a slit in the line segment connecting $z = 0$ and $z = -\frac{1}{a}$ on $\widehat{\mathbb{C}}_1$, and let the upper part be h_1 and the lower part be \tilde{h}_1 . Put in a slit in the line segment connecting $z = 0$ and $z = -\frac{1}{a}$ on $\widehat{\mathbb{C}}_2$, and let the upper part be h_2 and the lower part be \tilde{h}_2 . If \tilde{l}_1 is attached to l_2 , h_1 is attached to \tilde{h}_2 , \tilde{h}_1 is attached to h_2 and l_1 is attached to \tilde{l}_2 so that w_1 and w_2 are continuously connected, a Riemann surface M'_a homeomorphic to the torus can be obtained. Let $z_1 \in \widehat{\mathbb{C}}_1, z_2 \in \widehat{\mathbb{C}}_2$, and define the map $\phi : M'_a \rightarrow M_a$ by $z_1 \mapsto (z_1, w_1(z_1)), z_2 \mapsto (z_2, w_2(z_2))$. Let $z_1 \in l_1 \subset \widehat{\mathbb{C}}_1$ and $z_2 \in \tilde{l}_2 \subset \widehat{\mathbb{C}}_2$, and suppose $z_1 = z_2$ in M'_a . Then $\phi(z_1) = \phi(z_2)$ since $w_1(z_1) = w_2(z_2)$. The same holds at the other slits. Therefore, the map ϕ is well-defined. It is confirmed that ϕ is bijective. Identify M_a with M'_a by this bijection, and consider M_a as a Riemann surface.

3.2 Vector spaces $H(w)$ and $\widehat{H}(w)$

In this subsection, we describe the vector spaces, reviewed in the subsection 2.2, when the meromorphic function is $w : M_a \rightarrow \widehat{\mathbb{C}}$.

$z : M_a \ni (z, w) \mapsto z \in \widehat{\mathbb{C}}$ is a meromorphic function of degree 2 on M_a , has (∞, ∞) as a pole of order 2, and has $(0, 0)$ as a zero of order 2. dz is a meromorphic differential, has (∞, ∞) as a pole of order 3, and has $(0, 0), (a, 0), (-\frac{1}{a}, 0)$ as zeros of order 1, respectively.

$w : M_a \ni (z, w) \mapsto w \in \widehat{\mathbb{C}}$ is a meromorphic function of degree 3 on M_a , has (∞, ∞) as a pole of order 3, and has $(0, 0), (a, 0), (-\frac{1}{a}, 0)$ as zeros of order 1, respectively. dw is a meromorphic differential, has (∞, ∞) as a pole of order 4, and has $(A_1, \pm B_1), (A_2, \pm B_2)$ as zeros of order 1, respectively. Here, $A_1 = \frac{a - \frac{1}{a} + \sqrt{a^2 + \frac{1}{a^2} + 1}}{3}$, $A_2 = \frac{a - \frac{1}{a} - \sqrt{a^2 + \frac{1}{a^2} + 1}}{3}$, B_1 is the value $w_1(A_1)$ of w_1 at $z = A_1 \in \widehat{\mathbb{C}}_1$, and B_2 is the value $w_1(A_2)$ of w_1 at $z = A_2 \in \widehat{\mathbb{C}}_1$.

If $\frac{dz}{w}$ becomes zero, then dz becomes zero or w becomes pole, and $\frac{dz}{w}$ becomes pole, then dz becomes pole or w becomes zero. The possible points are the only four points $(0, 0), (a, 0), (-\frac{1}{a}, 0), (\infty, \infty)$. For example, considering at $(0, 0)$, since dz and w have $(0, 0)$ as a zero of order 1, these can be written as

$$dz = a_1 \zeta d\zeta + \cdots, \quad w = b_1 \zeta + b_2 \zeta^2 + \cdots, \quad a_1 \neq 0, \quad b_1 \neq 0$$

near $(0, 0)$. Therefore, $\frac{dz}{w} \sim \frac{a_1}{b_1} d\zeta$, and $\frac{dz}{w}$ has neither zero nor pole at $(0, 0)$ because $\frac{a_1}{b_1} \neq \infty, \frac{a_1}{b_1} \neq 0$. Similarly, it can be confirmed that $\frac{dz}{w}$ has neither zero nor pole at $(a, 0), (-\frac{1}{a}, 0), (\infty, \infty)$. Therefore, $\frac{dz}{w}$ has neither zero nor pole everywhere.

Using the nowhere vanishing holomorphic differential $\frac{dz}{w}$, $H(w)$ can also be written as follows.

$$H(w) = \left\{ f \frac{dz}{w} \mid f : M_a \rightarrow \widehat{\mathbb{C}} \text{ is a meromorphic function,} \right. \\ \left. D(f) + \widetilde{B}(w) \geq 0, \operatorname{Res}_{p_i} \left(f \frac{dz}{w} \right) = 0, i = 1, \dots, \mu, \right. \\ \left. \operatorname{Re} \int_{\alpha} ((1 - w^2), i(1 + w^2), 2w) \left(f \frac{dz}{w} \right) = 0, \forall \alpha \text{ closed curve} \right\},$$

where $D(f)$ is the divisor of f defined as follows.

Definition 3.1. Let f be a meromorphic function from a compact Riemann surface M to $\widehat{\mathbb{C}}$, p_1, \dots, p_m be the poles of order a_1, \dots, a_m of f , and q_1, \dots, q_n be the zeros of order b_1, \dots, b_m of f . Then

$$D(f) = - \sum_{i=1}^m a_i p_i + \sum_{j=1}^n b_j q_j$$

is called the *divisor* of f .

Similarly, $\widehat{H}(w)$ can also be written as follows.

$$\widehat{H}(w) = \left\{ f \frac{dz}{w} \mid f : M_a \rightarrow \widehat{\mathbb{C}} \text{ is a meromorphic function,} \right. \\ \left. D(f) + \widetilde{B}(w) \geq 0, \operatorname{Res}_{p_i} \left(f \frac{dz}{w} \right) = 0, i = 1, \dots, \mu \right\}.$$

4 Computation of $\text{Nul}(tw)$

In this section we compute the dimension of $H(tw)$ and, as a consequence, compute the nullity of tw . First, we find a basis of $\widehat{H}(w)$.

Lemma 4.1. $\widehat{H}(w)$ is a three dimensional complex vector space spanned by

$$\begin{aligned}\eta_1 &= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z - A_1)^2(z - A_2)^2} \frac{dz}{w}, \\ \eta_2 &= \frac{z^2 - A_1A_2}{(z - A_1)^2(z - A_2)^2} \frac{dz}{w}, \\ \eta_3 &= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z - A_1)^2(z - A_2)^2} dz.\end{aligned}$$

Proof. First, we prove that η_1, η_2, η_3 are elements of $\widehat{H}(w)$. Note that

$$\widetilde{B}(w) = 2(A_1, \pm B_1) + 2(A_2, \pm B_2) - 3(\infty, \infty)$$

by

$$B(w) = (A_1, \pm B_1) + (A_2, \pm B_2) + 2(\infty, \infty), \quad P(w) = 3(\infty, \infty).$$

We compute the divisors of

$$\begin{aligned}f_1 &= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z - A_1)^2(z - A_2)^2}, \\ f_2 &= \frac{z^2 - A_1A_2}{(z - A_1)^2(z - A_2)^2}, \\ f_3 &= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z - A_1)^2(z - A_2)^2} w.\end{aligned}$$

They are given by

$$\begin{aligned}D(f_1) &= -2(A_1, \pm B_1) - 2(A_2, \pm B_2) + 6(\infty, \infty) + \left(\frac{1}{2}(A_1 + A_2), \pm B_3\right), \\ D(f_2) &= -2(A_1, \pm B_1) - 2(A_2, \pm B_2) + 4(\infty, \infty) + (\sqrt{A_1A_2}, \pm B_4) + (-\sqrt{A_1A_2}, \pm B_4), \\ D(f_3) &= -2(A_1, \pm B_1) - 2(A_2, \pm B_2) + 3(\infty, \infty) + (0, 0) + \left(\frac{1}{2}(A_1 + A_2), \pm B_3\right) + (a, 0) \\ &\quad + \left(\frac{1}{a}, 0\right),\end{aligned}$$

where B_3 is the value of w_1 at $z = \frac{1}{2}(A_1 + A_2) \in \widehat{\mathbb{C}}_1$ and B_4 is the value of w_1 at $z = -\sqrt{A_1A_2} \in \widehat{\mathbb{C}}_1$. Therefore,

$$D(f_i) + \widetilde{B}(w) \geq 0, \quad i = 1, 2, 3.$$

We compute the residues of η_i , $i = 1, 2, 3$. Let $\zeta = z - A_1$ near $(A_1, \pm B_1)$. w can be written as

$$w = \pm B_1 + b_2 \zeta^2 + \cdots$$

near $\zeta = 0$. If we compute by using them, we obtain

$$\begin{aligned}\eta_1 &\sim \frac{1}{\pm 2B_1(A_1 - A_2)\zeta^2} d\zeta, \\ \eta_2 &\sim \frac{A_1(A_1 + A_2)}{\pm B_1(A_1 - A_2)^2 \zeta^2} d\zeta, \\ \eta_3 &\sim \frac{1}{2(A_1 - A_2)\zeta^2} d\zeta.\end{aligned}$$

Therefore,

$$\text{Res}(\eta_i, (A_1, \pm B_1)) = 0, \quad i = 1, 2, 3.$$

Similarly, we obtain

$$\text{Res}(\eta_i, (A_2, \pm B_2)) = 0, \quad i = 1, 2, 3.$$

Next, we prove that η_1, η_2, η_3 are linearly independent. For $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$ we write

$$\gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3 = 0. \quad (4.1)$$

If we substitute $z = \frac{1}{2}(A_1 + A_2)$ into (4.1), we obtain $\gamma_2 = 0$. Therefore, (4.1) becomes

$$\gamma_1 \eta_1 + \gamma_3 \eta_3 = 0. \quad (4.2)$$

(4.2) can be written as

$$\gamma_1(c_0 + c_1 \zeta + \cdots) + \gamma_3(d_1 \zeta + \cdots) = 0, \quad c_0 \neq 0, d_1 \neq 0, \text{ near } (0, 0).$$

$\gamma_1 c_0 = 0$ when $\zeta \rightarrow 0$. $\gamma_1 = 0$ by $c_0 \neq 0$. Therefore, $\gamma_3 = 0$. Thus, η_1, η_2, η_3 are linearly independent.

Next, we prove that the complex dimension of $\widehat{H}(w)$ is 3. If, $\dim_{\mathbb{C}} \widehat{H}(w) > 3$, then $\dim_{\mathbb{R}} \widehat{H}(w) \geq 8$. Since there are only 6 simultaneous equations from the period condition, the dimension of $H(w)$ is 2 or more. By Corollary 2.16, $\text{Nul}(tw)$ is 5 or more for all t . On the other hand, by a result of Nayatani [10, p518, THEOREM 2], to be reviewed below as Theorem 4.2, the nullity of tw is 3 when t is sufficiently small, which is a contradiction. Therefore, $\dim_{\mathbb{C}} \widehat{H}(w) = 3$. \square

Theorem 4.2 (Nayatani [10]). Let $g: M \rightarrow \widehat{\mathbb{C}}$ be a nonconstant meromorphic function of degree d . Let ν be the number of distinct poles of g . Then the following estimates hold for all sufficiently small t :

$$\begin{aligned}\text{Ind}(tg) &\geq 2d - \nu, \\ \text{Ind}(tg) + \text{Nul}(tg) &\leq 2d + \nu + 1, \\ \text{Nul}(tg) &\leq 2\nu + 1.\end{aligned}$$

In particular, if $\nu = 1$, then we have

$$\text{Ind}(tg) = 2d - 1 \quad \text{and} \quad \text{Nul}(tg) = 3$$

for all sufficiently small t .

In the proof of the next lemma, we use the perfect elliptic integrals. So we recall the definition of these integrals.

Definition 4.3.

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

is called the *perfect elliptic integral of the first kind* and

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta$$

is called the *perfect elliptic integral of the second kind*.

Lemma 4.4. For each a in the range $1 \leq a \leq a_0$ (where a_0 can be numerically evaluated) there are positive real numbers $t_1(a), t_2(a)$ ($t_1(a) < t_2(a)$) such that

$$\dim_{\mathbb{R}} H(tw) = \begin{cases} 1, & t = t_1(a), t_2(a), \\ 0, & t > 0, t \neq t_1(a), t_2(a). \end{cases}$$

Therefore,

$$\text{Nul}(tw) = \begin{cases} 4, & t = t_1(a), t_2(a), \\ 3, & t \neq t_1(a), t_2(a). \end{cases}$$

Proof. Let α_1, α_2 be

$$\begin{aligned} \alpha_1 &= \left\{ \left(a - \frac{1}{4a} \right) + \left(a + \frac{1}{4a} \right) e^{i\theta} \in \widehat{\mathbb{C}}_1 \mid 0 \leq \theta \leq \pi \right\} \\ &\quad \cup \left\{ \left(a - \frac{1}{4a} \right) + \left(a + \frac{1}{4a} \right) e^{i\theta} \in \widehat{\mathbb{C}}_2 \mid \pi \leq \theta \leq 2\pi \right\}, \\ \alpha_2 &= \left\{ \left(-\frac{1}{a} + \frac{a}{4} \right) + \left(\frac{1}{a} + \frac{a}{4} \right) e^{i\theta} \in \widehat{\mathbb{C}}_1 \mid 0 \leq \theta \leq 2\pi \right\}. \end{aligned}$$

We take $\eta \in \widehat{H}(w)$ and express it as

$$\eta = \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3, \quad \gamma_j = x_j + iy_j, \quad x_j, y_j \in \mathbb{R}, \quad j = 1, 2, 3.$$

We will find the simultaneous equation that $\gamma_1, \gamma_2, \gamma_3$ must satisfy so that the condition

$$\operatorname{Re} \int_{\alpha} (1 - (tw)^2, i(1 + (tw)^2), 2tw)\eta = 0 \text{ for } \forall \alpha \text{ closed curve in } M_a$$

holds for this η .

To do this, we first compute the integrals of $\eta_i, w\eta_i, w^2\eta_i$, $i = 1, 2, 3$, on α_1, α_2 . When we actually compute, we change α_1, α_2 and we compute along the closed intervals $[0, a], [-\frac{1}{a}, 0]$ on the real axis, respectively. Since the denominator of η_i , $i = 1, 2, 3$, has $z - A_1, z - A_2$ and these integrals diverge at $z = A_1, A_2$, we subtract from η_i the differentials of meromorphic functions f_i with poles at most order one at $(z, w) = (A_1, B_1), (A_2, B_2)$ on the Riemann surface M_a so that the integrals of the meromorphic differentials $\eta_i - df_i$ converge.

When we consider η_1 , we obtain

$$\begin{aligned} \eta_1 &= \frac{1}{2B_1^2} d \frac{w}{(z - A_1)(z - A_2)} \\ &= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z - A_1)^2(z - A_2)^2} \frac{dz}{w} - \frac{1}{2B_1^2} \left(\frac{dw}{(z - A_1)(z - A_2)} - \frac{(z - A_1)w + (z - A_2)w}{(z - A_1)^2(z - A_2)^2} dz \right) \\ &= -\frac{3}{4B_1^2} \frac{dz}{w} + \frac{1}{B_1^2} \frac{(z - \frac{1}{2}(A_1 + A_2))(w^2 + B_1^2)}{(z - A_1)^2(z - A_2)^2} \frac{dz}{w} \\ &= -\frac{3}{4B_1^2} \frac{dz}{w} + \frac{1}{B_1^2} \frac{(z - \frac{1}{2}(A_1 + A_2))(z + 2A_1 - (a - \frac{1}{a}))}{(z - A_2)^2 w} dz \\ &= -\frac{3}{4B_1^2} \frac{dz}{w} + \frac{1}{B_1^2} \frac{dz}{w} + \frac{(A_1 - A_2)^2}{4B_1^2(z - A_2)^2 w} dz \\ &= \frac{1}{4B_1^2} \frac{dz}{w} - \frac{(A_1 - A_2)^2}{4B_1^2(z - A_2)^2 w} dz. \end{aligned}$$

Therefore, by $\int_{\alpha_1} d \frac{w}{(z - A_1)(z - A_2)} = 0$,

$$\begin{aligned} &\int_{\alpha_1} \eta_1 \\ &= \frac{1}{4B_1^2} \int_{\alpha_1} \frac{dz}{w} - \frac{(A_1 - A_2)^2}{4B_1^2} \int_{\alpha_1} \frac{1}{(z - A_2)^2 w} dz \\ &= \frac{1}{4B_1^2} \left(\int_a^0 \frac{dt}{i\sqrt{t(a-t)(\frac{1}{a} + t)}} + \int_0^a \frac{dt}{-i\sqrt{t(a-t)(\frac{1}{a} + t)}} \right) \\ &\quad - \frac{(A_1 - A_2)^2}{4B_1^2} \left(\int_a^0 \frac{dt}{(t - A_2)^2 i\sqrt{t(a-t)(\frac{1}{a} + t)}} - \int_0^a \frac{dt}{(t - A_2)^2 i\sqrt{t(a-t)(\frac{1}{a} + t)}} \right) \\ &= \frac{i}{2B_1^2} \int_0^a \frac{dt}{\sqrt{t(a-t)(\frac{1}{a} + t)}} - \frac{i(A_1 - A_2)^2}{2B_1^2} \int_0^a \frac{dt}{(t - A_2)^2 \sqrt{t(a-t)(\frac{1}{a} + t)}}. \quad (4.3) \end{aligned}$$

We compute the rightmost side of (4.3). By the definition of the perfect elliptic integral,

$$E(ia) = \int_0^{\frac{\pi}{2}} \sqrt{1 + a^2 \sin^2 \theta} d\theta, \quad (4.4)$$

$$K(ia) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + a^2 \sin^2 \theta}}. \quad (4.5)$$

Let $a \sin^2 \theta = t$. Then $d\theta = \frac{dt}{2\sqrt{t(a-t)}}$, and (4.4), (4.5) becomes

$$E(ia) = \int_0^a \frac{\sqrt{1 + at}}{2\sqrt{t(a-t)}} dt, \quad (4.6)$$

$$K(ia) = \int_0^a \frac{dt}{2\sqrt{t(a-t)(1+at)}}. \quad (4.7)$$

Also,

$$E(ia) - K(ia) = \int_0^a \frac{at}{2\sqrt{t(a-t)(1+at)}} dt \quad (4.8)$$

holds. By (4.7), the first definite integral of the rightmost side of (4.3) is

$$\int_0^a \frac{dt}{\sqrt{t(a-t)\left(\frac{1}{a} + t\right)}} = 2\sqrt{a}K(ia). \quad (4.9)$$

Next, we compute the second definite integral of the rightmost side of (4.3). If we set

$$\begin{aligned} \varphi(t) &= t^3 + \left(\frac{1}{a} - a\right)t^2 - t, \\ I[m] &= \int_0^a \frac{t^m dt}{\sqrt{\varphi(t)}}, \\ J[m] &= \int_0^a \frac{dt}{(t - A_2)^m \sqrt{\varphi(t)}}, \end{aligned}$$

the recurrence formula

$$\begin{aligned} &2m\varphi(A_2)J[m+1] + (2m-1)\varphi'(A_2)J[m] + (m-1)\varphi''(A_2)J[m-1] \\ &+ \frac{2m-3}{6}\varphi'''(A_2)J[m-2] + \frac{m-2}{12}\varphi''''(A_2)J[m-3] = 0 \end{aligned}$$

holds (see [8]). Since

$$\varphi(A_2) = B_2^2, \quad \varphi'(A_2) = 0, \quad \varphi'''(A_2) = 6, \quad \varphi''''(A_2) = 0,$$

this recurrence formula becomes

$$2mB_2^2J[m+1] + (2m-3)J[m-2] = 0. \quad (4.10)$$

Substituting $m = 1$ into (4.10), we obtain the following equation

$$2B_2^2J[2] = J[-1]. \quad (4.11)$$

On the other hand,

$$J[-1] = I[1] - A_2I[0]. \quad (4.12)$$

By (4.6) and (4.8), we obtain

$$I[0] = \int_0^a \frac{dt}{\sqrt{t(a-t)\left(\frac{1}{a}+t\right)}} = 2\sqrt{a}K(ia), \quad (4.13)$$

$$I[1] = \int_0^a \frac{t}{\sqrt{t(a-t)\left(\frac{1}{a}+t\right)}} dt = \frac{2(E(ia) - K(ia))}{\sqrt{a}}. \quad (4.14)$$

Substituting (4.13) and (4.14) into (4.12), we obtain the following equation

$$J[-1] = 2\frac{E(ia) - K(ia) - aA_2K(ia)}{\sqrt{a}}. \quad (4.15)$$

Since the second definite integral of the rightmost side of (4.3) is $J[2]$,

$$\int_0^a \frac{dt}{(t-A_2)^2\sqrt{t(a-t)\left(\frac{1}{a}+t\right)}} = \frac{E(ia) - K(ia) - aA_2K(ia)}{\sqrt{a}B_2^2}$$

by (4.11) and (4.15). Using this, the integral of η_1 on α_1 is obtained as

$$\int_0^a \eta_1 = \frac{\sqrt{a}}{B_1^2}iK(ia) - \frac{(A_1 - A_2)^2}{2B_1^2} \frac{\frac{1}{\sqrt{a}}(E(ia) - K(ia)) - \sqrt{a}A_2K(ia)}{B_2^2} i.$$

Other integrals can be computed similarly. The results are as follows.

$$\begin{aligned} \int_{\alpha_1} \eta_1 &= \frac{\sqrt{a}}{B_1^2}iK(ia) - \frac{(A_1 - A_2)^2}{2B_1^2}iI_2(a), & \int_{\alpha_1} w\eta_1 &= 0, & \int_{\alpha_1} w^2\eta_1 &= 3\sqrt{a}iK(ia), \\ \int_{\alpha_1} \eta_2 &= \frac{2A_1\sqrt{a}}{B_1^2}iK(ia) + \frac{A_1 + A_2}{3B_1^2}iI_2(a), & \int_{\alpha_1} w\eta_2 &= 0, & \int_{\alpha_1} w^2\eta_2 &= \frac{6}{\sqrt{a}}iI_1(a), \\ \int_{\alpha_1} \eta_3 &= 0, & \int_{\alpha_1} w\eta_3 &= 3\sqrt{a}iK(ia), & \int_{\alpha_1} w^2\eta_3 &= 0, \end{aligned}$$

$$\begin{aligned}
\int_{\alpha_2} \eta_1 &= -\frac{1}{\sqrt{a}B_2^2}K\left(\frac{i}{a}\right) + \frac{(A_1 - A_2)^2}{2B_2^2}J_2(a), & \int_{\alpha_2} w\eta_1 &= 0, & \int_{\alpha_2} w^2\eta_1 &= \frac{3}{\sqrt{a}}K\left(\frac{i}{a}\right), \\
\int_{\alpha_2} \eta_2 &= -\frac{2A_2}{\sqrt{a}B_2^2}K\left(\frac{i}{a}\right) - \frac{A_1 + A_2}{3B_2^2}J_2(a), & \int_{\alpha_2} w\eta_2 &= 0, & \int_{\alpha_2} w^2\eta_2 &= -6\sqrt{a}J_1(a), \\
\int_{\alpha_2} \eta_3 &= 0, & \int_{\alpha_2} w\eta_3 &= \frac{3}{\sqrt{a}}K\left(\frac{i}{a}\right), & \int_{\alpha_2} w^2\eta_3 &= 0.
\end{aligned}$$

Here,

$$\begin{aligned}
I_1(a) &= E(ia) - K(ia), & I_2(a) &= \frac{1}{B_2^2} \left(\frac{1}{\sqrt{a}}I_1(a) - \sqrt{a}A_2K(ia) \right), \\
J_1(a) &= E\left(\frac{i}{a}\right) - K\left(\frac{i}{a}\right), & J_2(a) &= \frac{1}{B_1^2} \left(\sqrt{a}J_1(a) + \frac{1}{\sqrt{a}}A_1K\left(\frac{i}{a}\right) \right).
\end{aligned}$$

Now we return to the period condition

$$\operatorname{Re} \int_{\alpha} (1 - (tw)^2, i(1 + (tw)^2), 2tw)\eta = 0$$

for $\eta = \gamma_1\eta_1 + \gamma_2\eta_2 + \gamma_3\eta_3 \in \widehat{H}(w)$ and closed curves $\alpha = \alpha_1, \alpha_2$ on M_a . First we obtain

$$\begin{aligned}
\operatorname{Re} \int_{\alpha_1} w\eta &= 0 \iff \operatorname{Im}(\gamma_3) = 0, \\
\operatorname{Re} \int_{\alpha_2} w\eta &= 0 \iff \operatorname{Re}(\gamma_3) = 0.
\end{aligned}$$

Therefore, $\gamma_3 = 0$.

Next, we have

$$\begin{aligned}
\operatorname{Re} \int_{\alpha} (1 - w^2)\eta &= \operatorname{Re} \int_{\alpha} i(1 + w^2)\eta = 0 \\
\iff \int_{\alpha} \eta &= \overline{\int_{\alpha} w^2\eta}. \tag{4.16}
\end{aligned}$$

When we compute the left and right sides of (4.16) on α_1, α_2 , we obtain

$$\begin{aligned}
& \int_{\alpha_1} \eta \\
&= \gamma_1 \left(\frac{\sqrt{a}}{B_1^2}iK(ia) - \frac{(A_1 - A_2)^2}{2B_1^2}iI_2(a) \right) + \gamma_2 \left(\frac{2A_1\sqrt{a}}{B_1^2}iK(ia) + \frac{A_1 + A_2}{3B_1^2}iI_2(a) \right), \\
& \int_{\alpha_1} w^2\eta \\
&= -3t^2\overline{\gamma_1}\sqrt{a}iK(ia) - \frac{6}{\sqrt{a}}t^2\overline{\gamma_2}iI_1(a),
\end{aligned}$$

$$\begin{aligned}
& \int_{\alpha_2} \eta \\
&= \gamma_1 \left(\frac{-1}{\sqrt{a}B_2^2} K\left(\frac{i}{a}\right) + \frac{(A_1 - A_2)^2}{2B_2^2} J_2(a) \right) + \gamma_2 \left(\frac{-2A_2}{\sqrt{a}B_2^2} K\left(\frac{i}{a}\right) - \frac{A_1 + A_2}{3B_2^2} J_2(a) \right), \\
& \int_{\alpha_2} w^2 \eta \\
&= 2t^2 \overline{\gamma_1} \frac{3}{2\sqrt{a}} K\left(\frac{i}{a}\right) - 6\sqrt{at^2} \overline{\gamma_2} J_1(a).
\end{aligned}$$

Therefore, (4.16) with $\alpha = \alpha_1, \alpha_2$ becomes

$$\left\{ \begin{aligned}
& \gamma_1 \left(\frac{\sqrt{a}}{B_1^2} K(ia) - \frac{(A_1 - A_2)^2}{2B_1^2} I_2(a) \right) + \gamma_2 \left(\frac{2A_1 \sqrt{a}}{B_1^2} K(ia) + \frac{A_1 + A_2}{3B_1^2} I_2(a) \right) \\
& \qquad \qquad \qquad = -3t^2 \overline{\gamma_1} \sqrt{a} K(ia) - \frac{6t^2}{\sqrt{a}} \overline{\gamma_2} I_1(a), \\
& \gamma_1 \left(-\frac{1}{\sqrt{a}B_2^2} K\left(\frac{i}{a}\right) + \frac{2(A_1 - A_2)^2}{4B_2^2} J_2(a) \right) + \gamma_2 \left(-\frac{2A_2}{\sqrt{a}B_2^2} K\left(\frac{i}{a}\right) - \frac{A_1 + A_2}{3B_2^2} J_2(a) \right) \\
& \qquad \qquad \qquad = t^2 \overline{\gamma_1} \frac{3}{\sqrt{a}} K\left(\frac{i}{a}\right) - 6\sqrt{at^2} \overline{\gamma_2} J_1(a).
\end{aligned} \right. \quad (4.17)$$

Let

$$U = \begin{pmatrix} \frac{1}{B_1^2} K(ia) - \frac{(A_1 - A_2)^2}{2\sqrt{a}B_1^2} I_2(a) + 3t^2 K(ia) & \frac{2A_1}{B_1^2} K(ia) + \frac{A_1 + A_2}{3B_1^2 \sqrt{a}} I_2(a) + \frac{6t^2}{a} I_1(a) \\ \frac{1}{B_2^2} K\left(\frac{i}{a}\right) - \frac{\sqrt{a}(A_1 - A_2)^2}{2B_2^2} J_2(a) + 3t^2 K\left(\frac{i}{a}\right) & \frac{2A_2}{B_2^2} K\left(\frac{i}{a}\right) + \frac{A_1 + A_2}{3B_2^2} \sqrt{a} J_2(a) - 6at^2 J_1(a) \end{pmatrix}$$

and

$$V = \begin{pmatrix} \frac{1}{B_1^2} K(ia) - \frac{(A_1 - A_2)^2}{2\sqrt{a}B_1^2} I_2(a) - 3t^2 K(ia) & \frac{2A_1 K(ia)}{B_1^2} + \frac{A_1 + A_2}{3\sqrt{a}B_1^2} I_2(a) - \frac{6t^2}{a} I_1(a) \\ \frac{1}{B_2^2} K\left(\frac{i}{a}\right) - \frac{\sqrt{a}(A_1 - A_2)^2}{2B_2^2} J_2(a) - 3t^2 K\left(\frac{i}{a}\right) & \frac{2A_2 K\left(\frac{i}{a}\right)}{B_2^2} + \frac{A_1 - A_2}{3\sqrt{a}B_2^2} J_2(a) + 6at^2 J_1(a) \end{pmatrix}.$$

If we use these matrices, then (4.17) becomes

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = 0, \quad (4.18)$$

where $\gamma_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2$. (4.18) has a nontrivial solution if and only if $\det U = 0$ or $\det V = 0$. The conditional equation $\det U = 0$ is a quadratic equation of $x = t^2$. This has only one positive real solution for each a . If we denote the positive square root of this positive real solution by $t_1(a)$, then this is the only positive real t that satisfies $\det U = 0$. The conditional equation $\det V = 0$ is also a quadratic

equation of $x = t^2$. This has only one positive real solution for each a . If we denote the positive square root of this positive real solution by $t_2(a)$, then this is the only positive real t that satisfies $\det V = 0$. If we draw the graphs of $t_1(a), t_2(a)$ in the range $1 \leq a \leq a_0$ (where a_0 can be numerically evaluated) by Mathematica, we see that $t_1(a) < t_2(a)$. When $t = t_1(a)$, $\det U = 0$, but $\det V \neq 0$. Then there is only one nontrivial solution of (4.18) up to real multiple, and this nontrivial solution is written as $(x_1, x_2, y_1, y_2) = (a_1, a_2, 0, 0)$. When $t = t_2(a)$, $\det V = 0$, but $\det U \neq 0$. Then there is only one nontrivial solution of (4.18) up to real multiple, and this nontrivial solution is written as $(x_1, x_2, y_1, y_2) = (0, 0, b_1, b_2)$. Therefore, the basis of $H(t_1(a)w)$ is $\eta = a_1\eta_1 + a_2\eta_2$ when $t = t_1(a)$ and the basis of $H(t_2(a)w)$ is $\eta = ib_1\eta_1 + ib_2\eta_2$ when $t = t_2(a)$. Thus $\dim_{\mathbb{R}} H(tw) = 1$ when $t = t_1(a), t_2(a)$, and $\dim_{\mathbb{R}} H(tw) = 0$ when $t \neq t_1(a), t_2(a)$. Therefore, $\text{Nul}(tw) = 4$ when $t = t_1(a), t_2(a)$, and $\text{Nul}(tw) = 3$ when $t \neq t_1(a), t_2(a)$. \square

Remark 4. By using Mathematica, we can check that $t_1(a) < t_2(a)$ for small values of a . In fact, as the graphs of Figure 1 suggest, the constant a_0 in the statement of Lemma 4.4 is surely larger than 5. On the other hand, the graphs of Figure 2 suggest that the values of $t_1(a)$ and $t_2(a)$ become close to each other rather quickly as the parameter a becomes bigger.

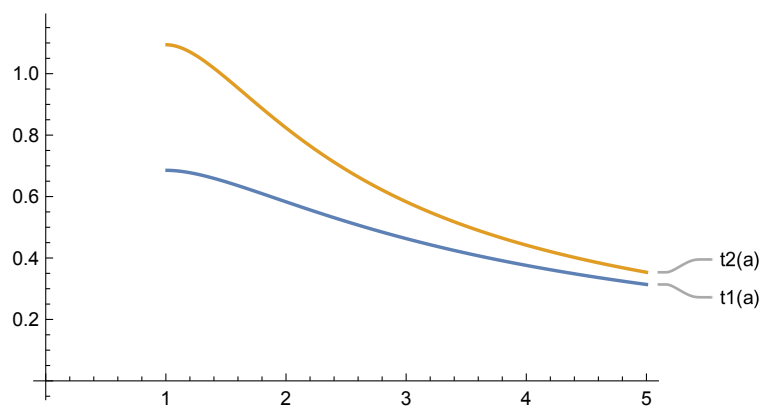


Figure 1: Graphs of t_1, t_2 for $a \leq 5$.

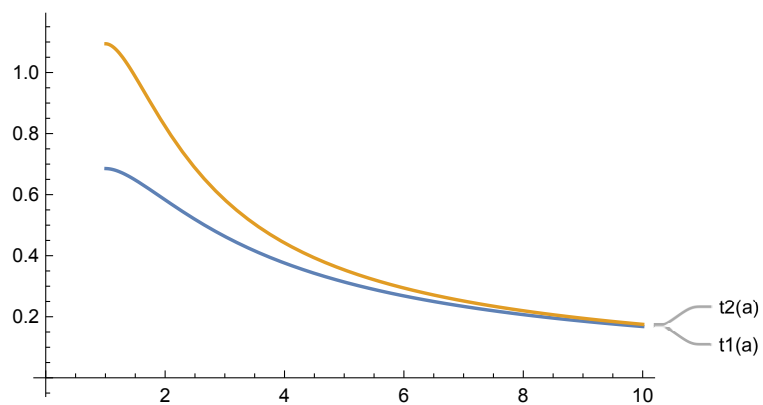


Figure 2: Graphs of t_1, t_2 for $a \leq 10$.

5 Computation of $\text{Ind}(tw)$

In this section we compute the index of tg_a for all t and a in the range $1 \leq a \leq a_0$, where a_0 is as in Section 4.

As mentioned in Introduction, Nayatani [10] computed the index and nullity of $t\wp'$, where \wp is the Weierstrass \wp -function corresponding to the square lattice $\mathbb{Z} \oplus i\mathbb{Z}$. The Riemann surface $\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$ is isomorphic to

$$M_1 = \left\{ (z, w) \in \widehat{\mathbb{C}}^2 \mid w^2 = z(z^2 - 1) \right\}$$

and \wp' coincides with $w: M_1 \rightarrow \widehat{\mathbb{C}}$ up to a multiplicative positive real constant. Since we use Nayatani's result in the proof of Theorem 5.2, we state his result in our setting.

Theorem 5.1 (Nayatani[10]). For the meromorphic function $w: M_1 \ni (z, w) \mapsto w \in \widehat{\mathbb{C}}$, we have

$$\text{Ind}(tw) = \begin{cases} 5, & 0 < t \leq t_1(1), t_2(1) \leq t, \\ 6, & t_1(1) < t < t_2(1), \end{cases}$$

and

$$\text{Nul}(tw) = \begin{cases} 4, & t = t_1(1), t_2(1), \\ 3, & t \neq t_1(1), t_2(1). \end{cases}$$

Theorem 5.2. If $t_1(a)$ and $t_2(a)$ are as in Lemma 4.4, then

$$\text{Ind}(tw) = \begin{cases} 5, & 0 < t \leq t_1(a), t_2(a) \leq t, \\ 6, & t_1(a) < t < t_2(a) \end{cases}$$

for any a in the range $1 \leq a \leq a_0$ (where a_0 can be numerically evaluated).

Proof. Let $g = tw$. We consider at $t = t_1(a)$. What we already know is $\text{Nul}(g) = 4, \text{Ind}(g) = 5$ when $t = t_1(1)$. That is, there are exactly 5 eigenvalues smaller than 2 of $-\Delta_g$. If a moves in the range $1 \leq a \leq a_0$, then $\text{Nul}(g) = 4$ forever. We arrange the eigenvalues of $-\Delta_g$ from the smallest, and we write the i -th eigenvalue as $\lambda_i(a)$, $i = 1, 2, \dots$. When a moves from 1 to a_0 , $\lambda_i(a)$ changes continuously, so if $\text{Ind}(g)$ changes, then $\text{Nul}(g)$ also changes. Therefore $\text{Ind}(g) = 5$ does not change. When $t = t_2(a)$, $\text{Ind}(g)$ can also be determined similarly.

Now, take one a in the range from 1 to a_0 , and we write it as $a = \tilde{a}_0$. We move t . When $t = t_1(\tilde{a}_0), t_2(\tilde{a}_0)$, then $\text{Nul}(g) = 4, \text{Ind}(g) = 5$. We consider $t = \frac{1}{2}t_1(\tilde{a}_0)$ in the

range from 0 to $t_1(\tilde{a}_0)$. When $t = \frac{1}{2}t_1(1)$, then $\text{Nul}(g) = 3, \text{Ind}(g) = 5$. If a moves from 1 to a_0 , $\text{Ind}(g) = 5$ also does not change since $\text{Nul}(g) = 3$ does not change in the range from 0 to $t_1(a)$. Therefore, when $t = \frac{1}{2}t_1(\tilde{a}_0)$, then $\text{Nul}(g) = 3, \text{Ind}(g) = 5$. $\text{Ind}(g) = 5$ also does not change since $\text{Nul}(g) = 3$ does not change in the range from 0 to $t_1(\tilde{a}_0)$. Therefore, when $0 < t < t_1(\tilde{a}_0)$, then $\text{Ind}(g) = 5$. We consider $t = \frac{1}{2}(t_1(\tilde{a}_0) + t_2(\tilde{a}_0))$ in the range from $t_1(\tilde{a}_0)$ to $t_2(\tilde{a}_0)$. When $t = \frac{1}{2}(t_1(1) + t_2(1))$, then $\text{Nul}(g) = 3, \text{Ind}(g) = 6$. If a moves from 1 to a_0 , $\text{Ind}(g) = 6$ also does not change since $\text{Nul}(g) = 3$ does not change in the range from $t_1(a)$ to $t_2(a)$. Therefore, When $t = \frac{1}{2}(t_1(\tilde{a}_0) + t_2(\tilde{a}_0))$, then $\text{Nul}(g) = 3, \text{Ind}(g) = 6$. $\text{Ind}(g) = 6$ also does not change since $\text{Nul}(g) = 3$ does not change in the range from $t_1(\tilde{a}_0)$ to $t_2(\tilde{a}_0)$. Therefore, when $t_1(\tilde{a}_0) < t < t_2(\tilde{a}_0)$, then $\text{Ind}(g) = 6$. We consider $t = 2t_2(\tilde{a}_0)$ when $t < t_2(\tilde{a}_0)$. If $t = 2t_2(1)$, then $\text{Nul}(g) = 3, \text{Ind}(g) = 5$. If a moves from 1 to a_0 , $\text{Ind}(g) = 5$ also does not change since $\text{Nul}(g) = 3$ does not change when $t < t_2(a)$. Therefore, When $t = 2t_2(\tilde{a}_0)$, then $\text{Nul}(g) = 3, \text{Ind}(g) = 5$. When $t < t_2(\tilde{a}_0)$, $\text{Ind}(g) = 5$ also does not change since $\text{Nul}(g) = 3$ does not change. Therefore, when $t < t_2(\tilde{a}_0)$, then $\text{Ind}(g) = 5$. \square

We close this section with concluding remarks.

Remark 5. Since $H(tw) \neq \{0\}$ for $t = t_1(a), t_2(a)$, there exists a (possibly branched) complete orientable minimal surface in \mathbb{R}^3 whose extended Gauss map is tw and all of whose ends are planer for each of $t = t_1(a), t_2(a)$. In particular, the Morse indices of these minimal surfaces are both 5. If $t \neq t_1(a), t_2(a)$, tw is still the extended minimal surface of *some* complete orientable minimal surfaces in \mathbb{R}^3 , and Theorem 5.2 computes the Morse indices of these minimal surfaces.

In the case that the Riemann surface has genus zero, it is a remarkable result of Ejiri-Kotani [2] and Montiel-Ros [7] that the index of a generic meromorphic function of degree d has index $2d - 1$. On the other hand, in the higher-genus case, there are not so many complete orientable minimal surfaces nor meromorphic functions whose indices are computed. Theorem 5.2 should be of some interest as it provides new examples of meromorphic functions on compact Riemann surfaces of genus 1 whose indices are computable.

6 Appendix

We write computations omitted in Section 4 here.

We compute the integrals of $\eta_i, w\eta_i, w^2\eta_i$, $i = 1, 2, 3$, on α_1, α_2 . When we actually compute, we change α_1, α_2 and we compute along the closed intervals $[0, a], [-\frac{1}{a}, 0]$ on the real axis, respectively. Since the denominator of η_i , $i = 1, 2, 3$, has $z - A_1, z - A_2$ and these integrals diverge at $z = A_1, A_2$, we subtract from η_i differentials of meromorphic functions f_i with poles at most order one at $(z, w) = (A_1, B_1), (A_2, B_2)$ on the Riemann surface M_a so that the integrals of the meromorphic differentials $\eta_i - df_i$ converge.

We compute the integrals of $w\eta_1$ on α_1 . First we compute

$$\begin{aligned} & w\eta_1 + \frac{1}{2}d\frac{1}{(z-A_1)(z-A_2)} \\ &= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z-A_1)^2(z-A_2)^2}dz - \frac{z - \frac{1}{2}(A_1 + A_2)}{(z-A_1)^2(z-A_2)^2}dz \\ &= 0. \end{aligned}$$

Therefore, by $\int_{\alpha_1} \frac{1}{2}d\frac{1}{(z-A_1)(z-A_2)} = 0$,

$$\int_{\alpha_1} w\eta_1 = 0.$$

We compute the integrals of $w^2\eta_1$ on α_1 . First we compute

$$\begin{aligned} & w^2\eta_1 + \frac{1}{2}d\frac{w}{(z-A_1)(z-A_2)} \\ &= \frac{(z - \frac{1}{2}(A_1 + A_2))w}{(z-A_1)^2(z-A_2)^2}dz + \frac{dw}{2(z-A_1)(z-A_2)} - \frac{(z-A_1)w + (z-A_2)w}{2(z-A_1)^2(z-A_2)^2}dz \\ &= \frac{3dz}{4w}. \end{aligned}$$

Therefore, by $\int_{\alpha_1} d\frac{w}{(z-A_1)(z-A_2)} = 0$,

$$\int_{\alpha_1} w^2\eta_1 = \frac{3}{4} \int_{\alpha_1} \frac{dz}{w} = 3\sqrt{ai}K(ia).$$

We compute the integrals of η_2 on α_1 . First we compute

$$\begin{aligned}
& \eta_2 - \frac{A_1}{B_1^2} d \frac{w}{(z-A_1)(z-A_2)} \\
&= \frac{z^2 - A_1 A_2}{(z-A_1)^2(z-A_2)^2 w} dz - \frac{A_1}{B_1^2} d \frac{w}{(z-A_1)(z-A_2)} \\
&= \frac{z^2 - A_1 A_2}{(z-A_1)^2(z-A_2)^2 w} dz - \frac{A_1}{B_1^2} \left(\frac{dw}{(z-A_1)(z-A_2)} - \frac{(z-A_1)w + (z-A_2)w}{(z-A_1)^2(z-A_2)^2} dz \right) \\
&= -\frac{3A_1}{2B_1^2} \frac{dz}{w} + \frac{B_1^2(z^2 - A_1 A_2) + A_1(2z - (A_1 + A_2))w^2}{B_1^2(z-A_1)^2(z-A_2)^2 w} dz \\
&= -\frac{3A_1}{2B_1^2} \frac{dz}{w} + \frac{B_1^2(z^2 - A_1 A_2) + (z^2 - A_1 A_2)w^2 - (z^2 - 2zA_1 + A_1^2)w^2}{B_1^2(z-A_1)^2(z-A_2)^2 w} dz \\
&= -\frac{3A_1}{2B_1^2} \frac{dz}{w} + \frac{(z^2 - A_1 A_2)(w^2 + B_1^2)}{B_1^2(z-A_1)^2(z-A_2)^2 w} dz - \frac{w}{B_1^2(z-A_2)^2} dz \\
&= -\frac{3A_1}{2B_1^2} \frac{dz}{w} + \frac{(z^2 - A_1 A_2)(z + 2A_1 - (a - \frac{1}{a}))}{B_1^2(z-A_2)^2 w} dz - \frac{w}{B_1^2(z-A_2)^2} dz \\
&= -\frac{3A_1}{2B_1^2} \frac{dz}{w} + \frac{2A_1 z^2 + \frac{2}{3}(2z + A_1) - \frac{1}{3}(a - \frac{1}{a})}{B_1^2(z-A_2)^2 w} dz \\
&= \frac{A_1}{2B_1^2} \frac{dz}{w} + \frac{A_1 + A_2}{6B_1^2} \frac{dz}{(z-A_2)^2 w}.
\end{aligned}$$

Therefore, by $\int_{\alpha_1} d \frac{w}{(z-A_1)(z-A_2)} = 0$,

$$\begin{aligned}
\int_{\alpha_1} \eta_2 &= \frac{A_1}{2B_1^2} \int_{\alpha_1} \frac{dz}{w} + \frac{A_1 + A_2}{6B_1^2} \int_{\alpha_1} \frac{dz}{(z-A_2)^2 w} \\
&= \frac{2A_1 \sqrt{a}}{B_1^2} iK(ia) + \frac{A_1 + A_2}{3B_1^2} iI_2(a).
\end{aligned}$$

We compute the integrals of $w\eta_2$ on α_1 . First we compute

$$\begin{aligned}
& w\eta_2 + d \frac{z}{(z-A_1)(z-A_2)} \\
&= \frac{z^2 - A_1 A_2}{(z-A_1)^2(z-A_2)^2} dz + d \frac{z}{(z-A_1)(z-A_2)} \\
&= \frac{z^2 - A_1 A_2}{(z-A_1)^2(z-A_2)^2} dz + \frac{1}{(z-A_1)(z-A_2)} dz - \frac{z(z-A_1) + z(z-A_2)}{(z-A_1)^2(z-A_2)^2} dz \\
&= 0.
\end{aligned}$$

Therefore, by $\int_{\alpha_1} d \frac{z}{(z-A_1)(z-A_2)} = 0$,

$$\int_{\alpha_1} w\eta_2 = 0.$$

We compute the integrals of $w^2\eta_2$ on α_1 . First we compute

$$\begin{aligned}
& w^2\eta_2 + d\frac{zw}{(z-A_1)(z-A_2)} \\
&= \frac{w(z^2 - A_1A_2)}{(z-A_1)^2(z-A_2)^2}dz + d\frac{zw}{(z-A_1)(z-A_2)} \\
&= \frac{w(z^2 - A_1A_2)}{(z-A_1)^2(z-A_2)^2}dz + \frac{w}{(z-A_1)(z-A_2)}dz \\
&\quad + \frac{zdw}{(z-A_1)(z-A_2)} - \frac{(z-A_1)w + (z-A_2)w}{(z-A_1)^2(z-A_2)^2}dz \\
&= \frac{w(z^2 - A_1A_2)}{(z-A_1)^2(z-A_2)^2}dz + \frac{3z}{2w}dz \\
&\quad + \frac{w(z^2 - (A_1 + A_2)z + A_1A_2 - 2z^2 + (A_1 + A_2)z)}{(z-A_1)^2(z-A_2)^2}dz \\
&= \frac{3z}{2w}dz.
\end{aligned}$$

Therefore, by $\int_{\alpha_1} d\frac{zw}{(z-A_1)(z-A_2)} = 0$,

$$\int_{\alpha_1} w^2\eta_2 = \int_{\alpha_1} \frac{3z}{2w}dz = \frac{6}{\sqrt{a}}iI_1(a).$$

We compute the integrals of η_3 on α_1 . First we compute

$$\begin{aligned}
& \eta_3 + \frac{1}{2}d\frac{1}{(z-A_1)(z-A_2)} \\
&= \frac{z - \frac{1}{2}(A_1 + A_2)}{(z-A_1)^2(z-A_2)^2}dz - \frac{z - \frac{1}{2}(A_1 + A_2)}{(z-A_1)^2(z-A_2)^2}dz \\
&= 0.
\end{aligned}$$

Therefore, by $\int_{\alpha_1} d\frac{1}{(z-A_1)(z-A_2)} = 0$,

$$\int_{\alpha_1} \eta_3 = 0.$$

We compute the integrals of $w\eta_3$ on α_1 . First we compute

$$\begin{aligned}
& w\eta_3 + \frac{1}{2}d\frac{w}{(z-A_1)(z-A_2)} \\
&= \frac{(z - \frac{1}{2}(A_1 + A_2))w}{(z-A_1)^2(z-A_2)^2}dz + \frac{dw}{2(z-A_1)(z-A_2)} - \frac{(z-A_1)w + (z-A_2)w}{2(z-A_1)^2(z-A_2)^2}dz \\
&= \frac{3dz}{4w}.
\end{aligned}$$

Therefore, by $\int_{\alpha_1} d\frac{w}{(z-A_1)(z-A_2)} = 0$,

$$\int_{\alpha_1} w\eta_3 = \frac{3}{4} \int_{\alpha_1} \frac{dz}{w} = \frac{6\sqrt{a}}{2}iK(ia).$$

We compute the integrals of $w^2\eta_3$ on α_1 . First we compute

$$\begin{aligned}
& w^2\eta_3 + \frac{B_1^2}{2} d\frac{1}{(z-A_1)(z-A_2)} \\
&= \frac{(z - \frac{1}{2}(A_1 + A_2))(w^2 - B_1^2)}{(z - A_1)^2(z - A_2)^2} dz \\
&= \frac{(z - \frac{1}{2}(A_1 + A_2))(z + 2A_1 - (a - \frac{1}{a}))}{(z - A_1)^2(z - A_2)^2} dz \\
&= dz - \frac{(A_1 - A_2)^2}{4} \frac{dz}{(z - A_2)^2}.
\end{aligned}$$

Therefore, by $\int_{\alpha_1} d\frac{1}{(z-A_1)(z-A_2)} = 0$,

$$\int_{\alpha_1} w^2\eta_3 = \int_{\alpha_1} dz - \frac{(A_1 - A_2)^2}{4} \int_{\alpha_1} \frac{dz}{(z - A_2)^2} = 0.$$

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