別紙４

報告番号	※　　　　第　　　　号

主　論　文　の　要　旨

論 文 題 目 Arithmetic progressions of Piatetski-Shapiro sequences and related problems (Piatetski-Shapiro 列からなる等差数列と関連する問題)

氏　　　名 齋藤 耕太

論　文　内　容　の　要　旨

This thesis is a summary of the works of the author. We discuss a problem whether a given set contains arithmetic progressions or not. Many researchers have been approaching this problem by Ramsey theory. This theory focuses on the existence of given structures in a set with large size. In 1927, van der Waerden firstly gave the Ramsey theoretical result on arithmetic progressions. He showed that if the set of all positive integers are finitely colored, then there exists a monochromatic set which contains arbitrarily long arithmetic progressions. By this theorem, we guess that if a given set had large density, then we would be able to find a long arithmetic progression of this set. Erdös and Turán started to study this implication in 1936. They conjectured that if any subset of positive integers with positive upper asymptotic density contains an arithmetic progression of length 3. This conjecture was solved by Roth in 1954. Furthermore, in 1973, Szemerédi showed that any set with positive upper asymptotic density contains arbitrarily long arithmetic progressions. However, it is still difficult to find long arithmetic progressions of a sparse set which has upper asymptotic density 0. Surprisingly, in 2008, Green and Tao showed that the set of all prime numbers contains arbitrarily long arithmetic progressions. Remark that the set of all prime numbers has asymptotic density 0 by the prime number theorem.

In this thesis, we mainly investigate arithmetic progressions of Piatetski-Shapiro sequences and more general sequences which also have upper asymptotic density 0. Here for every non-integral $\alpha > 1$, the sequence of the integer parts of n^α $(n = 1,2,\dots)$ is called the Piatetski-Shapiro sequence with exponent α, and let $\mathrm{PS}(\alpha)$ be the set of those terms. We present three main results.

Firstly, in the case $1 < \alpha < 2$, for fixed all integers $k \geq 3$ and $r \geq 1$, we study the set of $n \in \mathbb{N}$ such that the integer parts of $(n + rj)^\alpha$ $(j = 0, 1, \dots, k-1)$ forms an arithmetic progression. It essentially follows from the result of Frantzikinakis and

Wierdl in 2009 that the number of such n is infinite. In this work, we reveal that the density of the set is equal to $1/(k-1)$ which is independent of α and r. We also get an extended result for more general sequences. This work is collaborated with Yuuya Yoshida.

Secondary, we study arithmetic progressions of $\mathrm{PS}(\alpha)$ for $\alpha > 2$. In general, fix $a, b, c \in \mathbb{N}$. For every $2 < s < t$, we study the set of $\alpha \in [s, t]$ such that the Diophantine equation $ax + by = cz$ has infinitely many solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$ with x, y, z pairwise distinct. We show that the Hausdorff dimension of the set is greater than or equal to $1/s^3$. As a consequence, there are uncountably many $\alpha > 2$ such that $\mathrm{PS}(\alpha)$ contains infinitely many arithmetic progressions of length 3. This work is collaborated with Toshiki Matsusaka.

Thirdly, we investigate Diophantine linear equations with two variables in Piatetski-Shapiro sequences. Let $a, b \in \mathbb{R}$ with $0 \leq b < a$ and $a \neq 1$ satisfying that $y = ax + b$ has infinitely many solutions of positive integers. By the result of Glasscock in 2017, for Lebesgue almost all $\alpha > 1$, the equation $y = ax + b$ has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$ or not, according as $1 < \alpha < 2$ or $\alpha > 2$. We get an improvement of his result in this thesis. We discern more details concerning the geometric structure of the set $\alpha \in [s, t]$ such that $y = ax + b$ has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$. We reveal that the Hausdorff dimension of this set is coincident with $2/s$ for all $2 < s < t$. Further, we show that for all $1 < \alpha < 2$, $y = ax + b$ has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$. As a consequence, we obtain a partial result of the existence of a perfect Euler brick.