Arithmetic progressions of Piatetski-Shapiro sequences and related problems

(Piatetski-Shapiro 列からなる等差数列と関連する問題)

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Abstract

This thesis is a summary of the works of the author. We discuss problems whether a given set contains arithmetic progressions or not. By Szemerédi's theorem, any subset of positive integers with positive upper asymptotic density contains arbitrarily long arithmetic progressions. However, it is still difficult to find long arithmetic progressions of a sparse set. We mainly investigate arithmetic progressions of Piatetski-Shapiro sequences and more general sequences which have upper asymptotic density 0. Here for every non-integral $\alpha > 1$, the sequence of the integer parts of n^α $(n = 1, 2, \ldots)$ is called the Piatetski-Shapiro sequence with exponent α, and let $\mathrm{PS}(\alpha)$ be the set of those terms. We present three main results.

Firstly, in the case $1 < \alpha < 2$, we reveal the explicit density of the set of $n \in \mathbb{N}$ such that the integer parts of $(n + rj)^\alpha$ $(j = 0, 1, \ldots, k-1)$ forms an arithmetic progression for all integers $k \geq 3$ and $r \geq 1$. This density is equal to $1/(k-1)$ which is independent of α and r. We get an extended result for more general sequences. This work is collaborated with Yuuya Yoshida.

Secondary, fix $a, b, c \in \mathbb{N}$. For every $2 < s < t$, we study the set of $\alpha \in [s, t]$ such that the Diophantine equation $ax + by = cz$ has infinitely many solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$ with x, y, z pairwise distinct. We show that the Hausdorff dimension of the set is greater than or equal to $1/s^3$. As a consequence, there are uncountably many $\alpha > 2$ such that $\mathrm{PS}(\alpha)$ contains infinitely many arithmetic progressions of length 3. This work is collaborated with Toshiki Matsusaka.

Thirdly, we investigate Diophantine linear equations with two variables in Piatetski-Shapiro sequences. Let $a, b \in \mathbb{R}$ with $0 \leq b < a$ and $a \neq 1$ satisfying that $y = ax + b$ has infinitely many solutions of positive integers. Then we reveal the Hausdorff dimension of the set of $\alpha \in [s, t]$ such that $y = ax + b$ has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$. This dimension is coincident with $2/s$ for all $2 < s < t$. Furthermore, we show that for all $1 < \alpha < 2$, $y = ax + b$ has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$. As a consequence, we obtain a partial result of the existence of a perfect Euler brick.

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Chapter 1

Introduction

This thesis is a summary of the articles [SY21] (collaborated with Yuuya Yoshida), [MS20] (collaborated with Toshiki Matsusaka), and [Sai20]. The article [SY21] is on the distribution of arithmetic progressions of Piatetski-Shapiro sequences and more general sequences. We will define Piatetski-Shapiro sequences in Section 1.2. The article [MS20] is on linear Diophantine equations with three variables in Piatetski-Shapiro sequences. The article [Sai20] is on them with two variables. In this chapter, we present backgrounds of these researches.

1.1 Problems on the existence of arithmetic progressions

Let $\mathbb{N}$ be a set of all positive integers. Let $k \in \mathbb{N}$. We say that a sequence of real numbers $(a_i)_{i=0}^{k-1}$ is an *arithmetic progression of length* k (k*-AP* for short) if there exist $a \in \mathbb{R}$ and $d > 0$ such that

$$a_i = a + id$$

for all $i = 0, 1, \ldots, k-1$. We mainly discuss the following problems:

Problem 1.1.1. *Fix any $k \geq 3$. If a subset of real numbers is given, then does the set contain an AP of length k, or not?*

For example, if the set of all perfect squares is given, then this set consists of the following elements:

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad 81, \quad \ldots$$

We can find that $(1, 25, 49)$ is a 3-AP. Hence the set of all perfect squares contains infinitely many 3-APs since $(n^2, 25n^2, 49n^2)$ is a 3-AP of perfect squares for every $n \in \mathbb{N}$. Euler showed that the set of all perfect squares does not contain any 4-APs in 1780, according to the book written by Dickson [Dic66, pp.440 and 635].

Many researchers have been approaching Problem 1.1.1 by Ramsey theory. This theory focuses on the existence of given structures in a set with large size. For example, van der Waerden firstly gave the Ramsey theoretical result on APs in 1927. To assert this result, for all sets A and B we say $A \sqcup B := A \cup B$ if $A \cap B = \emptyset$, and define $[n] = \{1, \dots, n\}$ for all $n \in \mathbb{N}$.

Theorem 1.1.2 ([Van27]). *Let $r \in \mathbb{N}$. If $\mathbb{N} = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_r$, then there exists at least one $i \in [r]$ such that C_i contains arbitrarily long APs.*

Theorem 1.1.2 is called van der Waerden's theorem. A proof of this theorem also can be seen in [GRS90, Theorem 1 in Chapter 2]. The case when $r = 2$ was conjectured by Baudet in 1926. By this theorem, we may expect that if a given set had large density, then we would be able to find long APs of the set. In 1936, Erdős and Turán started to study this implication. More precisely, for all $N, k \in \mathbb{N}$, they defined

$$r_k(N) = \max\{\#A \colon A \subseteq [N] \text{ does not contain any APs of length } k\},$$

where $\#X$ denotes the number of elements in a set X. They proved that

Theorem 1.1.3 ([ET36, Theorem II]). *For all $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon) > 0$ such that for all integers $N \geq N_0$, we have $r_3(N) < (4/9 + \varepsilon)N$.*

Note that $r_1(N) = 0$ and $r_2(N) = 1$ for all $N \in \mathbb{N}$. Thus we assume $k \geq 3$. This is a non-trivial case. In the same article [ET36], Erdős-Turán conjectured that $r_3(N)/N \to 0$ as $N \to \infty$. This conjecture was affirmatively solved by Roth [Rot52]. He applied the Hardy-Littlewood circle method. Szemerédi studied more general cases when the length of APs is longer than 3. He reached at the following celebrated result which is called Szemerédi's theorem.

Theorem 1.1.4 ([Sze75]). *For all $k \in \mathbb{N}$,*

$$\lim_{N \to \infty} \frac{r_k(N)}{N} = 0. \tag{1.1.1}$$

Hence, Szemerédi affirmatively solved generalizations of the Erdős-Turán conjecture for arbitrary length. Further, Theorem 1.1.4 immediately implies

Theorem 1.1.5. *If $A \subseteq \mathbb{N}$ satisfies*

$$\overline{\lim_{N\to\infty}} \frac{\#(A \cap [N])}{N} > 0, \tag{1.1.2}$$

then A contains arbitrarily long APs.

Proof. Fix an arbitrary integer $k \geq 3$. Let $\delta = \overline{\lim}_{N\to\infty} \#(A \cap [N])/N$. Then there exists a sequence of positive integers $N_1 < N_2 < \cdots$ such that

$$\#(A \cap [N_i]) \geq \frac{\delta}{2} N_i$$

for all $i \in \mathbb{N}$. By Theorem 1.1.4, there exists $i_0 = i_0(\delta, k) \in \mathbb{N}$ such that $\#(A \cap [N_i]) \geq r_k(N_i)$ for all $i \geq i_0$. This implies that A contains a k-AP. □

The inverse implication is also true, that is, Theorem 1.1.5 implies Theorem 1.1.4. Thus Theorem 1.1.4 and Theorem 1.1.5 are equivalent. Theorem 1.1.5 is also called Szemerédi's theorem. The left-hand side of (1.1.2) is called the *upper asymptotic density of A*.

From Theorem 1.1.5, we can find arbitrarily long APs of a set with positive upper asymptotic density. For example, let A be the set of all square-free integers. Then A contains arbitrarily long APs since

$$\lim_{N\to\infty} \frac{\#(A \cap [N])}{N} = \frac{6}{\pi^2} > 0.$$

However, it is still difficult to find long APs in a sparse set which has upper asymptotic density zero. For example, let $\mathcal{P}$ be the set of all prime numbers. By the prime number theorem, there exists an absolute constant $C > 0$ such that

$$\frac{\#(\mathcal{P} \cap [N])}{N} \leq C \frac{1}{\log N}$$

for all $N \in \mathbb{N}$. Thus the upper asymptotic density of $\mathcal{P}$ is zero. It was a long standing open problem whether $\mathcal{P}$ contains arbitrarily long APs or not. Surprisingly, Green and Tao gave affirmative solution to this problem.

Theorem 1.1.6 ([GT08, Theorem 1.1]). *The set of all prime numbers contains arbitrarily long APs.*

More precisely, Green and Tao showed a much stronger result as follows:

Theorem 1.1.7 ([GT08, Theorem 1.2]). *Let $A \subseteq \mathcal{P}$. If A satisfies*

$$\overline{\lim_{N\to\infty}} \frac{\#(A \cap [N])}{\#(\mathcal{P} \cap [N])} > 0, \tag{1.1.3}$$

then A contains arbitrarily long APs.

This theorem is called Szemerédi's theorem in the primes. In this thesis, we also focus on the existence of APs of sparse sets. We investigate to find APs of Piatetski-Shapiro sequences and some general sequences which also have upper asymptotic density zero.

1.2 Piatetski-Shapiro sequences

For all $x \in \mathbb{R}$, let us $\lfloor x \rfloor$ denote the integer part of x, and $\{x\}$ denote the fractional part of x.

Definition 1.2.1. *For all non-integral $\alpha > 1$, we say that $(\lfloor n^\alpha \rfloor)_{n=1}^\infty$ is the Piatetski-Shapiro sequence with exponent α. Define*

$$\mathrm{PS}(\alpha) = \{\lfloor n^\alpha \rfloor : n \in \mathbb{N}\}.$$

A sequence $(a_n)_{n=1}^\infty$ of integers is called a Piatetski-Shapiro sequence if there exists a non-integral $\alpha > 1$ such that $a_n = \lfloor n^\alpha \rfloor$ for all $n \in \mathbb{N}$.

This sequence is named in honor of Ilya Piatetski-Shapiro. He showed

Theorem 1.2.2 ([PS53]). *For all $1 < \alpha < 12/11 = 1.0909\cdots$, we have*

$$\#\{p \in \mathcal{P} \cap [1,x] : \text{there exists } n \in \mathbb{N} \text{ such that } p = \lfloor n^\alpha \rfloor\} \sim \frac{x^{1/\alpha}}{\log x} \tag{1.2.1}$$

as $x \to \infty$.

A prime number formed as $\lfloor n^\alpha \rfloor$ is called a *Piatetski-Shapiro prime with exponent α*. We refer [GK91, Section 4.6] to the readers who want to study this proof in English. Many researchers extended the range $1 < \alpha < 12/11$ in Theorem 1.2.2 [Kol67, Lei80, HB83, Kol85, LR92]. Most recently, Rivat and Sargos proved that for all $1 < \alpha < 2817/2426 = 1.1617\cdots$ (1.2.1) is true [RS01]. Interestingly, Deshouillers [Des76] showed that for Lebesgue almost all $\alpha > 1$

$$\#\{p \in \mathcal{P}\colon \text{there exists } n \in \mathbb{N} \text{ such that } p = \lfloor n^\alpha \rfloor\} = \infty. \tag{1.2.2}$$

It is conjectured that the range of the exponent in Theorem 1.2.2 would be $1 < \alpha < 2$ but we have not reached it. Remark that in the case $\alpha = 2$, there is no prime number p such that $p = n^2$ for some $n \in \mathbb{N}$.

In 1933, Segel started researches on additive structure of Piatetski-Shapiro sequences earlier than the work by Piatetski-Shapiro.

Theorem 1.2.3 ([Seg33])**.** *Let $\alpha > 1$. If $k \geq \alpha^2 2^\alpha$, then for every sufficiently large $N \in \mathbb{N}$ there exist $x_1, \dots, x_k \in \mathbb{N} \cup \{0\}$ such that*

$$\lfloor x_1^\alpha \rfloor + \cdots + \lfloor x_k^\alpha \rfloor = N.$$

This is an analogue of Waring's problems. Many researchers studied Waring's problem on Piatetski-Shapiro sequences. For example, see [AG16, AZ84, Des73, Lis02]. We will discuss linear Diophantine equations in Piatetski-Shapiro sequences with three variables and two variables in Chapters 4 and 5, respectively.

1.3 APs of Piatetski-Shapiro sequences and main results

Let $\alpha > 1$ be a non-integral real number. For all $N \in \mathbb{N}$, we have

$$\#(\mathrm{PS}(\alpha) \cap [N]) = \#\{n \in \mathbb{N}\colon \lfloor n^\alpha \rfloor \leq N\} \leq \sum_{n^\alpha \leq N+1} 1 \leq (N+1)^{1/\alpha}.$$

Therefore $\mathrm{PS}(\alpha)$ has upper asymptotic density zero. We can not apply Szemerédi's theorem to Piatetski-Shapiro sequences. Let us discuss the following problem:

Problem 1.3.1. *Given an integer $k \geq 3$. Which does a non-integral $\alpha > 1$ satisfy that* $\mathrm{PS}(\alpha)$ *contain k-APs?*

Remark that $\mathrm{PS}(\alpha) = \mathbb{N}$ for every $0 < \alpha \leq 1$. Thus it is trivial that $\mathrm{PS}(\alpha)$ contains infinitely long APs in this case. Further, it is known that $\mathrm{PS}(\alpha)$ contains arbitrarily long APs for every $1 < \alpha < 2$. More strongly, we have the following:

Theorem 1.3.2. *Let $1 < \alpha < 2$. For all integers $k \geq 3$ and $r > 0$, there exist infinitely many $n \in \mathbb{N}$ such that $(\lfloor (n+jr)^\alpha \rfloor)_{j=0}^{k-1}$ is an AP.*

This result essentially follows from the work of Frantikinakis and Wierdl [FW09, Proposition 5.1]. In the case when $1 < \alpha < 2$, we can analyze more details on APs of Piatetski-Shapiro sequences. For example, we will give the distribution of $n \in \mathbb{N}$ satisfying $(\lfloor (n+jr)^\alpha \rfloor)_{j=0}^{k-1}$ is an AP as follows:

Theorem 1.3.3 ([SY21, Corollary 1.2])**.** *For all $1 < \alpha < 2$, all integers $k \geq 3$, and $r \in \mathbb{N}$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \in [N] \colon (\lfloor (n+jr)^\alpha \rfloor)_{j=0}^{k-1} \text{ is an AP}\} = \frac{1}{k-1}. \tag{1.3.1}$$

We will prove this theorem in Chapter 3. Remark that the right-hand side of (1.3.1) is independent of α and r. Further, in the case $\alpha = 1$, the right-hand side of (1.3.1) is equal to 1 since $\{n \in \mathbb{N} \colon (\lfloor (n+jr)^\alpha \rfloor)_{j=0}^{k-1} \text{ is an AP}\} = \mathbb{N}$. In the case $\alpha = 2$, there is no $n \in \mathbb{N}$ such that $(\lfloor (n+jr)^\alpha \rfloor)_{j=0}^{k-1}$ is an AP since

$$(n+r)^2 - n^2 = 2r + r^2 \neq 2r + 3r^2 = (n+2r)^2 - (n+r)^2.$$

Thus, the right-hand side of (1.3.1) is equal to 0 in this case. Hence we may expect that the limit (1.3.1) would be decreasing with respect to α, but actually, that is constant.

In the previous research of the author and Yoshida, they presented

Theorem 1.3.4 ([SY19, Theorem 4])**.** *Let A be a subset of $\mathbb{N}$ with positive upper asymptotic density. Then for all integers $k \geq 3$, there exists $(a_0, \dots, a_{k-1}) \in A^k$ satisfying*

$$(a_i)_{i=0}^{k-1} \text{ is a } k\text{-AP} \quad \text{and} \quad (\lfloor a_i^\alpha \rfloor)_{i=0}^{k-1} \text{ is also a } k\text{-AP}. \tag{1.3.2}$$

Theorem 1.3.4 can be considered as Szemerédi's theorem on Piatetski-Shapiro sequences. Indeed, Theorem 1.3.4 implies that

Theorem 1.3.5. *Let $1 < \alpha < 2$, and let $A \subseteq \mathrm{PS}(\alpha)$. If A satisfies*

$$\varlimsup_{N\to\infty} \frac{\#(A \cap [N])}{\#(\mathrm{PS}(\alpha) \cap [N])} > 0, \tag{1.3.3}$$

then A contains arbitrarily long APs.

Proof. Fix an arbitrary $A \subseteq \mathrm{PS}(\alpha)$ with (1.3.3). Let B be the set of all $n \in \mathbb{N}$ such that $\lfloor n^\alpha \rfloor \in A$. By (1.3.3), there exist $\delta > 0$ and a sequence of integers $N_1 < N_2 < \cdots$ such that

$$\#(A \cap [N_j]) \geq \delta \cdot \#(\mathrm{PS}(\alpha) \cap [N_j])$$

for every $j \in \mathbb{N}$. Without loss of generality, we may assume that $N_j \in \mathrm{PS}(\alpha)$ for every $j \in \mathbb{N}$. Let n_j be the integer such that $\lfloor n_j^\alpha \rfloor = N_j$ for every $j \in \mathbb{N}$. Then for every $j \in \mathbb{N}$ we have

$$\begin{aligned}\#(B \cap [n_j]) &= \sum_{\lfloor n^\alpha \rfloor \in A, n \leq n_j} 1 = \#(A \cap [\lfloor n_j^\alpha \rfloor]) = \#(A \cap [N_j]) \\ &\geq \delta \cdot \#(\mathrm{PS}(\alpha) \cap [N_j]) = \delta n_j.\end{aligned}$$

This yields that B has positive upper asymptotic density. By Theorem 1.3.4, A contains arbitrarily long APs. □

Note that Theorem 1.3.5 is an analogue of the Green-Tao theorem (Theorem 1.1.7) for Piatetski-Shapiro sequences. It is elementary to prove Theorem 1.3.4 but the idea of this proof is important for the study of this thesis. For all finite sequences of integers $P = (a(j))_{j=0}^{k-1}$ and for all functions $f : \mathbb{N} \to \mathbb{N}$, the author and Yoshida defined the semi-norm

$$N_P(f) = \sum_{j=0}^{k-2} |(f \circ a)(j+2) - 2(f \circ a)(j+1) + (f \circ a)(j)|. \tag{1.3.4}$$

If $N_P(f(n)) = 0$ and f is strictly increasing, then $((f \circ a)(j))_{j=0}^{k-1}$ should be a k-AP. Here, $N_P(\cdot)$ satisfies the triangle inequality. Therefore if we set $f(n) = \lfloor n^\alpha \rfloor$, $f_1(n) = n^\alpha$, and $f_2(n) = \{n^\alpha\}$, then we have

$$N_P(f) \leq N_P(f_1) + N_P(f_2).$$

By this inequality, if $N_P(f_1)$ and $N_P(f_2)$ are small, then $(\lfloor n^\alpha \rfloor)_{n \in P}$ is a k-AP. By this discussion, it is important to control the continuous part $N_P(f_1)$ and the fractional part $N_P(f_2)$. In Chapters 3–5, we will use this idea.

In the previous article [SY19], we control the continuous part by Szemerédi's theorem and Taylor's expansion. To see this, we now evaluate $N_P(f_1)$. Let A be a subset of integers with positive upper density. Then by Szemerédi's theorem (Theorem 1.1.5) A contains infinitely many APs of length k, where $k \geq 3$ is a fixed arbitrary integer. Let P be such an AP. Then $P = (dj+e)_{j=0}^{k-1}$ for some $d > 0$ and $e \geq 0$. By Taylor's expansion,

$$f_1(dj+e) = f_1(e) + djf_1'(e) + O_k(df''(e)),$$

which implies that

$$\begin{aligned} N_P(f_1) &= \sum_{j=0}^{k-2} |f_1(d(j+2)+e) - 2f_1(d(j+1)+e) + f_1(dj+e)| \\ &= O_{k,\alpha}\left(\sum_{j=0}^{k-2} de^{\alpha-2}\right) = O_{k,\alpha}(de^{\alpha-2}). \end{aligned}$$

Therefore, there exists $C_{k,\alpha} > 0$ such that $N_P(f_1) \leq C_{k,\alpha}de^{\alpha-2}$. Recall that $1 < \alpha < 2$. Hence $0 \leq N_P(f_1) < 1/2$ holds if we may take an AP $P = (dj+e)_{j=0}^{k-1}$ of A such that $e \geq (2C_{k,\alpha}d)^{1/(2-\alpha)}$. Actually, we can find such an AP but it is slightly complicated. Thus we refer [SY19] to the readers who want to know more details.

From this observation, one of important conditions is $\alpha < 2$ in this discussion since the error term $O_{k,\alpha}(de^{\alpha-2})$ must be large for $\alpha \geq 2$. Moreover, in the case $\alpha = 2$, we mentioned that PS(2) (=the set of all squares) does NOT contain any 4-APs in Section 1.1. Hence it seems that there is some barrier at $\alpha = 2$. Additionally, in the case $\alpha \geq 3$ and $\alpha \in \mathbb{N}$, it is known that PS(α) (=the set of all αth powers) does not contain any 3-APs. This result was partially solved by Euler (according to [Dic66, pp. 572-573]) and Dénes [Dén52], and finally solved by Darmon and Merel [DM97]. From those reasons, the author and Yoshida proposed the following question:

Question 1.3.6 ([SY19, Question 13])**.** *Is it true that*

$$\sup\{\alpha \geq 1 \colon \mathrm{PS}(\alpha) \textit{ contains arbitrarily long arithmetic progressions}\} = 2?$$

We do not get any answer to this question here. Remark that we can find 4-APs of Piatetski-Shapiro sequences with $\alpha > 2$ by calculation. For example,

$$\begin{gathered} (\lfloor 2^{2.2} \rfloor, \lfloor 11^{2.2} \rfloor, \lfloor 15^{2.2} \rfloor, \lfloor 18^{2.2} \rfloor), \\ (\lfloor 14^{2.655015} \rfloor, \lfloor 39^{2.655015} \rfloor, \lfloor 50^{2.655015} \rfloor, \lfloor 58^{2.655015} \rfloor), \\ (\lfloor 27^{2.720398} \rfloor, \lfloor 89^{2.720398} \rfloor, \lfloor 114^{2.720398} \rfloor, \lfloor 132^{2.720398} \rfloor). \end{gathered}$$

We did not have any theoretical approaches to find APs of Piatetski-Shapiro sequences with $\alpha > 2$ even if the length of APs is 3. However, in Chapter 4, we will show

Theorem 1.3.7 ([MS20, Corollary 1.3]). *For all $2 < s < t$, there are uncountably many $\alpha \in [s,t]$ such that $\mathrm{PS}(\alpha)$ contains infinitely many 3-APs.*

Surprisingly, by Theorem 1.3.7, the supremum of α such that $\mathrm{PS}(\alpha)$ contains infinitely many 3-APs is positive infinity. Further, we will discuss more general linear Diophantine equations $ax+by=cz$ where $a,b,c \in \mathbb{N}$. Precisely, we will show the positiveness of the Hausdorff dimension of the set of $\alpha \in [s,t]$ such that $ax+by=cz$ has infinitely many solutions $(x,y,z) \in \mathrm{PS}(\alpha)^3$. Remark that (x,z,y) is a 3-AP if and only if $x+y=2z$, and x,z,y are distinct. Hence results on 3-APs are special cases of results on $ax+by=cz$.

Glasscock studied linear equations with two variables in Piatetski-Shapiro sequences. Before stating, for all polynomials $f(x_1,\dots,x_n)$ with real coefficients and for all sets $X \subseteq \mathbb{R}$, we say that $f(x_1,\dots,x_n)=0$ is *solvable* in X if there are infinitely many $(x_1,\dots,x_n) \in X^n$ such that

$$f(x_1,\dots,x_n)=0 \quad \text{and} \quad x_1,\dots,x_n \text{ are pairwise distinct.}$$

Let $E \subseteq \mathbb{R}$, and let $S(x)$ be a statement depending on $x \in E$. We say that $S(x)$ holds for *Lebesgue almost all* $x \in E$ if $\{x \in E \colon S(x) \text{ does not hold}\}$ has 1-dimensional Lebesgue measure 0.

Theorem 1.3.8 ([Gla17, Gla20]). *Suppose $a,b \in \mathbb{R}$, $a \notin \{0,1\}$, are such that $y=ax+b$ is solvable in $\mathbb{N}$. For Lebesgue almost all $\alpha > 1$, the equation $y=ax+b$ is solvable or not in $\mathrm{PS}(\alpha)$ according as $\alpha < 2$ or $\alpha > 2$.*

In addition, he gave the following result by Theorem 1.3.8.

Theorem 1.3.9 ([Gla17, Corollary 1]). *For Lebesgue almost all $1 < \alpha < 2$, there exist infinitely many $(k,m,\ell) \in \mathrm{PS}(\alpha)^3$ such that all of*

$$k, \quad m, \quad \ell, \quad k+m, \quad m+\ell, \quad \ell+k, \quad k+m+\ell \tag{1.3.5}$$

are in $\mathrm{PS}(\alpha)$.

This result is related with a long-standing open problem whether there exists $(k,\ell,m) \in \mathbb{N}^3$ such that all of (1.3.5) are in $\mathrm{PS}(2)$ which is the set of all squares. If there was such a tuple $(k,\ell,m) \in \mathbb{N}^3$, we would prove the existence of a perfect Euler brick that is a rectangular cuboid of which all the edges, face diagonals, and body diagonal have integral lengths. In Chapter 5, we will discern more details of geometric structure of the set of $\alpha \in [s,t]$ such that the equation $y = ax + b$ is solvable in $\mathrm{PS}(\alpha)$. We will show that the Hausdorff dimension of this set is coincident with $2/s$ for all real numbers $2 < s < t$ if $0 \leq b < a$ and $a \neq 1$. Further, by Dirichlet's approximation theorem, we will show that Theorem 1.3.9 is still true if we replace "for Lebesgue almost all" with "for all", that is,

Theorem 1.3.10 ([Sai20, Corollary 1.3]). *For all $1 < \alpha < 2$, there exist infinitely many $(k, m, \ell) \in \mathrm{PS}(\alpha)^3$ such that all of (1.3.5) are in $\mathrm{PS}(\alpha)$.*

The rest of the thesis is organized as follows. First in Chapter 2, we introduce notions of the uniform distribution and the Hausdorff dimension, and describe some known useful results. In Chapter 3, we discuss Theorem 1.3.3 and related results. In Chapter 4, we investigate linear Diophantine equations with three variables in Piatetski-Shapiro sequences, and present a proof of Theorem 1.3.7. Finally we study linear Diophantine equations with two variables in Piatetski-Shapiro sequences, and give a proof of Theorem 1.3.10.

Notations

- Let $\mathbb{Z}$ be the set of all integers, $\mathbb{Q}$ be the set of all rational numbers, $\mathbb{R}$ be the set of all real numbers, and $\mathbb{C}$ be the set of all complex numbers.
- For all intervals I of real numbers, $I_{\mathbb{Z}}$ denotes $I \cap \mathbb{Z}$.
- For $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the minimum integer n such that $x \leq n$.
- Let $\sqrt{-1}$ denote the imaginary unit, and define $\mathrm{e}(x) = e^{2\pi\sqrt{-1}x}$ for all $x \in \mathbb{R}$.
- We write $O(1)$ for a bounded quantity. If this bound depends only on some parameters $a_1, \ldots, a_n$, then for instance we write $O_{a_1, a_2, \ldots, a_n}(1)$. As is customary, we often abbreviate $O(1)X$ and $O_{a_1,\ldots,a_n}(1)X$ to $O(X)$ and $O_{a_1,\ldots,a_n}(X)$ respectively for a non-negative quantity X. We also say $f(X) \ll g(X)$ and $f(X) \ll_{a_1,\ldots,a_n} g(X)$ as $f(X) = O(g(X))$ and $f(X) = O_{a_1,\ldots,a_n}(g(X))$ respectively, where $g(X)$ is non-negative.
- The class of C^k denotes the set of all real functions which can be k-th order continuously differentiable.

Chapter 2

Preparations

2.1 Uniform distribution modulo 1 and Hardy fields

To prove main results, the theory of uniform distribution modulo 1 is one of key points. For $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, define the notation

$$\{\mathbf{x}\} = (\{x_1\}, \{x_2\}, \ldots, \{x_d\}).$$

Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence of $\mathbb{R}^d$. We say that $(\mathbf{x}_n)_{n=1}^{\infty}$ is *uniformly distributed modulo* 1 if every convex set $\mathcal{C} \subseteq [0,1)^d$ satisfies that

$$\lim_{N\to\infty} \frac{1}{N} \#\big\{n \in [N] : \{\mathbf{x}_n\} \in \mathcal{C}\big\} = \mu(\mathcal{C}), \tag{2.1.1}$$

where μ denotes the Lebesgue measure on $\mathbb{R}^d$. It is known that $(\mathbf{x}_n)_{n=1}^{\infty}$ is uniformly distributed modulo 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(\langle \mathbf{h}, \mathbf{x}_n \rangle) = 0 \tag{2.1.2}$$

for all non-zero $\mathbf{h} \in \mathbb{Z}^d$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$. One can also say that $(\mathbf{x}_n)_{n=1}^{\infty}$ is uniformly distributed modulo 1 if and only if $(\langle \mathbf{h}, \mathbf{x}_n \rangle)_{n=1}^{\infty}$ is uniformly distributed modulo 1 for all non-zero $\mathbf{h} \in \mathbb{Z}^d$. Due to this equivalence, the following facts hold: if a sequence $(\mathbf{x}_n)_{n=1}^{\infty} = \big((x_{1,n}, \ldots, x_{d,n})\big)_{n=1}^{\infty}$ is uniformly distributed modulo 1, then

- so is the sequence $(\mathbf{x}_n\mathbf{A})_{n=1}^{\infty}$ for every integer matrix $\mathbf{A}$ of order d and rank d;
- so is the sequence $(x_{i,n})_{n=1}^{\infty}$ for every $i \in [d]$.

For the details, see [KN74, Theorem 6.2].

Next, we describe *Hardy fields* which are convenient to extend Piatetski-Shapiro sequences to more general ones. Let $\mathcal{B}$ be the set of all real-valued functions on intervals $[x_0, \infty)$, where the real numbers x_0 depend on the functions. The set $\mathcal{B}$ forms a ring under the induced addition and multiplication by the following equivalence relation: two functions $f_1, f_2 \in \mathcal{B}$ are equivalent to each other if and only if there exists $x_0' \in \mathbb{R}$ such that $f_1(x) = f_2(x)$ for all $x \geq x_0'$. Using this equivalence relation, we define Hardy fields as follows.

Definition 2.1.1. *A subfield of the ring $\mathcal{B}$ closed under differentiation is called a Hardy field. We denote by $\mathcal{H}$ the union of all Hardy fields.*

The notion of Hardy fields was first introduced by Bourbaki [Bou61], and has been used in analysis, e.g., differential equations [Bos81, Bos82, Bos87, Ros83a, Ros83b], difference and functional equations [Bos84b, Bos84a], and uniform distribution modulo 1 [Bos94, BKS19, Fra09]. The set $\mathcal{H}$ is so rich that $\mathcal{H}$ contains the set $\mathcal{LE}$ of all *logarithmico-exponential functions*. A logarithmico-exponential function, which was introduced by Hardy [Har24, Har12], is defined by a finite combination of the ordinary algebraic symbols (viz. $+, -, \times, \div$) and the functional symbols $\log(\cdot)$ and $\exp(\cdot)$ operating on a real variable x and on real constants. For instance, the function $x^{\alpha} = e^{\alpha \log x}$ belongs to $\mathcal{LE}$ for all $\alpha \in \mathbb{R}$.

To investigate uniform distribution modulo 1, we need to estimate exponential sums in general. However, if a function $f \in \mathcal{H}$ is *subpolynomial*, i.e., $f(x) \ll x^n$ for some $n \in \mathbb{N}$, then it is easy to investigate whether the sequence $(f(n))_{n=n_0}^{\infty}$ is uniformly distributed modulo 1.

Proposition 2.1.2 (Boshernitzan [Bos94])**.** *Let $n_0 \in \mathbb{N}$. For every subpolynomial $f \in \mathcal{H}$ defined on the interval $[n_0, \infty)$, the following conditions are equivalent.*

- *$(f(n))_{n=n_0}^{\infty}$ is uniformly distributed modulo 1.*
- *For every polynomial $p(x) \in \mathbb{Q}[x]$, the ratio $(f(x)-p(x))/\log x$ diverges to positive or negative infinity as $x \to \infty$, where the sign of infinity depends on p.*

The next corollary is a simple application of Proposition 2.1.2.

Corollary 2.1.3. *Let* $n_0 = \lceil e^e \rceil = 16$*, and let* f *be a function in* (3.2.3)*. Then the sequence* $\big((f(n), f'(n))\big)_{n=n_0}^{\infty}$ *is uniformly distributed modulo* 1*.*

Proof. Take a non-zero $(h_0, h_1) \in \mathbb{Z}^2$ arbitrarily. All we need is to show that the sequence $(h_0 f(n) + h_1 f'(n))_{n=n_0}^{\infty}$ is uniformly distributed modulo 1. It can be easily checked that for every $p(x) \in \mathbb{Q}[x]$ the ratio $(h_0 f(x) + h_1 f'(x) - p(x))/\log x$ diverges to positive or negative infinity as $x \to \infty$. Since the function $h_0 f + h_1 f'$ belongs to $\mathcal{H}$ and is subpolynomial, Proposition 2.1.2 implies that the sequence $(h_0 f(n) + h_1 f'(n))_{n=n_0}^{\infty}$ is uniformly distributed modulo 1. Therefore, we conclude this corollary. $\square$

For the function $f(x) = x^{\alpha}$ with $\alpha \in (d, d+1)$ and $d \in \mathbb{N}$, it can be proved that the sequence $\big((f(n), f'(n), f''(n)/2!, \dots, f^{(d)}(n)/d!)\big)_{n=1}^{\infty}$ is uniformly distributed modulo 1 in the same way as the above corollary.

2.2 Discrepancy and exponential sums

Let $(\mathbf{x}_n)_{n=1}^{N}$ be a sequence composed of $\mathbf{x}_n \in \mathbb{R}^d$ for all $n \in [N]$. We define the *discrepancy* $\mathcal{D}((\mathbf{x}_n)_{n=1}^{N})$ of the sequence $(\mathbf{x}_n)_{n=1}^{N}$ by

$$\sup_{\substack{0 \le a_i < b_i \le 1 \\ i \in [d]}} \left| \frac{\#\left\{ n \in [N] \colon \{\mathbf{x}_n\} \in \prod_{i \in [d]} [a_i, b_i) \right\}}{N} - \prod_{i \in [d]} (b_i - a_i) \right|.$$

In order to evaluate an upper bound for the discrepancy, we use the following inequality which is shown by Koksma [Kok50] and Szüsz [Szü52] independently: there exists $C_d > 0$ which depends only on d such that for all $K \in \mathbb{N}$, we have

$$\mathcal{D}((\mathbf{x}_n)_{n=1}^{N}) \le C_d \left(\frac{1}{K} + \sum_{\substack{0 < \|\mathbf{k}\|_\infty \le K \\ \mathbf{k} \in \mathbb{Z}^d}} \frac{1}{\nu(\mathbf{k})} \left| \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(\langle \mathbf{k}, \mathbf{x}_n \rangle) \right| \right), \qquad (2.2.1)$$

where we define

$$\|\mathbf{k}\|_\infty = \max(|k_1|, \dots, |k_d|), \quad \nu(\mathbf{k}) = \prod_{i=1}^{d} \max(1, |k_i|).$$

This inequality is sometimes reffered as the *Erdős-Turán-Koksma inequality*. We refer [DT97, Theorem 1.21] to the readers for more details on discrepancies and a proof of (2.2.1). In particular, if $d = 1$, then for all $K \in \mathbb{N}$

$$\mathcal{D}((x_n)_{n=1}^N) \ll \frac{1}{K} + \sum_{h=1}^{K} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(hx_n) \right|. \tag{2.2.2}$$

This is called the *Erdős-Turán inequality* of which a proof can be found in [KN74, Theorem 2.5 in Chapter 2]. These inequalities reduce the estimate of the discrepancy to that of exponential sums. Furthermore, the exponential sum is evaluated by the following lemmas.

Lemma 2.2.1 (Kusmin-Landau)**.** *Let $I = [a, b)$ be an interval with $b-a \geq 1$, and $f\colon I \to \mathbb{R}$ be a C^1-function such that f' is monotone. Let $\lambda_1 > 0$. Suppose that*

$$\lambda_1 \leq \min\{|f'(x) - n| : n \in \mathbb{Z}\}$$

for all $x \in I$. Then

$$\sum_{n \in I_{\mathbb{Z}}} \mathrm{e}(f(n)) \ll \lambda_1^{-1}.$$

Proof. See the book written by Graham and Kolesnik [GK91, Theorems 2.1]. □

Lemma 2.2.2 (van der Corput)**.** *Let $I = [a, b)$ be an interval with $b - a \geq 1$ and $f\colon I \to \mathbb{R}$ be a C^2-function. Let $T > 0$ and $\lambda_2 > 0$. Suppose that*

$$\lambda_2 \leq |f''(x)| \leq T\lambda_2$$

for all $x \in I$. Then

$$\sum_{n \in I_{\mathbb{Z}}} \mathrm{e}(f(n)) \ll_T (b - a)\lambda_2^{1/2} + \lambda_2^{-1/2},$$

Proof. See the book written by Graham and Kolesnik [GK91, Theorems 2.2]. □

Lemma 2.2.3 (Sargos-Gritsenko)**.** *Let $I = [a, b)$ be an interval with $b-a \geq 1$ and $f\colon I \to \mathbb{R}$ be a C^3-function. Let $T > 0$ and $\lambda_3 > 0$. Suppose that*

$$\lambda_3 \leq |f'''(x)| \leq T\lambda_3$$

for all $x \in I$. Then

$$\sum_{n \in I_{\mathbb{Z}}} \mathrm{e}(f(n)) \ll_T (b - a)\lambda_3^{1/6} + \lambda_3^{-1/3}.$$

Lemma 2.2.3 was shown by Sargos [Sar95, Corollary 4.2] and Grisenko [Gri96, Theorem] independently. In general, we have

Lemma 2.2.4 (van der Corput's k-th derivative test)**.** *Let $I = [a, b)$ be an interval with $b - a \geq 1$. Let $f\colon I \to \mathbb{R}$ be a C^k-function, where $k \geq 4$. Let $T > 0$ and $\lambda_k > 0$. Suppose that*

$$\lambda_k \leq |f^{(k)}(x)| \leq T\lambda_k$$

for all $x \in I$. Then

$$\left|\sum_{n\in I_{\mathbb{Z}}} \mathrm{e}(f(n))\right| \ll_{T,k} \left((b-a)\lambda_k^{1/(2^k-2)} + (b-a)^{1-2^{2-k}}\lambda_k^{-1/(2^k-2)}\right).$$

Proof. See Titchmarsh's book [Tit86, Theorem 5.13]. □

Using Lemmas 2.2.1–2.2.4, we will evaluate discrepancies.

2.3 Hausdorff dimension

We next introduce the Hausdorff dimension. For every $U \subseteq \mathbb{R}$, we define the diameter of U by $\mathrm{diam}(U) = \sup_{x,y\in U} |x - y|$. Fix $\delta > 0$. For all $F \subseteq \mathbb{R}$ and $s \in [0, 1]$, we define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{j=1}^\infty \mathrm{diam}(U_j)^s\colon F \subseteq \bigcup_{j=1}^\infty U_j,\ (\forall j \in \mathbb{N})\ \mathrm{diam}(U_j) \leq \delta\right\},$$

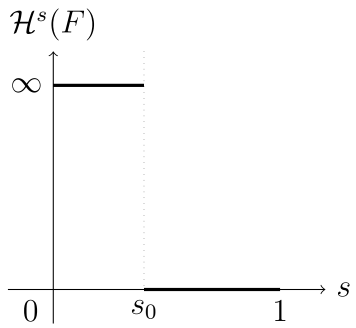
and $\mathcal{H}^s(F) = \lim_{\delta\to+0} \mathcal{H}^s_\delta(F)$ is called the *s-dimensional Hausdorff measure* of F. Remark that for all $0 \leq s < t \leq 1$, we have $\mathcal{H}^t(F) = 0$ if $\mathcal{H}^s(F) < \infty$. Indeed, we assume that $\mu = \mathcal{H}^s(F) < \infty$. Let $0 < \delta < 1/2$ be an arbitrary small real number. We take a family $(U_j)_{j=1}^\infty$ of subsets of real numbers such that $F \subseteq \bigcup_{j=1}^\infty U_j$, $\mathrm{diam}(U_j) \leq \delta$ for all $j \in \mathbb{N}$, and

$$\sum_{j=1}^\infty \mathrm{diam}(U_j)^s \leq \mu + 1.$$

We observe that

$$\sum_{j=1}^\infty \mathrm{diam}(U_j)^t = \sum_{j=1}^\infty \mathrm{diam}(U_j)^s \mathrm{diam}(U_j)^{t-s} \leq (\mu + 1)\delta^{t-s}.$$

This implies that $\mathcal{H}^t_\delta(F) \leq (\mu+1)\delta^{t-s} \to 0$ as $\delta \to +0$. Hence $\mathcal{H}^t(F) = 0$. By this discussion, the set $\{(s, \mathcal{H}^s(F))\colon 0 \leq s \leq 1\}$ should be described as



Remark that this graph has the unique singularity at $s = s_0$. We call this point the Hausdorff dimension of F. More precisely, we define the *Hausdorff dimension* of F by

$$\dim_{\mathrm{H}} F = \inf\{s \in [0,1]\colon \mathcal{H}^s(F) = 0\}.$$

Note that the Hausdorff dimension can be defined on all metric spaces, but we use only $\mathbb{R}$ in this article. By the definition, the following basic properties hold:

- (Monotonicity) for all $F \subseteq E \subseteq \mathbb{R}$, we have $\dim_{\mathrm{H}} F \leq \dim_{\mathrm{H}} E$;
- (Countable stability) if $F_1, F_2, \ldots \subseteq \mathbb{R}$ is a countable sequence of sets, then we have $\dim_{\mathrm{H}} \bigcup_{n=1}^{\infty} F_n = \sup_{n\in\mathbb{N}} \dim_{\mathrm{H}} F_n$.
- (Bi-Lipschitz invariance) let $F \subseteq \mathbb{R}$, and let $f : F \to \mathbb{R}$ be a bi-Lipschitz map, that is, there exist $C_1, C_2 > 0$ such that

  $$C_1|x-y| \leq |f(x) - f(y)| \leq C_2|x-y|$$

  for all $x, y \in F$. Then $\dim_{\mathrm{H}} F = \dim_{\mathrm{H}} f(F)$.

We refer the book written by [Fal14] for the readers who want to know more details on fractal dimensions. In this book [Fal14, (4.3)], we can see a general construction of Cantor sets and a technique to evaluate the Hausdorff dimension of them as follows: Let $[0,1] = E_0 \supseteq E_1 \supseteq E_2 \cdots$ be a decreasing sequence of sets, with each E_k a union of a finite number of disjoint closed intervals called k-th *level basic intervals*, with each interval of E_k containing at least two intervals of E_{k+1}, and the maximum length of k-th level intervals tending to 0 as $k \to \infty$. Then let

$$F = \bigcap_{k=0}^{\infty} E_k. \tag{2.3.1}$$

Lemma 2.3.1 ([Fal14, Example 4.6 (a)])**.** *Suppose in the general construction* (2.3.1) *each* $(k-1)$*-st level interval contains at least* $m_k \geq 2$ k*-th level intervals* $(k = 1, 2, \ldots)$ *which are separated by gaps of at least* δ_k*, where* $0 < \delta_{k+1} < \delta_k$ *for each* k*. Then*

$$\dim_{\mathrm{H}} F \geq \varliminf_{k\to\infty} \frac{\log m_1 \cdots m_{k-1}}{-\log(m_k \delta_k)}.$$

Since the Hausdorff dimension is stable under similarity transformations, the initial interval E_0 may be taken to be an arbitrary closed interval. Moreover, let E_k° be the set of interior points of E_k for all $k \in \mathbb{N}$. Then the Hausdorff dimension of $\bigcap_{k=0}^{\infty} E_k^\circ$ is equal to that of $\bigcap_{k=0}^{\infty} E_k$. To see why, let $\mathcal{N}_k$ be the boundary of E_k, that is, the set of all end points of k-th level intervals. We easily see that

$$\mathcal{N} := F \setminus \left(\bigcap_{k=0}^{\infty} E_k^\circ \right) \subset \bigcup_{k=0}^{\infty} \mathcal{N}_k =: \mathcal{N}_\infty.$$

Since each $\mathcal{N}_k$ is a finite set, the set $\mathcal{N}_\infty$ is a countable set. By monotonicity, and the fact that the Hausdorff dimension of a countable set is 0, we get

$$0 \leq \dim_{\mathrm{H}} \mathcal{N} \leq \dim_{\mathrm{H}} \mathcal{N}_\infty = 0,$$

that is, $\dim_{\mathrm{H}} \mathcal{N} = 0$. Therefore by countable stability, we have

$$\dim_{\mathrm{H}} F = \max \left\{ \dim_{\mathrm{H}} \left(\bigcap_{k=0}^{\infty} E_k^\circ \right), \dim_{\mathrm{H}} \mathcal{N} \right\} = \dim_{\mathrm{H}} \left(\bigcap_{k=0}^{\infty} E_k^\circ \right).$$

By summarizing this discussion, we have the following:

Lemma 2.3.2. *Let* E_0 *be any open interval, and let* $E_0 \supseteq E_1 \supseteq E_2 \cdots$ *be a decreasing sequence of sets, with each* E_k *a union of a finite number of disjoint open intervals, and the maximum length of* k*-th level intervals tending to* 0 *as $k \to \infty$. Suppose each $(k-1)$-st level interval contains at least $m_k \geq 2$ k-th level intervals ($k = 1, 2, \ldots$) which are separated by gaps of at least δ_k, where $0 < \delta_{k+1} < \delta_k$ for each k. Then*

$$\dim_{\mathrm{H}} \bigcap_{k=0}^{\infty} E_k \geq \varliminf_{k\to\infty} \frac{\log m_1 \cdots m_{k-1}}{-\log(m_k \delta_k)}.$$

By the bi-Lipschitz invariance of the Hausdorff dimension and the mean value theorem, we immediately obtain

Lemma 2.3.3. *Let $U \subseteq \mathbb{R}$ be an open set and let $V \subseteq U$ be a compact set. Let $f : U \to \mathbb{R}$ be a continuously differentiable function satisfying $|f'(x)| > 0$ for all $x \in V$. Then for all $F \subseteq V$, $\dim_{\mathrm{H}} f(F) = \dim_{\mathrm{H}} F$.*

For all $\gamma \geq 2$ and sets $X \subseteq \mathbb{R}$, define

$$\mathcal{A}(X, \gamma) = \left\{ x \in X : \text{there are infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N} \right. \\ \left. \text{such that } \left| x - \frac{p}{q} \right| \leq \frac{1}{q^{\gamma}} \right\}.$$

In particular, if $X = \mathbb{R}$ and $\gamma = 2$, we know that $\mathcal{A}(\mathbb{R}, 2) = \mathbb{R}$. This result is referred as Dirichlet's approximation theorem. In addition, in the case when $\gamma > 2$ and $X = [0, 1]$, the following result is known:

Theorem 2.3.4 (Jarník's theorem)**.** *For all $\gamma > 2$, we have*

$$\dim_{\mathrm{H}} \mathcal{A}([0, 1], \gamma) = 2/\gamma.$$

Proof. See [Fal14, Theorem 10.3]. □

Lemma 2.3.5. *For all non-empty and bounded open intervals $J \subseteq \mathbb{R}$, we have*

$$\dim_{\mathrm{H}} \mathcal{A}(J, \gamma) = 2/\gamma.$$

Proof. There exist $m \in \mathbb{Z}$ and $h \in \mathbb{N}$ such that $J \subseteq [m, m+h]$. When $x \in \mathcal{A}(J, \gamma)$, there are infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that $|x - p/q| \leq q^{-\gamma}$. Then for all $\varepsilon > 0$, and for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$,

$$\left| \frac{x-m}{h} - \frac{p-mq}{qh} \right| \leq \frac{1}{hq^{\gamma}} \leq \frac{1}{q^{\gamma-\varepsilon}}.$$

Thus $f(\mathcal{A}(J,\gamma)) \subseteq \mathcal{A}([0,1],\gamma-\varepsilon)$ where $f(x) = (x-h)/m$. By the bi-Lipschitz invariance and monotonicity of the Hausdorff dimension and Theorem 2.3.4, we obtain

$$\dim_{\mathrm{H}}\mathcal{A}(J,\gamma) = \dim_{\mathrm{H}} f(\mathcal{A}(J,\gamma)) \le \dim_{\mathrm{H}} \mathcal{A}([0,1],\gamma-\varepsilon) = \frac{2}{\gamma-\varepsilon}.$$

By taking $\varepsilon \to +0$, $\dim_{\mathrm{H}}\mathcal{A}(J,\gamma) \le 2/\gamma$.

We next show that $\dim_{\mathrm{H}}\mathcal{A}(J,\gamma) \ge 2/\gamma$. There exist $\ell \in \mathbb{Z}$ and $M \in \mathbb{N}$ such that $J \supseteq [\ell/M, (\ell+1)/M]$. Take such ℓ and M. Then for all $x \in \mathcal{A}([0,1],\gamma)$, there are infinitely many $(p,q) \in \mathbb{Z}\times\mathbb{N}$ such that $|x - p/q| \le q^{-\gamma}$. Then for infinitely many $(p,q) \in \mathbb{Z}\times\mathbb{N}$, we have

$$\left|\frac{\ell+x}{M} - \frac{\ell q + p}{qM}\right| \le \frac{1}{M}\frac{1}{q^\gamma} \le \frac{1}{q^\gamma}.$$

This inequality and $(\ell+x)/M \in J$ imply that $g(\mathcal{A}([0,1],\gamma)) \subseteq \mathcal{A}(J,\gamma)$ where $g(x) = (\ell+x)/M$. By the monotonicity and bi-Lipschitz invariance of the Hausdorff dimension and Theorem 2.3.4, we have $\dim_{\mathrm{H}}\mathcal{A}(J,\gamma) \ge 2/\gamma$. □

Chapter 3

Distribution of finite sequences represented by polynomials in Piatetski-Shapiro sequences

This chapter is based on [SY21]. We define the d-th order difference operator of sequences by

$$\Delta_r a(n) \coloneqq a(n+r) - a(n), \quad \Delta_r^m \coloneqq \Delta_r \circ \Delta_r^{m-1} \quad (m = 2, 3, \dots).$$

for all finite sequences $(a(n))_{n=0}^{k-1}$ of integers. A subset A of $\mathbb{N}$ is naturally identified with a strictly increasing sequence of $\mathbb{N}$, and *vice versa*. We study the sets

$$\mathcal{P}_{k,d} \coloneqq \left\{ \begin{array}{c} (a(n))_{n=0}^{k-1} \subseteq \mathbb{N} \\ \text{strictly increasing} \end{array} : (\Delta_1^d a(n))_{n=0}^{k-d-1} \text{ is a constant sequence} \right\}$$

with integers $d \geq 1$ and $k \geq d+2$. A sequence $(a(n))_{n=0}^{k-1}$ of $\mathbb{N}$ belongs to $\mathcal{P}_{k,d}$ if and only if $(a(n))_{n=0}^{k-1}$ is represented as $a(n) = p(n)$, $n \in [0,k) \cap \mathbb{Z}$, by using some polynomial $p(x) \in \mathbb{Q}[x]$ of degree at most d. In particular, when $d = 1$, a sequence belonging to $\mathcal{P}_{k,1}$ is a k-AP.

3.1 Results on Piatetski-Shapiro sequences

We find that every $\mathrm{PS}(\alpha)$ with $\alpha \in (d, d+1)$ and $d \in \mathbb{N}$ contains infinitely many sequences belonging to $\mathcal{P}_{k,d}$. This fact can be deduced from the work of Frantzikinakis and Wierdl [FW09]. Precisely speaking, for all $d \in \mathbb{N}$, $\alpha \in (d, d+1)$, and integers $k \geq d+2$ and $r \geq 1$, there exist infinitely many $n \in \mathbb{N}$ such that $(\lfloor (n+rj)^\alpha \rfloor)_{j=0}^{k-1}$ belongs to $\mathcal{P}_{k,d}$. However, the asymptotic density of such numbers n was not known. In this chapter, we show the asymptotic density, which can be expressed as the volume of a convex set of $\mathbb{R}^{d+1}$.

Theorem 3.1.1 ([SY21, Theorem 1.1])**.** *Let $d \in \mathbb{N}$. For all $\alpha \in (d, d+1)$ and all integers $k \geq d+2$ and $r \geq 1$,*

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \in [N] \colon (\lfloor (n+rj)^\alpha \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} = \mu(\mathcal{C}_{k,d+1}),$$

where μ denotes the Lebesgue measure on $\mathbb{R}^{d+1}$ and the convex set $\mathcal{C}_{k,d+1}$ of $\mathbb{R}^{d+1}$ is defined as

$$\mathcal{C}_{k,d+1} = \left\{ (y_i)_{i=0}^d \in \mathbb{R}^{d+1} : 0 \leq y_0 < 1,\ 0 \leq \sum_{i=0}^d \binom{j}{i} y_i < 1 \ (\forall j \in [k-1]) \right\}. \tag{3.1.1}$$

Also, $\mu(\mathcal{C}_{k,d+1})$ is bounded below by $1/\prod_{i=1}^d \binom{k-1}{i}$.

Note that for integers $n, l \geq 0$ the binomial coefficient $\binom{n}{l}$ is defined as

$$\binom{n}{l} = \frac{(n)_l}{l!},$$

where $(x)_l$ denotes the falling factorial: $(x)_l = x(x-1)\cdots(x-l+1)$ if $l \in \mathbb{N}$, and $(x)_l = 1$ if $l = 0$. Hence, $\binom{n}{l} = 0$ if $0 \leq n < l$. From the last sentence in Theorem 3.1.1, it follows that $\mu(\mathcal{C}_{k,d+1})$ is positive. When $d = 1$, Theorem 3.1.1 implies Theorem 1.3.3.

Proof of Theorem 1.3.3 *assuming Theorem* 3.1.1. Since the convex set $\mathcal{C}_{k,2}$ is equal to $\{(y_0, y_1) \in \mathbb{R}^2 : 0 \leq y_0 < 1,\ 0 \leq y_0 + (k-1)y_1 < 1\}$, Theorem 3.1.1 implies Theorem 1.3.3. □

The lower bound $1/\prod_{i=1}^d \binom{k-1}{i}$ of $\mu(\mathcal{C}_{k,d+1})$ is not equal to $\mu(\mathcal{C}_{k,d+1})$ in general, although the two values are equal to each other when $d = 1$. Also, the volume $\mu(\mathcal{C}_{k,d+1})$ can be computed by using a convex hull algorithm if necessarily. The definition of Piatetski-Shapiro sequences uses the function x^α, which is generalized to a function f with certain properties (Theorems 3.2.1 and 3.2.2).

Theorem 1.3.3 and Theorem 3.1.1 and can be regarded as the case when the common difference r is fixed. We next consider the case when the common difference r is not fixed.

Theorem 3.1.2 ([SY21, Theorem 1.3]). *Let $d \in \mathbb{N}$. For all $\alpha \in (d, d+1)$ and all integers $k \geq d+2$, there exist $A_{\alpha,k}, B_{\alpha,k} > 0$ and $N_{\alpha,k} \in \mathbb{N}$ such that for all integers $N \geq N_{\alpha,k}$,*

$$\begin{aligned} A_{\alpha,k} N^{2-\alpha/(d+1)} &\leq \#\{P \subseteq [N] : P \in \mathcal{P}_{k,1},\ (\lfloor n^\alpha \rfloor)_{n \in P} \in \mathcal{P}_{k,d}\} \\ &\leq B_{\alpha,k} N^{2-\alpha/(d+1)}. \end{aligned} \tag{3.1.2}$$

Since the number of k-APs contained in $[N]$ is about $N^2/2(k-1)$, the asymptotic density of the set in (3.1.2) is zero. We give explicit values of $A_{\alpha,k}$ and $B_{\alpha,k}$ in Section 3.4.

3.2 Results on Hardy fields

Recall that $\mathcal{H}$ denotes the union of all Hardy fields. The function x^α used in Theorems 3.1.1 and 3.1.2 is generalized to a function $f \in \mathcal{H}$ with $x^d \log x \prec f(x) \prec x^{d+1}$. Such a function f satisfies that $f'(x) \geq 1$ for every sufficiently large $x > 0$, since the relation $f(x) \succ x^d \log x$ implies $f'(x) \succ x^{d-1} \log x$ (see Section 3.3). From now on, we assume that a differentiable function $f \colon [n_0, \infty) \to \mathbb{R}$ satisfies $\inf_{x \geq n_0} f'(x) \geq 1$ in order to make the sequence $(\lfloor f(n) \rfloor)_{n=n_0}^\infty$ an increasing sequence. However, this assumption is not essential in any proofs of theorems.

Theorem 3.2.1 ([SY21, Theorem 2.2]). *Let $n_0, d \in \mathbb{N}$, and let $f \colon [n_0, \infty) \to \mathbb{R}$ be a differentiable function in $\mathcal{H}$ satisfying that*

(a1) $x^d \log x \prec f(x) \prec x^{d+1}$;

(a2) $\inf_{x \geq n_0} f'(x) \geq 1$.

Then, for all integers $k \geq d+2$ and $r \geq 1$,

$$\lim_{N \to \infty} \frac{1}{N} \#\{n \in [n_0, N]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} = \mu(\mathcal{C}_{k,d+1}), \tag{3.2.1}$$

where μ denotes the Lebesgue measure on $\mathbb{R}^{d+1}$ and the convex set $\mathcal{C}_{k,d+1}$ of $\mathbb{R}^{d+1}$ is defined as (3.1.1). Also, $\mu(\mathcal{C}_{k,d+1})$ is bounded below by $1/\prod_{i=1}^d \binom{k-1}{i}$.

Theorem 3.2.2 ([SY21, Theorem 2.3]). *Let $n_0, d \in \mathbb{N}$, and let $f\colon [n_0,\infty) \to \mathbb{R}$ be the same as Theorem 3.2.1. Then, for every integer $k \geq d+2$,*

$$\begin{aligned}&\#\{P \subseteq [n_0, N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor f(n)\rfloor)_{n\in P} \in \mathcal{P}_{k,d}\} \\ &\asymp_{c(\cdot),k,d} N f^{(d+1)}(N)^{-1/(d+1)} \quad (N \to \infty).\end{aligned} \tag{3.2.2}$$

When $d = 1$, one can apply Theorems 3.2.1 and 3.2.2 to the following functions:

$$x^\alpha, \quad x(\log x)^\beta, \quad \frac{x^2}{(\log x)^\gamma}, \quad \frac{x^2}{(\log\log x)^\gamma}, \tag{3.2.3}$$

where $\alpha \in (1,2)$, $\beta > 1$ and $\gamma > 0$. Note that all the above functions belong to $\mathcal{LE}$ and *a fortiori* $\mathcal{H}$. Hence, Theorems 3.1.1 and 3.1.2 are special cases of Theorems 3.2.1 and 3.2.2, respectively. Also, the implicit constants of (3.2.2) only depend on $c(\cdot)$, k and d. This fact is seen in Section 3.4 by giving explicit values of the implicit constants. For special $c(\cdot)$, the explicit values can be simplified, e.g., the case when $f(x) = x^\alpha$ with $\alpha \in (d, d+1)$. For details, see Remarks 3.4.6 and 3.4.7.

Finally, let us focus on PS(α) with $\alpha \in (1,2)$. Recall that the asymptotic density (1.3.1) is equal to $1/(k-1)$. However, Theorem 1.3.3 does not give us any information about convergence speed. The convergence speed of (1.3.1) is estimated as follows.

Theorem 3.2.3 ([SY21, Theorem 2.4]). *For all $\alpha \in (1,2)$ and all integers $k \geq 3$ and $r \geq 1$,*

$$\begin{aligned}&\frac{1}{N}\#\{n \in [N] : (\lfloor (n+rj)^\alpha \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,1}\} \\ &= \frac{1}{k-1} + O_{\alpha,k,r}(F(N)) \quad (N \to \infty),\end{aligned}$$

where

$$F(x) := \begin{cases} x^{(1-\alpha)/2} & \alpha \in (1, 5/4), \\ x^{(\alpha-3)/14}(\log x)^{1/2} & \alpha \in [5/4, 11/6), \\ x^{(\alpha-2)/6}(\log x)^{1/2} & \alpha \in [11/6, 2). \end{cases}$$

Theorem 3.2.3 gives an upper bound for the convergence speed of (1.3.1). We show an extended statement (Proposition 3.5.1) in Section 3.5, which can be applied to a short interval $[N, N+L)$. Theorem 3.2.3 is derived from the extended statement.

So far, we have stated only asymptotic results. In general, an asymptotic result does not give the information how long an interval containing no numbers n in the set in (1.3.1) is. Hence, we need a non-asymptotic result in order to know such information. To state a non-asymptotic result, let us define the minimum length $L_{\alpha,k,r}(x)$ as

$$L_{\alpha,k,r}(x) = \min\{y \geq 0 : \exists n \in [x, x+y]_{\mathbb{Z}},\ (\lfloor (n+rj)^\alpha \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,1}\}$$

for $\alpha \in (1,2)$, $x \geq 1$, and integers $k \geq 3$ and $r \geq 1$. The following theorem gives an upper bound for $L_{\alpha,k,r}(x)$.

Theorem 3.2.4 ([SY21, Theorem 2.5]). *For all $\alpha \in (1,2)$ and all integers $k \geq 3$ and $r \geq 1$, we have $L_{\alpha,k,r}(x) \ll_{\alpha,k,r} (x^{2-\alpha})$ for all $x \geq 1$.*

At glance, the growth rate $O_{\alpha,k,r}(x^{2-\alpha})$ is strange because it becomes smaller when α increases. However, for all $\alpha \in (1,2)$ and all integers $k \geq 4$ and $r \geq 1$, the growth rate $O_{\alpha,k,r}(x^{2-\alpha})$ is best in a certain meaning. When $k = 3$, we expect that $L_{\alpha,3,r}(x) = O_{\alpha,r}(x^{1-\alpha/2})$ for all $\alpha \in (1,2)$ and $r \in \mathbb{N}$. For details, see Section 3.6.

3.3 Key propositions

By using uniform distribution modulo 1, we obtain two propositions that imply Theorems 3.2.1 and 3.2.2.

Proposition 3.3.1 ([SY21, Proposition 3.3]). *Let $n_0, d \in \mathbb{N}$, and let $f\colon [n_0,\infty) \to \mathbb{R}$ be a $(d+1)$-times differentiable function satisfying that*

(A1) The $(d+1)$-st derivative $f^{(d+1)}(x)$ vanishes as $x \to \infty$;

(A2) $\big((f(n), f'(n), f''(n)/2!, \dots, f^{(d)}(n)/d!)\big)_{n=1}^{\infty}$ is uniformly distributed modulo 1;

(A3) $\inf_{x \geq n_0} f'(x) \geq 1$.

Then, for all integers $k \geq d+2$ and $r \geq 1$, the equality (3.2.1) holds. Also, $\mu(\mathcal{C}_{k,d+1})$ is bounded below by $1/\prod_{i=1}^{d} \binom{k-1}{i}$.

Proposition 3.3.2 ([SY21, Proposition 3.4]). *Let $n_0, d \in \mathbb{N}$, and let $f\colon [n_0,\infty) \to \mathbb{R}$ be a $(d+1)$-times differentiable function satisfying that*

(B1) *The* $(d+1)$*-st derivative* $f^{(d+1)}(x)$ *eventually decreases, and vanishes as* $x \to \infty$*;*

(B2) $\lim_{x\to\infty} x^{d+1} f^{(d+1)}(x) = \infty$*;*

(B3) *For every* $\delta \in (0,1)$*, there exist* $c(\delta) \geq 1$ *and* $x_0(\delta) \geq n_0/\delta$ *such that every* $x \geq x_0(\delta)$ *satisfies* $f^{(d+1)}(\delta x) \leq c(\delta) f^{(d+1)}(x)$*;*

(B4) $(f(n))_{n=n_0}^{\infty}$*,* $(f'(n))_{n=n_0}^{\infty}$*,* $(f''(n)/2!)_{n=n_0}^{\infty}$*, ...,* $(f^{(d)}(n)/d!)_{n=n_0}^{\infty}$ *are uniformly distributed modulo* 1*;*

(B5) $\inf_{x\geq n_0} f'(x) \geq 1$*.*

Then, for every integer $k \geq d+2$*, the equality* (3.2.2) *holds.*

The above propositions do not use the notion of Hardy fields, but uniform distribution modulo 1 is used instead. In general, it is not so easy to investigate uniform distribution modulo 1, but it is easy for $f \in \mathcal{H}$ as stated in Proposition 2.1.2. This is why we have used the notion of Hardy fields in Theorems 3.2.1 and 3.2.2. Propositions 3.3.1 and 3.3.2 are proved in Section 3.4.

Before proving Theorems 3.2.1 and 3.2.2 while assuming Propositions 3.3.1 and 3.3.2, we remark some properties of functions in $\mathcal{H}$ [FW09]:

(H1) Every $f \in \mathcal{H}$ has eventually constant sign;

(H2) Every $f \in \mathcal{H}$ is eventually monotone;

(H3) For every $f \in \mathcal{H}$, the limit $\lim\limits_{x\to\infty} f(x)$ exists as an element of $\mathbb{R} \cup \{\pm\infty\}$;

(H4) If $f \in \mathcal{H}$ and if $g \in \mathcal{LE}$ is eventually non-zero, then $f/g \in \mathcal{H}$;

(H5) For every $f \in \mathcal{H}$ and every $g \in \mathcal{LE}$ that is eventually non-zero, the limit $\lim\limits_{x\to\infty} f(x)/g(x)$ exists as an element of $\mathbb{R} \cup \{\pm\infty\}$;

(H6) If eventually positive $f \in \mathcal{H}$ and $g \in \mathcal{LE}$ satisfy $f(x) \succ g(x)$ (resp. $f(x) \prec g(x)$) and if $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$, then $f'(x) \succ g'(x)$ (resp. $f'(x) \prec g'(x)$).

(H7) If $f \in \mathcal{H}$ is eventually positive, then $\log f(\cdot) \in \mathcal{H}$.

Property (H1) is derived from the fact that $f \in \mathcal{H}$ is eventually zero or has a reciprocal. Property (H2) follows from (H1) by considering the derivative $f' \in \mathcal{H}$. Property (H3) follows from (H2) and the monotone convergence theorem. Property (H5) follows from (H3) and (H4). Property (H6) follows from (H5) and L'Hospital's rule. For (H7), see [Bos81, Theorem 5.3]. The remaining (H4) is verified as follows. The set $\mathcal{LE}$ is a Hardy field by the equivalence relation in Section 3.2 [Har24, Har12], and is contained in every maximal Hardy field (a Hardy field $\mathcal{F}$ is called *maximal* if there are not any Hardy fields strictly containing $\mathcal{F}$) [Bos81, Bos82]. Also, for every Hardy field $\mathcal{F}$, there exists a maximal Hardy field containing $\mathcal{F}$ (use Zorn's lemma). Therefore, for $f \in \mathcal{H}$ and $g \in \mathcal{LE}$ in (H4), the ratio f/g belongs to $\mathcal{H}$.

Proof of Theorems 3.2.1 *and* 3.2.2 *assuming Propositions* 3.3.1 *and* 3.3.2. Let $f\colon [n_0, \infty) \to \mathbb{R}$ be a differentiable function in $\mathcal{H}$ and satisfy (a1) and (a2). All we need is to show (B1)–(B4) and (A2).

Proof of (B1) and (B2). The relation $x^{-1} \prec f^{(d+1)}(x) \prec 1$ follows from (a1) and (H6). Thus, $f^{(d+1)}(x)$ converges to $+0$ as $x \to \infty$. This and (H2) imply (B1). Also, since the relation $f^{(d+1)}(x) \succ x^{-1}$ yields that $xf^{(d+1)}(x)$ diverges to positive infinity as $x \to \infty$, so does $x^{d+1} f^{(d+1)}(x)$.

Proof of (A2) and (B4). Properties (a1) and (H6) imply that $x^{d-i} \log x \prec f^{(i)}(x) \prec x^{d+1-i}$ for all $i \in [0, d]_{\mathbb{Z}}$. This fact and Proposition 2.1.2 imply (A2). Finally, (B4) follows from (A2) immediately.

Proof of (B3). All we need is to show that for every $\delta \in (0, 1)$,

$$\varlimsup_{x\to\infty} \frac{f^{(d+1)}(\delta x)}{f^{(d+1)}(x)} < \infty. \tag{3.3.1}$$

Let $g(x) = 1/f^{(d+1)}(x)$. Instead of (3.3.1), we show that for every $\beta > 1$,

$$\varlimsup_{x\to\infty} \frac{g(\beta x)}{g(x)} < \infty, \tag{3.3.2}$$

which is equivalent to (3.3.1). First, the relation $1 \prec g(x) \prec x$ follows from $x^{-1} \prec f^{(d+1)}(x) \prec 1$, and moreover the function $\log g(\cdot)$ belongs to $\mathcal{H}$ due to (H7). These facts and (H5) imply that the ratio $\log g(x)/\log x$ converges to some finite $\gamma \in [0, 1]$ as $x \to \infty$. Since both $\log g(x)$ and $\log x$ diverge to positive infinity, L'Hospital's rule and (H5) yield that

$$\lim_{x\to\infty} \frac{xg'(x)}{g(x)} = \lim_{x\to\infty} \frac{g'(x)/g(x)}{1/x} = \lim_{x\to\infty} \frac{\log g(x)}{\log x} = \gamma.$$

Thus, there exists $x_0 > 0$ such that $xg'(x) \le (\gamma+1)g(x)$ for all $x \ge x_0$. Also, since the relation $g'(x) \prec 1$ holds due to (H6), the derivative g' is eventually decreasing due to (H2).

Let $\beta > 1$. The mean value theorem implies that $g(\beta x) - g(x) = (\beta - 1)xg'(\beta' x)$ for some $\beta' = \beta'(x) \in (1, \beta)$. Since g' is eventually decreasing, every sufficiently large $x \ge x_0$ satisfies

$$g(\beta x) - g(x) = (\beta - 1)xg'(\beta' x) \le (\beta - 1)xg'(x) \le (\beta - 1)(\gamma + 1)g(x).$$

Therefore, the left-hand side in (3.3.2) is bounded above by $(\beta - 1)(\gamma + 1) + 1$. □

3.4 Proofs of Propositions 3.3.1 and 3.3.2

First, we begin with the proof of Proposition 3.3.1, which is a basis of subsequent proofs.

Proof of Proposition 3.3.1. Without loss of generality, we may assume $n_0 = 1$. Fix integers $k \ge d + 2$ and $r, d \ge 1$. Taylor's theorem implies that for every $n \in \mathbb{N}$ and $j \in [1, k)_{\mathbb{Z}}$ there exists $\theta = \theta(n, j) \in (n, n + rj)$ such that

$$f(n + rj) = \sum_{l=0}^{d} \frac{(rj)^l}{l!} f^{(l)}(n) + \frac{(rj)^{d+1}}{(d+1)!} f^{(d+1)}(n + \theta). \tag{3.4.1}$$

The falling factorials satisfy the formula $x^n = \sum_{i=0}^{n} S(n, i)(x)_i$, where $S(n, i)$, $i \in [0, n]_{\mathbb{Z}}$, denote the Stirling numbers of the second kind. Thus, (3.4.1) can be rewritten as

$$f(n + rj) = \sum_{i=0}^{d} a_i \binom{j}{i} + \frac{(rj)^{d+1}}{(d+1)!} f^{(d+1)}(n + \theta),$$

where

$$a_i = a_i(n) := \sum_{l=i}^{d} \frac{r^l}{l!} f^{(l)}(n) S(l, i) i!. \tag{3.4.2}$$

For convenience, we set $s_0 = 0$ in this proof. For every $\mathbf{s} = (s_i)_{i=1}^{d} \in \mathbb{Z}^d$, $n \in \mathbb{N}$ and $j \in [1, k)_{\mathbb{Z}}$, we have

$$f(n + rj) = \sum_{i=0}^{d} (\lfloor a_i \rfloor - s_i) \binom{j}{i} + \delta_{\mathbf{s}}, \tag{3.4.3}$$

where

$$\delta_{\mathbf{s}} = \delta_{\mathbf{s}}(n,j) \coloneqq \sum_{i=0}^{d} (\{a_i\} + s_i) \binom{j}{i} + \frac{(rj)^{d+1}}{(d+1)!} f^{(d+1)}(n+\theta).$$

Let $\varepsilon \in (0, 1/2)$ be arbitrary. Thanks to (A1), we can take $x_0 \geq 1$ such that every $x \geq x_0$ satisfies

$$\frac{(r(k-1))^{d+1}}{(d+1)!} \left| f^{(d+1)}(x) \right| \leq \varepsilon.$$

Now, let us show that

$$\varliminf_{N\to\infty} \frac{1}{N} \#\{n \in [1,N]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} \geq \mu(\mathcal{C}_{k,d+1}^{-}(\varepsilon)), \quad (3.4.4)$$

where the convex set $\mathcal{C}_{k,d+1}^{-}(\varepsilon)$ is defined as

$$\mathcal{C}_{k,d+1}^{-}(\varepsilon) = \left\{ (y_i)_{i=0}^{d} \in \mathbb{R}^{d+1} : 0 \leq y_0 < 1,\ \varepsilon \leq \sum_{i=0}^{d} \binom{j}{i} y_i < 1-\varepsilon \ (\forall j \in [1,k)_{\mathbb{Z}}) \right\}. \quad (3.4.5)$$

If the relations $\mathbf{s} = (s_i)_{i=1}^{d} \in \mathbb{Z}^d$, $n \geq x_0$, and $(\{a_i(n)\} + s_i)_{i=0}^{d} \in \mathcal{C}_{k,d+1}^{-}(\varepsilon)$ hold, then $0 \leq \delta_{\mathbf{s}}(n,j) < 1$ and

$$\lfloor f(n+rj) \rfloor = \sum_{i=0}^{d} (\lfloor a_i(n) \rfloor - s_i) \binom{j}{i}$$

for all $j \in [1,k)_{\mathbb{Z}}$. This implies the inclusion relation

$$\bigcup_{s_1,\ldots,s_d \in \mathbb{Z}} \{n \in [x_0,\infty)_{\mathbb{Z}} : (\{a_i(n)\} + s_i)_{i=0}^{d} \in \mathcal{C}_{k,d+1}^{-}(\varepsilon)\} \quad (3.4.6)$$
$$\subset \{n \in \mathbb{N} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\}.$$

The union (3.4.6) is disjoint because

1. the vectors $(\binom{j}{i})_{i=1}^{d} \in \mathbb{R}^d$, $j \in [1,k)_{\mathbb{Z}}$, span $\mathbb{R}^d$;

2. thus, if $(s_i)_{i=1}^{d}, (s_i')_{i=1}^{d} \in \mathbb{Z}^d$ are not equal to each other, then $\sum_{i=1}^{d} \binom{j}{i}(s_i - s_i')$ is a non-zero integer for some $j \in [1,k)_{\mathbb{Z}}$.

Also, the vectors $\mathbf{a}(n) \coloneqq (a_0(n), a_1(n), \ldots, a_d(n))$, $n \in \mathbb{N}$, can be expressed as

$$\mathbf{a}(n) = (f(n), f'(n), f''(n)/2!, \ldots, f^{(d)}(n)/d!)\mathbf{A}$$

by using the integer matrix $\mathbf{A} = (a_{ij})_{0\le i,j\le d}$ whose entry a_{ij} is equal to $r^i S(i,j)j!$ if $i \ge j$, and zero if $i < j$. Note that $\mathbf{A}$ has full rank. Since $(\mathbf{a}(n))_{n=1}^{\infty}$ is uniformly distributed modulo 1 thanks to (A2), it turns out that

$$\begin{aligned}
&\varliminf_{N\to\infty} \frac{1}{N}\#\{n \in [1,N]_{\mathbb{Z}} : (\lfloor f(n+rj)\rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} \\
&\ge \varliminf_{N\to\infty} \sum_{s_1,\ldots,s_d\in\mathbb{Z}} \frac{1}{N}\#\{n \in [x_0,N]_{\mathbb{Z}} : (\{a_i(n)\} + s_i)_{i=0}^{d} \in \mathcal{C}_{k,d+1}^{-}(\varepsilon)\} \\
&= \sum_{s_1,\ldots,s_d\in\mathbb{Z}} \mu\Big(\mathcal{C}_{k,d+1}^{-}(\varepsilon) \cap \prod_{i=0}^{d}[s_i, s_i+1)\Big) = \mu(\mathcal{C}_{k,d+1}^{-}(\varepsilon)),
\end{aligned} \tag{3.4.7}$$

where all the sums in (3.4.7) are finite sums because of the boundedness of $\mathcal{C}_{k,d+1}^{-}(\varepsilon)$. Therefore, (3.4.4) holds.

Next, let us show that

$$\varlimsup_{N\to\infty} \frac{1}{N}\#\{n \in [1,N]_{\mathbb{Z}} : (\lfloor f(n+rj)\rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} \le \mu(\mathcal{C}_{k,d+1}^{+}(\varepsilon)), \tag{3.4.8}$$

where the convex set $\mathcal{C}_{k,d+1}^{+}(\varepsilon)$ is defined as

$$\mathcal{C}_{k,d+1}^{+}(\varepsilon) = \left\{(y_i)_{i=0}^{d} \in \mathbb{R}^{d+1} : 0 \le y_0 < 1,\ -\varepsilon \le \sum_{i=0}^{d} \binom{j}{i} y_i < 1+\varepsilon\ (\forall j \in [1,k)_{\mathbb{Z}})\right\}. \tag{3.4.9}$$

Take an arbitrary integer $m \ge x_0$ such that $(\Delta_r^d \lfloor f(m+rj)\rfloor)_{j=0}^{k-d-1}$ is a constant sequence. Then the sequence $(\lfloor f(m+rj)\rfloor)_{j=0}^{k-1}$ is expressed as

$$\lfloor f(m+rj)\rfloor = \sum_{i=0}^{d} \Delta_r^i \lfloor f(m)\rfloor \cdot \binom{j}{i}$$

due to Newton's forward difference formula. Recalling the definition of $a_i(m)$ and putting $s_i = \lfloor a_i(m)\rfloor - \Delta_r^i \lfloor f(m)\rfloor$ for $i \in [1,d]_{\mathbb{Z}}$, we have that $\lfloor a_0(m)\rfloor = \lfloor f(m)\rfloor$ and

$$\lfloor f(m+rj)\rfloor = \sum_{i=0}^{d} (\lfloor a_i(m)\rfloor - s_i)\binom{j}{i} \quad (\forall j \in [0,k)_{\mathbb{Z}}).$$

This and (3.4.3) imply that $f(m+rj) = \lfloor f(m+rj) \rfloor + \delta_{\mathbf{s}}(m,j)$ and $0 \le \delta_{\mathbf{s}}(m,j) < 1$ for all $j \in [1,k)_{\mathbb{Z}}$, whence $(\{a_i(m)\} + s_i)_{i=0}^d \in \mathcal{C}_{k,d+1}^+(\varepsilon)$. Therefore, we obtain the inclusion relation

$$\begin{aligned} &\{n \in [x_0, \infty)_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} \\ \subset &\bigcup_{s_1,\dots,s_d \in \mathbb{Z}} \{n \in \mathbb{N} : (\{a_i(n)\} + s_i)_{i=0}^d \in \mathcal{C}_{k,d+1}^+(\varepsilon)\}. \end{aligned}$$

Since $(\mathbf{a}(n))_{n=1}^\infty$ is uniformly distributed modulo 1 thanks to (A2), it turns out that

$$\begin{aligned} &\overline{\lim_{N\to\infty}} \frac{1}{N} \#\{n \in [1,N]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} \\ &= \overline{\lim_{N\to\infty}} \frac{1}{N} \#\{n \in [x_0,N]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} \\ &\le \overline{\lim_{N\to\infty}} \sum_{s_1,\dots,s_d \in \mathbb{Z}} \frac{1}{N} \#\{n \in [1,N]_{\mathbb{Z}} : (\{a_i(n)\} + s_i)_{i=0}^d \in \mathcal{C}_{k,d+1}^+(\varepsilon)\} \qquad (3.4.10) \\ &= \sum_{s_1,\dots,s_d \in \mathbb{Z}} \mu\Big(\mathcal{C}_{k,d+1}^+(\varepsilon) \cap \prod_{i=0}^d [s_i, s_i+1)\Big) = \mu(\mathcal{C}_{k,d+1}^+(\varepsilon)), \end{aligned}$$

which is just (3.4.8).

Finally, once letting $\varepsilon \to +0$ in (3.4.4) and (3.4.8), we conclude the limit in Proposition 3.3.1. Also, the inequality $\mu(\mathcal{C}_{k,d+1}) \ge 1/\prod_{i=1}^d \binom{k-1}{i}$ is derived from the lemma below. □

Lemma 3.4.1 ([SY21, Lemma 4.1])**.** *Let $k \ge d+2$ and $d \ge 1$ be integers. Then $\mu(\mathcal{C}_{k,d+1}) \ge 1/\prod_{i=1}^d \binom{k-1}{i}$.*

Proof. Define the convex set $\mathcal{C}'_{k,d+1}$ as

$$\mathcal{C}'_{k,d+1} = \left\{ (y_0, y_1, \dots, y_d) \in \mathbb{R}^{d+1} : 0 \le y_0 < 1,\ 0 \le \sum_{i=0}^j \binom{k-1}{i} y_i < 1\ (\forall j \in [1,d]_{\mathbb{Z}}) \right\}.$$

We show the inclusion relation $\mathcal{C}'_{k,d+1} \subset \mathcal{C}_{k,d+1}$. Let $(y_0, y_1, \dots, y_d) \in \mathcal{C}'_{k,d+1}$ and $j \in [1,k)_{\mathbb{Z}}$. Set the real numbers $c_0, c_1, \dots, c_d \ge 0$ as

$$c_l = \begin{cases} \binom{j}{l}\binom{k-1}{l}^{-1} - \binom{j}{l+1}\binom{k-1}{l+1}^{-1} & l \in [0,d)_{\mathbb{Z}}, \\ \binom{j}{d}\binom{k-1}{d}^{-1} & l = d. \end{cases}$$

Then the inequality $0 \leq \sum_{i=0}^{d} \binom{j}{i} y_i < 1$ in the definition of $\mathcal{C}_{k,d+1}$ is equal to the sum of the inequalities $0 \leq \sum_{i=0}^{l} \binom{k-1}{i} y_i < 1$, $l \in [0,d]_{\mathbb{Z}}$, multiplied by c_l:

$$\sum_{l=0}^{d} c_l \sum_{i=0}^{l} \binom{k-1}{i} y_i = \sum_{i=0}^{d} \binom{k-1}{i} y_i \sum_{l=i}^{d} c_l$$
$$= \sum_{i=0}^{d} \binom{k-1}{i} y_i \binom{j}{i} \binom{k-1}{i}^{-1} = \sum_{i=0}^{d} \binom{j}{i} y_i.$$

Since $j \in [1,k)_{\mathbb{Z}}$ is arbitrary, the point $(y_0, y_1, \ldots, y_d)$ lies in $\mathcal{C}_{k,d+1}$. Therefore, $\mathcal{C}'_{k,d+1} \subset \mathcal{C}_{k,d+1}$. Finally, we conclude that $\mu(\mathcal{C}_{k,d+1}) \geq \mu(\mathcal{C}'_{k,d+1}) = 1/\prod_{i=1}^{d} \binom{k-1}{i}$ by easy calculation. □

Remark 3.4.2 ([SY21, Remark 4.2])**.** Let $f(x) = x \log x$. Then the sequence $\big((f(n), f'(n))\big)_{n=1}^{\infty}$ is not uniformly distributed modulo 1 because $(f'(n))_{n=1}^{\infty}$ does not satisfy the second condition in Proposition 2.1.2. However, one can show that for every convex set $\mathcal{C} \subset [0,1)^2$ and every $r \in \mathbb{N}$,

$$\frac{1}{N} \#\{n \in [1,N]_{\mathbb{Z}} : (\{f(n)\}, \{rf'(n)\}) \in \mathcal{C}\}$$
$$\to \iint_{\mathcal{C}} \Big(\mathbf{1}_{\leq \{r \log N\}}(y) + \frac{1}{e^{1/r}-1}\Big) \frac{e^{(y-\{r\log N\})/r}}{r} \, dxdy \quad (N \to \infty),$$

where $\mathbf{1}_{\leq c}(y) = 1$ if $y \leq c$, and $\mathbf{1}_{\leq c}(y) = 0$ if $y > c$. This implies that for every convex set $\mathcal{C} \subset [0,1)^2$ and every $r \in \mathbb{N}$,

$$\frac{1}{(e^{1/r}-1)r} \mu(\mathcal{C}) \leq \varliminf_{N\to\infty} \frac{1}{N} \#\{n \in [1,N]_{\mathbb{Z}} : (\{f(n)\}, \{rf'(n)\}) \in \mathcal{C}\}$$
$$\leq \varlimsup_{N\to\infty} \frac{1}{N} \#\{n \in [1,N]_{\mathbb{Z}} : (\{f(n)\}, \{rf'(n)\}) \in \mathcal{C}\} \leq \frac{e^{1/r}}{(e^{1/r}-1)r} \mu(\mathcal{C}).$$

Since the equalities $a_0(n) = f(n)$ and $a_1(n) = rf'(n)$ hold due to (3.4.2) with $d = 1$, it follows that for all integers $k \geq 3$ and $r \geq 1$,

$$\frac{1}{(e^{1/r}-1)r(k-1)} \leq \varliminf_{N\to\infty} \frac{1}{N} \#\{n \in [1,N]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,1}\}$$
$$\leq \varlimsup_{N\to\infty} \frac{1}{N} \#\{n \in [1,N]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,1}\} \leq \frac{e^{1/r}}{(e^{1/r}-1)r(k-1)}$$

in the same way as the proof of Proposition 3.3.1. The above both-hand sides converge to $1/(k-1)$ as $r \to \infty$.

Next, to prove Proposition 3.3.2, we need to evaluate exponential sums $\sum_{r=1}^{R} \mathrm{e}(p(r))$ for polynomials $p(x)$. Such an evaluation is achieved by induction on the degree of $p(x)$. The following lemma is often used to make the degree of a polynomial decrease.

Lemma 3.4.3. *Let $z_1, z_2, \dots, z_N \in \mathbb{C}$ and $H \in [1, N]_{\mathbb{Z}}$. Then*

$$\left|\sum_{n=1}^{N} z_n\right|^2 \le \frac{N+H-1}{H^2}\Big(H\sum_{n=1}^{N}|z_n|^2 + 2\sum_{h=1}^{H-1}(H-h)\Re\sum_{n=1}^{N-h} z_{n+h}\overline{z}_n\Big).$$

Proof. See [KN74, Lemma 3.1]. □

Lemma 3.4.4 ([SY21, Lemma 4.4])**.** *Let $N_m, R_m \in \mathbb{N}$ diverge to positive infinity as $m \to \infty$, and $d \ge 0$ be an integer. For $n \in \mathbb{N}$, let $q_n(x)$ be a polynomial of degree less than d; let $c_n \in \mathbb{R}$ and $p_n(x) = c_n x^d + q_n(x)$. If $(c_n)_{n=1}^{\infty}$ is uniformly distributed modulo 1, then*

$$\lim_{m\to\infty} \frac{1}{N_m R_m}\sum_{n=1}^{N_m}\sum_{r=1}^{R_m} e(p_n(r)) = 0.$$

Proof. We show the desired statement by induction on d. First, assume $d = 0$. Then $p_n(x) = c_n$ for all $n \in \mathbb{N}$, and thus the uniform distribution modulo 1 of $(c_n)_{n=1}^{\infty}$ implies that

$$\lim_{m\to\infty} \frac{1}{N_m R_m}\sum_{n=1}^{N_m}\sum_{r=1}^{R_m} e(p_n(r)) = \lim_{m\to\infty}\frac{1}{N_m}\sum_{n=1}^{N_m} e(c_n) = 0.$$

Next, assuming that the desired statement is true for $d-1$ with $d \ge 1$, we show that the desired statement is also true for d. Take an arbitrary $H \in \mathbb{N}$. Lemma 3.4.3 yields the inequality

$$\left|\sum_{r=1}^{R_m} e(p_n(r))\right|^2 \le \frac{R_m+H-1}{H^2}\Big(HR_m + 2\sum_{h=1}^{H-1}(H-h)\Re\sum_{r=1}^{R_m-h} e(\Delta_h p_n(r))\Big).$$

The above and Cauchy-Schwarz inequalities imply that

$$\begin{aligned}
&\left|\frac{1}{N_m R_m}\sum_{n=1}^{N_m}\sum_{r=1}^{R_m} e(p_n(r))\right|^2 \le \frac{1}{N_m R_m^2}\sum_{n=1}^{N_m}\left|\sum_{r=1}^{R_m} e(p_n(r))\right|^2\\
&\le \frac{R_m+H-1}{H^2 N_m R_m^2}\left(HN_mR_m + 2\sum_{h=1}^{H-1}(H-h)\Re\sum_{n=1}^{N_m}\sum_{r=1}^{R_m-h} e(\Delta_h p_n(r))\right)\\
&\le \frac{R_m+H}{H^2 N_m R_m^2}\left(HN_mR_m + 2H\sum_{h=1}^{H-1}\left|\sum_{n=1}^{N_m}\sum_{r=1}^{R_m-h} e(\Delta_h p_n(r))\right|\right)\\
&\le \frac{R_m+H}{HR_m}\left(1 + 2\sum_{h=1}^{H-1}\left|\frac{1}{N_m R_m}\sum_{n=1}^{N_m}\sum_{r=1}^{R_m-h} e(\Delta_h p_n(r))\right|\right).
\end{aligned} \tag{3.4.11}$$

Now, for all $h, n \in \mathbb{N}$, the polynomial $\Delta_h p_n(x)$ is expressed as $dhc_n x^{d-1} + q_{h,n}(x)$, where the degree of $q_{h,n}(x)$ is less than $d-1$. Since $(c_n)_{n=1}^\infty$ is uniformly distributed modulo 1, so is $(dhc_n)_{n=1}^\infty$ for every $h \in \mathbb{N}$. Thus, the hypothesis by induction implies that for every $h \in \mathbb{N}$

$$\lim_{m\to\infty}\frac{1}{N_m R_m}\sum_{n=1}^{N_m}\sum_{r=1}^{R_m-h} e(\Delta_h p_n(r)) = 0.$$

It follows from (3.4.11) that

$$\overline{\lim_{m\to\infty}}\left|\frac{1}{N_m R_m}\sum_{n=1}^{N_m}\sum_{r=1}^{R_m} e(p_n(r))\right|^2 \le 1/H.$$

Due to the arbitrariness of H, we find that the desired statement is true for d. □

Lemma 3.4.5 ([SY21, Lemma 4.5]). *Let $d \ge 0$ be an integer and $\mathbf{A}$ be an integer matrix of order $d+1$ and rank $d+1$; let $\mathbf{x}(n) = (x_0(n), x_1(n), \ldots, x_d(n)) \in \mathbb{R}^{d+1}$ and*

$$\mathbf{y}(n,r) = (y_0(n,r), y_1(n,r), \ldots, y_d(n,r)) = (x_0(n), rx_1(n), \ldots, r^d x_d(n))\mathbf{A}$$

for $n, r \in \mathbb{N}$. If $N_m, R_m \in \mathbb{N}$ diverge to positive infinity as $m \to \infty$ and if each entry $(x_i(n))_{n=1}^\infty$ of $(\mathbf{x}(n))_{n=1}^\infty$ is uniformly distributed modulo 1, then *for every convex set* $\mathcal{C} \subset [0,1)^{d+1}$,

$$\lim_{m\to\infty} \frac{\#\big\{(n,r) \in [1,N_m]_{\mathbb{Z}} \times [1,R_m]_{\mathbb{Z}} : \{\mathbf{y}(n,r)\} \in \mathcal{C}\big\}}{N_m R_m} = \mu(\mathcal{C}),$$

where μ *denotes the Lebesgue measure on* $\mathbb{R}^{d+1}$.

Proof. If the following criterion holds, Lemma 3.4.5 follows in the same way as Weyl's theorem on uniform distribution. *Weyl's criterion: for every non-zero* $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$,

$$\lim_{m\to\infty} \frac{1}{N_m R_m} \sum_{n=1}^{N_m} \sum_{r=1}^{R_m} e(\langle \mathbf{y}(n,r), \mathbf{h}\rangle) = 0. \tag{3.4.12}$$

Hence, taking a non-zero $\mathbf{h} \in \mathbb{Z}^{d+1}$ arbitrarily, we show (3.4.12). For $i,j \in [0,d]_{\mathbb{Z}}$, denote the (i,j)-th entry of $\mathbf{A}$ by a_{ij}. For $n \in \mathbb{N}$, regard $\langle \mathbf{y}(n,r), \mathbf{h}\rangle$ as a polynomial $p_n(r)$ of r:

$$p_n(r) = \langle \mathbf{y}(n,r), \mathbf{h}\rangle = \sum_{j=0}^{d} y_j(n,r) h_j = \sum_{i=0}^{d} r^i x_i(n) \sum_{j=0}^{d} a_{ij} h_j.$$

Take the maximum number i_0 of all $i \in [0,d]_{\mathbb{Z}}$ such that $\sum_{j=0}^{d} a_{ij} h_j$ is not zero (such a number i exists because the square matrix $\mathbf{A}$ has full rank). Then, for every $n \in \mathbb{N}$, the degree of $p_n(x)$ is at most i_0. Since $(x_{i_0}(n))_{n=1}^{\infty}$ is uniformly distributed modulo 1, so is the sequence $(x_{i_0}(n)\sum_{j=0}^{d} a_{i_0 j} h_j)_{n=1}^{\infty}$. Therefore, Lemma 3.4.4 implies (3.4.12), and we obtain Lemma 3.4.5. □

Now, let us show Proposition 3.3.2. Since (3.2.2) consists of the following inequalities:

$$\text{(liminf)} \qquad \varliminf_{N\to\infty} \frac{\#\{P \subset [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor f(n) \rfloor)_{n\in P} \in \mathcal{P}_{k,d}\}}{N f''(N)^{-1/(d+1)}} > 0,$$

$$\text{(limsup)} \qquad \varlimsup_{N\to\infty} \frac{\#\{P \subset [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor f(n) \rfloor)_{n\in P} \in \mathcal{P}_{k,d}\}}{N f''(N)^{-1/(d+1)}} < \infty,$$

we prove the above inequalities. Also, note that $f^{(d+1)}(x) > 0$ for every sufficiently large $x > 0$ because of (B2).

Proof of Proposition 3.3.2 *(liminf).* Without loss of generality, we may assume $n_0 = 1$. Fix integers $k \geq d+2$ and $d \geq 1$, and let $N \in \mathbb{N}$ be sufficiently large. Take arbitrary $\varepsilon \in (0,1)$ and $0 < \delta_1 < \cdots < \delta_t < \delta_{t+1} = 1$. Put

$$R_i = R_i(N) = \left\lfloor \frac{(\varepsilon(d+1)!)^{1/(d+1)}}{k-1} f^{(d+1)}(\delta_i N)^{-1/(d+1)} \right\rfloor$$

for $i \in [t]$ and $N \in \mathbb{N}$. Then all $x \geq \delta_i N$ and $r \in [1, R_i]_{\mathbb{Z}}$ satisfy

$$0 < \frac{(r(k-1))^{d+1}}{(d+1)!} f^{(d+1)}(x) \overset{\text{(B1)}}{\leq} \frac{(R_i(k-1))^{d+1}}{(d+1)!} f^{(d+1)}(\delta_i N) \leq \varepsilon. \tag{3.4.13}$$

Now, the following inequality holds:

$$\begin{aligned} &\#\{P \subset [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor f(n) \rfloor)_{n \in P} \in \mathcal{P}_{k,d}\} \\ &\geq \#\{(n,r) \in [1, N-(k-1)R_t]_{\mathbb{Z}} \times [1, R_t]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} \\ &\geq \#\{(n,r) \in [1, N]_{\mathbb{Z}} \times [1, R_t]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} - (k-1)R_t^2 \\ &\overset{(a)}{\geq} \sum_{i=1}^{t} M_i(N) - (k-1)R_t^2, \end{aligned} \tag{3.4.14}$$

where for $i \in [1,t]_{\mathbb{Z}}$ and $N \in \mathbb{N}$ the value $M_i(N)$ is defined as

$$M_i(N) = \#\{(n,r) \in (\delta_i N, \delta_{i+1} N]_{\mathbb{Z}} \times [1, R_i]_{\mathbb{Z}} : (\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\},$$

and the monotonicity $R_1 \leq R_2 \leq \cdots \leq R_t$ is used to obtain (a). For $n, r \in \mathbb{N}$ and $i \in [0,d]_{\mathbb{Z}}$, define the real number $a_i = a_i(n,r)$ as the right-hand side in (3.4.2). Then the vectors $\mathbf{a}(n,r) \coloneqq (a_0(n,r), a_1(n,r), \dots, a_d(n,r))$, $n, r \in \mathbb{N}$, can be expressed as

$$\mathbf{a}(n,r) = (f(n), rf'(n), r^2 f''(n)/2!, \dots, r^d f^{(d)}(n)/d!)\mathbf{A},$$

where the integer matrix $\mathbf{A} = (a_{ij})_{0 \leq i,j \leq d}$ is defined as $a_{ij} = S(i,j)j!$ if $i \geq j$, and $a_{ij} = 0$ if $i < j$. Also, define the convex set $\mathcal{C}_{k,d+1}^{-}(\varepsilon)$ as

$$\mathcal{C}_{k,d+1}^{-}(\varepsilon) = \left\{ (y_i)_{i=0}^{d} \in \mathbb{R}^{d+1} : 0 \leq y_0 < 1,\ 0 \leq \sum_{i=0}^{d} \binom{j}{i} y_i < 1 - \varepsilon\ (\forall j \in [1,k)_{\mathbb{Z}}) \right\}.$$

Due to (3.4.13), the same argument as the proof of Proposition 3.3.1 implies that if integers $n \geq \delta_i N$ and $r \in [1, R_i]_{\mathbb{Z}}$ and a vector $(s_j)_{j=1}^{d} \in \mathbb{Z}^d$ satisfy

$(\{a_j(n,r)\}+s_j)_{j=0}^{d} \in \mathcal{C}_{k,d+1}^{-}(\varepsilon)$, then $(\lfloor f(n+rj)\rfloor)_{j=0}^{k-1}$ belongs to $\mathcal{P}_{k,d}$, where $s_0 \coloneqq 0$. Thus,

$$\begin{aligned}
&\frac{\#\{P\subset[1,N]_{\mathbb{Z}} : P\in\mathcal{P}_{k,1},\ (\lfloor f(n)\rfloor)_{n\in P}\in\mathcal{P}_{k,d}\}}{Nf^{(d+1)}(N)^{-1/(d+1)}}\\
&\overset{(3.4.14)}{\geq} \sum_{i=1}^{t}\frac{M_i(N)}{Nf^{(d+1)}(N)^{-1/(d+1)}}-\frac{(k-1)R_t^2}{Nf^{(d+1)}(N)^{-1/(d+1)}}\\
&\geq \sum_{i=1}^{t}\sum_{s_1,\dots,s_d\in\mathbb{Z}}\frac{M_i'(\delta_i,\delta_{i+1},N)}{Nf^{(d+1)}(N)^{-1/(d+1)}}-\frac{(k-1)R_t^2}{Nf^{(d+1)}(N)^{-1/(d+1)}}, \qquad (3.4.15)
\end{aligned}$$

where for $i\in[1,t]_{\mathbb{Z}}$, $N\in\mathbb{N}$ and $y>x\geq 0$ the value $M_i'(x,y,N)$ is defined as

$$M_i'(x,y,N)=\#\{(n,r)\in(xN,yN]_{\mathbb{Z}}\times[1,R_i]_{\mathbb{Z}} : (\{a_j(n,r)\}+s_j)_{j=0}^{d}\in\mathcal{C}_{k,d+1}^{-}(\varepsilon)\}.$$

The absolute value of the second term of (3.4.15) is bounded above by

$$\begin{aligned}
&\frac{(k-1)R_t^2}{Nf^{(d+1)}(N)^{-1/(d+1)}} \leq \frac{k-1}{Nf^{(d+1)}(N)^{-1/(d+1)}}\cdot\frac{(\varepsilon(d+1)!)^{2/(d+1)}}{(k-1)^2}f^{(d+1)}(\delta_t N)^{-2/(d+1)}\\
&\overset{(B1)}{\leq} \frac{f^{(d+1)}(N)^{-2/(d+1)}}{Nf^{(d+1)}(N)^{-1/(d+1)}}\cdot\frac{(\varepsilon(d+1)!)^{2/(d+1)}}{k-1} \leq \frac{1}{Nf^{(d+1)}(N)^{1/(d+1)}}\cdot\frac{(d+1)!}{k-1}\xrightarrow[(B2)]{N\to\infty}0.
\end{aligned}$$

Also, the following inequality holds:

$$\begin{aligned}
&\frac{M_i'(\delta_i,\delta_{i+1},N)}{Nf^{(d+1)}(N)^{-1/(d+1)}} \overset{(B3)}{\geq} \frac{M_i'(\delta_i,\delta_{i+1},N)}{c(\delta_i)^{1/(d+1)}Nf^{(d+1)}(\delta_i N)^{-1/(d+1)}}\\
&=\frac{M_i'(0,\delta_{i+1},N)-M_i'(0,\delta_i,N)}{c(\delta_i)^{1/(d+1)}Nf^{(d+1)}(\delta_i N)^{-1/(d+1)}}\\
&=\frac{M_i'(0,\delta_{i+1},N)}{\delta_{i+1}NR_i}\cdot\frac{c(\delta_i)^{-1/(d+1)}\delta_{i+1}R_i}{f^{(d+1)}(\delta_i N)^{-1/(d+1)}}-\frac{M_i'(0,\delta_i,N)}{\delta_i NR_i}\cdot\frac{c(\delta_i)^{-1/(d+1)}\delta_i R_i}{f^{(d+1)}(\delta_i N)^{-1/(d+1)}}.
\end{aligned}$$

Once taking the limit $N\to\infty$ in the above inequality, Lemma 3.4.5 implies that

$$\begin{aligned}
&\varliminf_{N\to\infty}\frac{M_i'(\delta_i,\delta_{i+1},N)}{Nf^{(d+1)}(N)^{-1/(d+1)}}\\
&\geq \mu\Big(\mathcal{C}_{k,d+1}^{-}(\varepsilon)\cap\prod_{j=0}^{d}[s_j,s_j+1)\Big)\cdot\frac{(\varepsilon(d+1)!)^{1/(d+1)}c(\delta_i)^{-1/(d+1)}}{k-1}(\delta_{i+1}-\delta_i).
\end{aligned}$$

Therefore, letting $N \to \infty$ in (3.4.15), we obtain

$$
\begin{aligned}
&\varliminf_{N\to\infty} \frac{\#\{P \subset [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor f(n) \rfloor)_{n\in P} \in \mathcal{P}_{k,d}\}}{N f^{(d+1)}(N)^{-1/(d+1)}} \\
&\geq \sum_{i=1}^{t} \sum_{s_1,\dots,s_d\in\mathbb{Z}} \mu\Big(\mathcal{C}_{k,d+1}^{-}(\varepsilon) \cap \prod_{j=0}^{d}[s_j, s_j+1)\Big) \cdot \frac{(\varepsilon(d+1)!)^{1/(d+1)} c(\delta_i)^{-1/(d+1)}}{k-1}(\delta_{i+1}-\delta_i) \\
&= \mu(\mathcal{C}_{k,d+1}^{-}(\varepsilon))(\varepsilon(d+1)!)^{1/(d+1)} \sum_{i=1}^{t} \frac{c(\delta_i)^{-1/(d+1)}}{k-1}(\delta_{i+1}-\delta_i) > 0,
\end{aligned}
$$

where the last inequality is derived from $\mu(\mathcal{C}_{k,d+1}^{-}(\varepsilon)) \geq (1-\varepsilon)^{d+1}/\prod_{i=1}^{d}\binom{k-1}{i} > 0$ (see Lemma 3.4.1). □

Remark 3.4.6 ([SY21, Remark 4.6])**.** Let us consider the special case $f(x) = x^{\alpha}$ with $\alpha \in (d, d+1)$. Then we can take $c(\delta)$ in (B3) as $c(\delta) = \delta^{\alpha-d-1}$. Thus,

$$
\begin{aligned}
&\varliminf_{N\to\infty} \frac{\#\{P \subseteq [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor n^{\alpha} \rfloor)_{n\in P} \in \mathcal{P}_{k,d}\}}{((\alpha)_{d+1})^{-1/(d+1)} N^{2-\alpha/(d+1)}} \\
&\geq \mu(\mathcal{C}_{k,d+1}^{-}(\varepsilon))(\varepsilon(d+1)!)^{1/(d+1)} \sum_{i=1}^{t} \delta_i^{1-\alpha/(d+1)}(\delta_{i+1}-\delta_i).
\end{aligned}
$$

Since $\varepsilon \in (0,1)$ and $0 < \delta_1 < \cdots < \delta_t < \delta_{t+1} = 1$ are arbitrary, we obtain

$$
\begin{aligned}
&\varliminf_{N\to\infty} \frac{\#\{P \subseteq [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor n^{\alpha} \rfloor)_{n\in P} \in \mathcal{P}_{k,d}\}}{N^{2-\alpha/(d+1)}} \\
&\geq C_{k,d}\Big(\frac{(d+1)!}{(\alpha)_{d+1}}\Big)^{1/(d+1)} \int_0^1 x^{1-\alpha/(d+1)}\,dx \\
&= C_{k,d}\Big(\frac{(d+1)!}{(\alpha)_{d+1}}\Big)^{1/(d+1)} \frac{1}{2-\alpha/(d+1)} =: \tilde{A}_{\alpha,k},
\end{aligned}
$$

where

$$
C_{k,d} = \sup_{0<x<1} \mu(\mathcal{C}_{k,d+1}^{-}(x)) x^{1/(d+1)} \geq \frac{\sup_{0<x<1}(1-x)^{d+1} x^{1/(d+1)}}{\prod_{i=1}^{d}\binom{k-1}{i}}
$$

because the inequality $\mu(\mathcal{C}_{k,d+1}^{-}(x)) \geq (1-x)^{d+1}/\prod_{i=1}^{d}\binom{k-1}{i}$ is derived from Lemma 3.4.1. Therefore, the constant $A_{\alpha,k}$ in Theorem 3.1.2 is an arbitrary value in the interval $(0, \tilde{A}_{\alpha,k})$.

Proof of Proposition 3.3.2 *(limsup).* Without loss of generality, we may assume $n_0 = 1$. Fix integers $k \geq d+2$ and $d \geq 1$, and take an arbitrary $\beta > 1$. Due to (B2), we can take an integer $N_0 \geq x_0(1/\beta)$ such that every $x \geq N_0$ satisfies that $f^{(d+1)}(x) > 0$ and $1 + (k-1)R(x)/x < \beta$, where

$$R(x) \coloneqq \Big(\frac{2^d c(1/\beta)}{k-d-1}\Big)^{1/(d+1)} f^{(d+1)}(x)^{-1/(d+1)}.$$

First, we show that if integers $n \geq N_0$ and $r \geq 1$ satisfy $(\lfloor f(n+rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}$, then $r < R(n)$ by contradiction. Suppose that integers $m_0 \geq N_0$ and $r_0 \geq 1$ satisfied that $(\lfloor f(m_0 + r_0 j) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}$ and $r_0 \geq R_0 \coloneqq R(m_0)$. The derivative of the function

$$\begin{aligned} g(x) &\coloneqq \Delta_x^d f(m_0 + (k-d-1)x) - \Delta_x^d f(m_0) \\ &= \sum_{i=0}^{d} \binom{d}{i} (-1)^i f(m_0 + (k-1-i)x) - \sum_{i=0}^{d} \binom{d}{i} (-1)^i f(m_0 + (d-i)x) \end{aligned}$$

is equal to

$$\begin{aligned} g'(x) = &\sum_{i=0}^{d} \binom{d}{i} (-1)^i (k-1-i) f'(m_0 + (k-1-i)x) \\ &- \sum_{i=0}^{d} \binom{d}{i} (-1)^i (d-i) f'(m_0 + (d-i)x) \\ = &\,(k-1)\Delta_x^d f'(m_0 + (k-d-1)x) + \sum_{i=0}^{d} \binom{d}{i} (-1)^{i+1} i f'(m_0 + (k-1-i)x) \\ &- d\Delta_x^d f'(m_0) - \sum_{i=0}^{d} \binom{d}{i} (-1)^{i+1} i f'(m_0 + (d-i)x). \end{aligned}$$

Using the equality $\binom{d}{i}i = d\binom{d-1}{i-1}$, we have

$$\begin{aligned}
g'(x) &= (k-1)\Delta_x^d f'(m_0 + (k-d-1)x) + d\sum_{i=1}^{d}\binom{d-1}{i-1}(-1)^{i+1}f'(m_0 + (k-1-i)x) \\
&\quad - d\Delta_x^d f'(m_0) - d\sum_{i=1}^{d}\binom{d-1}{i-1}(-1)^{i+1}f'(m_0 + (d-i)x) \\
&= (k-1)\Delta_x^d f'(m_0 + (k-d-1)x) + d\sum_{i=0}^{d-1}\binom{d-1}{i}(-1)^{i}f'(m_0 + (k-2-i)x) \\
&\quad - d\Delta_x^d f'(m_0) - d\sum_{i=0}^{d-1}\binom{d-1}{i}(-1)^{i}f'(m_0 + (d-1-i)x) \\
&= (k-1)\Delta_x^d f'(m_0 + (k-d-1)x) + d\Delta_x^{d-1} f'(m_0 + (k-d-1)x) \\
&\quad - d\Delta_x^d f'(m_0) - d\Delta_x^{d-1} f'(m_0) \\
&= (k-1)\Delta_x^d f'(m_0 + (k-d-1)x) + d\Delta_x^{d-1} f'(m_0 + (k-d-1)x) \\
&\quad - d\Delta_x^{d-1} f'(m_0 + x).
\end{aligned}$$

The mean value theorem implies that for all $x > 0$ there exist $\theta_1, \ldots, \theta_d \in (0, x)$, $\theta_1' \in [0, (k-d-2)x]$ and $\theta_2' \ldots, \theta_d' \in (0, x)$ such that

$$\begin{aligned}
g'(x) &= (k-1)x\Delta_x^{d-1} f''(m_0 + (k-d-1)x + \theta_1) \\
&\quad + d(k-d-2)x\Delta_x^{d-1} f''(m_0 + x + \theta_1') \\
&= \cdots \\
&= (k-1)x^d f^{(d+1)}(m_0 + (k-d-1)x + \theta) \\
&\quad + d(k-d-2)x^d f^{(d+1)}(m_0 + x + \theta') \\
&\overset{(a)}{>} 0,
\end{aligned}$$

where $\theta = \theta_1 + \cdots + \theta_d$ and $\theta' = \theta_1' + \cdots + \theta_d'$; the inequality (a) follows from the fact that $f^{(d+1)}(y) > 0$ for all $y \geq N_0$. Thus, $g'(x)$ is positive, and $g(x)$

increases. Recalling that $(\lfloor f(m_0 + r_0 j)\rfloor)_{j=0}^{k-1}$ belongs to $\mathcal{P}_{k,d}$, we have

$$\begin{aligned}
2^d &= \Delta_{r_0}^d \lfloor f(m_0 + (k-d-1)r_0)\rfloor - \Delta_{r_0}^d \lfloor f(m_0)\rfloor + 2^d \\
&> \Delta_{r_0}^d f(m_0 + (k-d-1)r_0) - \Delta_{r_0}^d f(m_0) \\
&\overset{(b)}{\geq} \Delta_{R_0}^d f(m_0 + (k-d-1)R_0) - \Delta_{R_0}^d f(m_0) \\
&\overset{(c)}{=} (k-d-1)R_0 \Delta_{R_0}^d f'(m_0 + \theta_0) \\
&\overset{(c)}{=} \cdots \overset{(c)}{=} (k-d-1)R_0^{d+1} f^{(d+1)}(m_0 + \theta) \\
&= 2^d c(1/\beta) \frac{f^{(d+1)}(m_0+\theta)}{f^{(d+1)}(m_0)},
\end{aligned} \tag{3.4.16}$$

where the monotonicity of g and the inequality $r_0 \geq R_0$ have been used to obtain (b); the mean value theorem has been used to obtain (c); $\theta_0 \in (0, (k-d-1)R_0)$, $\theta_1, \ldots, \theta_d \in (0, R_0)$, and $\theta = \theta_0 + \theta_1 + \cdots + \theta_d$. Put $\beta_0 = 1 + (k-1)R_0/m_0$. Since the inequality $\beta_0 < \beta$ holds due to $m_0 \geq N_0$, it follows that

$$\begin{aligned}
f^{(d+1)}(m_0 + \theta) &\overset{\text{(B1)}}{\geq} f^{(d+1)}(m_0 + (k-1)R_0) = f^{(d+1)}(\beta_0 m_0) \\
&\geq f^{(d+1)}(\beta m_0) \overset{\text{(B3)}}{\geq} c(1/\beta)^{-1} f^{(d+1)}(m_0).
\end{aligned} \tag{3.4.17}$$

Thus, (3.4.16) and (3.4.17) yield that

$$2^d > 2^d c(1/\beta)\frac{f^{(d+1)}(m_0+\theta)}{f^{(d+1)}(m_0)} \geq 2^d c(1/\beta)\frac{c(1/\beta)^{-1} f^{(d+1)}(m_0)}{f^{(d+1)}(m_0)} = 2^d,$$

which is a contradiction. Therefore, if $(\lfloor f(n+rj)\rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}$, $n \geq N_0$ and $r \geq 1$, then $r < R(n)$.

Next, we show Proposition 3.3.2 (limsup). Let $N \in \mathbb{N}$ be sufficiently large. Since the inequality

$$\begin{aligned}
&\#\{P \subset [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor f(n)\rfloor)_{n\in P} \in \mathcal{P}_{k,d}\} \\
&\leq \#\{(n,r) \in [1,N]_{\mathbb{Z}} \times [1,N]_{\mathbb{Z}} : (\lfloor f(n+rj)\rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} \\
&\leq \#\{(n,r) \in [1,N_0]_{\mathbb{Z}} \times [1,N]_{\mathbb{Z}}\} + \#\{(n,r) \in [N_0,N]_{\mathbb{Z}} \times [1,N]_{\mathbb{Z}} : r < R(n)\} \\
&\leq N_0 N + \sum_{n=N_0}^{N} R(n)
\end{aligned}$$

holds, it follows that

$$\begin{aligned}
&\varlimsup_{N\to\infty} \frac{\#\{P \subset [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor f(n) \rfloor)_{n\in P} \in \mathcal{P}_{k,d}\}}{N f^{(d+1)}(N)^{-1/(d+1)}} \\
&\le \varlimsup_{N\to\infty} \frac{f^{(d+1)}(N)^{1/(d+1)}}{N} \sum_{n=N_0}^{N} R(n) \overset{\text{(B1)}}{\le} \varlimsup_{N\to\infty} f^{(d+1)}(N)^{1/(d+1)} R(N) \\
&= \Big(\frac{2^d c(1/\beta)}{k-d-1}\Big)^{1/(d+1)} < \infty.
\end{aligned}$$

□

Remark 3.4.7 ([SY21, Remark 4.7])**.** Let us consider the special case $f(x) = x^\alpha$ with $\alpha \in (d, d+1)$. Then we can take $c(\delta)$ in (B3) as $c(\delta) = \delta^{\alpha-d-1}$. Thus,

$$\begin{aligned}
&\varlimsup_{N\to\infty} \frac{\#\{P \subset [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor n^\alpha \rfloor)_{n\in P} \in \mathcal{P}_{k,d}\}}{((\alpha)_{d+1})^{-1/(d+1)} N^{2-\alpha/(d+1)}} \\
&\le \varlimsup_{N\to\infty} \frac{f^{(d+1)}(N)^{1/(d+1)}}{N} \sum_{n=1}^{N} \Big(\frac{2^d c(1/\beta)}{k-d-1}\Big)^{1/(d+1)} f^{(d+1)}(n)^{-1/(d+1)} \\
&= \varlimsup_{N\to\infty} N^{\alpha/(d+1)-2} \sum_{n=1}^{N} \Big(\frac{2^d \beta^{d+1-\alpha}}{k-d-1}\Big)^{1/(d+1)} n^{1-\alpha/(d+1)} \\
&= \Big(\frac{2^d \beta^{d+1-\alpha}}{k-d-1}\Big)^{1/(d+1)} \frac{1}{2-\alpha/(d+1)}.
\end{aligned}$$

The arbitrariness of $\beta > 1$ yields

$$\begin{aligned}
&\varlimsup_{N\to\infty} \frac{\#\{P \subset [1,N]_{\mathbb{Z}} : P \in \mathcal{P}_{k,1},\ (\lfloor n^\alpha \rfloor)_{n\in P} \in \mathcal{P}_{k,d}\}}{N^{2-\alpha/(d+1)}} \\
&\le \Big(\frac{2^d}{(\alpha)_{d+1}(k-d-1)}\Big)^{1/(d+1)} \frac{1}{2-\alpha/(d+1)} =: \tilde{B}_{\alpha,k}.
\end{aligned}$$

Therefore, the constant $B_{\alpha,k}$ in Theorem 3.1.2 is an arbitrary value in the interval $(\tilde{B}_{\alpha,k}, \infty)$.

3.5 Further analysis: discrepancy and short intervals

In this section, we show Theorems 3.2.3 and 3.2.4. These theorems are derived from the following proposition.

Proposition 3.5.1 ([SY21, Proposition 5.1]). *Let $\alpha \in (1,2)$ and $c > 0$, and let $k \geq 3$ and $r \geq 1$ be integers. Then, there exists $N_0 = N_0(\alpha, k, r) \in \mathbb{N}$ such that for all $N \in [N_0, \infty)_{\mathbb{Z}}$ and $L \in [1, cN]_{\mathbb{Z}}$,*

$$\frac{1}{L}\#\{n \in [N, N+L)_{\mathbb{Z}} : (\lfloor (n+rj)^{\alpha} \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,1}\} - \frac{1}{k-1}$$
$$\ll_{\alpha,k,r,c} \begin{cases} N^{(\alpha-2)/6}(\log N)^{1/2} + N^{(2-\alpha)/2}/L^{1/2} & \alpha \in (1,2), \\ N^{(\alpha-3)/14}(\log N)^{1/2} + N^{(2-\alpha)/2}/L^{1/2} & \alpha \in (1,3/2), \\ (N^{(\alpha-3)/14} + N^{(3-\alpha)/6}/L^{1/2})(\log N)^{1/2} & \alpha \in [3/2, 11/6). \end{cases} \tag{3.5.1}$$

Remark 3.5.2 ([SY21, Remark 5.2]). The first one of (3.5.1) is the best of the three cases when $\alpha \in (1, 5/4] \cup [11/6, 2)$; the second one of (3.5.1) is the best of the three cases when $\alpha \in (5/4, 3/2)$. However, it depends on the growth rate of L whether the third one of (3.5.1) is the best of the three cases when $\alpha \in [3/2, 11/6)$. For instance, if $L = N$ and $\alpha \in [3/2, 11/6)$, then the third one of (3.5.1) is the best of the three cases; but if $\varepsilon \in (0, (2-\alpha)/3)$, $L = N^{2-\alpha+\varepsilon}$ and $\alpha \in [3/2, 11/6)$, then the first one of (3.5.1) is the best of the three cases.

Proposition 3.5.1 is an asymptotic formula for the number of integers $n \geq 1$ in a short interval such that $(\lfloor (n+rj)^{\alpha} \rfloor)_{j=0}^{k-1}$ is an AP. We prove Proposition 3.5.1 at the end of this section. Note that (3.5.1) is meaningless when $L = L(N)$ is sufficiently smaller than N. This is because in the case, the right-hand side in (3.5.1) diverges to positive infinity as $N \to \infty$. Before proving Proposition 3.5.1, let us show Theorems 3.2.3 and 3.2.4 by using Proposition 3.5.1.

Proof of Theorem 3.2.3 *assuming Proposition* 3.5.1. Let $\alpha \in (1,2)$, and let $k \geq 3$ and $r \geq 1$ be integers. Also, define the set $\mathcal{Q}$ as

$$\mathcal{Q} = \{n \in \mathbb{N} : (\lfloor (n+rj)^{\alpha} \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,1}\}. \tag{3.5.2}$$

Then Proposition 3.5.1 implies that

$$\frac{\#(\mathcal{Q} \cap [x, 2x))}{x} = \frac{1}{k-1} + O_{\alpha,k,r}(F_0(x)),$$

where

$$F_0(x) := \begin{cases} x^{(\alpha-2)/6}(\log x)^{1/2} + x^{(1-\alpha)/2} & \alpha \in (1, 5/4) \cup [11/6, 2), \\ x^{(\alpha-3)/14}(\log x)^{1/2} + x^{(1-\alpha)/2} & \alpha \in [5/4, 3/2), \\ (x^{(\alpha-3)/14} + x^{-\alpha/6})(\log x)^{1/2} & \alpha \in [3/2, 11/6). \end{cases}$$

Noting the ranges of α, we have $F_0(x) \ll F(x)$, where F is defined in Theorem 3.2.3. Let $N \in \mathbb{N}$ be sufficiently large and take $M \in \mathbb{N}$ with $2^M \leq N < 2^{M+1}$. Then

$$0 \leq \frac{\#(\mathcal{Q} \cap [1,N])}{N} - \frac{1}{N}\sum_{m=1}^{M} \#(\mathcal{Q} \cap [2^{-m}N, 2^{1-m}N)) \leq 2/N,$$

$$\frac{1}{N}\sum_{m=1}^{M} \#(\mathcal{Q} \cap [2^{-m}N, 2^{1-m}N)) = \sum_{m=1}^{M} \frac{2^{-m}}{k-1} + O_{\alpha,k,r}\Big(\sum_{m=1}^{M} 2^{-m}F(2^{-m}N)\Big)$$
$$= \frac{1-2^{-M}}{k-1} + O_{\alpha,k,r}(F(N)) = \frac{1}{k-1} + O_{\alpha,k,r}(1/N + F(N)).$$

Therefore, Theorem 3.2.3 holds. □

Proof of Theorem 3.2.4 *assuming Proposition* 3.5.1. Let $\alpha \in (1,2)$, and let $k \geq 3$ and $r \geq 1$ be integers. Define the set $\mathcal{Q}$ as (3.5.2). Thanks to the first inequality of (3.5.1), there exist constants $C = C(\alpha,k,r) > 0$ and $N_0 = N_0(\alpha,k,r) \in \mathbb{N}$ such that for all $N \in [N_0,\infty)_\mathbb{Z}$ and $L \in [1,N]_\mathbb{Z}$,

$$\left|\frac{\#(\mathcal{Q} \cap [N, N+L))}{L} - \frac{1}{k-1}\right| \leq CE_0(N,L), \tag{3.5.3}$$

where $E_0(N,L) \coloneqq N^{(\alpha-2)/6}(\log N)^{1/2} + N^{(2-\alpha)/2}/L^{1/2}$. Without loss of generality, we may assume that $N^{(\alpha-2)/6}(\log N)^{1/2} < 1/2C(k-1)$ for every integer $N \geq N_0$. Putting $L = L(N) = \lceil 4C^2(k-1)^2 N^{2-\alpha}\rceil$, we have

$$E_0(N,L) < \frac{1}{2C(k-1)} + \frac{1}{2C(k-1)} = \frac{1}{C(k-1)}$$

for every integer $N \geq N_0$. Therefore, for every integer $N \geq N_0$, the left-hand side in (3.5.3) is less than $1/(k-1)$, whence $\#(\mathcal{Q} \cap [N, N+L)) > 0$. Finally, the length $L' = L'(N) \coloneqq \max\{N_0 + L(N_0), L\} = O_{\alpha,k,r}(N^{2-\alpha})$ satisfies that $\#(\mathcal{Q} \cap [N, N+L')) > 0$ for all $N \in \mathbb{N}$. □

To prove Proposition 3.5.1, we need to estimate the convergence speed of (2.1.1) for a uniformly distributed sequence. For this purpose, let us apply discrepancies. For a sequence $(\mathbf{x}_n)_{n=1}^N$ of $\mathbb{R}^d$, define the *isotropic discrepancy* J_N as

$$J_N = \mathcal{J}((\mathbf{x}_n)_{n=1}^N) = \sup_{\substack{\mathcal{C}\subseteq[0,1)^d\\ \text{convex}}} \left|\frac{\#\{n \in [N] \colon \{\mathbf{x}_n\} \in \mathcal{C}\}}{N} - \mu(\mathcal{C})\right|,$$

where μ denotes the Lebesgue measure on $\mathbb{R}^d$. Let D_N be the discrepancy of $(\mathbf{x}_n)_{n=1}^N$. Although the inequality $D_N \leq J_N$ is trivial, the following reverse inequality holds [KN74, Theorem 1.6, Chapter 2]:

$$J_N \leq (4d\sqrt{d}+1)D_N^{1/d} \tag{3.5.4}$$

for every $d, N \in \mathbb{N}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^d$. Thanks to (3.5.4), it suffices to give an upper bound for the discrepancy in order to estimate the convergence speed of (2.1.1). The Erdős-Turán-Koksma inequality (2.2.1) is useful to evaluate discrepancies. Thanks to this inequality, it suffices to evaluate exponential sums in order to find upper bounds for discrepancies.

Lemma 3.5.3 ([SY21, Lemma 5.7])**.** *Let $\alpha \in (1,2)$, $r \in \mathbb{N}$, and $c > 0$. Then, there exists $N_0 = N_0(\alpha, r) \in \mathbb{N}$ such that for all $N \in [N_0, \infty)_{\mathbb{Z}}$ and $L \in [1, cN]_{\mathbb{Z}}$, the discrepancy $D(N,L)$ of the sequence $\big((n^\alpha, r\alpha n^{\alpha-1})\big)_{n=N}^{N+L-1}$ satisfies*

$$D(N,L) \ll_{\alpha,c} \begin{cases} N^{(\alpha-2)/3}\log N + N^{2-\alpha}/L & \alpha \in (1,2), \\ N^{(\alpha-3)/7}\log N + N^{2-\alpha}/L & \alpha \in (1,3/2), \\ (N^{(\alpha-3)/7} + N^{(3-\alpha)/3}/L)\log N & \alpha \in [3/2, 11/6). \end{cases}$$

Proof. Let $f(x) = x^\alpha$. The inequality (2.2.1) with $d = 2$ implies that for all $L, N, K \in \mathbb{N}$,

$$D(N,L) \ll \frac{1}{K} + \sum_{\substack{|h_0|,|h_1| \leq K \\ (h_0,h_1)\neq(0,0)}} \frac{1}{\nu(h_0,h_1)} \left| \frac{1}{L}\sum_{n=N}^{N+L-1} \mathrm{e}(h_0 f(n) + h_1 r f'(n)) \right|.$$

Taking an integer

$$\begin{aligned} N_0 &= N_0(\alpha, r) \\ &\geq \max\{(2r)^{3/(1+\alpha)}, 2^{3/(2-\alpha)}, (4r)^{3/2(2-\alpha)},\ (2r)^{7/(4+\alpha)}, 2^{7/(3-\alpha)}, (4r)^{7/(11-6\alpha)}\}, \end{aligned}$$

we evaluate the right-hand side above in two ways.

Step 1. Let us show that for all $N \in [N_0, \infty)_{\mathbb{Z}}$ and $L \in [1, cN]_{\mathbb{Z}}$,

$$D(N,L) \ll_{\alpha,c} N^{(\alpha-2)/3}\log N + N^{2-\alpha}/L. \tag{3.5.5}$$

Take $N \in [N_0, \infty)_{\mathbb{Z}}$ and $L \in [1, cN]_{\mathbb{Z}}$ arbitrarily, and put $K = \lfloor N^{(2-\alpha)/3} \rfloor$. Then, note that $rK/N \leq rN^{-(1+\alpha)/3} \leq rN_0^{-(1+\alpha)/3} \leq 1/2$ and $\log K \geq \log 2$.

Consider the case when $|h_0|, |h_1| \le K$ and $h_0 \ne 0$. When $x \in [N, N+L-1]$, the function $g(x) = h_0 f(x) + h_1 r f'(x)$ satisfies that

$$\begin{aligned} |g''(x)| &\le |h_0| \, f''(x)(1 + rK \, |f'''(x)/f''(x)|) \\ &\ll |h_0| \, N^{\alpha-2}(1 + rK/N) \ll |h_0| \, N^{\alpha-2}, \\ |g''(x)| &\ge |h_0| \, f''(x)(1 - rK \, |f'''(x)/f''(x)|) \\ &\gg_\alpha |h_0| \, (N+L)^{\alpha-2}(1 - rK/N) \gg_c |h_0| \, N^{\alpha-2}. \end{aligned}$$

Thus, Lemma 2.2.2 implies that

$$\frac{1}{L} \sum_{n=N}^{N+L-1} \mathrm{e}(h_0 f(n) + h_1 r f'(n)) \ll_{\alpha,c} |h_0|^{1/2} \, N^{(\alpha-2)/2} + |h_0|^{-1/2} \, N^{(2-\alpha)/2}/L.$$

Therefore, it follows that

$$\begin{aligned} &\sum_{\substack{|h_0|,|h_1| \le K \\ h_0 \ne 0}} \frac{1}{\nu(h_0,h_1)} \left| \frac{1}{L} \sum_{n=N}^{N+L-1} \mathrm{e}(h_0 f(n) + h_1 r f'(n)) \right| \\ &\ll_{\alpha,c} \sum_{\substack{|h_0|,|h_1| \le K \\ h_0 \ne 0}} \frac{|h_0|^{1/2} \, N^{(\alpha-2)/2} + |h_0|^{-1/2} \, N^{(2-\alpha)/2}/L}{\nu(h_0,h_1)} \\ &\ll \Big(\sum_{h_1=1}^{K} \frac{1}{h_1} + 1 \Big) \sum_{h_0=1}^{K} \big(h_0^{-1/2} N^{(\alpha-2)/2} + h_0^{-3/2} N^{(2-\alpha)/2}/L \big) \\ &\ll (\log K)(K^{1/2} N^{(\alpha-2)/2} + N^{(2-\alpha)/2}/L) \ll (N^{(\alpha-2)/3} + N^{(2-\alpha)/2}/L) \log N. \end{aligned}$$

Next, consider the case when $1 \le |h_1| \le K$ and $h_0 = 0$. When $x \in [N, N+L-1]$, the function $g(x) = h_1 r f'(x)$ satisfies that

$$\begin{gathered} |g'(x)| = r \, |h_1| \, f''(x) \le 2rKN^{\alpha-2} \le 2rN^{(2/3)(\alpha-2)} \le 2rN_0^{(2/3)(\alpha-2)} \le 1/2, \\ |g'(x)| \gg_\alpha |h_1| \, (N+L)^{\alpha-2} \gg_c |h_1| \, N^{\alpha-2}. \end{gathered}$$

This yields that $\min\{|g'(x) - m| : m \in \mathbb{Z}\} = |g'(x)|$ for all $x \in [N, N+L-1]$. Thus, Lemma 2.2.1 implies that

$$\frac{1}{L} \sum_{n=N}^{N+L-1} \mathrm{e}(h_1 r f'(n)) \ll_{\alpha,c} |h_1|^{-1} \, N^{2-\alpha}/L.$$

Therefore, it follows that

$$\begin{aligned} &\sum_{\substack{1\le|h_1|\le K\\ h_0=0}} \frac{1}{\nu(h_0,h_1)} \left| \frac{1}{L} \sum_{n=N}^{N+L-1} \mathrm{e}(h_0 f(n) + h_1 r f'(n)) \right| \\ &\ll_{\alpha,c} \sum_{1\le|h_1|\le K} \frac{|h_1|^{-1} N^{2-\alpha}/L}{|h_1|} \ll N^{2-\alpha}/L. \end{aligned} \tag{3.5.6}$$

Summarizing the above two cases, we have

$$\begin{aligned} D(N,L) &\ll \frac{1}{K} + \sum_{\substack{|h_0|,|h_1|\le K\\ (h_0,h_1)\ne(0,0)}} \frac{1}{\nu(h_0,h_1)} \left| \frac{1}{L} \sum_{n=N}^{N+L-1} \mathrm{e}(h_0 f(n) + h_1 r f'(n)) \right| \\ &\ll_{\alpha,c} N^{(\alpha-2)/3} + (N^{(\alpha-2)/3} + N^{(2-\alpha)/2}/L)\log N + N^{2-\alpha}/L \\ &\ll N^{(\alpha-2)/3}\log N + N^{2-\alpha}/L, \end{aligned}$$

which is just (3.5.5).

Step 2. Assume $\alpha \in (1, 11/6)$. Let us show that for all $N \in [N_0,\infty)_{\mathbb{Z}}$ and $L \in [1, cN]_{\mathbb{Z}}$,

$$D(N,L) \ll_{\alpha,c} \begin{cases} N^{(\alpha-3)/7}\log N + N^{2-\alpha}/L & \alpha \in (1,3/2), \\ (N^{(\alpha-3)/7} + N^{(3-\alpha)/3}/L)\log N & \alpha \in [3/2, 11/6). \end{cases} \tag{3.5.7}$$

Take $N \in [N_0,\infty)_{\mathbb{Z}}$ and $L \in [1, cN]_{\mathbb{Z}}$ arbitrarily, and put $K = \lfloor N^{(3-\alpha)/7} \rfloor$. Then, note that $rK/N \le rN^{-(4+\alpha)/7} \le rN_0^{-(4+\alpha)/7} \le 1/2$ and $\log K \ge \log 2$. Consider the case when $|h_0|, |h_1| \le K$ and $h_0 \ne 0$. When $x \in [N, N+L-1]$, the function $g(x) = h_0 f(x) + h_1 r f'(x)$ satisfies that

$$\begin{aligned} |g'''(x)| &\le |h_0 f'''(x)| \left(1 + rK \left|f''''(x)/f'''(x)\right|\right) \\ &\ll |h_0| N^{\alpha-3}(1 + rK/N) \ll |h_0| N^{\alpha-3}, \\ |g'''(x)| &\ge |h_0| f'''(x)(1 - rK \left|f''''(x)/f'''(x)\right|) \\ &\gg_\alpha |h_0| (N+L)^{\alpha-3}(1 - rK/N) \gg_c |h_0| N^{\alpha-3}. \end{aligned}$$

Since $0 < |h_0| N^{\alpha-3} \le KN^{\alpha-3} \le N^{(6/7)(\alpha-3)} < 1$, Lemma 2.2.3 implies that

$$\frac{1}{L} \sum_{n=N}^{N+L-1} \mathrm{e}(h_0 f(n) + h_1 r f'(n)) \ll_{\alpha,c} |h_0|^{1/6} N^{(\alpha-3)/6} + |h_0|^{-1/3} N^{(3-\alpha)/3}/L.$$

Therefore, it follows that

$$\sum_{\substack{|h_0|,|h_1|\le K\\ h_0\neq 0}} \frac{1}{\nu(h_0,h_1)}\left|\frac{1}{L}\sum_{n=N}^{N+L-1} \mathrm{e}(h_0 f(n) + h_1 r f'(n))\right|$$
$$\ll_{\alpha,c} \sum_{\substack{|h_0|,|h_1|\le K\\ h_0\neq 0}} \frac{|h_0|^{1/6} N^{(\alpha-3)/6} + |h_0|^{-1/3} N^{(3-\alpha)/3}/L}{\nu(h_0,h_1)}$$
$$\ll \Big(\sum_{h_1=1}^{K}\frac{1}{h_1}+1\Big)\sum_{h_0=1}^{K}\big(h_0^{-5/6}N^{(\alpha-3)/6} + h_0^{-4/3}N^{(3-\alpha)/3}/L\big)$$
$$\ll (\log K)(K^{1/6}N^{(\alpha-3)/6} + N^{(3-\alpha)/3}/L) \ll (N^{(\alpha-3)/7} + N^{(3-\alpha)/3}/L)\log N.$$

Next, consider the case when $1 \le |h_1| \le K$ and $h_0 = 0$. When $x \in [N, N+L-1]$, the function $g(x) = h_1 r f'(x)$ satisfies that

$$|g'(x)| = r\,|h_1|\,f''(x) \le 2rKN^{\alpha-2} \le 2rN^{(6\alpha-11)/7} \le 2rN_0^{(6\alpha-11)/7} \le 1/2,$$
$$|g'(x)| \gg_{\alpha,c} |h_1|\,N^{\alpha-2}.$$

This yields that $\min\{|g'(x)-m| : m \in \mathbb{Z}\} = |g'(x)|$ for all $x \in [N, N+L-1]$. From the same calculation as Step 1, the inequality (3.5.6) follows. Summarizing the above two cases, we have

$$D(N,L) \ll \frac{1}{K} + \sum_{\substack{|h_0|,|h_1|\le K\\ (h_0,h_1)\neq(0,0)}} \frac{1}{\nu(h_0,h_1)}\left|\frac{1}{L}\sum_{n=N}^{N+L-1} \mathrm{e}(h_0 f(n) + h_1 r f'(n))\right|$$
$$\ll_{\alpha,c} N^{(\alpha-3)/7} + (N^{(\alpha-3)/7} + N^{(3-\alpha)/3}/L)\log N + N^{2-\alpha}/L$$
$$\ll (N^{(\alpha-3)/7} + N^{(3-\alpha)/3}/L)\log N + N^{2-\alpha}/L$$
$$\ll \begin{cases} N^{(\alpha-3)/7}\log N + N^{2-\alpha}/L & \alpha \in (1,3/2),\\ (N^{(\alpha-3)/7} + N^{(3-\alpha)/3}/L)\log N & \alpha\in[3/2,11/6), \end{cases}$$

which is just (3.5.7).

Finally, combining (3.5.5) and (3.5.7), we obtain Lemma 3.5.3. □

Proof of Proposition 3.5.1. Take $N_0 = N_0(\alpha, r) \in \mathbb{N}$ in Lemma 3.5.3. Let $f(x) = x^\alpha$,

$$N_0' = N_0'(\alpha,k,r) = \max\Big(N_0, \Big\lceil \big(r^2(k-1)^2\alpha(\alpha-1)\big)^{1/(2-\alpha)}\Big\rceil\Big),$$

$N \in [N_0', \infty)_{\mathbb{Z}}$ and $L \in [1, cN]_{\mathbb{Z}}$. Then

$$\varepsilon = \varepsilon(N) \coloneqq \frac{r^2(k-1)^2}{2} f''(N) \in (0, 1/2).$$

The discrepancy and isotropic discrepancy of the sequence $\big((a_0(n), a_1(n))\big)_{n=N}^{N+L-1}$ are denoted by $D(N,L)$ and $J(N,L)$ respectively, where $a_0(n)$ and $a_1(n)$ are defined by (3.4.2) with $d = 1$. Note that $a_0(n) = f(n)$ and $a_1(n) = rf'(n)$. Also, define the set $\mathcal{Q}$ as (3.5.2). Recall the proof of Proposition 3.3.1. The sets $\mathcal{C}_{k,2}^{\mp}(\varepsilon)$ defined by (3.4.5) and (3.4.9) with $d = 1$ satisfy the inclusion relations

$$\begin{aligned} \bigcup_{s_1 \in \mathbb{Z}} \{n \in [N, \infty)_{\mathbb{Z}} : (\{a_0(n)\}, \{a_1(n)\} + s_1) \in \mathcal{C}_{k,2}^{-}(\varepsilon)\} \subset \mathcal{Q}, \\ \mathcal{Q} \cap [N, \infty) \subset \bigcup_{s_1 \in \mathbb{Z}} \{n \in \mathbb{N} : (\{a_0(n)\}, \{a_1(n)\} + s_1) \in \mathcal{C}_{k,2}^{+}(\varepsilon)\}. \end{aligned} \tag{3.5.8}$$

Thus, we have that

$$\begin{aligned} &\frac{\#(\mathcal{Q} \cap [N, N+L))}{L} \\ &\geq \sum_{s_1 \in \mathbb{Z}} \frac{\#\{n \in [N, N+L)_{\mathbb{Z}} : (\{a_0(n)\}, \{a_1(n)\} + s_1) \in \mathcal{C}_{k,2}^{-}(\varepsilon)\}}{L} \\ &\geq \sum_{s_1 \in \mathbb{Z}} \Big(\mu\Big(\mathcal{C}_{k,2}^{-}(\varepsilon) \cap \big([0,1) \times [s_1, s_1+1)\big)\Big) - J(N,L)\Big) \geq \mu(\mathcal{C}_{k,2}^{-}(\varepsilon)) - C_k^{-} J(N,L) \end{aligned}$$

and

$$\begin{aligned} &\frac{\#(\mathcal{Q} \cap [N, N+L))}{L} \\ &\leq \sum_{s_1 \in \mathbb{Z}} \frac{\#\{n \in [N, N+L)_{\mathbb{Z}} : (\{a_0(n)\}, \{a_1(n)\} + s_1) \in \mathcal{C}_{k,2}^{+}(\varepsilon)\}}{L} \\ &\leq \sum_{s_1 \in \mathbb{Z}} \Big(\mu\Big(\mathcal{C}_{k,2}^{+}(\varepsilon) \cap \big([0,1) \times [s_1, s_1+1)\big)\Big) + J(N,L)\Big) \leq \mu(\mathcal{C}_{k,2}^{+}(\varepsilon)) + C_k^{+} J(N,L) \end{aligned}$$

for some $C_k^{\mp} \in \mathbb{N}$, since all the above sums are finite sums. (Indeed, we can take $C_k^{\mp} = 2$, but this fact is not used here). Now, the sets $\mathcal{C}_{k,2}^{\mp}(\varepsilon)$ are simplified as

$$\mathcal{C}_{k,2}^{-}(\varepsilon) = \{(y_0, y_1) \in \mathbb{R}^2 : 0 \le y_0 < 1,\ \varepsilon \le y_0 + (k-1)y_1 < 1-\varepsilon\},$$
$$\mathcal{C}_{k,2}^{+}(\varepsilon) = \{(y_0, y_1) \in \mathbb{R}^2 : 0 \le y_0 < 1,\ -\varepsilon \le y_0 + (k-1)y_1 < 1+\varepsilon\},$$

whence $\mu(\mathcal{C}_{k,2}^{\mp}(\varepsilon)) = (1 \mp 2\varepsilon)/(k-1)$. Thus,

$$\left|\frac{\#(\mathcal{Q} \cap [N, N+L))}{L} - \frac{1}{k-1}\right| \le \frac{2\varepsilon}{k-1} + \max\{C_k^{\mp}\} \cdot J(N, L).$$

Using the inequality (3.5.4) and Lemma 3.5.3, we obtain

$$\begin{aligned}
&\left|\frac{\#(\mathcal{Q} \cap [N, N+L))}{L} - \frac{1}{k-1}\right| \le \frac{2\varepsilon}{k-1} + \max\{C_k^{\mp}\} \cdot 2(8\sqrt{2}+1)D(N, L)^{1/2} \\
&\ll_{k,r} N^{\alpha-2} + D(N, L)^{1/2} \\
&\ll_{\alpha,c} \begin{cases} N^{(\alpha-2)/6}(\log N)^{1/2} + N^{(2-\alpha)/2}/L^{1/2} & \alpha \in (1,2), \\ N^{(\alpha-3)/14}(\log N)^{1/2} + N^{(2-\alpha)/2}/L^{1/2} & \alpha \in (1,3/2), \\ (N^{(\alpha-3)/14} + N^{(3-\alpha)/6}/L^{1/2})(\log N)^{1/2} & \alpha \in [3/2, 11/6), \end{cases}
\end{aligned}$$

where the inequality $(x+y)^{1/2} \le x^{1/2} + y^{1/2}$ for $x, y \ge 0$ has been used to obtain the last inequality. □

3.6 Optimality of the growth rate $O_{\alpha,k,r}(x^{2-\alpha})$

Throughout this appendix, let $f(x) = x^\alpha$. As stated in Theorem 3.2.3, the relation $L_{\alpha,k,r}(x) = O_{\alpha,k,r}(x^{2-\alpha})$ holds. We show that the growth rate $O_{\alpha,k,r}(x^{2-\alpha})$ is best for every $k \ge 4$ in the following meaning.

Proposition 3.6.1 ([SY21, Proposition A.1]). *For all $\alpha \in (1,2)$ and all integers $k \ge 4$ and $r \ge 1$,*

$$\varlimsup_{x\to\infty} \frac{L_{\alpha,k,r}(x)}{x^{2-\alpha}} \ge \frac{k-3}{\alpha(\alpha-1)r(k-1)}. \tag{3.6.1}$$

Proof. Let $k \ge 4$ and $r \ge 1$ be integers, and let $\alpha \in (1,2)$ and $\beta \in (0, k-3)$. Since $(rf'(n))_{n=1}^\infty$ is uniformly distributed modulo 1 and the inequality $1/(k-1) < 1-(\beta+1)/(k-1)$ holds, there exist infinitely many $N \in \mathbb{N}$ such that

$$\frac{1}{k-1} \le \{rf'(N)\} \le 1 - \frac{\beta+1}{k-1}. \tag{3.6.2}$$

Take a sufficiently large $N \in \mathbb{N}$ that satisfies (3.6.2) and

$$\varepsilon = \varepsilon(N) := \frac{(k-1)^2 r^2}{2} f''(N) \in (0,1). \tag{3.6.3}$$

Now, define the set $\mathcal{Q}$ as (3.5.2), and take $m \in [0, L_{\alpha,k,r}(N)]_{\mathbb{Z}}$ such that $N + m \in \mathcal{Q}$. Recall the proof of Proposition 3.3.1. The set

$$\mathcal{C}_{k,2}^{+}(\varepsilon) = \{(y_0, y_1) \in \mathbb{R}^2 : 0 \le y_0 < 1,\ -\varepsilon \le y_0 + (k-1)y_1 < 1\} \tag{3.6.4}$$

satisfies the inclusion relation (3.5.8), where $a_0(n)$ and $a_1(n)$ are defined by (3.4.2) with $d = 1$. Note that $a_0(n) = f(n)$ and $a_1(n) = rf'(n)$. Due to (3.5.8), the vector $(\{f(N+m)\}, \{rf'(N+m)\} + s_1)$ lies in $\mathcal{C}_{k,2}^{+}(\varepsilon)$ for some $s_1 \in \mathbb{Z}$. The integer s_1 is equal to 0 or -1, which is proved at the end of this proof.

If $s_1 = -1$, then the inequalities $-\varepsilon \le \{f(N+m)\} + (k-1)(\{rf'(N+m)\} - 1) < 1$ and (3.6.2) and the mean value theorem imply that

$$\begin{aligned} 1 - \frac{1+\varepsilon}{k-1} \le \{rf'(N+m)\} &\le \{rf'(N)\} + rmf''(N) \\ &\le 1 - \frac{\beta+1}{k-1} + rL_{\alpha,k,r}(N) f''(N), \end{aligned}$$

whence $L_{\alpha,k,r}(N) f''(N) \ge (\beta - \varepsilon)/r(k-1)$. If $s_1 = 0$, then the inequalities $-\varepsilon \le \{f(N+m)\} + (k-1)\{rf'(N+m)\} < 1$ and (3.6.2) yield

$$\{rf'(N+m)\} < \frac{1}{k-1} \le \{rf'(N)\} \le 1 - \frac{\beta+1}{k-1}.$$

Since f' and f'' are increasing and decreasing functions respectively, the mean value theorem implies that

$$\frac{\beta+1}{k-1} \le rf'(N+m) - rf'(N) \le rmf''(N) \le rL_{\alpha,k,r}(N) f''(N),$$

whence $L_{\alpha,k,r}(N) f''(N) \ge \beta / r(k-1)$. Since $\varepsilon = \varepsilon(N)$ vanishes as $N \to \infty$, it turns out that

$$\varlimsup_{x \to \infty} \frac{L_{\alpha,k,r}(x)}{x^{2-\alpha}} \ge \frac{\beta}{\alpha(\alpha-1)r(k-1)}.$$

Letting $\beta \to k-3$, we obtain (3.6.1).

We show that if $(x_0, x_1 + s_1) \in \mathcal{C}^+_{k,2}(\varepsilon)$, $(x_0, x_1) \in [0,1)^2$ and $s_1 \in \mathbb{Z}$, then $s_1 \in \{0, -1\}$. (The assumption $k \geq 3$ suffices here.) The definition of $\mathcal{C}^+_{k,2}(\varepsilon)$ yields that

$$\begin{aligned}
(k-1)s_1 \leq x_0 + (k-1)(x_1 + s_1) < 1 + \varepsilon < 2,\\
0 \leq x_0 + (k-1)(x_1 + s_1) < 1 + (k-1)(1 + s_1),
\end{aligned}$$

whence $-3/2 \leq -k/(k-1) < s_1 < 2/(k-1) \leq 1$. Therefore, the integer s_1 is equal to 0 or -1. □

When $k = 3$, the above proof does not work well, since there does not exist $N \in \mathbb{N}$ satisfying (3.6.2). The relation $L_{\alpha,3,r}(x) = O_{\alpha,r}(x^{1-\alpha/2})$ probably holds, but we do not have its proof. However, if $L_{\alpha,3,r}(x) = O_{\alpha,r}(x^{1-\alpha/2})$ holds, then the growth rate $O_{\alpha,r}(x^{1-\alpha/2})$ is best in the following meaning.

Proposition 3.6.2 ([SY21, Proposiiton A.2])**.** *For all $\alpha \in (1,2)$ and $r \geq \mathbb{N}$,*

$$\varlimsup_{x\to\infty} \frac{L_{\alpha,3,r}(x)}{x^{1-\alpha/2}} \geq \frac{\sqrt{2}-1}{\sqrt{\alpha(\alpha-1)r}}.$$

To prove Proposition 3.6.2, we need to choose infinitely many $N \in \mathbb{N}$ with certain properties instead of (3.6.1). For this purpose, let us show the following lemmas.

Lemma 3.6.3 ([SY21, Lemma A.3])**.** *Let $\alpha \in (1,2)$ and $r \in \mathbb{N}$. Then there exist infinitely many $N \in \mathbb{N}$ such that $0 \leq \{f'(N)\} - 1/2r < f''(N-1)$.*

Proof. Take an arbitrary $N \in \mathbb{N}$ such that $f''(N) < 1/2r$ and $Nf''(2N) > 1$. Since the inequality $f'(2N) - f'(N) > Nf''(2N) > 1$ holds, some $m \in \mathbb{Z}$ satisfies $f''(N) < 1/2r + m < f''(2N)$. Also, the sequence $(f'(N+n))_{n=0}^N$ increases and the difference $f'(N+n+1) - f'(N+n)$ is bounded above by $f''(N) < 1/2r$. Thus, we can take the minimum $n \in [1,N]_{\mathbb{Z}}$ such that $f'(N+n-1) < 1/2r + m \leq f'(N+n) < 1+m$. Then it follows that

$$0 \leq \{f'(N+n)\} - 1/2r < f'(N+n) - f'(N+n-1) < f''(N+n-1).$$

The arbitrariness of N implies Lemma 3.6.3. □

Lemma 3.6.4 ([SY21, Lemma A.4])**.** *Let $\alpha \in (1,2)$ and $r \in \mathbb{N}$. For all $c_0 > 2r^{1/2}$ and $c_1 > r^{-1/2}$, there exist infinitely many $N \in \mathbb{N}$ such that $\{f(N)\} < c_1 f''(N)^{1/2}$ and $0 \leq \{f'(N)\} - 1/2r < c_0 f''(N)^{1/2}$.*

Proof. Let $c_0 > 2r^{1/2}$ and $c_1 > r^{-1/2}$. Take a sufficiently large $N \in \mathbb{N}$ such that $0 \leq \{f'(N)\} - 1/2r < f''(N-1)$ (see Lemma 3.6.3). Also, take $s \in [1, 2r]_{\mathbb{Z}}$ such that $-1/2r < \{f(N)\} - s/2r \leq 0$. Defining $n_m = 2rm - s$ and $x_m = f(N + n_m) - m - n_m \lfloor f'(N) \rfloor$ for $m \in [1, M+1]_{\mathbb{Z}}$, we verify the following facts.

1. $0 < x_{m+1} - x_m < 2rf''(N-1) + 4r^2(M+1)f''(N)$ for all $m \in [1, M]_{\mathbb{Z}}$.
2. $x_{M+1} - x_1 > 2r^2M^2 f''(N + 2r(M+1)) - 2r^2 f''(N)$.
3. $-1/2r < x_1 - \lfloor f(N) \rfloor < 2rf''(N-1) + 2r^2 f''(N)$.

Fact (1):

$$\begin{aligned}
x_{m+1} - x_m &> 2rf'(N) - 1 - 2r\lfloor f'(N) \rfloor = 2r\{f'(N)\} - 1 \geq 0, \\
x_{m+1} - x_m &< 2rf'(N + n_{m+1}) - 1 - 2r\lfloor f'(N) \rfloor \\
&< 2r(f'(N) + n_{m+1}f''(N)) - 1 - 2r\lfloor f'(N) \rfloor \\
&< 2r\{f'(N)\} - 1 + 2rn_{m+1}f''(N) \\
&< 2rf''(N-1) + 4r^2(M+1)f''(N).
\end{aligned}$$

Fact (2):

$$\begin{aligned}
x_{M+1} - x_1 &> f(N + n_{M+1}) - f(N + n_1) - M - 2rM\lfloor f'(N) \rfloor \\
&= f(N + n_{M+1}) - f(N + n_1) - 2rMf'(N) + M(2r\{f'(N)\} - 1) \\
&\geq f(N + n_{M+1}) - f(N + n_1) - 2rMf'(N) \\
&> \Big(f(N) + n_{M+1}f'(N) + \frac{n_{M+1}^2}{2} f''(N + n_{M+1}) \Big) \\
&\quad - \Big(f(N) + n_1 f'(N) + \frac{n_1^2}{2} f''(N) \Big) - 2rMf'(N) \\
&= \frac{n_{M+1}^2}{2} f''(N + n_{M+1}) - \frac{n_1^2}{2} f''(N) \\
&> 2r^2M^2 f''(N + 2r(M+1)) - 2r^2 f''(N).
\end{aligned}$$

Fact (3):

$$\begin{aligned}
x_1 - \lfloor f(N) \rfloor &= f(N + n_1) - 1 - n_1\lfloor f'(N) \rfloor - \lfloor f(N) \rfloor \\
&> f(N) + n_1 f'(N) - 1 - n_1\lfloor f'(N) \rfloor - \lfloor f(N) \rfloor \\
&= \{f(N)\} + n_1\{f'(N)\} - 1 \\
&\geq \{f(N)\} + n_1/2r - 1 = \{f(N)\} - s/2r > -1/2r
\end{aligned}$$

and

$$\begin{aligned} x_1 - \lfloor f(N) \rfloor &= f(N+n_1) - 1 - n_1 \lfloor f'(N) \rfloor - \lfloor f(N) \rfloor \\ &< f(N) + n_1 f'(N) + \frac{n_1^2}{2} f''(N) - 1 - n_1 \lfloor f'(N) \rfloor - \lfloor f(N) \rfloor \\ &= \{f(N)\} + n_1 \{f'(N)\} + \frac{n_1^2}{2} f''(N) - 1 \\ &< \{f(N)\} + n_1 (1/2r + f''(N-1)) + \frac{n_1^2}{2} f''(N) - 1 \\ &< \{f(N)\} - s/2r + n_1 f''(N-1) + \frac{n_1^2}{2} f''(N) \\ &\leq n_1 f''(N-1) + \frac{n_1^2}{2} f''(N) < 2r f''(N-1) + 2r^2 f''(N). \end{aligned}$$

Now, we have the following two cases:

1. $x_1 - \lfloor f(N) \rfloor \geq 0$,
2. $x_1 - \lfloor f(N) \rfloor < 0$.

Case (1). The sufficiently large N satisfies that

$$\begin{aligned} \{f(N+n_1)\} &= \{x_1\} \overset{\text{Fact (3)}}{<} 2r f''(N-1) + 2r^2 f''(N) < c_0 f''(N+n_1)^{1/2}, \\ \{f'(N+n_1)\} &< \{f'(N)\} + n_1 f''(N) < 1/2r + f''(N-1) + 2r f''(N) \\ &< 1/2r + c_1 f''(N+n_1)^{1/2}, \\ \{f'(N+n_1)\} &> \{f'(N)\} \geq 1/2r. \end{aligned}$$

Case (2). Take $1 < \beta < \beta' = \min\{c_0/2r^{1/2}, c_1/r^{-1/2}\}$ and put $M = \lceil \beta f''(N)^{-1/2}/2r^{3/2} \rceil = O(N^{1-\alpha/2})$. Since the sufficiently large N satisfies

$$\begin{aligned} x_{M+1} - x_1 &> \frac{f''(N+2r(M+1))}{2r f''(N)} - 2r^2 f''(N) \\ &= \frac{\beta}{2r} \Big(\frac{N}{N+2r(M+1)} \Big)^{2-\alpha} - 2r^2 f''(N) > \frac{1}{2r}, \end{aligned}$$

we can take the minimum $m \in [1, M]_{\mathbb{Z}}$ such that $x_{m+1} - \lfloor f(N) \rfloor \geq 0$. Then the sufficiently large N satisfies that

$$\begin{aligned} \{f(N+n_{m+1})\} &= \{x_{m+1}\} < x_{m+1} - x_m \overset{\text{Fact (2)}}{<} 2r f''(N-1) + 4r^2 (M+1) f''(N) \\ &< 2r^{1/2} \beta' f''(N+n_{m+1})^{1/2} \leq c_0 f''(N+n_{m+1})^{1/2} \end{aligned}$$

and

$$\{f'(N+n_{m+1})\} < \{f'(N)\} + n_{m+1}f''(N) < 1/2r + f''(N-1) + 2r(M+1)f''(N)$$
$$< 1/2r + r^{-1/2}\beta' f''(N+n_{m+1})^{1/2} \leq 1/2r + c_1 f''(N+n_{m+1})^{1/2},$$
$$\{f'(N+n_{m+1})\} > \{f'(N)\} \geq 1/2r.$$

Therefore, Lemma 3.6.4 holds. □

Proof of Proposition 3.6.2. Let $c_0 > 2r^{1/2}$, $c_1 > r^{-1/2}$ and $0 < c_2 < \sqrt{c_1^2 + 1/r} - c_1$. Thanks to Lemma 3.6.4, we can take a sufficiently large $N \in \mathbb{N}$ such that

1. $\{f(N)\} < c_0 f''(N)^{1/2}$,
2. $0 \leq \{f'(N)\} - 1/2r < c_1 f''(N)^{1/2}$,
3. $rc_1 f''(N)^{1/2} < 1/2$.

Moreover, the inequality

4. $0 \leq \{rf'(N)\} - 1/2 < rc_1 f''(N)^{1/2}$

follows from (2) and (3). Set $\varepsilon = \varepsilon(N) = 2r^2 f''(N) \in (0,1)$, which is just (3.6.3) with $k = 3$. We show that $L_{\alpha,3,r}(N) > c_2 f''(N)^{-1/2}$ by contradiction. Suppose that $L_{\alpha,3,r}(N) \leq c_2 f''(N)^{-1/2}$. Take $m \in [0, L_{\alpha,3,r}(N)]_{\mathbb{Z}}$ such that $(\lfloor f(N+m+rj)\rfloor)_{j=0}^2$ is an AP. Since the set $\mathcal{C}_{k,2}^{+}(\varepsilon)$ defined by (3.6.4) satisfies the inclusion relation (3.5.8), the vector $(\{f(N+m)\}, \{rf'(N+m)\} + s_1)$ lies in $\mathcal{C}_{k,2}^{+}(\varepsilon)$ for some $s_1 \in \mathbb{Z}$. The integer s_1 is equal to 0 or -1 (see the end of the proof of Proposition 3.6.1).

If $s_1 = 0$, then the inequalities $-\varepsilon \leq \{f(N+m)\} + 2\{rf'(N+m)\} < 1$, $m \leq L_{\alpha,3,r}(N) \leq c_2 f''(N)^{-1/2}$ and (4) yield that

$$\{rf'(N+m)\} < 1/2 \leq \{rf'(N)\} < 1/2 + rc_1 f''(N)^{1/2}$$

and thus

$$1/2 - rc_1 f''(N)^{1/2} < rf'(N+m) - rf'(N) \leq rmf''(N) \leq rc_2 f''(N)^{1/2},$$

which is a contradiction because N is sufficiently large.

Next, consider the case $s_1 = -1$. Then the inequalities $-\varepsilon \le \{f(N + m)\} + 2(\{rf'(N+m)\} - 1) < 1$, $m \le L_{\alpha,3,r}(N) \le c_2 f''(N)^{-1/2}$ and (4) yield that

$$\begin{aligned} 1 - \frac{\{f(N+m)\} + \varepsilon}{2} &\le \{rf'(N+m)\} \le \{rf'(N)\} + rmf''(N) \\ &< 1/2 + rc_1 f''(N)^{1/2} + rc_2 f''(N)^{1/2}, \end{aligned}$$

whence

$$\{f(N+m)\} > 1 - \varepsilon - 2r(c_1 + c_2) f''(N)^{1/2}. \tag{3.6.5}$$

Since Taylor's theorem implies that

$$f(N+m) = f(N) + mf'(N) + \frac{m^2}{2} f''(N+\theta)$$

for some $\theta \in [0, m]$, the inequalities (1) and $m \le L_{\alpha,3,r}(N) \le c_2 f''(N)^{-1/2}$ yield that

$$\begin{aligned} \{f(N+m)\} &\le \{f(N)\} + \{mf'(N)\} + \frac{m^2}{2} f''(N+\theta) \\ &< c_0 f''(N)^{1/2} + \{mf'(N)\} + c_2^2/2. \end{aligned} \tag{3.6.6}$$

Also, the inequalities (2) and $m \le L_{\alpha,3,r}(N) \le c_2 f''(N)^{-1/2}$ yield that

$$0 \le m\{f'(N)\} - m/2r < c_1 m f''(N)^{1/2} \le c_1 c_2,$$

whence

$$\{mf'(N)\} \le \{m/2r\} + c_1 c_2 \le 1 - 1/2r + c_1 c_2. \tag{3.6.7}$$

Recall the definition of $\varepsilon = \varepsilon(N)$. Using (3.6.5)–(3.6.7), we have

$$\begin{aligned} 1 - 2r^2 f''(N) - 2r(c_1 + c_2) f''(N)^{1/2} &< \{f(N+m)\} \\ < c_0 f''(N)^{1/2} + (1 - 1/2r + c_1 c_2) + c_2^2/2&, \end{aligned}$$

whence

$$1/2r - c_1 c_2 - c_2^2/2 < 2r^2 f''(N) + (c_0 + 2r(c_1 + c_2)) f''(N)^{1/2}. \tag{3.6.8}$$

Since the assumption $0 < c_2 < \sqrt{c_1^2 + 1/r} - c_1$ implies $1/2r - c_1 c_2 - c_2^2/2 > 0$, the inequality (3.6.8) is a contradiction because N is sufficiently large. Therefore,

$$\varlimsup_{x \to \infty} \frac{L_{\alpha,3,r}(x)}{x^{1-\alpha/2}} \ge \frac{c_2}{\sqrt{\alpha(\alpha - 1)}}.$$

Finally, letting $c_2 \to \sqrt{c_1^2 + 1/r} - c_1$ and $c_1 \to r^{-1/2}$, we obtain Proposition 3.6.2. $\square$

Finally, let us show the following proposition that supports $L_{\alpha,3,r}(x) = O_{\alpha,r}(x^{1-\alpha/2})$.

Proposition 3.6.5 ([SY21, Proposition A.5])**.** *Let $\alpha \in (1,2)$ and $r \in \mathbb{N}$, and let $w(x)$ be an arbitrary positive-valued function such that $x^{\alpha/2-1}w(x) \to 0$ and $w(x) \to \infty$ as $x \to \infty$. Then*

$$\lim_{M\to\infty} \frac{\#\{N \in [1,M]_{\mathbb{Z}} : L_{\alpha,3,r}(N) \le N^{1-\alpha/2}w(N)\}}{M} = 1.$$

Proof. For $N, L \in \mathbb{N}$, define $D(N,L)$ as the discrepancy of the sequence $(f(n))_{n=N}^{N+L-1}$. Let $L = L(N) = \lceil N^{1-\alpha/2}w(N)\rceil$ and $K = K(N) = \lceil N^{(2-\alpha)/3}\rceil$. The inequality (2.2.2) and Lemma 2.2.2 imply that for every $N \in \mathbb{N}$,

$$\begin{aligned} D(N,L) &\ll \frac{1}{K} + \frac{1}{L}\sum_{h=1}^{K}\frac{1}{h}\left|\sum_{n=N}^{N+L-1} \mathrm{e}(hf(n))\right| \\ &\ll_{\alpha,w(\cdot)} 1/K + K^{1/2}N^{\alpha/2-1} + N^{1-\alpha/2}/L \ll N^{(\alpha-2)/3} + 1/w(N). \end{aligned}$$

Thus, there exists $C > 0$ such that for every $N \in \mathbb{N}$,

$$D(N,L) \le C(N^{(\alpha-2)/3} + 1/w(N)).$$

Now, let $\varepsilon \in (0, 1/6)$ be arbitrary. Define the sets $\mathcal{C}_{3,2}^{-}(\varepsilon)$, $\mathcal{V}_0$, $\mathcal{V}_1$ and $\mathcal{V}$ as

$$\begin{aligned} \mathcal{C}_{3,2}^{-}(\varepsilon) &= \{(y_0,y_1) \in \mathbb{R}^2 : 0 \le y_0 < 1,\ 0 \le y_0 + 2y_1 < 1-\varepsilon\}, \\ \mathcal{V}_0 &= \big\{N \in \mathbb{N} : \{rf'(N)\} < 1/2 - 3\varepsilon\big\}, \\ \mathcal{V}_1 &= \big\{N \in \mathbb{N} : 1/2 + \varepsilon < \{rf'(N)\} < 1-\varepsilon\big\}, \\ \mathcal{V} &= \{N \in \mathbb{N} : L_{\alpha,3,r}(N) \le N^{1-\alpha/2}w(N)\}. \end{aligned}$$

Due to the assumptions $x^{\alpha/2-1}w(x) \to 0$ and $w(x) \to \infty$, we can taking a positive number x_0 such that

1. $C(x^{(\alpha-2)/3} + 1/w(x)) < 2\varepsilon$ for all $x \ge x_0$,
2. $r\alpha(\alpha-1)x^{\alpha/2-1}w(x) < \varepsilon$ for all $x \ge x_0$,
3. $2r^2 f''(x) \le \varepsilon$ for all $x \ge x_0$.

Let us show the inclusion relation $(\mathcal{V}_0 \cup \mathcal{V}_1) \cap [x_0, \infty) \subset \mathcal{V}$ below.

First, assume $N \in \mathcal{V}_0 \cap [x_0, \infty)$. Then the set $\mathcal{W}_0 := \{n \in [0, L)_{\mathbb{Z}} : \varepsilon < \{f(N+n)\} < 3\varepsilon\}$ satisfies

$$\#\mathcal{W}_0/L \geq 2\varepsilon - D(N, L) = 2\varepsilon - C(N^{(\alpha-2)/3} + 1/w(N)) \overset{(1)}{>} 0.$$

Take an element $m \in \mathcal{W}_0 \neq \emptyset$. Then the assumption $N \in \mathcal{V}_0$ implies that

$$\begin{aligned}\{rf'(N+m)\} &\leq \{rf'(N)\} + rmf''(N) < 1/2 - 3\varepsilon + r\alpha(\alpha-1)(L-1)N^{\alpha-2} \\ &< 1/2 - 3\varepsilon + r\alpha(\alpha-1)N^{\alpha/2-1}w(N) \overset{(2)}{<} 1/2 - 2\varepsilon.\end{aligned}$$

Thus,

$$0 \leq \{f(N+m)\} + 2\{rf'(N+m)\} < 3\varepsilon + 2(1/2 - 2\varepsilon) = 1 - \varepsilon,$$

whence $(\{f(N+m)\}, \{rf'(N+m)\}) \in \mathcal{C}_{3,2}^{-}(\varepsilon)$. Therefore, $(\lfloor f(N+m+rj) \rfloor)_{j=0}^{2}$ is an AP (see the proof of Proposition 3.3.1). Since the inequality $L_{\alpha,3,r}(N) \leq m < L$ holds, it turns out that N lies in $\mathcal{V}$.

Next, assume $N \in \mathcal{V}_1 \cap [x_0, \infty)$. The set $\mathcal{W}_1 := \{n \in [0, L)_{\mathbb{Z}} : 1 - 2\varepsilon < \{f(N+n)\} < 1 - \varepsilon\}$ is also not empty in the same way as $\mathcal{W}_0 \neq \emptyset$. Take an element $m \in \mathcal{W}_1$. Since the difference $rf'(N+m) - rf'(N)$ is bounded above by

$$rmf''(N) \leq r\alpha(\alpha-1)(L-1)N^{\alpha-2} < r\alpha(\alpha-1)N^{\alpha/2-1}w(N) \overset{(2)}{<} \varepsilon,$$

the assumption $N \in \mathcal{V}_1$ implies $\{rf'(N+m)\} \geq \{rf'(N)\} > 1/2 + \varepsilon$. This and $1 - 2\varepsilon < \{f(N+m)\} < 1 - \varepsilon$ yield that

$$\begin{aligned}\{f(N+m)\} + 2(\{rf'(N+m)\} - 1) &< 1 - \varepsilon, \\ \{f(N+m)\} + 2(\{rf'(N+m)\} - 1) &> 1 - 2\varepsilon + 2(1/2 + \varepsilon - 1) = 0,\end{aligned}$$

whence $(\{f(N+m)\}, \{rf'(N+m)\} - 1) \in \mathcal{C}_{3,2}^{-}(\varepsilon)$. Therefore, $(\lfloor f(N+m+rj) \rfloor)_{j=0}^{2}$ is an AP (see the proof of Proposition 3.3.1). Since the inequality $L_{\alpha,3,r}(N) \leq m < L$ holds, it turns out that N lies in $\mathcal{V}$.

The inclusion relation $(\mathcal{V}_0 \cup \mathcal{V}_1) \cap [x_0, \infty) \subset \mathcal{V}$ has been proved above. Since the sequence $(rf'(N))_{N=1}^{\infty}$ is uniformly distributed modulo 1 and the sets $\mathcal{V}_0$ and $\mathcal{V}_1$ are disjoint, it follows that

$$\begin{aligned}\varliminf_{M\to\infty} \frac{\#(\mathcal{V} \cap [1, M])}{M} &\geq \varliminf_{M\to\infty} \frac{\#(\mathcal{V}_0 \cap [1, M])}{M} + \varliminf_{M\to\infty} \frac{\#(\mathcal{V}_1 \cap [1, M])}{M} \\ &\geq (1/2 - 3\varepsilon) + (1/2 - 2\varepsilon) = 1 - 5\varepsilon.\end{aligned}$$

Letting $\varepsilon \to +0$, we obtain Proposition 3.6.5. □

Chapter 4

Linear Diophantine equations with three variables in Piatetski-Shapiro sequences

This chapter is based on [MS20]. We investigate the solvability in $\mathrm{PS}(\alpha)$ of linear Diophantine equations with three variables.

4.1 Results for $\alpha > 2$

For any fixed $a, b, c \in \mathbb{N}$, does the equation

$$ax + by = cz \tag{4.1.1}$$

have infinitely many pairwise distinct solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$, where $\alpha > 2$? By the result of Glasscock [Gla17, Gla20] (Theorem 1.3.8), if the equation $y = \theta x + \eta$ has infinitely many solutions $(x, y) \in \mathbb{N}^2$, then for Lebesgue-a.e. $\alpha > 1$ it is solvable or not in $\mathrm{PS}(\alpha)$ according as $\alpha < 2$ or $\alpha > 2$. As a direct consequence, for Lebesgue-a.e. $1 < \alpha < 2$, the equation $z = (a/c)x + (b/c)$ is solvable in $\mathrm{PS}(\alpha)$ for all $a, b, c \in \mathbb{N}$ with $\gcd(a, c) | b$. In other words, the equation (4.1.1) with $\gcd(a, c) | b$ is solvable in $\mathrm{PS}(\alpha)$. On the other hand, for $\alpha > 2$, we did not know at all whether the equation (4.1.1) is solvable in $\mathrm{PS}(\alpha)$ or not.

Our main result provides an answer to this question. We consider the set of α in a short interval $[s, t] \subset (2, \infty)$ so that (4.1.1) is solvable. The following theorem asserts that its Hausdorff dimension is positive.

Theorem 4.1.1 ([MS20, Theorem 1.1]). *Let $a, b, c \in \mathbb{N}$. For all real numbers $2 < s < t$, we have*

$$\dim_{\mathrm{H}}(\{\alpha \in [s,t] \colon ax + by = cz \textit{ is solvable in } \mathrm{PS}(\alpha)\})$$
$$\geq \begin{cases} \left(s + \dfrac{s^3}{(2 + \{s\} - 2^{1-\lfloor s \rfloor})(2 - \{s\})} \right)^{-1} & \textit{if } a = b = c \\ 2\left(s + \dfrac{s^3}{(2 + \{s\} - 2^{1-\lfloor s \rfloor})(2 - \{s\})} \right)^{-1} & \textit{otherwise.} \end{cases}$$

Note that the lower bound in either case is greater than $1/s^3$ for all $2 < s < t$. The positiveness of the Hausdorff dimension implies that this set is uncountable for any closed interval $[s,t] \subset (2, \infty)$. Moreover, we can easily conclude the following.

Corollary 4.1.2 ([MS20, Corollary 1.2]). *For any closed interval $I \subset (2, \infty)$, the set of $\alpha \in I$ such that $ax + by = cz$ is solvable in $\mathrm{PS}(\alpha)$ is uncountable and dense in I.*

In particular, for $a = b = 1, c = 2$, a pairwise distinct tuple (x, z, y) satisfying (4.1.1) forms an arithmetic progression of length 3. Therefore Corollary 4.1.2 implies Theorem 1.3.7. Glasscock posed a related question to the equation (4.1.1) for $a = b = c = 1$.

Question 4.1.3 ([Gla17, Question 6]). *Does there exist an $\alpha_S > 1$ with the property that for Lebesgue-a.e. or all $\alpha > 1$, the equation $x + y = z$ is solvable or not in $\mathrm{PS}(\alpha)$ according as $\alpha < \alpha_S$ or $\alpha > \alpha_S$?*

By Corollary 4.1.2, the case with "all $\alpha > 1$" in Question 4.1.3 is false since the supremum of $\alpha > 0$ such that (4.1.1) is solvable in $\mathrm{PS}(\alpha)$ is positive infinity. However, the case with "Lebesgue-a.e." in Question 4.1.3 is still open.

4.2 Lemmas I

Let us consider the solvability of the equation (4.1.1). In this and subsequent sections, we fix $a, b, c, d \in \mathbb{N}$ with $d \geq 2$ and $\beta, \gamma \in \mathbb{R}$ with $d < \beta < \gamma < d+1$. Unless it causes confusion, we do not indicate their dependence hereinafter.

Take a large parameter $x_0 = x_0(a,b,c,d,\beta,\gamma) > 0$. For all integer $x \geq x_0$, we define

$$J_{a,b,c}(x) = \begin{cases} \left(\left(\dfrac{b}{cx^2 \log x} + \dfrac{a}{c} \right)^{1/\gamma} x,\ \left(\dfrac{a}{c} \right)^{1/\beta} x \right)_{\mathbb{N}} \setminus (x\mathbb{N}) & \text{if } c < a, \\ \left(\left(\dfrac{a}{c - b(x^2 \log x)^{-1}} \right)^{1/\beta} x,\ \left(\dfrac{a}{c} \right)^{1/\gamma} x \right)_{\mathbb{N}} & \text{if } a < c, \\ \left(2^{1/\gamma} \left(x + \dfrac{1}{x \lceil \log x \rceil} \right),\ 2^{1/\beta} x \right)_{\mathbb{N}} & \text{if } a = b = c, \end{cases}$$

where let $I_{\mathbb{N}}$ denote $I \cap \mathbb{N}$ for all intervals I of real numbers, and let $x\mathbb{N} = \{xn \colon n \in \mathbb{N}\}$. Note that $J_{a,b,c}(x)$ is non-empty if x_0 is sufficiently large. In the case when $a = c$ and $b \neq c$, $J_{a,b,c}(x)$ is not defined above, however this case comes down to the case when $a \neq c$ by switching the roles of (a, x) and (b, y). Thus the three cases in the definition of $J_{a,b,c}(x)$ are sufficient.

Lemma 4.2.1 ([MS20, Lemma 3.1])**.** *Assume that $a \neq c$. Then there exists $C > 0$ such that for all integers $x \geq x_0$ and for all $z \in J_{a,b,c}(x)$, we can find $\alpha = \alpha(x,z) \in (\beta, \gamma)$ so that $ax^\alpha + b = cz^\alpha$, and*

$$\left| \alpha - \frac{\log(a/c)}{\log(z/x)} \right| \leq \frac{C}{x^2 \log x}. \tag{4.2.1}$$

Proof. Fix any $x \geq x_0$ and $z \in J_{a,b,c}(x)$. For all $u \in \mathbb{R}$, define a continuous function $f(u) = ax^u + b - cz^u$. We prove the claim by considering two cases $a > c$ and $c > a$.

Case $a > c$. Let

$$\alpha_0 = \frac{\log(a/c)}{\log(z/x)}, \quad \alpha_1 = \frac{\log(a/c + b/(cx^2 \log x))}{\log(z/x)}.$$

Then, $z \in J_{a,b,c}(x)$ implies $\beta < \alpha_0 < \alpha_1 < \gamma$. It follows that $f(\alpha_0) = b > 0$. If necessary, by taking a larger x_0, we have

$$f(\alpha_1) = x^{\alpha_1}(a + bx^{-\alpha_1} - c(z/x)^{\alpha_1}) \leq x^{\alpha_1}(a + b/(x^2 \log x) - c(z/x)^{\alpha_1}) = 0.$$

Therefore, by the intermediate value theorem, there exists a zero $\alpha = \alpha(x,z)$ of f such that $\beta < \alpha_0 \le \alpha \le \alpha_1 < \gamma$. Since $\log(1+u) \le u$ for all $u \in (-1,\infty)$, we have

$$|\alpha_1 - \alpha_0| = \frac{\log(1 + b/(ax^2\log x))}{\log(z/x)} \le \frac{b}{ax^2\log x} \cdot \frac{1}{\log(z/x)}.$$

By this inequality and $1/\log(z/x) \ll_{a,c,\gamma} 1$, we obtain (4.2.1).

Case $c > a$. Let

$$\alpha_0 = \frac{\log(c/a)}{\log(x/z)}, \quad \alpha_1' = \frac{\log(c/a - b/(ax^2\log x))}{\log(x/z)}.$$

Since $z \in J_{a,b,c}(x)$, $\beta < \alpha_1' < \alpha_0 < \gamma$ and $x \ll_{a,b,c,\beta,\gamma} z$ hold. Then by the calculation in Case $a > c$, $f(\alpha_0) = b > 0$. Further $x \ll z$ implies $z^{-\alpha_1'} \le z^{-\beta} \ll x^{-\beta}$. Thus if x_0 is sufficiently large, we have $z^{-\alpha_1'} \le 1/(x^2\log x)$, which yields that

$$f(\alpha_1') = z^{\alpha_1'}(a(x/z)^{\alpha_1'} + bz^{-\alpha_1'} - c) \le z^{\alpha_1'}(a(x/z)^{\alpha_1'} + b/(x^2\log x) - c) = 0.$$

Therefore, by the intermediate value theorem, there exists a zero $\alpha = \alpha(x,z)$ of f such that $\beta < \alpha_1' \le \alpha \le \alpha_0 < \gamma$. Since $|\log(1-u)| \le 2u$ for all $u \in (0,1/2)$,

$$|\alpha_0 - \alpha_1'| = \frac{|\log(1 - b/(cx^2\log x))|}{\log(x/z)} \le \frac{2b}{cx^2\log x} \cdot \frac{1}{\log(x/z)}$$

provided x_0 is sufficiently large. By this inequality and $1/\log(x/z) \ll_{a,c,\gamma} 1$, we obtain (4.2.1). □

Lemma 4.2.2 ([MS20, Lemma 3.2]). *There exists $C > 0$ such that for all integers $x \ge x_0$ and $z \in J_{1,1,1}(x)$, we can find $\alpha = \alpha(x,z) \in (\beta,\gamma)$ so that $x^\alpha + (x + (x\lceil\log x\rceil)^{-1})^\alpha = z^\alpha$, and*

$$\left|\alpha - \frac{\log 2}{\log(z/x)}\right| \le \frac{C}{x^2\log x}. \tag{4.2.2}$$

Proof. Take any $x \ge x_0$ and $z \in J_{1,1,1}(x)$. For all $u \in \mathbb{R}$, define a continuous function $f(u) = x^u + (x + (x\lceil\log x\rceil)^{-1})^u - z^u$, and set

$$\alpha_0 = \frac{\log 2}{\log(z/x)}, \quad \alpha_1 = \frac{\log 2}{\log\left(\dfrac{z}{x + (x\lceil\log x\rceil)^{-1}}\right)}.$$

By $z \in J_{1,1,1}(x)$, $\beta < \alpha_0 < \alpha_1 < \gamma$ holds. By the definitions of α_0 and α_1, we have

$$f(\alpha_0) > z^{\alpha_0}\left(\frac{1}{2}+\frac{1}{2}-1\right) = 0, \quad f(\alpha_1) < z^{\alpha_1}\left(\frac{1}{2}+\frac{1}{2}-1\right) = 0.$$

Therefore, by the intermediate value theorem, there exists a zero $\alpha = \alpha(x, z)$ of f such that $\alpha_0 \leq \alpha \leq \alpha_1$. Further, we conclude (4.2.2) since

$$|\alpha_1 - \alpha_0| \leq \frac{\gamma^2}{\log 2}\log\left(1+\frac{1}{x^2\log x}\right) \leq \frac{\gamma^2}{\log 2}\cdot\frac{1}{x^2\log x}.$$

□

Lemma 4.2.3 ([MS20, Lemma 3.3]). *Let $\varepsilon > 0$ be an arbitrarily small real number. For all $X, Y, Z \in \mathbb{N}$, and $\alpha \in \mathbb{R}$ with $\beta < \alpha < \gamma$, if we have*

$$aX^\alpha + bY^\alpha = cZ^\alpha, \tag{4.2.3}$$

then there exists $n_0 \in \mathbb{N}$ such that

$$a\lfloor (n_0X)^\alpha\rfloor + b\lfloor (n_0Y)^\alpha\rfloor = c\lfloor (n_0Z)^\alpha\rfloor, \tag{4.2.4}$$

$$\max(\{(n_0X)^\alpha\}, \{(n_0Y)^\alpha\}, \{(n_0Z)^\alpha\}) < \frac{1}{2}, \tag{4.2.5}$$

$$n_0 \ll_\varepsilon (X+Y)^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\}))+\varepsilon}. \tag{4.2.6}$$

Proof. Choose $X, Y, Z \in \mathbb{N}$ and α with $\beta < \alpha < \gamma$ satisfying (4.2.3). For all $n \in \mathbb{N}$,

$$c\lfloor (nZ)^\alpha\rfloor = c(nZ)^\alpha - c\{(nZ)^\alpha\} = a\lfloor (nX)^\alpha\rfloor + b\lfloor (nY)^\alpha\rfloor + \delta(n),$$

where define $\delta(n) = a\{(nX)^\alpha\} + b\{(nY)^\alpha\} - c\{(nZ)^\alpha\}$. Let

$$A = \left\{n \in \mathbb{N} \colon |\delta(n)| < 1,\ \max(\{(nX)^\alpha\}, \{(nY)^\alpha\}, \{(nZ)^\alpha\}) < \frac{1}{2}\right\},$$

and note that any $n \in A$ satisfies (4.2.4) and (4.2.5). Let us show the existence of $n \in A$ satisfying (4.2.6). Take a small $\xi = \xi(d, \beta, \gamma, \varepsilon) > 0$ and take a sufficiently large parameter $R = R(a, b, c, d, \beta, \gamma, \varepsilon)$. Set

$$N = \left\lceil R(X+Y)^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\}))+\varepsilon}\right\rceil, \tag{4.2.7}$$

and set $\psi = \{\beta\} - 2 + (2^{d+2} - 2)(1/2^d - 2\xi)$. Since this is reformulated to

$$\psi = 2 + \{\beta\} - 2^{1-\lfloor\beta\rfloor} + O(\xi), \tag{4.2.8}$$

we have $0 < \psi < \beta < \alpha$ for a small enough ξ. Moreover, we let $L(h_1, h_2) = (h_1 X^\alpha + h_2 Y^\alpha)/c$.

Case 1. We firstly discuss the case when

$$|L(h_1, h_2)| \geq N^{-\psi} \tag{4.2.9}$$

holds for all $h_1, h_2 \in \mathbb{Z}$ with $0 < \max\{|h_1|, |h_2|\} \leq N^\xi$. In this case, define

$$A_1 = \left\{ n \in \mathbb{N} \colon 0 \leq \{(nX)^\alpha/c\} < \frac{1}{4ac}, \quad 0 \leq \{(nY)^\alpha/c\} < \frac{1}{4bc} \right\}. \tag{4.2.10}$$

Then we have $A_1 \subseteq A$. Indeed, take any $n \in A_1$. We see that

$$(nX)^\alpha = c\lfloor (nX)^\alpha/c \rfloor + c\{(nX)^\alpha/c\}. \tag{4.2.11}$$

Since the first term on the right-hand side of (4.2.11) is an integer and the second term belongs to $[0,1)$ by $n \in A_1$, we get $\{(nX)^\alpha\} = c\{(nX)^\alpha/c\}$. This yields that $\{(nX)^\alpha\} < 1/(4a)$. Similarly, $\{(nY)^\alpha\} < 1/(4b)$ holds. Further,

$$\{(nZ)^\alpha\} = \{a(nX)^\alpha/c + b(nY)^\alpha/c\} \leq a\{(nX)^\alpha/c\} + b\{(nY)^\alpha/c\} < \frac{1}{2c}.$$

Hence we have

$$|\delta(n)| \leq a\{(nX)^\alpha\} + b\{(nY)^\alpha\} + c\{(nZ)^\alpha\} < \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1.$$

Therefore $A_1 \subseteq A$ holds.

We now evaluate the distribution of A_1. Let $D_1(N)$ be the discrepancy of the sequence $((nX)^\alpha/c,\ (nY)^\alpha/c)_{N<n\leq 2N}$. The inequality (2.2.1) with $K = \lfloor N^\xi \rfloor$ implies that

$$D_1(N) \ll N^{-\xi} + \sum_{0<\|(h_1,h_2)\|_\infty \leq N^\xi} \frac{1}{\nu(h_1,h_2)} \left| \frac{1}{N} \sum_{N<n\leq 2N} e(L(h_1,h_2)n^\alpha) \right|.$$

For all $u \in \mathbb{R}$, define $f(u) = L(h_1, h_2)u^\alpha$. For each $N < u \leq 2N$,

$$|L(h_1,h_2)|N^{\alpha-(d+2)} \ll |f^{(d+2)}(u)| \ll |L(h_1,h_2)|N^{\alpha-(d+2)}.$$

Therefore Lemma 2.2.4 with $k = d + 2$ yields that

$$\begin{aligned}
&\frac{1}{N} \sum_{N<n\leq 2N} \mathrm{e}(L(h_1,h_2)n^\alpha) \\
&\ll (|L(h_1,h_2)|N^{\alpha-(d+2)})^{1/(2^{d+2}-2)} + \frac{(|L(h_1,h_2)|N^{\alpha-(d+2)})^{-1/(2^{d+2}-2)}}{N^{1/2^d}} \\
&\ll (L(N^\xi,N^\xi)N^{\{\gamma\}-2})^{1/(2^{d+2}-2)} + \frac{N^{(2-\{\beta\}+\psi)/(2^{d+2}-2)}}{N^{1/2^d}},
\end{aligned}$$

where in the last inequality we used that $\alpha - d < \{\gamma\}$ and $d+2-\alpha < 2-\{\beta\}$. By the definition of ψ, it follows that $(2-\{\beta\}+\psi)/(2^{d+2}-2) - 1/2^d = -2\xi$. Then

$$\frac{1}{N} \sum_{N<n\leq 2N} \mathrm{e}(L(h_1,h_2)n^\alpha) \ll \left((X+Y)^\gamma N^{\{\gamma\}-2+\xi}\right)^{1/(2^{d+2}-2)} + N^{-2\xi}.$$

Therefore, since

$$\sum_{0<\|(h_1,h_2)\|_\infty \leq N^\xi} \frac{1}{\nu(h_1,h_2)} \ll (\log N^\xi)^2 \ll_\xi N^{\xi/(2^{d+2}-2)},$$

we have

$$D_1(N) \ll_\xi N^{-\xi} + \left((X+Y)^\gamma N^{\{\gamma\}-2+2\xi}\right)^{1/(2^{d+2}-2)}. \tag{4.2.12}$$

Let $E_1(N)$ be the right-hand side of (4.2.12). By the definition of the discrepancy,

$$\frac{\#(A_1 \cap (N,2N])}{N} = \frac{1}{16abc^2} + O_\xi\left(E_1(N)\right).$$

By (4.2.7), we have

$$(X+Y)^\gamma N^{\{\gamma\}-2+2\xi} \ll R^{\{\gamma\}-2+2\xi}(X+Y)^e. \tag{4.2.13}$$

Here the exponent e of $(X+Y)$ on the right-hand side of (4.2.13) is negative since

$$\begin{aligned}
e &= \gamma + (\{\gamma\}-2+2\xi)\left(\frac{\gamma^2}{(2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\})} + \varepsilon\right) \\
&= \gamma\left(1 - \frac{\gamma}{2+\{\beta\}-2^{1-\lfloor\beta\rfloor}}\right) - \varepsilon(2-\{\gamma\}) + O(\xi) \\
&\leq \gamma \cdot \frac{2+\{\beta\}-\gamma}{2+\{\beta\}-2^{1-\lfloor\beta\rfloor}} - \varepsilon(2-\{\gamma\}) + O(\xi) < 0
\end{aligned}$$

holds for a small enough ξ. This yields that

$$E_1(N) \ll_\xi R^{-\xi} + R^{(\{\gamma\}-2+2\xi)/(2^{d+2}-2)}.$$

Therefore if ξ is sufficiently small and R is sufficiently large, then the following holds:

$$\frac{1}{16abc^2} + O_\xi\left(E_1(N)\right) \geq \frac{1}{32abc^2}.$$

Hence, in this case, $\#(A \cap (N, 2N]) \geq \#(A_1 \cap (N, 2N]) \geq N/(32abc^2) > 0$, which implies that there exists $n_0 \in A$ satisfying (4.2.6).

Case 2. We next discuss the case when (4.2.9) is false, that is to say, there exist $h_1, h_2 \in \mathbb{Z}$ with $0 < \max\{|h_1|, |h_2|\} \leq N^\xi$ such that

$$|L(h_1, h_2)| < N^{-\psi}. \tag{4.2.14}$$

We observe that h_1 and h_2 are non-zero and that h_1 has the opposite sign of h_2, since if not, $1/c \leq |L(h_1, h_2)| < N^{-\psi}$ holds, which causes a contradiction when R is sufficiently large. Thus we may assume that $h_1 < 0 < h_2$ by multiplying the both sides of (4.2.14) by $|(-1)|$ if necessary. Let $h_1' = -h_1$, and $\theta = L(h_1, h_2)/h_2$.

In the case $\theta \geq 0$, by letting

$$A_2 = \left\{ n \in [1, N^{\psi/\alpha}/(8bc)^{1/\alpha}] \cap \mathbb{N} \colon 0 \leq \{(nX)^\alpha/(ch_2)\} < \frac{1}{8abcN^\xi} \right\}, \tag{4.2.15}$$

$A_2 \subseteq A$ holds. To see why, suppose $n \in A_2$. Then $(nX)^\alpha/c = h_2\lfloor (nX)^\alpha/(ch_2)\rfloor + h_2\{(nX)^\alpha/(ch_2)\}$, of which the first term is an integer and the second term belongs to $[0, 1)$. This yields that $\{(nX)^\alpha/c\} = h_2\{(nX)^\alpha/(ch_2)\}$. Thus we obtain $0 \leq \{(nX)^\alpha/c\} < 1/(4ac)$. Further, since

$$(nY)^\alpha/c = \frac{h_1'}{ch_2}(nX)^\alpha + n^\alpha\theta = h_1'\lfloor (nX)^\alpha/(ch_2)\rfloor + h_1'\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta,$$
$$h_1'\lfloor (nX)^\alpha/(ch_2)\rfloor \in \mathbb{Z}, \quad 0 \leq h_1'\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta < \frac{1}{8bc} + \frac{1}{8bc} = \frac{1}{4bc},$$

we have $\{(nY)^\alpha/c\} = h_1'\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta$ and $0 \leq \{(nY)^\alpha/c\} < 1/(4bc)$. Hence, we obtain $A_2 \subseteq A_1 \subseteq A$.

We next evaluate the distribution of A_2. Let $V = N^{\psi/\alpha}/(2(8bc)^{1/\alpha})$, and $D_2(N)$ be the discrepancy of the sequence $((nX)^\alpha/(ch_2))_{V<n\le 2V}$. Then by the inequality (2.2.1) with $K = \lfloor N^{2\xi} \rfloor$,

$$D_2(N) \ll \frac{1}{N^{2\xi}} + \sum_{0<|h|\le N^{2\xi}} \frac{1}{|h|} \left| \frac{1}{V} \sum_{V<n\le 2V} \mathrm{e}((h/(ch_2))X^\alpha n^\alpha) \right|.$$

From Lemma 2.2.4 with $k = d+2$, the following holds:

$$\begin{aligned} D_2(N) &\ll \frac{1}{N^{2\xi}} \\ &+ \sum_{0<|h|\le N^{2\xi}} \frac{1}{|h|} \left(\left(\frac{|h|X^\alpha}{ch_2} V^{\alpha-d-2} \right)^{1/(2^{d+2}-2)} + \frac{\left(\frac{|h|X^\alpha}{ch_2} V^{\alpha-d-2} \right)^{-1/(2^{d+2}-2)}}{V^{1/2^d}} \right). \end{aligned}$$

We see that

$$\begin{aligned} &\sum_{0<|h|\le N^{2\xi}} \frac{1}{|h|} \left(\frac{|h|X^\alpha}{ch_2} V^{\alpha-d-2} \right)^{1/(2^{d+2}-2)} \\ &\le (X^\gamma V^{\{\gamma\}-2})^{1/(2^{d+2}-2)} \cdot 2 \sum_{1\le h\le N^{2\xi}} h^{-1+1/(2^{d+2}-2)} \\ &\ll (X^\gamma V^{\{\gamma\}-2})^{1/(2^{d+2}-2)} \cdot N^{2\xi/(2^{d+2}-2)}. \end{aligned}$$

In addition, by $d - \alpha < 0$ and $h_2 \le N^\xi$, we see that

$$\begin{aligned} &\sum_{0<|h|\le N^{2\xi}} \frac{1}{|h|} \cdot \frac{\left(\frac{|h|X^\alpha}{ch_2} V^{\alpha-d-2} \right)^{-1/(2^{d+2}-2)}}{V^{1/2^d}} \\ &\le \left(\frac{ch_2}{X^\alpha} \right)^{1/(2^{d+2}-2)} V^{(2+d-\alpha)/(2^{d+2}-2)-1/2^d} \cdot 2 \sum_{h=1}^{\infty} h^{-1-1/(2^{d+2}-2)} \\ &\ll N^\xi \cdot V^{1/(2^{d+1}-1)-1/2^d} = N^\xi V^{(-1+2^{-d})/(2^{d+1}-1)}. \end{aligned}$$

Hence we have

$$\begin{aligned} D_2(N) &\ll \frac{1}{N^{2\xi}} + \left(X^\gamma N^{2\xi} V^{\{\gamma\}-2} \right)^{1/(2^{d+2}-2)} + N^\xi V^{(-1+2^{-d})/(2^{d+1}-1)} \\ &\ll \frac{1}{N^{2\xi}} + \left(X^\gamma N^{2\xi+\psi(\{\gamma\}-2)/\gamma} \right)^{1/(2^{d+2}-2)} + N^{\xi+\psi(-1+2^{-d})/(\gamma(2^{d+1}-1))}. \end{aligned}$$

Let $E_2(N)$ be the most right-hand side. Now by (4.2.7), we have

$$X^{\gamma}N^{2\xi+\psi(\{\gamma\}-2)/\gamma} \ll R^{2\xi+\psi(\{\gamma\}-2)/\gamma}(X+Y)^{e'}. \tag{4.2.16}$$

The exponent e' of $(X+Y)$ on the right-hand side of (4.2.16) is equal to

$$\begin{aligned} e' &= \gamma + \left(2\xi + \frac{\psi}{\gamma}(\{\gamma\}-2)\right)\left(\frac{\gamma^2}{(2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\})} + \varepsilon\right) \\ &= \gamma - \gamma\cdot\frac{2+\{\beta\}-2^{1-\lfloor\beta\rfloor}+O(\xi)}{2+\{\beta\}-2^{1-\lfloor\beta\rfloor}} - \varepsilon\cdot\frac{\psi}{\gamma}(2-\{\gamma\}) + O(\xi) \\ &= -\varepsilon\cdot\frac{\psi}{\gamma}(2-\{\gamma\}) + O(\xi), \end{aligned}$$

where we used (4.2.8). This yields that for a small enough ξ,

$$\begin{aligned} E_2(N) \ll N^{-2\xi} &+ (R^{2\xi+\psi(\{\gamma\}-2)/\gamma}(X+Y)^{e'})^{1/(2^{d+2}-2)} \\ &+ N^{\xi+\psi(-1+2^{-d})/(\gamma(2^{d+1}-1))} \\ \ll N^{-2\xi}. \end{aligned}$$

Therefore, if necessary, by making ξ smaller and R larger, we get

$$\frac{\#(A_2\cap(V,2V])}{V} = \frac{1}{8abcN^{\xi}} + O(E_2(N)) \geq \frac{1}{16abcN^{\xi}} > 0.$$

Hence, there exists $n_0 \in A$ such that

$$n_0 \ll_{\varepsilon} ((X+Y)^{\psi/\alpha})^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor\beta\rfloor})(2-\{\gamma\}))+\varepsilon},$$

which implies the inequality (4.2.6) since $\psi < \alpha$. In the case $\theta < 0$, let $\theta' = L(h_1,h_2)/h_1 > 0$. By switching the roles of (θ, X^{α}) and (θ', Y^{α}) and repeating a similar argument to the case $\theta \geq 0$, we also find $n_0 \in A$ satisfying (4.2.6). □

Lemma 4.2.4 ([MS20, Lemma 3.4]). *For all $\alpha > 0$ and $X, Y, Z \in \mathbb{N}$, define*

$$\eta(\alpha, X, Y, Z) = \min\left\{\frac{\log((\lfloor W^{\alpha}\rfloor+1)W^{-\alpha})}{\log W} : W = X, Y, Z\right\}.$$

For all $\alpha > 0$ and $X, Y, Z \in \mathbb{N}$, if $a\lfloor X^{\alpha}\rfloor + b\lfloor Y^{\alpha}\rfloor = c\lfloor Z^{\alpha}\rfloor$ holds, then for all $\tau \in (\alpha, \alpha+\eta(\alpha, X, Y, Z))$, we have

$$a\lfloor X^{\tau}\rfloor + b\lfloor Y^{\tau}\rfloor = c\lfloor Z^{\tau}\rfloor.$$

Proof. The claim is clear since we observe that

$$\lfloor X^{\alpha}\rfloor = \lfloor X^{\tau}\rfloor, \quad \lfloor Y^{\alpha}\rfloor = \lfloor Y^{\tau}\rfloor, \quad \lfloor Z^{\alpha}\rfloor = \lfloor Z^{\tau}\rfloor$$

for all $\tau \in (\alpha, \alpha+\eta(\alpha, X, Y, Z))$. □

4.3 Lemmas II

Let $2 \le \beta < \gamma$, and let $a, b, c \in \mathbb{N}$ as in the previous section. Let $x_0 > 0$ be a large parameter. For each $x \ge x_0$, let $K(x) \subseteq \mathbb{N}$ be a non-empty finite set. For each $x \ge x_0$ and $z \in K(x)$, let $\theta(x,z)$ and $\ell(x,z)$ be positive real numbers, and define an interval $I(x,z) = (\theta(x,z),\ \theta(x,z)+\ell(x,z))$. For each $x \ge x_0$, define

$$G_x = \bigcup_{z \in K(x)} I(x,z).$$

Let us consider the following conditions:

(C1) for all integer $x \ge x_0$, $K(x)$ does not contain any multiples of x;

(C2) for all integers $x \ge x_0$ and $z \in K(x)$, if $z \ne \max K(x)$, then $z+1 \in K(x)$ or $z+2 \in K(x)$;

(C3) there exists $Q_1 > 0$ such that for all $x \ge x_0$,

$$\max\left(\inf\{|\beta - \alpha| \colon \alpha \in G_x\},\ \inf\{|\gamma + x^{-2} - \alpha| \colon \alpha \in G_x\}\right) \le Q_1 x^{-1};$$

(C4) there exists a real number $\kappa \in (0,\infty) \setminus \{1\}$ such that for all $x \ge x_0$ and $z \in K(x)$,

$$\theta(x,z) = \frac{\log \kappa}{\log(z/x)} + O\left(\frac{1}{x^2 \log x}\right);$$

(C5) there exist $Q_2, Q_3 > 0$ and $q > 2$ such that for all $x \ge x_0$ and $z \in K(x)$,

$$Q_2 x^{-q} \le \ell(x,z) \le Q_3 x^{-\beta};$$

(C6) for all integer $x \ge x_0$, $G_x \subseteq (\beta, \gamma + x^{-2})$ holds;

(C7) for all integers $x \ge x_0$ and $z \in K(x)$, there exists a pairwise distinct tuple $(X(x,z), Y(x,z), Z(x,z)) \in \mathbb{N}^3$ such that for all $\tau \in I(x,z)$,

$$a\lfloor X(x,z)^\tau \rfloor + b\lfloor Y(x,z)^\tau \rfloor = c\lfloor Z(x,z)^\tau \rfloor, \quad X(x,z) \ge x.$$

Proposition 4.3.1 ([MS20, Proposition 4.1]). *Suppose that there exist x_0, $K(x)$, $\theta(x,z)$, and $\ell(x,z)$ satisfying* (C1) *to* (C7). *Let q be as in* (C5). *Then we have*

$$\dim_{\mathrm{H}}(\{\alpha \in [\beta,\gamma] \colon ax+by=cz \text{ is solvable in } \mathrm{PS}(\alpha)\}) \ge \frac{2}{q}.$$

Remark 4.3.2 ([MS20, Remark 4.2]). The idea of the proof of Proposition 4.3.1 comes from the proof of Jarník's theorem (Theorem 2.3.4) in the book written by Falconer [Fal14, Theorem 10.3].

The goal of this section is to prove Proposition 4.3.1. Suppose that there exist x_0, $K(x)$, $\theta(x,z)$, and $\ell(x,z)$ satisfying the conditions (C1) to (C7), and choose such x_0, $K(x)$, $\theta(x,z)$, and $\ell(x,z)$. Take constants Q_1, Q_2, Q_3, κ, q which appear in the conditions (C3) to (C5). Let $x_1 > 0$ and $U_1 > 0$ be large parameters depending on a, b, c, d, β, γ, Q_1, Q_2, Q_3, κ, x_0, q. We do not indicate the dependence of those parameters, hereinafter. Let p denote a variable running over prime numbers.

Lemma 4.3.3 ([MS20, Lemma 4.3]). *There exists $B_1 > 0$ such that for all $p \geq x_1$ and distinct $z, z' \in K(p)$, two intervals $I(p,z)$ and $I(p,z')$ are separated by a gap of at least $B_1 p^{-1}$ if x_1 is sufficiently large.*

Proof. By the conditions (C4) and (C6), for all $p \geq x_1$ and $z \in K(p)$, we have
$$\frac{\beta}{2} \leq \frac{\log \kappa}{\log(z/p)} \leq 2\gamma \tag{4.3.1}$$
if x_1 is sufficiently large. This implies that
$$p \ll z \ll p. \tag{4.3.2}$$
By the condition (C4) and the inequalities (4.3.1) and (4.3.2), there exists $B_0 > 0$ such that
$$\begin{aligned}
|\theta(p,z) - \theta(p,z')| &= \left| \frac{\log \kappa}{\log \frac{z}{p}} - \frac{\log \kappa}{\log \frac{z'}{p}} + O\left(\frac{1}{p^2 \log p} \right) \right| \\
&\geq \frac{|\log \kappa| |\log \frac{z'}{z}|}{|\log \frac{z}{p}| |\log \frac{z'}{p}|} + O\left(\frac{1}{p^2 \log p} \right) \\
&\geq \frac{\beta^2}{4 |\log \kappa|} \log \left(\frac{z+1}{z} \right) + O\left(\frac{1}{p^2 \log p} \right) \geq B_0 p^{-1}
\end{aligned}$$
for all $p \geq x_1$ and all $z, z' \in K(p)$ with $z < z'$. Further, since $\ell(p,z) \leq Q_3 p^{-2}$ holds by (C5), there exists $B_1 > 0$ such that for all $p \geq x_1$ and distinct $z, z' \in K(p)$, two intervals $I(p,z)$ and $I(p,z')$ are separated by a gap of at least
$$B_0 p^{-1} - Q_3 p^{-2} \geq B_1 p^{-1} \tag{4.3.3}$$
if x_1 is sufficiently large. □

Now we call the open interval $I(p,z)$ $(z \in K(p))$ a basic interval of G_p for all $p \geq x_1$. For each $U \geq U_1$, define

$$H_U = \bigcup_{\substack{U<p\leq 2U \\ p:\ \text{prime}}} G_p.$$

For all $U < p \leq 2U$, we also call the basic interval of G_p basic interval of H_U.

Lemma 4.3.4 ([MS20, Lemma 4.4]). *There exist $B_2, B_3 > 0$ such that for any $U \geq U_1$, all distinct basic intervals of H_U are separated by gaps of at least $B_2 U^{-2}$, and the length of each basic interval of H_U is at least $B_3 U^{-q}$ if U_1 is sufficiently large.*

Proof. We take distinct prime numbers p and p' with $U < p, p' \leq 2U$. Then, for all $z \in K(p)$ and $z' \in K(p')$, the condition (C4), the inequality (4.3.1), and the mean value theorem imply that

$$\begin{aligned} |\theta(p,z) - \theta(p',z')| &\geq \left| \frac{\log\kappa}{\log(z/p)} - \frac{\log\kappa}{\log(z'/p')} \right| + O\left(\frac{1}{U^2 \log U}\right) \\ &\geq \frac{\beta^2}{4|\log\kappa|} \left| \frac{z}{p} - \frac{z'}{p'} \right| \min\left\{\frac{p}{z}, \frac{p'}{z'}\right\} + O\left(\frac{1}{U^2\log U}\right). \end{aligned}$$

We may assume that $p'/z' > p/z$. By the condition (C1), z and p are coprime, which yields that $|zp' - z'p| \geq 1$. Therefore we obtain

$$\left|\frac{z}{p} - \frac{z'}{p'}\right| \min\left\{\frac{p}{z}, \frac{p'}{z'}\right\} = \left|\frac{z}{p} - \frac{z'}{p'}\right| \frac{p}{z} \geq \frac{1}{p'z} \gg U^{-2}$$

by the inequalities (4.3.2) and $U < p, p' \leq 2U$. Therefore for all $U \geq U_1$, we have

$$|\theta(p,z) - \theta(p',z')| \gg \frac{1}{U^2} \tag{4.3.4}$$

if U_1 is sufficiently large. Further, for all $U < p \leq 2U$ and $z \in K(p)$, we have by (C5) that $\ell(p,z) \ll U^{-\beta}$, where $\beta \geq 2$. Hence there exists $D_1 > 0$ such that for all distinct prime numbers $U < p, p' \leq 2U$, $z \in K(p)$, and $z' \in K(p')$, the intervals $I(p,z)$ and $I(p',z')$ are separated by gaps of at least $D_1 U^{-2}$. By combining with Lemma 4.3.3, there exists $D_2 > 0$ such that distinct basic intervals of H_U are separated by gaps of at least $D_2 U^{-2}$. Furthermore by (C5), for all $U < p \leq 2U$ and $z \in K(p)$, we have $Q_2 \cdot 2^{-q} U^{-q} \leq \ell(p,z)$. In conclusion, we find that all distinct basic intervals of H_U are separated by gaps of at least $B_2 U^{-2}$, and have length of at least $B_3 U^{-q}$, where we let $B_2 = D_2$ and $B_3 = Q_2 \cdot 2^{-q}$. □

Lemma 4.3.5 ([MS20, Lemma 4.5])**.** *There exists $B_4 > 0$ such that the following statement holds: for every $U \geq U_1$, if an open interval $I \subset (\beta, \gamma + p^{-2})$ satisfies*

$$3B_4/\mathrm{diam}(I) < U < p \leq 2U, \tag{4.3.5}$$

then the open interval I completely includes at least

$$\frac{U^2}{6B_4 \log U} \cdot \mathrm{diam}(I) \tag{4.3.6}$$

basic intervals of H_U.

Proof. By (C4), (4.3.1), and (4.3.2), there exists $D_3 > 0$ such that for every $z \in K(p)$ and the minimum $z' \in K(p)$ with $z' > z$,

$$\begin{aligned}|\theta(p,z) - \theta(p,z')| &= \left| \frac{\log \kappa}{\log(z/p)} - \frac{\log \kappa}{\log(z'/p)} + O\left(\frac{1}{p^2 \log p}\right) \right| \\ &\leq \frac{4\gamma^2}{|\log \kappa|} \cdot \frac{1}{z} \cdot |z - z'| + O\left(\frac{1}{p^2 \log p}\right) \leq D_3 p^{-1}. \end{aligned} \tag{4.3.7}$$

Here we apply (C2) when we deduce the last inequality. From (C3) ,(C6) and (4.3.7), there exists $B_4 > 0$ such that

$$\begin{aligned}(\beta, \gamma + p^{-2}) \subseteq \left(\beta, \beta + B_4 p^{-1}\right) \cup \left(\bigcup_{z \in K(p)} \left(\theta(p,z), \theta(p,z) + B_4 p^{-1}\right) \right) \\ \cup \left(\gamma + p^{-2} - B_4 p^{-1}, \gamma + p^{-2}\right).\end{aligned}$$

Therefore for all $U \geq U_1$ and $U < p \leq 2U$, any open interval $I \subset (\beta, \gamma + p^{-2})$ satisfying (4.3.5) completely includes at least $B_4^{-1} p \cdot \mathrm{diam}(I) - 2 \geq (3B_4)^{-1} U \cdot \mathrm{diam}(I)$ basic intervals of G_p. Hence, by the prime number theorem, the open interval I completely includes at least (4.3.6) basic intervals of H_U for a large enough U_1. □

Proof of Proposition 4.3.1. Let B_3 and B_4 be constants as in Lemma 4.3.4 and Lemma 4.3.5, respectively. Let $u_1 = \max(U_1, 2)$. For every $k = 2, 3, \ldots$, we put

$$u_k = \max\{u_{k-1}^k, \ \lceil 3(B_4/B_3) u_{k-1}^q \rceil\},$$

and $B_5 = B_3/(6B_4)$. Let E_1 be the open interval $(\beta, 2\gamma)$. For every $k = 2, 3, \ldots$, let E_k be the union of basic intervals of H_{u_k} which are completely included by E_{k-1}. Let F be the intersection of all E_k's. Define $m_1 = 1$, and for $k \geq 2$, define

$$m_k = \frac{u_k^2}{6B_4 \log u_k} B_3 u_{k-1}^{-q} = B_5 \frac{u_k^2 u_{k-1}^{-q}}{\log u_k}.$$

Lemma 4.3.4 implies that each $(k-1)$-st level interval of F has length at least $B_3 u_{k-1}^{-q}$. Then, by Lemma 4.3.5, each $(k-1)$-st level interval of F contains at least m_k k-th level intervals. In addition, by Lemma 4.3.4, disjoint k-th level intervals of F are separated by gaps of at least $\delta_k = B_2 u_k^{-2}$. Therefore, Lemma 2.3.2 implies that

$$\begin{aligned}
&\dim_{\mathrm{H}} F \\
&\geq \varliminf_{k\to\infty} \frac{\log(m_1 m_2 \cdots m_{k-1})}{-\log(\delta_k m_k)} \\
&= \varliminf_{k\to\infty} \frac{2\log u_{k-1} + \log\left(B_5^{k-2} u_1^{-q} (u_2 \cdots u_{k-2})^{2-q} (\log u_2)^{-1} \cdots (\log u_{k-1})^{-1}\right)}{q \log u_{k-1} + \log\log u_k - \log(B_2 B_5)}.
\end{aligned}$$

Since $u_k \geq u_{k-1}^k$ for all $k \geq 2$, we have $\log u_k \geq k! \log u_1$ and $u_k \geq u_{k-1}$. Further, for a large enough $k \geq 1$, $u_k = u_{k-1}^k$ holds. Thus for a large enough $k \geq 1$, we see that

$$2\log u_{k-1} = 2k^{-1}\log u_k, \quad q\log u_{k-1} = qk^{-1}\log u_k,$$
$$\left|\log\left(B_5^{k-2} u_1^{-q} (u_2 \cdots u_{k-2})^{2-q} (\log u_2)^{-1} \cdots (\log u_{k-1})^{-1}\right)\right| \ll \log u_{k-2}.$$

Therefore, since $\log u_{k-2} / \log u_k = 1/(k(k-1)) \to 0$ as $k \to \infty$, we get

$$\dim_{\mathrm{H}} \left(\bigcap_{k=1}^{\infty} E_k \right) \geq \frac{2}{q}.$$

We finally show that for any $\tau \in F$, the equation $ax + by = cz$ is solvable in $\mathrm{PS}(\tau)$ and $\tau \in [\beta, \gamma]$. If this claim is true, we get the conclusion of Proposition 4.3.1 by the monotonicity of the Hausdorff dimension.

Take any $\tau \in F$. It is clear that $\tau \in [\beta, \gamma]$ since the condition (C6) yields $H_{u_k} \subseteq (\beta, \gamma + u_k^{-2})$, which implies $F \subseteq [\beta, \gamma]$. Further, by (C7), for all $k > 1$, there exist a prime number $u_k < p_k \leq 2u_k$ and $z_k \in K(p_k)$ such that we find a pairwise distinct tuple $(X(p_k, z_k), Y(p_k, z_k), Z(p_k, z_k)) \in \mathbb{N}^3$ such that

$$a\lfloor X(p_k, z_k)^\tau \rfloor + b\lfloor Y(p_k, z_k)^\tau \rfloor = c\lfloor Z(p_k, z_k)^\tau \rfloor, \quad X(p_k, z_k) \geq p_k.$$

Since $X(p_k, z_k) \geq p_k \geq u_k \to \infty$ as $k \to \infty$, the equation $ax + by = cz$ is solvable in $\mathrm{PS}(\tau)$. $\square$

4.4 Proof of Theorem 4.1.1

Fix any $a, b, c \in \mathbb{N}$. Without loss of generality, we may assume that either $a \neq c$ or $a = b = c = 1$. Let $\varepsilon > 0$ be an arbitrarily small real number. Let $d = \lfloor s \rfloor$ and choose real numbers β, γ with $d \leq s < \beta < \gamma < \min\{t, d+1\}$. Let $x_0 = x_0(a, b, c, d, \beta, \gamma)$ be from Section 4.2. By the monotonicity of the Hausdorff dimension, we have

$$\begin{aligned}&\dim_{\mathrm{H}}(\{\alpha \in [s,t] \colon ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\})\\ &\geq \dim_{\mathrm{H}}(\{\alpha \in [\beta,\gamma] \colon ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\}).\end{aligned} \tag{4.4.1}$$

Take $\alpha(x, z)$ as in Lemma 4.2.1 and Lemma 4.2.2. Let $K(x) = J_{a,b,c}(x)$, $\theta(x, z) = \alpha(x, z)$. We give $\ell(x, z)$ later. Let us check the conditions (C1) to (C7), and apply Proposition 4.3.1.

Case $a > c$. By Lemma 4.2.1, for all $x \geq x_0$ and $z \in J_{a,b,c}(x)$,

$$ax^{\alpha(x,z)} + b = cz^{\alpha(x,z)}$$

holds. Thus by Lemma 4.2.3, there exists $n_0 = n_0(x, z) \in \mathbb{N}$ such that

$$a\lfloor (n_0 x)^\alpha \rfloor + b\lfloor n_0^\alpha \rfloor = c\lfloor (n_0 z)^\alpha \rfloor, \tag{4.4.2}$$

$$\max(\{(n_0 x)^\alpha\}, \{(n_0)^\alpha\}, \{(n_0 z)^\alpha\}) < 1/2, \tag{4.4.3}$$

$$n_0 \ll_\varepsilon x^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor \beta \rfloor})(2-\{\gamma\}))+\varepsilon}. \tag{4.4.4}$$

Define η as in Lemma 4.2.4. Let $\ell(x, z) = \eta(\alpha(x, z), n_0 x, n_0, n_0 z)$. The condition (C1) is clear from the definition of $J_{a,b,c}(x)$. The condition (C2) is also clear since we find at most one multiple of x in any 3-consective integers if $x_0 \geq 3$. Lemma 4.2.1 implies (C4). By Lemma 4.2.4, for each $x \geq x_0$ and $z \in J_{a,b,c}(x)$, each $\tau \in (\alpha(x, z), \alpha(x, z) + \ell(x, z))$ satisfies

$$a\lfloor (n_0 x)^\tau \rfloor + b\lfloor n_0^\tau \rfloor = c\lfloor (n_0 z)^\tau \rfloor, \quad n_0 x \geq x.$$

Therefore we have (C7). Let us show (C3), (C5), (C6).

We show (C3). Let x be an integer with $x \geq x_0$. For each $i \in \{1, 2\}$, let

$$z_{1,i} = \left\lfloor \left(\frac{b}{cx^2 \log x} + \frac{a}{c} \right)^{1/\gamma} x \right\rfloor + i, \quad z_{2,i} = \lfloor (a/c)^{1/\beta} x \rfloor - i.$$

Note that $J_{a,b,c}(x)$ does not contain multiples of x. Thus we do not know whether $z_{1,i}, z_{2,i} \in J_{a,b,c}(x)$ for each $i \in \{1,2\}$. However, by (C2), there exist $i_1, i_2 \in \{1,2\}$ such that $z_{1,i_1}, z_{2,i_2} \in J_{a,b,c}(x)$. Lemma 4.2.1 implies that

$$\alpha(x, z_{1,i_1}) = \frac{\log(a/c)}{\log(z_{1,i_1}/x)} + O\left(\frac{1}{x^2 \log x}\right).$$

Here we have

$$\begin{aligned}\log(z_{1,i_1}/x) &= \log\left(\left(\frac{b}{cx^2\log x} + \frac{a}{c}\right)^{1/\gamma} + O(x^{-1})\right)\\ &= \frac{1}{\gamma}\log(a/c) + \log\left(1 + O\left(\frac{b}{a\gamma x^2 \log x}\right) + O(x^{-1})\right)\\ &= \frac{1}{\gamma}\log(a/c) + O(x^{-1}).\end{aligned}$$

Therefore

$$\alpha(x, z_{1,i_1}) = \frac{\log(a/c)}{\frac{1}{\gamma}\log(a/c) + O(x^{-1})} + O\left(\frac{1}{x^2\log x}\right) = \gamma + O(x^{-1}).$$

Similarly, we have $\alpha(x, z_{2,i_2}) = \beta + O(x^{-1})$. Hence we obtain (C3).

We next show (C5). For all $x \geq x_0$ and $z \in J_{a,b,c}(x)$, $x < z$ holds by the definition of $J_{a,b,c}(x)$. Recall that

$$\ell(x,z) = \eta(\alpha(x,z), n_0x, n_0, n_0z) = \frac{\log\left((\lfloor W^\alpha \rfloor + 1)W^{-\alpha}\right)}{\log W},$$

where W is one of n_0x, n_0, or n_0z. By $\beta < \alpha(x,z)$, we have $\ell(x,z) \leq \log(1+(n_0x)^{-\beta}) \leq x^{-\beta}$. Further, by the facts (4.4.3), (4.4.4), $1 < x < z \ll x$, and $\alpha < \gamma$, we have

$$\ell(x,z) \geq \frac{\log(1 + 2^{-1}W^{-\alpha})}{\log W} \gg \frac{1}{(n_0z)^\gamma \log(n_0z)} \gg_\varepsilon x^{-q},$$

where let

$$q = q(\beta,\gamma,\varepsilon) = (\gamma+\varepsilon)\left(\frac{\gamma^2}{(2 + \{\beta\} - 2^{1-\lfloor\beta\rfloor})(2 - \{\gamma\})} + 1 + \varepsilon\right).$$

Therefore (C5) holds (with $Q_3 = 1$). The remaining condition (C6) is clear since $\beta < \alpha(x,z) < \gamma$ and $\alpha(x,z) + \ell(x,z) < \gamma + x^{-2}$ by (C5) (with $Q_3 = 1$).

Case $c > a$. Define $n_0 = n_0(x,z)$, $\ell(x,z)$, $q(\beta,\gamma,\varepsilon)$ by the same way in Case $a > c$. The condition (C1) is clear since $z < x$ by the definition of $J_{a,b,c}(x)$. The condition (C2) is also clear since $J_{a,b,c}(x)$ forms a set of consecutive integers. Lemma 4.2.1 implies (C4). Similarly to the discussion in Case $a > c$, we have (C5), (C6), and (C7). Let us show the remaining condition (C3). Let x be an integer with $x \geq x_0$. Let

$$z_1 = \left\lfloor \left(\frac{a}{c - b(x^2 \log x)^{-1}} \right)^{1/\beta} x \right\rfloor + 1, \quad z_2 = \lfloor (a/c)^{1/\gamma} x \rfloor - 1.$$

We observe that $z_1, z_2 \in J_{a,b,c}(x)$ if x_0 is sufficiently large. Lemma 4.2.1 implies that $\alpha(x,z_1) = \beta + O(x^{-1})$ and $\alpha(x,z_2) = \gamma + O(x^{-1})$. Therefore we have (C3).

Case $a = b = c = 1$. By Lemma 4.2.2, for all $x \geq x_0$ and $z \in J_{1,1,1}(x)$, by letting $X = X(x,z) = x^2 \lceil \log x \rceil$, $Y = Y(x,z) = x^2 \lceil \log x \rceil + 1$, $Z = Z(x,z) = zx \lceil \log x \rceil$, we have

$$X^{\alpha(x,z)} + Y^{\alpha(x,z)} = Z^{\alpha(x,z)}.$$

Therefore, from Lemma 4.2.3, there exists $n_0 = n_0(x,z) \in \mathbb{N}$ such that

$$\begin{gathered} \lfloor (n_0 X)^{\alpha} \rfloor + \lfloor (n_0 Y)^{\alpha} \rfloor = \lfloor (n_0 Z)^{\alpha} \rfloor, \\ \max(\{(n_0 X)^{\alpha}\}, \{(n_0 Y)^{\alpha}\}, \{(n_0 Z)^{\alpha}\}) < 1/2, \\ n_0 \ll_{\varepsilon} (X+Y)^{\gamma^2/((2+\{\beta\}-2^{1-\lfloor \beta \rfloor})(2-\{\gamma\}))+\varepsilon}. \end{gathered} \tag{4.4.5}$$

Defining $r = r(\gamma,\beta,\varepsilon) = \gamma^2/((2+\{\beta\}-2^{1-\lfloor \beta \rfloor})(2-\{\gamma\})) + \varepsilon$, we obtain

$$n_0 \ll_{\varepsilon} x^{(2+\varepsilon)r}. \tag{4.4.6}$$

Let $\ell(x,z) = \eta(\alpha(x,z), n_0 X, n_0 Y, n_0 Z)$ from Lemma 4.2.4.

The condition (C1) is clear since $x < z < 2x$ by the definition of $J_{1,1,1}(x)$. The condition (C2) is also clear since $J_{1,1,1}(x)$ forms a set of consecutive integers. Lemma 4.2.2 implies (C4). By Lemma 4.2.4, for all $x \geq x_0$, $z \in J_{1,1,1}(x)$, each $\tau \in (\alpha(x,z), \alpha(x,z) + \ell(x,z))$ satisfies

$$\lfloor (n_0 X)^{\tau} \rfloor + \lfloor (n_0 Y)^{\tau} \rfloor = \lfloor (n_0 Z)^{\tau} \rfloor, \quad n_0 X \geq x.$$

Therefore (C7) holds. Therefore it suffices to show (C3), (C5), and (C6).

Let us show (C3). Take any integer $x \geq x_0$. Let

$$z_1 = \lfloor 2^{1/\gamma}(x + (x\lceil \log x \rceil)^{-1}) \rfloor + 1, \quad z_2 = \lfloor 2^{1/\beta} x \rfloor - 1.$$

It follows that $z_1, z_2 \in J_{1,1,1}(x)$ if x_0 is sufficiently large. Then Lemma 4.2.2 implies that $\alpha(x, z_1) = \gamma + O(x^{-1})$ and $\alpha(x, z_2) = \beta + O(x^{-1})$. Therefore we have (C3).

We next show (C5). Let x be an integer with $x \geq x_0$ and $z \in J_{1,1,1}(x)$. It is clear that $x < z$ and $X(x,z) < Y(x,z) < Z(x,z)$. Recall that

$$\ell(x,z) = \eta(\alpha(x,z), n_0X, n_0Y, n_0Z) = \frac{\log\left((\lfloor W^\alpha \rfloor + 1)W^{-\alpha}\right)}{\log W},$$

where W is one of n_0X, n_0Y, or n_0Z. Therefore, by $\beta < \alpha$, we have $\ell(x,z) \leq \log(1 + (n_0Z)^{-\beta}) \leq Z^{-\beta} \leq x^{-\beta}$. Further, by the facts in (4.4.5) and (4.4.6) and $\alpha < \gamma$, we obtain

$$\ell(x,z) \geq \frac{\log(1 + 2^{-1}W^{-\alpha})}{\log W} \gg \frac{1}{(n_0Z)^\gamma \log(n_0Z)} \gg_\varepsilon x^{-(2+\varepsilon)(\gamma+\varepsilon)(r+1)}.$$

Hence, (C5) holds. The condition (C6) is clear since $\beta < \alpha(x,z) < \gamma$ and $\alpha(x,z) + \ell(x,z) < \gamma + x^{-2}$ by (C5).

By summarizing the above discussion, define

$$D_{a,b,c}(\beta,\gamma,\varepsilon) = \begin{cases} \dfrac{2}{(2+\varepsilon)(\gamma+\varepsilon)(r(\beta,\gamma,\varepsilon)+1)} & \text{if } a = b = c, \\ \dfrac{2}{q(\beta,\gamma,\varepsilon)} & \text{otherwise.} \end{cases}$$

Case $a > c$, Case $c > a$, Case $a = b = c = 1$, and Proposition 4.3.1 imply that

$$\dim_{\mathrm{H}}(\{\alpha \in [\beta,\gamma] \colon ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\}) \geq D_{a,b,c}(\beta,\gamma,\varepsilon).$$

Therefore, by (4.4.1) and by letting $\varepsilon \to +0$, $\gamma \to \beta$, $\beta \to s$, we have

$$\dim_{\mathrm{H}}(\{\alpha \in [s,t] \colon ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\}) \geq D_{a,b,c}(s,s,0).$$

By the definitions of q and r, we get the conclusion of Theorem 4.1.1.

Chapter 5

Linear Diophantine equations with two variables in Piatetski-Shapiro sequences

This chapter is based on [Sai20]. We investigate the solvability in $\mathrm{PS}(\alpha)$ of linear Diophantine equations with two variables.

5.1 Improvements on Glasscock's results

Recall that Glasscock studied the solvability in $\mathrm{PS}(\alpha)$ of the equation

$$y = ax + b \tag{5.1.1}$$

for fixed $a, b \in \mathbb{R}$ with $a \notin \{0, 1\}$. As a result, he reached Theorem 1.3.8.

The goal of this chapter is to put forward the following theorem which is an improvement on Glasscock's result in the case when $0 \leq b < a$.

Theorem 5.1.1 ([Sai20, Theorem 1.1])**.** *Let $a, b \in \mathbb{R}$, with $a \neq 1$ and $0 \leq b < a$. Assume that the equation $y = ax + b$ is solvable in $\mathbb{N}$. Then for all $1 < \alpha < 2$, the equation $y = ax + b$ is solvable in $\mathrm{PS}(\alpha)$. Moreover, for all $s, t \in \mathbb{R}$ with $2 < s < t$, we have*

$$\dim_{\mathrm{H}}\{\alpha \in (s, t) \colon y = ax + b \textit{ is solvable in } \mathrm{PS}(\alpha)\} = 2/s.$$

We avail of two main improvements when $0 \leq b < a$. Firstly, in the case when $1 < \alpha < 2$, we arrive at the same conclusion as Glasscock's result even if we replace "for Lebesgue almost every" with "for all". Secondly, in the case when $\alpha > 2$, his result is equivalent to stating that the set $\{\alpha \in (s,t)\colon y = ax+b \text{ is solvable in } \mathrm{PS}(\alpha)\}$ has Lebesgue measure 0 for all $2 < s < t$. However, from Theorem 5.1.1, we find that the set has a Hausdorff dimension of exactly $2/s$. Hence we can discern more details concerning the geometric structure of the set. We will show Theorem 5.1.1 in Section 5.4.

From the first improvement, we obtain Theorem 1.3.10.

Proof of Theorem 1.3.10 assuming Theorem 5.1.1 . Fix an arbitrary $\alpha \in (1,2)$. By Theorem 5.1.1, the equation $y = 2x$ is solvable in $\mathrm{PS}(\alpha)$. Let $x_1 < x_2 < \cdots$ and $y_1 < y_2 < \cdots$ be solutions $(x,y) = (x_n, y_n) \in \mathrm{PS}(\alpha)^2$ to $y = 2x$. By [FW09, Proposition 5.1], for every sufficiently large n, there exists $a_n \in \mathrm{PS}(\alpha)$ such that all $a_n,\ a_n + x_n,\ a_n + 2x_n$ are in $\mathrm{PS}(\alpha)$. Therefore, for every sufficiently large n, by substituting $(k_n, m_n, \ell_n,) = (a_n, x_n, x_n)$, all

$$k_n, \quad m_n, \quad \ell_n, \quad k_n + m_n, \quad m_n + \ell_n, \quad \ell_n + k_n, \quad k_n + m_n + \ell_n$$

belong to $\mathrm{PS}(\alpha)$. □

Remark that the method of this proof can be seen in [Gla17, Corollary 1]. We next discuss the solvability in $\mathrm{PS}(\alpha)$ of the equation

$$ax + by = cz \tag{5.1.2}$$

for fixed $a, b, c \in \mathbb{N}$. As a corollary of Theorem 5.1.1, the following holds:

Corollary 5.1.2 ([Sai20, Corollary 1.3])**.** *Let $a, b, c \in \mathbb{N}$ with $\gcd(a,c)|b$, $a > b$ and $a \neq c$. Then, for all $1 < \alpha < 2$, the equation $ax + by = cz$ is solvable in $\mathrm{PS}(\alpha)$. Further, for all $2 < s < t$, we have*

$$\dim_{\mathrm{H}}\{\alpha \in (s,t)\colon ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\} \geq \frac{2}{s}. \tag{5.1.3}$$

Indeed, from the condition $\gcd(a,c)|b$, the equation $ax + b = cz$ is solvable in $\mathbb{N}$. By dividing both sides by c, we have the equation $z = (a/c)x + (b/c)$ whose coefficients a/c and b/c satisfy the conditions in Theorem 5.1.1. Moreover, if the equation $ax + b = cz$ is solvable in $\mathrm{PS}(\alpha)$, then by letting $y = 1 = \lfloor 1^\alpha \rfloor$, we see that the equation $ax + by = cz$ is solvable in $\mathrm{PS}(\alpha)$. Therefore we conclude Corollary 5.1.2 from Theorem 5.1.1.

The lower bounds (5.1.3) in Corollary 5.1.3 are better than Theorem 4.1.1 for all $2 < s < t$. In particular, we find that the left-hand side of (5.1.3) goes to 1 as $s \to 2 + 0$ from Corollary 5.1.3 if a, b, c are restricted.

5.2 Lemmas

The goal of this section is to show a series of lemmas so as to evaluate discrepancies and calculate the Hausdorff dimension.

Lemma 5.2.1 ([Sai20, Lemma 3.1]). *For every non-integral $\alpha > 1$, integer $k \geq 4$, and real numbers $\eta > 0$ and $V \geq 1$, if $\eta V^{\alpha-k} < 1$ holds, then we have*

$$\mathcal{D}((\eta n^{\alpha})_{V<n\leq 2V}) \ll_{\alpha,k} (\eta V^{\alpha-k})^{1/(2^k-1)} + \eta^{-1/(2^k-2)} V^{(k-\alpha)/(2^k-2)-2^{2-k}}.$$

Proof. Fix any α, k, η, V given in Lemma 5.2.1 which satisfy $\eta V^{\alpha-k} < 1$. Let $f_h(x) = h\eta x^{\alpha}$ for every $h \in \mathbb{N}$ and $x > 0$. Then we have

$$h\eta V^{\alpha-k} \ll_{\alpha,k} |f_h^{(k)}(x)| \ll_{\alpha,k} h\eta V^{\alpha-k}$$

for all $V < x \leq 2V$. Therefore, the following holds from the Erdős-Turán inequality (2.2.2) and Lemma 2.2.4 with $I = (V, 2V]$, $f = f_h$: for all $K \in \mathbb{N}$,

$$\begin{aligned}
&\mathcal{D}((\eta n^{\alpha})_{V<n\leq 2V}) \\
&\ll K^{-1} + \sum_{h=1}^{K} \frac{1}{h} \left| \frac{1}{V} \sum_{V<n\leq 2V} \mathrm{e}(h\eta n^{\alpha}) \right| \\
&\ll_{\alpha,k} K^{-1} + \sum_{h=1}^{K} \frac{1}{h} \left| \left(h\eta V^{\alpha-k}\right)^{1/(2^k-2)} + V^{-2^{2-k}} \left(h\eta V^{\alpha-k}\right)^{-1/(2^k-2)} \right| \\
&\ll_{\alpha,k} K^{-1} + (K\eta V^{\alpha-k})^{1/(2^k-2)} + \eta^{-1/(2^k-2)} V^{(k-\alpha)/(2^k-2)-2^{2-k}}.
\end{aligned}$$

Hence by substituting $K = \lceil (\eta^{-1} V^{k-\alpha})^{1/(2^k-1)} \rceil$, we get the lemma. □

Lemma 5.2.2 ([Sai20, Lemma 3.2]). *Let $\alpha > 1$ be a non-integral real number, $\gamma \in \mathbb{R}$ with $0 < \gamma - \alpha < 1$, and let $A > 0$ be a real number. Then there exist $Q_0 = Q_0(\alpha, \gamma, A) > 0$, $\xi_0 = \xi_0(\alpha, \gamma) > 0$, and $\psi = \psi(\alpha, \gamma) < 0$ such that for all $Q \geq Q_0$ and $0 < \xi \leq \xi_0$, we have*

$$\mathcal{D}((AQ^{\alpha} n^{\alpha})_{V<n\leq 2V}) \ll_{\alpha,\gamma,A} Q^{\psi}$$

where $V = Q^{(\gamma-\alpha-\xi)/\alpha}$.

Proof. Fix any α, γ, A given in Lemma 5.2.2. Let $Q_0 = Q_0(\alpha, \gamma, A) > 0$ be a sufficiently large parameter, and $\xi_0 = \xi_0(\alpha, \gamma) > 0$ be a sufficiently small parameter. Take arbitrary real numbers $0 < \xi \le \xi_0$ and $Q \ge Q_0$. Then there exists an integer $k = k(\alpha, \gamma) \ge 4$ such that

$$\frac{\gamma(k-3)}{\gamma+k-3} < \alpha < \frac{\gamma k}{\gamma+k}. \tag{5.2.1}$$

Indeed, let $g(k) = \gamma(k-3)/(\gamma+k-3)$. Then $g(k)$ is strictly increasing for all $k \ge 4$. Since $g(4) = \gamma/(\gamma+1)$ and $\lim_{k\to\infty} g(k) = \gamma$, we have

$$\alpha \in (1, \gamma) \subseteq \bigcup_{k=4}^{\infty} (g(k), g(k+3)).$$

Therefore, there exists an integer $k \ge 4$ satisfying (5.2.1). Let us fix such an integer as $k = k(\alpha, \gamma) \ge 4$. By the condition $V = Q^{(\gamma-\alpha-\xi)/\alpha}$, we observe that

$$AQ^\alpha V^{\alpha-k} = AQ^{\psi_1}$$

where $\psi_1 := \alpha + (\gamma - \alpha - \xi)(\alpha - k)/\alpha$. Then we have

$$\psi_1 = ((\gamma+k)\alpha - \gamma k)/\alpha - \xi(\alpha-k)/\alpha \le ((\gamma+k)\alpha - \gamma k)/(2\alpha) < 0$$

by $\alpha < \gamma k/(\gamma+k)$, $0 < \xi \le \xi_0$, and the assumption that $\xi_0 = \xi_0(\alpha, \gamma)$ is sufficiently small. Therefore $AQ^\alpha V^{\alpha-k} < 1$ holds since Q_0 is sufficiently large and $Q \ge Q_0$. Thus we may apply Lemma 5.2.1 with $\eta = AQ^\alpha$ and $V = Q^{(\gamma-\alpha-\xi)/\alpha}$ to obtain

$$\begin{aligned}\mathcal{D}((AQ^\alpha n^\alpha)_{V<n\le 2V}) &\ll_{\alpha,\gamma,A} (Q^\alpha V^{\alpha-k})^{1/(2k-1)} + Q^{-\alpha/(2^k-2)} V^{(k-\alpha)/(2^k-2)-2^{2-k}} \\ &= Q^{\psi_1/(2^k-1)} + Q^{\psi_2},\end{aligned}$$

where

$$\psi_2 := -\frac{\alpha}{2^k-2} + \frac{\gamma-\alpha-\xi}{\alpha}\left(\frac{k-\alpha}{2^k-2} - \frac{4}{2^k}\right).$$

Then we have

$$\begin{aligned}\psi_2 &= \frac{-\alpha^2 2^k + (\gamma-\alpha)(k-\alpha)2^k - 4(\gamma-\alpha)(2^k-2)}{2^k(2^k-2)\alpha} + O_{\alpha,\gamma}(\xi) \\ &= \frac{(\gamma k - (\gamma+k-4)\alpha - 4\gamma)2^k + 8(\gamma-\alpha)}{2^k(2^k-2)\alpha} + O_{\alpha,\gamma}(\xi).\end{aligned}$$

Therefore the inequalities $\alpha > \gamma(k-3)/(\gamma+k-3)$, $k \geq 4$, $\gamma - \alpha < 1$, $1 < \alpha < \gamma$ and $0 < \xi \leq \xi_0$ imply that for sufficiently small $\xi_0 > 0$,

$$\begin{aligned}\psi_2 &< \frac{-2^k \cdot \gamma^2/(\gamma+k-3)+8}{2^k(2^k-2)\alpha} + O_{\alpha,\gamma}(\xi_0)\\ &< \frac{-2^k/(k-2)+8}{2^{k+1}(2^k-2)\alpha} \leq 0.\end{aligned}$$

Therefore, there exists $\psi = \psi(\alpha, k) < 0$ so that

$$\mathcal{D}((AQ^\alpha n^\alpha)_{V<n\leq 2V}) \ll_{\alpha,\gamma,A} Q^\psi.$$

□

We next present lemmas on the Hausdorff dimension. Recall the definition of $\mathcal{A}(J,\gamma)$ in Section 2.3

Lemma 5.2.3 ([Sai20, Lemma 3.4]). *Let $I \subseteq (1,\infty)$ be a non-empty and bounded open interval, and let $\gamma > 2$ and $a > 0$ be real numbers with $a \neq 1$. Define*

$$\begin{aligned}\mathcal{E}(I,\gamma;a) = \Big\{&\alpha \in I\colon \text{there are infinitely many } (p,q) \in \mathbb{Z}\times\mathbb{N}\\ &\text{such that } \left|a^{1/\alpha} - \frac{p}{q}\right| \leq \frac{1}{q^\gamma}\Big\}.\end{aligned}$$

Then we have $\dim_{\mathrm{H}}\mathcal{E}(I,\gamma;a) = 2/\gamma$.

Proof. For all $u > 0$, let $f(u) = a^{1/u}$. Fix a compact set $V \subseteq \mathbb{R}$ with $I \subseteq V$. Clearly, f is continuously differentiable and $|f'(u)| > 0$ for all $u \in V$. By the definitions, $f(\mathcal{E}(I,\gamma;a)) = \mathcal{A}(f(I),\gamma)$. Since $f(I)$ is also a bounded open interval, Lemma 2.3.3 and Lemma 2.3.5 imply that

$$\dim_{\mathrm{H}}\mathcal{E}(I,\gamma;a) = \dim_{\mathrm{H}} f(\mathcal{E}(I,\gamma;a)) = \dim_{\mathrm{H}}\mathcal{A}(f(I),\gamma) = \frac{2}{\gamma}.$$

□

5.3 Key Propositions

In this section, we show two key propositions by applying rational approximations.

Proposition 5.3.1 ([Sai20, Proposition 4.1]). *Let $a, b \in \mathbb{R}$ with $a \neq 1$ and $a > 0$. For all $1 \leq \beta < \gamma$, we have*

$$\{\alpha \in (\beta, \gamma) \colon y = ax + b \textit{ is solvable in } \mathrm{PS}(\alpha)\} \subseteq \mathcal{E}((\beta, \gamma), \beta; a).$$

Proof. Fix $\beta, \gamma \in \mathbb{R}$ with $1 \leq \beta < \gamma$. Take any $\alpha \in (\beta, \gamma)$ such that the equation $y = ax + b$ is solvable in $\mathrm{PS}(\alpha)$. Then there are infinitely many $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $\lfloor p^\alpha \rfloor = a \lfloor q^\alpha \rfloor + b$, which implies that

$$\frac{p}{q} = \left(a + \frac{b + \{p^\alpha\} - a\{q^\alpha\}}{q^\alpha} \right)^{1/\alpha} = a^{1/\alpha} + O_{a,b}(q^{-\alpha}).$$

Hence, there exist $C = C(a, b) > 0$ such that for infinitely many $(p, q) \in \mathbb{N}^2$,

$$\left| a^{1/\alpha} - \frac{p}{q} \right| \leq \frac{C}{q^\alpha} \leq \frac{1}{q^\beta}.$$

This yields that $\alpha \in \mathcal{E}((\beta, \gamma), \beta; a)$. □

Proposition 5.3.2 ([Sai20, Proposition 4.2]). *Let $a, b \in \mathbb{R}$ with $a \neq 1$ and $0 \leq b < a$. Suppose that $y = ax + b$ is solvable in $\mathbb{N}$. Then for all $1 \leq \beta < \gamma$ with $\lfloor \beta \rfloor < \beta < \gamma < \lfloor \beta \rfloor + 1$, we have*

$$\mathcal{E}((\beta, \gamma), \gamma; a) \subseteq \{\alpha \in (\beta, \gamma) \colon y = ax + b \textit{ is solvable in } \mathrm{PS}(\alpha)\}.$$

Proof. Since the equation $y = ax + b$ is solvable in $\mathbb{N}$, there exist distinct solutions $(x_1, y_1), (x_2, y_2) \in \mathbb{N}^2$ of the equation. Then since $y_2 - y_1 = a(x_2 - x_1)$ and $(x_1, y_1) \neq (x_2, y_2)$, we have $a \in \mathbb{Q}$. In addition, $b \in \mathbb{Q}$ holds from $b = y_1 - ax_1$. Thus we may let $a = a_1/a_2$, $b = b_1/b_2$, $(a_1, a_2, b_2 \in \mathbb{N}$, $b_1 \in \mathbb{N} \cup \{0\})$. By letting $c = a_2 b_2$, $d = a_1 b_2$, $e = a_2 b_1$, a pair $(x, y) \in \mathbb{N}^2$ satisfies the equation $cy - dx = e$ if and only if (x, y) satisfies the equation $y = ax + b$. Therefore we now discuss the solvability in $\mathrm{PS}(\alpha)$ of the equation $cy - dx = e$. Take any $\alpha \in \mathcal{E}((\beta, \gamma), \gamma; a) = \mathcal{E}((\beta, \gamma), \gamma; d/c)$. Let us show that the equation $cy - dx = e$ is solvable in $\mathrm{PS}(\alpha)$.

By the definition, there is a sequence $((p_n, q_n))_{n=1}^{\infty} \in (\mathbb{Z} \times \mathbb{N})^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$\left|(d/c)^{1/\alpha} - p_n/q_n\right| < q_n^{-\gamma},$$

where $q_1 < q_2 < \cdots$. Since $(d/c)^{1/\alpha} > 0$ and $d/c \neq 1$, there exists $n_0 = n_0(d, c) \in \mathbb{N}$ such that for all $n \geq n_0$, we obtain $p_n > 0$ and $p_n \neq q_n$.

From the solvability in $\mathbb{N}$, there exist $u, v \in \mathbb{N}$ such that $cu - dv = e$. By the division algorithm, there exist $r, v' \in \mathbb{Z}$ such that $v = cr + v'$, $0 \leq v' < c$. Hence by replacing $u - dr$ and v' with u and v respectively, we obtain

$$cu - dv = e, \quad 0 \leq v < c. \tag{5.3.1}$$

Take a sufficiently small parameter $\xi = \xi(\alpha, \gamma) > 0$, and take a sufficiently large parameter $n_1 = n_1(\alpha, \gamma, c, d, \varepsilon) \in \mathbb{N}$. Let ε be a real number with $0 < \varepsilon < \min(1/2, (d - e)/c)$. Note that $b < a$ implies $e = a_2 b_1 < a_1 b_2 = d$. Hence we verify the existence of ε. Take $n \in \mathbb{N}$ with $n \geq n_1$. Let $V_n = q_n^{(\gamma - \alpha - \xi)/\alpha}$. Define

$$I = \left[\frac{v}{c}, \ \frac{v}{c} + \frac{1}{c}\right) \cap \left[\frac{u}{d} + \frac{\varepsilon}{d}, \ \frac{u}{d} + \frac{1}{d} - \frac{\varepsilon}{d}\right), \tag{5.3.2}$$

$$B_n = \left\{x \in \mathbb{N} \colon \left\{\frac{(q_n x)^\alpha}{c}\right\} \in I\right\}.$$

Here the interval

$$\left[\frac{u}{d} + \frac{\varepsilon}{d}, \ \frac{u}{d} + \frac{1}{d} - \frac{\varepsilon}{d}\right)$$

is non-empty by $\varepsilon < 1/2$. If n_1 is large enough and ξ is small enough, then by the definition of the discrepancy and Lemma 5.2.2 with $V = V_n$, $Q = q_n$, $A = 1/c$, there exists $\psi = \psi(\alpha, \gamma) < 0$ such that

$$\#(B_n \cap (V_n, 2V_n])/V_n = \operatorname{diam}(I \cap [0, 1)) + O_{\alpha,\gamma,c}(q_n^{\psi}).$$

Here we show $\operatorname{diam}(I \cap [0, 1)) > 0$. Indeed, by (5.3.1), we obtain

$$\frac{u}{d} + \frac{\varepsilon}{d} = \frac{v}{c} + \frac{e}{cd} + \frac{\varepsilon}{d} > \frac{v}{c}.$$

Moreover, the inequality $\varepsilon < (d - e)/c$ yields that

$$\frac{u}{d} + \frac{\varepsilon}{d} = \frac{v}{c} + \frac{1}{c} + \frac{e - d}{cd} + \frac{\varepsilon}{d} < \frac{v}{c} + \frac{1}{c} \leq 1.$$

Hence $\operatorname{diam}(I \cap [0,1)) > 0$. Therefore there exists a large enough $n_1 = n_1(\alpha, \gamma, c, d, e, \varepsilon) \in \mathbb{N}$ such that for all $n \geq n_1$, we get $\#(B_n \cap (V_n, 2V_n])/V_n \geq \operatorname{diam}(I \cap [0,1))/2$, which means that $B_n \cap (V_n, 2V_n]$ is non-empty.

Hence we may take $x \in B_n \cap (V_n, 2V_n]$ where $n \geq n_1$. Then

$$(q_n x)^\alpha = c \left\lfloor \frac{(q_n x)^\alpha}{c} \right\rfloor + c \left\{ \frac{(q_n x)^\alpha}{c} \right\} = \left(c \left\lfloor \frac{(q_n x)^\alpha}{c} \right\rfloor + v \right) + \left(c \left\{ \frac{(q_n x)^\alpha}{c} \right\} - v \right).$$

The first term on the most right-hand side is an integer, and the second is in $[0,1)$ from the definition of B_n. Therefore we have

$$\lfloor (q_n x)^\alpha \rfloor = c \left\lfloor \frac{(q_n x)^\alpha}{c} \right\rfloor + v.$$

Let $\theta = p_n/q_n - (d/c)^{1/\alpha}$. By the mean value theorem, there exist $C = C(c, d, \alpha) > 0$ and $\theta' \in \mathbb{R}$ with $|\theta'| \leq |\theta|$ such that $(p_n/q_n)^\alpha = ((d/c)^{1/\alpha} + \theta)^\alpha = d/c + C\theta'$. Therefore,

$$\begin{aligned} (p_n x)^\alpha &= \left(\frac{p_n}{q_n} \right)^\alpha (q_n x)^\alpha = d \left\lfloor \frac{(q_n x)^\alpha}{c} \right\rfloor + d \left\{ \frac{(q_n x)^\alpha}{c} \right\} + C\theta'(q_n x)^\alpha \\ &= \left(d \left\lfloor \frac{(q_n x)^\alpha}{c} \right\rfloor + u \right) + \left(d \left\{ \frac{(q_n x)^\alpha}{c} \right\} - u \right) + C\theta'(q_n x)^\alpha. \end{aligned}$$

The first term on the most right-hand side is an integer, and the second term is in $[\varepsilon, 1-\varepsilon)$ by $x \in B_n$. Further, if necessary, we replace n_1 with a larger one, and by $x \in (V_n, 2V_n]$, the third term is evaluated by

$$|C\theta'(q_n x)^\alpha| \leq 2^\alpha C \frac{q_n^\alpha q_n^{\gamma - \alpha - \xi}}{q_n^\gamma} \leq 2^\alpha C q_n^{-\xi} \leq 2^\alpha C q_{n_1}^{-\xi} < \varepsilon.$$

Hence we obtain

$$\lfloor (p_n x)^\alpha \rfloor = d \left\lfloor \frac{(q_n x)^\alpha}{c} \right\rfloor + u.$$

By the above discussion, if $x \in B_n \cap (V_n, 2V_n]$ and $n \geq n_1$, then

$$c\lfloor (p_n x)^\alpha \rfloor - d\lfloor (q_n x)^\alpha \rfloor = cd \left\lfloor \frac{(q_n x)^\alpha}{c} \right\rfloor + cu - dc \left\lfloor \frac{(q_n x)^\alpha}{c} \right\rfloor - dv = cu - dv = e,$$

which means that $(\lfloor (q_n x)^\alpha \rfloor, \lfloor (p_n x)^\alpha \rfloor) \in \mathbb{N}^2$ is a solution of the equation $cy - dx = e$. Therefore the equation $cy - dx = e$ is solvable in $\mathrm{PS}(\alpha)$ since $B_n \cap (V_n, 2V_n]$ is non-empty for all $n \geq n_1$. □

Remark 5.3.3 ([Sai20, Remark 4.3])**.** In the case when $0 \le a \le b$, the interval I in (5.3.2) should be empty. Indeed, $a \le b$ implies $d = a_1 b_2 \le a_2 b_1 = e$. Thus we observe that

$$\frac{u}{d} - \left(\frac{v}{c} + \frac{1}{c}\right) = \frac{e-d}{cd} \ge 0,$$

which implies $I = \emptyset$. Because of this technical problem, we restrict the coefficients a, b to $0 \le b < a$.

5.4 Proof of Theorem 5.1.1

Fix $a, b \in \mathbb{R}$ with $a \ne 1$ and $0 \le b < a$. In the case $\alpha \in (1,2)$, we apply Proposition 5.3.2 with $\beta = 1$ and $\gamma = 2$. Then

$$\mathcal{E}((1,2),2;a) \subseteq \{\alpha \in (1,2) \colon y = ax + b \text{ is solvable in } \mathrm{PS}(\alpha)\}.$$

By Dirichlet's approximation theorem, $\mathcal{E}((1,2),2;a) = (1,2)$. Therefore the equation $y = ax + b$ is solvable in $\mathrm{PS}(\alpha)$ for all $\alpha \in (1,2)$.

We next discuss the case when $\alpha > 2$. Fix $s, t \in \mathbb{R}$ with $2 < s < t$. By applying Proposition 5.3.1 with $\beta = s$ and $\gamma = t$, and applying Lemma 5.2.3, we have

$$\begin{aligned} &\dim_{\mathrm{H}}\{\alpha \in (s,t) \colon y = ax + b \text{ is solvable in } \mathrm{PS}(\alpha)\} \\ &\le \dim_{\mathrm{H}} \mathcal{E}((s,t),s;a) = \frac{2}{s}. \end{aligned} \tag{5.4.1}$$

Further, let $\delta > 0$ be an arbitrarily small parameter. By applying Proposition 5.3.2 with $\beta = s$ and $\gamma = \min\{s+\delta, \lfloor s \rfloor + 1, t\}$, and applying Lemma 5.2.3, we obtain

$$\begin{aligned} &\dim_{\mathrm{H}}\{\alpha \in (s,t) \colon y = ax + b \text{ is solvable in } \mathrm{PS}(\alpha)\} \\ &\ge \dim_{\mathrm{H}} \mathcal{E}((s,\gamma),\gamma;a) = \frac{2}{s+\delta} \end{aligned}$$

for every small enough $\delta > 0$. Therefore we get the theorem by taking $\delta \to +0$.

Remark 5.4.1 ([Sai20, Remark 5.1])**.** Let $\alpha \in (\beta, \gamma)$ where β and γ satisfy $1 \le \beta < \gamma$ and $\lfloor \beta \rfloor < \beta < \gamma < \lfloor \beta \rfloor + 1$. If $a^{1/\alpha} \in \mathbb{Q}$, then it is clear that for infinitely many $(p,q) \in \mathbb{Z} \times \mathbb{N}$ we have $|a^{1/\alpha} - p/q| \leq q^{-\gamma}$. By Proposition 5.3.2, the equation $y = ax + b$ is solvable in $\mathrm{PS}(\alpha)$. Therefore, for all $a, b \in \mathbb{R}$ with $a \neq 1$ and $0 \leq b < a$, and for all non-integral $\alpha > 1$ satisfying $a^{1/\alpha} \in \mathbb{Q}$, the equation $y = ax + b$ is solvable in $\mathrm{PS}(\alpha)$.

Remark 5.4.2 ([Sai20, Remark 5.2])**.** We apply Proposition 5.3.1 and Lemma 5.2.3 to show the inequality (5.4.1). Note that the condition $0 \leq b < a$ is not required in Proposition 5.3.1 and Lemma 5.2.3. Hence, for all $a, b \in \mathbb{R}$ with $a \neq 1$ and $a > 0$, and for all $s, t \in \mathbb{R}$ with $2 < s < t$, we obtain

$$\dim_{\mathrm{H}}\{\alpha \in (s,t) \colon y = ax + b \text{ is solvable in } \mathrm{PS}(\alpha)\} \leq \frac{2}{s}.$$

Chapter 6

Future works

In this chapter, we give some open problems related with this thesis.

6.1 The case when $1 < \alpha < 2$

We have investigated distributions of finite sequences represented by polynomials in $\mathrm{PS}(\alpha)$, and especially done the case $\alpha \in (1, 2)$ in detail. We have not proved the convergence in the proof of Theorem 3.1.2, but the middle-hand side in (3.1.2) divided by $N^{2-\alpha/(d+1)}$ probably converges to some positive number as $N \to \infty$. It is a future work. As other natural questions, we have the positive-density version and prime-number version.

Question 6.1.1 ([SY21, Question 6.1] Positive-density version). *Let $d \in \mathbb{N}$ and $\alpha \in (d, d+1)$; let $A \subset \mathbb{N}$ be a set with positive density, and $k \geq d+2$ and $r \geq 1$ be integers. Then does*

$$\#\{P \subset A \cap [1, N] : P \in \mathcal{P}_{k,1},\ (\lfloor n^\alpha \rfloor)_{n \in P} \in \mathcal{P}_{k,d}\} \asymp N^{2-\alpha/(d+1)} \quad (N \to \infty) \tag{6.1.1}$$

hold?

Question 6.1.2 ([SY21, Question 6.2] Prime-number version). *How about the case when A in Question 6.1.1 is replaced with the set of all prime numbers? In this case, what is suitable as the right-hand side in* (6.1.1)*?*

Actually, we can replace the first term n in (3.2.1) with a prime number p: for every $f \in \mathcal{H}$ that satisfies the same assumptions as Theorem 3.2.1,

$$\lim_{N \to \infty} \frac{1}{\pi(N)} \#\{p \in [n_0, N]_{\mathbb{Z}} : (\lfloor f(p + rj) \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,d}\} = \mu(\mathcal{C}_{k,d+1}), \tag{6.1.2}$$

where $\pi(N)$ denotes the number of prime numbers less than or equal to N. The proof of this statement is the same as that of Theorem 3.2.1 because for every subpolynomial $f \in \mathcal{H}$ defined on the interval $[n_0, \infty)$, the sequence $(f(p))_{p\,\mathrm{prime}\geq n_0}$ is uniformly distributed modulo 1 if and only if $(f(n))_{n=n_0}^{\infty}$ is uniformly distributed modulo 1 [BKS19]. In (6.1.2), it is only guaranteed that the first term p is prime. In order to make all terms $p, p+r, \dots, p+(k-1)r$ prime, we need to study whether $(f(p))_{p\in\mathcal{S}_{k,r}\cap[n_0,\infty)}$ is uniformly distributed modulo 1 or not, where $\mathcal{S}_{k,r}$ is the set of all prime numbers p such that all $p, p+r, \dots, p+(k-1)r$ are prime. Of course, r must be restricted to some extent depending on k. The set $\mathcal{S}_{k,r}$ is related to twin prime pairs (when $(k,r) = (2,2)$), sexy prime triplets (when $(k,r) = (3,6)$), and generally prime k-tuples. It is known that there exists an even number r such that $\mathcal{S}_{2,r}$ is infinite [Pol14, May15], but it is still open whether $\mathcal{S}_{k,r}$ is infinite for general k and admissible r.

Finally, we focus on an asymptotic formula when α runs over the interval $(1, 2)$.

Question 6.1.3 ([SY21, Question 6.3] Asymptotic formula when α running)**.** *Fix a sufficiently large $N \in \mathbb{N}$ and integers $k \geq 3$ and $r \geq 1$. Let*

$$D_{N,k,r}(\alpha) = \frac{1}{N}\#\{n \in [1, N]_{\mathbb{Z}} : (\lfloor (n+rj)^{\alpha} \rfloor)_{j=0}^{k-1} \in \mathcal{P}_{k,1}\}.$$

Can we find any asymptotic formulas of $D_{N,k,r}(\alpha)$ when α runs over the interval $(1, 2)$?

Figure 6.1 illustrates the behavior of $D_{N,k,r}(\alpha)$ by numerical computation, where the points $(\alpha, D_{N,k,r}(\alpha))$ are plotted for all $\alpha \in \{1 + 0.001i : i \in [0, 1000]_{\mathbb{Z}}\}$. In view of this figure, $D_{N,k,r}(\alpha)$ would be approximated by the sum of continuous waves and discrete errors. In order to theoretically observe a phenomenon like this figure, it is probably needed to further analyze the distribution of the sequence $\left((n^{\alpha}, \alpha n^{\alpha-1})\right)_{n=1}^{N}$ modulo 1.

6.2 The case when $\alpha > 2$

Question 6.2.1. *Does there exist $\alpha > 2$ such that the set of Piatetski-Shapiro primes with exponent α contains infinitely many 3-APs.*

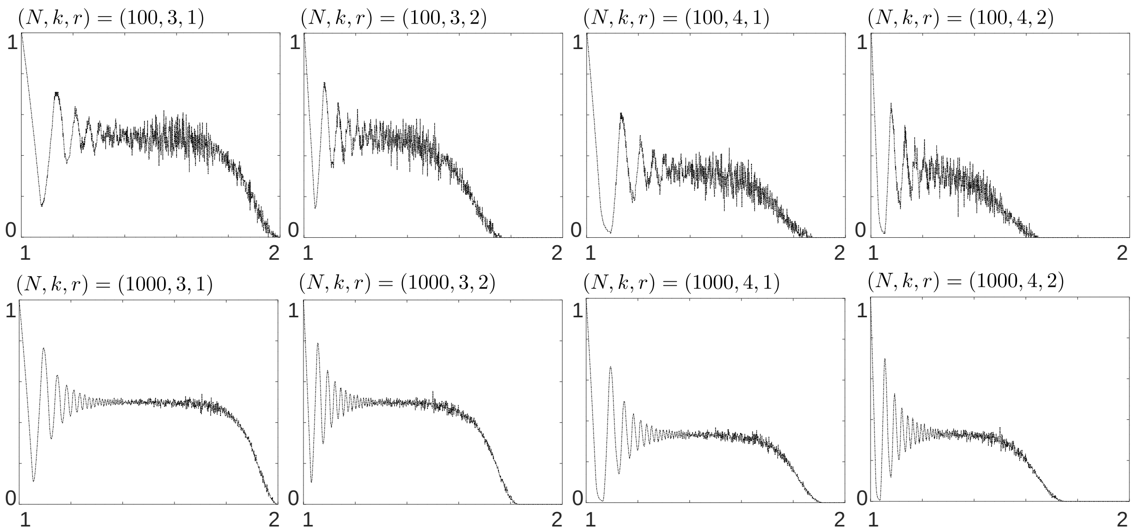


Figure 6.1: The behavior of $D_{N,k,r}(\alpha)$ for $(N, k, r) \in \{100, 1000\} \times \{3, 4\} \times \{1, 2\}$. The abscissa and ordinate denote values of α and $D_{N,k,r}(\alpha)$, respectively.

Recall that we say that a prime number p is a Piatetski-Shapiro prime with exponent α if there exists $n \in \mathbb{N}$ such that $p = \lfloor n^\alpha \rfloor$. That was introduced in Chapter 1. Mirek [Mir15] proved that for every $\alpha \in (1, 72/71)$, the set of all Piatetski-Shapiro primes with exponent α contains infinitely many 3-APs. Note that we know the existence of Piatetski-Shapiro primes with exponent $\alpha > 2$ from (1.2.2).

Question 6.2.2 (Roth's theorem on Piatetski-Shapiro sequences with $\alpha > 2$)**.** *Does there exist $\alpha > 2$ such that any $A \subseteq \mathrm{PS}(\alpha)$ with*

$$\varlimsup_{N\to\infty} \frac{\#(A \cap [N])}{\#(\mathrm{PS}(\alpha) \cap [N])} > 0$$

contains infinitely many 3-APs?

In the case when $1 < \alpha < 2$, the answer to Question 6.2.2 is "YES" by Theorem 1.3.5. By Fourier analytic methods, Green showed Roth's theorem on prime numbers, that is, if $A \subseteq \mathcal{P}$ satisfies (1.1.3), then A contains infinitely many 3-APs [Gre05].

Question 6.2.3. *Fix an arbitrary $a, b, c \in \mathbb{N}$. What is the exact Hausdorff dimension of the set*

$$\{\alpha \in [s, t] \colon ax + by = cz \text{ is solvable in } \mathrm{PS}(\alpha)\}? \tag{6.2.1}$$

We only have lower bounds of the Hausdorff dimension of the set (6.2.1). In general, the lower bounds are $1/s^3$ by Theorem 4.1.1. Further, by Corollary 5.1.2, the lower bounds become $2/s$ if a, b, and c are restricted. The author expects that we would get better lower bounds. The Hausdorff dimension of (6.2.1) would be $3/s$ for all $a, b, c \in \mathbb{N}$.

Question 6.2.4. *Define*

$$d_s(n) = \dim_{\mathrm{H}}\{\alpha \in [s, \infty) \colon x_1 + \cdots + x_{n-1} = x_n \text{ is solvable in } \mathrm{PS}(\alpha)\}$$

for all real numbers $s \geq 1$ and $n \in \mathbb{N}$. Fix any $s \geq 1$. Then is it true that $d_s(n)$ is increasing with respect to n?

This question is related with the Waring problems on Piatetski-Shapiro sequences. If we ignored pairwise distinctness in the definition of "solvable", we would have $d_s(n) = 1$ for every $n \geq (s+\varepsilon)^2 2^{s+\varepsilon} + 1$ by Theorem 1.2.3. By this observation, the answer to Question 6.2.4 would be "YES". The author expects that $d_s(n)$ would be 1 for all $1 \leq s < n$, and would be n/s for all $n \leq s$.

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