On the descendibility and extendability of homogeneous quasimorphisms

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Introduction

A real-valued function μ on a group Γ is called a *quasimorphism* if the condition

$$\sup_{\gamma_1,\gamma_2\in\Gamma} |\mu(\gamma_2) - \mu(\gamma_1\gamma_2) + \mu(\gamma_1)| < \infty$$

holds. A quasimorphism is called *homogeneous* if the condition

$$\mu(\gamma^n) = n\mu(\gamma)$$

is satisfied for any $\gamma \in \Gamma$ and $n \in \mathbb{Z}$. By definition, homomorphisms and bounded functions from Γ to $\mathbb{R}$ are quasimorphisms. We call a quasimorphism *trivial* if the quasimorphism is obtained as a sum of a homomorphism and a bounded function.

Quasimorphisms are related to various areas of mathematics, for example, Topology, Geometry, Algebra, and Dynamical Systems. The best known and earliest example of a homogeneous quasimorphism is the translation number, which was defined by Poincaré [**Poi85**] in his study on circle dynamics (see Example 1.4). Quasimorphisms also appeared in the study of characteristic classes. The Milnor-Wood inequality ([**Mil58**]) states that the Euler number of flat $SL(2,\mathbb{R})$-bundles over a surface has an upper bound, where a quasimorphism on the universal covering group $\widetilde{SL}(2,\mathbb{R})$ played a crucial role in the proof.

Quasimorphisms have also played an essential role in the study on bounded cohomology theory. Bounded cohomology of discrete groups was defined by Johnson [**Joh72**]. After that, in the celebrated paper [**Gro82**], Gromov introduced and studied bounded cohomology of topological spaces. In [**Gro82**], the Milnor-Wood inequality was reformulated as follows: the real Euler class is a bounded cohomology class and has the Gromov norm equal to $1/2$. Quasimorphisms can be used to show the non-triviality of the second bounded cohomology (see also ICM-2006 proceeding [**Mon06**]). If we can construct a non-trivial homogeneous quasimorphism on a group Γ, then we immediately obtain a non-zero second bounded cohomology class of Γ. By using this method, it has been proved that a number of groups that admit the second bounded cohomology of infinite dimension (for example, [**Mit84**] for the free groups and the non-amenable surface groups and [**EF97**] for the non-elementary word-hyperbolic groups; see also [**Mon06**]). In the last twenty years, many researchers have constructed homogeneous quasimorphisms on transformation groups (for example, [**EP03**], [**GG04**], [**She14**], and [**BHW19**]; see also ICM-2006 proceeding [**Ghy07**]).

Let us consider a group extension

$$1 \to K \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1.$$

Throughout this thesis, we identify the group K with the normal subgroup $i(K)$ of Γ under the map i. Let $Q(K)$, $Q(\Gamma)$, and $Q(G)$ be the space of the homogeneous quasimorphisms on K, Γ, and G, respectively. Then there is an exact sequence (Lemma 1.5 and Remark 1.7)

$$0 \to Q(G) \xrightarrow{p^*} Q(\Gamma) \xrightarrow{i^*} Q(K)^\Gamma,$$

where $Q(K)^\Gamma$ denotes the subspace of $Q(K)$ whose elements are invariant under the conjugation action by Γ (see Definition 1.6).

The main concerns of this thesis are the following two problems of homogeneous quasimorphisms: a descending problem and an extending problem. The descending problem asks whether a given element of $Q(\Gamma)$ defines that of $Q(G)$ or not, and the extending problem asks whether a given element of $Q(K)^\Gamma$ is a restriction of that of $Q(\Gamma)$ or not. To deal with the above problems, we introduce and study two spaces $\mathcal{ND}$ and $\mathcal{NE}$ defined below.

The space $\mathcal{ND}$ is defined by

$$\mathcal{ND} = Q(\Gamma) \,/ \big(p^* Q(G) + H^1(\Gamma)\big) \tag{0.0.1}$$

(see Definition 2.1). Here $H^1(\Gamma)$ is the first cohomology group of Γ with coefficients in $\mathbb{R}$, which is identified with the space of all homomorphisms from Γ to $\mathbb{R}$. Recall that a homomorphism is called a trivial homogeneous quasimorphism. Thus a non-trivial element in $\mathcal{ND}$ is represented by a non-trivial homogeneous quasimorphism on Γ which does not descend to G. In this sense, we call $\mathcal{ND}$ the space of *non-descendible homogeneous quasimorphisms*.

The space $\mathcal{NE}$ is the following (Definition 3.1):

$$\mathcal{NE} = Q(K)^\Gamma \,/ \big(i^* Q(\Gamma) + H^1(K)^\Gamma\big)\,. \tag{0.0.2}$$

Here $H^1(K)^\Gamma$ is the subspace of $H^1(K)$ whose elements are invariant under the conjugation action by Γ (see (1.2.3)). A non-trivial element in $\mathcal{NE}$ is represented by a Γ-invariant non-trivial homogeneous quasimorphism on K which does not extend to Γ. Thus we call $\mathcal{NE}$ the space of *non-extendable homogeneous quasimorphisms*. The triviality and non-triviality of this space has some applications to stable commutator length and the virtual splitting of the surjection $p\colon \Gamma \to G$ (see Section 3.1).

Let us briefly describe the content of each chapter.

Chapter 1. This chapter is devoted to preliminaries. We recall definitions and basic properties of quasimorphisms, group and bounded cohomology, characteristic classes, and transformation groups.

Chapter 2. This chapter is based on [**KM20**], which is a joint work with Morimichi Kawasaki. In this chapter, we mainly focus on the space $\mathcal{ND}$ of the group extension

$$0 \to \pi_1(G) \to \widetilde{G} \xrightarrow{p} G \to 1, \tag{0.0.3}$$

where G is a connected topological group, $p\colon \widetilde{G} \to G$ is the universal covering, and $\pi_1(G)$ is the fundamental group of G.

Let G^δ be the group G with the discrete topology and $\iota\colon G^\delta \to G$ the identity homomorphism. Then the homomorphism ι induces the continuous map $B\iota\colon BG^\delta \to BG$ between the classifying spaces. Thus we obtain the homomorphism

$$(B\iota)^*\colon H^\bullet_{\mathrm{top}}(BG) \to H^\bullet_{\mathrm{top}}(BG^\delta) \cong H^\bullet(G)$$

by pullback, where $H^\bullet(-)$ and $H^\bullet_{\mathrm{top}}(-)$ denote the cohomology of groups and topological spaces with coefficients in $\mathbb{R}$, respectively. In this thesis, we call a cohomology class $a \in H^\bullet(G)$ a *characteristic class of foliated G-bundles* if $a \in \mathrm{Im}(B\iota)^*$.

Let $H^\bullet_b(G)$ be the bounded cohomology of G with coefficients in $\mathbb{R}$ and

$$c_G\colon H^\bullet_b(G) \to H^\bullet(G)$$

the comparison map (see Subsection 1.2.2). A class $a \in H^\bullet(G)$ is called *bounded* if $a \in \mathrm{Im}(c_G)$. One of the main theorems of the celebrated paper [**Gro82**] by Gromov is the following:

THEOREM ([**Gro82**]). If G is an algebraic subgroup of $GL(n, \mathbb{R})$, then each characteristic class of foliated G-bundles is bounded, that is, the following holds:

$$\mathrm{Im}(B\iota)^* \subset \mathrm{Im}(c_G).$$

The above theorem does not necessarily hold for an arbitrary topological group. Therefore, when G is not an algebraic subgroup of $GL(n, \mathbb{R})$, one may ask whether a given characteristic class is bounded or not. If we ask the boundedness of characteristic classes of fiber bundles, then we need to deal with homeomorphism groups or diffeomorphism groups as G. However, when G is such a group, there are only a few characteristic classes that are known to be bounded or not (the known examples are listed in EXAMPLE 2.22).

Since the first real bounded cohomology is always trivial, the non-trivial boundedness problem occurs when the degree of the class is greater than or equal to two. So we consider classes in second cohomology groups. The study of the boundedness of characteristic classes of foliated G-bundles can be rephrased as the study of the space

$$\mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G) \subset H^2(G). \tag{0.0.4}$$

The following theorem clarifies a relation between the space (0.0.4) and the space $\mathcal{ND}$ of non-descendible homogeneous quasimorphisms on the universal covering group $\widetilde{G}$.

THEOREM A (THEOREM 2.17). For the space $\mathcal{ND}$ for the group extension $0 \to \pi_1(G) \to \widetilde{G} \to G \to 1$, there is an isomorphism

$$\mathcal{ND} \cong \mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G).$$

THEOREM A states that any bounded characteristic classes of foliated G-bundles are characterized by non-trivial homogeneous quasimorphisms on $\widetilde{G}$. In particular, if the space $Q(\widetilde{G})$ is trivial, then we can deduce that any non-zero characteristic classes of foliated G-bundles are unbounded. Note that many researchers have constructed non-descendible homogeneous quasimorphisms on the universal coverings of diffeomorphism groups (for example, [**Giv90**], [**Ost06**], [**OT09**], and [**FOOO19**]).

One of the applications is the following. Let $\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)$ and $\mathrm{Cont}_0(S^3, \xi)$ be the identity component of the symplectomorphism group and the contact diffeomorphism group, respectively (see Section 1.4 and Subsection 2.4.1 for the definitions). If $1 < \lambda \leq 2$, there are canonical characteristic classes

$$(\mathfrak{o}_{S^2 \times S^2})_{\mathbb{R}} \in H^2_{\mathrm{top}}(B\,\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)) \cong \mathbb{R}$$

and

$$(\mathfrak{o}_{S^3})_{\mathbb{R}} \in H^2_{\mathrm{top}}(B\,\mathrm{Cont}_0(S^3, \xi)) \cong \mathbb{R},$$

which are defined by the coefficients change of the primary obstructions to the construction of a cross-section.

COROLLARY (COROLLARY 2.23). The following properties hold:

(1) The class $(B\iota)^*(\mathfrak{o}_{S^2 \times S^2})_{\mathbb{R}}$ is bounded.
(2) The class $(B\iota)^*(\mathfrak{o}_{S^3})_{\mathbb{R}}$ is unbounded.

Chapter 3. In this chapter, Section 3.2 is based on [**Mar20**], and Sections 3.3 and 3.4 are based on [**KKM$^+$21**], where [**KKM$^+$21**] is a joint work with Morimichi Kawasaki, Mitsuaki Kimura, Takahiro Matsushita, and Masato Mimura. Our main concern of this chapter is the space $\mathcal{NE}$.

Recall that, for a group extension $1 \to K \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1$, there is an exact sequence

$$0 \to Q(G) \xrightarrow{p^*} Q(\Gamma) \xrightarrow{i^*} Q(K)^{\Gamma}.$$

Therefore, the non-descendibility of a homogeneous quasimorphism $\mu \in Q(\Gamma)$ is equivalent to the non-triviality of the restriction $\mu|_K = i^*\mu$. In contrast, it is difficult to show the non-extendability of a given Γ-invariant non-trivial homogeneous quasimorphism on K. In fact, there has been only one such example found to the author's knowledge. For the identity component $\mathrm{Symp}_0(\Sigma_g, \omega)$ of the symplectomorphism group and its normal subgroup $\mathrm{Ham}(\Sigma_g, \omega)$ called the Hamiltonian diffeomorphism group, Py constrcuted in [**Py06**] a $\mathrm{Symp}_0(\Sigma_g, \omega)$-invariant homogeneous quasimorphism on $\mathrm{Ham}(\Sigma_g, \omega)$. Kawasaki and Kimura showed in [**KK19**] that Py's homogeneous quasimorphism does not extend to $\mathrm{Symp}_0(\Sigma_g, \omega)$.

There is another example to explain the difficulty of the extending problem. In Section 3.2, we show the extendability of the Ruelle invariant. Let D be the closed unit disk and ω the standard symplectic form. Let $\Gamma = \mathrm{Symp}(D, \omega)$ be the symplectomorphism group and K the subgroup that preserves the boundary pointwise. The Ruelle invariant R is a homogeneous quasimorphism on K, which was defined in [**Rue85**]. It is known that the group K is contractible, and that the fundamental group of Γ is isomorphic to $\mathbb{Z}$.

The Ruelle invariant R is well defined since the group K is contractible and the same definition does not seem to be applicable to Γ since it is not. One may suspect that an extension of R does not exist on Γ; however we can prove the following:

Theorem B (Theorem 3.8). *There exists a homogeneous quasimorphism*

$$\mathfrak{R} \colon \Gamma \to \mathbb{R}$$

satisfying $\mathfrak{R}|_K = R$. In other words, the Ruelle invariant R is extendable to the group Γ.

As explained above, it is difficult to show the non-triviality of the space $\mathcal{NE}$. Note that the space $\mathcal{NE}$ is defined for a pair (Γ, K) of groups where K is a normal subgroup of Γ. To the best of the author's knowledge, there were no examples of finitely generated groups and its normal subgroups that admit non-trivial $\mathcal{NE}$. The following two theorems (THEOREM C and THEOREM D) give the first examples of such pairs of groups, which are derived from the low-dimensional hyperbolic geometry.

THEOREM C (THEOREM 3.2). Let Γ be the surface group of genus $g \geq 2$ and $K = [\Gamma, \Gamma]$ the commutator subgroup. Then the dimension of the space $\mathcal{NE}$ is equal to one.

By the theorem of [**Mit84**], the surface group Γ admits infinite dimensional $Q(\Gamma)$. Moreover, since $\Gamma/K \cong \mathbb{Z}^{2g}$, we have $Q(\Gamma/K) \cong H^1(\mathbb{Z}^{2g}) \cong \mathbb{R}^{2g}$ (see PROPOSITION 1.2). Together with the exact sequence

$$0 \to Q(\Gamma/K) \to Q(\Gamma) \to Q(K)^{\Gamma},$$

the space $Q(K)^{\Gamma}$ is infinite-dimensional. Therefore, both numerator and denominator of $\mathcal{NE}$ are infinite-dimensional. However, THEOREM C claims that the quotient space $\mathcal{NE}$ is finite-dimensional (more precisely, is just one-dimensional).

THEOREM D (THEOREM 3.3). Let Σ_g be a closed surface of genus $g \geq 2$ and $f \in \mathrm{Homeo}_+(\Sigma_g)$ an orientation preserving homeomorphism. Set $\Gamma = \pi_1(X_f)$ and $K = [\Gamma, \Gamma]$, where X_f is the mapping torus. If the mapping class $[f]$ is a pseudo-Anosov element and in the Torelli group $\mathcal{I}_g$, then the dimension of the space $\mathcal{NE}$ is equal to $2g + 1$.

For THEOREM D, the similar phenomenon to THEOREM C can be observed. By the theorem of Thurston ([**Thu86**], [**Ota96**]), the mapping class $[f]$ is a pseudo-Anosov element if and only if the mapping torus X_f admits a hyperbolic structure. Thus, by the theorem of [**EF97**], the fundamental group Γ of X_f has infinite dimensional $Q(\Gamma)$. Therefore, as the case of THEOREM C, the numerator and the denominator of $\mathcal{NE}$ are infinite dimensional. However, THEOREM D asserts that the quotient $\mathcal{NE}$ is finite-dimensional if the mapping class is in the Torelli group. The Torelli group is the subgroup of the mapping class group which act trivially on the homology of the surface. It is known that the Torelli group contains a pseudo Anosov element. Moreover, pseudo Anosov elements are generic in the Torelli group in the sense of Random Walk (see REMARK 3.4).

THEOREM C and THEOREM D are proved from a homological algebraic point of view, and we do not construct a non-extendable homogeneous quasimorphism in the proof. In Section 3.5, we give a description of a non-zero element of $\mathcal{NE}$ in Theorem C in terms of Poincaré's translation number (Theorem 3.30).

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CHAPTER 1

Preliminaries

1.1. Quasimorphism

In this section, we review some besic notions about quasimorphisms. We refer to [**Cal09**] and [**Fri17**] for details.

DEFINITION 1.1. A real-valued function μ on a group Γ is called a *quasimorphism* if

$$D(\mu) = \sup_{\gamma_1, \gamma_2 \in \Gamma} |\mu(\gamma_2) - \mu(\gamma_1\gamma_2) + \mu(\gamma_1)| < \infty$$

holds. The value $D(\mu)$ is called the *defect of* μ. A quasimorphism μ is *homogeneous* if the equality $\mu(\gamma^n) = n\mu(\gamma)$ holds for any $\gamma \in \Gamma$ and $n \in \mathbb{Z}$. Let $Q(\Gamma)$ denote the $\mathbb{R}$-vector space consisting of all homogeneous quasimorphisms on Γ.

Note that a homomorphism is a homogeneous quasimorphism of defect zero. We call a quasimorphism *trivial* if it is a sum of a homomorphism and a bounded function. The following fact is well known.

PROPOSITION 1.2. If Γ is amenable, then any homogeneous quasimorphism on Γ is a homomorphism. In particular, any homogeneous quasimorphism on an abelian group is a homomorphism.

For a quasimorphism $\mu \colon \Gamma \to \mathbb{R}$ and $\gamma \in \Gamma$, the limit

$$\overline{\mu} = \lim_{n \to \infty} \frac{\mu(\gamma^n)}{n}$$

always exists, and the difference $\overline{\mu} - \mu$ is a bounded function on Γ (see [**Cal09**, LEMMA 2.21]). By definition, the function $\overline{\mu} \colon \Gamma \to \mathbb{R}$ is a homogeneous quasimorphism. Namely, perturbing μ by a bounded function, we can obtain a homogeneous quasimorphism $\overline{\mu} \colon \Gamma \to \mathbb{R}$. This $\overline{\mu}$ is called the *homogenization* of μ.

EXAMPLE 1.3. The floor function $\lfloor \cdot \rfloor \colon \mathbb{R} \to \mathbb{R}$ is a quasimorphism. Moreover, the homogenization $\overline{\lfloor \cdot \rfloor} \colon \mathbb{R} \to \mathbb{R}$ is equal to the identity homomorphism on $\mathbb{R}$.

Example 1.4 (Poincaré's translation number). Let $\mathrm{Homeo}_+(S^1)$ denote the group of orientation preserving homeomorphisms of the circle. Here we consider $\mathrm{Homeo}_+(S^1)$ as a topological group with the compact-open topology. The universal covering group $\widetilde{\mathrm{Homeo}}_+(S^1)$ is identified with

$$\widetilde{\mathrm{Homeo}}_+(S^1) = \{\gamma \in \mathrm{Homeo}(\mathbb{R}) \mid \gamma \boldsymbol{t} = \boldsymbol{t}\gamma\},$$

where $\boldsymbol{t}\colon \mathbb{R} \to \mathbb{R}$ is the translation by one, that is, $\boldsymbol{t}(x) = x + 1$. Let us define a function $\mu\colon \widetilde{\mathrm{Homeo}}_+(S^1) \to \mathbb{R}$ by

$$\mu(\gamma) = \gamma(0).$$

Then it turns out that the map μ is a quasimorphism on $\widetilde{\mathrm{Homeo}}_+(S^1)$. The homogenization

$$\overline{\mu}(\gamma) = \lim_{n\to\infty} \frac{\gamma^n(0)}{n}$$

is called *Poincaré's translation number*. Note that Poincaré's translation number is not a homomorphism.

In considering homogeneous quasimorphisms on a group extension, we shall use the following left exactness theorem.

Lemma 1.5 ([**Cal09**, Remark 2.90]). Let $1 \to K \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1$ be an exact sequence of groups, then the induced sequence of homogeneous quasimorphisms

$$0 \to Q(G) \xrightarrow{p^*} Q(\Gamma) \xrightarrow{i^*} Q(K)$$

is exact.

Definition 1.6. Let K be a normal subgroup of a group Γ. An element $\mu \in Q(K)$ is called Γ-*invariant* if the equality

$$\mu(\gamma^{-1} k \gamma) = \mu(k)$$

holds for any $k \in K$ and $\gamma \in \Gamma$. Let $Q(K)^\Gamma$ denote the $\mathbb{R}$-vector space of the Γ-invariant homogeneous quasimorphisms on K.

Remark 1.7. The exact sequence in Lemma 1.5 can be refined as

$$0 \to Q(G) \xrightarrow{p^*} Q(\Gamma) \xrightarrow{i^*} Q(K)^\Gamma.$$

Indeed, any element $\mu \in Q(\Gamma)$ is Γ-invariant, that is, μ satisfies the equality

$$\mu(\gamma_1^{-1}\gamma_2\gamma_1) = \mu(\gamma_2)$$

for any $\gamma_1, \gamma_2 \in \Gamma$ (see [**Cal09**, Section 2.2.3]). Thus the image of the map $i^*\colon Q(\Gamma) \to Q(K)$ is contained in $Q(K)^\Gamma$.

The following property plays an essential role in Section 2.3.

PROPOSITION 1.8 ([**PR14**, Proposition 3.1.4]). For an element $\mu \in Q(\Gamma)$ and for any elements $\gamma_1, \gamma_2 \in \Gamma$ satisfying $\gamma_1\gamma_2 = \gamma_2\gamma_1$, the equality

$$\mu(\gamma_1\gamma_2) = \mu(\gamma_1) + \mu(\gamma_2)$$

holds.

1.2. Group cohomology and bounded cohomology

In this section, we recall the notion of (bounded) cohomology of discrete groups. We refer to [**Bro82**], [**Cal09**], and [**Fri17**] for details.

1.2.1. Group cohomology.

DEFINITION 1.9. Let G be a group and $(M, +)$ be a left G-module. The set of all maps

$$C^n(G; M) = \{c \colon G^n \to M : \text{map}\}$$

is called the *group n-cochains of G with coefficients in M*. For $n > 0$, the coboundary map $\delta \colon C^n(G; M) \to C^{n+1}(G; M)$ is defined by

$$\begin{aligned} \delta c(g_1, \ldots, g_{n+1}) = & g_1 \cdot c(g_2, \ldots, g_{n+1}) + \sum_{j=1}^{n} (-1)^j c(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} c(g_1, \ldots, g_n) \end{aligned}$$

and $\delta \colon C^0(G; M) \to C^1(G; M)$ by $\delta m(g) = g \cdot m - m$, where we identify $C^0(G; M)$ with M. The *group cohomology $H^\bullet(G; M)$ of G with coefficients in M* is the cohomology of the cochain complex $(C^\bullet(G; M), \delta)$.

NOTATION 1. In this thesis, if the coefficients is the trivial G-module $\mathbb{R}$, we abbreviate $H^\bullet(G; \mathbb{R})$ to $H^\bullet(G)$.

EXAMPLE 1.10. In this example, we consider the trivial G-module M. The 0-th cohomology $H^0(G; M)$ is isomorphic to M since $\delta = 0 \colon C^0(G; M) \to C^1(G; M)$ and $C^0(G; M) = M$. The first cohomology $H^1(G; M)$ is isomorphic to the homomorphisms $\mathrm{Hom}(G; M)$ from G to M since the coboundary $\delta \colon C^1(G; M) \to C^2(G; M)$ is defined by

$$\delta c(g_1, g_2) = c(g_2) - c(g_1 g_2) + c(g_1).$$

An exact sequence $1 \to M \xrightarrow{i} E \to G \to 1$ of groups is called a *central M-extension of G* if the image $i(M)$ is in the center of E. Two central M-extensions E_1 and E_2 are *equivalent* if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & M \arrow[r] & E_2 \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

The following fact is well known, which gives a classification of the equivalence classes of central M-extensions.

PROPOSITION 1.11. Let M be a trivial G-module. The second group cohomology $H^2(G;M)$ is bijective to the set of all equivalence classes of central M-extensions of G;

$$H^2(G;M) \cong \{\text{central } M\text{-extensions of } G\}/\sim .$$

For a central M-extension E of G, the corresponding cohomology class $e(E) \in H^2(G;M)$ is defined as follows (see **[Bro82]** for details). Take a set-theoretical section $s\colon G \to E$ of the projection $E \to G$. For any $g_1, g_2 \in G$, the value $s(g_1)s(g_2)s(g_1g_2)^{-1}$ is in $i(M) \cong M$. Thus we obtain a cochain $c \in C^2(G;M)$ by setting

$$c(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}. \tag{1.2.1}$$

It can be shown that the cochain c is a cocycle, and its cohomology class $[c]$ does not depend on the choice of the section s. We set $e(E) = [c]$. This correspondence between a central M-extension E and the second cohomology class $e(E)$ gives the bijection in PROPOSITION 1.11.

PROPOSITION 1.12. Let $0 \to M \to E \to G \to 1$ be a central M-extension and $e(E)$ the corresponding cohomology class in $H^2(G;M)$. Let H be a group and $f\colon H \to G$ a homomorphism. The pullback $f^*e(E)$ is equal to zero if and only if the homomorphism has a lift, that is, there exists a homomorphism $\widetilde{f}\colon H \to E$ such that the diagram

$$\begin{tikzcd} & E \arrow[d] \\ H \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & G \end{tikzcd}$$

commutes.

The proof of the following lemma is straightforward.

LEMMA 1.13. For a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f"] & E_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & M_2 \arrow[r] & E_2 \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

of central extensions, the cohomology class $f_* e(E_1) \in H^2(G; M_2)$ is equal to $e(E_2)$, where $f_*\colon H^2(G; M_1) \to H^2(G; M_2)$ is the change of coefficients homomorphism.

It is known that the group cohomology $H^\bullet(G; M)$ is canonically isomorphic to the cohomology $H^\bullet_{\mathrm{top}}(BG^\delta; M)$ of the classifying space of the discrete group G^δ. Here G^δ denotes the group G with the discrete topology and $H^\bullet_{\mathrm{top}}(-; M)$ the cohomology of topological space with coefficients in M. By this isomorphism, a group cohomology class gives a characteristic class of foliated bundles, and vice versa.

EXAMPLE 1.14. In this example, we only consider the case when the coefficients is the trivial module $\mathbb{R}$ (see also NOTATION 1). The above isomorphism $H^\bullet(G) \cong H^\bullet_{\mathrm{top}}(BG^\delta)$ allows us to calculate the cohomology of some groups.

(1) The classifying space $B(\mathbb{Z}^n)$ is the n-dimensional torus $(S^1)^n$. Thus the group cohomology $H^\bullet(\mathbb{Z}^n)$ is isomorphic to the cohomology $H_{\mathrm{top}}((S^1)^n)$. In particular, we have
$$H^1(\mathbb{Z}^n) \cong \mathbb{R}^n, \quad H^2(\mathbb{Z}^n) \cong \mathbb{R}^{n(n-1)/2}.$$
(2) The classifying space BF_n of the rank n free group is the bouquet $\bigvee_n S^1$. Thus the group cohomology $H^\bullet(F_n)$ is isomorphic to $H_{\mathrm{top}}(\bigvee_n S^1)$. In particular, we have
$$H^1(F_n) = \mathbb{R}^n, \quad H^2(F_n) = 0.$$
(3) Let Σ_g be a closed oriented surface of genus $g > 0$ and $\Gamma_g = \pi_1(\Sigma_g)$ the surface group. Then the classifying space $B\Gamma_g$ is homotopy equivalent to Σ_g. Thus the cohomology $H^\bullet(\Gamma_g)$ is isomorphic to $H^\bullet_{\mathrm{top}}(\Sigma)$. In particular, we have
$$H^1(\Gamma_g) \cong \mathbb{R}^{2g}, \quad H^2(\Gamma_g) \cong \mathbb{R}.$$
(4) Let X be a closed connected manifold. Let $\mathrm{Homeo}_0(X)$ be the identity component of the group of homeomorphisms of X with the compact-open topology, and $\mathrm{Homeo}_0(X)^\delta$ denotes the same group with the discrete topology. The identity homomorphism $\iota\colon \mathrm{Homeo}_0(X)^\delta \to \mathrm{Homeo}_0(X)$ induces a continuous map
$$B\iota\colon B\,\mathrm{Homeo}_0(X)^\delta \to B\,\mathrm{Homeo}_0(X).$$
By the theorem of Thurston [**Thu74**], the map $B\iota$ induces an isomorphism of cohomology.Let us consider the case when X is the circle S^1. Then the group $\mathrm{Homeo}_0(S^1)$ coincides with

the group $\mathrm{Homeo}_+(S^1)$ of orientation preserving homeomorphisms. By the fact that the group $\mathrm{Homeo}_+(S^1)$ is homotopy equivalent to S^1, we have

$$H^\bullet(\mathrm{Homeo}_+(S^1);\mathbb{Z}) \cong H^\bullet_{\mathrm{top}}(BS^1;\mathbb{Z}) = \mathbb{Z}[e],$$

where the class $e \in H^2_{\mathrm{top}}(BS^1;\mathbb{Z}) = H^2_{\mathrm{top}}(\mathbb{C}P^\infty;\mathbb{Z})$ is the Euler class.

For a group extension, there is a spectral sequence called the *Hochschild-Serre spectral sequence* (for simplicity, we only states the $\mathbb{R}$-coefficients case).

THEOREM 1.15 (**[HS53]**). For a group extension $1 \to K \to \Gamma \to G \to 1$, there exists a first quadrant spectral sequence $(E_r^{p,q}, d_r^{p,q})$ with $E_2^{p,q} \cong H^p(G; H^q(K))$ which converges to $H^{p+q}(\Gamma)$

Here the G-action on $H^q(K)$ is defined as follows. The group Γ acts on $C^q(K)$ by

$$({}^\gamma c)(k_1,\dots,k_q) = c(\gamma^{-1}k_1\gamma,\dots,\gamma^{-1}k_q\gamma). \tag{1.2.2}$$

This action induces a Γ-action on the cohomology $H^q(K)$. It is known that the action restricted to K is trivial, and therefore the action induces the G-action on $H^q(K)$. In this way, the cohomology $H^q(K)$ is considered as a left G-module. Let $H^q(K)^G$ denote the G-invariant part of $H^q(K)$. Then, in particular, any element f of $H^1(K)^G$ satisfies the equality

$$f(\gamma^{-1}k\gamma) = f(k) \tag{1.2.3}$$

for any $\gamma \in \Gamma$ and $k \in K$. Thus the space $H^1(K)^G$ is identified with the space of all Γ-invariant homomorphisms from K to $\mathbb{R}$.

REMARK 1.16. Under the canonical isomorphism between $H^\bullet(G)$ and $H^\bullet_{\mathrm{top}}(BG^\delta)$, the Hochschild-Serre spectral sequence of a group extension

$$1 \to K \to \Gamma \to G \to 1$$

is isomorphic to the Serre spectral sequence of the fibration

$$BK^\delta \to B\Gamma^\delta \to BG^\delta$$

of the classifying spaces.

For a first quadrant spectral sequence $(E_r^{p,q}, d_r^{p,q})$, we obtain the following exact sequence called the *seven-term exact sequence*:

$$\begin{aligned} 0 \to E_2^{1,0} \to E_\infty^1 \to E_2^{0,1} &\xrightarrow{d_2^{0,1}} E_2^{2,0} \\ &\to \mathrm{Ker}(E_\infty^2 \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}. \end{aligned}$$

By applying this to the Hochschild-Serre spectral sequence, we obtain the following.

THEOREM 1.17 (Seven-term exact sequence of group cohomology). For a group extension $1 \to K \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1$, there is an exact sequence

$$\begin{aligned}(1.2.4)\qquad 0 \to H^1(G) \to H^1(\Gamma) \to H^1(K)^G \xrightarrow{\tau} H^2(G)& \\ \to \mathrm{Ker}(i^*) \xrightarrow{\zeta} H^1(G; H^1(K)) \to H^3(G)&,\end{aligned}$$

where $i^*\colon H^2(\Gamma) \to H^2(K)$.

REMARK 1.18. By the definition of the G-action on $H^q(K)$, the G-invariant part $H^q(K)^G$ is the same as the Γ-invariant part $H^q(K)^\Gamma$. In our setting in Chapter 3, it is more convenient to consider the G-invariant part $H^1(K)^G$ in (1.2.4) as the Γ-invariant part $H^1(K)^\Gamma$.

REMARK 1.19. Other coefficients versions of the exact sequence (1.2.4) also hold. For example, there is an exact sequence

$$\begin{aligned}(1.2.5)\qquad 0 \to H^1(G;\mathbb{Z}) \to H^1(\Gamma;\mathbb{Z}) \to H^1(K;\mathbb{Z})^G \xrightarrow{\tau} H^2(G;\mathbb{Z})& \\ \to \mathrm{Ker}(i^*) \xrightarrow{\zeta} H^1(G; H^1(K;\mathbb{Z})) \to H^3(G;\mathbb{Z})&,\end{aligned}$$

where $i^*\colon H^2(\Gamma;\mathbb{Z}) \to H^2(K;\mathbb{Z})$.

The explicit descriptions of the maps τ and ζ are known.

PROPOSITION 1.20 ([**NSW08**, (1.6.6) Proposition and (2.4.3) Theorem]). For any G-invariant homomorphism $f \in H^1(K)^G$, there exists a one-cochain $F\colon \Gamma \to \mathbb{R}$ such that $i^*F = f$ and that $\delta F(\gamma_1, \gamma_2)$ depends only on $p(\gamma_1)$ and $p(\gamma_2)$, that is, there exists a cocycle $c \in C^2(G)$ satisfying

$$c(p(\gamma_1), p(\gamma_2)) = \delta F(\gamma_1, \gamma_2)$$

for any $\gamma_1, \gamma_2 \in \Gamma$. For such a cochain F, the equality

$$\tau(f) = [c] \in H^2(G)$$

holds.

PROPOSITION 1.21 (Section 10.3 of [**DHW12**]). For an element $c \in \mathrm{Ker}(i^*\colon H^2(\Gamma) \to H^2(K))$, take a representing two-cocycle $f \in C^2(\Gamma)$ satisfying $f|_{K\times K} = 0$. Then

$$\big(\zeta(c)(p(\gamma))\big)(k) = f(\gamma, \gamma^{-1}k\gamma) - f(k, \gamma).$$

1.2.2. Bounded cohomology. In this subsection, we assume that M is either the trivial G-module $\mathbb{Z}$ or $\mathbb{R}$.

DEFINITION 1.22. Let $(C_b^\bullet(G;M),\delta)$ be the subcomplex consisting of all bounded functions. The cohomology $H_b^\bullet(G;M)$ of the subcomplex $(C_b^\bullet(G;M),\delta)$ is called the *bounded cohomology of G with coefficients in M*. The inclusion $C_b^\bullet(G;M)\hookrightarrow C^\bullet(G;M)$ induces the map

$$c_G\colon H_b^\bullet(G;M)\to H^\bullet(G;M)$$

called the *comparison map*. The kernel of c_G is called the *exact bounded cohomology* and denoted by $EH_b^\bullet(G;M)$.

NOTATION 2. As NOTATION 1, we abbreviate $H_b^\bullet(G;\mathbb{R})$ to $H_b^\bullet(G)$ and $EH_b^2(G;\mathbb{R})$ to $EH_b^2(G)$.

EXAMPLE 1.23. The 0-th bounded cohomology $H_b^0(G;M)$ is isomorphic to M by definition. The first cohomology $H_b^1(G;M)$ is trivial since any bounded homomorphism to M is trivial.

For the triviality of bounded cohomology, the following is known.

PROPOSITION 1.24. If G is abelian, the bounded cohomology $H_b^n(G)$ is trivial for $n>0$.

EXAMPLE 1.25. By PROPOSITION 1.24, the second real bounded cohomology $H_b^2(\mathbb{Z})$ is trivial. In contrast, the second integral bounded cohomology $H_b^2(\mathbb{Z};\mathbb{Z})$ is isomorphic to $\mathbb{R}/\mathbb{Z}$ since there is an exact sequence

$$\to H_b^1(\mathbb{Z})=0\to H^1(\mathbb{Z};\mathbb{R}/\mathbb{Z})\to H_b^2(\mathbb{Z};\mathbb{Z})\to H_b^2(\mathbb{Z})=0\to$$

and $H^1(\mathbb{Z};\mathbb{R}/\mathbb{Z})\cong H^1_{\mathrm{top}}(S^1;\mathbb{R}/\mathbb{Z})\cong\mathbb{R}/\mathbb{Z}$.

For $\mu\in Q(G)$, the coboundary $\delta\mu$ is a bounded two-cocycle in $C_b^2(G)$ by the definition of quasimorphism. Note that the bounded cocycle $\delta\mu$ is not necessarily a coboundary as a bounded cochain since the homogeneous quasimorphism is not bounded if $\mu\neq 0$. The following fact is well known.

PROPOSITION 1.26. The following sequence is exact:

$$0\to H^1(G)\to Q(G)\xrightarrow{\delta_*} H_b^2(G)\xrightarrow{c_G} H^2(G), \tag{1.2.6}$$

where the map δ_* is given by $\delta_*(\mu)=[\delta\mu]$.

REMARK 1.27. The exact sequence (1.2.6) is equivalent to the following exact sequence:

$$0\to H^1(G)\to Q(G)\to EH_b^2(G)\to 0. \tag{1.2.7}$$

EXAMPLE 1.28. By the exact sequence (1.2.6), we can show that the second bounded cohomology is infinite dimensional for some groups.

(1) For the free group F_2, Brooks [**Bro81**] constructed infinitely many homogeneous quasimorphisms on F_2.
(2) Let X be a closed hyperbolic manifold and $\pi_1(X)$ the fundamental group. In [**EF97**], Epstein and Fujiwara constructed infinitely many homogeneous quasimorphisms on $\pi_1(X)$ (more generally, on the word-hyperbolic groups).

By using the above homogeneous quasimorphisms, it was shown that the dimension of $H_b^2(G)$ is the cardinal of continuum (in [**Mit84**] for $G = F_2$ and in [**EF97**] for $G = \pi_1(X)$). This implies that the bounded cohomology and the ordinary group cohomology are quite different.

THEOREM 1.29 ([**Bou95**]). For a group extension $1 \to K \to \Gamma \to G \to 1$, there is an exact sequence

$$0 \to H_b^2(G) \to H_b^2(\Gamma) \to H_b^2(K)^\Gamma \to H_b^3(G).$$

Here the Γ-action on $H_b^\bullet(K)$ is defined in the same way as in (1.2.2).

1.3. Characteristic classes

In this section, we recall the notion of characteristic class of fiber bundles.

DEFINITION 1.30. Let G be a topological group. A fiber bundle is called a *G-bundle* if the structure group of the bundle has a reduction to G. A *foliated G-bundle* is a fiber bundle whose structure group has a reduction to the discrete group G^δ.

Let BG be the classifying space of G. We can regard a cohomology class of BG as a characteristic class of G-bundles by the following theorem.

THEOREM 1.31. There is a bijective correspondence between characteristic classes of G-bundles and cohomology classes of BG.

The identity homomorphism $\iota\colon G^\delta \to G$ induces the continuous map $B\iota\colon BG^\delta \to BG$ and therefore the homomorphism

$$(B\iota)^*\colon H^\bullet_{\mathrm{top}}(BG; M) \to H^\bullet_{\mathrm{top}}(BG^\delta; M) \cong H^\bullet(G; M),$$

where M is a trivial G-module.

DEFINITION 1.32. An element $a \in H^\bullet(G; M)$ is called a *characteristic class of foliated G-bundles* if $a \in \mathrm{Im}(B\iota)^*$.

There is a well-known characteristic class called the *primary obstruction class*, which is defined as an obstruction to the construction of a cross-section. We briefly recall the definition of the obstruction class of fibrations via the Serre spectral sequence (see [**Whi78**] for details). Let $F \to E \to B$ be a fibration, and for simplicity, we assume that the base space B is one-connected, the fiber F is path-connected, and the fundamental group $\pi_1(F)$ is abelian. Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence with coefficients in $\pi_1(F)$. Since B is one-connected, any local coefficient system is simple, and thus we have

$$E_2^{p,q} \cong H^p_{\mathrm{top}}(B; H^q_{\mathrm{top}}(F; \pi_1(F))).$$

Therefore we have

$$E_2^{2,0} \cong H^2_{\mathrm{top}}(B; \pi_1(F))$$

and

$$E_2^{0,1} \cong H^1_{\mathrm{top}}(F; \pi_1(F)).$$

Note that the cohomology $H^1_{\mathrm{top}}(F; \pi_1(F))$ is isomorphic to the space of all self-homomorphisms $\mathrm{Hom}(\pi_1(F), \pi_1(F))$ on $\pi_1(F)$. Then the derivation map $d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}$ gives a map

$$d_2^{0,1}\colon \mathrm{Hom}(\pi_1(F), \pi_1(F)) \to H^2_{\mathrm{top}}(B; \pi_1(F))$$

(here we abuse the symbol $d_2^{0,1}$).

We are now ready to define the primary obstruction class of fibrations.

Definition 1.33. Let $F \to E \to B$ be a fibration such that B is one-connected, F is path-connected, and $\pi_1(F)$ is abelian. Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence. The cohomology class

$$\mathfrak{o} = -d_2^{0,1}(\mathrm{id}_{\pi_1(F)}) \in H^2_{\mathrm{top}}(B; \pi_1(F))$$

is called the *primary obstruction class*.

Remark 1.34. It is known that the above definition is equivalent to the classical definition of the obstruction to the construction of a cross-section (see, for example, [**Whi78**, (6.10) Corollary and (7.9*) Theorem]).

Remark 1.35. Let G be a connected topological group and $\widetilde{G}$ the universal covering group. By Theorem 1.31, the primary obstruction class of G-bundles defines a cohomology class $\mathfrak{o}$ in $H^2(BG; \pi_1(G))$. This class is also obtained as follows. By taking classifying spaces of the central extension

$$0 \to \pi_1(G) \to \widetilde{G} \to G \to 1,$$

we obtain the following fibration

$$B\pi_1(G) \to B\widetilde{G} \to BG. \tag{1.3.1}$$

Since we are assuming that G is connected, the base space BG is one-connected. Note that the fundamental group of $B\pi_1(G)$ is isomorphic to $\pi_1(G)$ and this is abelian. Then the primary obstruction class of the fibration (1.3.1) is the class $\mathfrak{o} \in H^2(BG; \pi_1(G))$.

Let $f\colon \pi_1(G) \to \mathbb{R}$ be a homomorphism and

$$f_*\colon H^\bullet_{\mathrm{top}}(-;\pi_1(G)) \to H^\bullet_{\mathrm{top}}(-)$$

the change of coefficients map. Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence of (1.3.1) with coefficients in $\mathbb{R}$. Since $E_2^{0,1} \cong H^1_{\mathrm{top}}(B\pi_1(G)) \cong \mathrm{Hom}(\pi_1(G))$ and $E_2^{2,0} \cong H^2_{\mathrm{top}}(BG)$, the derivation $d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}$ defines a homomorphism

$$d_2^{0,1}\colon \mathrm{Hom}(\pi_1(G), \mathbb{R}) \to H^2_{\mathrm{top}}(BG).$$

PROPOSITION 1.36. Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence of (1.3.1) with coefficients in $\mathbb{R}$. For a homomorphism $f\colon \pi_1(G) \to \mathbb{R}$, an equality

$$-d_2^{0,1}(f) = f_*\mathfrak{o} \in H^2_{\mathrm{top}}(BG)$$

holds.

PROOF. Let $(E_r'^{p,q}, d_r'^{p,q})$ be the Serre spectral sequence of (1.3.1) with coefficients in $\pi_1(G)$, then the equality $-d_2'^{0,1}(\mathrm{id}_{\pi_1(G)}) = \mathfrak{o}$ holds. Since the derivation maps in the Serre spectral sequence are compatible with the change of coefficients homomorphisms, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\pi_1(G), \pi_1(G)) \cong E_2'^{0,1} & \xrightarrow{d_2'^{0,1}} & E_2'^{2,0} \cong H^2_{\mathrm{top}}(BG; \pi_1(G)) \\
\downarrow f_* & & \downarrow f_* \\
\mathrm{Hom}(\pi_1(G), \mathbb{R}) \cong E_2^{0,1} & \xrightarrow{d_2^{0,1}} & E_2^{2,0} \cong H^2_{\mathrm{top}}(BG).
\end{array}$$

Since $f = f_*(\mathrm{id}_{\pi_1(G)}) \in \mathrm{Hom}(\pi_1(G), \mathbb{R})$, we obtain

$$-d_2^{0,1}(f) = -d_2^{0,1}(f_*(\mathrm{id}_{\pi_1(G)})) = f_*(-d_2'^{0,1}(\mathrm{id}_{\pi_1(G)})) = f_*\mathfrak{o}$$

and the proposition follows. □

1.4. Transformation groups

For a smooth manifold X, let $\mathrm{Homeo}(X)$ denote the group of homeomorphisms of X with the compact-open topology and $\mathrm{Diff}(X)$ the group of diffeomorphisms of X with the C^∞-topology. Let $\mathrm{Homeo}_0(X)$ and $\mathrm{Diff}_0(X)$ be the identity component of $\mathrm{Homeo}(X)$ and $\mathrm{Diff}(X)$, respectively.

A *symplectic manifold* is a pair (X, ω) of a smooth manifold X and a non-degenerate closed two-form $\omega \in \Omega^2(X)$ (called a *symplectic form*).

DEFINITION 1.37. Let (X, ω) be a symplectic manifold. A diffeomorphism $g\colon X \to X$ is called a *symplectomorphism* if $g^*\omega = \omega$ holds. The group

$$\mathrm{Symp}(X, \omega) = \{g \in \mathrm{Diff}(X) \mid g^*\omega = \omega\}$$

is called the *symplectomorphism group*.

REMARK 1.38. On a symplectic manifold, there is another natural transformation group $\mathrm{Ham}(X, \omega)$ called the *Hamiltonian diffeomorphism group* (see [**Ban97**], [**MS98**] for the definition). The Hamiltonian diffeomorphism group is a normal subgroup of $\mathrm{Symp}(M, \omega)$.

THEOREM 1.39 ([**Ban78**]). For a closed symplectic manifold (X, ω), the Hamiltonian diffeomorphism group $\mathrm{Ham}(X, \omega)$ and its universal covering group $\widetilde{\mathrm{Ham}}(X, \omega)$ are perfect.

REMARK 1.40. It is known that the Hamiltonian diffeomorphism group $\mathrm{Ham}(X, \omega)$ coincides with $\mathrm{Symp}_0(X, \omega)$ for a closed symplectic manifold (X, ω) whose first Betti number is equal to zero (see, for example, [**Ban97**]). Thus, for such a symplectic manifold, the groups $\mathrm{Symp}_0(X, \omega)$ and $\widetilde{\mathrm{Symp}}_0(X, \omega)$ are perfect.

In Section 2.4, we will consider the following symplectic manifold. On the direct product $S^2 \times S^2$, there is a symplectic form ω_λ defined by

$$\omega_\lambda = \mathrm{pr}_1^*\omega_0 + \lambda \cdot \mathrm{pr}_2^*\omega_0,$$

where ω_0 is the standard symplectic form on S^2, λ a non-zero real number, and $\mathrm{pr}_1, \mathrm{pr}_2\colon S^2 \times S^2 \to S^2$ are the first and second projection, respectively. Gromov showed in [**Gro85**] that the symplectomorphism group $\mathrm{Symp}_0(S^2 \times S^2, \omega_1)$ is homotopy equivalent to the group $SO(3) \times SO(3)$. If $1 < \lambda \leq 2$, the homotopy type of $\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)$ is determined in [**Anj02**].

THEOREM 1.41 ([**Anj02**]). If $1 < \lambda \leq 2$, the symplectomorphism group $\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)$ is homotopy equivalent to the space

$$\Omega(\Sigma(S^1 \vee SO(3))) \times S^1 \times SO(3) \times SO(3),$$

where the space $\Omega(\Sigma(S^1 \vee SO(3)))$ is the loop space of the suspension of the smash product $S^1 \vee SO(3)$.

REMARK 1.42. For $1 < \lambda \leq 2$, the fundamental group $\pi_1(\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda))$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Indeed, by THEOREM 1.41, the fundamental group is isomorphic to

$$\begin{aligned}&\pi_1(\Omega(\Sigma(S^1 \vee SO(3)))) \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\\ &\cong \pi_2(\Sigma(S^1 \vee SO(3))) \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

Since the smash product of connected spaces is one-connected, so is the space $S^1 \vee SO(3)$. Together with the Freudenthal suspension theorem, we have

$$\pi_2(\Sigma(S^1 \vee SO(3))) \cong \pi_1(S^1 \vee SO(3)) = 0.$$

A *contact manifold* is a pair (X, ξ) of a $(2n+1)$-dimensional smooth manifold X and a maximally non-integrable hyperplane field $\xi \subset TX$, that is, a defining local one-form α gives a local volume form $\alpha \wedge (d\alpha)^n$. The globally defined one-form α is called a *contact form*.

DEFINITION 1.43. For a contact manifold (X, ξ), a diffeomorphism $g\colon X \to X$ is called a *contactomorphism* if g preserves the contact structure, that is, $g_*\xi = \xi$ holds. Let $\mathrm{Cont}(X, \xi)$ denote the group of contactomorphisms.

THEOREM 1.44 ([**Ryb10**]). For a closed contact manifold (X, ξ), the group $\mathrm{Cont}_0(X, \xi)$ and its universal covering group $\widetilde{\mathrm{Cont}_0}(X, \xi)$ are perfect.

CHAPTER 2

Non-descendible homogeneous quasimorphisms

This chapter is a part of the joint work with Kawasaki [**KM20**]. Let Γ and G be groups and $p\colon \Gamma \to G$ a surjective homomorphism. In this section, we consider the homogeneous quasimorphisms on Γ which do not descend to G. To do this, we introduce the following space:

Definition 2.1. For groups Γ, G and a surjective homomorphism $p\colon \Gamma \to G$, set

$$\mathcal{ND} = Q(\Gamma) \big/ \big(p^* Q(G) + H^1(\Gamma)\big)\,.$$

In this chapter, we discuss the space $\mathcal{ND}$ and apply it to characteristic classes of foliated bundles.

2.1. General principle

Definition 2.2. A subspace $\mathcal{C}(\Gamma)$ of $C^1(\Gamma)$ is defined by

$$\begin{aligned} \mathcal{C}(\Gamma) = \{F \in & C^1(\Gamma) \\ & \mid F(k\gamma) = F(\gamma k) = F(\gamma) + F(k) \text{ for any } \gamma \in \Gamma, k \in K\}. \end{aligned} \tag{2.1.1}$$

We define a map $\mathfrak{D}\colon \mathcal{C}(\Gamma) \to C^2(G)$ by setting

$$\mathfrak{D}(F)(g_1, g_2) = F(\gamma_2) - F(\gamma_1\gamma_2) + F(\gamma_1),$$

where γ_j is an element of Γ satisfying $p(\gamma_j) = g_j$.

Lemma 2.3 ([**KM20**]). *The map $\mathfrak{D}\colon \mathcal{C}(\Gamma) \to C^2(G)$ is well defined.*

Proof. Let γ'_j be another element of Γ satisfying $p(\gamma'_j) = g_j$. Then there exist elements $k_1, k_2 \in K$ satisfying $\gamma'_1 = k_1\gamma_1$ and $\gamma'_2 = \gamma_2 k_2$. Then, by the definition of $\mathcal{C}(\Gamma)$, we have

$$\begin{aligned} & F(\gamma'_2) - F(\gamma'_1\gamma'_2) + F(\gamma'_1) \\ & = F(\gamma_2 k_2) - F(k_1\gamma_1\gamma_2 k_2) + F(k_1\gamma_1) \\ & = (F(\gamma_2) + F(k_2)) - (F(k_1) + F(\gamma_1\gamma_2) + F(k_2)) + (F(k_1) + F(\gamma_1)) \\ & = F(\gamma_2) - F(\gamma_1\gamma_2) + F(\gamma_1). \end{aligned}$$

This implies the well-definedness of the map $\mathfrak{D}$. $\square$

LEMMA 2.4 ([**KM20**]). For any $F \in \mathcal{C}(\Gamma)$, the cochain $\mathfrak{D}(F)$ is a cocycle.

PROOF. Since $p^*\mathfrak{D}(F) = \delta F$, we have

$$p^*(\delta\mathfrak{D}(F)) = \delta\delta F = 0.$$

By the surjectivity of $p\colon \Gamma \to G$, we have $\delta\mathfrak{D}(F) = 0$. $\square$

DEFINITION 2.5. A homomorphism $\mathfrak{d}\colon \mathcal{C}(\Gamma) \to H^2(G)$ is defined by

$$\mathfrak{d}(F) = [\mathfrak{D}(F)] \in H^2(G).$$

In the seven-term exact sequence (1.2.4), there is a map

$$\tau\colon H^1(K)^\Gamma \to H^2(G)$$

(see THEOREM 1.17 and PROPOSITION 1.20). Recall that an element f of $H^1(K)^G$ is a G-invariant homomorphism, that is, f satisfies $f(\gamma^{-1}k\gamma) = f(k)$ for any $k \in K$ and $\gamma \in \Gamma$. The homomorphism $\mathfrak{d}$ defined in Definition 2.5 is related to the map τ as the following commutative diagram (PROPOSITION 2.7)

$$\begin{tikzcd}
\mathcal{C}(\Gamma) \arrow[d, "i^*"] \arrow[dr, "\mathfrak{d}"] & \\
H^1(K)^G \arrow[r, "\tau"'] & H^2(G).
\end{tikzcd}$$

For an element F of $\mathcal{C}(\Gamma)$, the restriction $F|_K = i^*F$ is in $H^1(K)^G$ since we have

$$F(k) = F(k\gamma) - F(\gamma) = F(\gamma \cdot \gamma^{-1}k\gamma) - F(\gamma) = F(\gamma^{-1}k\gamma).$$

Thus we obtain a homomorphism $i^*\colon \mathcal{C}(\Gamma) \to H^1(K)^G$.

LEMMA 2.6 ([**KM20**]). The map $i^*\colon \mathcal{C}(\Gamma) \to H^1(K)^G$ is surjective.

PROOF. Let $s\colon G \to \Gamma$ be a section of $p\colon \Gamma \to G$ satisfying $s(\mathrm{id}_G) = \mathrm{id}_\Gamma$, where $\mathrm{id}_G \in G$ and $\mathrm{id}_\Gamma \in \Gamma$ be the unit elements of G and Γ, respectively. Then, since an element $\gamma \cdot s(p(\gamma))^{-1}$ is in $\mathrm{Ker}(p\colon \Gamma \to G)$, we regard the element $\gamma \cdot s(p(\gamma))^{-1}$ as that of K under the injection $i\colon K \to \Gamma$. For an element f of $H^1(K)^G$, define $f_s\colon \Gamma \to \mathbb{R}$ by

$$f_s(\gamma) = f(\gamma \cdot s(p(\gamma))^{-1}).$$

Note that the restriction of f_s to K is equal to f. Moreover, the equalities

$$\begin{aligned}
f_s(k\gamma) &= f(k\gamma \cdot s(p(k\gamma))^{-1}) = f(k\gamma \cdot s(p(\gamma))^{-1}) \\
&= f(k) + f(\gamma \cdot s(p(\gamma))^{-1}) = f_s(k) + f_s(\gamma)
\end{aligned}$$

and

$$\begin{aligned} f_s(\gamma k) &= f(\gamma k \cdot s(p(\gamma k))^{-1}) = f(\gamma k \cdot s(p(\gamma))^{-1}) \\ &= f(\gamma k \gamma^{-1}) + f(\gamma \cdot s(p(\gamma))^{-1}) \\ &= f(k) + f(\gamma \cdot s(p(\gamma))^{-1}) = f_s(k) + f_s(\gamma) \end{aligned}$$

hold, where we use the G-invariance of f. Thus the map f_s is an element of $\mathcal{C}(\Gamma)$ and the surjectivity follows. □

PROPOSITION 2.7 ([**KM20**]). The diagram

$$\begin{tikzcd} \mathcal{C}(\Gamma) \arrow[d, "i^*"] \arrow[dr, "\mathfrak{d}"] & \\ H^1(K)^G \arrow[r, "\tau"'] & H^2(G). \end{tikzcd}$$

commutes.

PROOF. By DEFINITION 2.5 and PROPOSITION 1.20, we obtain the proposition. □

2.2. A diagram via bounded cohomology and quasimorphism

In this section, we refine the commutative diagram in PROPOSITION 2.7 in view of bounded cohomology and homogeneous quasimorphisms. A cohomology class $a \in H^2(G)$ is called *bounded* if a is in the image of the comparison map $c_G : H_b^2(G) \to H^2(G)$.

PROPOSITION 2.8 ([**KM20**]). There is a commutative diagram

$$\begin{tikzcd} \mathcal{C}(\Gamma) \cap Q(\Gamma) \arrow[r, "\mathfrak{d}_b"] \arrow[d, "i^*"] \arrow[dr, "\mathfrak{d}"] & H_b^2(G) \arrow[d, "c_G"] \\ H^1(K)^G \arrow[r, "\tau"'] & H^2(G). \end{tikzcd}$$

PROOF. For an element $F \in \mathcal{C}(\Gamma) \cap Q(\Gamma)$, the cocycle $\mathfrak{D}(F) \in C^2(G)$ is bounded since

$$\mathfrak{D}(F)(g_1, g_2) = F(\gamma_2) - F(\gamma_1\gamma_2) + F(\gamma_1)$$

for any $g_1, g_2 \in G$ and F is a quasimorphism. Thus $\mathfrak{D}\colon \mathcal{C}(\Gamma) \to C^2(G)$ induces a homomorphism

$$\mathfrak{d}_b\colon \mathcal{C}(\Gamma) \to H_b^2(G)$$

satisfying $\mathfrak{d} = c_G \circ \mathfrak{d}_b\colon \mathcal{C}(\Gamma) \cap Q(\Gamma) \to H^2(G)$. □

REMARK 2.9. For a central extension

$$0 \to A \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1,$$

the space $Q(\Gamma)$ is contained in $\mathcal{C}(\Gamma)$. Indeed, by the definition of central extension, elements $a \in A$ and $\gamma \in \Gamma$ satisfy $a\gamma = \gamma a$. Thus, by PROPOSITION 1.8, any homogeneous quasimorphism $\mu \in Q(\Gamma)$ satisfies

$$\mu(a\gamma) = \mu(\gamma a) = \mu(a) + \mu(\gamma).$$

This implies $Q(\Gamma) \subset \mathcal{C}(\Gamma)$. Moreover, any homomorphism $f\colon A \to \mathbb{R}$ is G-invariant since $\gamma^{-1}a\gamma = a\gamma^{-1}\gamma = a$ for any $\gamma \in \Gamma$ and $a \in A$. Thus, together with PROPOSITION 2.8, we obtain a commutative diagram

$$\begin{tikzcd}
Q(\Gamma) \arrow[r, "\mathfrak{d}_b"] \arrow[d, "i^*"'] \arrow[dr, "\mathfrak{d}"] & H_b^2(G) \arrow[d, "c_G"] \\
H^1(A) \arrow[r, "\tau"'] & H^2(G).
\end{tikzcd}$$

REMARK 2.10. In this remark, we temporary use the symbol $Q'(\Gamma)$ to denote the set of all (not necessarily homogeneous) quasimorphisms. If we use $Q'(\Gamma)$ rather than $Q(\Gamma)$, the same statement as PROPOSITION 2.8 also holds.

LEMMA 2.11. If the pullback $i^*\mu$ of a homogeneous quasimorphism $\mu \in Q(\Gamma)$ is a homomorphism on K, then μ is contained in $\mathcal{C}(\Gamma)$.

PROOF. For any $\gamma \in \Gamma$, $k \in K$, and $n \in \mathbb{N}$, the equalities

$$(k\gamma)^n = k \cdot \gamma k \gamma^{-1} \cdot \gamma^2 k \gamma^{-2} \cdot \cdots \cdot \gamma^{n-1} k \gamma^{-(n-1)} \cdot \gamma^n$$

and

$$(\gamma k)^n = \gamma^n \cdot \gamma^{-(n-1)} k \gamma^{n-1} \cdot \cdots \cdot \gamma^{-2} k \gamma^2 \cdot \gamma^{-1} k \gamma \cdot k$$

hold. Since the pullback $i^*\mu$ is Γ-invariant homomorphism, we have

$$\mu(k \cdot \gamma k \gamma^{-1} \cdot \gamma^2 k \gamma^{-2} \cdot \cdots \cdot \gamma^{n-1} k \gamma^{-(n-1)}) = \mu(k^n)$$

and

$$\mu(\gamma^{-(n-1)} k \gamma^{n-1} \cdot \cdots \cdot \gamma^{-2} k \gamma^2 \cdot \gamma^{-1} k \gamma \cdot k) = \mu(k^n).$$

Thus we obtain

$$n \cdot |\mu(k\gamma) - \mu(k) - \mu(\gamma)| = |\mu((k\gamma)^n) - \mu(k^n) - \mu(\gamma^n)| \leq D(\mu)$$

and

$$n \cdot |\mu(\gamma k) - \mu(\gamma) - \mu(k)| = |\mu((\gamma k)^n) - \mu(\gamma^n) - \mu(k^n)| \leq D(\mu),$$

and these imply the equalities $\mu(k\gamma) = \mu(k) + \mu(\gamma)$ and $\mu(\gamma k) = \mu(\gamma) + \mu(k)$. $\square$

THEOREM 2.12 ([**KM20**]). For a group extension Γ of G, the homomorphism $\mathfrak{d}\colon \mathcal{C}(\Gamma) \to H^2(G)$ induces an isomorphism

$$(\mathcal{C}(\Gamma) \cap Q(\Gamma))/(H^1(\Gamma) + p^*Q(G)) \to \mathrm{Im}(\tau) \cap \mathrm{Im}(c_G).$$

PROOF. Let us consider the following commutative diagram whose rows and columns are exact:

$$\begin{tikzcd}
H^1(K)^\Gamma \arrow[r] & Q(K)^\Gamma \arrow[r] & H_b^2(K)^\Gamma & \\
H^1(\Gamma) \arrow[r] \arrow[u] & Q(\Gamma) \arrow[r, "\delta_*"] \arrow[u] & H_b^2(\Gamma) \arrow[r] \arrow[u] & H^2(\Gamma) \\
H^1(G) \arrow[r] \arrow[u] & Q(G) \arrow[r] \arrow[u, "p^*"] & H_b^2(G) \arrow[r, "c_G"] \arrow[u, "p^*"] & H^2(G) \arrow[u, "p^*"] \\
 & & & H^1(K)^\Gamma, \arrow[u, "\tau"]
\end{tikzcd}$$

where the exactness of each row and each column comes from REMARK 1.7, THEOREM 1.17, and THEOREM 1.29. By the definition of $\mathfrak{d}_b$, we have $p^*\mathfrak{d}_b(\mu) = \delta_*(\mu)$ for $\mu \in \mathcal{C}(\Gamma) \cap Q(\Gamma)$. Thus the map $p^*\colon H_b^2(G) \to H_b^2(\Gamma)$ gives an isomorphism

$$p^*\colon H_b^2(G) \xrightarrow{\cong} \delta_*(\mathcal{C}(\Gamma) \cap Q(\Gamma)).$$

Therefore, in this diagram, the map $\mathfrak{d}$ is given as the composite

$$c_G \circ (p^*)^{-1} \circ \delta_*\colon \mathcal{C}(\Gamma) \cap Q(\Gamma) \to H^2(G).$$

It is easily checked by the diagram chasing that the kernel $\mathrm{Ker}(\mathfrak{d})$ is equal to $H^1(\Gamma)+p^*Q(G)$. The surjectivity of the map $\mathfrak{d}\colon \mathcal{C}(\Gamma)\cap Q(\Gamma) \to \mathrm{Im}(\tau) \cap \mathrm{Im}(c_G)$ follows from LEMMA 2.11 and a diagram chasing argument. $\square$

REMARK 2.13. For a central extension Γ of G, the homomorphism $\mathfrak{d}\colon \mathcal{C}(\Gamma) \to H^2(G)$ induces an isomorphism

$$Q(\Gamma)/(H^1(\Gamma) + p^*Q(G)) \to \mathrm{Im}(\tau) \cap \mathrm{Im}(c_G).$$

since $\mathcal{C}(\Gamma) \cap Q(\Gamma) = Q(\Gamma)$ (see Remark 2.9). The domain is exactly the space $\mathcal{ND}$ of the non-descendible homogeneous quasimorphisms.

2.3. On universal covering groups

Let G be a connected topological group, $\widetilde{G}$ the universal covering group of G, and $\pi_1(G)$ the fundamental group of G. Then, these groups define a group extension

$$0 \to \pi_1(G) \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \to 1. \tag{2.3.1}$$

Because the exact sequence (2.3.1) is a central extension, there exists a commutative diagram

$$\begin{tikzcd} Q(\widetilde{G}) \arrow[r, "\mathfrak{d}_b"] \arrow[d, "i^*"] \arrow[dr, "\mathfrak{d}"] & H_b^2(G) \arrow[d, "c_G"] \\ H^1(\pi_1(G)) \arrow[r, "\tau"'] & H^2(G). \end{tikzcd} \tag{2.3.2}$$

by Remark 2.9. Thus the homomorphism $\mathfrak{d}\colon Q(\widetilde{G}) \to H^2(G)$ induces an isomorphism

$$\mathcal{ND} = Q(\widetilde{G})/(H^1(\widetilde{G}) + p^*Q(G)) \to \mathrm{Im}(\tau) \cap \mathrm{Im}(c_G) \tag{2.3.3}$$

by Remark 2.13.

Now we give a geometric meaning of the space $\mathrm{Im}(\tau) \cap \mathrm{Im}(c_G)$. By considering the classifying spaces of the central extension (2.3.1) with the discrete topology, we obtain the following commutative diagram of fibrations

$$\begin{tikzcd} B\pi_1(G) \arrow[r] \arrow[d, equal] & B\widetilde{G}^\delta \arrow[r] \arrow[d] & BG^\delta \arrow[d, "B\iota"] \\ B\pi_1(G) \arrow[r] & B\widetilde{G} \arrow[r] & BG. \end{tikzcd}$$

In what follows, we regard the pullback $(B\iota)^*$ as a homomorphism

$$(B\iota)^*\colon H^\bullet_{\mathrm{top}}(BG) \to H^\bullet(G)$$

under the isomorphism $H^\bullet_{\mathrm{top}}(BG^\delta) \cong H^\bullet(G)$.

Lemma 2.14 ([**KM20**]). *Let $(E_r^{p,q}, d_r^{p,q})$ be the cohomology Serre spectral sequence of the fibration $B\pi_1(G) \to B\widetilde{G} \to BG$. Then the equality*

$$(B\iota)^* \circ d_2^{0,1} = \tau\colon H^1(\pi_1(G)) \to H^2(G)$$

holds, where we identify $E_2^{0,1}$ with $H^1(\pi_1(G))$.

Proof. Let $({}^\delta E_r^{p,q}, {}^\delta d_r^{p,q})$ be the Hochschild-Serre spectral sequence of the central extension (2.3.1). Note that it is isomorphic to the Serre spectral sequence of the fibration $B\pi_1(G) \to B\widetilde{G}^\delta \to BG^\delta$ (Remark 1.16). Thus, by the naturality of the Serre spectral sequence, we obtain

$$(B\iota)^* \circ d_2^{0,1} = {}^\delta d_2^{0,1} = \tau$$

and the lemma follows. □

Remark 2.15. Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence used in Lemma 2.14, then the map $d_2^{0,1}$ is an isomorphism. Indeed, the spectral sequence induces an exact sequence

$$0 \to H^1_{\mathrm{top}}(BG) \to H^1_{\mathrm{top}}(B\widetilde{G}) \to H^1_{\mathrm{top}}(B\pi_1(G)) \\ \xrightarrow{d_2^{0,1}} H^2_{\mathrm{top}}(BG) \to H^2_{\mathrm{top}}(B\widetilde{G}).$$

Since $\widetilde{G}$ is one-connected, the classifying space $B\widetilde{G}$ is two-connected. Thus the cohomologies $H^1_{\mathrm{top}}(B\widetilde{G})$ and $H^2_{\mathrm{top}}(B\widetilde{G})$ are trivial, and this implies that the derivation map $d_2^{0,1}$ is an isomorphism.

Corollary 2.16 ([**KM20**]). If $H^1(\widetilde{G})$ is trivial, the homomorphism

$$(B\iota)^*\colon H^2_{\mathrm{top}}(BG) \to H^2(G)$$

is injective.

Proof. The map τ is injective since the sequence

$$0 \to H^1(G) \to H^1(\widetilde{G}) \to H^1(\pi_1(G)) \\ \xrightarrow{\tau} H^2(G) \to H^2(\widetilde{G})$$

is exact and $H^1(\widetilde{G})$ is trivial. Thus Lemma 2.14 and Remark 2.15 say that the map $(B\iota)^*$ is injective. □

The following is the main theorem of this section.

Theorem 2.17 ([**KM20**]). The homomorphism $\mathfrak{d}\colon Q(\widetilde{G}) \to H^2(G)$ induces an isomorphism

$$\mathcal{ND} = Q(\widetilde{G})/(H^1(\widetilde{G}) + p^*Q(G)) \to \mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G).$$

Proof. We have $\mathrm{Im}(B\iota)^* = \mathrm{Im}(\tau)$ by Lemma 2.14 and Remark 2.15. Thus, the isomorphism (2.3.3) implies the theorem. □

Corollary 2.18 ([**KM20**]). If the first cohomology $H^1(\widetilde{G})$ is trivial, then the homomorphism $\mathfrak{d}$ induces the isomorphism

$$Q(\widetilde{G})/p^*Q(G) \to \mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G).$$

In particular, if $\mu \in Q(\widetilde{G})$ does not descend to G, then the class $\mathfrak{d}(\mu) \in H^2(G)$ is non-trivial.

Corollary 2.19 ([**KM20**]). Let X be a closed manifold and $G = \mathrm{Homeo}_0(X)$ the identity component of $\mathrm{Homeo}(X)$, then there is an isomorphism

$$Q(\widetilde{G})/p^*Q(G) \to \mathrm{Im}(c_G).$$

Proof. Because the map $(B\iota)^*\colon H^\bullet(BG) \to H^\bullet(BG^\delta)$ is isomorphic [**Thu74**] and the universal covering group $\widetilde{G}$ is perfect [**KR11**], the corollary follows. $\square$

For a homogeneous quasimorphism, the corresponding characteristic class is given as follows.

Proposition 2.20 ([**KM20**]). For an element $\mu \in Q(\widetilde{G})$, the equality

$$\mathfrak{d}(\mu) = -(B\iota)^*(\mu|_{\pi_1(G)})_*\mathfrak{o}$$

holds, where $\mathfrak{o} \in H^2_{\mathrm{top}}(BG;\pi_1(G))$ is the primary obstruction class.

Proof. Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence of the fibration $B\pi_1(G) \to B\widetilde{G} \to BG$. By using Proposition 1.36, we obtain

$$(B\iota)^* d_2^{0,1}(\mu|_{\pi_1(G)}) = -(B\iota)^*(\mu|_{\pi_1(G)})_*\mathfrak{o}.$$

On the other hand, by using Lemma 2.14 and the commutative diagram (2.3.2), we obtain

$$(B\iota)^* d_2^{0,1}(\mu|_{\pi_1(G)}) = \tau(\mu|_{\pi_1(G)}) = \tau(i^*(\mu)) = \mathfrak{d}(\mu).$$

Therefore the equality $\mathfrak{d}(\mu) = -(B\iota)^*(\mu|_{\pi_1(G)})_*\mathfrak{o}$ holds. $\square$

Example 2.21. For the group $G = \mathrm{Homeo}_+(S^1)$, the spaces appearing in the isomorphism $\mathcal{ND} \cong \mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G) \subset H^2(G)$ are completely described as follows. By the theorem of Thurston and the isomorphism $H^2(BG) \cong \mathbb{R}[e]$, we have $\mathrm{Im}(B\iota)^* = H^2(G) \cong \mathbb{R}[e]$, where the class e is the Euler class. Since the Euler class is bounded and the comparison map $c_G\colon H_b^2(G) \to H^2(G)$ is injective by the theorem of Matsumoto-Morita [**MM85**], we have $\mathrm{Im}(c_G) = H^2(G)$. Therefore we have

$$\mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G) = H^2(G) = \mathbb{R}[e].$$

Since the group G is uniformly perfect and the universal covering $\widetilde{G}$ is perfect, we have $Q(G) = 0$ and $H^1(\widetilde{G}) = 0$. Moreover, by the theorem of Ghys [**Ghy01**], the space $Q(\widetilde{G})$ is isomorphic to $\mathbb{R}[\mu]$, where μ is Poincaré's translation number. Thus we have

$$\mathcal{ND} = Q(\widetilde{G}) \Big/ \Big(p^*Q(G) + H^1(\widetilde{G})\Big) = Q(\widetilde{G}) \cong \mathbb{R}[\mu],$$

and the isomorphism $\mathfrak{d}\colon \mathcal{ND} \to \mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G)$ sends μ to $-e$ by Proposition 2.20.

2.4. Applications

In this section, two applications of Theorem 2.17 and Proposition 2.20 are given.

It is an interesting and difficult problem to determine whether a given characteristic class is bounded or not. The Milnor-Wood inequality ([**Mil58**], [**Woo71**]) asserts that the Euler class of flat $SL(2,\mathbb{R})$-bundles is bounded. It was shown that any element of $\mathrm{Im}(B\iota)^*$ is bounded for any real algebraic subgroups of $GL(n,\mathbb{R})$ ([**Gro82**]).

To the best of the author's knowledge, the boundedness of characteristic classes for homeomorphism groups is known only for the following specific examples.

Example 2.22.

- The Euler class of $\mathrm{Homeo}_+(S^1)$ is bounded [**Woo71**].
- Any non-zero second cohomology class of $\mathrm{Homeo}_0(\mathbb{R}^2)$ is unbounded [**Cal04**].
- Any non-zero second cohomology class of $\mathrm{Homeo}_0(T^2)$ is unbounded, where T^2 is a two-dimensional torus [**MR18**].
- Let M be a closed Seifert-fibered 3-manifold such that the inclusion $SO(2) \to \mathrm{Homeo}_0(M)$ defined by the rotation of the fibers induces an inclusion of $\pi_1(SO(2))$ as a direct factor in $\pi_1(\mathrm{Homeo}_0(M))$. Then several second cohomology classes of $\mathrm{Homeo}_0(M)$ are unbounded [**Man20**].

2.4.1. On the boundedness of characteristic classes. By using Proposition 2.20 and Theorem 2.17, we show the boundedness and unboundedness of characteristic classes of foliated $\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)$-bundles and foliated $\mathrm{Cont}_0(S^3,\xi)$-bundles.

Let $(S^2 \times S^2, \omega_\lambda)$ be the symplectic manifold defined in Section 1.4. If $1 < \lambda \leq 2$, the fundamental group $\pi_1(\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda))$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (see Remark 1.42). Let

$$\mathfrak{o}_{S^2\times S^2} \in H^2(B\,\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda); \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

denote the primary obstruction class (see Remark 1.35). By the change of coefficients homomorphism induced from

$$\phi\colon \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{R}\,;\ (n,a,b) \mapsto n,$$

we obtain a cohomology class

$$(\mathfrak{o}_{S^2\times S^2})_{\mathbb{R}} = \phi_*(\mathfrak{o}_{S^2\times S^2}) \in H^2(B\,\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)).$$

Let us consider the contact manifold (S^3,ξ) with the standard contact structure ξ. Because the fundamental group $\pi_1(\mathrm{Cont}_0(S^3,\xi))$ is isomorphic to $\mathbb{Z}$ ([**Eli92**]), we obtain the primary obstruction class

$$\mathfrak{o}_{S^3} \in H^2(B\,\mathrm{Cont}_0(S^3,\xi);\mathbb{Z}).$$

By the change of coefficients homomorphism induced from the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, we obtain a cohomology class

$$(\mathfrak{o}_{S^3})_{\mathbb{R}} \in H^2(B\,\mathrm{Cont}_0(S^3,\xi)).$$

Now we have similar two characteristic classes $(\mathfrak{o}_{S^2\times S^2})_{\mathbb{R}}$ and $(\mathfrak{o}_{S^3})_{\mathbb{R}}$. By using Theorem 2.17 and Proposition 2.20, we can clarify the difference between these classes in terms of boundedness.

Corollary 2.23 ([**KM20**]). The following properties hold.

(1) The cohomology class
$$(B\iota)^*(\mathfrak{o}_{S^2\times S^2})_{\mathbb{R}} \in H^2(\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda))$$
is bounded.

(2) The cohomology class
$$(B\iota)^*(\mathfrak{o}_{S^3})_{\mathbb{R}} \in H^2(\mathrm{Cont}_0(S^3,\xi))$$
is unbounded.

Proof. (1) Ostrover introduced in [**Ost06**] a homogeneous quasimorphism μ^λ on $\widetilde{\mathrm{Symp}}_0(S^2\times S^2,\omega_\lambda)$ that does not descend to $\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda)$. Since $\pi_1(\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda)) \cong \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and the restriction $\mu^\lambda|_{\pi_1(\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda))}$ is a non-trivial homomorphism to $\mathbb{R}$, there exists a non-zero constant a such that

$$\phi = a\mu^\lambda|_{\pi_1(\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda))} \colon \pi_1(\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda)) \to \mathbb{R}.$$

Thus, by Proposition 2.20, we have

$$\begin{aligned}(B\iota)^*(\mathfrak{o}_{S^2\times S^2})_{\mathbb{R}} &= (B\iota)^*\phi_*(\mathfrak{o}_{S^2\times S^2})\\ &= (B\iota)^*(a\mu|_{\pi_1(\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda))})_*\mathfrak{o}_{S^2\times S^2}\\ &= -a\mathfrak{d}(\mu|_{\pi_1(\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda))})\end{aligned}$$

Since the class $\mathfrak{d}(\mu|_{\pi_1(\mathrm{Symp}_0(S^2\times S^2,\omega_\lambda))})$ is bounded, so is $(B\iota)^*(\mathfrak{o}_{S^2\times S^2})_{\mathbb{R}}$.

(2) It was shown in [**FPR18**] that there are no homogeneous quasimorhpisms on $\widetilde{\mathrm{Cont}}(S^3,\xi)_0$, that is, $Q(\widetilde{\mathrm{Cont}}(S^3,\xi)_0) = 0$. Thus, by Theorem 2.17, we have

$$\mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G) = 0.$$

By the perfectness of $\widetilde{\mathrm{Cont}}(S^3,\xi)_0$ and Corollary 2.16, the map $(Bi)^*$ is injective and thus the class $(B\iota)^*(\mathfrak{o}_{S^3})_{\mathbb{R}}$ is non-zero. This implies that the class $(B\iota)^*(\mathfrak{o}_{S^3})_{\mathbb{R}}$ is unbounded. $\square$

2.4.2. Non-extendability of homomorphisms on $\pi_1(G)$. In Subsection 2.4.1, we used the space $Q(\widetilde{G})$ to show the (un)boundedness of characteristic classes. In this section, on the contrary, we use the unboundedness of characteristic classes to study homogeneous quasi-morphisms on $\widetilde{G}$.

Let $T = S^1 \times S^1$ be the two-dimensional torus and $\mathrm{Homeo}_0(T)$ the identity component of the homeomorphism group. In [**Ham65**], it was shown that the fundamental group $\pi_1(\mathrm{Homeo}_0(T))$ is isomorphic to $\mathbb{Z}^2$.

Corollary 2.24 ([**KM20**]). Any non-zero homomorphism from $\pi_1(\mathrm{Homeo}_0(T)) \cong \mathbb{Z}^2$ to $\mathbb{R}$ cannot be extended to a homogeneous quasi-morphism on $\widetilde{\mathrm{Homeo}}_0(T)$.

Proof. It is enough to show that the space $Q(\widetilde{\mathrm{Homeo}}_0(T))$ is equal to $p^*Q(\mathrm{Homeo}_0(T))$, where $p\colon \widetilde{\mathrm{Homeo}}_0(T) \to \mathrm{Homeo}_0(T)$ is the universal covering. Because the universal covering $\widetilde{\mathrm{Homeo}}_0(T)$ is perfect [**KR11**], we have

$$Q(\widetilde{\mathrm{Homeo}}_0(T))/p^*Q(\mathrm{Homeo}_0(T)) = \mathrm{Im}(B\iota)^* \cap \mathrm{Im}(c_G)$$

by Corollary 2.18. Since the non-zero classes of $\mathrm{Im}(B\iota)^*$ are unbounded [**MR18**], we have

$$Q(\widetilde{\mathrm{Homeo}}_0(T))/p^*Q(\mathrm{Homeo}_0(T)) = 0$$

and the corollary holds. $\square$

CHAPTER 3

Non-extendable homogeneous quasimorphisms

In this chapter, Section 3.2 is based on [**Mar20**], and Section 3.3 and Section 3.4 are based on a part of the joint work with Morimichi Kawasaki, Mitsuaki Kimura, Takahiro Matsushita, and Masato Mimura [**KKM$^+$21**].

Let Γ be a group and K a normal subgroup of Γ. For a given homogeneous quasimorphism μ on K, it is natural to ask whether μ can be extended to that on Γ or not. Recall that the inclusion map $i\colon K \to \Gamma$ induces the map

$$i^*\colon Q(\Gamma) \to Q(K)^\Gamma,$$

where $Q(K)^\Gamma$ is the space of all Γ-invariant homogeneous quasimorphisms on K (see Remark 1.7). Therefore the Γ-invariance is a necessary condition for a homogeneous quasimorphism on K to be extended to that on Γ. In order to discuss the extension problem under the assumption of Γ-invariance, we introduce the following space.

Definition 3.1. For a group Γ and its normal subgroup K, set

$$\mathcal{NE} = Q(K)^\Gamma \big/ \left(i^*Q(\Gamma) + H^1(K)^\Gamma\right), \tag{3.0.1}$$

where $i^*\colon Q(\Gamma) \to Q(K)^\Gamma$ is the pullback by the inclusion $i\colon K \to \Gamma$.

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and $\Gamma_g = \pi_1(\Sigma_g)$ the surface group. The main results of this chapter are the following theorems.

Theorem 3.2. Let $\Gamma = \Gamma_g$ be the surface group of genus $g \geq 2$ and $K = [\Gamma, \Gamma]$ the commutator subgroup. Then the dimension of the space $\mathcal{NE}$ is equal to one.

For an orientation preserving homeomorphism $f \in \mathrm{Homeo}_+(\Sigma_g)$, let X_f denote the mapping torus.

Theorem 3.3. Set $\Gamma = \pi_1(X_f)$ and $K = [\Gamma, \Gamma]$. If the mapping class $[f]$ is a pseudo-Anosov element and in the Torelli group $\mathcal{I}_g$, then the dimension of the space $\mathcal{NE}$ is equal to $2g + 1$.

Here the Torelli group $\mathcal{I}_g$ is the subgroup of the mapping class group $\mathcal{M}_g$ consisting of the elements which act trivially on $H_\bullet^{\mathrm{top}}(\Sigma_g;\mathbb{Z})$. The Torelli group $\mathcal{I}_g$ is trivial when $g=1$, not finitely generated when $g=2$ ([**MM86**]), and finitely generated when $g \geq 3$ ([**Joh83**]).

REMARK 3.4. It is known that the Torelli group $\mathcal{I}_g$ contains pseudo-Anosov elements for $g \geq 2$ (see [**FM12**, Corollary 14.3] for example). Moreover, in the sense of Random Walk, pseudo-Anosov elements are generic in the Torelli group for $g \geq 3$ ([**LM12**] [**MS13**]).

REMARK 3.5. Three-manifolds we consider in THEOREM 3.3 are basic examples in the following sense: any closed hyperbolic three-manifold is obtained as a mapping torus up to finite-sheeted cover. This is known as the virtual fibering conjecture, which was proposed by Thurston [**Thu82**] and solved by Agol [**Ago13**].

3.1. Backgrounds and motivations

In this section, we explain why we consider the space $\mathcal{NE}$. The notions introduced in this subsection will not appear in the subsequent sections.

The extendability of Γ-invariant homogeneous quasimorphisms has been studied in [**Ish14**], [**Sht15**], [**KK19**] and [**KKMM20**]. Let Σ_g be a closed surface of genus $g \geq 2$ and ω a symplectic form. Py constructed in [**Py06**] a $\mathrm{Symp}_0(\Sigma_g,\omega)$-invariant homogeneous quasimorphism on the Hamiltonian diffeomorphism group $\mathrm{Ham}(\Sigma_g,\omega)$. In [**KK19**], Kawasaki and Kimura showed that Py's homogeneous quasimorphism is non-extendable, and this was the only known example of non-extendable homogeneous quasimorphisms at the present moment. THEOREM 3.2 and THEOREM 3.3 give many pair of groups that admit non-extendable homogeneous quasimorphisms.

The triviality and the non-triviality of the space

$$\mathcal{NE} = Q(K)^\Gamma \big/ \left(i^* Q(\Gamma) + H^1(K)^\Gamma\right). \tag{3.1.1}$$

have several applications below. Ishida showed in [**Ish14**] that any Γ-invariant homogeneous quasimorphism on K can be extended to Γ when $G=\Gamma/K$ is a finite group. In other words, the map $i^*\colon Q(\Gamma)\to Q(K)^\Gamma$ is surjective. In [**KKMM20**], Kawasaki, Kimura, Matsushita, and Mimura extended it as follows: if the projection $\Gamma \to G$ has a virtually splitting, that is, there exists a finite index subgroup Λ of G and a section homomorphism $s\colon \Lambda \to \Gamma$, then the map $i^*\colon Q(\Gamma)\to Q(K)^\Gamma$ is surjective. Thus, the non-triviality of the space (3.1.1) gives an obstruction to the existence of virtual splittings.

Another application is to the stable commutator length. Let $[\Gamma,\Gamma]$ be the commutator subgroup of Γ. The *commutator length* $\mathrm{cl}_\Gamma(x)$ of $x \in [\Gamma,\Gamma]$ is the minimal number of commutators whose product is equal to x. The *stable commutator length* $\mathrm{scl}_\Gamma(x)$ is defined as

$$\mathrm{scl}_\Gamma(x) = \lim_{n\to\infty} \frac{\mathrm{cl}_\Gamma(x^n)}{n}.$$

Note that the commutator length cl_Γ satisfies subadditivity, and thus the limit exists (see [**Cal09**] for details). By the pioneering work [**Bav91**] by Bavard, the following relation between scl_Γ and the space $Q(\Gamma)$ has been clarified.

THEOREM 3.6 (Bavard's duality theorem [**Bav91**]). For $x \in [\Gamma,\Gamma]$, the equality

$$\mathrm{scl}(x) = \sup_{[\mu]\in Q(\Gamma)/H^1(\Gamma)} \frac{|\mu(x)|}{2D(\mu)} \tag{3.1.2}$$

holds. If the space $Q(\Gamma)/H^1(\Gamma)$ is trivial, then we regard the right-hand side as zero.

In [**KK19**], a variant of the (stable) commutator length called the *mixed (stable) commutator length* was introduced. Let Γ be a group and K a normal subgroup of Γ. For $\gamma \in \Gamma$ and $k \in K$, we call an element $[\gamma,k] = \gamma k \gamma^{-1} k^{-1}$ a *mixed commutator*. Let $[\Gamma,K]$ denote the subgroup of Γ which is generated by the mixed commutators. The *mixed commutator length* $\mathrm{cl}_{\Gamma,K}(x)$ of $x \in [\Gamma,K]$ is the minimal number of mixed commutators whose product is equal to x, and the *mixed stable commutator length* $\mathrm{scl}_{\Gamma,K}(x)$ is defined by

$$\mathrm{scl}_{\Gamma,K}(x) = \lim_{n\to\infty} \frac{\mathrm{cl}_{\Gamma,K}(x^n)}{n}.$$

In [**KKMM20**], the following mixed version of Bavard's duality theorem was proven, which clarifies a relation between $scl_{\Gamma,K}$ and the space $Q(K)^\Gamma$.

THEOREM 3.7 ([**KKMM20**, Theorem 1.2]). For $x \in [\Gamma,K]$, the equality

$$\mathrm{scl}_{\Gamma,K}(x) = \sup_{[\mu]\in Q(K)^\Gamma/H^1(K)^\Gamma} \frac{|\mu(x)|}{2D(\mu)} \tag{3.1.3}$$

holds. If the space $Q(K)^\Gamma \setminus H^1(K)^\Gamma$ is trivial, then we regard the right-hand side as zero.

By definition, $\mathrm{scl}_{\Gamma,K}(x)$ is greater than or equal to $\mathrm{scl}_\Gamma(x)$ for $x \in [\Gamma, K]$. Moreover, by comparing Theorem 3.6 with Theorem 3.7, we may show the difference of scl_Γ and $\mathrm{scl}_{\Gamma,K}$ if there exists a Γ-invariant non-extendable homogeneous quasimorphism on K. In fact, Kawasaki and Kimura showed in **[KK19]** that there exists an element $x \in [\Gamma, K]$ such that $\mathrm{scl}_\Gamma(x) = 0$ and $\mathrm{scl}_{\Gamma,K}(x) > 0$ for $\Gamma = \mathrm{Symp}_0(\Sigma_g, \omega)$ and $K = \mathrm{Ham}(\Sigma_g, \omega)$ by using Py's homogeneous quasimorphism on $\mathrm{Ham}(\Sigma_g, \omega)$.

3.2. Extendable homogeneous quasimorphisms

To explain the difficulty of the extending problem, we present an example of an extendable homogeneous quasimorphism.

Let us consider the symplectomorphism group

$$\Gamma = \mathrm{Symp}(D, \omega)$$

of the closed unit disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ with the standard symplectic form $\omega = dx \wedge dy$. It is known that the homomorphism

$$p \colon \Gamma \to \mathrm{Diff}_+(S^1)$$

is surjective, which is obtained by restricting the domain to the boundary. Let K be the kernel of the map p, then we obtain a group extension

$$1 \to K \xrightarrow{i} \Gamma \xrightarrow{p} \mathrm{Diff}_+(S^1) \to 1.$$

On the group K, there is a homogeneous quasimorphism called the *Ruelle invariant* **[Rue85]**. In this section, we show that the Ruelle invariant is an extendable homogeneous quasimorphism.

We consider the C^∞-topology on the group K. Then it is known that the group K is contractible. Take an element k of K and a path $\{k_t\}_{t\in[0,1]}$ in K with $k_0 = \mathrm{id}$ and $k_1 = k$. For any point $x \in D$, let $u_t(x) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ denote the first column of the differential $dk_t(x) \in SL(2, \mathbb{R})$. Then the variation of the angle of $u_t(x)$ depends on x and the homotopy class of the path $\{k_t\}_{t\in[0,1]}$ relatively to the fixed ends. Since K is contractible, the variation of the angle of $u_t(x)$ depends only on the endpoint $k \in K$. Therefore we use the symbol $\mathrm{Ang}_k(x)$ to denote the variation of the angle. This defines a continuous function

$$\mathrm{Ang}_k \colon D \to \mathbb{R}. \tag{3.2.1}$$

For $k, l \in K$ and a path $\{l_t\}_{t\in[0,1]}$ which satisfies $l_0 = \mathrm{id}$ and $l_1 = l$, an inequality

$$|\,\mathrm{Ang}_{kl}(x) - \mathrm{Ang}_l(x) - \mathrm{Ang}_k(l(x))| < 1/2 \tag{3.2.2}$$

holds for any $x \in D$, where we consider the variation of the full rotation as one. Define a function $r\colon K \to \mathbb{R}$ by

$$r(k) = \int_D \mathrm{Ang}_k \cdot \omega,$$

where ω is the symplectic form. By the above inequality (3.2.2) and the condition $l^*\omega = \omega$ for $l \in K$, the function $r\colon K \to \mathbb{R}$ is a quasimorphism on K. Let $R = \overline{r}$ denote the homogenization of r. This homogeneous quasimorphism $R\colon K \to \mathbb{R}$ is called the *Ruelle invariant*.

As explained above, the definition of the Ruelle invariant R relies on the fact that the group K is contractible. Because the fundamental group of Γ is isomorphic to $\mathbb{Z}$, we cannot apply the definition of R to the group Γ. However, we will show the following.

THEOREM 3.8 ([**Mar20**, Theorem 4.4]). There exists a homogeneous quasimorphism

$$\mathfrak{R}\colon \Gamma \to \mathbb{R}$$

satisfying $\mathfrak{R}|_K = R$. In other words, the Ruelle invariant R is extendable to the group Γ.

To show THEOREM 3.8, we explicitly construct the homogeneous quasimorphism $\mathfrak{R}\colon \Gamma \to \mathbb{R}$. To do this, first we construct a homogeneous quasimorphism on the universal covering group $\widetilde{\Gamma}$ rather than Γ.

Let $q\colon \widetilde{\Gamma} \to \Gamma$ be the universal covering of Γ. Let us consider the universal covering group $\widetilde{\Gamma}$ as the group of homotopy classes (relative to the fixed ends) of paths on Γ starting at the identity. For an element α of $\widetilde{\Gamma}$, we can define a continuous function

$$\mathrm{Ang}_\alpha\colon D \to \mathbb{R}$$

in the same way as in (3.2.1). More precisely, we define Ang_α as follows. For a path $\{\alpha_t\}_{t\in[0,1]}$ representing $\alpha \in \widetilde{\Gamma}$ and a point $x \in D$, let $u_t(x)$ denote the first column of the differential $d\alpha_t(x) \in SL(2,\mathbb{R})$. Then we define $\mathrm{Ang}_\alpha(x)$ as the variation of the angle of the vector $u_t(x)$. Note that the value $\mathrm{Ang}_\alpha(x)$ is independent of the choice of the representing path $\{\alpha_t\}_t$. This function also satisfies the inequality

$$|\mathrm{Ang}_{\alpha\beta}(x) - \mathrm{Ang}_\beta(x) - \mathrm{Ang}_\alpha(\beta_1(x))| < 1/2,$$

where $\beta \in \widetilde{\Gamma}$ and $\beta_1 = q(\beta) \in \Gamma$ is the projection of β. Thus we obtain a quasimorphism $\widetilde{r}\colon \widetilde{\Gamma} \to \mathbb{R}$ by

$$\widetilde{r}(\alpha) = \int_D \mathrm{Ang}_\alpha \cdot \omega$$

and a homogeneous quasimorphism $\widetilde{R}\colon \widetilde{\Gamma} \to \mathbb{R}$ as the homogenization of $\widetilde{r}$.

LEMMA 3.9. The homogeneous quasimorphism $\widetilde{R}\colon \widetilde{\Gamma} \to \mathbb{R}$ does not descend to Γ, that is, $\widetilde{R}$ is not contained in the image of the pullback $q^*\colon Q(\Gamma) \to Q(\widetilde{\Gamma})$.

PROOF. By LEMMA 1.5, it is enough to show that the restriction $\widetilde{R}|_{\pi_1(\Gamma)}$ is non-zero. We define $\rho_{n,t}\colon D \to D$ by

$$\rho_{n,t}(z) = e^{2\pi i n t} \cdot z,$$

where we consider D as the subspace of $\mathbb{C}$. Then the path $\{\rho_{n,t}\}_{t\in[0,1]}$ in Γ defines an non-zero element $\rho_n = [\{\rho_{n,t}\}_{t\in[0,1]}]$ of $\pi_1(\Gamma)$. Moreover, the correspondence between the element $\rho_n \in \pi_1(\Gamma)$ and the integer $n \in \mathbb{Z}$ gives an isomorphism $\pi_1(\Gamma) \cong \mathbb{Z}$. For any $n \in \mathbb{Z}$, we have $\mathrm{Ang}_{\rho_n}(z) = n$ for any $z \in D$ by the definition of Ang, and therefore we have $\widetilde{r}(\rho_n) = 2\pi n$. By the definition of the homogenization, we have

$$\widetilde{R}(\rho) = \lim_{n\to\infty} \frac{\widetilde{r}(\rho_n)}{n} = 2\pi.$$

Thus the lemma follows. $\square$

PROOF OF THEOREM 3.8. Let us consider the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[d] & \\
1 \arrow[r] & \widetilde{K} = K \arrow[r, "\widetilde{i}"] \arrow[d, equal] & \widetilde{\Gamma} \arrow[r, "\widetilde{p}"] \arrow[d, "q"] & \widetilde{\mathrm{Diff}}_+(S^1) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K \arrow[r, "i"] & \Gamma \arrow[r, "p"] \arrow[d] & \mathrm{Diff}_+(S^1) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

where $\widetilde{p}\colon \widetilde{\Gamma} \to \widetilde{\mathrm{Diff}}_+(S^1)$ is defined by the restriction of a path $\alpha_t\colon D \to D$ to the boundary. Let $\mu\colon \widetilde{\mathrm{Diff}}_+(S^1) \to \mathbb{R}$ be Poincaré's translation number. Then the pullback $\widetilde{p}^*\mu\colon \widetilde{\Gamma} \to \mathbb{R}$ is a homogeneous quasimorphism which satisfies $\widetilde{p}^*\mu(\rho_n) = n$. Let us consider the homogeneous quasimorphism

$$\widetilde{R} - 2\pi\widetilde{p}^*\mu \colon \widetilde{\Gamma} \to \mathbb{R}.$$

Note that the homogeneous quasimorphism $\widetilde{R} - 2\pi\widetilde{p}^*\mu$ is equal to zero on $\pi_1(\Gamma) = \mathbb{Z}$. Therefore, by Lemma 1.5, there exists a homogeneous quasimorphism $\mathfrak{R} \in Q(\Gamma)$ such that the equality

$$q^*\mathfrak{R} = \widetilde{R} - 2\pi\widetilde{p}^*\mu$$

holds. By the definition of $\widetilde{R}$ and the Ruelle invariant R, we have $\widetilde{i}^*\widetilde{R} = R$. Moreover, by the exactness of $1 \to K \xrightarrow{\widetilde{i}} \widetilde{\Gamma} \xrightarrow{p} \Gamma \to 1$, we have

$$\widetilde{i}^*(2\pi\widetilde{p}^*\mu) = 0.$$

Therefore we obtain

$$i^*\mathfrak{R} = \widetilde{i}^*q^*\mathfrak{R} = \widetilde{i}^*(\widetilde{R} - 2\pi\widetilde{p}^*\mu) = R.$$

This implies that the Ruelle invariant is extendable to Γ. □

3.3. Exact sequences

In this section, we show the following:

Theorem 3.10 ([**KKM+21**]). For a group extension $1 \to K \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1$, there are two exact sequences

(1) $0 \to \mathcal{NE} \to EH_b^2(K)^\Gamma / i^*EH_b^2(\Gamma) \xrightarrow{\alpha} H^1(\Gamma; H^1(K))$,

(2) $H_b^2(G) \to \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \xrightarrow{\beta} EH_b^2(K)^\Gamma / i^*EH_b^2(\Gamma) \to H_b^3(G)$,

where $EH_b^2(K)^\Gamma$ is the Γ-invariant part of $EH_b^2(K)$, $c_\Gamma \colon H_b^2(\Gamma) \to H^2(\Gamma)$ is the comparison map, and $i^* \colon H^2(\Gamma) \to H^2(K)$.

If $H_b^2(G) = H_b^3(G) = 0$, then the two exact sequences (1) and (2) of Theorem 3.10 can be combined into one as follows.

Corollary 3.11 ([**KKM+21**]). If $H_b^2(G) = H_b^3(G) = 0$, there is an exact sequence

$$0 \to \mathcal{NE} \to \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \xrightarrow{\alpha\circ\beta} H^1(\Gamma; H^1(K)).$$

Remark 3.12. By Corollary 3.11, we have

$$\dim_{\mathbb{R}} \mathcal{NE} \leq \dim_{\mathbb{R}} H^2(\Gamma)$$

if $H_b^2(G) = H_b^3(G) = 0$. Thus, for a free group and its commutator subgroup, the space $\mathcal{NE}$ is trivial.

To prove THEOREM 3.10, we define a map

$$\varphi\colon EH_b^2(K)^\Gamma \to H^1(\Gamma; H^1(K)).$$

For an element $c \in EH_b^2(K)^\Gamma$, take a quasimorphism $\mu\colon K \to \mathbb{R}$ satisfying $c = [\delta\mu]$. Since c is Γ-invariant, the equality ${}^\gamma[\delta\mu] = [\delta\mu]$ holds for any $\gamma \in \Gamma$. Thus, for any $\gamma \in \Gamma$, there exists a bounded one-cochain $b_\gamma \in C_b^1(K)$ such that the equality

$$ {}^\gamma\delta\mu = \delta\mu + \delta b_\gamma \tag{3.3.1}$$

holds. The bounded one-cochain b_γ is unique. Indeed, if $\delta b_\gamma = \delta c_\gamma$, then $b_\gamma - c_\gamma$ is a bounded homomorphism to $\mathbb{R}$. This implies that $b_\gamma = c_\gamma$.

DEFINITION 3.13. A cochain $\varphi_\mu \in C^1(\Gamma; H^1(K))$ is defined by

$$\varphi_\mu(\gamma) = \mu - {}^\gamma\mu + b_\gamma.$$

Note that the cochain φ_μ is well defined by (3.3.1).

LEMMA 3.14 ([**KKM$^+$21**]). A map

$$\varphi\colon EH_b^2(K)^\Gamma \to H^1(\Gamma; H^1(K)); [\delta\mu] \mapsto [\varphi_\mu]$$

is well defined.

PROOF. For $\gamma_1, \gamma_2 \in \Gamma$, we have

$$\begin{aligned} \delta b_{\gamma_1\gamma_2} &= {}^{\gamma_1\gamma_2}\delta\mu - \delta\mu = {}^{\gamma_1}({}^{\gamma_2}\delta\mu - \delta\mu) + {}^{\gamma_1}\delta\mu - \delta\mu \\ &= {}^{\gamma_1}\delta b_{\gamma_2} + \delta b_{\gamma_1} = \delta({}^{\gamma_1}b_{\gamma_2} + b_{\gamma_1}), \end{aligned}$$

and therefore $b_{\gamma_1\gamma_2} - ({}^{\gamma_1}b_{\gamma_2} + b_{\gamma_1})$ is a bounded homomorphism, that is, the equality $b_{\gamma_1\gamma_2} = {}^{\gamma_1}b_{\gamma_2} + b_{\gamma_1}$ holds. Thus we obtain

$$\begin{aligned} \varphi_\mu(\gamma_1\gamma_2) &= \mu - {}^{\gamma_1\gamma_2}\mu + b_{\gamma_1\gamma_2} \\ &= {}^{\gamma_1}(\mu - {}^{\gamma_2}\mu + b_{\gamma_2}) + \mu - {}^{\gamma_1}\mu + b_{\gamma_1} \\ &= {}^{\gamma_1}(\varphi_\mu(\gamma_2)) + \varphi_\mu(\gamma_1), \end{aligned}$$

and this implies that the cochain $\varphi_\mu \in C^1(\Gamma; H^1(K))$ is a cocycle.

Next we show that the class $\varphi([\delta\mu]) = [\varphi_\mu]$ does not depend on the choice of μ. Take another quasimorphism $\mu'\colon K \to \mathbb{R}$ satisfying $[\delta\mu'] = [\delta\mu] = c$. Since $\mu - \mu' \in H^1(K) + C_b^1(K)$, there exist $h \in H^1(K)$ and $b \in C_b^1(K)$ satisfying $\mu - \mu' = h + b$. Then we have $\delta\mu = \delta\mu' + \delta b$, and hence

$${}^\gamma(\delta\mu') = {}^\gamma(\delta\mu) - {}^\gamma(\delta b) = \delta\mu + \delta b_\gamma - {}^\gamma(\delta b) = \delta\mu' + \delta(b_\gamma + b - {}^\gamma b).$$

Thus we obtain

$$\begin{aligned}(\varphi_{\mu'} - \varphi_\mu)(\gamma) &= (\mu' - {}^\gamma\mu' + (b_\gamma + b - {}^\gamma b)) - (\mu - {}^\gamma\mu + b_\gamma) \\ &= {}^\gamma h - h = \delta h(\gamma).\end{aligned}$$

This implies that the class $[\varphi_{\mu'}]$ is equal to $[\varphi_\mu]$ in $H^1(\Gamma; H^1(K))$. $\square$

LEMMA 3.15 ([**KKM$^+$21**]). The sequence

$$0 \to H^1(K)^\Gamma \to Q(K)^\Gamma \to EH_b^2(K)^\Gamma \xrightarrow{\varphi} H^1(\Gamma; H^1(K))$$

is exact.

Since the proof of the lemma is straightforward, we omit it.

PROOF OF THEOREM 3.10 (1). Let us consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^1(\Gamma) \arrow[r] \arrow[d, "i^*"] & Q(\Gamma) \arrow[r] \arrow[d, "i^*"] & EH_b^2(\Gamma) \arrow[r] \arrow[d, "i^*"] & 0 \arrow[d] \\ 0 \arrow[r] & H^1(K)^\Gamma \arrow[r] & Q(K)^\Gamma \arrow[r] & EH_b^2(K)^\Gamma \arrow[r, "\varphi"] & H^1(\Gamma; H^1(K)). \end{tikzcd}$$

Taking cokernel of the vertical maps, we obtain an exact sequence

$$\begin{aligned} &H^1(K)^\Gamma / i^* H^1(\Gamma) \to Q(K)^\Gamma / i^* Q(\Gamma) \\ &\qquad \to EH_b^2(K)^\Gamma / i^* EH_b^2(\Gamma) \to H^1(\Gamma; H^1(K)), \end{aligned}$$

where the exactness at $Q(K)^\Gamma / i^* Q(\Gamma)$ comes from the snake lemma and the exactness at $EH_b^2(K)^\Gamma / i^* EH_b^2(\Gamma)$ from a standard diagram chasing. Since the cokernel of the map $H^1(K)^\Gamma / i^* H^1(\Gamma) \to Q(K)^\Gamma / i^* Q(\Gamma)$ is equal to $Q(K)^\Gamma \big/ \big(i^* Q(\Gamma) + H^1(K)^\Gamma\big)$, we obtain the exact sequence

$$\begin{aligned} &0 \to Q(K)^\Gamma \big/ \big(i^* Q(\Gamma) + H^1(K)^\Gamma\big) \\ &\qquad \to EH_b^2(K)^\Gamma / i^* EH_b^2(\Gamma) \xrightarrow{\alpha} H^1(\Gamma; H^1(K)). \end{aligned}$$

$\square$

PROOF OF THEOREM 3.10 (2). Let us consider the following commutative diagram whose rows are exact:

$$\begin{tikzcd} 0 \arrow[r] & EH_b^2(\Gamma) \arrow[r] \arrow[d, "i^*"] & H_b^2(\Gamma) \arrow[r] \arrow[d, "i^*"] & \mathrm{Im}(c_\Gamma) \arrow[r] \arrow[d, "i^*"] & 0 \\ 0 \arrow[r] & EH_b^2(K)^\Gamma \arrow[r] & H_b^2(K)^\Gamma \arrow[r] & H^2(K)^\Gamma. \end{tikzcd}$$

By THEOREM 1.29, the kernel of the map $i^* \colon H_b^2(\Gamma) \to H_b^2(K)^\Gamma$ is isomorphic to $H_b^2(G)$, and there exists an injection from the cokernel of $i^*\colon H_b^2(\Gamma) \to H_b^2(K)^\Gamma$ to $H_b^3(G)$. Thus, by the snake lemma, we obtain the exact sequence

$$H_b^2(G) \to \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \xrightarrow{\beta} EH_b^2(K)^\Gamma / i^* EH_b^2(\Gamma) \to H_b^3(G).$$

□

To show the non-triviality of the space $\mathcal{NE}$, the following theorem is essential, which clarifies the relation between COROLLARY 3.11 and the seven-term exact sequence.

THEOREM 3.16 ([**KKM+21**]). *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \arrow[r, "\alpha\circ\beta"] \arrow[d, hook] & H^1(\Gamma; H^1(K)) \\ \mathrm{Ker}(i^*) \arrow[r, "\zeta"] & H^1(G; H^1(K)), \arrow[u, "p^*"'] \end{tikzcd} \tag{3.3.2}$$

where $\alpha \circ \beta$ is the map in COROLLARY 3.11 *and ζ in the seven-term exact sequence* (1.2.4).

PROOF. First we describe the map

$$\alpha \circ \beta\colon \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \to EH_b^2(K)^\Gamma / i^* EH_b^2(\Gamma) \to H^1(\Gamma; H^1(K))$$

explicitly. Let c be an element of $\mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma)$. Take a bounded representative $f \in C_b^2(\Gamma)$ of c. Since $i^*c = 0 \in H^2(K)$, we can take a one-cochain $\mu \in C^1(K)$ satisfying $f|_{K\times K} = \delta\mu$. Since f is bounded, μ is a quasimorphism on K. Then by the definition of connecting homomorphism in the snake lemma, we obtain $\beta(c) = [\delta\mu]$.

Recall that there exists a unique bounded function $b_\gamma\colon K \to \mathbb{R}$ satisfying

$$\delta b_\gamma = {}^\gamma\delta\mu - \delta\mu.$$

Next we show the equality $b_\gamma(k) = f(\gamma, \gamma^{-1}k\gamma) - f(k,\gamma)$. Set $a_\gamma(k) = f(\gamma, \gamma^{-1}k\gamma) - f(k,\gamma)$. Let k and l be elements of K. Since $\delta f = 0$, we have

$$\begin{aligned} \delta a_\gamma(k,l) &= \delta a_\gamma(k,l) + \delta f(\gamma, \gamma^{-1}k\gamma, \gamma^{-1}l\gamma) \\ &\qquad\qquad + \delta f(k,l,\gamma) - \delta f(k,\gamma,\gamma^{-1}l\gamma) \\ &= ({}^\gamma f)(k,l) - f(k,l) \\ &= ({}^\gamma(\delta\mu) - \delta\mu)(k,l) \\ &= \delta b_\gamma(k,l). \end{aligned}$$

By the uniqueness of b_γ, we have $a_\gamma = b_\gamma$.

Now we shall complete the proof of THEOREM 3.10 (3.3.2). For $c \in \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma)$, take a bounded 2-cocycle f of G and a quasimorphism $\mu \colon K \to \mathbb{R}$ such that $f|_{K\times K} = \delta\mu$. Define $\mu_\Gamma \in C^1(\Gamma)$ by

$$\mu_\Gamma(\gamma) = \begin{cases} \mu(\gamma) & \gamma \in K \\ 0 & \text{otherwise.} \end{cases}$$

Then $f - \delta\mu_\Gamma$ is a cocycle such that $(f - \delta\mu_\Gamma)|_{K\times K} = 0$. Thus PROPOSITION 1.21 implies

$$\begin{aligned} ((p^*\zeta(c))(\gamma))(k) =& (\zeta(c)(p(\gamma)))(k) \\ =& (f - \delta\mu_\Gamma)(\gamma, \gamma^{-1}k\gamma) - (f - \delta\mu_\Gamma)(k, \gamma) \\ =& f(\gamma, \gamma^{-1}k\gamma) - f(k, \gamma) - \mu_\Gamma(\gamma^{-1}k\gamma) + \mu_\Gamma(k\gamma) \\ & \qquad - \mu_\Gamma(\gamma) + \mu_\Gamma(\gamma) - \mu_\Gamma(k\gamma) + \mu_\Gamma(k) \\ =& \mu(k) - {}^\gamma\mu(k) + b_\gamma(k) \\ =& (\mu - {}^\gamma\mu + b_\gamma)(k) \\ =& \varphi_\mu(\gamma)(k). \end{aligned}$$

This implies $\alpha \circ \beta(c) = p^* \circ \zeta(c)$. □

3.4. Examples

In this section, we prove THEOREM 3.2 and THEOREM 3.3.

3.4.1. Preliminary. First we recall the definition of group homology.

DEFINITION 3.17. Let $C_n(G;\mathbb{Z})$ be the free $\mathbb{Z}$-module generated by G^n. The boundary operator $\partial \colon C_n(G;\mathbb{Z}) \to C_{n-1}(G;\mathbb{Z})$ is defined by

$$\begin{aligned} \partial(g_1, \ldots, g_n) =& (g_2, \ldots, g_n) + \sum_i (-1)^i (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) \\ & + (-1)^n (g_1, \ldots, g_{n-1}). \end{aligned}$$

The homology $H_\bullet(G;\mathbb{Z})$ of the chain complex $(C_\bullet(G;\mathbb{Z}), \partial)$ is called the *group homology of* G.

Note that the first group homology $H_1(G;\mathbb{Z})$ is isomorphic to the abelianization $G_{\mathrm{ab}} = G/[G,G]$.

Let n be an integer greater than one, and set

$$\mathbf{e}_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{Z}^n$$

for $1 \leq i \leq n$. For $1 \leq i < j \leq n$, let us define $A_{i,j} \in C^2(\mathbb{Z}^n)$ by

$$A_{i,j}\left(\sum_k m_k \mathbf{e}_k, \sum_l n_l \mathbf{e}_l\right) = m_i n_j. \tag{3.4.1}$$

It is easily shown that the cochain $A_{i,j}$ is a cocycle. Recall that the cohomology $H^2(\mathbb{Z}^n)$ is isomorphic to $\mathbb{R}^{n(n-1)/2}$ (see EXAMPLE 1.14).

LEMMA 3.18. The family $\{[A_{i,j}]\}_{1\leq i<j\leq n} \subset H^2(\mathbb{Z}^n) \cong \mathbb{R}^{n(n-1)/2}$ is a basis.

PROOF. For $1 \leq k < l \leq n$, let us define a group two-chain $\sigma_{k,l} \in C_2(\mathbb{Z}^n)$ by

$$\sigma_{k,l} = (\mathbf{e}_k, \mathbf{e}_l) + (\mathbf{e}_k + \mathbf{e}_l, -\mathbf{e}_k) - (\mathbf{e}_k, -\mathbf{e}_k) + (0, 0).$$

By definition, the chain $\sigma_{k,l}$ is a cycle. Moreover, we have

$$\langle A_{i,j}, \sigma_{k,l}\rangle = \begin{cases} 1 & (i,j) = (k,l) \\ 0 & \text{otherwise,} \end{cases}$$

where $\langle A_{i,j}, \sigma_{k,l}\rangle$ is the pairing. Thus the classes $[A_{i,j}]$ are non-zero and linearly independent, and the lamma follows. $\square$

REMARK 3.19. Since $A_{i,j}$ is also a $\mathbb{Z}$-coefficients group cocycle, it defines an element of $H^2(\mathbb{Z}^n; \mathbb{Z})$. Moreover, by the arguments same as in the proof of LEMMA 3.18, we can show that the classes $[A_{i,j}]$ are free generators of $H^2(\mathbb{Z}^n; \mathbb{Z}) \cong \mathbb{Z}^{n(n-1)/2}$.

We will use the following theorem in the proof of THEOREM 3.2 and THEOREM 3.3.

THEOREM 3.20 ([**Gro82**, 1.2.(C)]). Let X be a closed hyperbolic manifold, then the comparison map

$$c_{\pi_1(X)} \colon H_b^n(\pi_1(X)) \to H^n(\pi_1(X))$$

is surjective for $n \geq 2$.

3.4.2. On the surface groups. In this subsection, we prove THEOREM 3.2. Let

$$\Gamma_g = \langle a_1, \ldots, a_{2g} \mid [a_1, a_2] \ldots [a_{2g-1}, a_{2g}]\rangle$$

be the surface group. Set $\Gamma = \Gamma_g$ and $K = [\Gamma, \Gamma]$, then there is a group extension

$$1 \to K \xrightarrow{i} \Gamma \xrightarrow{p} \mathbb{Z}^{2g} \to 0, \tag{3.4.2}$$

where p is the abelianization homomorphism. Note that the map $p\colon \Gamma_g \to \mathbb{Z}^{2g}$ sends a_i to $\mathbf{e}_i$ for each $1 \le i \le 2g$. Recall that there is an exact sequence

$$H^2(\mathbb{Z}^{2g}) \xrightarrow{p^*} \mathrm{Ker}(i^*) \xrightarrow{\zeta} H^1(\mathbb{Z}^{2g}; H^1(K)), \tag{3.4.3}$$

which is a part of the seven-term exact sequence of (3.4.2). Here $\mathrm{Ker}(i^*)$ is a subspace of $H^2(\Gamma) \cong \mathbb{R}$.

LEMMA 3.21. The map $p^*\colon H^2(\mathbb{Z}^{2g}) \to \mathrm{Ker}(i^*)$ is non-zero. In particular, $\mathrm{Ker}(i^*)$ is isomorphic to $\mathbb{R}$.

PROOF. We show that the class $p^*[A_{1,2}] \in \mathrm{Ker}(i^*) \subset H^2(\Gamma)$ is non-zero. It is known that the generator of $H_2(\Gamma;\mathbb{Z}) \cong \mathbb{Z}$ is represented by the following group two-cycle:

(3.4.4)
$$\begin{aligned}\sigma =&(a_1,a_2) + (a_1a_2, a_1^{-1}) + (a_1a_2a_1^{-1}, a_2^{-1}) + (a_1a_2a_1^{-1}a_2^{-1}, a_3)\\ &+ \cdots + (a_1a_2a_1^{-1}\cdots a_{2g}, a_{2g-1}^{-1}) + (a_1a_2a_1^{-1}\cdots a_{2g}a_{2g-1}^{-1}, a_{2g}^{-1})\\ &- (2g+1)(\mathrm{id}_\Gamma, \mathrm{id}_\Gamma) - \sum_i (a_i, a_i^{-1})\end{aligned}$$

(see **[Dup78]** for example). The pushforward $p_*(\sigma) \in C_2(\mathbb{Z}^{2g})$ is equal to

$$\begin{aligned}&(\mathbf{e}_1,\mathbf{e}_2) + (\mathbf{e}_1+\mathbf{e}_2, -\mathbf{e}_1) + (\mathbf{e}_2, -\mathbf{e}_2) + (0, \mathbf{e}_3)\\ &+ \cdots + (\mathbf{e}_{2g-1}+\mathbf{e}_{2g}, -\mathbf{e}_{2g-1}) + (\mathbf{e}_{2g}, -\mathbf{e}_{2g})\\ &- (2g+1)(0,0) - \sum_i (\mathbf{e}_i, -\mathbf{e}_i).\end{aligned}$$

By definition of $A_{1,2}$, we have

$$\langle p^*A_{1,2}, \sigma\rangle = \langle A_{1,2}, p_*\sigma\rangle = 1.$$

Thus the class $p^*[A_{1,2}] \in \mathrm{Ker}(i^*)$ is non-zero, and the lemma follows. $\square$

REMARK 3.22. The map

$$p^*\colon H^1(\mathbb{Z}^{2g};\mathbb{Z}) \to \mathrm{Ker}(i^*\colon H^2(\Gamma;\mathbb{Z}) \to H^2(K;\mathbb{Z}))$$

sends $[A_{1,2}] \in H^2(\mathbb{Z}^{2g};\mathbb{Z})$ to the generator of $H^2(\Gamma;\mathbb{Z})$ by the arguments same as in the proof of LEMMA 3.21.

Proof of Theorem 3.2. By Corollary 3.11 and Theorem 3.16, we obtain commutative diagram whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N}\mathcal{E} \arrow[r] & \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \arrow[r, "\alpha\circ\beta"] & H^1(\Gamma; H^1(K)) \\
& H^2(\mathbb{Z}^{2g}) \arrow[r, "p^*"] & \mathrm{Ker}(i^*) \arrow[u, hook] \arrow[r, "\zeta"] & H^1(\mathbb{Z}^{2g}; H^1(K)) \arrow[u]
\end{tikzcd}$$

where $i^*\colon H^2(\Gamma) \to H^2(K)$. By Theorem 3.20, the comparison map $c_\Gamma\colon H^2_b(\Gamma) \to H^2(\Gamma)$ is surjective. Thus we have $\mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) = \mathrm{Ker}(i^*)$. By Lemma 3.21, the kernel $\mathrm{Ker}(i^*)$ is equal to $H^2(\Gamma) \cong \mathbb{R}$ and contained in the image of $p^*\colon H^2(\mathbb{Z}^{2g}) \to \mathrm{Ker}(i^*)$. Therefore the map ζ is zero. By the commutativity of the diagram above, the map $\alpha \circ \beta$ is also zero. By the exactness of the first low, we have $\mathcal{N}\mathcal{E} \cong \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \cong \mathbb{R}$. $\square$

3.4.3. On the fundamental group of mapping tori. In this subsection, we prove Theorem 3.3. Let Σ_g be a closed oriented surface of genus $g \geq 2$ and $\mathcal{M}_g = \pi_0(\mathrm{Homeo}_+(\Sigma_g))$ the mapping class group. For a homeomorphism $f \in \mathrm{Homeo}_+(\Sigma_g)$, let

$$X_f = \Sigma_g \times [0,1]/(x,0) \sim (f(x),1)$$

denote the mapping torus. Let Γ_g be the surface group and $f_*\colon \Gamma_g \to \Gamma_g$ the pushforward. Under the presentation

$$\Gamma_g = \langle a_1, \dots, a_{2g} \mid [a_1,a_2]\dots[a_{2g-1},a_{2g}]\rangle,$$

the presentation of the group Γ is given by

$$\Gamma = \langle a_1, \dots, a_{2g}, a_{2g+1} \mid [a_1,a_2]\dots[a_{2g-1},a_{2g}], \\ a_{2g+1}a_i = (f_*a_i)a_{2g+1} \text{ for all } 1 \leq i \leq 2g\rangle$$

since the fundamental group Γ of the mapping torus X_f is isomorphic to the semidirect product $\Gamma_g \rtimes_{f_*} \mathbb{Z}$.

By the Nielsen-Thurston classification, any element of $\mathcal{M}_g$ is periodic, reducible, or pseudo-Anosov. Moreover, this classification determines the geometry on the mapping torus. If the mapping class of f is pesudo-Anosov, the following holds (for other cases, see also [**FM12**]).

Theorem 3.23 ([**Thu86**], [**Ota96**]). *The mapping class $[f] \in \mathcal{M}_g$ is a pseudo-Anosov element if and only if the mapping torus X_f admits a hyperbolic structure.*

In this subsection, we set $\Gamma = \pi_1(X_f)$ and $K = [\Gamma, \Gamma]$.

LEMMA 3.24 ([**KKM$^+$21**]). If the mapping class $[f]$ is a pseudo-Anosov element and in the Torelli group $\mathcal{I}_g$, then the abelianization Γ/K is isomorphic to $\mathbb{Z}^{2g+1}$. In particular, the dimension of $H^2(\Gamma/K)$ is equal to $g(2g+1)$.

PROOF. By THEOREM 3.23, the mapping torus X_f is $K(\Gamma, 1)$-manifold, and thus the homology of X_f is isomorphic to the group homology of Γ. Since Γ/K is the abelianization of Γ, we have

$$\Gamma/K = H_1(\Gamma;\mathbb{Z}) \cong H_1^{\mathrm{top}}(X_f;\mathbb{Z}).$$

A part of the homology five-term exact sequence of the fibration

$$\Sigma_g \to X_f \to S^1$$

gives the following exact sequence:

$$0 \to H_1^{\mathrm{top}}(\Sigma_g;\mathbb{Z})^{\mathbb{Z}} \to H_1^{\mathrm{top}}(X_f;\mathbb{Z}) \to H_1^{\mathrm{top}}(S^1;\mathbb{Z}) \to 0.$$

Note that the $\mathbb{Z}$-action on $H_1^{\mathrm{top}}(\Sigma_g;\mathbb{Z})$ is defined by the pushforward $f_*\colon H_1^{\mathrm{top}}(\Sigma_g;\mathbb{Z}) \to H_1^{\mathrm{top}}(\Sigma_g;\mathbb{Z})$. Since the element $[f]$ is in the Torelli group, the action on $H_1(\Sigma_g;\mathbb{Z})$ is trivial, and therefore $H_1^{\mathrm{top}}(\Sigma_g;\mathbb{Z})^{\mathbb{Z}} = H_1^{\mathrm{top}}(\Sigma_g;\mathbb{Z})$. Thus we have $\Gamma/K \cong H_1^{\mathrm{top}}(X_f;\mathbb{Z}) = \mathbb{Z}^{2g+1}$. □

REMARK 3.25. The abelianization homomorphism $p\colon \Gamma \to \mathbb{Z}^{2g+1}$ sends a_i to $\mathbf{e}_i$ for each i.

By LEMMA 3.24, we have the group extension

$$1 \to K \xrightarrow{i} \Gamma \xrightarrow{p} \mathbb{Z}^{2g+1} \to 0. \tag{3.4.5}$$

Thus we obtain the exact sequence

$$H^2(\mathbb{Z}^{2g+1}) \xrightarrow{p^*} \mathrm{Ker}(i^*) \to H^1(\mathbb{Z}^{2g+1}; H^1(K))$$

by the seven-term exact sequence of (3.4.5). Note that the space $\mathrm{Ker}(i^*)$ is the subspace of $H^2(\Gamma) \cong \mathbb{R}^{2g+1}$.

LEMMA 3.26. The image of the map $p^*\colon H^2(\mathbb{Z}^{2g+1}) \to \mathrm{Ker}(i^*)$ is isomorphic to $\mathbb{R}^{2g+1}$. In particular, the map p^* is surjective.

PROOF. Let $A_{i,j} \in C^2(\mathbb{Z}^{2g+1})$ be the group two-cocycles defined by (3.4.1). We show that the classes $p^*[A_{1,2}]$ and $p^*[A_{i,2g+1}]$ for each $1 \le i \le 2g$ are non-zero and linearly independent. Let $\sigma \in C_2(\Gamma)$ be a group two-cycle defined by the formula (3.4.4). Then the class $p^*[A_{1,2}]$ is non-zero by the argument same as in the proof of LEMMA 3.21. Recall that the relation $a_{2g+1}a_i a_{2g+1}^{-1} = f_* a_i$ for each $1 \le i \le 2g$. Since the mapping class $[f]$ is in the Torelli group, the class $(f_* a_i) \cdot a_i^{-1}$ is in the commutator subgroup $[\Gamma_g, \Gamma_g]$, where we consider Γ_g as a normal subgroup of Γ. Thus there exists a group two-chain $u_i \in C_2(\Gamma_g)$ such that $\partial u_i = (f_* a_i) \cdot a_i^{-1}$. Let us define a group two-cycle $\sigma_i \in C_2(\Gamma)$ by

$$\sigma_i = (f_* a_i, a_{2g+1}) - (a_{2g+1}, a_i) + ((f_* a_i) \cdot a_i^{-1}, a_i) - u_i.$$

Since the mapping class $[f]$ is in the Torelli group, the projection $p_*(f_* a_i)$ is equal to $\mathbf{e_i}$ for each $1 \le i \le 2g$. Thus we have

$$p_* \sigma_i = (\mathbf{e}_i, \mathbf{e}_{2g+1}) - (\mathbf{e}_{2g+1}, \mathbf{e}_i) + (0, \mathbf{e}_i) - p_* u_i.$$

Note that $p_* u_i \in C_2(\mathbb{Z}^{2g+1})$ does not contain the term $\mathbf{e}_{2g+1}$. Thus we have

$$\langle A_{i,2g+1}, \sigma_j \rangle = \begin{cases} 1 & i = j \\ 0 & \text{otherwise}, \end{cases} \tag{3.4.6}$$

and therefore the class $p^*[A_{i,2g+1}] \in H^2(\mathbb{Z}^{2g+1})$ is non-zero. We now show that the classes $p^*[A_{1,2}]$ and $p^*[A_{i,2g+1}]$ are linearly independent. Assume that the class $rp^*[A_{1,2}] + \sum_i r_i p^*[A_{i,2g+1}]$ is equal to zero. By taking the pairing with the homology class $[\sigma] \in H_2(\Gamma)$ defined above, we have $r = 0$. Then the condition (3.4.6) implies the equality $r_i = 0$ for each i. $\square$

Proof of Theorem 3.3. By Corollary 3.11 and Theorem 3.16, we obtain the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{NE} \arrow[r] & \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \arrow[r, "\alpha\circ\beta"] & H^1(\Gamma; H^1(K)) \\
& H^2(\mathbb{Z}^{2g+1}) \arrow[r, "p^*"] & \mathrm{Ker}(i^*) \arrow[u, hook] \arrow[r, "\zeta"] & H^1(\mathbb{Z}^{2g+1}; H^1(K)) \arrow[u]
\end{tikzcd}$$

where $i^* \colon H^2(\Gamma) \to H^2(K)$. By Theorem 3.20 and Theorem 3.23, the comparison map $c_\Gamma \colon H^2_b(\Gamma) \to H^2(\Gamma)$ is surjective. Thus we have $\mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) = \mathrm{Ker}(i^*) \cong \mathbb{R}^{2g+1}$. Since the map p^* is surjective by Lemma 3.26, the map ζ is zero. By the commutativity of the diagram above, the map $\alpha \circ \beta$ is also zero. By the exactness of the first low, we have $\mathcal{NE} \cong \mathrm{Ker}(i^*) \cap \mathrm{Im}(c_\Gamma) \cong \mathbb{R}^{2g+1}$. $\square$

3.5. Explicit construction of a homogeneous quasimorphism

We return to the case of the surface group. Let $\Gamma = \Gamma_g$ be the surface group of genus $g \ge 2$ and $K = [\Gamma, \Gamma]$ the commutator subgroup. By Theorem 3.2, the dimension of the space

$$\mathcal{NE} = Q(K)^\Gamma \big/ \big(i^* Q(\Gamma) + H^1(K)^\Gamma\big)$$

is equal to one. In this section, we describe a non-zero element in $\mathcal{NE}$ in terms of Poincaré's translation number.

Let $e \in H^2(\mathrm{Homeo}_+(S^1); \mathbb{Z})$ be the Euler class, that is, the cohomology class corresponding to the central extension

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}}_+(S^1) \xrightarrow{\pi} \mathrm{Homeo}_+(S^1) \to 1$$

(see also PROPOSITION 1.11). Let $\rho\colon \Gamma \to \mathrm{Homeo}_+(S^1)$ be a homomorphism such that the class $\rho^* e \in H^2(\Gamma; \mathbb{Z})$ is non-zero. It is known that there exists such a homomorphism ρ, e.g., a maximal representation.

LEMMA 3.27. *Let $i\colon K \to \Gamma$ be the inclusion. Then the map $\rho \circ i\colon K \to \mathrm{Homeo}_+(S^1)$ has a lift $\widetilde{\rho}\colon K \to \widetilde{\mathrm{Homeo}}_+(S^1)$, that is, the diagram*

$$\begin{array}{ccc} K & \xrightarrow{\widetilde{\rho}} & \widetilde{\mathrm{Homeo}}_+(S^1) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \pi} \\ \Gamma & \xrightarrow{\rho} & \mathrm{Homeo}_+(S^1) \end{array} \tag{3.5.1}$$

commutes.

PROOF. By REMARK 3.22, we have $\mathrm{Ker}(i^*) = H^2(\Gamma_g; \mathbb{Z})$ for the map $i^*\colon H^2(\Gamma; \mathbb{Z}) \to H^2(K; \mathbb{Z})$. Thus the class $i^*\rho^* e$ is equal to zero. By PROPOSITION 1.12, there exists a lift $\widetilde{\rho}$ of $\rho \circ i$. $\square$

REMARK 3.28. We can also construct a lift $\widetilde{\rho}$ explicitly. Since the commutator subgroup $K = [\Gamma_g, \Gamma_g]$ is an infinite-rank free group, we take a generating set $\{a_j\}_{j \in \mathbb{N}}$ of K. Set

$$a_j = [\gamma_{1,j}, \gamma_{2,j}] \dots [\gamma_{n_j - 1, j}, \gamma_{n_j, j}].$$

Let $s\colon \mathrm{Homeo}_+(S^1) \to \widetilde{\mathrm{Homeo}}_+(S^1)$ be a section. We define a homomorphism $\widetilde{\rho}\colon K \to \widetilde{\mathrm{Homeo}}_+(S^1)$ by

$$\widetilde{\rho}(a_j) = [s\rho(\gamma_{1,j}), s\rho(\gamma_{2,j})] \dots [s\rho(\gamma_{n_j - 1, j}), s\rho(\gamma_{n_j, j})],$$

then this $\widetilde{\rho}$ is a lift of $\rho|_K\colon K \to \mathrm{Homeo}_+(S^1)$. Note that this lift $\widetilde{\rho}$ is independent of the choice of the section s.

Let $\mu \in Q(\widetilde{\mathrm{Homeo}}_+(S^1))$ denote Poincaré's translation number. By the lift $\widetilde{\rho}$, we obtain a homogeneous quasimorphism $\widetilde{\rho}^* \mu$ on K.

PROPOSITION 3.29. *There exists a homomorphism $h \in H^1(K)$ such that the sum $\widetilde{\rho}^* \mu + h$ is Γ-invariant.*

PROOF. By LEMMA 3.21 and the seven-term exact sequence of $1 \to K \to \Gamma \to \mathbb{Z}^{2g} \to 0$, we have $\zeta(\rho^* e_{\mathbb{R}}) = 0$. Here ζ is the map in the seven-term exact sequence and $e_{\mathbb{R}} \in H^2(\mathrm{Homeo}_+(S^1))$ is the real Euler class. Let $e_b \in H_b^2(\mathrm{Homeo}_+(S^1))$ be the bounded Euler class, that is, e_b is the element satisfies $c_{\mathrm{Homeo}_+(S^1)}(e_b) = e_{\mathbb{R}}$. It is known that Poincaré's translation number μ satisfies $[\delta\mu] = -\pi^* e_b$. Therefore we have

$$-i^*\rho^* e_b = -\widetilde{\rho}^*\pi^* e_b = [\delta(\widetilde{\rho}^*\mu)].$$

Note that the class $[\delta(\widetilde{\rho}^*\mu)]$ is Γ-invariant since it is in the image of $i^*\colon H_b^2(\Gamma) \to H_b^2(K)^\Gamma$. By THEOREM 3.10 (3.3.2), we have

$$\alpha([\delta(\widetilde{\rho}^*\mu)]) = p^*\zeta(\rho^* e_{\mathbb{R}}) = 0.$$

By the definitions of the maps φ and α (see LEMMA 2.3 and the proof of THEOREM 3.10 (1)), we have

$$0 = \alpha([\delta(\widetilde{\rho}^*\mu)]) = [\varphi_{\widetilde{\rho}^*\mu}] \in H^1(\Gamma, H^1(K)).$$

By the equality $[\varphi_{\widetilde{\rho}^*\mu}] = 0$, there exists an element $h \in H^1(K)$ such that the equality

$$\varphi_{\widetilde{\rho}^*\mu}(\gamma) = {}^\gamma h - h \tag{3.5.2}$$

holds for any $\gamma \in \Gamma$. By the definition of $\varphi_{\widetilde{\rho}^*\mu}$ (DEFINITION 3.13), for any $\gamma \in \Gamma_g$, there exists a unique bounded function $b_\gamma\colon K \to \mathbb{R}$ such that the equality

$$\varphi_{\widetilde{\rho}^*\mu}(\gamma) = \widetilde{\rho}^*\mu - {}^\gamma(\widetilde{\rho}^*\mu) + b_\gamma \tag{3.5.3}$$

holds. By (3.5.2) and (3.5.3), we obtain

$${}^\gamma(\widetilde{\rho}^*\mu + h) - (\widetilde{\rho}^*\mu + h) = b_\gamma. \tag{3.5.4}$$

Since the left-hand side of (3.5.4) is a homogeneous quasimorphism and the right-hand side of (3.5.4) is a bounded function, we have

$${}^\gamma(\widetilde{\rho}^*\mu + h) - (\widetilde{\rho}^*\mu + h) = 0,$$

and this implies that the homogeneous quasimorphism $\widetilde{\rho}^*\mu + h$ is Γ-invariant. $\square$

THEOREM 3.30. Let $\widetilde{\rho}^*\mu + h \in Q(K)^\Gamma$ be a Γ-invariant homogeneous quasimorphism, where $h \in H^1(K)$. Then $\widetilde{\rho}^*\mu + h$ gives a non-zero element of $\mathcal{NE}$.

PROOF. Assume that there exist elements $\mu' \in Q(\Gamma)$ and $h' \in H^1(K)^\Gamma$ such that the equality

$$\widetilde{\rho}^*\mu + h = i^*\mu' + h'$$

holds. Then we have $[\delta(\widetilde{\rho}^*\mu)] = i^*[\delta\mu'] \in H_b^2(K)$. Moreover, by the diagram (3.5.1), we have

$$[\delta(\widetilde{\rho}^*\mu)] = \widetilde{\rho}^*[\delta\mu] = -\widetilde{\rho}^*\pi^*e_b = i^*(-\rho^*e_b).$$

Thus we obtain $i^*[\delta\mu'] = i^*(-\rho^*e_b) \in H_b^2(K)$. By THEOREM 1.29 and the triviality of the second bounded cohomology of $\Gamma/K \cong \mathbb{Z}^{2g}$, the map $i^*\colon H_b^2(\Gamma) \to H_b^2(K)^\Gamma$ is injective. Thus the equality $[\delta\mu'] = -\rho^*e_b \in H_b^2(\Gamma)$ holds, and therefore the image $c_\Gamma(\rho^*e_b) \in H^2(\Gamma)$ is equal to zero. This contradicts the assumption $\rho^*e \neq 0$. Indeed, we have

$$c_{\Gamma_g}(\rho^*e_b) = \rho^*(c_{\mathrm{Homeo}_+(S^1)}e_b) = \rho^*e_\mathbb{R} \in H^2(\Gamma).$$

Moreover, since the change of coefficients map

$$f\colon H^2(\Gamma;\mathbb{Z}) \to H^2(\Gamma)$$

is injective, the class $\rho^*e_\mathbb{R} = f(\rho^*e)$ is non-zero. $\square$

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