Special Mathematics Lecture
Groups and their representations

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Handwritten notes taken by L. Zhang

Groups & Representations

I) Groups

I.1) Basic def

Def. A group is a set G together with
a map $G \times G \to G$ (denoted by "$\cdot$", " ", or "+") && satisfying:
$\forall\ a,b,c \in G$

1) $(ab)c = a(bc)$ ASSOCIATIVITY
2) $\exists e \in G : ea = ae = a$ IDENTITY ELEMENT
3) $\forall a \in G\ \exists a^{-1} \in G : aa^{-1} = e$ EXSISTANCE of INVERSE

⚠ If use "+" notation, $a^{-1} =: -a$, $e =: 0$; for "$\cdot$" notations $e =: 1$

Remark

1) $e^{-1} = e$, $a^{-1}a = e$, $(a^{-1})^{-1} = a$, $(ab)^{-1} = b^{-1}a^{-1}$, a^{-1}, e are unique.
2) If $ab = ac$ then $b = c$; If $ba = ca$ then $b = c$

Def.

G is Abelian (or commutative) if $\forall a,b \in G : ab = ba$

G is finite if containing only a finite number of elements, ($\Leftrightarrow$: $\overbrace{|G|}^{\text{number of elements of } G} < \infty$)

Examples

1) $(\mathbb{Z}, +)$ $(\mathbb{R}, +)$ $(\mathbb{R}_{\overset{:=(0,+\infty)}{+}}, \cdot)$
2) Cyclic group C_n with $C_n = \{e, a, a^2 = aa, a^3, a^4, \cdots, a^{n-1}\}$ $(e \equiv a^0 \equiv a^n)$
with $a^j a^k = a^{j+k \bmod n}$, $(a^j)^{-1} = a^{n-j}$ (Abelian group)
3) Symmetric group S_n = group of perm$\overset{u}{\text{a}}$tations of n elements
   - It contains $n!$ elements
   - Not Abelian (if $n \geqslant 3$)

   For example, $n = 3$, the elements are:

$$\underset{\overset{\shortparallel}{e}}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

with $\begin{pmatrix} \underset{\downarrow}{1} & \underset{\downarrow}{2} & \underset{\downarrow}{3} \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ \underline{1} & \underline{3} & \underline{2} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \underline{2} & \underline{3} & \underline{1} \end{pmatrix}$

4) $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$: $n \times n$ invertible matrices with multiplication

$SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$: invertible matrices with det = 1

$U(n)$: inv. matrices s.t. $U^* = U^{-1}$ $(U^* := \overline{U^T})$ → complex conjugate $(\Rightarrow |\det U| = 1)$

$SU(n)$: $\{M \in U(n) \mid \det M = 1\}$

$O(n) \subset GL(n,\mathbb{R})$ with $A^t = A^{-1}$ $(\Rightarrow \det A = \pm 1)$

$SO(n) \subset O(n)$ with $\det(A) = 1$

Def. A subgroup G_0 is a subset of group G which $\overset{(G_0)}{\text{is}}$ a group.

G_0 is proper if $G_0 \neq G$ and non-trivial ~~is~~ if $G_0 \neq \{e\}$

Check in examples: which are subgroups? Does C_n contain subgroups?

Def. For $a, b \in G$, a is conjugate to b if

$\exists c \in G : a = cbc^{-1}$

Then we white $a \sim b$ when a is conjugate to b.

Remark: $\sim$ is an equivalence relation, indeed:

1) $a \sim a$ (choose $c = e$) REFLEXIVITY

2) $b \sim a$ $(a = cbc^{-1} \Leftrightarrow c^{-1}ac = b)$ SYMMETRY

3) $b \sim d \Rightarrow a \sim d$ (check it) TRANSITIVITY $\left.\begin{matrix} a = mbm^{-1} \\ b = ndn^{-1} \end{matrix}\right\} \Rightarrow \begin{matrix} a = mndn^{-1}m^{-1} \\ = mnd(mn)^{-1} \end{matrix}$

Def. a, b are in the same equivalence class (or conjugacy class) if $a \sim b$.

Remark:

- Each element $a \in G$ is in a single conjugacy class.
- e generates a class on its own.
- If G is Abelian, each element generates its own class.

More generaly, let G_0 be subgroup of G

and set $cG_0c^{-1} := \{cac^{-1} \mid a \in G_0\}$, $(\forall a, b \in G : cac^{-1}cbc^{-1} = cabc^{-1} \in G_0)$

Then cG_0c^{-1} is also a subgroup of G

Def: cG_0c^{-1} is a subgroup conjugated to G_0

- If $\forall c \in G : cG_0c^{-1} = G_0$ then G_0 is called normal or invariant.

(written $G_0 \triangleleft G$)

Examples:

Ex. · Consider $(\mathbb{R},+)=G$, and $(\mathbb{Z},+)=G_0$ is a normal subgroup.

· $G=GL(n,\mathbb{C})$, $G_0=\mathbb{C}^*\mathbb{1}_{n\times n}$ is normal (and Abelian)

↳ not necessary for normal subgroup

Def. The center $Z(G)$ of a group G is defined by

$\{a\in G \mid ab=ba\ \forall b\in G\}$

Exercise: $Z(G)$ is an Abelian & normal subgroup. of G

Def. · G is simple if $\{e\}$ is the only proper normal subgroup of G.

· G is semi-simple if $\{e\}$ is the only proper normal Abelian subgroup of G.

Def. Let G_0 be a subgroup of G, and $a,b\in G$

We set $a\overset{L}{\sim}b$ iff $a^{-1}b\in G_0$, then observe

1) $a\overset{L}{\sim}a$

2) $b\overset{L}{\sim}a$ $(b^{-1}a=(a^{-1}b)^{-1}\in G_0)$

3) $b\overset{L}{\sim}c\Rightarrow a\overset{L}{\sim}c$ $(a^{-1}c=a^{-1}bb^{-1}c\in G_0)$

$\Rightarrow \overset{L}{\sim}$ is a equivalence relation.

Denote by ${}_{G_0}[a]$ the equivalence class of $\overset{L}{\sim}$ containing a

Indeed ${}_{G_0}[a]=aG_0=:$ Left Coset

$(b\in G_0\Rightarrow a^{-1}ab\in G_0\Rightarrow a\overset{L}{\sim}ab\Rightarrow ab\in{}_{G_0}[a])$

Similarly, $a\overset{r}{\sim}b$ iff $ba^{-1}\in G_0$

$\rightsquigarrow\overset{r}{\sim}$ is an equivalence relation with equivalence class $[a]_{G_0}=G_0a=:$Right Coset

⚠ aG_0 & G_0a are usually not subgroups of G.

Prop.

1) ${}_{G_0}[a]=[a]_{G_0}\Leftrightarrow G_0\triangleleft G$ (normal subgroup)

2) $G_0\triangleleft G\Rightarrow [a]_{G_0}[b]_{G_0}=[ab]_{G_0}$

This makes $\{[a]_{G_0}\mid a\in G\}$ a group.

This group is denoted by G/G_0 and called quotion group (or factor group)

Example: $G=(\mathbb{R},+)$, $G_0=(\mathbb{Z},+)$, then $G/G_0=([0,1),+\text{ mod }1)$

$\simeq\mathbb{S}:=(\{z\in\mathbb{C}\mid |z|=1\},\cdot)\simeq\mathbb{T}$

Prop. If $|G|=:g<\infty$, $G_0\triangleleft G$ with $|G_0|=:g_0\leq g$ then $|G/G_0|=g/g_0$

Def. Let G, G' be 2 groups

A homomorphism is a map $\phi : G \to G'$ $\overset{\text{such that}}{\text{s.t.}}$ $\phi(ab) = \phi(a)\phi(b)$

If ϕ is bijective, ϕ is called an isomorphism, and we write $G \simeq G'$

If $G = G'$, ϕ is called an endomorphism

and an automorphism if bijective (G = G' : elements & multiplication are same)

Prop. Let ϕ be a group homomorphism from G to G'

1) If G_0 a subgroup of G, then $\phi(G_0)$ is a subgroup of G'

2) $\phi(e_G) = e_{G'}$ and $\phi(a^{-1_G}) = (\phi(a))^{-1_{G'}}$

3) $\mathrm{Ker}(\phi) = \{a \in G \mid \phi(a) = e_{G'}\}$ is normal subgroup of G,

and $G/\mathrm{Ker}(\phi)$ is isomorphic to $\phi(G)$

the isomorphism $\tilde{\phi}([a]_{\mathrm{Ker}(\phi)}) := \phi(a)$

Note: For $m\times m$ matrices A, B,

$\det(AB) = \det(A)\det(B) \Rightarrow$ det is a homomorphism from $GL(n,\mathbb{C})$ to $\mathbb{C}^*$

⚠ $\det(A+B) \neq \det(A)+\det(B)$ generally

Question: How to construct 1 group from 2 groups?

Def. Let G, G' be 2 groups, and set

$G\otimes G' := \{a\otimes a' \mid a\in G, a'\in G'\}$

with $(a\otimes a')(b\otimes b') := ab\otimes a'b'$, $e = e\otimes e'$, $(a\otimes a')^{-1} = a^{-1}\otimes (a')^{-1}$

Then $G\otimes G'$ can be proved to be a group, called the DIRECT PRODUCT of G&G'

Conversely, if G is group and G_1, G_2 are subgroups of G with

1) $G_1\cap G_2 = \{e\}$

2) $\forall a_j\in G_j : a_1a_2 = a_2a_1$

3) $\forall a\in G\ \exists a_1\in G_1, a_2\in G_2 : a = a_1a_2$

Then $G \simeq G_1\otimes G_2$

Observe this G_1, G_2 is unique (hint: from $G_1\cap G_2 = \{e\}$)

and G_1, G_2 are normal.

Def. (INNER SEMI-DIRECT PRODUCT)

G is a semi-direct product if $\exists G_1, G_2$ subgroups with

1) G_1 normal 2) $G_1\cap G_2 = \{e\}$

3) $\forall a\in G$, $a = a_1a_2$ with $a_j\in G_j$ $\Big\}$ We write $G = G_1\odot G_2 = G_1\rtimes G_2$

Abstract (OUTER SEMI-DIRECT PRODUCT)

Let H, N be 2 groups and let

$\phi : H\to \mathrm{Aut}(N)$ be a homomorphism

↳ $=\{$automorphism on $N\}$ Check that $\mathrm{Aut}(N)$ is a group.

Then we set $N\rtimes_\phi H = \{(n,h)\in N\times H\}$ with the product

$(n_1,h_1)(n_2,h_2) = (n_1\phi(h_1)n_2, h_1h_2)$ and note that

$e = (e_N, e_H)$, $(n,h)^{-1} = (\phi(h^{-1})(n^{-1}), h^{-1})$

↳ For $h\in H$, $\phi(h)\in \mathrm{Aut}(N) \therefore \phi(h)$ is a map $N\to N$

Also, if we set $G_1 = \{(n, e_H) \mid n\in N\}$ and $G_2 = \{(e_N, h) \mid h\in H\}$

then G_1 is normal in $N\rtimes_\phi H$ and

$G_1\odot G_2 = N\rtimes_\phi H$.

Prop.

The map $R: SU(2) \longrightarrow SO(3)$ defined by

$R(U)_{jk} = \frac{1}{2} tr(\sigma_j U \sigma_k U^{-1})$ → trace

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

is a surjective map with $\ker(R) = \{\mathbb{1}, -\mathbb{1}\}$

↳(homomorphism)↑

(typical in quantum mechanics)

Lemma: $O(3) \simeq SO(3) \otimes J$ with $J = \{1, -1\}$

$SL(n, \mathbb{C}) \triangleleft GL(n, \mathbb{C})$

↳$\exists a$

($\because \det(cac^{-1}) = \det(a)$ $\therefore$ normal)

Back to examples

5) Euclidean group $E(n) = \{(A, b) | A \in O(n), b \in \mathbb{R}^n\}$

with $(A, b)(A', b') = (AA', b + Ab')$,

$e = (\mathbb{1}, \mathbf{0})$ and $(A, b)^{-1} = (A^{-1}, -A^{-1}b)$

Observe $E(n) = (\mathbb{1}, \mathbb{R}^n) \rtimes (O(n), \mathbf{0})$ (inner semi-direct product)

Exercise: represent it as an outer semi-direct product (find ϕ)

6) If $g = \begin{pmatrix} 1 & & & \\ & -1 & \bigcirc & \\ & \bigcirc & -1 & \\ & & & -1 \end{pmatrix} \in M_4(\mathbb{R})$, the Lorentz group L is given by

↳$\equiv M_{4\times4}(\mathbb{R}) \equiv \mathbb{R}^{4\times4}$

$L = \{\Lambda \in M_4(\mathbb{R}) | \Lambda^t g \Lambda = g\}$

7) Poincaré group $P = \{(\Lambda, b) | \Lambda \in L, b \in \mathbb{R}\}$

with $(\Lambda b)(\Lambda' b') = (\Lambda\Lambda', b + \Lambda b')$

$\Rightarrow P = (\mathbb{1}, \mathbb{R}^4) \rtimes (L, \mathbf{0})$

Def. A group of transformation of a set X is

a set X, a group G, and a map $\circ : G \times X \to X$ s.t.

$$a \circ \underbrace{(b \circ x)}_{\in X} = \underbrace{(ab)}_{\in G} \circ x, \text{ and } e \circ x = x$$

Also called: X is a G-set.

Def. $\forall x \in X$ The set $O(x) \equiv O_x := \{a \circ x \mid a \in G\} \subset X$ called the ORBIT of x.

$S(x) \equiv G_x := \{a \in G \mid a \circ x = x\} \subset G$, called the STABILIZER of x.

Lemma:

1) The set of orbits defines a partition of X.
2) $S(x)$ is a subgroup of G
3) If $x' \in O(x)$ then $S(x) \simeq S(x')$

Lemma:

Let G be a finite group of transformation of X, then

$\forall x \in X : |S(x)| \cdot |O(x)| = |G|$

Remark:

The Euclidean group is the group of transformation of $\mathbb{R}^n$ leaving $\|x-y\|$ invarignt

The Poincaré group is " " of $\mathbb{R}^4$

leaving $x^\nu y_\nu := x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4$ invariant.

Remark: $O(3) = SO(3) \sqcup \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} SO(3)$ ($\sqcup$: disjoint union) $A \sqcup B$ means $A \cup B$ with $A \cap B = \emptyset$

$\forall R \in SO(3) \exists n \in \mathbb{R}^3$ with $\|n\| = 1$: s.t. $Rn = n$

Then in a basis (n, e_2, e_3) R takes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\delta) & -\sin(\delta) \\ 0 & \sin(\delta) & \cos(\delta) \end{pmatrix} \exists \delta \in [0, 2\pi)$$

We can parameterize R with n and δ.

I.2 Crystallographic Groups

(linearly independent)

Def. Let b_1, b_2, b_3 be 3 lin. indep. vectors in $\mathbb{R}^3$, and set

$$\mathcal{L} = \{m_1 b_1 + m_2 b_2 + m_3 b_3 \mid m_j \in \mathbb{Z}\} \Rightarrow (0,0,0) \in \mathcal{L}$$

$\mathcal{L}$ is called a LATTICE in $\mathbb{R}^3$. (used for discribing a crystal)

Def. A CRYSTALLOGRAPHIC POINT GROUP is a subgroup of $O(3)$ which leaves a lattice $\mathcal{L}$ invariant.

↳ $\forall A \in CPG : A\mathcal{L} = \mathcal{L}$

(We'll find 32 such groups)

Remark: Given a c.p.g. the lattice $\mathcal{L}$ is not arbitrary

⇝ determination of all finite subgroups of $O(3)$ and $SO(3)$

Let's construct ↑

Consider G finite non-trivial subgroup of $SO(3)$, and let's identify $\{n \in \mathbb{R}^3 \mid \|n\| = 1\}$ with $\mathbb{S}^2$

We set $X = \{n \in \mathbb{S}^2 \mid Rn = n \ \exists R \in G \setminus \{\mathbb{1}\}\}$

At most $|X| \leq 2(g-1)$ with $g = |G|$

↳ because of identity

Observe that G is a group of transformation of X.

Take $n \in X : Rn = n$. Consider $R'n$ and check $R'n \in X$.

Indeed, $\underbrace{(R'RR'^{-1})}_{\in G} R'n = R'Rn = R'n \Rightarrow R'n \in X$

$\Rightarrow X = O_1 \sqcup O_2 \sqcup \cdots \sqcup O_r$ with each O_j the orbit of a point $n \in X$.

Let S_j be the stabilizer for one point of the orbit O_j, then

$$2 \leq |S_j| =: g_j \leq g$$

↳ identity + at least one R

Aim: find all finite subgroups of $O(3)$

$\forall n \in O_j \subset X$, the number of $R(\neq \mathbb{1})$ which leaves n invariant is $g_j - 1$

Let $p_j := |O_j|$. Then $\sum_{j=1}^{r} p_j (g_j - 1) = 2(g-1)$ (☆)

↳ # of elements of S^2 invariant by ≥ 1 element of G, multiplicity counted

$$\Leftrightarrow \sum_{j=1}^{r} g - p_j = 2(g-1)$$

$$\Leftrightarrow rg - \sum_{j=1}^{r} p_j = 2(g-1)$$

$$\Leftrightarrow 2 - \frac{2}{g} = r - \sum_{j=1}^{r} \frac{1}{g_j} \ \sharp \qquad (*)$$

Since $g_j \geq 2$, $\sum_{j=1}^{r} \frac{1}{g_j} \leq \frac{r}{2} \Leftrightarrow -\sum_{j=1}^{r} \frac{1}{g_j} \geq -\frac{r}{2}$

$$\Rightarrow 2 - \frac{2}{g} = r - \sum_{j=1}^{r} \frac{1}{g_j} \geq r - \frac{r}{2} = \frac{r}{2}$$

$\Rightarrow r < 4$. At most X is decomposed to 3 orbits.

Also: $r > 1$. Since otherwise $2 - \frac{2}{g} = 1 - \frac{1}{g_1} \Leftrightarrow \frac{1}{g_1} = \frac{2}{g} - 1 \leq 0$

$\Rightarrow r \in \{2, 3\}$

If $r=2$: $2 - \frac{2}{g} = 2 - \frac{1}{g_1} - \frac{1}{g_2} \Leftrightarrow \frac{2}{g} = \frac{1}{g_1} + \frac{1}{g_2}$ (*)

We have $g_j \leq g \Rightarrow g_1 = g_2 = g$

$\Rightarrow$ Each element of X is invariant under all elements of G

$\Rightarrow G \simeq C_g$ = Cyclic group: rotations by $\frac{2\pi}{g}k$ for $k \in \{0, \cdots, g-1\}$

If $r=3$: (*) $\Leftrightarrow 2 - \frac{2}{g} = 3 - \frac{1}{g_1} - \frac{1}{g_2} - \frac{1}{g_3} \Leftrightarrow \frac{1}{g_1} + \frac{1}{g_2} + \frac{1}{g_3} = 1 + \frac{2}{g}$

Wlog (Without loss of generality) $g_1 \leq g_2 \leq g_3 \Rightarrow \frac{1}{g_1} \geq \frac{1}{g_2} \geq \frac{1}{g_3}$

If $g_1 > 2$ ($\Leftrightarrow g_1 \geq 3$) $\Rightarrow \frac{1}{g_1} + \frac{1}{g_2} + \frac{1}{g_3} \leq \frac{3}{g_1} \leq 1 \Rightarrow g < 0$ contradiction

$\Rightarrow g_1 = 2$. (*) $\Leftrightarrow \frac{1}{g_2} + \frac{1}{g_3} = \frac{1}{2} + \frac{2}{g}$

If $g_2 \geq 4 \Rightarrow \frac{1}{g_2} + \frac{1}{g_3} \leq \frac{2}{g_2} \leq \frac{1}{2} \Rightarrow g < 0$ contradiction again

$\Rightarrow g_2 \in \{2, 3\}$.

Similarly, if $g_2 = 3$, then $g_3 \leq 5$

$\Rightarrow$ All possibilities of (g_1, g_2, g_3):

$(2, 2, l)$ for $l \in \mathbb{N}_+ \setminus \{1\}$; $\Rightarrow g = 2l$ group D_l (dehedral)

$(2, 3, 3)$ $\Rightarrow g = 12$ group T (tetrahedral)

$(2, 3, 4)$ $\Rightarrow g = 24$ group O (octahedral)

$(2, 3, 5)$ $\Rightarrow g = 60$ group I (icosahedral)

Lemma: Let G be a cpg and consider $\{R\equiv(n,\frac{2\pi}{l}k) \mid k\in\{0,1,\cdots,l-1\}\}\subset G$ be subgroup.
Then $l\in\{1,2,3,4,6\}$

Proof: Compute trace(R) in two different bases.
According to prop of trace, they are equal.

$$\text{In 1 basis } R=\begin{pmatrix}1&0&0\\0&\cos\delta&-\sin\delta\\0&\sin\delta&\cos\delta\end{pmatrix}\Rightarrow tr(R)=1+2\cos\delta$$

Recall $\mathcal{L}=\{c_1b_1+c_2b_2+c_3b_3 \mid c_j\in\mathbb{Z}\}$ and $b_1\sim b_3$ 3 lin. indep. elements of $\mathbb{R}^3$

Then $Rb_j=\sum_{k=1}^{3}c_{jk}b_k$ with $c_{jk}\in\mathbb{Z}$

$\Rightarrow tr(R)=c_{11}+c_{22}+c_{33}\in\mathbb{Z}$

$\Rightarrow 1+2\cos\delta\in\mathbb{Z}\Rightarrow 2\cos\delta\in\mathbb{Z}\Rightarrow\delta\in\{\frac{2\pi}{l}k \mid k=\{0,\cdots,l-1\}, l\in\{1,2,3,4,6\}\}$ □

Remark: Same result for matrix $-\mathbb{1}R$ for $R\in SO(3)$

In summary: The cpg of 1st type are

$C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O$

$\downarrow$
trivial group

(I has a rotation with $\ell=5$)

Let us now study the finite subgroups of O(3).

Recall $O(3)=SO(3)\otimes\{\mathbb{1},-\mathbb{1}\}=SO(3)\sqcup(-\mathbb{1})\,SO(3)$

det ±1 +1 −1 $\quad\hookrightarrow=:\Pi$

$O(3)$ $SO(3)$ $O(3)\backslash SO(3)$

$\cap$ $\cap$ $\cap$

2 cases: Either $G=G_+\sqcup G_-$ contains Π or does not contain Π

1) If $\Pi\in G$ then $\Pi\in G_-$ and $G_-=\Pi G_+$ by using $\det(AB)=\det(A)\det(B)$

$\Rightarrow$ 11 new cpg made of $G_+\sqcup\Pi G_+$ with G_+ in the list of 1st type.
They are called cpg of 2nd type with inversion.

2) If $\Pi\notin G$

Let us set $\phi: G\to SO(3)$, $\phi(R)=\begin{cases}R & \text{if } R\in G_+\\ \Pi R & \text{if } R\in G_-\end{cases}$ and set $\mathbb{G}:=\phi(G)$.

Observe $G_+\cap\Pi G_-=\phi$. (empty set)

(Indeed if $G_+\ni a=\Pi b$, $b\in G_-$ then $\Pi=ab^{-1}\in G$)

and also ϕ is a homomorphism and an isomorphism between $G\leftrightarrow\mathbb{G}$

Also $|G_+|=|G_-|$ (exercise) $\quad\nearrow\mathbb{G}=G_+\sqcup\phi(G_-), |\mathbb{G}|=2g$

$\Rightarrow\mathbb{G}$ is a finite subgroup of SO(3) containing a subgroup G_+ and a subset $\phi(G_-)$

By inspection, the possible pairs (G_+, G) are

$$\left.\begin{array}{ll} (C_n, C_{2n}) & \text{for } n\in\{1,2,3\} \\ (C_n, D_{2n}) & \quad '' \\ (D_n, D_{2n}) & \quad '' \\ (T, O) & \end{array}\right\} \text{10 solutions}$$

Implicit: Column 1 are subgroups of column 2

Now we have all 32 finite subgroups of $O(3)$ which leave a lattice invariant

Next step: for a given subgroup, find the corresponding invariant lattice

→ 7 lattice systems
14 Bravais lattices

Exercise: Do the same thing with $E(3)$ (containing translation) instead of $O(3)$

→ 230 finite subgroups leaving a lattice invariant.

Or for $O(2)$

Chapter II: Linear representation

II.1 Generalities

Def. A Hilbert space is a complex vector space together with an inner product $\langle \cdot, \cdot \rangle$ (linear in the second argument) and complete for the norm $\|f\| = \sqrt{\langle f, f\rangle}$

Example: ($\mathbb{R}^n$ "real Hilbert space")

- $\mathbb{C}^n$ with $a = (a_1, \cdots, a_n), b = (b_1, \cdots, b_n) \in \mathbb{C}^n$ $\langle a, b\rangle = \sum_{j=1}^{n} \overline{a_j}\, b_j$
- $L^2(\mathbb{R}^n)$ with $\langle f, g\rangle = \int_{\mathbb{R}^n} \overline{f(x)}\, g(x)\, dx$
- $\ell^2(\mathbb{Z}^n)$ with $\langle a, b\rangle = \sum_{j\in\mathbb{Z}^n} \overline{a_j}\, b_j$

Remark: $\langle a, b\rangle = \overline{\langle b, a\rangle}$ and $\langle a, a\rangle \geqslant 0$ with equality iff $a = \mathbf{0}$

II. Linear representation

Def. A bounded linear operator is a $\overset{\text{linear}}{\text{map}}$ $T:\mathcal{H}\to\mathcal{H}$ s.t.

$\exists c<\infty\ \forall f\in\mathcal{H}: \|Tf\|\leq c\|f\|$

The infimum over c is called the NORM of $T =: \|T\|$

$B(\mathcal{H})$ $(\equiv M_n(\mathbb{C}))$:= the set of all b. l. op. on $\mathcal{H}$. (It is a group).

Def. Linear map T^* satisfying $\langle T^*f, g\rangle = \langle f, Tg\rangle\ \forall f,g\in\mathcal{H}$

is called the ADJOINT of T. (always exist)

※In finite dimension, $[T^*] = [\overline{T^t}]$ → complex conjugate → transposition

Def. If $\langle f, Tf\rangle \geq 0\ \forall f\in\mathcal{H}$, T is POSITIVE. (then $T^* = T$)

Def. Let $T\in B(\mathcal{H})$

1) T is UNITARY if $T^*T = TT^* = \mathbb{1}$.

2) T is an ORTHOGONAL PROJECTION if $\overset{\to T\cdot T}{T^2} = T = T^*$

3) T is INVERTIBLE (in $B(\mathcal{H})$) if

$T:\mathcal{H}\to\mathcal{H}$ is bijective $\Leftrightarrow \exists T^{-1}\in B(\mathcal{H}): TT^{-1} = T^{-1}T = \mathbb{1}$

For linear representation

Def. Let G be a group and $\mathcal{H}$ a hilbert space.

A [LINEAR] REPRESENTATION of G in $\mathcal{H}$ is a hormomorphism

$U: G\mapsto B(\mathcal{H})$

it means $U(ab) = U(a)U(b)$ and $U(e) = \mathbb{1} \Rightarrow U(a^{-1}) = U(a)^{-1}$

Remark: This definition can be generized to non-linear or to projective, …, representati

If $U(a)$ is $\overset{a}{\text{unitery}}$, $\forall a\in G$, we speak about a unitary representation.

$U: G\to \mathcal{U}(\mathcal{H}) \subset B(\mathcal{H})$

Def. · $U: G\mapsto B(\mathcal{H})$ is TRIVIAL if $U(a) = \mathbb{1}\ \forall a\in G$

· $U: G\mapsto B(\mathcal{H})$ is FAITHFUL if $U(a)\neq\mathbb{1}\ \forall a\in G\setminus\{e\} \Rightarrow$ injective

· The DIMENSION of the representation is the dimension of $\mathcal{H}$.

Lemma $U: G\mapsto B(\mathcal{H})$.

1) If $G_0 \triangleleft G$, and if $V_0: G/G_0 \mapsto B(\mathcal{H})$ is a representation

Then $V(a) :=$ $([a]_{G_0})$ defines a representation of G in $\mathcal{H}$ $(a\in G)$

2) The set $G_0 := \{a\in G \mid U(a) = \mathbb{1}\} \triangleleft G$.

3) In particular if G is simple, then all non-trivial representations are faithful.

see lecture 1

Def. Let G be a group, $U: G \mapsto \mathcal{B}(\mathcal{H})$ and $U': G \mapsto \mathcal{B}(\mathcal{H}')$ 2 representations of G.

They are SIMILAR or EQUIVALENT if

$\exists S: \mathcal{H} \mapsto \mathcal{H}'$ a linear operator which is bijective and s.t. ~~$\forall$~~

$U'(a) = S \cdot U(a) \cdot S^{-1} \; \forall a \in G.$

They are UNITARILY EQUIVALENT if in addition

$S^* = S^{-1}$

Thm. Let G be finite and $U: G \mapsto \mathcal{B}(\mathcal{H})$ be a linear representation

Then U is similar to a unitary representation $U': G \mapsto \underset{\substack{\uparrow \\ \text{set of unitary elements in } \mathcal{H}}}{\mathcal{U}}(\mathcal{H})$

II.2 Reducible or Irreducible Representation

Recall if $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$, there exists $\mathcal{H}_1$ also closed subspace of $\mathcal{H}$ such that $\mathcal{H}_0 \underset{\substack{\uparrow \\ \text{orthogonal sum}}}{\oplus} \mathcal{H}_1 = \mathcal{H}$

If $T \in \mathcal{B}(\mathcal{H})$ then T can be written as

$$\begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}$$

(always true for matrices, but be careful when inf~~e~~initely dimensional)

Def. Let $(G, U, \mathcal{H})$ be a group and a linear representation.

- A closed subspace $\mathcal{H}_0 \subset \mathcal{H}$ is INVARIANT under the representation if
  $U(a)\mathcal{H}_0 \subset \mathcal{H}_0$ ($\Leftrightarrow \forall f \in \mathcal{H}_0 : U(a)f \in \mathcal{H}_0$) $\exists a \in G$
- $\mathcal{H}_0$ is PROPER if $\mathcal{H}_0 \neq \mathcal{H}$ and NON-TRIVAL if $\mathcal{H}_0 \neq \{0\}$
- $\mathcal{H}_0$ is MINIMAL if $\nexists \mathcal{H}_1 : \{0\} \neq \mathcal{H}_1 \subsetneq \mathcal{H}_0$ with $\mathcal{H}_1$ invariant
- $(U, \mathcal{H})$ (≡ the linear representation) is IRREDUCIBLE if
  $\{0\}$ and $\mathcal{H}$ are the only invariant closed subspaces.
  (≡ $\mathcal{H}$ is minimal) Otherwise it is REDUCIBLE.

Lemma: If G is finite, and $(U, \mathcal{H})$ is an irreducible representation of G, then $\dim \mathcal{H} \leq |G|$. Proof as exercise

Observasion: Consider $(G, U, \mathcal{H})$ and $\mathcal{H}_0 \subset \mathcal{H}$ invariant, then

$$\forall a \in G : U(a) = \begin{pmatrix} U(a)_{00} & U(a)_{01} \\ 0 & U(a)_{11} \end{pmatrix} \text{ in } \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$$

↱ not implying the existance of $\mathcal{H}_0$

Def. $(G, U, \mathcal{H})$ is COMPLETELY REDUCIBIE if $\forall \mathcal{H}_0 \subset \mathcal{H}$ invariant one has

$$U(a) = \begin{pmatrix} U(a)_{00} & 0 \\ 0 & U(a)_{11} \end{pmatrix} \forall a \in G \quad (\Leftrightarrow \mathcal{H}_0^{\perp} \text{ is invariant too})$$

Thm: 1) If $(G, U, \mathcal{H})$ is a unitary representation then it's completely reducible.

2) If G is finite then $(G, U, \mathcal{H})$ is completely reducible.

Shrur Lemma

Let $(G, U, \mathcal{H})$ be a finite dimensional irreducible rep of G,

$(U', \mathcal{H}')$ be another (maybe $\dim = \infty$) irreducible rep of G.

Let $Z : \mathcal{H} \to \mathcal{H}'$ be linear satisfying $ZU(a) = U'(a)Z \ \forall a \in G$

Then either $Z = 0$ or Z is a similarity transformation. ($\Rightarrow U$ and U' are similar)

Corollory

1) Let $(G, U, \mathcal{H})$ be a finite dimensional irreducible rep of G

Let $T \in \mathcal{B}(\mathcal{H})$ s.t. $U(a)T = TU(a) \ \forall a \in G$

Then $T = \lambda \mathbb{1} \ \exists \lambda \in \mathbb{C}$. (and λ is the eigenvalue of T)

Proof

Since $\dim \mathcal{H} < \infty$, T is a matrix and has at least one eigenvalue λ.

Then $(T - \lambda\mathbb{1})U(a) = U(a)(T - \lambda\mathbb{1}) \ \forall a \in G$ ↳$\Rightarrow (T - \lambda\mathbb{1})v = 0$ $\Rightarrow$ not inversible

By Shrur Lemma, since $T - \lambda\mathbb{1}$ cannot be a similarity transformation, (since $T - \lambda\mathbb{1}$ is not inversible), one has

$T - \lambda\mathbb{1} = 0$ □

2) If G is abelian, any finite dimensional inreducible rep of G is of dimension 1. (Proof as exercise)

Prop: Let G be a finite group and G_0 an abelian subgroup.

Then any finite dim. inreducible rep. of G is of dimension $\leq \frac{|G|}{|G_0|}$

II.3 Representations of Finite Group

Prop. $(U,\mathcal{H})$ $(U',\mathcal{H}')$ be 2 lin. rep. of a finite group G, and let

$T : \mathcal{H} \mapsto \mathcal{H}'$ bounded

$$Z_T := \frac{1}{|G|} \sum_{a \in G} U'(a) T U(a)^{-1}$$

Then

1) If $(U,\mathcal{H}) \overset{\text{not similar}}{\not\sim} (U',\mathcal{H}')$ then $Z_T = 0$

2) If $(U,\mathcal{H}) = (U',\mathcal{H}')$, then $Z_T = \frac{1}{n} \overset{\text{trace}}{\mathrm{tr}}(T) \mathbb{1}$ with $n = \dim \mathcal{H}$.

Proof: 1) One checks $U'(b) Z_T = Z_T U(b) \quad \forall b \in G$

By Schur Lemma $\Rightarrow Z_T = 0$.

2) By the previous corollary, $Z_T = \lambda \mathbb{1}$. Then

$$\mathrm{tr}(Z_T) = \lambda \, \mathrm{tr}(\mathbb{1}) = \lambda n$$

$$\parallel$$

$$\mathrm{tr}\Big(\frac{1}{|G|} \sum_{a \in G} U(a) T U(a)^{-1}\Big) = \frac{1}{|G|} \sum_{a \in G} \mathrm{tr}(T \cancel{U(a)^{-1}} \cancel{U(a)}) = \frac{1}{|G|} \sum_{a \in G} \mathrm{tr}(T) = \mathrm{tr}(T)$$

$\Rightarrow \lambda = \frac{\mathrm{tr}(T)}{n} \Rightarrow Z_T = \frac{1}{n} \mathrm{tr}(T) \mathbb{1}$ □

Let $(U^k, \mathcal{H}^k)$ be a unitary and irreducible representation of G,

and $\{\gamma^k\}_k$ with $\gamma^k := \underbrace{[(U^k, \mathcal{H}^k)]}_{\leftarrow \text{equivalence class}}$ be an enumeration of such rep.

Let $\{e_j^k\}_{j=1}^{n_k}$ be $\underbrace{\text{a basis}}_{\text{orthonormal}}$ of $\mathcal{H}^k$, with $n_k = \dim \mathcal{H}^k$

Let us set

$U_j^k(a) = \langle e_i^k, U^k(a) e_j^k \rangle$

and $(U_{ij}(a))_{a\in G} \in \ell^2(G) \longrightarrow := \{ \overbrace{(a_1, \cdots, a_{|G|})}^{\overset{\mathbf{a}}{\shortparallel}} \mid a_j \in \mathbb{C}, \sum_{j=1}^{|G|} |a_j|^2 < \infty \}$

$\ell^2(G)$ is a $\mathcal{H}$ with $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{|G|} \overline{a_j} b_j$. $\dim \ell^2(G) = |G| =: g$

Consider $T := |e_s^\ell\rangle\langle e_j^k| \in \mathcal{B}(\mathcal{H}^k, \mathcal{H}^l)$

If $l \neq k$ then $Z_T = 0$

$$\Rightarrow 0 = \langle \underline{e_r^\ell}, Z_T \underline{e_i^k} \rangle = \frac{1}{g} \sum_a \langle \underline{e_r^\ell}, U^\ell(a) e_s^\ell \rangle \langle e_j^k, U^k(a)^{-1} \underline{e_i^k} \rangle$$

$$= \frac{1}{g} \sum_a U_{rs}^\ell(a) \overline{U_{ij}^k(a)}$$

$$\Rightarrow (U_{ij}^k(a))_{a\in G} \perp (U_{rs}^\ell(a))_{a\in G}$$

If $l = k$ then $\underline{Z_T = \frac{1}{n_k}} \mathrm{tr}(|e_s^k\rangle\langle e_j^k|) \mathbb{1} = \frac{1}{n_k} \delta_{sj} \mathbb{1}$

$$\Rightarrow \frac{1}{g} \sum_a U_{rs}^\ell(a) \overline{U_{ij}^k(a)} = \langle e_r^k, Z_T e_i^k \rangle = \frac{1}{n_k} \delta_{sj} \delta_{ri}$$

In summary, $\frac{1}{g} \sum_a U_{rs}^\ell(a) \overline{U_{ij}^k(a)} = \frac{1}{n_k} \delta_{k\ell} \delta_{sj} \delta_{ri}$ (∗)

Corollary: $\exists N < \infty$ inequivalent unitary irreducible representations of G

with $\sum_{k=1}^{N} r_k^2 = g$

Proof of $\leq$

For each rep. $(U^k, \mathcal{H}^k)$ one has n_k^2 elements of $\ell^2(G)$ which are orthogonal $\rightarrow$ of dimension g

$\Rightarrow \sum_{k=1}^{N} r_k^2 \leq \dim \ell^2(G) = g$ □

For any finite dimensional representation of G we set $\chi(a) := \mathrm{tr}(U(a))$

$\{\chi(a)\}_{a\in G}$ is called the set of the characters of G. in $\mathcal{H}$.

Again $\chi(\cdot) \in \ell^2(G)$

Corollary: Let $(U^k, \mathcal{H}^k)$ $(U^\ell, \mathcal{H}^l)$ be 2 unitary irreducible rep of G then

$$\frac{1}{g} \sum_a \chi^l(a) \overline{\chi_k(a)} = \begin{cases} 1 & \text{if } (U^k, \mathcal{H}^k) \sim (U^\ell, \mathcal{H}^\ell) \\ 0 & \text{otherwise} \end{cases}$$

Proof

Observe that the character depends only on the equivalent class of rep.

and not on the representative.

Since $\chi^k(a) = \sum_{j=1}^{n_k} U^k_{jj}(a)$ one has

$$\frac{1}{g}\sum_{a\in G}\chi^\ell(a)\,\overline{\chi^k(a)} = \underbrace{\frac{1}{g}\sum_{a\in G}}\sum_{j=1}^{n_k}\sum_{r=1}^{n_\ell}\underbrace{U^\ell_{rr}(a)\,\overline{U^k_{jj}(a)}} = \sum_{j=1}^{n_k}\frac{1}{n_k}\delta_{k\ell} = \delta_{k\ell} \qquad \square$$

$$\hookrightarrow \frac{1}{n_k}\delta_{k\ell}\,\delta_{rj}\,\delta_{rj} \leftarrow$$

If $(U,\mathcal{H})$ is a completely reducible and finite dim rep of G, then

$$\mathcal{H} = \bigoplus_{k=1}^{N} \nu_k \mathcal{H}^k \qquad , U = \bigoplus_{k=1}^{N} \nu_k U^k$$

$\uparrow$ modulo rearrangement $\qquad \hookrightarrow \in \mathbb{N}$ $\qquad \downarrow$ multiplicity of the representation of type η^k

$$U = \begin{pmatrix} \boxed{U_1} & & & \\ & \boxed{U_1} & & \\ & & \boxed{U_2} & \\ & & & \boxed{U_5} \end{pmatrix} \quad \begin{array}{l} \nu_1 = 2 \\ \nu_2 = \nu_5 = 1 \\ \nu_3 = \nu_4 = 0 \end{array}$$

Thm. Let $(U,\mathcal{H})$ be a finite dim rep of G, then

1) $\nu_k := \frac{1}{g}\sum_{a\in G}\overline{\chi(a)}\,\chi^k(a)$ ← character in rep $\mathcal{H}^k$
2) This rep is irreducible iff $\frac{1}{g}\sum|\chi(a)|^2 = 1$
3) If $(U',\mathcal{H}')$ is another finite dim rep of G, then $(U,\mathcal{H})\sim(U,\mathcal{H})$ iff their χ are equal.

We introduce the regular representation of G

Def. Let G be finite group and set $\mathcal{H}^{reg} := \ell^2(G)$
and $[U^{reg}(a)f](b) = f(a^{-1}b)$ for $f \in \mathcal{H}^{reg}$

Exercise: Check if it is a representation.

$(U^{reg},\mathcal{H}^{reg})$ is called the regular representation.

This representation is completely reducible since G is finite.

$\Rightarrow \mathcal{H}^{reg} = \bigoplus_{k=1}^{N}\nu_k\mathcal{H}^k$, $U^{reg} = \bigoplus_{k=1}^{N}\nu^k U^k$ and $\sum_{k=1}^{N}\nu_k n_k = g = \dim\mathcal{H}^{reg}$

Let us define $\delta_a \in \ell^2(G)$ by $\delta_a(b) = 1$ if $a=b$ and $\delta_a(b)=0$ otherwise
and $\{\delta_a\}_{a\in G}$ is an orthonormal basis of $\mathcal{H}^{reg}$

One has $U^{reg}_{bc}(a) = \langle\delta_b, U^{reg}(a)\delta_c\rangle = \langle\delta_b, \delta_c(a^{-1}\bullet)\rangle = \sum_{d\in G}\delta_b(d)\delta_c(a^{-1}d) = \delta_c(a^{-1}b) = \begin{cases}1 & \text{if } c=a^{-1} \\ 0 & \text{otherwis}\end{cases}$

In particular if $b=c$, $U^{reg}_{bb}(a) = \begin{cases}1 & \text{if } b=a^{-1}b \\ 0 & \text{otherwise}\end{cases} = \begin{cases}1 & \text{if } a=e \\ 0 & \text{otherwise}\end{cases} \Rightarrow \chi^{reg}(a) = \begin{cases}g & \text{if } a\in e \\ 0 & \text{otherwise}\end{cases}$

Thm. 1) $\sum_{k=1}^{N} n_k^2 = g$
2) $U^{reg} = \bigoplus_{k=1}^{N} n_k U^k$
$\hookrightarrow$ multiplicity = $\dim\mathcal{H}^k$ (only true for regular rep)

Proof: $\nu_k = \frac{1}{g}\sum_{a\in G}\overline{\chi^{reg}(a)}\,\chi^k(a) = \frac{1}{g}\,g\,\chi^k(e) = n_k$ $\qquad \square$

⇒ The regular representation contains all irreducible representations.

Lemma

Let $C^1, \cdots, C^M$ be the list of conjugacy classes of finite group G with $d_\ell := |C^\ell|$. Then

$$\frac{1}{g} d_\ell \sum_{k=1}^{N} \underbrace{\overline{\chi^k(C^\ell)}}_{\rightarrow \text{character on any element of } C^\ell \text{ (same)}} \chi^k(C^i) = \delta_{i\ell}$$

Thm. For any finite group $N = M$

Proof:

$$\sum_{\ell=1}^{M} \frac{1}{g} d_\ell \sum_{k=1}^{N} |\chi_k(C^\ell)|^2 = M$$

Also

$$\sum_{a\in G} = \sum_{\ell=1}^{M} \sum_{a\in C^\ell}, \text{ then}$$

$$N = \sum_{k=1}^{N} \frac{1}{g} \sum_{a\in G} \overline{\chi^k(a)}\, \chi^k(a) = \sum_{k=1}^{N} \frac{1}{g} \sum_{\ell=1}^{M} \sum_{a\in C^\ell} |\chi^k(a)|^2 = \sum_{k=1}^{N} \frac{1}{g} \sum_{\ell=1}^{M} d_\ell |\chi^k(C^\ell)|^2$$

$$= \sum_{\ell=1}^{M} \frac{1}{g} d_\ell \sum_{k=1}^{N} |\chi^k(C^\ell)|^2 = M \qquad \square$$

II.4 Tensor Product of Representations

Let $\mathcal{H}_1, \mathcal{H}_2$ 2 Hilbert space, let $\varphi_1 \in \mathcal{H}_1$; $\varphi_2 \in \mathcal{H}_2$

Set $\varphi_1 \otimes \varphi_2 : \mathcal{H}_1 \times \mathcal{H}_2 \mapsto \mathbb{C}$, $\varphi_1 \otimes \varphi_2 (\psi_1, \psi_2) := \langle \psi_1, \varphi_1 \rangle_{\mathcal{H}_1} \langle \psi_2, \varphi_2 \rangle_{\mathcal{H}_2}$

Then $\varphi_1 \otimes \varphi_2$ is a bi-antilinear map.

($\Leftrightarrow \varphi_1 \otimes \varphi_2 (\psi_1 + \lambda \psi_1', \psi_2) = \varphi_1 \otimes \varphi_2 (\psi_1, \psi_2) + \bar{\lambda} \varphi_1 \otimes \varphi_2 (\psi_1', \psi_2)$ and same for ψ_2)

Let ε be the set of finite linear combination of $\varphi_1 \otimes \varphi_2$ $\left(= \left\{ \sum_{j=1}^{N} \lambda_j \varphi_{1,j} \otimes \varphi_{2,j} \middle| N \in \mathbb{N}^*, \lambda_j \in \mathbb{C} \right\}\right)$

(Observe ε is a Hilbert space)

and define the scalar product $\langle \varphi_1 \otimes \varphi_2, \varphi_1' \otimes \varphi_2' \rangle_\varepsilon = \langle \varphi_1, \varphi_1' \rangle_{\mathcal{H}_1} \langle \varphi_2, \varphi_2' \rangle_{\mathcal{H}_2}$

Lemma: $\langle \bullet, \bullet \rangle_\varepsilon$ is well-defined and positive definite.

$\hookrightarrow :\Leftrightarrow \langle \varphi_1 \otimes \varphi_2, \varphi_1 \otimes \varphi_2 \rangle_\varepsilon \geq 0$

Def. $\boxed{\mathcal{H}_1 \otimes \mathcal{H}_2}$ is the completion of ε with respect to the norm associated with $\langle \cdot, \cdot \rangle_\varepsilon$.

$\hookrightarrow$ called the tensor product of $\mathcal{H}_1$ with $\mathcal{H}_2$.

Lemma: If $\{\varphi_{1,j}\}_j$ & $\{\varphi_{2,k}\}_k$ are orthonormal bases of $\mathcal{H}_1$ & $\mathcal{H}_2$, respectively, then $\{\varphi_{1,j} \otimes \varphi_{2,k}\}_{j,k}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Let $A_j \in \mathcal{B}(\mathcal{H}_j)$, and set $A_1 \otimes A_2 \in \underbrace{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$ with $\rightarrow$ Exercise: Proove it's bounded

$(A_1 \otimes A_2)(\varphi_1 \otimes \varphi_2) := A_1 \varphi_1 \otimes A_2 \varphi_2$ and then by linearlity we define $A_1 \otimes A_2$ on any element of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

If $A_1 \otimes A_2$, $B_1 \otimes B_2 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_2)$ Then $(A_1 \otimes A_2)(B_1 \otimes B_2) = A_1 B_1 \otimes A_2 B_2$

and if $\mathcal{H}_1$ & $\mathcal{H}_2$ are finite dimensional then $\mathrm{tr}(A_1 \otimes A_2) = \mathrm{tr}_{\mathcal{H}_1}(A_1) \mathrm{tr}_{\mathcal{H}_2}(A_2)$

Let $(G, U, \mathcal{H})$ $(G', U', \mathcal{H}')$ be 2 representations of 2 groups

For $(a, a') \in G \otimes G'$ we set $\mathcal{U}(a, a') = U(a) \otimes U(a') \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$

Then $(G \otimes G, \mathcal{U}, \mathcal{H} \otimes \mathcal{H}')$ is a linear representation

And if $\mathcal{H}, \mathcal{H}'$ are finite dimensional, $\chi_{\mathcal{U}}((a, a')) = \chi_{\mathcal{U}}(a) \chi_{\mathcal{U}}(a')$

Prop. If $(G, U, \mathcal{H})$ $(G', U', \mathcal{H}')$ are irreducible representations of finite groups then

1) $(G \otimes G', \mathcal{U}, \mathcal{H} \otimes \mathcal{H}')$ is irreducible.

2) All irreducible rep. of $G \otimes G$ is of this form.

Consider now the rep of a single group.

Let $\overset{\mathcal{U}:}{} G \ni a \overset{\forall}{\mapsto} U(\underline{a}) \otimes U'(\underline{a}) \in \mathcal{H} \mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$

This defines a rep of G, and if U, U' are irreducibe then $(\mathcal{U}, \mathcal{H} \otimes \mathcal{H}')$ might be reducible

If $(\mathcal{U},\mathcal{H})$ $(\mathcal{U}',\mathcal{H}')$ are finite dim, then since $\chi_{\mathcal{U}}(a) = \chi_{U}(a)\ \chi_{U'}(a)$ one gets $U \otimes U'$ is equivalent to $U' \otimes U$. The decomposition of $\mathcal{U} = \bigoplus_k \nu_k \mathcal{U}^k$ can be computed with $\nu_k = \frac{1}{g} \sum_a \overline{\chi_U(a)\ \chi_{U'}(a)}\ \chi^k(a)$

Remark

Consider 2 irreducible representation $(U^k, \mathcal{H}^k)(U^j, \mathcal{H}^j)$

Since $\mathcal{H}^j \otimes \mathcal{H}^k = \bigoplus_\ell \nu_\ell \mathcal{H}^\ell$ and $U^j \otimes U^k = \bigoplus_\ell \nu_\ell U^\ell$

One can express a suitable basis of $\bigoplus_\ell \nu_\ell \mathcal{H}^\ell$ in terms of the basis $\{e^j_r \otimes e^k_s\}_{r,s}$

The coeffecients relate to the change of basis are called the

CLEBSCH-GORDAN COEFFICIENTS.

Selection Rules

Let G be a group and $(U,\mathcal{H})$ one representation.

Let $\mathcal{U}: G \mapsto \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ (automorphism 自同構)

with $\mathcal{U}(a)T := U(a)TU(a)^{-1}$ ~~for~~ $\forall\ T \in \mathcal{B}(\mathcal{H})$

one has $\mathcal{U}(ab) = \mathcal{U}(a)\mathcal{U}(b)$, $\mathcal{U}(e)T = \mathcal{U}(e)T\mathcal{U}(e)^{-1} = T$

We have a linear rep of G on a vector space $\mathcal{B}(\mathcal{H})$

⚠ $\mathcal{B}(\mathcal{H})$ is not a Hilbert space!

If $\mathcal{H}$ is $\overset{\Leftrightarrow \mathcal{H}=\mathbb{C}^n}{\text{finite}}$ then $\overset{=\mathbb{C}^{n\times n}}{\mathcal{B}(\mathcal{H})}$ is of finite dimension n^2 and then

$\mathcal{B}(\mathcal{H})$ can be decomposed $= \bigoplus_k \nu_k \mathcal{L}^k$ and $\mathcal{U} = \bigoplus_k \nu_k \mathcal{U}^k$

is the decomposition into irreducible rep. with $(\mathcal{U}^k, \mathcal{L}^k) \in \imath^k = [(U^k, \mathcal{H}^k)]$

It means $\exists \phi: \mathcal{H}^k \to \mathcal{L}^k$ bjective: $\phi(U^k(a)f) = \mathcal{U}^k(a)\underset{\hookrightarrow \in \mathcal{L}^k}{\phi(f)} = U(a)\phi(f)U(a)^{-1}$

Thus we have a decomposition of $\mathcal{B}(\mathcal{H})$

which is based on irreducible rep. of G.

Thm (Selection Rule)

Let $(G, U, \mathcal{H})$ be a unitary rep and consider $\mathcal{H} = \bigoplus_k \nu_k \mathcal{H}^k$, $U = \bigoplus_k \nu_k U^k$

Let $(G, U^j, \mathcal{H}^j)$ be one irred. rep. of G and

Let $\phi: \mathcal{H}^j \mapsto \mathcal{B}(\mathcal{H})$ with $\phi(U^j(a)f_j) = U(a)\phi(f_j)U(a)^{-1}\ \forall f_j \in \mathcal{H}^j$

Then $\forall f_k \in \mathcal{H}^k \subset \mathcal{H}\ \forall f_i \in \mathcal{H}^i \subset \mathcal{H}: \langle f_i, \phi(f_j) f_k \rangle = 0$

EXCEPT if $(\mathcal{H}^i, U^i)$ appears in the decomposition of $(\mathcal{H}^j \otimes \mathcal{H}^k)$ & $(U^j \otimes U^k)$

in a sum of irred. rep.

Remark: This result is also related to the Clebsch-Gordan coef. (Wigner-Eckant Thm.)

II.5 Symmetries and Projective Representation

Consider $\mathcal{H}$ a Hilbert space, and $\hat{\mathcal{H}} := \mathcal{H}/\mathbb{C}$

which means $\hat{\mathcal{H}} \ni \hat{\psi} = \{ \psi \in \mathcal{H} \mid \varphi = \alpha \psi \ \exists \alpha \in \mathbb{C} \}$

This interest of $\hat{\mathcal{H}}$ is that its elements are in bijection with all pure states,

which means with all 1D prejection of the form

$$P_{\hat{\psi}} = |\varphi\rangle\langle\varphi| \text{ with } \varphi \in \hat{\psi} \text{ and } \|\varphi\| = 1.$$

For 2 such pure states $P_{\hat{\psi}}$ and $P_{\hat{\varphi}}$

the transition probabily is defined by

$$Tr(P_{\hat{\psi}} P_{\hat{\varphi}}) = |\langle\varphi, \psi\rangle|^2 \text{ with } \varphi \in \hat{\varphi},\ \psi \in \hat{\psi},\ \|\varphi\| = 1 = \|\psi\|$$

Def. A SYMMETRY is a map

$S : \hat{\mathcal{H}} \mapsto \hat{\mathcal{H}}$ satisfying

$$Tr(P_{S\hat{\psi}} P_{S\hat{\varphi}}) = Tr(p_{\hat{\psi}} P_{\hat{\varphi}})$$

Consider U a unitary operator $\overset{\text{in}}{\cancel{\text{on}}}$ $\mathcal{H}$ ($\Leftrightarrow$ $U \in B(\mathcal{H}), U^* = U^{-1}$)

And set $S_U : \hat{\mathcal{H}} \mapsto \hat{\mathcal{H}}$, $S_U \hat{\varphi} = \widehat{U\psi}$ with $\psi \in \hat{\psi}$

Then S_U is a symmetry. Indeed

Take $f = \alpha U \psi$ with $\cancel{1}\, 1 = \|f\| = |\alpha| \, \|U\psi\| = |\overset{\alpha}{\cancel{\beta}}| \, \|\psi\|$

$g = \beta U \varphi$ with $1 = \|g\| = |\beta| \, \|\varphi\|$

Then $T_2(P_{S\hat{\varphi}} P_{S\hat{\varphi}}) = |\langle f, g\rangle|^2 = |\langle \alpha U\psi, \beta U\varphi\rangle| = |\langle \alpha\psi, \beta\varphi\rangle| = T_r(P_{\hat{\varphi}} P_{\hat{\varphi}})$

The same holds if U is anti-unitary

(which means $U(f + \alpha g) = Uf + \bar{\alpha} Ug$, $\langle Uf, Ug\rangle = \overline{\langle f, g\rangle}$

Thm. (Wigner's Thm)

Let $S : \hat{\mathcal{H}} \mapsto \hat{\mathcal{H}}$ be a symmetry.

Then $\exists U : \mathcal{H} \mapsto \mathcal{H}$ ~~with~~ which is either unitary or anti-unitary s.t. $S = S_U$.

U is unique $\overset{\text{(up to)}}{\text{modulo}}$ $\alpha \in \mathbb{C}$ with $|\alpha| = 1$,

since U and αU define the same symmetry.

Now consider a group of symmetries, which means a

homomorphism $S : G \mapsto \{\text{symmetries on } \mathcal{H}\}$

(each $S(a) : \mathcal{H} \mapsto \mathcal{H}$)

s.t. $S(a)S(b) = S(ab)$ $\forall a, b \in G$; $S(e) = \mathbb{1}$

By Wigner's Thm, $\forall a \in G \, \exists U(a)$ unitary or anti-unitary s.t. $S(a) = S_{U(a)}$.

Suppose all $U(a)$ are unitary.

Natual question: $U(a)U(b) = U(ab)$? No in general (problem of phase)

↳ Projective projections

Indeed if we fix $U(\cdot)$ we usually have

$U(a)U(b) = \omega(a,b) U(ab)$ with $\omega(a,b) \in \mathbb{C}$, $|\omega(a,b)| = 1$

and if we set $U'(a) = \rho(a) U(a)$ with $\rho : G \mapsto \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$

then $U'(a)U'(b) = \rho(a)U(a)\rho(b)U(b) = \rho(a)\rho(b)\omega(a,b)U(ab)$

$$\downarrow \qquad = \frac{\rho(a)\rho(b)}{\rho(ab)} \omega(a,b) \, U'(ab)$$

$$= \omega'(a,b) \, U'(ab)$$

$$\Rightarrow \omega'(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)} \omega(a,b)$$

Def. Let V be a complex vector space and let G be a group.
A PROJECTIVE REPRESENTATION ~~is~~ of G in V is a map
$U : G \mapsto GL(V)$ ($\rightarrow$ set of invertible operators in V) s.t.
$U(a)U(b) = \omega(a,b)\, U(ab)$ with $\omega(a,b) \in \mathbb{C}^*$ and
$U(e) = 1$. called 2-cocycle (Ma) or Schur (Ph) multiplier.

Def. Two 2-cocycles $\omega, \omega' : G \times G \mapsto \mathbb{C}^*$ are EQUIVALENT if
$\exists \rho : G \mapsto \mathbb{C}^* : \omega'(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}\, \omega(a,b) \ \forall a,b \in G$
ω is TRIVIAL if it is equivalent to 1, which means
$\exists \rho : G \mapsto \mathbb{C}^* : \omega(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}$

Remarks

- Projective representations are essential in QM.
- A linear representation ~~if~~ is a special case of a proj. rep.
- If ω is trivial, then by setting $U'(a) = \rho(a)^{-1}U(a)$ we get a linear representation. Indeed,

$$U'(a)U'(b) = \underbrace{\rho(ab)\rho(a)^{-1}\rho(b)^{-1}\omega(ab)}_{\hookrightarrow = 1}\rho(ab)^{-1}U(ab) = U'(ab)$$

- If $(U, \mathcal{H}), (U', \mathcal{H})$ are proj. rep. and if $U'(a) = \rho(a)U(a)$ $\exists \rho : G \mapsto \mathbb{C}^*$ then the 2 proj. rep. are EQUIVILENT.

Question: Can one always trivialize a 2-cocycle?
($\Leftrightarrow$ Do we always have $\exists \rho : G \mapsto \mathbb{C}^* : \omega(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}$?) No
For a given group, the answer is in the study of GROUP COHOMOLOGY.

What is often useful is to consider a larger group and its lin. rep.

Example: Consider G and a second group $\tilde{G}$ with a normal subgroup $\tilde{G}_0$ s.t.
$\tilde{G}/\tilde{G}_0 \simeq G$ (Let's denote by $\phi : \tilde{G}/\tilde{G}_0 \mapsto G$ the isomorphism)
Let $(\tilde{U}, \mathcal{H})$ be a linear rep. of $\tilde{G}$ s.t.
$\forall a \in \tilde{G}_0 : \tilde{U}(a) = \sigma(a)\mathbb{1}$ $\exists \sigma(a) \in \mathbb{C}^*$ ($\Leftrightarrow \tilde{U}(\tilde{G}_0) \subset \mathbb{C}^*\mathbb{1}$)
$\forall a \in G$ let's choose $\tilde{a} \in \tilde{G}$ with $\phi([\tilde{a}]_{G_0}) = a$ and ~~$U : G \mapsto \mathcal{B}(\mathcal{H})$ with $U(a) :=$~~
$U : G \mapsto \mathcal{B}(\mathcal{H})$ with $U(a) := \tilde{U}(\tilde{a})$.
Then U defines a projective rep. of G.

Indeed let $a, b \in G$ and set $d := ab$. Con~~d~~sider $\tilde{a}, \tilde{b}, \tilde{d}$ in $\tilde{G}$. Then

$$\phi([\tilde{d}]_{\tilde{G}_0}) = d = ab = \phi([\tilde{a}]_{\tilde{G}_0})\, \phi([\tilde{b}]_{\tilde{G}_0}) = \phi([\tilde{a}]_{\tilde{G}_0}[\tilde{b}]_{\tilde{G}_0}) = \phi([\tilde{a}\tilde{b}]_{\tilde{G}_0})$$

$\Rightarrow$ $[\tilde{d}]_{\tilde{G}_0} = [\tilde{a}\tilde{b}]_{\tilde{G}_0}$ (for ϕ is an isomorphism $\Rightarrow$ bijective)

$\Rightarrow \exists c \in \tilde{G}_0 : \tilde{a}\tilde{b} = c\tilde{d}$ (for they are the same equivalent class)

↳ depends on a and b

Finally $\underline{U(a)U(b)} = \tilde{U}(\tilde{a})\tilde{U}(\tilde{b}) = \tilde{U}(\tilde{a}\tilde{b}) = \tilde{U}(c\tilde{d}) = \tilde{U}(c)\tilde{U}(\tilde{d})$ ~~$= \sigma(c)U(\tilde{d})$~~

$$= \sigma(c)\tilde{U}(\tilde{d}) = \sigma(c)U(d) = \underline{\sigma(c)U(ab)} \quad \exists\, \sigma(c) \in C^*$$ □

Remark: If $\tilde{U}$ is unitary then U is also unitary.

Question: Can we always do this construction? No

but if $\tilde{G}$ exists ($\Leftrightarrow$ any proj. rep. of G is induced by a lin. rep. of $\tilde{G}$)

then we call $\tilde{G}$ the UNIVERSAL COVER of G.

(Important in many places)

Remark: If ~~f~~ G is finite, a universal cover exists.

The universal cover of $SO(3)$ is $SU(2)$.

We are going to constract the universal cover of finite groups.

Observe that if $U(a)U(b) = \omega(a,b)U(ab)$

and $a=e$ or $b=e$, then we know $\omega(a,e) = \omega(e,b) = 1$ ①

Also $U(a)U(b)U(c) = \omega(a,b)U(ab)U(c) = \omega(a,b)\omega(ab,c)U(abc)$

↳ $= U(a)\omega(b,c)U(bc) = \omega(a,bc)\omega(b,c)U(abc)$

$\Rightarrow \omega(a,b)\omega(ab,c) = \omega(a,bc)\omega(b,c)$ (2-cocycle relation) ②

Def. A 2-COCYCLE on G is map $\omega : G \times G \mapsto \mathbb{C}^*$ satisfying ①②.

$\omega \sim \omega'$ (EQUIVALENCE) if $\exists \rho : G \mapsto C^* : \omega'(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}\omega(a,b)$

and we denote by $[\omega]$ the equivalence class containing ω.

We can check $[\omega][\ddot{\omega}] := [\omega\ddot{\omega}]$ defines a PRODUCT ON THE EQUIVALENCE CLASSES of 2-cocycles on G.

Prop. 1) $\{[\omega]\}$ with the above multiplication is a Abelian group, denoted by $M(G)$

called the SCHUR MULTIPLIER or the SECOND COHOMOLOGY GROUP $H^2(G, \mathbb{C}^*)$.

2) If G is finite, then $M(G)$ is also finite, and

$\forall [\omega] \in M(G)\ \exists \underline{\omega} \in [\omega], k \in \mathbb{N} : \underline{\omega}(a,b) \in \{e^{i2\pi n/k} \mid n \in \{0, 1, \cdots, k-1\}\}\ \forall a,b \in G$

↳ called the ORDER of $[\omega]$.

Proof as exercise

Let G be a $\overbrace{\text{group}}^{\text{finite}}$ and let $\overbrace{\xi, \eta, \zeta}^{\to \text{equivalence classes}}$ denote the elements of M(G) (Abelian)

Let $\{\xi_1, \cdots, \xi_{\tilde{n}}\}^{\subset M(G)}$ be ~~the~~ a minimal generating set of M(G), which means

$\forall \xi \in M(G) : \xi = \xi_1^{n_1} \xi_2^{n_2} \cdots \xi_{\tilde{n}}^{n_{\tilde{n}}} \quad \exists n_1, \cdots, n_{\tilde{n}} \in \mathbb{N}$

and one cannot do it with less elements.

For each ξ_s one sets K_s for the K(order) defind by Prop (*) for ξ_s.

and set $S_s := e^{2\pi i / k_s}$

In addition $\exists \underline{\omega_s} \in \xi_s$ s.t. $\omega_s(a,b) = \varepsilon_s^{\,n(a,b)} \quad \exists n : G \times G \mapsto \{0, 1, \cdots, k_s - 1\}$

By the 2-cocycles relations, one has

$$n(a,e) = 0 = n(e,b)$$
$$n(a,b) + n(ab,c) = n(a,bc) + n(b,c)$$

Finally we set $\phi : G \times G \mapsto M(G)$ by

$$\phi(a,b) = \prod_{s=1}^{\tilde{n}} \xi_s^{n_s(a,b)} \in M(G)$$

Thm. Let G be a finite group and M(G) its Schur multiplier

Let $\tilde{G} := \{(\xi, a) \in M(G) \times G\}$ with the product

$(\xi, a) \cdot (\eta, b) = (\xi \eta \phi(a,b), ab)$. ⊼

Then

1) $\tilde{G}$ with the above multiplication is a group;

2) $\tilde{G}_0 := \{(\xi, e) \mid \xi \in M(G)\} \triangleleft \tilde{G}$, and $\tilde{G}/\tilde{G}_0 \simeq G$. Proof as exercise

2) If $(U, \mathcal{H})$ is a proj. rep. of G then

$\exists p : G \mapsto C^*$ and

$(\tilde{U}, \mathcal{H})$ a linear rep. of G s.t. $U(a) = \overset{\downarrow \text{This defines an equivalent proj. rep}}{p(a)\tilde{U}([1], a)}$ Proof as exercise

III. Lie groups

III.1: Main Definitions and Properties

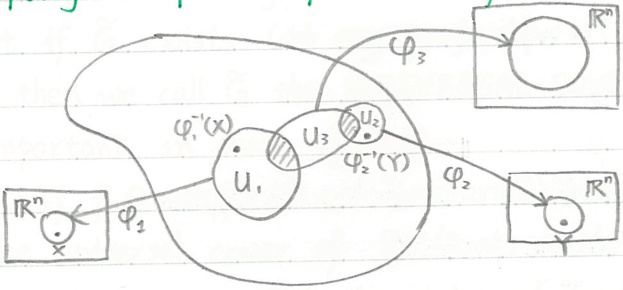
Examples: $O(3), SU(n), SO(25)$ are Lie groups.

Roughly, a lie group is a group together with a differiential structure compatible with the group operation and with taking the inverse.

Def. G is a SMOOTH MANIFOLD if G is a second-countable Hausdalf topologica space with local bijective and bi-continuous maps from the manifold to $\mathbb{R}$ s.t. $\varphi_j^{-1}\circ\varphi_i$ and $\varphi_i^{-1}\circ\varphi_j$ are C^∞ where def. ↳: M and M^{-1} are continuous

Hausdalf: 2 points always have disjoint neighborhood ⊗ ⊗

Topological space: Open sets defined



$\varphi_1^{-1}\circ\varphi_3$ defined on a subset of $\varphi_1(U_1)$

A manifold is not necessarily a group.

Also (U_j, φ_j) is called a LOCAL CHART, $\{(U_j,\varphi_j)\}$ with the compatibility ($\varphi_j^{-1}\circ\varphi_i, \varphi_i^{-1}\circ\varphi_j \in C^\infty$) conditions is called an ATLAS, and we have

$G = \bigcup_j U_j$.

$\Rightarrow$ Locally the manifold can be parameterized by n real parameters.

When G is also a group, and the map

$G\times G \ni (a,b) \mapsto ab \in G$ and $G \ni a \mapsto a^{-1} \in G$ are smooth,

It means $\forall (U_j, \varphi_j)$ for $j\in\{1,2,3\}$

1) $\Omega_{123} := \{(X,Y)\in\varphi_1(U_1)\times\varphi_2(U_2) \mid \varphi_1^{-1}(X)\varphi_2^{-1}(Y)\in U_3\} \subset \mathbb{R}^{2n}$ and then

$\Omega_{123} \ni (X,Y) \mapsto \varphi_3(\varphi_1^{-1}(X)\varphi_2^{-1}(Y)) \in \mathbb{R}^n$ is smooth;

2) $\Omega_{12} := \{X \in \varphi_1(U_1) \mid (\varphi_1^{-1}(X))^{-1} \in U_2\} \subset \mathbb{R}^n$ and then

$\Omega_{12} \ni X \mapsto \varphi_2((\varphi_1^{-1}(X))^{-1}) \in \mathbb{R}^n$ is smooth.

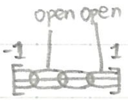
Then G is a LIE GROUP.

Examples

$(\mathbb{R},+), (\mathbb{R}_+,\cdot), (\mathbb{T},\cdot)$ are Lie groups, with $\mathbb{T} = \{z\in\mathbb{C} \mid |z|=1\}$

Def. G is a COMPACT LIE GROUP if it is a group and every cover of G by open sets admits a finite subcover.

For subsets of $\mathbb{R}^n$, compactness means bounded and open.



$\Rightarrow$ $(\mathbb{T}, \cdot)$ is a compact Lie group but $(\mathbb{R}, +)$ and $(\mathbb{R}_+, \cdot)$ are not.

A fundamental property of compact Lie group G:

Prop. $C(G) := \{f : G \mapsto \mathbb{C} \mid f \text{ is continuous}\}$

$\exists I : C(G) \mapsto \mathbb{C}$ s.t.

1) $I(f_1 + \alpha f_2) = I(f_1) + \alpha I(f_2)$ $\forall f_j \in C(G), \alpha \in G$ Linearity

2) $f \geqslant 0 \Rightarrow I(f) \geqslant 0$ and $f = 1 \Rightarrow I(f) = 1$ Positivity & Normalization

3) $I(f(a\cdot)) = I(f) = I(f(\cdot a))$ Invariance under Left & Right Multiplication

4) $I(f(\cdot^{-1})) = I(f)$ Invariance under Taking the Inverse

$\Rightarrow$ This map I corresponds to "integration" and is often denoted by

$\int_G f(a)\, da$ (called the Haar measure)

The Haar measure can be explicitely constructed locally if G is not compact (but ~~not~~ locally compact), and such a measure also exists but with less properties

($f = 1 \not\Rightarrow I(f) = 1$;

(3)(4) are not true and it is divided to left Haar & right Haar measures).

A Haar measure is very convenient because

$$\frac{1}{g} \sum_{a \in G} \underbrace{U(a)\, T\, U(a)^{-1}}_{\hookrightarrow\, \in \mathcal{B}(\mathcal{H})} \text{ (if } G \text{ is finite)} \rightsquigarrow \int_G \underbrace{U(a)\, T\, U(a)^{-1}}_{\hookrightarrow\, \in L(V) \text{ linear maps on a vector space}} da$$

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Date 2018 6 6

Lie groups and Lie algebras are useful in QM and in particle physics.

Lemma: Let $T: G \mapsto \mathcal{B}(\mathcal{H})$ with G a compact group, s.t.

this map is weakly continuous.

Weakly continuous $:\Leftrightarrow G \ni a \mapsto \overbrace{\langle f, T(a)g \rangle}^{\text{inner product on Hilbert space } \mathcal{H}} \in \mathbb{C}$ is continuous $\forall f, g \in \mathcal{H}$)

Then $\exists \mathbf{T} \in \mathcal{B}(\mathcal{H}): \langle f, \mathbf{T}g \rangle = \int_G \langle f, T(a)g \rangle \, da \ \forall f, g \in \mathcal{H}$.

It means that $\mathbf{T} = \int_G T(a) \, da$ ↘ different from the integral introduced in the last lecture? $\left\{ \begin{array}{l} \text{on } C(G) \\ \text{on representa} \end{array} \right.$

Idea of the proof:

Choose $\{f_i\}_{i \in \mathbb{N}}$ basis of $\mathcal{H}$ and compute $\alpha_{jk} := \int \langle f_i, T(a) f_k \rangle \, da$,

and we set $\langle f_i, \mathbf{T} f_k \rangle := \alpha_{jk}$. Then we should check $\mathbf{T}$ is a bounded operator.

To give a meaning of $\int_G U(a) T U(a)^{-1} \, da$

For Lie group G we shall consider unitary representations

(it means $\forall a \in G : U(a) \in \mathcal{B}(\mathcal{H})$ is unitary)

which are strongly continuous, it means $G \ni a \mapsto U(a) f \in \mathcal{H}$ is continuous

$:\Leftrightarrow \lim_{b \to a} \| U(b) f - U(a) f \| = 0 \ \forall f \in \mathcal{H}$

↑ When using ε-δ, δ can be dependent on f.

(If continuous on norm, δ is independent on f)

└ can be defined by the local charts

Corollary: If $U: G \mapsto \mathcal{U}(\mathcal{H})$ set of unitary operators on $\mathcal{H}$ is strongly continuous

Proof as exercise

and G a compact Lie group,

Then for any $T \in \mathcal{B}(\mathcal{H})$: the map $G \ni a \mapsto \overset{\text{SC}}{U(a)} \overset{\text{any}}{T} U(a)^{-1}$ is strongly continuou

($\Rightarrow$ weakly continuous) and

then $M_G(T) := \int U(a) T U(a)^{-1} \, da \in \mathcal{B}(\mathcal{H})$

$M_G(T)$ satisfies $U(b) M_G(T) U(b)^{-1} = M_G(T) \ \forall b \in G$

A deep thm (Peter-Weyl theorem (Part II))

Let $U: G \mapsto \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation

of a compact Lie group G.

Then $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$ with $\dim \mathcal{H}_n < \infty$ ¿Est-ce possible qu'il y a infini d'$\mathcal{H}_n$?

and $U|_{\mathcal{H}_n}$ is an irreducible representation of G.

$\Rightarrow$ All irreducible representations of a compact Lie group are finite dimensional.

We denote by $\eta_K = [(U_K, \mathcal{H}_K)]$ the equivalence class containing the irreducible representation $(U_K, \mathcal{H}_K)$. ~~We set~~

Prop. Let $(U, \mathcal{H})$ be any finite dim rep of G. proof as exercise

Recall the character $\chi(a) := \mathrm{tr}(U(a))$

1) $\mathcal{H} = \oplus \nu_K \mathcal{H}_K$ with $\nu_K = \int_G \overline{\chi(b)} \chi_K(b)\, db$

2) $(U, \mathcal{H})$ is irreducible iff $\int |\chi(b)|^2 db = 1$

3) $\int \overline{\chi_K(b)} \chi_\ell(b)\, db = \delta_{K\ell}$

4) $\int U(b)^{\ell}_{st} U(b)^{k}_{pq}\, db = \delta_{\ell k} \delta_{sp} \delta_{tq}$

These relations can still be interpreted as orthogonal relations in $L^2(G)$

$L^2(G) := \{ f : G \mapsto \mathbb{C} \mid \int |f(b)|^2 db < \infty \}$ (L^2-integrable on G)

Note that $\dim(L^2(G)) = \infty$

Now we pay attention to the props around the identity of a Lie group. (which leads to Lie-Algebra)

Recall that G is CONNECTED if any $a, b \in G$ can be connected by a continuous path, it means $\exists f : [0,1] \mapsto G$ continuous with $f(0) = a$, $f(1) = b$.

G is SIMPLY CONNECTED if any closed curve can be deformed to a point in G.

In a Lie group, the IDENTITY COMPONENT $G_0 \equiv G_e$ is the set of all elements of G which are connected to e.

Prop. 1) $G_0 \triangleleft G$ proof as exercise

2) If $(U, \mathcal{H})$ is a representation or a projective representation of G, Then G_0 is always represented by unitary operators.

Proof of 2)

$\forall a \in G_e : a = a_1^2 a_2^2 \cdots a_N^2$ for a finite family of elements in G_0. will be shown later

Then we observe $U(a_1^2) = w(a_1, a_1) U(a_1) U(a_1)$ with

$$U(a_1) \text{ is } \begin{cases} \text{unitary} & :\Leftrightarrow \langle Uf, Ug \rangle = \langle f, g \rangle \Rightarrow U(a_1)U(a_1) \text{ is unitary} \\ \text{or} & \\ \text{anti-unitary} & :\Leftrightarrow \langle Uf, Ug \rangle = \langle g, f \rangle \Rightarrow U(a_1)U(a_1) \text{ is unitary} \end{cases}$$

$\Rightarrow U(a)$ is unitary. □

III.2 Linear (or matrix) Lie groups and Lie algebra

⚠ Most (if not all) statements are true for general Lie groups, but the statements are more complicated.

Def. A LINEAR (or MATRIX) LIE GROUP is a Lie subgroup of $GL(n,\mathbb{C})$ or $GL(n,\mathbb{R})$

⚠ Even in $GL(n,\mathbb{C})$ we consider some parametrization with real coefficients. (up to $2n^2$ parameters)

Topology on $GL(n,\mathbb{C})$ is induced by the distance $(\|\cdot\|_2)$

$$d: GL(n,\mathbb{C})\times GL(n,\mathbb{C}),\ d(A,B) := \Big(\sum_{j,k=1}^{n} |a_{jk}-b_{jk}|^2\Big)^{\frac{1}{2}} \text{ with } A=(a_{jk}),\ B=(b_{jk})$$

Let G be a Lie subgroup of $GL(n,\mathbb{C})$ and let (U_0,φ_0) be a chart near $e=\mathbb{1}\in G$.

Suppose (for simplicity) $\varphi_0(e) = \mathbf{0}\in\mathbb{R}^m$. Let us set

$$Y_j := \lim_{\varepsilon\to 0}\frac{\varphi_0^{-1}(\varepsilon E_j)-\mathbb{1}}{\varepsilon} \in M_n(\mathbb{C}) \qquad \text{for } j=1,\cdots,m$$

Here the linear assumption plays a role:

$$(Y_j)_{\ell m} = [\partial_j \underbrace{(\varphi_0^{-1})_{\ell m}}_{\hookrightarrow :\ \mathbb{R}^m\to\mathbb{C}}](0)$$

Facts:
proof as exercise

$\hookrightarrow \alpha Y_j+\beta Y_k = 0 \,\&\&\, \alpha,\beta\in\mathbb{R}$
$\Leftrightarrow \alpha=\beta=0$

1) The matrices $Y_1,\cdots,Y_m$ are linearly independent (over $\mathbb{R}$)

(1 and i are linearly independent over $\mathbb{R}$ but not over $\mathbb{C}$)

but each Y_j can be made of complex numbers.

2) If $(-1,1)\ni t\mapsto X(t)\in\mathbb{R}^m$ with $X(0)=0$ is smooth

Then $(-1,1)\ni t\mapsto \varphi_0^{-1}\circ X(t)\in G$ is a smooth map and

$$\varphi_0^{-1}\circ X(0)=e \text{ and } \Big[\frac{d}{dt}\varphi_0^{-1}\circ X(t)\Big](0) = \sum_{j=1}^{m} Y_j X_j'(0)$$

It means that the derivative at 0 of any smooth curve in G passing to e at 0 is a linear combination (over $\mathbb{R}$) of $\{Y_j\}_{j=1}^m$.

The vector space generated by $\{Y_j\}$ is called the TANGENT SPACE at e.

Note: Tangent space is defined in any Lie group.

In compact Lie groups, like finite groups, all representations are equivelant to some unitary groups; but if the Lie group is not compact, not true.

So we consider not only unitary valued but more general representations with values in GL(V)

example: $(\mathbb{R},+) \ni x \mapsto e^x \in M_1(\mathbb{R})$

In Ⅲ,2 last time,

$$Y_j := \lim_{\varepsilon \to 0} \frac{\varphi_0^{-1}(\varepsilon E_j) - \mathbb{1}}{\varepsilon} \in M_n(\mathbb{C})$$

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$.

Def. A LIE ALGEBRA $\mathcal{L}$ over $\mathbb{K}$ is a (finite dim.) vector space over $\mathbb{K}$ with a composition $[\cdot,\cdot] : \mathcal{L} \times \mathcal{L} \mapsto \mathcal{L}$ s.t. $\forall X, Y, Z \in \mathcal{L}, \alpha, \beta \in \mathbb{K}$:

1) $[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z]$
2) $[X, Y] = -[Y, X]$
3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobian identity)

Remark: if $X, Y, Z \in M_n(\mathbb{C})$ and $[X, Y] := XY - YX$ (COMMUTATOR) then the 3 conditions are satisfied.

So we only have to check that

$X, Y \in \mathcal{L} \Rightarrow [X, Y] \in \mathcal{L}$

Def. Given a basis $\{Y_1, \cdots, Y_n\}$ of $\mathcal{L}$, the coefficients $\{c_{ij}^k\}_{i,j,k=1}^n$ defiend by $[Y_i, Y_j] = \sum_{k=1}^n c_{ij}^k Y_k$ are called STRUCTURE CONSTANTS or STRUCTURE COEFFICIENTS of $\mathcal{L}$.

$\Rightarrow c_{ij}^k = -c_{ji}^k$ and relation from Jacobian identity.

Lemma: Let G be a linear Lie group and $\mathcal{L}(G)$ be its tangent space. Then $\mathcal{L}(G)$ is a Lie algebra of the same dimension. (Proof as exercise)

Sketch of the proof: One just has to show $[Y_i, Y_j] \in \mathcal{L}(G)$.

Consider smooth curves

$(-1,1) \ni t \mapsto A(t) \in G$ with $A(0) = \mathbb{1}$ and $A'(0) = Y_i$
$(-1,1) \ni t \mapsto B(t) \in G$ with $B(0) = \mathbb{1}$ and $B'(0) = Y_j$

Consider $(-1,1) \ni t \mapsto A(\sqrt{t})B(\sqrt{t})A(\sqrt{t})^{-1}B(\sqrt{t})^{-1}$ (*)

Then $A(\sqrt{t}) = \mathbb{1} + \sqrt{t}Y_i + \cdots$, $B(\sqrt{t}) = \mathbb{1} + \sqrt{t}Y_j + \cdots$, $A(\sqrt{t})^{-1} = \mathbb{1} - \sqrt{t}Y_i + \cdots$, $B(\sqrt{t})^{-1} = \mathbb{1} - \sqrt{t}Y_j + \cdots$

Then $(*) = \mathbb{1} + t[Y_i, Y_j] + \cdots \Rightarrow (*)'(0) = [Y_i, Y_j]$ □

Important properties of G or $\mathcal{L}(G)$:

Some proofs are very nice but too long.

Let G be a Lie group and $\mathcal{L}(G)$ be its Lie algebra.

1) $\forall X \in \mathcal{L}(G),\ t \in \mathbb{R} : \exp(tX) := \sum_{j=0}^{\infty} \frac{1}{j!}(tX)^j \in G$

2) $\exp(sX)\exp(tX) \overset{\text{multiplication}}{\text{in } G} = \exp((s+t)X)$; $\exp(tX)^{-1}_{\text{in } G} = \exp(-tX)$

3) $t \mapsto \exp(tX)$ is the only one-parameter subgroup of G satisfying

$$\frac{d}{dt}\exp(tX)\Big|_{t=0} = X.$$

Prop. (Same framework)

a) $\exists$ an open set $\mathcal{V} \subset G$ containing $\mathbb{1}$ s.t.

1) $\forall A \in \mathcal{V}\ \exists X \in \mathcal{L}(G) : A = \exp(X)$

2) $\forall A \in \mathcal{V}\ \exists B \in \mathcal{V} : A = B^2 := BB$

b) $\forall A \in G_0\ \exists X_1, \cdots, X_N \in \mathcal{L}(G) : A = \exp(X_1)\exp(X_2)\cdots\exp(X_n)$ (not always unique)

c) If G is compact, we can choose $N = 1$ ($\Leftrightarrow A = \exp(X_1)$)

⚠ $\exp(X_1)\cdots\exp(X_n) \neq \exp(X_1 + \cdots + X_n)$ in general

In fact $\exp(X)\exp(Y) = \exp(f(X,Y))$

with $f(X,Y)$ called the CAMPBELL-BAKER-HAUSDORFF FORMULA. ($\in \mathcal{L}(G)$)

Exercise : find $f(X,Y)$. (a series)

What about the relation between representations of G and ones of $\mathcal{L}(G)$?

Def : a REPRESENTASION of a Lie algebra $\mathcal{L}$ is

a pair (h, V) with V a vector space and $h : \mathcal{L} \mapsto \overset{\text{linear map}}{L(V)}$ a homomorphism

It means $\begin{cases} h(X+\alpha Y) = h(X) + \alpha h(Y) \\ h([X,Y]) = h(X)h(Y) - h(Y)h(X) \\ h(\mathbf{0}) = 0 \end{cases}$ $\begin{pmatrix} \mathbb{1}f = f\ \forall f \in V \\ 0f = 0\ \forall f \in V \end{pmatrix}$

Lemma : Let (U, V) be a representation of a Lie group G

in a finite dimensional vector space V.

Then $\Gamma : \mathcal{L}(G) \mapsto L(V)$, $\Gamma(X) := \frac{d}{ds} U(\exp(SX))\big|_{s=0}$

defines a representation of $\mathcal{L}(G)$.

In addition $\underbrace{\exp\underbrace{(S\Gamma(X))}_{\in L(V)}}_{\in GL(V)} = \underbrace{U(\exp(SX))}_{\in GL(V)}$ (*

Proof as exercise

⚠ A kind of converse is not true:
a representation of a Lie algebra does not define
a representation of a unique Lie group by (*).
If two Lie groups are isomorphic close to the identity,
then the corresponding Lie algebras are isomorphic.

Application self-adjoint operator (extension of Hermitian matrix into ∞-dim Hilbert spaces)

If $\{e^{-itH}\}_{t\in\mathbb{R}}$ with $e^{-itH}\in\mathcal{B}(\mathcal{H})$ describes the evolution of a quantum system

And if ∃ a Lie group G and unitary rep. $(U,\mathcal{H})$ s.t.

$$U(a)e^{-itH}=e^{-itH}U(a)\ \forall a\in G$$

Then any $X\in\mathcal{L}(G)$ defines a constant of motion.

More precisely $\Gamma(X):\mathcal{H}\mapsto\mathcal{H}$ which satisfies (not necessarily $\Gamma(X)\in\mathcal{B}(\mathcal{H})$)

$$e^{-itH}\Gamma(X)=\Gamma(X)e^{-itH}\Leftrightarrow e^{-itH}\Gamma(X)e^{itH}=\Gamma(X)$$

III.3 SO(3), O(3) and SU(2)

Recall that $\exists\phi: SU(2)\mapsto SO(3)$ surjective and with kernal $\{\mathbb{1},-\mathbb{1}\}$

Prop. (proofs as exercises)

1) O(3), SO(3) and SU(2) are $\overbrace{\text{compact}}^{\text{maybe difficult to show}}$ Lie groups.
2) O(3) is not connected.
3) SO(3) is connected but not simply connected.
4) SU(2) is simply connected.
5) SO(3) and SU(2) isomorphic $\overbrace{\text{near the identity}}^{\text{the neighborhood}}$ (see the note for metric & topological spaces in the Tue seminar last term) $(\mathcal{L}(SO(3))\simeq\mathcal{L}(SU(2)))$
6) The Pauli matrices $\sigma_1,\sigma_2,\sigma_3$ generate SU(2) in the following sense:

$$\{\underbrace{-\tfrac{i}{2}\sigma_1}_{Y_1},\underbrace{-\tfrac{i}{2}\sigma_2}_{Y_2},\underbrace{-\tfrac{i}{2}\sigma_3}_{Y_3}\}\text{ generate } SU(2) \text{ with } [Y_i,Y_j]=\sum_k\varepsilon_{ijk}Y_k \qquad (*)$$

with ~~ε_{ijk}~~ $=\begin{cases}\varepsilon_{123}=\varepsilon_{231}=\varepsilon_{312}=1\\ \varepsilon_{132}=\varepsilon_{213}=\varepsilon_{321}=-1\\ \varepsilon_{ijk}=0 \text{ otherwise}\end{cases}$

7) The following 3 matrices define the Lie algebra of SO(3)

$$Y_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ Y_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\ Y_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$$

with the same relations of (*)

IV Semisimple theory

$\rightsquigarrow SU(n)$

IV. 1) Complexification and linear independence

Note: $SU(n)$ is a Lie group and $su(n)$ is a Lie algebra.

Recall that a basis of $su(2)$ is given by

$$\left\{\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\right\}$$ They are lin. indep. on $\mathbb{R}$ but also on $\mathbb{C}$

On the other hand, a basis for $sl(2,\mathbb{C})$ is

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}\right\}$$

and they are lin. indep. on $\mathbb{R}$ but not on $\mathbb{C}$.

Let us define the complexification of any real Lie algebra $\mathcal{L}$ of dim n.

Consider $\mathcal{L} \oplus \mathcal{L}$ with

$$(\lambda + i\mu)\overset{\text{"}X+iY\text{"}}{(X,Y)} := (\lambda X - \mu Y, \mu X + \lambda Y) \quad \forall \lambda, \mu \in \mathbb{R}, X, Y \in \mathcal{L}.$$

Exercise: Check that this defines a complex vector space of dim n with $(X_1, 0), \cdots, (X_n, 0)$ with $\{X_1, \cdots, X_n\}$ a basis of $\mathcal{L}$.

$$[(X,Y),(X',Y')] := ([X,X'] - [Y,Y'], [X,Y'] + [Y,X'])$$

Lemma: $\mathcal{L} \oplus \mathcal{L}$ with the above scalar mult. & the above $[\ ,\ ]$ is a Lie algebra over $\mathbb{C}$ of dim n.

Proof as exercise

Def. This $\mathcal{L} \oplus \mathcal{L}$ is called the COMPLEXIFICATION of $\mathcal{L}$ and denoted by $\mathcal{L}_{\mathbb{C}}$.

Lemma: Iff $\mathcal{L}$ has a basis also $\overset{\text{lin}}{\text{indep.}}$ over $\mathbb{C}$ and is a real linear Lie algebra then the map

$$\phi : \mathcal{L}_{\mathbb{C}} \mapsto M_n(\mathbb{C}),\ \phi(X,Y) = X + iY$$

defines an injective homomorphism, and it is an isomorphism on its image on $M_n(\mathbb{C})$.

Proof as exercise

Exercise

$gl(n,\mathbb{R})_{\mathbb{C}} \simeq gl(n,\mathbb{C})$ $\qquad$ $su(n)_{\mathbb{C}} \simeq sl(n,\mathbb{C})$

$u(n)_{\mathbb{C}} \simeq gl(n,\mathbb{C})$ $\qquad$ $sl(n,\mathbb{R})_{\mathbb{C}} \simeq sl(n,\mathbb{C})$

IV. 2

Exercise

1) $X \in su(n) \Leftrightarrow X^* = -X$ and $tr(X) = 0$ (us exponensial)

2) $su(n)$ is of real dimension $n^2 - 1$

3) Any basis of $su(n)$ is lin. indep. over $\mathbb{C}$

⇝ the second lemma applies

IV. 2) Properties of Lie algebra

Def. 1) a SUBALGEBRA of a Lie algebra of $\mathcal{L}$ is a subspace $\mathcal{L}^1$ α s.t.

$$[X,Y] \in \mathcal{L}^1 \Leftarrow X, Y \in \mathcal{L}^1$$

2) A subspace $\mathcal{L}'$ of $\mathcal{L}$ is an IDEAL if

$$[X,Y] \in \mathcal{L}' \ \forall X \in \mathcal{L}' \text{ and } Y \in \mathcal{L}$$

It is a PROPER IDEAL if $\mathcal{L}' \neq \mathcal{L}$

3) The CENTER of $\mathcal{L}$ is defined by

$$\{X \in \mathcal{L} \mid [X,Y] = 0 \ \forall Y \in \mathcal{L}\}$$

Note that the center is always an abelian ideal.

Def. A Lie algebra $\mathcal{L}$ with $\dim \mathcal{L} > 1$ is SIMPLE if

$\{0\}$ is the only proper ideal of $\mathcal{L}$.

And $\mathcal{L}$ with $\dim \mathcal{L} > 1$ is SEMI-SIMPLE if

$\{0\}$ is the only abelian proper ideal of $\mathcal{L}$.

Def. A connected Lie group is SIMPLE if

it does not contain any proper normal Lie subgroup. $\mathbb{R} \rhd \mathbb{Z}$ but $\mathbb{Z}$ is not a Lie gro
$\mathbb{R}$ is still simple

A connected Lie group is SEMI-SIMPLE if

it deos not contain any normal abelian proper Lie subgroup.

Lemma. Let G be a connected linear Lie group and $\mathcal{L}(G)$ its Lie algebra (over $\mathbb{R}$)

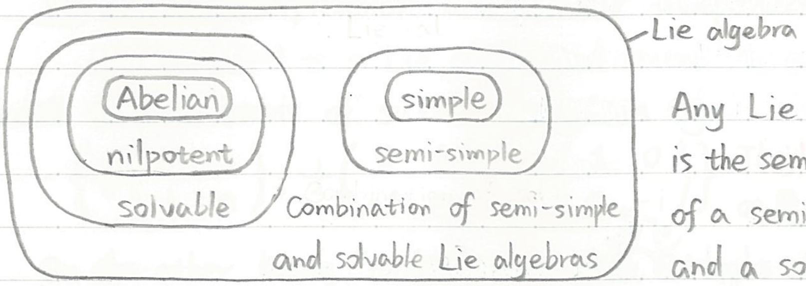
1) A Lie subgroup G' is normal iff $\mathcal{L}(G')$ is an ideal in $\mathcal{L}(G)$
2) G is simple iff $\mathcal{L}(G)$ is simple
3) G is semi-simple iff $\mathcal{L}(G)$ is semi-simple

} small exercises

4) $\mathcal{L}(G)_{\mathbb{C}}$ is semi-simple iff ↑
5) If $\mathcal{L}(G)_{\mathbb{C}}$ is simple then $\mathcal{L}(G)$ is simple

} more deep

Remark (task for mathematitians)



Any Lie algebra is the semi-direct product of a semi-simple Lie algebra and a solvable Lie algebra

Let $\mathcal{L}$ be a Lie algebra and consider its adjoint representation defined by

$$\begin{array}{lll} \mathrm{ad}: & \mathcal{L} & \mapsto L(\mathcal{L}) \\ & \rotatebox{90}{$\in$} & \\ & X & \mapsto \mathrm{ad}_X \quad \text{with} \quad \mathrm{ad}_X(Y) = [X,Y] \end{array}$$

linear operators on the vector space $\mathcal{L}$

indeed $\mathrm{ad}_X \in L(\mathcal{L})$ since

$$\mathrm{ad}_X(Y+\lambda Z) = [X, Y+\lambda Z] = [X,Y] + \lambda [X,Z]$$

It is a representation since

- $\mathrm{ad}_0 = 0$
- $\mathrm{ad}_{X+\alpha Y} = \mathrm{ad}_X + \alpha\, \mathrm{ad}_Y$ (since $[\ ,\ ]$ is bi-linear) (*)
- $\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X$ (use Jacobian identity)

Note that

$$\mathrm{ad}_X([Y,Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$$ because of Jacobian identity

Def. the KILLING FORM of $\mathcal{L}$ is the symmetric bi-linear map
↑(*)

$$K: \mathcal{L} \times \mathcal{L} \mapsto \mathbb{C}, \quad K(\overset{\curvearrowleft\curvearrowright}{X,Y}) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y) \in \mathbb{C}$$

Exercise: 0) Illustrate the $\overset{\text{in this page}}{\text{theory}}$ with $\mathcal{L} = su(2)$

1) If $\{Y_1, \cdots, Y_n\}$ is a basis of $\mathcal{L}$ with

$$[Y_i, Y_j] = \sum_{k=1}^{n} c_{ij}^{k} Y_k$$

Then $g_{ij} := K(Y_i, Y_j) = \sum_{k,r} c_{ir}^{k} c_{jk}^{r}$

2) $K([X,Y], Z) = K(X, [Y,Z]) \ \forall X, Y, Z \in \mathcal{L}$

3) $\left[K(X,Y) = 0 \ \forall Y \in \mathcal{L} \Rightarrow X = 0\right] \Leftrightarrow \underset{\uparrow}{\det}\left((g_{ij})_{i,j=1}^{n}\right) \neq 0$

In such a case we say that the Killing form is non degenerate

A quite important thm (Cartan's criterion) :

A Lie algebra $\mathcal{L}$ is semi-simple iff its Killing form is non-degenerate.

Lemma: A semi-simple connected Lie group is compact iff the Killing form of its Lie algebra is negative definite. It means

$$K(X,X) < 0 \quad \forall X \in \mathcal{L},\ X \neq 0$$

Example: su(n), one has $K(X,Y) = 2n \cdot \mathrm{tr}(XY)$ and it follows that

$$K(X,X) = 2n \cdot \mathrm{tr}(XX) = -2n \cdot \mathrm{tr}(X^* X) \ \#$$
$$= -2n \sum_{j=1}^{n} \langle e_j , X^* X e_j \rangle = -2n \sum_{j=1}^{n} \langle X e_j ; X e_j \rangle = -2n \sum_{j=1}^{n} \| X e_j \|^2 < 0$$

By knowing that SU(n) is semi-simple, it follows that it is compact.

No. 12

Date 2018 6 27

IV.3

Thm. $\mathcal{L}$ is semi-simple iff $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_N$ with each $\mathcal{L}_i$ simple Lie algebra

$\downarrow \quad \downarrow \quad \downarrow$ They don't speak to each other

Example: For $\mathcal{L}_1 \oplus \mathcal{L}_2$

$[X_1 + X_2, Y_1 + Y_2] := [X_1, Y_1] + [X_2, Y_2]$

Corollory: Any semi-simple Lie algebra of dim 2 or 3 is simple.

(Since any simple Lie algebra is of dim >1)

IV.3 Roots of semi-simple complex Lie algebra

Recall that any real Lie algebra can be complexified.

Recall that for any $X \in \mathcal{L}$, $ad_X : \mathcal{L} \mapsto \mathcal{L}$ linear

$\Rightarrow$ We can look at eigenvalues of ad_X, which means

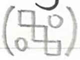
$\lambda \in \mathbb{C} : [X, Y] = ad_X(Y) = \lambda Y \quad \exists Y \in \mathcal{L}$

If $\mathcal{L}$ is of dim n, then $\det(ad_X - \lambda \mathbb{1})$ is a polynomial with deg n, with n roots

Clearly 0 is an eigenvalue since $ad_X(X) = [X, X] = 0 = 0X$

Recall also that ad_X admits n generalized eigenvectors

(Jordan normal form of a matrix)



Def. Let $\mathcal{L}$ be a semi-simple complex Lie algebra.

A CARTAN SUBALGEBRA $\mathcal{L}_0$ of $\mathcal{L}$ is a maximal abelian subalgebra of $\mathcal{L}$ with all $ad_X \{$with $X \in \mathcal{L}_0\}$ simultaneously diagonalizable.

It means $\mathcal{L}_0$ is a complex vector space s.t.

1) $\forall X_1, X_2 \in \mathcal{L}_0 : [X_1, X_2] = 0$

2) If for $Y \in \mathcal{L}$, $[X, Y] = 0 \ \forall X \in \mathcal{L}_0$, then $Y \in \mathcal{L}_0$

3) $\forall X \in \mathcal{L}_0$: ad_X is diagonalizable

Remark: One should show that for semi-simple Lie algebras, such Cartan subalgebra always exists; and if there are >1, then they have the same dimension.

We call the RANK of $\mathcal{L} =: n_0 < n$ the dim of $\mathcal{L}_0$

↳ because of semi-simple

for ad_X is diagonalizable ③

Let us fix $\mathcal{L}_0$ a Cartan subalgebra in $\mathcal{L}$, and choose a basis $\{Y_1, \cdots, Y_n\}$ of $\mathcal{L}$ s.t. $ad_X(Y_j) = \lambda_j(X) Y_j \ \forall X \in \mathcal{L}_0$.

Let us observe that if $X, X' \in \mathcal{L}_0$ and $\alpha \in \mathbb{C}$ then $X + \alpha X' \in \mathcal{L}_0$, so

$$\lambda_j(X+\alpha X')Y_j = ad_{X+\alpha X'}(Y_j) = [X+\alpha X', Y_j] = [X, Y_j] + \alpha [X', Y_j]$$
$$= ad_X(Y_j) + \alpha\, ad_{X'}(Y_j) = (\lambda_j(X) + \alpha \lambda_j(X'))Y_j$$

$\Rightarrow \lambda_j(X+\alpha X') = \lambda_j(X) + \alpha \lambda_j(X') \quad \Rightarrow \lambda_j : \mathcal{L} \mapsto \mathbb{C}$ is linear $\forall j = 1, \cdots, n$

$:\Leftrightarrow \lambda_j \in \mathcal{L}_0^*$ (DUAL of $\mathcal{L}_0$) (of dim n_0)

Remark: Since $ad_X(Y) = 0$ if $X, Y \in \mathcal{L}_0$, we can choose a basis of $\mathcal{L}$ s.t.

$$\{\underbrace{Y_1, \cdots, Y_{n_0}}_{\in \mathcal{L}_0}, Y_{n_0+1}, \cdots, Y_n\}$$

Remark: Observe that $\lambda_j(X) = 0$ if $j \in \{1, \cdots, n_0\}$ and $X \in \mathcal{L}_0$

Is it possible that $\lambda_j(X) = 0 \ \forall j \in \{1, \cdots, n\}$ and $X \in \mathcal{L}_0$?

No. Because any $Y \in \mathcal{L}_0^\perp$ cannot commute with all elements of $\mathcal{L}_0$ (maximality assumption) ②

Def. (independent of the choice of a basis)

A ROOT of $\mathcal{L}$ (relative to a fixed Cartan subalgebra $\mathcal{L}_0$) is an element $\alpha \in \overset{\rightarrow \Leftrightarrow \alpha : \mathcal{L}_0 \mapsto \mathbb{C}}{\mathcal{L}_0^*}$, $\alpha \neq 0$ s.t. $\exists Y \in \mathcal{L} \setminus \{0\} : ad_X(Y) = \alpha(X) Y \ \forall X \in \mathcal{L}_0$

The set of all roots is denoted by $R \subset \mathcal{L}_0^*$

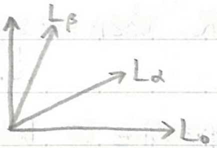
It corresponds to the generalization of an eigenvalue.

For any $\alpha \in R$ we set

$$\mathcal{L}_\alpha := \{Y \in \mathcal{L} \mid ad_X(Y) = \alpha(X) Y \ \forall X \in \mathcal{L}_0\} \neq \mathcal{L}_0 \text{ (or } \alpha = 0)$$

$\Rightarrow$ Since all ad_X can be diagonalized simultaneously, one infers that

$$\mathcal{L} = \mathcal{L}_0 \oplus \left(\bigoplus_{\alpha \in R} \mathcal{L}_\alpha \right)$$



⚠ No notion of orthogonality

In this representation $ad_X = 0 \oplus \bigoplus_{\alpha \in R} \alpha(X)$

Exercise: think about this.

We now generalize $\mathcal{L}_\alpha = \begin{cases} \mathcal{L}_\alpha & \text{if } \alpha \in R \\ \{0\} & \text{if } \alpha \notin R \text{ but } \alpha \in \mathcal{L}_0^* \end{cases}$

Lemma: $\forall \alpha, \beta \in \mathcal{L}_0^*, X_\alpha \in \mathcal{L}_\alpha, X_\beta \in \mathcal{L}_\beta$:

$[X_\alpha, X_\beta] \in \mathcal{L}_{\alpha+\beta}$ Do the several cases separately. ~~and~~ Proof as exercise, use Jacobian identity.

$\Rightarrow$ If $\alpha, \beta \in R$ but $\alpha + \beta \notin R$ then $[X_\alpha, X_\beta] = 0$; but if $\alpha + \beta = 0$ then $[X_\alpha, X_\beta] \in \mathcal{L}_0$

Prop. $^{1)}$ If $\alpha \in R$ then $-\alpha \in R$
2) $\dim \mathcal{L}_\alpha = 1$
3) $\mathrm{span}(\alpha \mid \alpha \in R) = \mathcal{L}_0^*$

$\left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\}$ Book of Hall, P165~166

With an additional change of basis, one can construct a basis
$\{H_1, \cdots, H_{n_0}, E_\alpha, \cdots, E_{\gamma}\}$ of $\mathcal{L}$ s.t.

1) $[H_j, H_k] = 0$

2) $[H_j, E_\alpha] = \alpha(H_j) E_\alpha$ with $\alpha(H_j) \in \mathbb{R}$

3) $$[E_\alpha, E_\beta] = \begin{cases} \sum_{j=1}^{n_0} \alpha(H_j) H_j & \text{if } \alpha+\beta = 0 \\ J_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \in R \\ 0 & \text{otherwise} \end{cases}$$

In this basis $(g_{ij})_{ij} = (\mathrm{tr}(\mathrm{ad}_{x_i} \mathrm{ad}_{x_j}))_{ij} =$

$$n_0\left\{\begin{matrix}\\ \\ \\ \end{matrix}\right.\begin{pmatrix} \overbrace{\begin{matrix}1 & & 0\\ & 1 & \\ 0 & \ddots & 1\end{matrix}}^{n_0} & & & & 0 \\ & \begin{matrix}0&1\\1&0\end{matrix} & & & \\ & & \begin{matrix}0&1\\1&0\end{matrix} & & 0 \\ 0 & & & \ddots & \\ & & 0 & & \begin{matrix}0&1\\1&0\end{matrix}\end{pmatrix} \in M_n(\mathbb{R})$$

Def. Since $\alpha(H_j) \in \mathbb{R}$ in this basis, we say that the root
α is POSITIVE ($:\Leftrightarrow \alpha \in R_+$) if the first non-zero entry of $\alpha(H_j)$ is positive ↱ for $j \in \{1, \cdots, n_0\}$
α is NEGATIVE ($:\Leftrightarrow \alpha \in R_-$) if 〃 negative

$\Rightarrow R = R_+ \sqcup R_-$

If $\alpha, \beta \in R$, then $\alpha > \beta :\Leftrightarrow \alpha - \beta \in R_+$
↳ called the LEXICOGRAPHIC ORDER on $\mathbb{R}^n$

Def. With respective to this basis a root is SIMPLE
if it cannot be expressed as a linear combination of other positive roots.

In def of POSITIVE (NEGATIVE) ROOT, the "first non-zero" entry is in the reverse order: $\alpha(H_{n_0}), \alpha(H_{n_0-1}), \cdots, \alpha(H_1)$

Prop. Let $\mathcal{L}$ be a semi-simple complex Lie algebra, and consider the canonical basis $\{H_1, \cdots, H_{n_0}, E_\alpha, \cdots, E_\gamma\}$ or $\mathcal{L}$.

1) There are n_0 simple roots $\alpha^1, \cdots, \alpha^{n_0}$ ⚠"n_0" ≠ "no"

2) These n_0 roots generate $\mathcal{L}_0^*$ ⇒ they are lin. indep.

3) If $\beta \in \mathcal{R}$ not simple, then $\exists a_1, \cdots, a_{n_0} \in \mathbb{Z} : \beta = a_1\alpha^1 + \cdots + a_{n_0}\alpha^{n_0}$ with either $a_1, \cdots, a_{n_0} \geq 0$ or $a_1, \cdots, a_{n_0} \leq 0$

Recall that $su(n) = \{X \in M_n(\mathbb{C}) \mid X^* = \overset{\rightarrow \text{not } X^{-1}! \text{ (unitary is in SU(n))}}{-X} \text{ and } tr(X) = 0\}$ and $\dim(su(n)) = n^2 - 1$ (over $\mathbb{R}$)

What is the dimension of any Cartan subalgebra in $su(n)_{\mathbb{C}}$?

Observation: if $X \in su(n)$ then $(iX)^* = -i(-X) = iX$ ⇒ iX is Hermitian

How many elements in $\{Y \in M_n(\mathbb{C}) \mid Y^* = Y \text{ and } tr(Y) = 0\}$ are diagonal & lin. indep?

Answer: $n - 1 =: n_0$

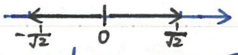
Example:

1) $su(2)$ of dim 3, and $n_0 = 1$. A basis for $su(2)_{\mathbb{C}}$ is given by the Pauli matrices. We choose the following basis:

$$H = \frac{1}{2\sqrt{2}}\sigma_3 = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & -\frac{1}{2\sqrt{2}} \end{pmatrix}; \underset{(E_\alpha)}{E_+} = \frac{1}{4}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}; \underset{(E_{-\alpha})}{E_-} = \frac{1}{4}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$$

$\sigma_1, \sigma_2, \sigma_3$: are Pauli matrices

In this basis $\alpha = \alpha(H) = \frac{1}{\sqrt{2}}$, $ad_H(E_\alpha) = \frac{1}{\sqrt{2}}E_\alpha$, and $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

The roots are  ; $[E_\alpha, E_{-\alpha}] = \frac{1}{\sqrt{2}}H$

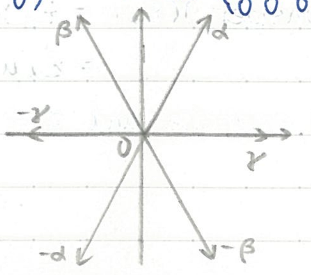
2) $su(3)$ of dim 8 and $n_0 = 2$. $\mathcal{R}$ contains 6 elements. Choose

$$H_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad E_\alpha = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad E_\beta = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad E_\gamma = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}; \quad E_{-\alpha} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad E_{-\beta} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad E_{-\gamma} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In this basis g has expected form and

$$\underbrace{\underset{\hookrightarrow = \beta + \gamma}{\alpha = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)}; \quad \underset{\hookrightarrow \text{simple}}{\beta = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)}; \quad \underset{\hookrightarrow \text{simple}}{\gamma = \left(\frac{1}{\sqrt{3}}, 0\right)}}_{\text{positive roots}}$$



$\underline{\text{IV}}$. 4 Weights of semi-simple complex Lie algebras

Let $\mathcal{L}$ be a complex Lie algebra, and let

$(h, \mathcal{V})$ be a representation in $\mathcal{V}$ of finite dimension. It means

$h : \mathcal{L} \mapsto L(\mathcal{V})$ s.t. ~~k~~

$$h(\lambda X + Y) = \lambda h(X) + h(Y) \,;\; h([X,Y]) = h(X)h(Y) - h(Y)h(X) \,;\; h(0) = 0.$$

Def. As for adjoint map, we look for $v \in \mathcal{V}$ s.t.

$$v \neq 0 \text{ and } h(H)v = \overset{\in \mathbb{C}}{\mu(H)}\, v \;\; \forall H \in \mathcal{L}_0$$

In this case, v is called a WEIGHT VECTOR; and

| the map $\mu : \mathcal{L}_0 \mapsto \mathbb{C}$ a WEIGHT

In fact $\mu \in \mathcal{L}_0^*$ ($\Leftrightarrow \mu(H_1 + \lambda H_2) = \mu(H_1) + \lambda \mu(H_2)$)

More generally, $\forall \mu \in \mathcal{L}_0^*$ we set $\mathcal{V}_\mu = \{ v \in \mathcal{V} \mid \underbrace{\underbrace{h(\overset{\in\mathcal{L}}{H})}_{\in L(\mathcal{V})} v}_{\in \mathcal{V}} = \underbrace{\mu(H)}_{\in \mathbb{C}} v \}$ and

$\dim(\mathcal{V}_\mu) =:$ MULTIPLICITY of μ

Remark : Roots are special weights when $\mathcal{V} = \mathcal{L}$. ($\Rightarrow$ multiplicity $\in \{0, 1\}$)

Choose again the canonical basis $\{H_1, \cdots, H_{n_0}, E_\alpha, \cdots, E_\gamma\}$ of $\mathcal{L}$ and set

$h(H_j) =: \mathcal{H}_j \,;\; h(E_\alpha) =: \mathcal{E}_\alpha \,;\; h(H) =: \mathcal{H}$ for $H \in \mathcal{L}_0$

$$\Rightarrow \begin{cases} [\mathcal{H}, \mathcal{E}_\alpha] = \alpha(H)\mathcal{E}_\alpha \,; \cancel{\mathcal{E}} \\ [\mathcal{H}_i, \mathcal{H}_j] = 0 \,; \\ [\mathcal{E}_\alpha, \mathcal{E}_\beta] = \begin{cases} \sum_{j=1}^{n_0} \alpha(H_j)\mathcal{H}_j & \text{if } \alpha + \beta = 0 \\ J_{\alpha\beta}\, \mathcal{E}_{\alpha+\beta} & \text{if } \alpha + \beta \in R \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

⚠ $\{\mathcal{H}_i, \mathcal{E}_\alpha\}$ are not always lin. indep.

Prop. Let $(h, \mathcal{V})$ be rep. of $\mathcal{L}$; ~~Let~~

Let μ be a weight with a weight vector $v \in \mathcal{V}_\mu \setminus \{0\}$.

1) $\mathcal{E}_\alpha v \in \mathcal{V}_{\alpha+\mu}$, and $\mu + \alpha$ is a weight if $\dim(\mathcal{V}_{\mu+\alpha}) \neq 0$;

2) The weight vectors associated with different weights are lin. indep.;

3) There exists $\leq \dim(\mathcal{V})$ weights for $(h, \mathcal{V})$.

Proof: 1) Consider $\mathcal{H}\mathcal{E}_\alpha v = \mathcal{E}_\alpha \mathcal{H} v + [\mathcal{H}, \mathcal{E}_\alpha] v$

$$= \mathcal{E}_\alpha \mu(H) v + \alpha(H)\mathcal{E}_\alpha v = \overset{\in \mathcal{L}_0^*}{(\mu + \alpha)}(\mathcal{H})\, \mathcal{E}_\alpha v \in \mathcal{V}_{\mu+\alpha}. \qquad \square$$

2) as exercise and 2) $\Rightarrow$ 3)

Def. For $\forall \alpha \in R$ and any weight $\mu \in \mathcal{L}_0^*$, we set

$$\alpha \cdot \mu := {}^T\alpha g \mu \in \mathbb{C}$$ (g : upper part of the Killing form)

Exercise:

1) $\alpha \cdot \mu$ is independent of the choice of a basis in $\mathcal{L}_0$

2) If we choose $\mathcal{V} = \mathcal{L}$, then $\|\alpha\|^2 := \alpha \cdot \alpha = \sum \alpha(H_j)^2 > 0$

Question: For which $k \in \mathbb{Z}$ one has $\mu + k\alpha$ is a weight
($\Leftrightarrow \mathcal{V}_{\mu + k\alpha} \neq \{0\}$) ?

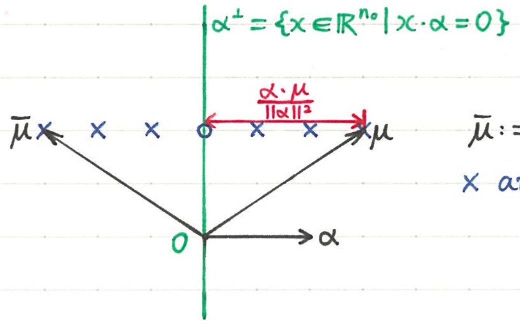
Lemma (technical but based only on $[H_j, \mathcal{E}_\alpha] = \cdots$ and $[\mathcal{E}_\alpha, \mathcal{E}_\beta] = \cdots$)

1) $\alpha \cdot \mu \in \mathbb{R}$

2) $N := -2 \frac{\alpha \cdot \mu}{\|\alpha\|^2} \in \mathbb{Z}$ (can be positive or negative)
↳ in this case

3) $\forall k \in \mathbb{Z} \cap [0, N]$: $\mu + k\alpha$ is a weight ⟵ replace $[0,N]$ by $[N,0]$

4) In the standard basis with $\alpha_j := \alpha(H_j) \in \mathbb{R}$, we have $\mu_j := \mu(H_j) \in \mathbb{R}$ ($\Rightarrow$ (1))



$\bar{\mu} := \mu - 2 \frac{\alpha \cdot \mu}{\|\alpha\|^2}$

× are weights ; o is sometimes a weight

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Example for $su(2)_{\mathbb{C}}$

Let $(h, \mathcal{V})$ be a finite irreducible rep. for $su(2)_{\mathbb{C}}$, and

and let $\underset{max}{\mu}$ be the maximal weight (since $n_0 = 1$, $\mathcal{L}_0^* = \mathbb{C}$ and all $\mu \in \mathbb{R}$)

Since $\alpha = \pm\frac{1}{\sqrt{2}}$, $\Rightarrow \|\alpha\|^2 = \frac{1}{2}$ and for $\alpha = -\frac{1}{\sqrt{2}}$ one has

$$N = -2\,\frac{-\mu/\sqrt{2}}{1/2} = 2\sqrt{2}\,\mu \in \mathbb{N} \Rightarrow \mu \in \frac{\sqrt{2}}{4}\mathbb{N}$$

The other possible weights are

$$\mu,\ \mu - \frac{1}{\sqrt{2}},\ \mu - \frac{2}{\sqrt{2}},\ \cdots,\ \mu - \frac{1}{\sqrt{2}}(2\sqrt{2}\,\mu) = -\mu$$

Or equivalently if $j := \sqrt{2}\,\mu$ then the possible values of $\sqrt{2}$(weight) are

$$j,\ j-1,\ j-2,\ \cdots,\ -j \text{ with } 2j \in \mathbb{N}$$

The missing argument:

Ł. Lemma: for any $d \in \mathbb{N}^*$ there exists a unique irreducible representation of $su(2)$ (modulo equivalence), and in such a representation, $j = \frac{d-1}{2}$.

(This is usually in quantum mechanics)

In the setting we consider $\mathcal{L}$ with the canonical basis and let $(h, \mathcal{V})$ be an irreducible map of $\mathcal{L}$.

Let μ_{max} be the maximal weight (once $\mathbb{R}^{n_0} = (\mu_1, \cdots, \mu_{n_0})$ with $\mu_j = \mu(H_j)$ is endowed with the lexicographic order).

Clearly if α is a positive root then $\mathcal{L}_{\mu_{max}+\alpha} = \{0\}$, otherwise contradiction.

Exercise If $(h, \mathcal{V})$ is an irreducible rep. of $\mathcal{L}$, and and μ is a weight with with $v \in \mathcal{L}_\mu$.

1) $\operatorname{span}\{v, \mathcal{E}_\alpha v, \mathcal{E}_\alpha \mathcal{E}_\beta v, \cdots \mid \alpha, \beta, \cdots \in R\} = \mathcal{V}$. ($\alpha = \beta$ also considered)

2) $\mu = \mu_{max} \Rightarrow \operatorname{span}\{v, \mathcal{E}_\alpha v, \mathcal{E}_\alpha \mathcal{E}_\beta v, \cdots \mid \alpha, \beta, \cdots \in R_-\} = \mathcal{V}$

$\Rightarrow \operatorname{span}\{v, \mathcal{E}_\alpha v, \mathcal{E}_\alpha \mathcal{E}_\beta v, \cdots \mid \alpha, \beta, \cdots \in R_+ \text{ simple}\} = \mathcal{V}$

In addition one has

Prop: Let $(h, \mathcal{V})$ be an irreducible map of $\mathcal{L}$ and μ_{max} the maximal weight.

1) $\forall \mu$ weight: $\mu = \mu_{max} - \sum_{\substack{\alpha \in \mathbb{R}_- \\ \alpha \text{ simple}}} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$;

2) $\dim \mathcal{V} = \sum_{\mu \text{ weight}} \dim \mathcal{L}_\mu$;

3) $\dim \mathcal{L}_{\mu_{max}} = 1$.

Remark: In order to get all irreducible representations of $\mathcal{L}$, we we should know all possible μ_{max}. Such formulae exist and

(IV.5)

for a given μ_{max} one has

$$\dim \mathcal{V} = \frac{\prod_{\alpha\in R_+} \alpha\cdot(\mu_{max}+\delta)}{\prod_{\alpha\in R_+}\alpha\cdot\delta} \quad \text{with } \delta = \frac{1}{2}\sum_{\alpha\in R_+}\alpha \quad \text{(Weyl Formula)}$$

and the multiplicity of each weight can be computed by the so-called Kostant's formula.

IV.5 Representations of $su(3)_{\mathbb{C}}$

Recall that $\dim(su(3)) = 8$ and $n_0 = 2$, and the matrices $H_1, H_2, E_{\pm\alpha}, E_{\pm\beta}, E_{\pm\gamma}$ are exibited in IV. 3

The roots are $\alpha = (\frac{1}{2\sqrt{3}}, \frac{1}{2})$, $\beta = (-\frac{1}{2\sqrt{3}}, \frac{1}{2})$, $\gamma = (\frac{1}{\sqrt{3}}, 0)$

We set

$I_3 := \sqrt{3}\, H_1 \qquad I_\pm := \sqrt{6}\, E_{\pm\gamma}$ (I-spin)

$U_3 := \frac{3}{2} H_2 - \frac{\sqrt{3}}{2} H_1 \qquad U_\pm := \sqrt{6}\, E_{\pm\beta}$ (U-spin)

$V_3 := -\frac{3}{2} H_2 - \frac{\sqrt{3}}{2} H_1 \qquad V_\pm := \sqrt{6}\, E_{\mp\alpha}$ (V-spin)

Then

$U_\pm, U_3$ leave $\{k\begin{pmatrix}1\\0\\0\end{pmatrix} \mid k\in\mathbb{C}\}$ invariant

$V_\pm, V_3$ " $\begin{pmatrix}0\\1\\0\end{pmatrix}$ "

$I_\pm, I_3$ " $\begin{pmatrix}0\\0\\1\end{pmatrix}$ "

These 3 triples generate 3 representations of $su(2)_{\mathbb{C}}$ which are not irreducible since there is an invariant subspace

Consider a $<\infty$ dim. irred. rep. $(h, \mathcal{V})$ of $su(3)_{\mathbb{C}}$, and set

$\mathcal{H}_j := h(H_j)$, $\mathcal{E}_\alpha := h(E_\alpha)$; $\mathcal{I}_3 := h(I_3)$, $\mathcal{U}_3 := h(U_3)$, $\mathcal{V}_3 := h(V_3)$

As for $su(2)$ before, $\mathcal{I}_3$, $\mathcal{U}_3$ and $\mathcal{V}_3$ are diagonalizable with evals in $\mathbb{Z}/2$ (like j before)

$\Rightarrow$ if μ is a weight, $\sqrt{3}\,\mu_1 = \sqrt{3}\,\mu(H_1) \in \mathbb{Z}/2$ and

$\frac{3}{2}\mu_2 - \frac{\sqrt{3}}{2}\mu_1 \in \mathbb{Z}/2 \Leftrightarrow 3\mu_2 \in \mathbb{Z} + \sqrt{3}\,\mu_1 \subset \mathbb{Z}/2$

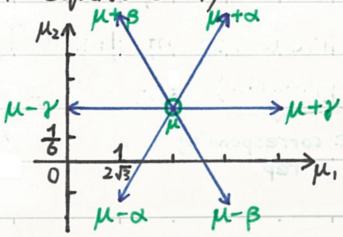
Thus $\mu_1 \in \mathbb{Z}/2\sqrt{3}$ and $\mu_2 \in \mathbb{Z}/6$

and $\mu\pm\alpha = (\mu_1 \pm \frac{1}{\sqrt{3}}\frac{1}{2}, \mu_2 \pm \frac{1}{2})$

$\mu\pm\beta = (\mu_1 \mp \frac{1}{\sqrt{3}}\frac{1}{2}, \mu_2 \pm \frac{1}{2})$

$\mu\pm\gamma = (\mu_1 \pm \frac{1}{\sqrt{3}}1, \mu_2)$

Or equivalently



Weight diagram (for $su(3)$)

$\hat{\nu} = \nu_{max}$

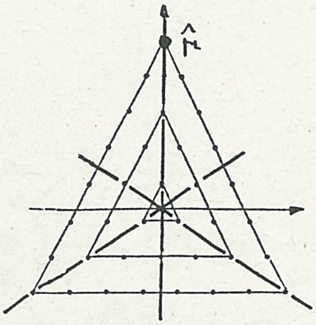


Figure 5.11(a)

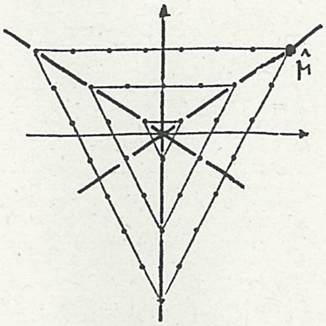


Figure 5.11(b)

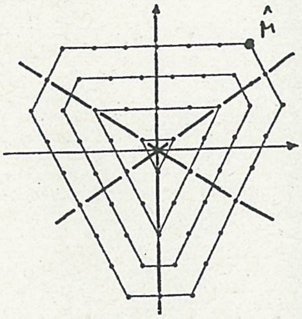


Figure 5.11(c)

If we represent the maximal weights on a so-called weight diagram, because of symmetry there are only 3 different types of position (see picture).

For simplicity we introduce the notation $(K_1, K_2) \in \mathbb{N} \times \mathbb{N}$ (?) ~~with~~

with $K_1 = 2\sqrt{3}\,\mu_{max}(H_1)$, $K_2 = 3\mu_{max}(H_2) - \sqrt{3}\,\mu_{max}(H_1)$

$\Leftrightarrow \mu_{max} = \left(\frac{K_1}{2\sqrt{3}}, \frac{K_1 + 2K_2}{6}\right)$

From any $(K_1, K_2) \in \mathbb{N} \times \mathbb{N}$ we can generate a weight diagram, and thus a irred. rep. of $su(3)_{\mathbb{C}}$

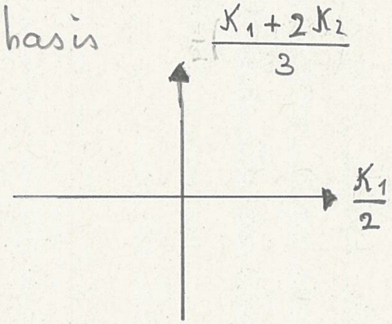
It means all irred. rep. of finite dim. are indexed by the 2 integers (K_1, K_2) and $\dim(\underbrace{D^{(K_1,K_2)}}_{\text{the corresponding rep.}}) =: n = \frac{1}{2}(K_1+1)(K_2+1)(K_1+K_2+2)$

(f~~e~~rom Weyl formula)

$D^{(K_1, K_2)}$ with

$$\nu_{max} = \left(\frac{K_1}{2\sqrt{3}} , \frac{K_1 + 2K_2}{6} \right).$$

Representation in the basis



● = ν_{max}

⊙ = weight of multiplicity 2.

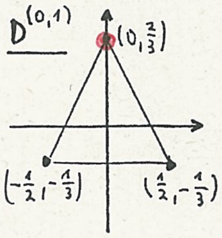


Figure 5.13(a)

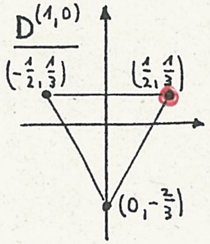


Figure 5.13(b)

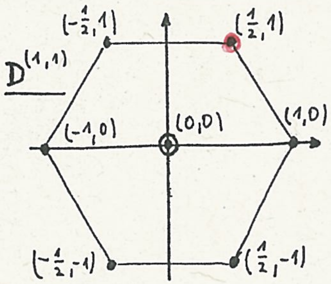


Figure 5.13(c)

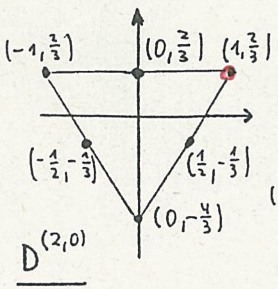


Figure 5.13(d)

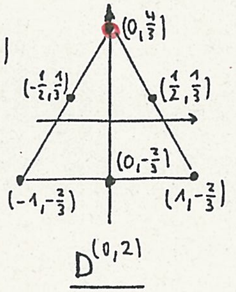


Figure 5.13(e)

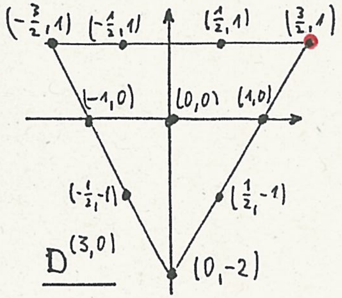


Figure 5.13(f)

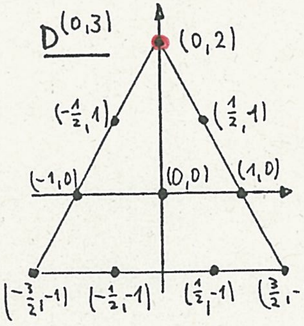


Figure 5.13(g)

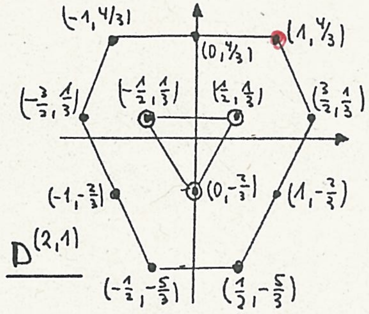


Figure 5.13(h)

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Lemma for $su(2)_{\mathbb{C}}$ last time

$\forall d \in \mathbb{N}^* \, \exists !$ irreducible unitary rep. of $su(2)$ in the space of dim. d

In rep. (h, V) $h(H)$ has eigenvalues $j, j-1, \cdots, -j$ for $j = \frac{d-1}{2}$ ⚠

Since we have, rep. of $su(3)$ of dim. $1, 3, \bar{3}, 6, \bar{6}, 8, 10, \overline{10}$

Recall that if $(h_1, V_1)(h_2, V_2)$ are irred. rep. of G, then

then $(h_1 \otimes h_2, V_1 \otimes V_2)$ is usually not an irred. rep. ⇒ can be decomposed

E.g. $3 \otimes 3 = 6 \oplus \bar{3}$; $3 \otimes \bar{3} = 8 \oplus 1$; $6 \otimes 3 = 10 \oplus 8$;

$3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = (6 \otimes 3) \oplus (\bar{3} \otimes 3) = 10 \oplus 2 \cdot 8 \oplus 1$

(unique modulo commutation and unitary equivalence)

Remark: Each semi-simple complex Lie algebra with a Cartan subalgebra of dim n_0 has n_0 indep. Casimir operators.

They are denoted by $C_2, C_3, \cdots, C_{n_0+1}$ and can be constructed with elem. of $\mathcal{L}$.

They don't $\in \mathcal{L}$ but they commute with each element of $\mathcal{L}$, it means

$[C_j, Y] = 0 \; \forall Y \in \mathcal{L}$ (one has to give a meaning to $[C_j, Y]$)

In addition, in any finite rep. of $\mathcal{L}$, one has

$h(C_j) = c_j \mathbb{1}_V$ with $c_j \in \mathbb{C}$

E.g. for $su(2)$, $C_2 = J^2 = J_1^2 + J_2^2 + J_3^2$ and $c_2 = j(j+1)$

For $su(3)$, one has 2 Casimir operators

$C_2 := H_1^2 + H_2^2 + E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha + E_\beta E_{-\beta} + E_{-\beta} E_\beta + E_\gamma E_{-\gamma} + E_{-\gamma} E_\gamma$

and in $D^{(\kappa_1, \kappa_2)}$ one has

kappa

$c_2 = \frac{1}{9}(\kappa_1^2 + \kappa_1 \kappa_2 + \kappa_2^2) + \frac{1}{3}(\kappa_1 + \kappa_2)$

C_3 is a polynomial of degree 3 in the generators of the algebra, (too complicated)

and in $D^{(\kappa_1, \kappa_2)}$ one has

$c_3 = \frac{1}{9}(\kappa_1 - \kappa_2)(2\kappa_1 + \kappa_2 + 3)(\kappa_1 + 2\kappa_2 + 3)$

IV.6 Application of SU(3) to physics

Since the elements of a Cartan subalgebra can be diagonalized simultaneously, they are often used to index families of particles.

In particular su(3) is often used. One has

$I_3 := \sqrt{3} H_1$ (isospin) and $Y := 2H_2$ (hypercharge)

People have observed that particles with similar properties gather by families of 1, 8, or 10 members, (see figures 5.19)

Such families can be generated by

$3 \otimes \bar{3} = 8 \oplus 1$ or $3 \otimes 3 \otimes 3 = 10 \oplus 2 \cdot 8 \oplus 1$

$\xrightarrow{\text{idea}}$ Basic building block of the theory should be 3 quarks and 3 antiquarks (see figure 5.20)

We give the names u, d, s or $\bar{u}, \bar{d}, \bar{s}$ for weights of 3 and $\bar{3}$

With this idea, Figure 5.19 (a) ocorresponds to the decomposision $8 \oplus 1$ of $3 \otimes \bar{3}$ or more precisely it is called the family of MESONS of spin 0 made of 1 quark and 1 $\overline{\text{quark}}$. (see figure 5.21)

Figure 5.19 (b) : BARYON DECUPLET made of 3 quarks u, d, s.

" (c) : BARYON OCTET " "

Recall that a basis of $V_1 \otimes V_2$ is given by $x \otimes y$ for x a basis of V_1 and y a basis of V_2.

In picture 5.21, one uses the notation $d\bar{s}$ for $d \otimes \bar{s}$, etc.

Then for $3 \otimes \bar{3}$ the symmetric element $\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ corresponds to the representation 1 in $8 \oplus 1$.

Nowadays, models are much more complicated than this.

Also used SU(5), ⋯, SO(10)

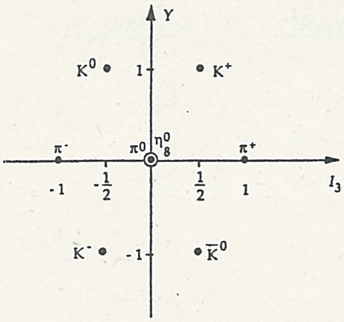


Figure 5.19(a)

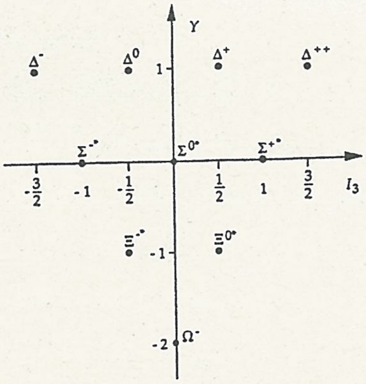


Figure 5.19(b)

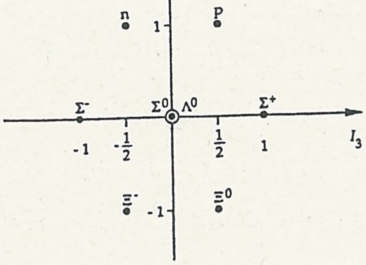


Figure 5.19(c)

Figure 5.20

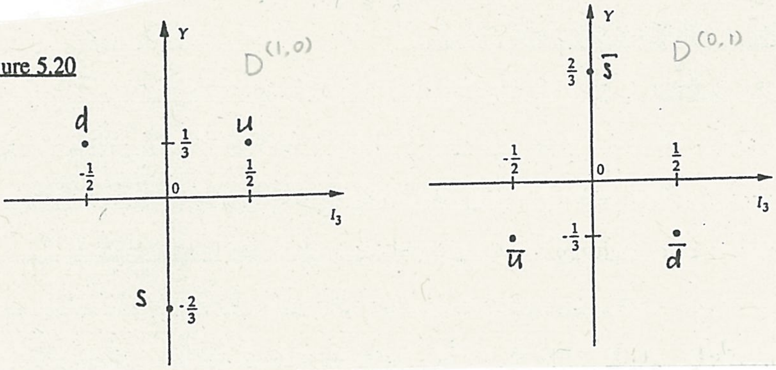
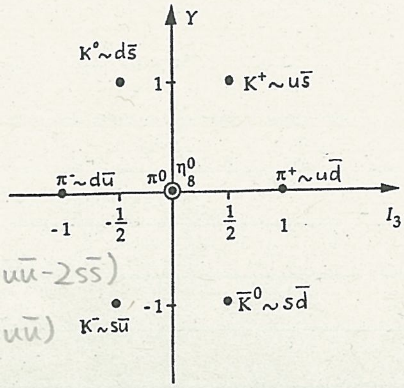


Figure 5.21



$\eta^0_8 \sim \frac{1}{\sqrt{6}}(d\bar{d} - u\bar{u} - 2s\bar{s})$

$\pi_0 \sim \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$

IV.7 Classification thm

Recall that semi-simple Lie algebras consist of sum of simple Lie algebras

⇒ The building blocks are simple complex Lie algebras
↳ {0} is the only proper ideal

Recall also that roots have very special properties. In the standard basis

$-2\frac{\alpha\cdot\beta}{\|\alpha\|^2}\in\mathbb{Z}$, $-2\frac{\alpha\cdot\beta}{\|\beta\|^2}\in\mathbb{Z}$ for any roots (⇒ weights) α, β

$\Leftrightarrow 2\alpha\cdot\beta = N_1\|\alpha\|^2$ and $2\alpha\cdot\beta = N_2\|\beta\|^2$ $\exists N_1, N_2 \in \mathbb{Z}$

$\Rightarrow \frac{\|\alpha\|}{\|\beta\|} = \sqrt{\frac{N_2}{N_1}}$ and $\frac{|\alpha\cdot\beta|^2}{\|\alpha\|^2\|\beta\|^2} = \frac{N_1N_2}{4}$; $N_1 = 0$ iff $N_2 = 0$

↳ $=: |\cos(\phi_{\alpha\beta})|^2 \in [0,1] \Rightarrow N_1N_2 \in [0,4]$

Possible (N_1, N_2):

$N_2 \backslash N_1$	0	1	2	3	4
0		×	×	×	×
1	×				
2	×			×	×
3	×		×	×	×
4	×		×	×	×

} $N_1N_2 > 4$

⇓

(N_1,N_2)	$(2,2)$	$(1,3)$	$(1,2)$	$(1,4)$	$(0,0)$	$(1,1)$
$\phi_{\alpha,\beta}$	0° 180°	30° →150°	45° →135°	0° 180°	→90°	60° →120°
	$\|\alpha\|=\|\beta\|$ $\Rightarrow \alpha=\beta$ we don't want to consider (trivial)			$\|\alpha\|=2\|\beta\|$ $\Rightarrow \alpha=2\beta$ impossible for simple roots		

One more information: $\alpha\cdot\beta \leq 0$ Prop. 8.11 in [Hall]

⇒ only possible angles are

90°, 120°, 135°, 150°

Then if we consider complex simple Lie algebras $\mathfrak{L}$ and denote by ○ any simple root of $\mathfrak{L}$

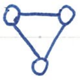
α β ,

$\|\alpha\|^2$↗ ○ ○ if $\phi_{\alpha\beta} = 90°$

○—○ if $\phi_{\alpha\beta} = 120°$

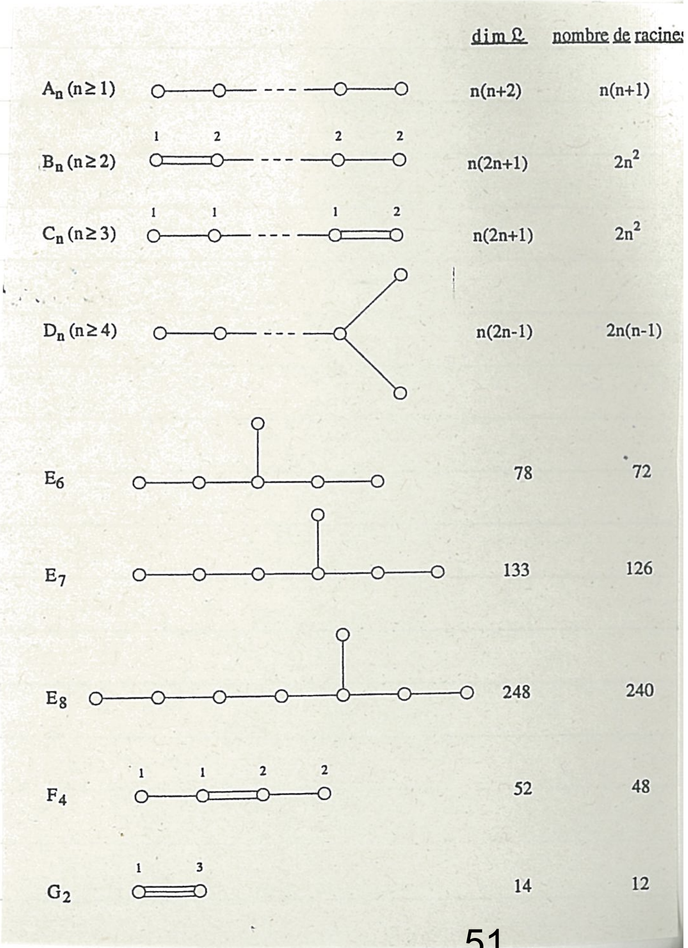
○=○ if $\phi_{\alpha\beta} = 135°$

○≡○ if $\phi_{\alpha\beta} = 150°$



cannot exist (see book)

By using this notation and that simple roots are linear indep, one can get the list of all simple complex Lie algebras →



	dim 𝔏	nombre de racine
A_n (n≥1)	n(n+2)	n(n+1)
B_n (n≥2)	n(2n+1)	2n^2
C_n (n≥3)	n(2n+1)	2n^2
D_n (n≥4)	n(2n-1)	2n(n-1)
E_6	78	72
E_7	133	126
E_8	248	240
F_4	52	48
G_2	14	12

These algebras have been extensively studied and have applications in physics and in mathematics (see Wikipedia on E_8)

Conclusion

We have seen many concepts which can be used in physics (QM) but also in mathematics. Please remember all these, and come back to the literature as often as possible.