Stability Enhancement with Stochastic Delay Switching

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Abstract

We propose a simple delay differential equation with a stochastic delay switching. In this model, the delay is a stochastic variable taking two values across the stability boundary. With this stochastic switching of the delay, the region of stability is enhanced. We also show that this is in contrast to the analogous case of stochastic switching of coefficient parameters in the equation.We can view this model with these associated behaviors as yet another example showing the intricate interplay between delay and stochasticity.

1 Introduction

Time delays are known to induce rather intricate behaviors in simple dynamical systems. For example, even for first-order ordinary differential equations, the stability of the fixed point changes by increasing delay. As time delays are almost ubiquitous on feedback control systems or multi-body interacting systems, the delay differential equations have been actively employed in theoretical modelings in various fields[1, 2, 3, 4, 5, 6, 7, 8, 9]. Mathematical aspects of delay differential equations have also been investigated. Due to the non-linearity introduced by the delay, however, there still are many unsolved problems analytically.

Also, when we introduce stochasticity additionally, the study of the dynamics becomes more challenging.Even though there has been a series of investigations on such delayed stochastic systems[10, 11, 12, 13, 14, 15], understandings of the interplay between delay and stochasticity are yet to come.

What we propose here is a slightly modified simple delay differential equation that gives rise to yet another curious behavior. In our model, the delay is a stochastic variable taking two values: one in the stability and the other in the unstable regions of a fixed point. It is observed that the region of stability is

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enhanced by this stochastic switching of delay. We show that this is in contrast to the analogous case of stochastic switching of coefficient parameters in the equation.

2 Model

The basic equation we study is the following simple delay differential equation:

$$\frac{dX(t)}{dt} = aX(t-\tau) \tag{1}$$

where a is a real parameter τ is the delay. This equation is known as a special case of Hayes equation[1], which has been much investigated. It is known that non-zero delay induces oscillations, and for $a < 0$ the asymptotic stability of the origin $X = 0$ is lost for the delay larger than the critical value

$$\tau_c = -\frac{\pi}{2a}. \tag{2}$$

The stability boundary is shown in Figure 1.

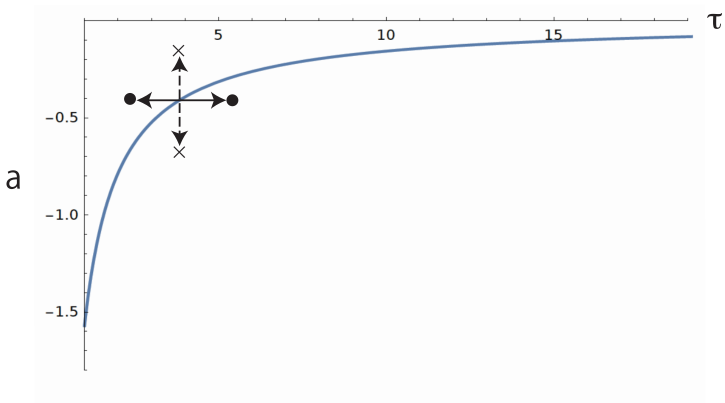

Figure 1: The stability boundary of the origin of Equation (1). The area with the curve and the axes is the stable region. Also, the schematics of the stochastic switchings are shown. Delay stochastic switching is between the two dot points indicated by the solid horizontal arrows, while the switching for the coefficients a is indicated by the two cross points with the dashed vertical arrows.

We now extend this model so that the delay is changed to a real stochastic variable $\hat{\tau}$ in such a way that it takes two values across the critical delay. Specifically, it is given by the following

$$\hat{\tau} = \tau_c(1 + \mu\xi). \tag{3}$$

where $\mu \in (0,1)$ is a real parameter, and ξ is a stochastic variable taking $+1$ and -1 with probabilities p and $1-p$ respectively. Thus,

$\hat{\tau} > \tau_c$ (with the probability p),
$\hat{\tau} < \tau_c$ (with the probability 1-p).

So, as mentioned, the delay switches stochastically across the critical value of the delay as indicated by the solid horizontal arrows in Figure 1.

We study this equation with stochastic delay switching numerically. The notable observation is that this switching enhances the stability of the origin: The origin can be asymptotically stable even when the average delay value, $\langle\hat{\tau}\rangle$, is larger than τ_c. Also, we compare these results with the case of the stochastic stability crossing by switching two values of a. Namely, we consider the following as a comparison,

$$\frac{dX(t)}{dt} = \hat{a}X(t-\tau) \tag{4}$$

where $\hat{a}$ is now the stochastic variable taking two values across the stability boundary with a given delay as indicated by the dashed vertical arrows in Figure 1. This type of equation is studied under the name of delay differential equation with a parametric noise. For our interest, we observe that that the analogous stability enhancement does not occur in contrast to the case of the stochastic delay switching.

3 Stability Chart by Numerical simulations

We have performed the numerical simulation of equation (1), and constructed a stability diagram as shown in Figure 2. The bottom figure is the log-log plot. As in Figure 1, The solid line is the stability boundary without stochastic switching. Here, the average delay value for the stochastic delay switching, $\langle\hat{\tau}\rangle_c$ are numerically estimated and indicated with the solid points on the plot. The enhancement of stability region by stochastic delay switching is clearly observed as $\tau_c < \langle\hat{\tau}\rangle_c$. Also, as we mentioned we have performed the stochastic stability crossing by switching two values of a with Equation (4). Representative results are also shown in Figure 2 with estimated average critical values $\langle\hat{a}\rangle_c$ indicated by the crosses. We see that the stability enhancement does not occur. In concrete, the values of the average critical values of $\hat{a}$ lie on the solid line so that $\langle\hat{a}\rangle_c \approx a_c$.

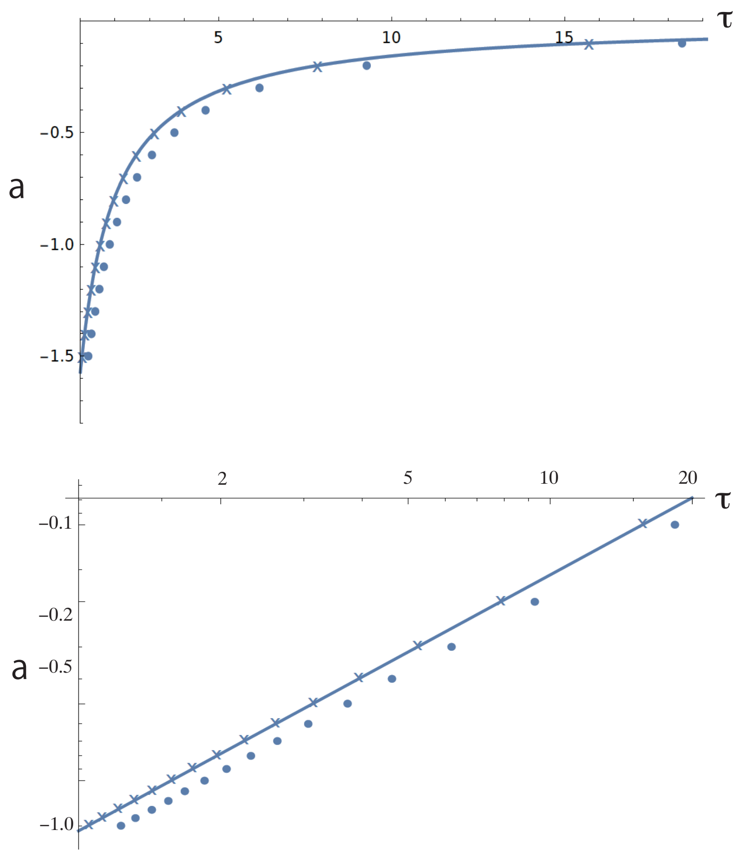

Figure 2: A representative plot of the stability boundary of the origin of Equation (1) with the stochastic switching. The bottom figure is the log-log plot. Also, the results of stochastic switching of the coefficients a are indicated by the crosses that shows no stability enhancement. (The parameter value of the switching amplitude is set as $\mu = 0.5$ for both cases)

4 Discussion

In this paper, we proposed a simple delay differential equation with stochastic switching of delays across the stability boundary. With such a mechanism, the stability region was enhanced in the sense that the stability of the fixed point holds even with the value of the average delay exceeding the normal critical delay value. Analogous stochastic switching with the coefficient, however, does not show such enhancement. Thus, how we cross the stability boundary by stochastic switching can affect this enhancement phenomena. Whether there exists an optimal way to cross the boundary to have the maximal stability enhancement is an interesting question to explore.

We can view this model with associated behaviors as yet another example showing the intricate interplay between delay and stochasticity. As the model is simple, it is hoped that some theoretical understanding can be obtained in the future.

Acknowledgment

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