

# Nontrivial black hole solutions in $f(R)$ gravitational theory

G. G. L. Nashed<sup>\*</sup>

*Centre for Theoretical Physics, The British University in Egypt,  
P.O. Box 43, El Sherouk City, Cairo 11837, Egypt*

S. Nojiri<sup>†</sup>

*Kobayashi-Maskawa Institute for the Origin of Particles and the Universe,  
Nagoya University, Nagoya 464-8602, Japan*



(Received 10 August 2020; accepted 12 November 2020; published 8 December 2020)

Recent observation shows that general relativity (GR) is not valid in the strong regime.  $f(R)$  gravity, where  $R$  is the Ricci scalar, is regarded to be a good candidate able to cure the anomalies that appear in conventional general relativity. In this realm, we apply the equations of motion of  $f(R)$  gravity to a spherically symmetric spacetime with two unknown functions and derive original black hole (BH) solutions without any constraints on the Ricci scalar as well as on the form of  $f(R)$  gravity. Those solutions depend on a convolution function and are deviating from the Schwarzschild solution of the Einstein GR. These solutions are characterized by the gravitational mass of the system and the convolution function that in the asymptotic form gives extra terms that are responsible to make such BHs different from GR. Also, we show that these extra terms make the singularities of the invariants much weaker than those of the GR BH. We analyze such BHs using the trend of thermodynamics and show their consistency with well-known quantities in thermodynamics like Hawking radiation, entropy, and quasilocal energy. We also show that our BH solutions satisfy the first law of thermodynamics. Moreover, we study the stability analysis using the odd-type mode and show that all the derived BHs are stable and have radial speed equal to one. Finally, using geodesic deviations, we derive the stability conditions of these BHs.

DOI: [10.1103/PhysRevD.102.124022](https://doi.org/10.1103/PhysRevD.102.124022)

## I. INTRODUCTION

More than one decade ago, Newton gravity failed to investigate some issues like the advances of Mercury besides the Michelson-Morley experiment [1]. In 1915, Einstein constructed his famous theory, the general theory of relativity (GR), that was able to resolve the issue of Mercury [2]. After that, most researchers trust GR as a successful theory of the gravitational field. However, in recent times, this theory, GR, failed to be consistent with observation and was not able to describe the dark energy and dark matter that are confirmed by observations [3–8]. Also, GR gives a violation of the Chandrasekhar mass limit for white dwarfs of the super-Chandrasekhar as well as white dwarfs of the sub-Chandrasekhar limiting mass [9–16].

The Einstein theory of GR has many forms of modifications, like modifications of GR action that can be achieved to derive different kinds of modification theories of gravity like  $f(T)$  gravity [17–20], where  $T$  is the torsion scalar in teleparallelism;  $f(R)$  gravity [21–25] with  $R$  the

scalar curvature;  $f(G)$  gravity with  $G$  the Gauss-Bonnet invariant [26];  $f(R, T)$  gravity, where  $T$  the trace of the energy-momentum tensor of matter [27,28];  $f(T, T)$  gravity, where  $T$  is the torsion scalar in teleparallelism and  $T$  the trace of the energy-momentum tensor of matter [29]; etc. All of these modified theories have received much attention to investigate the shortcomings of GR like the accelerated expansion of our Universe, investigation of flat rotation curves of galaxies, wormhole behavior, and other ambiguous phenomena near black holes (BHs) [30–34]. The first time to use a quadratic form of the Ricci scalar was given by Starobinskii [35]. It was shown that the higher-order curvature of  $f(R)$  gravity can solve the issue of the massive neutron stars [36–40]. It is well known that  $f(R)$  gravity consists of an arbitrary function whose first order is the Ricci scalar. The equations of motion of  $f(R)$  gravity have higher degrees and supply substantial classes of solutions that are different from GR. In the frame of  $f(R)$  gravity, the dynamical behavior of the matter field and dark energy has been studied [41–44]. From the viewpoint of cosmology, many researchers have carried out their studies from different directions [45–58]. A spherically symmetric vacuum BH solution in  $f(R)$  gravity has been derived in Refs. [59–62]. Using Noether

<sup>\*</sup>nashed@bue.edu.eg

<sup>†</sup>nojiri@gravity.phys.nagoya-u.ac.jp

symmetry, Capozziello *et al.* have derived spherically symmetric solutions in the frame of  $f(R)$  gravity [63,64]. Using the same techniques, Noether symmetry, axially symmetric vacuum BH solutions are derived [65]. Nontrivial spherically symmetric BH solutions for a specific class of  $f(R)$  gravity are derived [66–68]. Because of the fact of the existence of higher-order curvature terms in  $f(R)$  gravity, one can discuss potentially the importance of strong gravitational background in local objects. In this frame, many researchers concentrate to study spherically symmetric, static BHs [69–75], and neutron star [76–88] solutions in the quadratic model of  $f(R)$  gravity. We note that  $f(R)$  gravity is equivalent to the Brans–Dicke theories [89] with a scalar potential of gravitational origin [90–93]. It is the purpose of this manuscript to derive original spherically symmetric BHs in  $f(R)$  gravity without assuming any constraints on the Ricci scalar nor on the form of  $f(R)$  gravity and study the relevant physics of those BHs.

The structure of this study is as follows: In Sec. II, we give the fundamentals of  $f(R)$  gravity. In Sec. III, we apply the field equations of  $f(R)$  gravity to a spherically symmetric line element having unequal metric potentials. We derive the system of differential equations that have three unknown functions and derive different solutions of this system that is characterized by a convolution function. If this convolution function is vanishing, we return to the BH of GR, the Schwarzschild solution. So this convolution function appears as the effect of higher-order curvature that characterizes  $f(R)$  gravity. Moreover, we give the asymptote form of this convolution function up to a certain order and show the trace of the higher-order curvature. Also, we calculate the Kretschmann scalar, the Ricci tensor square, and the Ricci scalar and show the trace of  $f(R)$  gravity on such invariants that makes the singularity weaker than those of GR BHs. In Sec. IV, we calculate the above-mentioned thermodynamical quantities to be compared with their previous findings. In Sec. V, we use the odd-type method and study the stability of these BHs. Also in Sec. V, we use the geodesic deviation to derive the condition of stability for such BHs derived in Sec. III. In the final section, we give our concluding remarks.

## II. FUNDAMENTALS OF $f(R)$ GRAVITATIONAL THEORY

In this section, we consider a four-dimensional action of  $f(R)$  gravity where  $f(R)$  is an arbitrary differential function. It is important to stress the fact that  $f(R)$  gravity

is a modification of GR and coincides with the Einstein GR at lower order, i.e.,  $f(R) = R$ . When  $f(R) \neq R$ , then we have a theory that is different from GR. The action of  $f(R)$  gravity can take the form (cf. [22–24,94–98])

$$\mathcal{I} := \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R), \quad (1)$$

where  $\kappa$  is Newton’s gravitational constant and  $g$  is the determinant of the metric.

The use of the variational principle to action (1) gives the vacuum field equations [99]

$$\mathcal{I}_{\mu\nu} = R_{\mu\nu} f_R - \frac{1}{2} g_{\mu\nu} f(R) + [g_{\mu\nu} \square - \nabla_\mu \nabla_\nu] f_R \equiv 0, \quad (2)$$

such that  $\square$  is the d’Alembertian operator and  $f_R = \frac{df}{dR}$ . The trace of the field equations (2) takes the form

$$\mathcal{I} = 3\square f_R + R f_R - 2f(R) \equiv 0. \quad (3)$$

From Eq. (3), one can obtain  $f(R)$  in the form

$$f(R) = \frac{1}{2} [3\square f_R + R f_R]. \quad (4)$$

Using Eq. (4) in Eq. (2), we get [100]

$$\mathcal{I}_{\mu\nu} = R_{\mu\nu} f_R - \frac{1}{4} g_{\mu\nu} R f_R + \frac{1}{4} g_{\mu\nu} \square f_R - \nabla_\mu \nabla_\nu f_R = 0. \quad (5)$$

Accordingly, it is important to examine Eqs. (3) and (5) to a spherically symmetric spacetime having two unknown functions.

## III. SPHERICALLY SYMMETRIC BH SOLUTIONS

To study the equations of motion (3) and (5) in order to derive a general form of the arbitrary function  $f(R)$  without assuming any restrictions on the Ricci scalar, we use a spherically symmetric spacetime having two unknown functions of the following form:

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (6)$$

with  $A(r)$  and  $B(r)$  functions depending on radial coordinate  $r$ . The Ricci scalar of the metric (6) is figured out as

$$R(r) = \frac{r^2 B A'^2 - r^2 A A' B' - 2r^2 A B A'' - 4rA [B A' - A B'] + 4A^2(1 - B)}{2r^2 A^2}, \quad (7)$$

where  $A \equiv A(r)$ ,  $B \equiv B(r)$ ,  $A' = \frac{dA}{dr}$ ,  $A'' = \frac{d^2A}{dr^2}$ , and  $B' = \frac{dB}{dr}$ . Plugging Eqs. (3) and (5) into Eq. (6) and using Eq. (7), we get

$$\begin{aligned}
\mathcal{I}_{t^t} &= \frac{1}{8r^2N^2} \{r^2[BFN'^2 - 3NFB'N' - 2NBFN'' - 2N^2FB'' - 3NBN'F' - 2N^2B'F' + 2BN^2F''] \\
&\quad - 4rNB[FN' - NF'] - 4N^2F[1 - B]\} = 0, \\
\mathcal{I}_{r^r} &= \frac{1}{8r^2N^2} \{r^2[BFN'^2 - 3NFB'N' - 2NBFN'' - 2N^2FB'' + NBN'F' - 2N^2B'F' - 6BN^2F''] \\
&\quad + 4rNB[FN' + NF'] - 4N^2F[1 - B]\} = 0, \\
\mathcal{I}_{\theta^\theta} = \mathcal{I}_{\phi^\phi} &= \frac{1}{8r^2N^2} \{r^2[3NFB'N' + 2NBFN'' + 2N^2FB'' - BFN'^2 + NBN'F' + 2N^2B'F' + 2BN^2F''] \\
&\quad - 4rN^2BF' + 4N^2F[1 - B]\} = 0, \\
\mathcal{I} &= \frac{1}{2r^2N^2} \{r^2[6N^2B'F' - 3NFB'N' - 2NBFN'' - 2N^2FB'' + BFN'^2 + 3NBN'F' + 6BN^2F''] \\
&\quad + 4rN[3NBF' - FBN' - 2FNB'] + 4N^2F[1 - B] - 4r^2N^2f(r)\} = 0, \tag{8}
\end{aligned}$$

where  $N(r) = \frac{A(r)}{B(r)}$ ,  $F \equiv F(r) = \frac{df(R(r))}{dR(r)}$ ,  $F' = \frac{dF(r)}{dr}$ ,  $F'' = \frac{d^2F(r)}{dr^2}$ , and  $F''' = \frac{d^3F(r)}{dr^3}$ . Since we are dealing with spherical symmetry, we take  $f(R) = f(r)$ . It is of interest to mention here that the above system of differential equations given by (8) is identical with the differential equations given in Ref. [101].

Equations (8), except the trace part, can be rewritten in the following form:

$$0 = r^2[BFN'^2 - 3NFB'N' - 2NBFN'' - 2N^2FB'' - 3NBN'F' - 2N^2B'F' + 2BN^2F''] - 4rNB[FN' - NF'] - 4N^2F[1 - B], \tag{9}$$

$$0 = r^2[BFN'^2 - 3NFB'N' - 2NBFN'' - 2N^2FB'' + NBN'F' - 2N^2B'F' - 6BN^2F''] + 4rNB[FN' + NF'] - 4N^2F[1 - B], \tag{10}$$

$$0 = r^2[-BFN'^2 + 3NFB'N' + 2NBFN'' + 2N^2FB'' + NBN'F' + 2N^2B'F' + 2BN^2F''] - 4rN^2BF' + 4N^2F[1 - B]. \tag{11}$$

By using Eqs. (9) and (10) [(9) minus (10)], we obtain

$$0 = r^2[-4NBN'F' + 8BN^2F''] - 8rNN'BF. \tag{12}$$

On the other hand, by using Eqs. (9) and (11) [(9) plus (11)], we obtain

$$0 = -2r^2NBN'F' + 4r^2BN^2F'' - 4rNN'BF, \tag{13}$$

which is identical with (12), and, therefore, only two equations in (9)–(11) are independent. For example, Eq. (9) is equal to minus Eq. (10) minus 2 times Eq. (11). Then, for example, we can choose Eqs. (9) and (13) as independent equations. Because we have three unknown functions  $B$ ,  $N$ , and  $F$ , we cannot determine one function.

As an example, we assume the Schwarzschild-type solution

$$N = 1. \tag{14}$$

By assuming  $B \neq 0$  almost everywhere, which is physically required, Eq. (12) gives

$$F'' = 0; \quad \text{that is, } F = F_0 + F_1r. \tag{15}$$

Equation (9) has the following form:

$$\begin{aligned}
0 &= r^2[-2FB'' - 2B'F'] + 4rBF' - 4F[1 - B] \\
&= -2r^2(F_0 + F_1r)B'' - 2r^2F_1B' \\
&\quad + 4(F_0 + 2F_1r)B - 4(F_0 + F_1r). \tag{16}
\end{aligned}$$

In case  $F_1 = 0$ , Eq. (16) reduces to

$$0 = -r^2B'' + 2B - 2, \tag{17}$$

whose solution is given by

$$B = 1 + \frac{B_0}{r} + B_1r^2. \tag{18}$$

Here  $B_0$  and  $B_1$  are constants. Of course, the solution (18) expresses the Schwarzschild–(anti–)de Sitter spacetime.

We may also consider the case  $F_0 = 0$ . Then Eq. (16) becomes

$$0 = -r^2 B'' - rB' + 4B - 2. \quad (19)$$

The solution is given by

$$B = \frac{1}{2} + \tilde{B}_0 r^2 + \tilde{B}_1 r^{-2}. \quad (20)$$

Here  $\tilde{B}_0$  and  $\tilde{B}_1$  are constants. The solution (20) corresponds to the solution given before in Refs. [66,68].

In case either of  $F_0$  and  $F_1$  does not vanish, when  $r$  is small, the  $F_0$  term in (16) dominates and the solution should behave as in (18). On the other hand when  $r$  is large,

the  $F_1$  term in (16) dominates and the solution should behave as (20). Then there should exist a solution connecting the solution in (18) in the small  $r$  region and the solution (20) in the large  $r$  region.

We may consider a more general case. By assuming  $B \neq 0$ , again, Eq. (12) can be rewritten as

$$N = \exp\left(\int dr \frac{2rF''}{rF' + 2F}\right). \quad (21)$$

We now rewrite Eq. (9) as follows:

$$0 = -2r^2 N^2 F B'' + r^2 (-3NN'F - 2N^2 F') B' + [r^2 (FN'^2 - 2NN''F - 3NN'F' + 2N^2 F'') - 4rN(FN' - NF') + 4N^2 F] B - 4N^2 F. \quad (22)$$

By substituting (21), we obtain

$$0 = -2r^2 F B'' + r^2 \left(-\frac{6rFF''}{rF' + 2F} - 2F'\right) B' + \left[4F + 4rF' - 4r^2 F'' - 2r^3 F''' + \frac{6r^3 F' F'' + 2r^4 F' F'''}{rF' + 2F} - \frac{6r^4 F'^2 F''}{(rF' + 2F)^2}\right] B - 4F. \quad (23)$$

Equation (23) is the linear inhomogeneous differential equation for  $B$  when  $F$  is given. For example, we consider the case  $F \propto r^n$  with a constant  $n$ . Then Eq. (23) reduces to

$$0 = -2r^2 B'' + r \frac{n(-8n+6)}{n+2} B' + \frac{8(-n^4 + n^3 + 3n^2 + 4n + 2)}{(n+2)^2} B - 4. \quad (24)$$

Let the solution of the following algebraic equation for a constant  $\alpha$  be  $\alpha_{\pm}$ :

$$0 = -2\alpha(\alpha - 1) + \frac{n(-8n+6)}{n+2} \alpha + \frac{8(-n^4 + n^3 + 3n^2 + 4n + 2)}{(n+2)^2}; \quad (25)$$

that is,

$$\alpha_{\pm} \equiv \frac{-2n^2 + 2n + 1 \mp \sqrt{-4n^3 + 12n^2 + 20n + 9}}{n+2}. \quad (26)$$

Then the solution of Eq. (24) is given by

$$B = B_+ r^{\alpha_+} + B_- r^{\alpha_-} + C. \quad (27)$$

Here  $C$  is a constant given by

$$C = \frac{(n+2)^2}{2(-n^4 + n^3 + 3n^2 + 4n + 2)}. \quad (28)$$

If  $C = 1$ , we have two real solutions  $n = 0$  and  $n = 2.39356\dots$  and two complex solutions  $n = -0.696781\dots \pm i0.591668\dots$ . The solution  $n = 0$  gives  $\alpha_{\pm} = -1, 2$ , and it corresponds to the Schwarzschild-(anti-)de Sitter spacetime in (18), but other cases correspond to new kinds of spherically symmetric solutions.

### A. New BH

As we discussed above, we have three unknowns in two independent differential equations. Thus, to be able to solve these differential equations, we assume in this study the unknown function  $F$  to have the form

$$F = 1 + \frac{c_1}{r^2}. \quad (29)$$

Equation (29) shows that when  $c_1 = 0$  we return to the case of GR, since in that case  $f(R) = \text{cons}$ :

$$B(r) = \frac{e^{3c_1/2r^2}}{r} \left\{ \mathbb{H}c_2 + \mathbb{H}_1 r^3 c_3 + 2\mathbb{H}_1 r^3 \int \frac{e^{-3c_1/2r^2} \mathbb{H}}{r[(2c_1 \mathbb{H}_2 - 3r^2 \mathbb{H}) \mathbb{H}_1 - 2c_1 \mathbb{H} \mathbb{H}_3]} dr - 2\mathbb{H} \int \frac{e^{-3c_1/2r^2} r^2 \mathbb{H}_1}{(2c_1 \mathbb{H}_2 - 3r^2 \mathbb{H}) \mathbb{H}_1 - 2c_1 \mathbb{H} \mathbb{H}_3} dr \right\},$$

$$N(r) = ce^{-3c_1/r^2}, \quad A(r) = N(r)B(r), \quad F = 1 + \frac{c_1}{r^2}, \quad (30)$$

where  $\mathbb{H} = \text{HeunC}(\frac{3}{2}, \frac{3}{2}, 0, \frac{3}{8}, \frac{9}{8}, -\frac{c_1}{r^2})$ ,  $\mathbb{H}_1 = \text{HeunC}(\frac{3}{2}, -\frac{3}{2}, 0, \frac{3}{8}, \frac{9}{8}, -\frac{c_1}{r^2})$ ,  $\mathbb{H}_2 = \text{HeunCPrime}(\frac{3}{2}, \frac{3}{2}, 0, \frac{3}{8}, \frac{9}{8}, -\frac{c_1}{r^2})$ , and  $\mathbb{H}_3 = \text{HeunCPrime}(\frac{3}{2}, -\frac{3}{2}, 0, \frac{3}{8}, \frac{9}{8}, -\frac{c_1}{r^2})$ .<sup>1</sup> Using Eq. (30) in the trace equation, i.e., the fourth equation of Eq. (8), we get  $f(r)$  in the form

$$f(r) = -\frac{2e^{3c_1/2r^2}}{r^7} \left\{ 2r^3(3r^2[r^2 + c_1] \mathbb{H}_1 + 2c_1[r^2 + 3c_1] \mathbb{H}_3) \int \frac{e^{-3c_1/2r^2} \mathbb{H}}{r[(2c_1 \mathbb{H}_2 - 3r^2 \mathbb{H}) \mathbb{H}_1 - 2c_1 \mathbb{H} \mathbb{H}_3]} dr \right. \\ \left. + 4c_1(3r^2 \mathbb{H} - [3c_1 + r^2] \mathbb{H}_2) \int \frac{e^{-3c_1/2r^2} \mathbb{H}_1 r^2}{(2c_1 \mathbb{H}_2 - 3r^2 \mathbb{H}) \mathbb{H}_1 - 2c_1 \mathbb{H} \mathbb{H}_3} dr + 3c_3 r^5 [r^2 + c_1] \mathbb{H}_1 \right. \\ \left. + 2c_1 c_3 r^3 [r^2 + 3c_1] \mathbb{H}_3 + 2c_1 c_2 [r^2 + 3c_1] \mathbb{H}_2 - 6c_1 c_2 r^2 \mathbb{H} - r^3 e^{-3c_1/2r^2} [r^2 + c_1] \right\}, \quad (32)$$

where  $c$ ,  $c_1$ ,  $c_2$ , and  $c_3$  are constants. Using Eq. (32) in Eq. (7), we get

$$R = -\frac{2e^{3c_1/2r^2}}{r^5(r^2 + c_1)} \left\{ 2r^3(3r^2[2r^2 + c_1] \mathbb{H}_1 + 2c_1[2r^2 + 3c_1] \mathbb{H}_3) \int \frac{e^{-3c_1/2r^2} \mathbb{H}}{r[(2c_1 \mathbb{H}_2 - 3r^2 \mathbb{H}) \mathbb{H}_1 - 2c_1 \mathbb{H} \mathbb{H}_3]} dr \right. \\ \left. + 4c_1(3r^2 \mathbb{H} - [3c_1 + 2r^2] \mathbb{H}_2) \int \frac{e^{-3c_1/2r^2} \mathbb{H}_1 r^2}{(2c_1 \mathbb{H}_2 - 3r^2 \mathbb{H}) \mathbb{H}_1 - 2c_1 \mathbb{H} \mathbb{H}_3} dr + 3c_3 r^5 [2r^2 + c_1] \mathbb{H}_1 \right. \\ \left. + 2c_1 c_3 r^3 [2r^2 + 3c_1] \mathbb{H}_3 + 2c_1 c_2 [2r^2 + 3c_1] \mathbb{H}_2 - 6c_1 c_2 r^2 \mathbb{H} - 2r^3 e^{-3c_1/2r^2} [r^2 + c_1] \right\}. \quad (33)$$

Equations (30), (32), and (33) show that when  $c_1 = 0$  we get

$$N(r) = c, \quad A(r) = B(r), \quad \text{and} \quad F(r) = 1. \quad (34)$$

Equation (34) shows that when  $F(r) = 1$  this gives  $f(R) = R$ , and in that case  $A(r) = B(r) = 1 + \frac{c}{r}$  provided that  $c_3 = 0$ . All the above data ensure that when  $c_1 = 0$  we return to the GR BHs.<sup>2</sup>

<sup>1</sup>The HeunC function is the solution of the Heun confluent equation, which is defined as

$$X''(r) - \frac{1 + \beta - (\alpha - \beta - \gamma - 2)r - r^2 \alpha}{r(r-1)} X'(r) - \frac{\alpha(1 + \beta) - \gamma - 2\eta - (1 + \gamma)\beta - r(2\delta + [2 + \gamma + \beta])}{2r(r-1)} X(r) = 0. \quad (31)$$

The solution of the above differential equation is defined HeunC( $\alpha, \beta, \gamma, \delta, \eta, r$ ); for more details, interested readers can check Refs. [102,103]. The HeunCPrime is the derivative of the Heun confluent function.

<sup>2</sup>Note that when  $c_1 = 0$  we get  $\mathbb{H} = \mathbb{H}_1 = \text{HeunC}(\frac{3}{2}, \frac{3}{2}, 0, \frac{3}{8}, \frac{9}{8}, 0) = \text{HeunC}(\frac{3}{2}, -\frac{3}{2}, 0, \frac{3}{8}, \frac{9}{8}, 0) = 1$ ,  $\mathbb{H}_2 = \text{HeunCPrime}(\frac{3}{2}, \frac{3}{2}, 0, \frac{3}{8}, \frac{9}{8}, -\frac{c_1}{r^2}) = 0$ , and  $\mathbb{H}_3 = \text{HeunCPrime}(\frac{3}{2}, -\frac{3}{2}, 0, \frac{3}{8}, \frac{9}{8}, -\frac{c_1}{r^2}) = -\frac{3}{2}$  [102,103].

### B. Physical properties of the BH (30)

We are going, in this section, to understand the physical properties of solution (30). To that aim, we write the asymptote behaviors of the metric potentials,  $A(r)$  and  $B(r)$ , given by Eq. (30) and get

$$\begin{aligned} A(r) &\approx 1 - \frac{2M}{r} - \frac{b^2}{r^2} + \frac{8b^4}{5r^3} + \frac{6b^2M}{r^3} - \frac{52b^4}{21r^4} + \dots, \\ B(r) &\approx 1 - \frac{2M}{r} + \frac{5b^2}{r^2} + \frac{8b^4}{5r^3} - \frac{6b^2M}{r^3} + \frac{200b^4}{21r^4} + \dots, \end{aligned} \quad (35)$$

where we have assumed  $c = 1$ ,  $c_1 = 2b^2$ ,  $c_2 = -2M$ , and  $c_3 = 0$ . Using Eq. (35) in (6), we get

$$ds^2 = - \left[ 1 - \frac{2M}{r} - \frac{b^2}{r^2} + \frac{8b^4}{5r^3} + \frac{6b^2M}{r^3} - \frac{52b^4}{21r^4} \right] dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{5b^2}{r^2} + \frac{8b^4}{5r^3} - \frac{6b^2M}{r^3} + \frac{200b^4}{21r^4}} + r^2(d\theta^2 + \sin^2 d\phi^2). \quad (36)$$

The line element (36) is asymptotically approaching a flat spacetime and does not coincide with the Schwarzschild spacetime due to the contribution of the extra terms that come mainly from the constant parameter  $b$ , whose source is the effect of higher-order curvature terms of  $f(R)$ . As one can check easily, when these extra terms equal zero, one can smoothly return to the Schwarzschild spacetime [104]. We assume the constant  $c_3 = 0$  in the asymptote of the metric potential (35). This assumption makes the metric potentials asymptote to flat spacetime. When the constant  $c_3 \neq 0$ , the metric potentials will not asymptote to flat spacetime but to AdS/dS spacetime as follows:

$$\begin{aligned} A(r) &\approx \pm \Lambda_{\text{eff}} r^2 + 1 - \frac{2M}{r} - \frac{b^2}{r^2} + \frac{8b^4}{5r^3} + \frac{6b^2M}{r^3} - \frac{52b^4}{21r^4} + \dots, \\ B(r) &\approx \pm \Lambda_{\text{eff}} r^2 + 1 \pm 2 - \frac{2M}{r} + \frac{(5 \pm 6)b^2}{r^2} + \frac{8b^4}{5r^3} - \frac{6b^2M}{r^3} + \frac{(200 \pm 252)b^4}{21r^4} + \dots, \end{aligned} \quad (37)$$

where  $\Lambda_{\text{eff}} = c_3 = \pm \frac{1}{3b^2}$ . Using Eq. (37) in (6), we get

$$\begin{aligned} ds^2 = & - \left[ \pm \Lambda_{\text{eff}} r^2 + 1 - \frac{2M}{r} - \frac{b^2}{r^2} + \frac{8b^4}{5r^3} + \frac{6b^2M}{r^3} - \frac{52b^4}{21r^4} \right] dt^2 \\ & + \frac{dr^2}{\pm \Lambda_{\text{eff}} + 1 \pm 2 - \frac{2M}{r} + \frac{(5 \pm 6)b^2}{r^2} + \frac{8b^4}{5r^3} - \frac{6b^2M}{r^3} + \frac{(200 \pm 252)b^4}{21r^4}} + r^2(d\theta^2 + \sin^2 d\phi^2). \end{aligned} \quad (38)$$

The line element (36) asymptotically approaches AdS/dS spacetime according to the sign of  $\Lambda_{\text{eff}}$ .

Now we are going to use Eq. (38) in Eq. (7) and get

$$\begin{aligned} R(r) &\approx -12c_3 - \frac{18c_1c_3}{r^2} + \frac{6c_1}{r^4} + \dots \equiv -12\Lambda_{\text{eff}} \mp \frac{12}{r^2} + \frac{12b^2}{r^4} + \dots, \\ r(R) &= \pm \frac{\sqrt{2(\mp 3 \pm \sqrt{9 \pm 12 + 3R})}}{\sqrt{12\Lambda_{\text{eff}} + R}} \approx \pm 0.88954361b \mp 0.019200037b^3R \pm 0.0013073405b^5R^2 \mp 0.0001157435b^7R^3 \\ &+ \dots, \end{aligned} \quad (39)$$

where we have put  $c_3 = \frac{1}{3b^2}$  which correspond to AdS spacetime and the other two roots are neglected because they give imaginary quantities. Equation (39) shows that when the constant  $c_3 = 0$  we have a nontrivial value of the Ricci scalar which contributes to higher-order curvature, and when  $c_1 = 0$  we get a trivial value of the Ricci scalar which corresponds to a GR BH. The asymptote form of  $f(r)$ , given by Eq. (32), has the form

$$\begin{aligned} f(r) &\approx -6\Lambda_{\text{eff}} - \frac{16}{r^2} - \frac{16b^2}{r^3} - \frac{50b^2}{r^4} - \frac{201.6b^4}{r^5} - \frac{48b^2M}{r^5} - \frac{60b^4}{r^6} + \dots, \quad \text{for } c_3 = \frac{1}{3b^2}, \\ &\approx 6\Lambda_{\text{eff}} + \frac{16}{r^2} + \frac{16b^2}{r^3} + \frac{70b^2}{r^4} + \frac{182b^4}{r^5} - \frac{48b^2M}{r^5} + \frac{228b^4}{r^6} + \dots, \quad \text{for } c_3 = \frac{-1}{3b^2}. \end{aligned} \quad (40)$$

Using the second equation of (39) in (40), we get

$$f(R) \approx C_1 + C_2 R + C_3 R^2 + C_4 R^3, \quad (41)$$

with  $C_i$ ,  $i = 1 \dots 4$ , constants that their values depend on the sign of  $c_3 \pm \frac{1}{3b^2}$ .

Now use Eq. (30) in order to calculate the invariants to obtain

$$\begin{aligned} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= 24\Lambda_{\text{eff}}^2 + \frac{16}{b^2 r^2} + \frac{32}{r^4} + \frac{64M}{r^5} + \frac{64b^2}{15r^5} \dots, \\ R_{\mu\nu}R^{\mu\nu} &= 36\Lambda_{\text{eff}}^2 + \frac{24}{b^2 r^2} + \frac{24}{r^4} + \frac{96M}{r^5} + \frac{32b^2}{5r^5} \dots, \\ R &= -12\Lambda_{\text{eff}} - \frac{12}{r^2} + \frac{12b^2}{r^4} - \frac{48b^2 M}{r^5} - \frac{16b^4}{5r^5} \dots, \end{aligned} \quad (42)$$

with  $(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R)$  the Kretschmann scalar, the Ricci tensor square, and the Ricci scalar, respectively, and all of them have a true singularity at  $r = 0$ . Moreover, the above equations show that  $b^2 = c_1/2$  must not equal zero. It is important to stress the fact that the constant  $c_1$  is the main source for the deviation of the above results from GR that has the following values:  $(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R) = (16[2\Lambda^2 r^6 + 9M]/[3r^6], 16\Lambda^2, \mp 8\Lambda)$ . Equation (42) indicates that the leading term of the invariants  $(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R)$  is  $(\frac{1}{r^2}, \frac{1}{r^2}, \frac{1}{r^2})$ , which is different from the Schwarzschild BH, which gives the leading term of the Kretschmann scalar as  $\frac{1}{r^6}$  and the other invariants  $R_{\mu\nu}R^{\mu\nu} = R = \text{const}$ . Therefore, Eq. (42) indicates that Kretschmann singularity is milder than the Schwarzschild BH of GR.

#### IV. THERMODYNAMICS OF THE BH

We are going in this section to study the properties of the BHs (35) and (37) from the viewpoint of thermodynamics. To this aim, we will write the basic definitions of the

quantities of thermodynamics that we will use. The surface gravity of a spacetime having two horizons is defined as

$$\chi_{\pm} = \frac{r_{\pm} - r_{\mp}}{2r_{\pm}^2}, \quad (43)$$

where  $r_{\pm}$  are the inner and outer horizons of the spacetime. The Hawking temperature is given by [105–110]

$$T_{\pm} = \frac{\chi_{\pm}}{2\pi} = \frac{r_{\pm} - r_{\mp}}{4\pi r_{\pm}^2}. \quad (44)$$

The semiclassical Bekenstein-Hawking entropy of the horizons is defined as

$$\delta_{\pm} = \frac{1}{4}(A_{\pm})f_R, \quad (45)$$

with  $A_{\pm}$  being the area of the horizons. The quasilocal energy is figured out as [105–108,111,112]

$$\begin{aligned} E(r_{\pm}) &= \frac{1}{4} \int [2f_R(r_{\pm}) + r_{\pm}^2 \{f(R(r_{\pm})) \\ &\quad - R(r_{\pm})f_R(r_{\pm})\}] dr_{\pm}. \end{aligned} \quad (46)$$

Finally, the Gibbs free energy is figured out as [112,113]

$$G(r_{\pm}) = E(r_{\pm}) - T(r_{\pm})S(r_{\pm}). \quad (47)$$

#### A. Thermodynamics of solution (35) that has a flat spacetime

The BH (35) derived in the previous section is portrayed by the mass of the BH  $M$  and the parameter  $b$ , and when the parameter  $b$  is vanishing we get the Schwarzschild spacetime which corresponds to GR. To find the horizons of this BH, (35), we put  $A(r) = 0$ . This gives four roots; two of them are real, and the others are imaginary. The real roots have the form

$$r_{\pm} = \frac{[14910M^{4/3} + \sqrt{710}X_1^2]X_1 \pm \sqrt{710}\sqrt{Y_1X_1^2 + 7.008182433 \times 10^7 M(5M^3 - 5Mb^2 - 8b^4)}}{29820M^{1/3}X_1}, \quad (48)$$

where  $X_1 = [993.4163240(112M + 1625)b^{8/3} + 313110M^{8/3} + 1043700b^2M^{2/3} + 9.398818978 \times 10^5 b^{4/3}M^{4/3} + 4.172348561 \times 10^5 b^{2/3}M^2]^{1/4}$  and  $Y_1 = [-993.4163240(112M + 1625)b^{8/3} + 626220M^{8/3} + 2087400b^2M^{2/3} - 9.398818978 \times 10^5 b^{4/3}M^{4/3} - 4.172348561 \times 10^5 b^{2/3}M^2]$ . Equation (48),  $r_{\pm}$ , puts the following constraint to have a real value:

<sup>3</sup>The constants  $C_i$ ,  $i = 1 \dots 4$ , have different values depending on the sign of  $c_3 = \pm \frac{1}{3b^2}$ . For example, when  $c_3 = \frac{1}{3b^2}$ , the constants  $C_i$  take the values  $C = -(669.5280519\Lambda_{\text{eff}} \pm \frac{384.6857554}{b} \pm \frac{86.17970730M}{b^3})$ ,  $C_1 = -(23.45043154 \pm 40.53430523b \pm \frac{9.300575885M}{b})$ ,  $C_2 = (\pm 0.1670765328b^3 \pm 0.03104615112qM + 0.01169075762b^2)$ , and  $C_3 = -(\pm 0.02091872050b^5 \pm 0.004384004185b^3M + 0.01057683849b^4)$ .

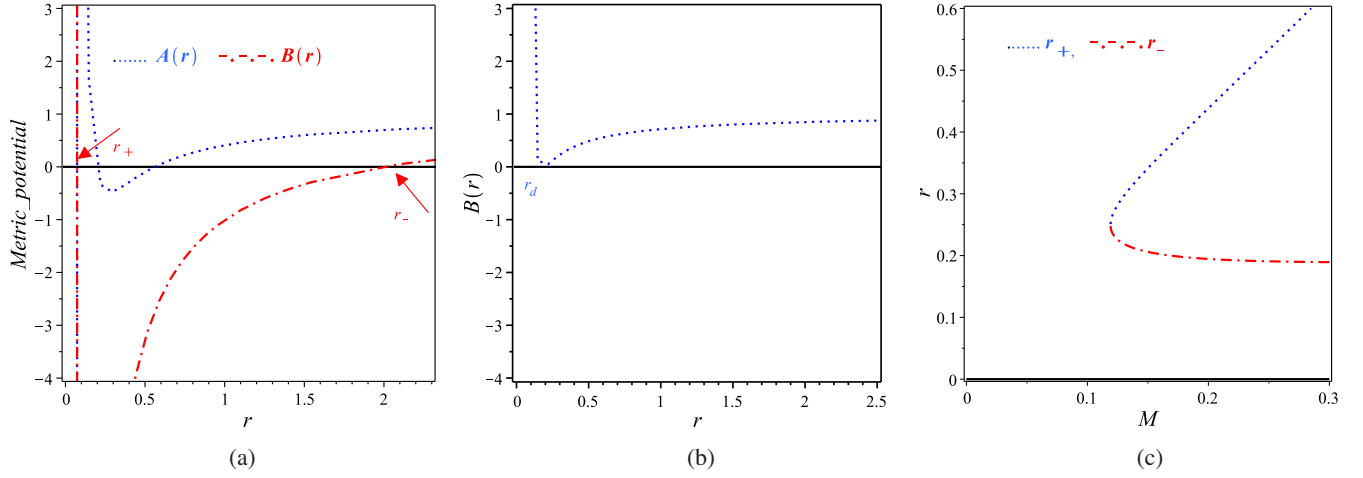


FIG. 1. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) The metric potential of BH (35), (b) The metric potential of BH (35), and (c) Hawking temperature of BH (35).

$$Y_1 X_1^2 + 7.008182433 \times 10^7 (5M^3 - 5Mq^2 - 8b^4) > 0. \quad (49)$$

The metric potentials of the BH (35) are drawn in Fig. 1(a). From Fig. 1(a), we can easily see the two horizons of the metric potentials  $A(r)$  and  $B(r)$ . Also, the behavior of the horizons given by Eq. (48) are drawn in Fig. 1(c). It is easy to check that the degenerate horizon for the metric potential  $B(r)$  happens for a specific value of  $(b, M, r) \equiv (0.1, 0.163, 0.1647332393)$ , respectively, which correspond to the Nariai BH. The degenerate behavior is shown in Fig. 1(b). Figure 1(c) shows the horizon  $r_+$  increasing with  $M$  while  $r_-$  is decreasing. Using Eq. (44), the Hawking temperature can be calculated as

$$T_{\pm} = \pm \frac{14058.36M^{1/3} \sqrt{X_1^2 Y_1 + [350409121.7M^4 - 350409121.7M^2 b^2 - 560654594.7b^4 M] X_1^{1/4}}}{((4970M^{4/3} + 8.88194173X_2^2)X_1 \pm 8.88 \sqrt{Y_1 X_1^2 + 350409121.7M^4 - 350409121.7M^2 b^2 - 560654594.7b^4 M})^2}. \quad (50)$$

The behavior of the Hawking temperature given by Eq. (50) is drawn in Fig. 2(a), which shows that  $T_+ > T_-$ . As Fig. 2(a) shows, the  $T_+$  has an increasing positive temperature while  $T_-$  has a decreasing negative temperature. Figure 2(a) indicates that  $T_+$  has a vanishing value at  $r_+ = r_d$ . Moreover, when  $r_+ < r_d$ ,  $T_-$  becomes negative and an ultracold BH is formed. Also, Davies [114] clarified that there is no clear reason from thermodynamical effects to prevent the BH temperature from being below absolute zero, and in that case a naked singularity is formed. Figure 2(a) shows the Davies argument at the  $r_+ < r_{\min}$  region.

Using Eq. (45), we get the entropy of BH (35) in the form

$$\delta_{\pm} = \frac{3.531135393([14910M^{4/3} + \sqrt{710}X_1^2]X_1 \pm \sqrt{710} \sqrt{Y_1 X_1^2 + 7.008182433 \times 10^7 M(5M^3 - 5Mq^2 - 8b^4)})^2}{10^9 M^{2/3} X_1^2} + \pi b^2. \quad (51)$$

The entropy behavior is given in Fig. 2(b) that indicates an increasing value for  $\delta_+$  and a decreasing value for  $\delta_-$ . From Eq. (46), the quasilocal energy takes the form

$$E_{\pm} = \frac{[14910M^{4/3} + \sqrt{710}X_1^2]X_1 \pm \sqrt{710} \sqrt{Y_1 X_1^2 + 7.008182433 \times 10^7 M(5M^3 - 5Mb^2 - 8b^4)}}{59640M^{1/3}X_1} - \frac{29820M^{1/3}X_1 b^2}{2\{[14910M^{4/3} + \sqrt{710}X_1^2]X_1 \pm \sqrt{710} \sqrt{Y_1 X_1^2 + 7.008182433 \times 10^7 M(5M^3 - 5Mb^2 - 8b^4)}\}}. \quad (52)$$

The quasilocal energy behaviors are shown in Fig. 3(a), which also shows a positive increasing value for  $E_+$  and a positive decreasing value for  $E_-$ . Finally, we use Eqs. (50)–(52) in Eq. (47) to calculate the Gibbs free energies. The behavior of these free energies is in Fig. 3(b), which shows positive increasing for  $G_+$  and positive decreasing for  $G_-$ .



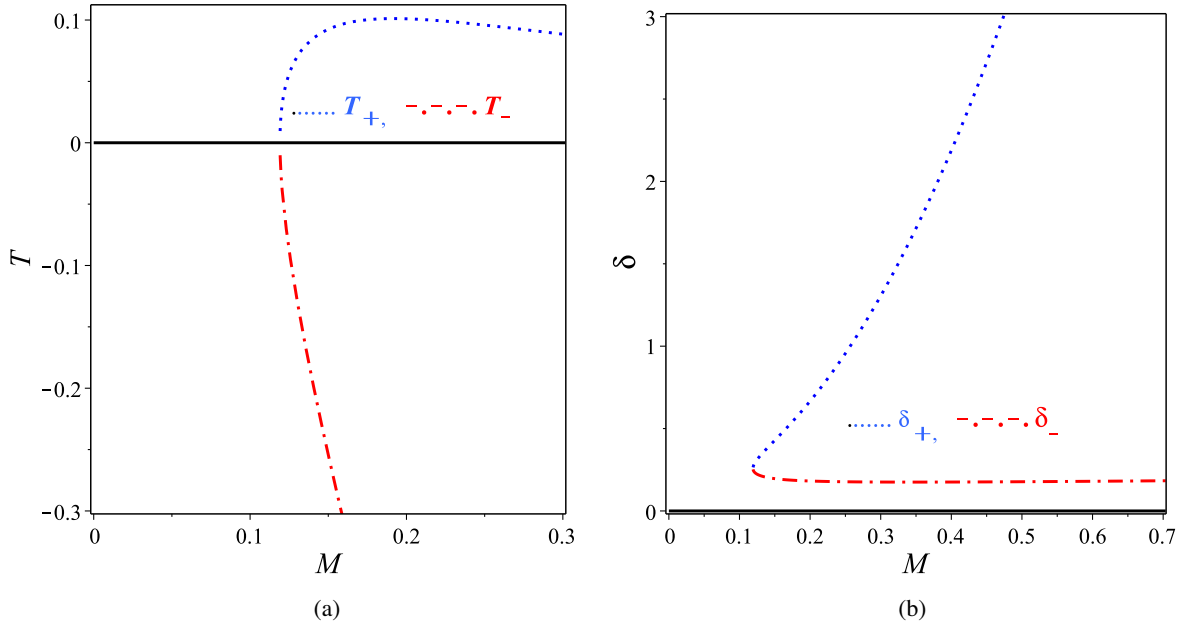


FIG. 2. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) Hawking temperature of BH (35) and (b) Entropy of BH (35).

**B. Thermodynamics of the BH (37) that has asymptote flat AdS**

In this subsection, we are going to study the BH (37) which is characterized by the mass of the BH  $M$ , the parameter  $b$ , and a positive cosmological effective constant. The metric potential when  $\Lambda_{\text{eff}} = \frac{1}{3b^2}$  takes the form

$$\begin{aligned}
 A(r) &\approx \Lambda_{\text{eff}} r^2 + 1 - \frac{2M}{r} - \frac{b^2}{r^2} + \frac{8b^4}{5r^3} + \frac{6b^2M}{r^3} - \frac{52b^4}{21r^4} + \dots, \\
 B(r) &\approx \Lambda_{\text{eff}} r^2 + 3 - \frac{2M}{r} + \frac{11b^2}{r^2} + \frac{8b^4}{5r^3} - \frac{6b^2M}{r^3} + \frac{452b^4}{21r^4} + \dots.
 \end{aligned}
 \tag{53}$$

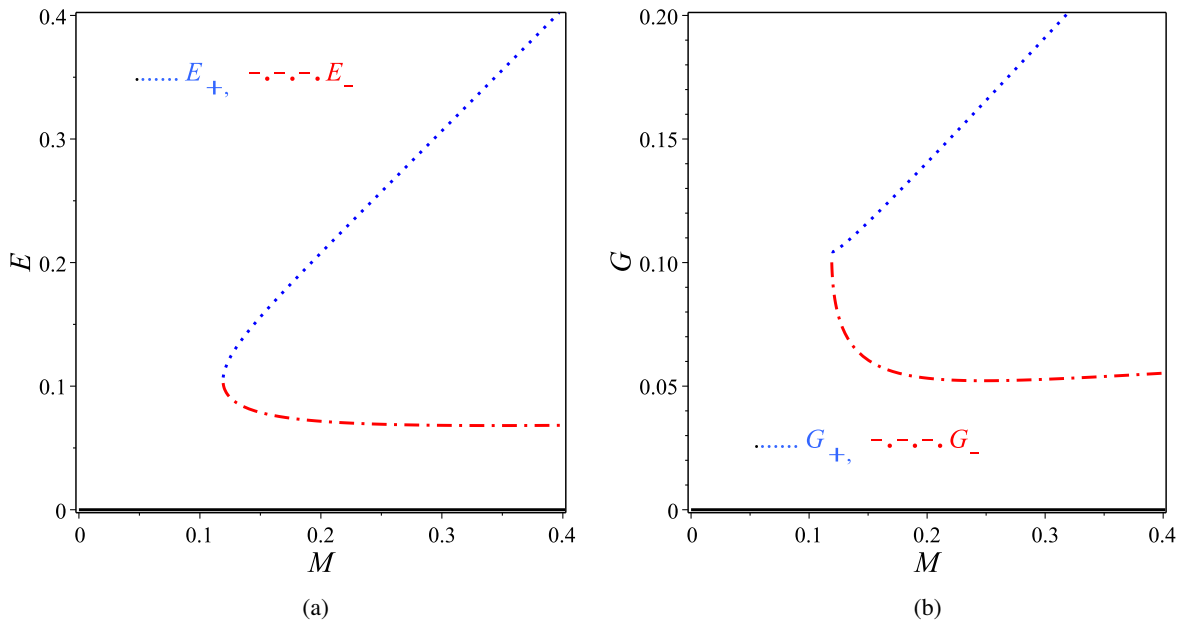


FIG. 3. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) Quasilocal energy of BH (35) and (b) Gibbs energy of BH (35).

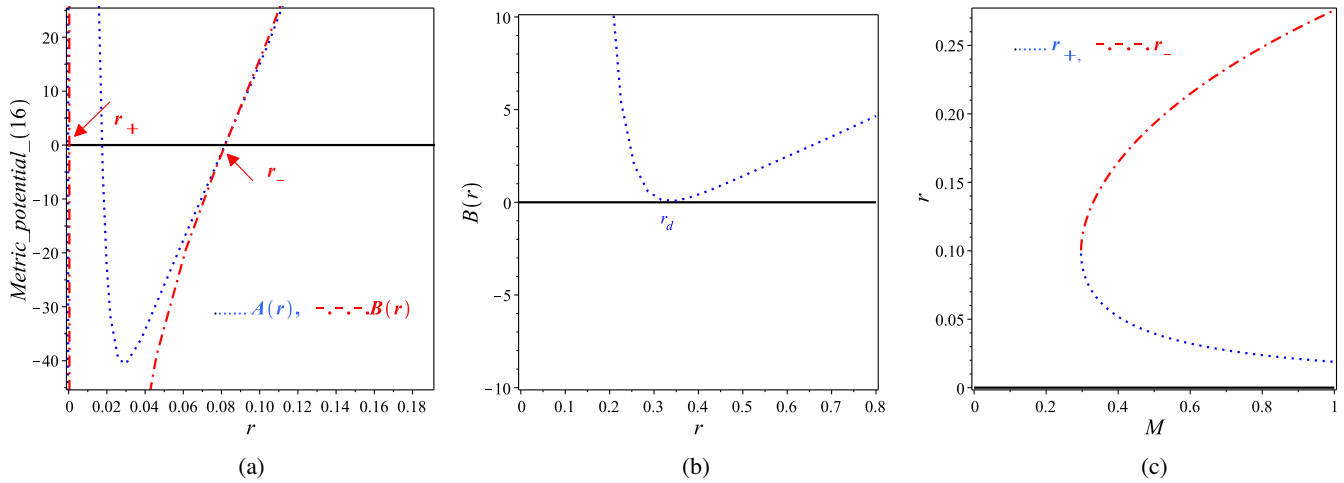


FIG. 4. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) The metric potential of BH (53), (b) The metric potential of BH (53), and (c) Hawking temperature of BH (53).

When the parameter  $b$  is vanishing, we get the Schwarzschild AdS spacetime which corresponds to the Einstein GR. The metric potentials of the BH (53) are drawn in Fig. 4(a). From Fig. 4(a), we can easily see the two horizons of the metric potentials  $A(r)$  and  $B(r)$ . To find the horizons of this BH, (53), we put  $A(r) = 0$  in Eq. (53) [109]. This gives six roots; two of them are real, and the others are imaginary. These real roots are lengthy; however, their behavior is drawn in Fig. 4(c). It is easy to check that the degenerate horizon for the metric potential  $B(r)$  given by Eq. (53) happens for specific values for  $(b, M, r) \equiv (0.242, 1, 0.333717999)$ , respectively, which corresponds to the Nariai BH. The degenerate behavior is shown in Fig. 4(b). Figure 4(c) shows the horizon  $r_-$  increasing with

$M$  while  $r_+$  is decreasing. Using Eq. (44), we draw the behavior of the Hawking temperatures in Fig. 5(a), which shows that  $T_- > T_+$ . As Fig. 5(a) shows,  $T_-$  has an increasing positive temperature while  $T_+$  has a decreasing negative temperature. Figure 5(a) shows that  $T_- = 0$  at  $r_+ = r_d$ . At  $r_+ < r_d$ ,  $T_- < 0$  and an ultracold BH is formed.

Using Eq. (45), we draw the entropy in Fig. 5(b), showing an increasing value for  $\delta_-$  and a decreasing value for  $\delta_+$ . Using Eq. (46), we draw the quasilocal energy in Fig. 6(a) and show that it has a positive increasing value for  $E_{\pm}$ . Finally, we use Eqs. (50)–(52) in Eq. (47) to calculate the Gibbs free energies. The behavior of these free energies is shown in Fig. 6(b), which shows positive increasing values for  $G_{\pm}$ .

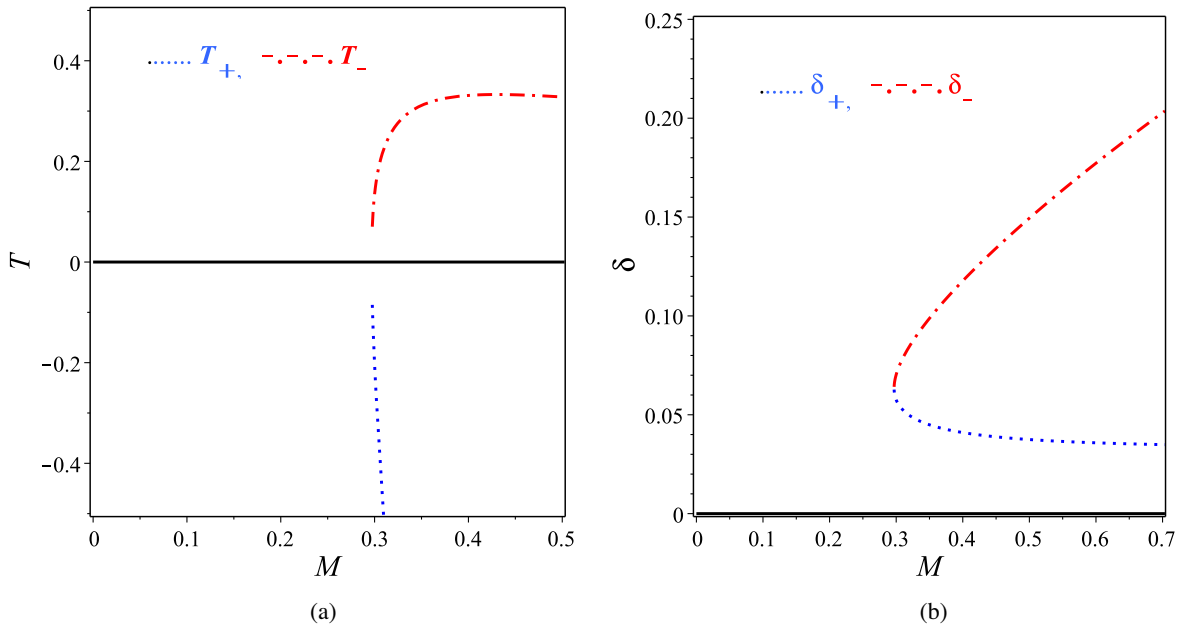


FIG. 5. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) Hawking temperature of BH (53), (b) Entropy of BH (53).

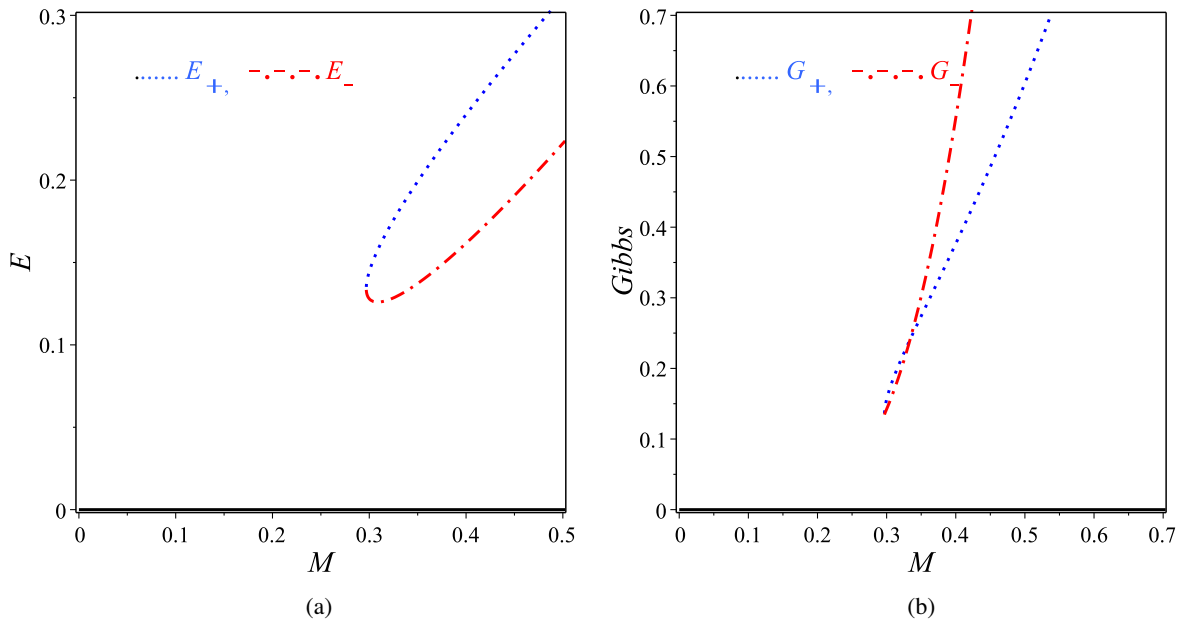


FIG. 6. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) Quasilocal energy of BH (53) and (b) Gibbs energy of BH (53).

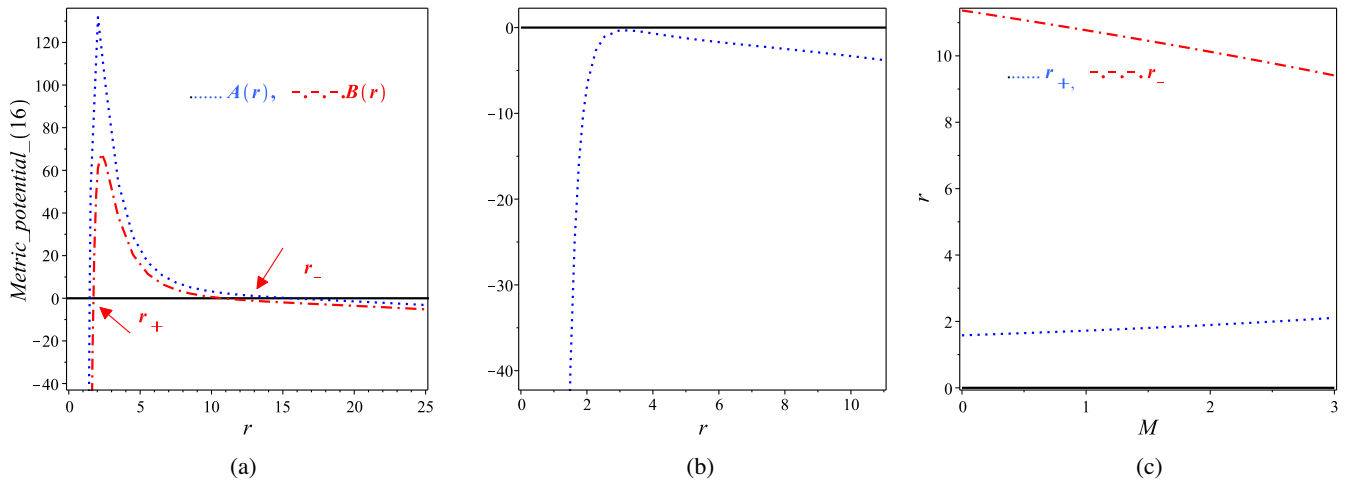


FIG. 7. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) The metric potential of BH (54), (b) The metric potential of BH (54), and (c) Hawking temperature of BH (54).

### C. Thermodynamics of the BH (37) that has asymptote flat dS

The BH (54) is characterized by the mass of the BH  $M$ , the parameter  $b$ , and a cosmological effective constant.<sup>4</sup>The metric potentials of the BH (54) are figured in Fig. 7(a). From Fig. 7(a), we can easily see the two horizons of the

<sup>4</sup>The metric potential when  $\Lambda_{\text{eff}} = \frac{1}{3b^2}$  takes the form

$$\begin{aligned}
 A(r) &\approx -\Lambda_{\text{eff}} r^2 + 1 - \frac{2M}{r} - \frac{b^2}{r^2} + \frac{8b^4}{5r^3} + \frac{6b^2 M}{r^3} - \frac{52b^4}{21r^4} + \dots, \\
 B(r) &\approx -\Lambda_{\text{eff}} r^2 - 1 - \frac{2M}{r} - \frac{b^2}{r^2} + \frac{8b^4}{5r^3} - \frac{6b^2 M}{r^3} - \frac{52b^4}{21r^4} + \dots.
 \end{aligned} \quad (54)$$

metric potentials  $A(r)$  and  $B(r)$ . To find the horizons of this BH, (54), we put  $A(r) = 0$  in Eq. (54) [109]. This gives six roots; two of them are real, and the others are imaginary. These real roots are lengthy; however, their behavior is drawn in Fig. 7(c). It is easy to check that the degenerate horizon for the metric potential  $B(r)$  given by Eq. (54) happens for a specific value for  $(b, M, r) \equiv (0.242, 1, 0.333717999)$ , respectively, which corresponds to a Nariai BH. The degenerate behavior is shown in Fig. 7(b). Figure 7(c) shows the horizon  $r_-$  increasing with  $M$  while  $r_+$  is decreasing. Using Eq. (44), we draw the behavior of the Hawking temperatures in Fig. 8(a), which shows that  $T_+ < T_-$ . As Fig. 8(a) shows, the  $T_-$  has an increasing

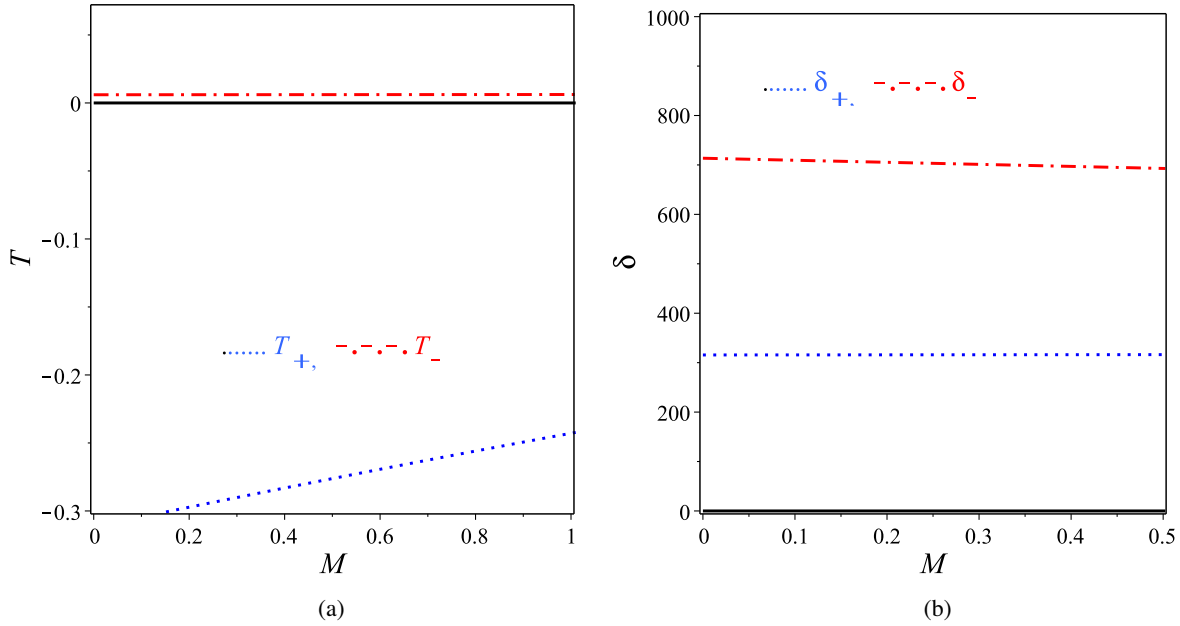


FIG. 8. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) Hawking temperature of BH (53) and (b) Entropy of BH (54).

positive temperature while  $T_+$  has a decreasing negative temperature.

Using Eq. (45), we draw the entropy in Fig. 8(b), showing that it has a positive value for  $\delta_{\pm}$ . From Eq. (46), we calculate and draw the quasilocal energy in Fig. 9(a), showing that it has a positive increasing value for  $E_{\pm}$ . Finally, we use Eqs. (50)–(52) in Eq. (47) to calculate the Gibbs free energies. The behavior of these free energies

is shown in Fig. 9(b), which shows positive increasing values for  $G_{\pm}$ .

**D. First law of thermodynamics of the BHs (35) and (37)**

It is worth examining the verification of the first law for the BHs (35) and (37). Accordingly, we will apply this law on  $f(R)$  using the following form [115]:

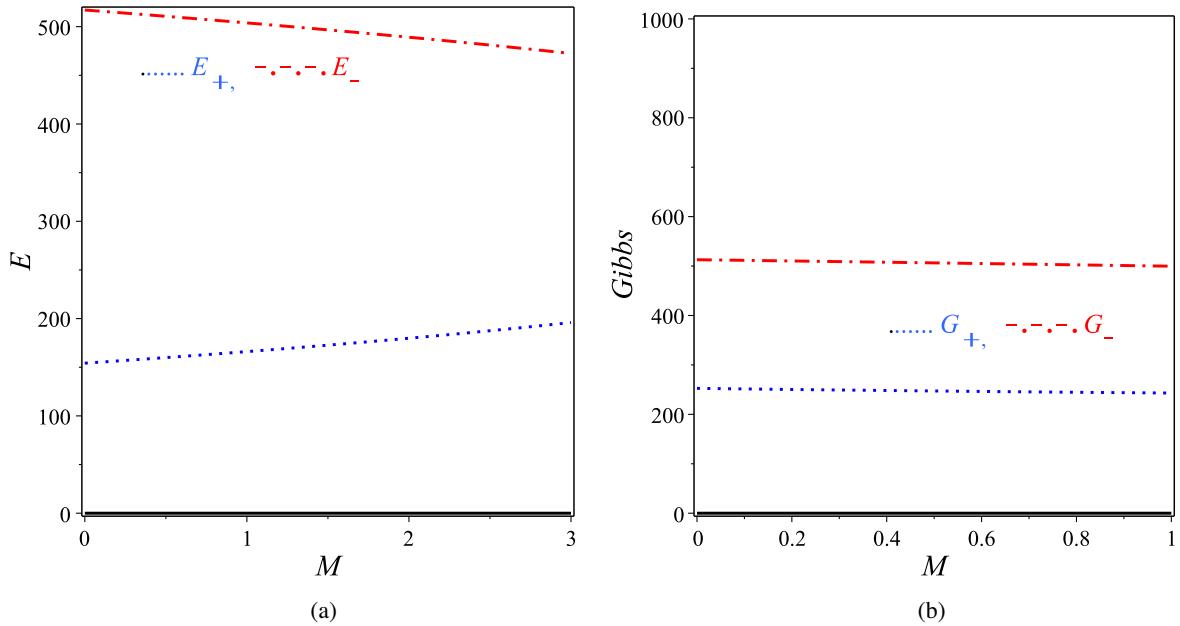


FIG. 9. Plot of the horizons given by Eq. (48) using  $b = 0.1$ . (a) Quasilocal energy of BH (54) and (b) Gibbs energy of BH (54).

$$dE = Td\delta - PdV, \quad (55)$$

where  $E$  is the quasilocal energy,  $\delta$  is the Bekenstein-Hawking entropy,  $T$  is the Hawking temperature,  $P$  is the radial component of the stress-energy tensor that serves as a thermodynamic pressure  $P = T_r{}^r|_{\pm}$ , and  $V$  is the geometric volume. In the frame of  $f(R)$  gravitational theory, the pressure is defined as [115]

$$P = -\frac{1}{8\pi} \left\{ \frac{F}{r_{\pm}^2} + \frac{1}{2}(f - RF) \right\} + \frac{1}{4} \left( \frac{2F}{r_{\pm}} + F' \right) T. \quad (56)$$

For the flat spacetime (35), if we neglect  $O(\frac{1}{r^4})$  to make the calculations more applicable, we get<sup>5</sup>

$$r = \frac{2M}{3} + \frac{\sqrt[3]{1000m^3 - 2700b^4 - 9000mb^2 + 75\mathfrak{U}}}{15} + \frac{5(3b^2 + 4m^2)}{3\sqrt[3]{1000m^3 - 2700b^4 - 9000mb^2 + 75\mathfrak{U}}}, \quad E = \frac{r^4 - b^2r^2 + 4b^4}{2r^3},$$

$$\delta = \pi(r^2 + 2b^2), \quad T = \frac{5mr^2 + 5b^2r - 45b^2m - 12b^4}{10\pi r^4}, \quad P = \frac{10mr^2 - 5r^3 + 5b^2r - 90mb^2 - 24b^4}{40\pi r^5}, \quad (57)$$

where  $\mathfrak{U} = \sqrt{14100m^2b^4 - 75b^6 - 3600m^4b^2 + 8640mb^6 + 1296b^8 - 960m^3b^4}$ . Substituting the form of  $r$  into the thermodynamical quantities, we get

$$dE = \frac{4m^4 + b^2m^2 + b^4}{8m^4}, \quad \delta = \frac{\pi(40m^4 - 20m^2b^2 - 8mb^4 - 25b^4)}{10m^3}, \quad T = \frac{4mb^4 - 20m^4 + 35b^4 + 15m^2b^2}{160\pi m^5},$$

$$P = -\frac{b^2(30m^2 + 75b^2 + 8mb^2)}{640\pi m^6}. \quad (58)$$

If we use Eq. (58) in (56), we can verify the first law of thermodynamics for the BH (35).

If one repeats the same procedure for the BH (37), one can verify the first law of thermodynamics provided that we neglect all the quantities containing  $b$  to make the calculation easier to carry out.

## V. THE STABILITY OF THE BHS IN $f(R)$ GRAVITY

To study the stability of the above solutions, we recast  $f(R)$  gravity using scalar-tensor theory. The action given by Eq. (1) can be recast as [116]

$$\mathcal{I} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\psi \mathbf{R} - V(\psi)], \quad (59)$$

where  $\psi$  is a scalar field coupled to the Ricci scalar ( $\mathbf{R}$ ) and  $V(\psi)$  is the potential (see Refs. [23,116] for details). To discuss the perturbation, we use the spherically symmetric metric as

$$ds^2 = g_{\mu\nu}^0 dx^\mu dx^\nu = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (60)$$

with  $g_{\mu\nu}^0$  being the background metric. We are going to check BH solutions, (35) and (37), using the linear

<sup>5</sup>When we neglect the terms of the order of  $O(\frac{1}{r^4})$  and when  $A(r) = 0$ , we get three roots; one of them has only positive value, while the other two are imaginary.

perturbations, if their background metrics are stable or not. Moreover, we are going to investigate the value of the speed of propagation. For this theory, the background equation reads

$$V = -\frac{4B\psi'}{r} - \frac{2\psi BA'}{Ar} - \frac{\psi'BA'}{A} + \frac{2\psi}{r^2} - \frac{2B\psi}{r^2},$$

$$\psi'' = -\frac{\psi'B'}{2B} - \frac{\psi B'}{rB} + \frac{\psi'A'}{2A} + \frac{\psi A'}{rA}, \quad \mathbf{R} = \frac{dV}{d\psi}, \quad (61)$$

where a prime means the differentiation with respect to the radial coordinate  $r$ .

### A. Brief description of Regge-Wheeler-Zerilli construction

Following the workings by Regge, Wheeler [117], and Zerilli [118], we decompose the metric perturbations subject to the transformation properties of two-dimensional rotations. Because of the fact that Regge, Wheeler, and Zerilli studied the perturbations of the Schwarzschild BH in GR, we can apply it to the BHs of  $f(R)$ .

We will assume the perturbation of  $g_{\mu\nu}$  as

$$g_{\mu\nu} = g_{\mu\nu}^0 + h_{\mu\nu},$$

with  $h_{\mu\nu}$  being small quantities. In the lower order, we assume the perturbations to be small, i.e.,  $g_{\mu\nu}^0 \gg h_{\mu\nu}$ . Therefore, on two-dimensional rotations,  $h_{tt}$ ,  $h_{rr}$ , and  $h_{rr}$  transform as  $h_{ta}$ ,  $h_{ra}$  transforms as a vector, and  $h_{ij}$

transforms as a tensor (where  $i$  and  $j$  are either  $\theta$  or  $\phi$ ). It is well known that  $\Phi$  can be expressed as

$$\Theta(t, r, \theta, \phi) = \sum_{\ell, m} \Theta_{\ell m}(t, r) Y_{\ell m}(\theta, \phi). \quad (62)$$

Thus, the apparent solution becomes independent of the index  $m$  and takes the form

$$\Delta_{\theta, \phi} Y_{\ell}(\theta, \phi) = -\ell(\ell + 1) Y_{\ell}(\theta, \phi). \quad (63)$$

As for the vector  $C_a$ , one can decompose it into two parts, a divergence part and a nondivergence one, as

$$C_a(t, r, \theta, \phi) = \nabla_a \Theta_1 + E_a^b \nabla_b \Theta_2, \quad (64)$$

such that  $\Theta_1$  and  $\Theta_2$  are two scalars and  $E_{ab} \equiv \sqrt{\det \Omega} \epsilon_{ab}$ , where  $\Omega_{ab}$  is the two-dimensional metric and  $\epsilon_{ab}$  is completely skew symmetric with  $\epsilon_{\theta\phi} = 1$ . In this study,  $\nabla_a$  is the covariant derivative with respect to the metric  $\Omega_{ab}$ . Since  $C_a$  is a two-component vector, it can be described by  $\Theta_1$  and  $\Theta_2$ . Thus, the scalar decomposition (62) can be applied to  $\Theta_1$  and  $\Theta_2$  to decompose  $C_a$ .

For  $S_{ab}$ , which is symmetric, one can decompose it as

$$S_{ab}(t, r, \theta, \phi) = \nabla_a \nabla_b \Theta_1 + \gamma_{ab} \Theta_2 + \frac{1}{2} (E_a^c \nabla_c \nabla_b \Theta_3 + E_b^c \nabla_c \nabla_a \Theta_3), \quad (65)$$

with  $\Theta_1$ ,  $\Theta_2$ , and  $\Theta_3$  scalars. Since  $S_{ab}$  have three independent components that completely describe  $S_{ab}$ , one can use the scalar decomposition (62) to  $\Theta_1$ ,  $\Theta_2$ , and  $\Theta_3$  to decompose  $S_{ab}$ . We mention  $O_{ab}$  by the odd-type variables and the rest by even-type ones. The use of these

methods has an advantage that in the linearized form the odd type and even type are separated, which makes us study them separately.

## B. Perturbations of $f(R)$ gravity using the odd modes

Using the Regge-Wheeler formalism, the odd-type metric perturbations can have the form

$$h_{tt} = 0, \quad h_{tr} = 0, \quad h_{rr} = 0, \quad (66)$$

$$h_{ta} = \sum_{\ell, m} h_{0, \ell m}(t, r) E_{ab} \partial^b Y_{\ell m}(\theta, \phi), \quad (67)$$

$$h_{ra} = \sum_{\ell, m} h_{1, \ell m}(t, r) E_{ab} \partial^b Y_{\ell m}(\theta, \phi), \quad (68)$$

$$h_{ab} = \frac{1}{2} \sum_{\ell, m} h_{2, \ell m}(t, r) [E_a^c \nabla_c \nabla_b Y_{\ell m}(\theta, \phi) + E_b^c \nabla_c \nabla_a Y_{\ell m}(\theta, \phi)]. \quad (69)$$

Thus, implementing the gauge transformation  $x^\mu \rightarrow x^\mu + \xi^\mu$ , we can put some of the metric perturbations equal to zero due to the fact that all of them need not be physical ones. We use the following transformation for the odd-type perturbation:

$$\xi_t = \xi_r = 0, \quad \xi_a = \sum_{\ell, m} \Lambda_{\ell m}(t, r) E_a^b \nabla_b Y_{\ell m}, \quad (70)$$

where  $\Lambda_{\ell m}$  can put  $h_{2, \ell m}$  equal to zero (the Regge-Wheeler gauge). Using this method, we can fix  $\Lambda_{\ell m}$ , and the action of odd modes takes the form [117]

$$I_{\text{odd}} = \frac{1}{2\kappa} \sum_{\ell, m} \int dt dr \mathcal{I}_{\text{odd}} = \frac{1}{4\kappa} \sum_{\ell, m} \int dt dr j^2 \left[ \psi \sqrt{\frac{B}{A}} (\dot{h}_1 - h'_0)^2 + \frac{4h_0 \dot{h}_1 \psi}{r} \sqrt{\frac{B}{A}} + \frac{h_0^2}{r^2} \left[ 2 \left( r \left\{ \psi \sqrt{\frac{B}{A}} \right\}' + \psi \sqrt{\frac{B}{A}} \right) + \frac{(j^2 - 2)\psi}{\sqrt{AB}} \right] - \frac{(j^2 - 2)\sqrt{AB}\psi h_1^2}{r^2} \right], \quad (71)$$

with  $j^2 = l(l + 1)$ . Equation (71) is independent of  $\dot{h}_0$ , and, therefore, the variation of it with respect to  $h_0$  gives

$$\left[ \psi (h'_0 - \dot{h}_1) \sqrt{\frac{B}{A}} \right]' = \frac{1}{r^2} \left[ 2 \left( r \left\{ \psi \sqrt{\frac{B}{A}} \right\}' + \psi \sqrt{\frac{B}{A}} \right) + \frac{(j^2 - 2)\psi}{\sqrt{AB}} \right] h_0 + \frac{2\sqrt{\frac{B}{A}} \psi \dot{h}_1}{r}. \quad (72)$$

Equation (72) has no solution for  $h_0$ ; therefore, (71) turns to be

$$I_{\text{odd}} = \frac{j^2 \psi \sqrt{\frac{B}{A}}}{2} \left( \dot{h}_1 - h'_0 + \frac{2h_0}{r} \right)^2 - \frac{j^2 \left( \psi \sqrt{\frac{B}{A}} + r \left[ \sqrt{\frac{B}{A}} \psi \right]' \right) h_0^2}{r^2} + \frac{j^2 h_0^2}{2r^2} \left[ 2 \left( r \left\{ \psi \sqrt{\frac{B}{A}} \right\}' + \psi \sqrt{\frac{B}{A}} \right) + \frac{(j^2 - 2)\psi}{\sqrt{AB}} \right] - \frac{j^2 (j^2 - 2) \sqrt{AB} \psi h_1^2}{2r^2}. \quad (73)$$

Using the Lagrange multiplier  $q$ , Eq. (73) can be rewritten as follows:

$$I_{\text{odd}} = \frac{j^2 \psi \sqrt{\frac{B}{A}}}{2} \left[ 2q \left( \dot{h}_1 - \dot{h}_0 + \frac{2h_0}{r} \right) - q^2 \right] - \frac{j^2 \left( \psi \sqrt{\frac{B}{A}} + r \left[ \sqrt{\frac{B}{A}} \psi \right]' \right) h_0^2}{r^2} + \frac{j^2 h_0^2}{2r^2} \left[ 2 \left( r \left\{ \psi \sqrt{\frac{B}{A}} \right\}' + \psi \sqrt{\frac{B}{A}} \right) \right. \\ \left. + \frac{(j^2 - 2)\psi}{\sqrt{AB}} \right] - \frac{j^2 (j^2 - 2) \sqrt{AB} \psi h_1^2}{2r^2}. \quad (74)$$

Equation (74) can be written as

$$h_1 = -\frac{r^2 \dot{q}}{(j^2 - 2)A}, \quad (75)$$

$$h_0 = \frac{r \left[ \left\{ \psi \sqrt{\frac{B}{A}} + 2r \left[ \sqrt{\frac{B}{A}} \psi \right]' \right\} q + 2q' r \psi \sqrt{\frac{B}{A}} \right]}{2j^2 \left[ \psi \sqrt{\frac{B}{A}} + r \left[ \sqrt{\frac{B}{A}} \psi \right]' - \left( r \left\{ \psi \sqrt{\frac{B}{A}} \right\}' + \psi \sqrt{\frac{B}{A}} + \frac{(j^2 - 2)\psi}{2\sqrt{AB}} \right) \right]}. \quad (76)$$

Equation (75) relates the physical modes  $h_0$  and  $h_1$  to  $q$ . Therefore, when the auxiliary field  $q$  is known, the quantities  $h_0$  and  $h_1$  are also known. Using the physical quantities  $h_0$  and  $h_1$  in Eq. (71) and by carrying out the integration by parts to the term proportional to  $q'q$ , we get

$$I_{\text{odd}} = \frac{j^2 r^2 \psi \sqrt{\frac{B}{A^3}}}{2(j^2 - 2)} \dot{q}^2 - \frac{j^2 B \psi^2 q'^2}{4A \left[ \left( r \left\{ \psi \sqrt{\frac{B}{A}} \right\}' + \psi \sqrt{\frac{B}{A}} + \frac{(j^2 - 2)\psi}{2\sqrt{AB}} \right) - \psi \sqrt{\frac{B}{A}} + r \left[ \sqrt{\frac{B}{A}} \psi \right]' \right]} - \alpha^2 q^2, \quad (77)$$

where

$$\alpha^2 = \frac{a_1 r^2 [r^2 a_1' a_3' - r^2 a_1'' a_3 + 2a_1 a_3 + 4a_1'^2 + r^2 a_3^2 - 2a_1 a_1'' + 2ra_1 a_3' - 4rs_1' a_3]}{(2a_1 + 2ra_1' - r^2 a_3)^2} \quad (78)$$

and

$$a_1 = \frac{j^2 \psi \sqrt{B}}{2\sqrt{A}}, \quad a_2 = \frac{2j^2 \psi (j^2 - 2) \sqrt{AB}}{r^2}, \quad a_3 = \frac{j^2}{r^2} \left( \frac{\psi \sqrt{B}}{\sqrt{A}} + r \left\{ \frac{\psi \sqrt{B}}{\sqrt{A}} \right\}' + \frac{(j^2 - 2)\psi}{2\sqrt{AB}} \right).$$

Equation (77) shows that it is free from ghosts and gives

$$j^2 \geq 2 \quad \text{and} \quad \psi \sqrt{\frac{B}{A^3}} \geq 0.$$

Therefore, for solutions that proportional to  $e^{i(\omega t - kr)}$ , where  $k$  and  $\omega$  are large, the radial dispersion relation is regarded as

$$\omega^2 = ABk^2.$$

Thus, the radial speed reads

$$c_{\text{odd}}^2 = \left( \frac{dr_*}{d\tau} \right)^2 = 1,$$

where we used the radial tortoise coordinate ( $dr_*^2 = dr^2/B$ ) and the proper time ( $d\tau^2 = Adt^2$ ).

### C. Geodesic deviation

The trajectories of a particle in a gravitational background are prescribed by

$$\frac{d^2 x^\sigma}{d\tau^2} + \{\sigma_{\mu\nu}\} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (79)$$

where  $\tau$  is an affine parameter along the geodesic. Equation (79) is the geodesic equations, and their deviation take the form [119]

$$\frac{d^2 \xi^\sigma}{d\tau^2} + 2\{\sigma_{\mu\nu}\} \frac{dx^\mu}{d\tau} \frac{d\xi^\nu}{ds} + \{\sigma_{\mu\nu}\}_{,\rho} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \xi^\rho = 0, \quad (80)$$

where  $\xi^\rho$  is the deviation 4-vector. Using Eqs. (79) and (80) into the line element (6), we get

$$\frac{d^2 t}{d\tau^2} = 0, \quad \frac{1}{2}A'(r)\left(\frac{dt}{d\tau}\right)^2 - r\left(\frac{d\phi}{d\tau}\right)^2 = 0, \quad \frac{d^2 \theta}{d\tau^2} = 0, \quad \frac{d^2 \phi}{d\tau^2} = 0, \quad (81)$$

and

$$\begin{aligned} \frac{d^2 \xi^1}{d\tau^2} + B(r)A'(r)\frac{dt}{d\tau}\frac{d\xi^0}{d\tau} - 2rB(r)\frac{d\phi}{d\tau}\frac{d\xi^3}{d\tau} + \left[\frac{1}{2}(A'(r)B'(r) + B(r)A''(r))\left(\frac{dt}{d\tau}\right)^2 - (B(r) + rB'(r))\left(\frac{d\phi}{d\tau}\right)^2\right]\xi^1 &= 0, \\ \frac{d^2 \xi^0}{d\tau^2} + \frac{A'(r)}{A(r)}\frac{dt}{d\tau}\frac{d\xi^1}{d\tau} = 0, \quad \frac{d^2 \xi^2}{d\tau^2} + \left(\frac{d\phi}{d\tau}\right)^2 \xi^2 = 0, \quad \frac{d^2 \xi^3}{d\tau^2} + \frac{2}{r}\frac{d\phi}{d\tau}\frac{d\xi^1}{d\tau} = 0, \end{aligned} \quad (82)$$

where  $A(r)$  and  $B(r)$  are defined by Eq. (35) or (37). Equations (81) and (82) are the geodesic and geodesic deviations, respectively. Using the circular orbit

$$\theta = \frac{\pi}{2}, \quad \frac{d\theta}{d\tau} = 0, \quad \frac{dr}{d\tau} = 0, \quad (83)$$

we get

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{A'(r)}{r(2A(r) - rA'(r))}, \quad \left(\frac{dt}{d\tau}\right)^2 = \frac{2}{2A(r) - rA'(r)}. \quad (84)$$

Equations (82) can be rewritten as

$$\begin{aligned} \frac{d^2 \xi^1}{d\phi^2} + B(r)A'(r)\frac{dt}{d\phi}\frac{d\xi^0}{d\phi} - 2rB(r)\frac{d\xi^3}{d\phi} + \left[\frac{1}{2}(A'(r)B'(r) + B(r)A''(r))\left(\frac{dt}{d\phi}\right)^2 - (B(r) + rB'(r))\right]\xi^1 &= 0, \\ \frac{d^2 \xi^2}{d\phi^2} + \xi^2 = 0, \quad \frac{d^2 \xi^0}{d\phi^2} + \frac{A'(r)}{A(r)}\frac{dt}{d\phi}\frac{d\xi^1}{d\phi} = 0, \quad \frac{d^2 \xi^3}{d\phi^2} + \frac{2}{r}\frac{d\xi^1}{d\phi} = 0. \end{aligned} \quad (85)$$

The second equation of (85) shows that it possesses a simple harmonic motion, which means it has a stable motion. We can assume the solution of the remaining of Eq. (85) to be

$$\xi^0 = \zeta_1 e^{i\sigma\phi}, \quad \xi^1 = \zeta_2 e^{i\sigma\phi}, \quad \text{and} \quad \xi^3 = \zeta_3 e^{i\sigma\phi}, \quad (86)$$

where  $\zeta_1$ ,  $\zeta_2$ , and  $\zeta_3$  are constants and  $\phi$  should be determined. Substituting (86) in (85), we get

$$\frac{3ABA' - \omega^2 A'A - 2rBA'^2 + rABA''}{AB'} > 0, \quad (87)$$

which is the stability condition. Equation (87) for the BH (35) can be rewritten as

$$24M^2 r b^2 + 84b^2 M^3 - 29Mr^2 b^2 + Mr^4 + 4M^2 r^3 - 12M^3 r^2 + 2b^2 r^3 > 0, \quad (88)$$

which is the stability condition for the solution (35), and when  $b = 0$  we get  $r > 2M$ , which is the stability condition of the Schwarzschild spacetime [104].

## VI. DISCUSSION AND CONCLUSIONS

Spherically symmetric spacetime is considered as an ingredient tool for BH physics due to the fact that its basic properties can be investigated easily [120]. Previously, there were many spherically symmetric BH solutions provided a specific form of  $f(R)$  gravity and equal metric

potential [66,68]. In the present study, we use a spherically symmetric spacetime having unequal metric potentials and without assuming any form of the metric potential. Before we continue, we emphasize the following points:

- (i) From the trace equation of the field equation of  $f(R)$ , we isolate the form of  $f(R)$  in one side.
- (ii) We use this form of  $f(R)$  in the field equations (5) and (3) and got a form of the field equations that contains only the first derivative of  $f(R)$  with respect to  $R$ . Thus, we have applied the above-mentioned form as on Eq. (6) and derived the differential



equations that governed such a system. We have succeeded to solve such a system analytically and derive the form of the metric potentials in addition to the form of  $f(R)$ .

Moreover, it has been revealed that the associated BH solution depends only on a convolution function that it is responsible to make it different from the GR BH solution. Under some constraints, if this convolution function is equal to zero, we discover the Schwarzschild BH of the Einstein GR. Therefore, the effect of higher curvature of  $f(R)$  gravity is restored on this convolution function. To understand the physics of this original solution, we asymptote the convolution function up to a certain order, fourth order. The most beautiful thing in this asymptote is the fact that we get two constants; one can be related to the cosmological constant, and the other is the one responsible for making a deviation from a GR BH. So we classified our asymptote to two classes with a constant that is related to the cosmological constant and the other without this constant. As for the form of the metric potentials without the cosmological constant, we restore all the higher-order correction to one constant and show that the metric potentials asymptote as a flat spacetime. Also, we show that if this constant is vanishing we discover the Schwarzschild BH of GR. As for the second asymptote that includes the cosmological constant, the line element behaves asymptotically as AdS/dS spacetime. It is important to compare our results with the ones obtained by Jaime, Patino, and Salgado [101]: They studied  $f(R)$  models that fulfill two conditions:  $\frac{\partial f(R)}{R} > 0$  and  $\frac{\partial^2 f(R)}{R^2} > 0$ . The motivation for the assumption of these two conditions was the fact that the authors' focus was to obtain solutions of relativistic extended objects with external matter fields and not vacuum black hole solutions. However, the black hole solutions we obtain do not satisfy these conditions.

The nonexistence of the Birkhoff theorem in  $f(R)$  gravitational theories is studied in Ref. [121]. Recently, several authors have tried to investigate if the Birkhoff theorem is valid or not in the conformal frame [122–129]. In this study, we did not assume any approximation or carry out a conformal transformation to obtain the analytic solutions (30). Our results confirm that the Birkhoff theorem is not valid for  $f(R)$  gravity theories [130]. It is well known that the Birkhoff theorem is valid in GR due to the absence of spin-0 modes in the linearized field equations. When spin 0 is absent, the spherically symmetric spacetime cannot couple to higher-spin excitations [104,121]. Therefore, in the case of  $f(R)$  gravitational theories, the differential equation satisfied by the Ricci scalar  $R$  plays the role of spin-0 modes. Hence, a nontrivial dependence between the metric and the Ricci scalar, in general, leads to the breaking of the Birkhoff theorem in

$f(R)$ . This is exactly the case of our analytic solution given by Eq. (39), which gives a nontrivial value of the Ricci scalar  $R$ .

We study the physics of those BHs by calculating the invariants of them and show that all the invariants behave up to the leading order as  $(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R) \approx (\frac{1}{r^2}, \frac{1}{r^2}, \frac{1}{r^2})$ , which are different from the Schwarzschild BH which gives the leading term of the Kretschmann scalar as  $\frac{1}{r^6}$  and the other invariants  $R_{\mu\nu}R^{\mu\nu} = R = \text{const}$ . This means that the singularity of our BH for the Kretschmann scalar is much softer than that of GR. We must emphasize that such merit is due to the contribution of the higher-order curvature of  $f(R)$ .

To continue our investigation of these BHs, we calculate the thermodynamical quantities like the Hawking temperature, entropy, quasilocal energy, and Gibbs free energy. We show in detail the BH without the cosmological constant that all thermodynamical quantities are consistent with the literature. Essentially, we show that the Hawking temperature depends on the degenerate horizon; if the temperature becomes less than the degenerate horizon, we got a negative temperature, and if it is greater, we got a positive value. We repeat our calculations with the line element that contains the cosmological constant and carried out our calculation for the AdS and dS spacetimes separately. We show that the degenerate horizons of those spacetimes play an important role to make the Hawking temperature have a positive value. Meanwhile, we show that our black hole satisfies the first law of thermodynamics.

Finally, we have studied the stability of these BHs. For this aim, we write the Lagrangian of the  $f(R)$  gravitational theory as a scalar field that is coupled with the Ricci scalar. Using the odd-type procedure, we have derived the gradient instability condition and the radial propagation speed that is equal to one for our BHs. Moreover, we have examined the stability conditions for those types of BH as shown in (88).

Finally, we stress the fact that our BH solution given by Eq. (29) is not a general solution for the  $f(R)$  gravitational theory. This is because of the fact that in this study we have assumed  $F = 1 + \frac{c_1}{r^2}$  to get the BH (30). When we change the form of  $F$  given by Eq. (29), we will get a new BH different from the one given by Eq. (30).

## ACKNOWLEDGMENTS

This work is partially supported by MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas ‘‘Cosmic Acceleration’’ No. 15H05890 (S. N.) and the JSPS Grant-in-Aid for Scientific Research (C) No. 18K03615 (S. N.).

- [1] C. Eisele, A. Y. Nevsky, and S. Schiller, *Phys. Rev. Lett.* **103**, 090401 (2009).
- [2] J. A. Wheeler, *A Journey into Gravity and Space-Time* (W H Freeman & Co, New York, 1990), ISBN 0716760347, 9780716760344.
- [3] S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), *Astrophys. J.* **517**, 565 (1999).
- [4] A. G. Riess *et al.* (Supernova Search Team), *Astron. J.* **116**, 1009 (1998).
- [5] A. G. Riess *et al.* (Supernova Search Team), *Astrophys. J.* **607**, 665 (2004).
- [6] K. Hirata *et al.* (Kamiokande-II Collaboration), *Phys. Rev. Lett.* **58**, 1490 (1987).
- [7] S. Dodelson and L. M. Widrow, *Phys. Rev. Lett.* **72**, 17 (1994).
- [8] S. Cole, A. Aragon-Salamanca, C. S. Frenk, J. F. Navarro, and S. E. Zepf, *Mon. Not. R. Astron. Soc.* **271**, 781 (1994).
- [9] D. A. Howell *et al.* (SNLS Collaboration), *Nature (London)* **443**, 308 (2006).
- [10] R. A. Scalzo *et al.*, *Astrophys. J.* **713**, 1073 (2010).
- [11] A. V. Filippenko *et al.*, *Astron. J.* **104**, 1543 (1992).
- [12] P. A. Mazzali, N. Chugai, M. Turatto, L. B. Lucy, I. J. Danziger, E. Cappellaro, M. D. Valle, and S. Benetti, *Mon. Not. R. Astron. Soc.* **284**, 151 (1997).
- [13] M. Turatto, A. Piemonte, S. Benetti, E. Cappellaro, P. M. Mazzali, I. J. Danziger, and F. Patat, *Astron. J.* **116**, 2431 (1998).
- [14] M. Modjaz, W. Li, A. V. Filippenko, J. Y. King, D. C. Leonard, T. Matheson, R. R. Treffers, and A. G. Riess, *Astron. Soc. Pac.* **113**, 308 (2001).
- [15] P. M. Garnavich *et al.*, *Astrophys. J.* **613**, 1120 (2004).
- [16] S. Taubenberger *et al.*, *Mon. Not. R. Astron. Soc.* **385**, 75 (2008).
- [17] Y.-F. Cai, S. Capozziello, M. De Laurentis, and E. N. Saridakis, *Rep. Prog. Phys.* **79**, 106901 (2016).
- [18] A. Awad, W. El Hanafy, G. G. L. Nashed, and E. N. Saridakis, *J. Cosmol. Astropart. Phys.* **02** (2018) 052.
- [19] A. Awad, W. El Hanafy, G. G. L. Nashed, S. D. Odintsov, and V. K. Oikonomou, *J. Cosmol. Astropart. Phys.* **07** (2018) 026.
- [20] M. Zubair and G. Abbas, *Astrophys. Space Sci.* **361**, 27 (2016).
- [21] A. De Felice and S. Tsujikawa, *Living Rev. Relativity* **13**, 3 (2010).
- [22] S. Nojiri and S. D. Odintsov, *Phys. Rep.* **505**, 59 (2011).
- [23] S. Capozziello and M. De Laurentis, *Phys. Rep.* **509**, 167 (2011).
- [24] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, *Phys. Rep.* **692**, 1 (2017).
- [25] V. Faraoni and S. Capozziello, *Beyond Einstein Gravity*, Vol. 170 (Springer, Dordrecht, 2011).
- [26] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, *Phys. Rev. D* **73**, 084007 (2006).
- [27] T. Harko, F. S. N. Lobo, S. Nojiri, and S. D. Odintsov, *Phys. Rev. D* **84**, 024020 (2011).
- [28] M. Zubair, G. Abbas, and I. Noureen, *Astrophys. Space Sci.* **361**, 8 (2016).
- [29] R. Saleem, F. Kramat, and M. Zubair, *Phys. Dark Universe* **30**, 100592 (2020).
- [30] A. De Felice and S. Tsujikawa, *Living Rev. Relativity* **13**, 3 (2010).
- [31] S. Capozziello and M. D. Laurentis, *Phys. Rep.* **509**, 167 (2011).
- [32] S. Nojiri and S. D. Odintsov, *Int. J. Geom. Methods Mod. Phys.* **04**, 115 (2007).
- [33] I. De Martino, M. De Laurentis, and S. Capozziello, *Universe* **1**, 123 (2015).
- [34] K. Bamba, S. Capozziello, S. Nojiri, and S. D. Odintsov, *Astrophys. Space Sci.* **342**, 155 (2012).
- [35] A. A. Starobinskii, *Pis'ma Zh. Eksp. Teor. Fiz.* **30**, 719 (1979) [*JETP Lett.* **30**, 682 (1979)], <http://www.jetpletters.ac.ru/ps/1370/index.shtml>.
- [36] A. V. Astashenok, S. Capozziello, and S. D. Odintsov, *J. Cosmol. Astropart. Phys.* **12** (2013) 040.
- [37] A. V. Astashenok, S. Capozziello, and S. D. Odintsov, *Phys. Rev. D* **89**, 103509 (2014).
- [38] A. V. Astashenok, S. Capozziello, and S. D. Odintsov, *J. Cosmol. Astropart. Phys.* **01** (2015) 001.
- [39] A. V. Astashenok, S. D. Odintsov, and Á. de la Cruz-Dombriz, *Classical Quantum Gravity* **34**, 205008 (2017).
- [40] A. V. Astashenok, *Int. J. Mod. Phys. Conf. Ser.* **41**, 1660130 (2016).
- [41] D. C. Rodrigues, P. L. de Oliveira, J. C. Fabris, and G. Gentile, *Mon. Not. R. Astron. Soc.* **445**, 3823 (2014).
- [42] S. Capozziello, T. Harko, T. S. Koivisto, F. S. Lobo, and G. J. Olmo, *J. Cosmol. Astropart. Phys.* **07** (2013) 024.
- [43] Y. Shirasaki, M. Akiyama, T. Nagao, Y. Toba, W. He, M. Ohishi, Y. Mizumoto, S. Miyazaki, A. J. Nishizawa, and T. Usuda, *Publ. Astron. Soc. Jpn.* **70**, S30 (2017).
- [44] S. Nojiri, S. D. Odintsov, and M. Sami, *Phys. Rev. D* **74**, 046004 (2006).
- [45] P. Shah and G. C. Samanta, *Eur. Phys. J. C* **79**, 414 (2019).
- [46] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, *Nucl. Phys.* **B941**, 11 (2019).
- [47] S. D. Odintsov and V. K. Oikonomou, *Classical Quantum Gravity* **36**, 065008 (2019).
- [48] S. D. Odintsov and V. K. Oikonomou, *Phys. Rev. D* **99**, 064049 (2019).
- [49] J. R. Nascimento, G. J. Olmo, P. J. Porfirio, A. Yu. Petrov, and A. R. Soares, *Phys. Rev. D* **99**, 064053 (2019).
- [50] T. Miranda, C. Escamilla-Rivera, O. F. Piattella, and J. C. Fabris, *J. Cosmol. Astropart. Phys.* **05** (2019) 028.
- [51] A. V. Astashenok, K. Mosani, S. D. Odintsov, and G. C. Samanta, *Int. J. Geom. Methods Mod. Phys.* **16**, 1950035 (2019).
- [52] E. Elizalde, S. D. Odintsov, T. Paul, and D. S.-C. Gómez, *Phys. Rev. D* **99**, 063506 (2019).
- [53] E. Elizalde, S. D. Odintsov, V. K. Oikonomou, and T. Paul, *J. Cosmol. Astropart. Phys.* **02** (2019) 017.
- [54] L. Chen, *Phys. Rev. D* **99**, 064025 (2019).
- [55] F. Sbisà, O. F. Piattella, and S. E. Jorás, *Phys. Rev. D* **99**, 104046 (2019).
- [56] F. Bombacigno and G. Montani, *Eur. Phys. J. C* **79**, 405 (2019).
- [57] S. Capozziello, C. A. Mantica, and L. G. Molinari, *Int. J. Geom. Methods Mod. Phys.* **16**, 1950008 (2019).
- [58] G. C. Samanta and N. Godani, *Eur. Phys. J. C* **79**, 623 (2019).
- [59] T. Multamäki and I. Vilja, *Phys. Rev. D* **74**, 064022 (2006).
- [60] G. G. L. Nashed, *Eur. Phys. J. Plus* **133**, 18 (2018).
- [61] G. G. L. Nashed, *Int. J. Mod. Phys. D* **27**, 1850074 (2018).
- [62] G. G. L. Nashed, *Adv. High Energy Phys.* **2018**, 1 (2018).

- [63] S. Capozziello, A. Stabile, and A. Troisi, *Classical Quantum Gravity* **24**, 2153 (2007).
- [64] S. Capozziello, N. Frusciante, and D. Vernieri, *Gen. Relativ. Gravit.* **44**, 1881 (2012).
- [65] S. Capozziello, M. D. Laurentis, and A. Stabile, *Classical Quantum Gravity* **27**, 165008 (2010).
- [66] E. Elizalde, G. G. L. Nashed, S. Nojiri, and S. D. Odintsov, *Eur. Phys. J. C* **80**, 109 (2020).
- [67] G. Nashed, W. El Hanafy, S. Odintsov, and V. Oikonomou, *Int. J. Mod. Phys. D* **29**, 2050090 (2020).
- [68] G. G. L. Nashed and S. Capozziello, *Phys. Rev. D* **99**, 104018 (2019).
- [69] J. Sultana and D. Kazanas, *Gen. Relativ. Gravit.* **50**, 137 (2018).
- [70] P. Cañate, *Classical Quantum Gravity* **35**, 025018 (2018).
- [71] S. Yu, C. Gao, and M. Liu, *Res. Astron. Astrophys.* **18**, 157 (2018).
- [72] P. Cañate, L. G. Jaime, and M. Salgado, *Classical Quantum Gravity* **33**, 155005 (2016).
- [73] A. Kehagias, C. Kounnas, D. Lüst, and A. Riotto, *J. High Energy Phys.* **05** (2015) 143.
- [74] W. Nelson, *Phys. Rev. D* **82**, 104026 (2010).
- [75] A. de la Cruz-Dombriz, A. Dobado, and A. L. Maroto, *Phys. Rev. D* **80**, 124011 (2009); **83**, 029903(E) (2011).
- [76] W.-X. Feng, C.-Q. Geng, W. F. Kao, and L.-W. Luo, *Int. J. Mod. Phys. D* **27**, 1750186 (2018).
- [77] M. Aparicio Resco, L. de la Cruz-Dombriz, F. J. Llanes Estrada, and V. Zapatero Castrillo, *Phys. Dark Universe* **13**, 147 (2016).
- [78] S. Capozziello, M. De Laurentis, R. Farinelli, and S. D. Odintsov, *Phys. Rev. D* **93**, 023501 (2016).
- [79] K. V. Staykov, D. Popchev, D. D. Doneva, and S. S. Yazadjiev, *Eur. Phys. J. C* **78**, 586 (2018).
- [80] D. D. Doneva and S. S. Yazadjiev, *J. Cosmol. Astropart. Phys.* **11** (2016) 019.
- [81] S. S. Yazadjiev, D. D. Doneva, and D. Popchev, *Phys. Rev. D* **93**, 084038 (2016).
- [82] S. S. Yazadjiev, D. D. Doneva, and K. D. Kokkotas, *Phys. Rev. D* **91**, 084018 (2015).
- [83] S. S. Yazadjiev, D. D. Doneva, K. D. Kokkotas, and K. V. Staykov, *J. Cosmol. Astropart. Phys.* **06** (2014) 003.
- [84] A. Ganguly, R. Gannouji, R. Goswami, and S. Ray, *Phys. Rev. D* **89**, 064019 (2014).
- [85] A. V. Astashenok, S. Capozziello, and S. D. Odintsov, *J. Cosmol. Astropart. Phys.* **12** (2013) 040.
- [86] M. Orellana, F. Garcia, F. A. Teppa Pannia, and G. E. Romero, *Gen. Relativ. Gravit.* **45**, 771 (2013).
- [87] A. S. Arapoglu, C. Deliduman, and K. Y. Eksi, *J. Cosmol. Astropart. Phys.* **07** (2011) 020.
- [88] A. Cooney, S. DeDeo, and D. Psaltis, *Phys. Rev. D* **82**, 064033 (2010).
- [89] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961).
- [90] T. Chiba, *Phys. Lett. B* **575**, 1 (2003).
- [91] J. O'Hanlon, *Phys. Rev. Lett.* **29**, 137 (1972).
- [92] S. Chakraborty and S. SenGupta, *Eur. Phys. J. C* **77**, 573 (2017).
- [93] S. Chakraborty and S. SenGupta, *Eur. Phys. J. C* **76**, 552 (2016).
- [94] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, *Phys. Rev. D* **70**, 043528 (2004).
- [95] H. A. Buchdahl, *Mon. Not. R. Astron. Soc.* **150**, 1 (1970).
- [96] S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **68**, 123512 (2003).
- [97] S. Capozziello, V. F. Cardone, S. Carloni, and A. Troisi, *Int. J. Mod. Phys. D* **12**, 1969 (2003).
- [98] S. Capozziello, *Int. J. Mod. Phys. D* **11**, 483 (2002).
- [99] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, *J. Cosmol. Astropart. Phys.* **2** (2005) 010.
- [100] S. Kalita and B. Mukhopadhyay, *Eur. Phys. J. C* **79**, 877 (2019).
- [101] L. G. Jaime, L. Patino, and M. Salgado, *Phys. Rev. D* **83**, 024039 (2011).
- [102] A. Ronveaux, *Appl. Math. Comput.* **141**, 177 (2003), advanced Special Functions and Related Topics in Differential Equations, Third Melfi Workshop, Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics.
- [103] R. S. Maier, *J. Diff. Equations* **213**, 171 (2005).
- [104] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
- [105] A. Sheykhi, *Phys. Rev. D* **86**, 024013 (2012).
- [106] A. Sheykhi, *Eur. Phys. J. C* **69**, 265 (2010).
- [107] S. H. Hendi, A. Sheykhi, and M. H. Dehghani, *Eur. Phys. J. C* **70**, 703 (2010).
- [108] A. Sheykhi, M. H. Dehghani, and S. H. Hendi, *Phys. Rev. D* **81**, 084040 (2010).
- [109] Y.-Q. Wang, Y.-X. Liu, and S.-W. Wei, *Phys. Rev. D* **99**, 064036 (2019).
- [110] A. Zakria and A. Afzal, arXiv:1808.04361.
- [111] G. Cognola, O. Gorbunova, L. Sebastiani, and S. Zerbini, *Phys. Rev. D* **84**, 023515 (2011).
- [112] Y. Zheng and R.-J. Yang, *Eur. Phys. J. C* **78**, 682 (2018).
- [113] W. Kim and Y. Kim, *Phys. Lett. B* **718**, 687 (2012).
- [114] P. C. W. Davies, *Proc. R. Soc. A* **353**, 499 (1977).
- [115] Y. Zheng and R. Yang, *Eur. Phys. J. C* **78**, 682 (2018).
- [116] A. De Felice, T. Suyama, and T. Tanaka, *Phys. Rev. D* **83**, 104035 (2011).
- [117] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
- [118] F. J. Zerilli, *Phys. Rev. Lett.* **24**, 737 (1970).
- [119] R. D'Inverno, *Introducing Einstein's relativity* (Oxford University Press, 1992), ISBN 978-0-19-859686-8.
- [120] S. Chakraborty and S. SenGupta, *J. Cosmol. Astropart. Phys.* **07** (2017) 045.
- [121] R. J. Riegert, *Phys. Rev. Lett.* **53**, 315 (1984).
- [122] T. P. Sotiriou and V. Faraoni, *Phys. Rev. Lett.* **108**, 081103 (2012).
- [123] L. Sebastiani and S. Zerbini, *Eur. Phys. J. C* **71**, 1591 (2011).
- [124] S. E. Perez Bergliaffa and Y. E. C. de Oliveira Nunes, *Phys. Rev. D* **84**, 084006 (2011).
- [125] C. Gao and Y.-G. Shen, *Gen. Relativ. Gravit.* **48**, 131 (2016).
- [126] Z. Amirabi, M. Halilsoy, and S. Habib Mazharimousavi, *Eur. Phys. J. C* **76**, 338 (2016).
- [127] M. Calzà, M. Rinaldi, and L. Sebastiani, *Eur. Phys. J. C* **78**, 178 (2018).
- [128] J. Oliva and S. Ray, *Classical Quantum Gravity* **28**, 175007 (2011).
- [129] S. Capozziello and D. Sáez-Gómez, *Ann. Phys. (Amsterdam)* **524**, 279 (2012).
- [130] S. Xavier, J. Mathew, and S. Shankaranarayanan, *Classical Quantum Gravity* **37**, 225006 (2020).