The dark universe future and singularities: the account of thermal and quantum effects

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The knowledge of the universe future is of fundamental importance for any advanced civilization. We study the future of the singular dark universe where thermal effects due to the Hawking radiation on the apparent horizon of the FRW universe are taken in the consideration. It is shown that the dark universe which ends up as finite-time Type I and Type III singularity or the infinite-time Little Rip singularity transits to the finite-time Type II singularity thanks to account of the thermal effects. However, Type II and IV singular universe does not change the qualitative behavior. The combined account of the quantum and thermal effects shows that depending on the specific features of the universe only one of the effects is dominant. When (conformal matter) quantum effects are dominant, the future singularity is usually removed while for dominant thermal effects the universe final state is Type II singularity.

PACS numbers:

I. INTRODUCTION

Theoretical discovery of accelerating dark energy universe significally changed our knowledge about the universe future. It is rather well-known that the dark energy epoch may be qualitatively understood as universe filled by the exotic effective fluid with negative pressure. Depending on its structure and in correspondence with observable bounds, current dark universe may show phantom ($w_{\rm eff} < -1$), de Sitter ($w_{\rm eff} = -1$) or quintessential ($-1 < w_{\rm eff} < -1/3$) behaviour where $w_{\rm eff}$ is the effective universe EoS parameter. So far the precise understanding in which dark era we live is still lacking. However, for big sub-class of phantom or quintessential universes it turns out that the universe ends up in some sort of future singularity. It is fundamentally important for any advanced civilization to know what happens with the universe in the future even this event is rather distant (few dozen billion years).

The most known type of finite-time future singularity is related with phantom evolution and is called Big Rip singularity. In this case the universe ends up very rapid expansion and any extended object will be destroyed by the tidal force some million years before reaching the Rip time [1]. Apart from this most strong singularity there are few soft singularities which are classified as Type II, III, and IV singularities. For the Big Rip singularity, the Hubble rate H diverges in a finite time and the time $t_{\rm Rip}$ that the divergence occurs is called the Big Rip time. Note that this is the classical consideration. However, taking into account different related effects (like quantum effects, strong electromagnetic fields, condensation, etc.) may qualitatively change the classical consideration and give the realistic picture of the future universe. First of all, let us remind that the large Hubble rate H means the large temperature of the universe. The Hawking radiation effectively should be generated at the apparent horizon of the FRW universe [2, 3]. Eventually, it should give the important contribution to the energy-density of late-time universe, especially right before the Rip time. In other words, at large temperature which may even diverge at the Rip time, there should appear thermal radiation. Recently in Ref. [4], it was argued that the kind of cyclic cosmology might be realized instead of the Big Rip singularity due to effect of thermal radiation. The purpose of this paper is to study what could really happen with the future singular dark universe when the effect of thermal radiation is included. In the next section we consider the dark universe with Type I, II, III, and IV finite-time future singularities as well as the infinite-time Little Rip singularity in the presence of thermal effects due to the Hawking radiation. We demonstrate that for Type II and IV singular universe there is no qualitative effect to its final state due to the thermal radiation. Type I and III singular universes as well as the Little Rip universe ends up as Type II singularity due to thermal effects. The third section is devoted to the account of both, quantum and thermal effects to future singularities. We show that quantum effects as a rule remove future singularity creating the non-singular universe. When quantum and thermal effects are taken into consideration, depending on the specific features of the theory (particles content, fluids, temperature, etc.) only one of the effects becomes dominant. For instance, when the thermal effects are dominant, the universe ends up at Type II singularity in the same way as without quantum effects. Finally, the summary and some outlook are given in the last section.

II. FINITE-TIME FUTURE SINGULARITIES IN THE DARK UNIVERSE: THE ACCOUNT OF THERMAL EFFECTS

We start from a spatially-flat FRW universe,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 . \quad (1)$$

Here $a(t)$ is a scale factor. We consider dark energy epoch when the effective equation of state (EoS) of the universe is around -1. In this case, the accelerating universe may evolve in one of the following ways: phantom evolution, quintessential evolution, and de Sitter expansion ($w_{\text{eff}} = -1$). What happens with such universe in the future? In principle, depending on specific aspects of time-dependent effective EoS of the universe any future is possible including deccelerating or ever-expanding universe. Let us consider here the sub-class of dark energy universes which lead to finite-time future singularities.

Let us remind that the FRW equations for general relativity (GR) coupled with general perfect fluid with the pressure p and the energy-density ρ are given by

$$\rho = \frac{3}{\kappa^2} H^2 , \quad p = -\frac{1}{\kappa^2} \left(3H^2 + 2\dot{H}\right) . \quad (2)$$

Here $H \equiv \dot{a}/a$.

For the dark energy universes ending at finite-time singularity it has been developed the classification of such singularities in Ref. [5] (see also [6]):

- Type I ("Big Rip") Ref. [1]: This is a characteristic crushing type singularity, for which as $t \to t_s$, the scale factor $a(t)$, the total effective pressure p_{eff} and the total effective energy density ρ_{eff} diverge strongly, that is, $a \to \infty$, $\rho_{\text{eff}} \to \infty$, and $|p_{\text{eff}}| \to \infty$. For works on this type of singularity, see Refs. [7–20].

- Type II ("sudden"): This type of singularity is milder than the Big Rip scenario, and it is also known as a pressure singularity, firstly studied in Refs. [21, 22], and later developed in [23–32], see also [33, 34]. Here, only the total effective pressure diverges as $t \to t_s$, and the total effective energy density and the scale factor remain finite, that is, $a \to a_s$, $\rho_{\text{eff}} \to \rho_s$ and $|p_{\text{eff}}| \to \infty$.

- Type III : In this type of singularity, both the total effective pressure and the total effective energy density diverge as $t \to t_s$, but the scale factor remains finite, that is, $a \to a_s$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$. Then Type III singularity is milder than Type I (Big Rip) but stronger than Type II (sudden).

- Type IV : This type of singularity is the mildest from a phenomenological point of view. It was discovered in Ref. [35] and further investigated in [5, 36–42]. In this case, all the aforementioned physical quantities remain finite as $t \to t_s$, that is, $a \to a_s$, $\rho_{\text{eff}} \to 0$, $|p_{\text{eff}}| \to 0$, but higher derivatives of the Hubble rate, $H^{(n)}$ ($n \geq 2$) diverge. This singularity may be related with the inflationary era, since the Universe may smoothly pass through this singularity without any catastrophic implications on the physical quantities. As was shown in [43], the graceful exit from the inflationary era may be triggered by this type of soft singularity.

Here, ρ_{eff} and p_{eff} are defined by

$$\rho_{\text{eff}} \equiv \frac{3}{\kappa^2} H^2 , \quad p_{\text{eff}} \equiv -\frac{1}{\kappa^2} \left(2\dot{H} + 3H^2\right) . \quad (3)$$

Note that ρ_{eff} and p_{eff} are different from ρ and p in (2). For example, ρ_{eff} and p_{eff} may include the contribution from the modified gravity. Then Eq. (3) shows that for Type I and III singularities, H diverges but for Type II and IV, H is finite. However, in Type II singularity $\dot{H}$ diverges. We will be interesting in the account of thermal effects especially for Type I and III singularities.

There is also a scenario called the Little Rip cosmology [44–46] where the universe enters to singular state at infinite future. In this scenario, the Hubble rate is finite in the finite time but it becomes infinite in the infinite future. We show that thermal radiation will become important in the far future for the Little Rip evolution much before the arrival to the infinite Rip time.

A. Big Rip with thermal effects: transition to Type II singularity

The Big Rip singularity of the universe can be generated by the cosmic fluid, which is often called "phantom", with the equation of state (EoS) parameter w, which is defined by

$$w = \frac{p}{\rho}\,, \tag{4}$$

for the pressure p and the energy density ρ for general cosmic fluid, it is less than -1, $w < -1$. By assuming the conservation law,

$$0 = \dot\rho + 3H\left(\rho + p\right)\,, \tag{5}$$

we find

$$\rho = \rho_0 a^{-3(1+w)}\,, \tag{6}$$

with a positive constant ρ_0. Then in case of the phantom, because $-3\left(1+w\right) > 0$, the energy density dominates at late time where a becomes large. Then by using the FRW equations (2), we find H behaves as

$$H \propto \frac{1}{t_{\mathrm{Rip}} - t}\,, \tag{7}$$

and H diverges at $t = t_{\mathrm{Rip}}$, which is the Big Rip singularity.

Near the Big Rip singularity, the temperature of the universe becomes large and we may expect the generation of the thermal radiation as in the case of the Hawking radiation. The Hawking temperature T is proportional to the inverse of the radius r_{H} of the apparent horizon and the radius r_{H} is proportional to the inverse of the Hubble rate H. Therefore the temperature T is proportional to the Hubble rate H. As well-known in the statistical physics, the energy-density $\rho_{\mathrm{t\_rad}}$ of the thermal radiation is proportional to the fourth power of the temperature. Then when H is large enough, we may assume that the energy-density of the thermal radiation is given by

$$\rho_{\mathrm{t\_rad}} = \alpha H^4\,, \tag{8}$$

with a positive constant α. At the late time, the FRW equation (2) should be modified by the account of thermal radiation,

$$\frac{3}{\kappa^2}H^2 = \rho_0 a^{-3(1+w)} + \alpha H^4\,. \tag{9}$$

Here we assume $w < -1$. At the late time but much before the Big Rip time, the first term in the equation (9) dominates and therefore the universe expands to the Big Rip singularity, where the Hubble rate H behaves as in (7). Then near the Big Rip time t_{Rip}, the second term in (9) should dominate and we obtain

$$\frac{3}{\kappa^2}H^2 \sim \alpha H^4\,, \tag{10}$$

whose non-trivial solution is given by

$$H^2 = H_{\mathrm{crit}}^2 \equiv \frac{3}{\kappa^2\alpha}\,. \tag{11}$$

As H goes to a constant, we might expect that the space-time goes to the asymptotically de Sitter space-time but it is not true. Even in the de Sitter space-time, the scale factor a becomes larger and larger as an exponential function of t, then the first term in the equation (9) should dominate finally. The Hubble rate H is, however, already larger than H_{crit}, there is no solution of (9). Then the universe should end up at finite time with some kind of the singularity.

For more quantitative analysis, we solve (9), with respect to H^2 as follows,

$$H^2 = \frac{\frac{3}{\kappa^2} \pm \sqrt{\frac{9}{\kappa^4} - 4\alpha\rho_0 a^{-3(1+w)}}}{2\alpha}\,. \tag{12}$$

Because H^2 is a real number, we find that there is a maximum for the scale factor a,

$$a \leq a_{\mathrm{max}} \equiv \left(\frac{9}{4\kappa^4\alpha\rho_0}\right)^{-\frac{1}{3(1+w)}}\,. \tag{13}$$

Then we consider the behavior of a or H around the maximal $a = a_{\rm max}$ by writing the scale factor a as

$$a = a_{\rm max} \mathrm{e}^N \, . \tag{14}$$

Here N corresponds to the e-folding number but N should be negative because $a < a_{\rm max}$. Furthermore because we are interested in the region $a \sim a_{\rm max}$, we assume $|N| \ll 1$. Then by using $H = \frac{dN}{dt}$, Eq. (12) can be rewritten as

$$\left(1 \mp \frac{1}{2} \sqrt{3\left(1+w\right) N} \right) dN \sim dt \sqrt{\frac{3}{2\alpha\kappa^2}} \, , \tag{15}$$

which can be integrated as

$$N \mp \frac{1}{3} \left(-N\right)^{\frac{3}{2}} \sqrt{-3\left(1+w\right)} \sim - \left(t_{\rm max} - t\right) \sqrt{\frac{3}{2\alpha\kappa^2}} \, . \tag{16}$$

Here $a = a_{\rm max}$ when $t = t_{\rm max}$. Because we are assuming $|N| \ll 1$, Eq. (16) can be rewritten as

$$N \sim - \left(t_{\rm max} - t\right) \sqrt{\frac{3}{2\alpha\kappa^2}} \mp \frac{\sqrt{-3\left(1+w\right)}}{3} \left(\left(t_{\rm max} - t\right) \sqrt{\frac{3}{2\alpha\kappa^2}} \right)^{\frac{3}{2}} \, . \tag{17}$$

Because $H = \frac{dN}{dt}$, we find

$$\begin{aligned} H \sim & \sqrt{\frac{3}{2\alpha\kappa^2}} \mp \frac{\sqrt{-3\left(1+w\right)}}{2} \left(\sqrt{\frac{3}{2\alpha\kappa^2}} \right)^{\frac{3}{2}} \left(t_{\rm max} - t\right)^{\frac{1}{2}} \, , \\ \dot H \sim & \mp \frac{\sqrt{-3\left(1+w\right)}}{4} \left(\sqrt{\frac{3}{2\alpha\kappa^2}} \right)^{\frac{3}{2}} \left(t_{\rm max} - t\right)^{-\frac{1}{2}} \, . \end{aligned} \tag{18}$$

Then in the limit $t \to t_{\rm max}$, although H is finite but $\dot H$ diverges. Therefore the universe ends up with Type II singularity at $t = t_{\rm max}$ and the cyclic cosmology does not occur. Thus, we demonstrated that the account of thermal effects near the Big Rip singularity changes the universe evolution to the finite-time Type II singularity.

B. Type III singularity with account of thermal effects: transition to Type II singularity

The scale factor which generates Type III singularity can be expressed as

$$a(t) = a_s \mathrm{e}^{-\beta (t_s - t)^\gamma} \, , \tag{19}$$

with positive constants a_s, t_s, β, and γ. In order to generate Type III singularity we restrict the value of γ as

$$0 < \gamma < 1 \, . \tag{20}$$

Then the Hubble rate H is given by

$$H = \beta\gamma \left(t_s - t\right)^{\gamma - 1} \, . \tag{21}$$

Hence, in the limit $t \to t_s$, H diverges but the scale factor a is finite. From Eq. (3) it follows

$$\rho_{\rm eff} = \frac{3}{\kappa^2} \beta^2 \gamma^2 \left(t_s - t\right)^{2(\gamma - 1)} \, , \quad p_{\rm eff} = - \frac{1}{\kappa^2} \left(-2\beta\gamma\left(\gamma - 1\right) \left(t_s - t\right)^{\gamma - 2} + 3\beta^2\gamma^2 \left(t_s - t\right)^{2(\gamma - 1)} \right) \, . \tag{22}$$

By deleting $(t_s - t)$, we find the following equation of state,

$$p_{\rm eff} = -\rho_{\rm eff} + \frac{2\beta\gamma\left(\gamma - 1\right)}{\kappa^2} \left(\frac{\kappa^2 \rho_{\rm eff}}{3\beta^2\gamma^2} \right)^{\frac{\gamma - 2}{2(\gamma - 1)}} \, . \tag{23}$$

Using the conservation law (5) or directly using (19) and (22), one gets

$$\rho_{\rm eff} = \frac{3}{\kappa^2} \beta^2 \gamma^2 \left(\frac{1}{\beta} \ln \left(\frac{a_s}{a(t)} \right) \right)^{\frac{2(\gamma - 1)}{\gamma}} = \frac{3\gamma^2 \beta^{\frac{2}{\gamma}}}{\kappa^2} \left(\ln \left(\frac{a_s}{a(t)} \right) \right)^{\frac{2(\gamma - 1)}{\gamma}} \, . \tag{24}$$

With the account of the thermal radiation, instead of (9), we have

$$\frac{3}{\kappa^2}H^2 = A\left(\ln\left(\frac{a_s}{a(t)}\right)\right)^{-B} + \alpha H^4\,, \quad A \equiv \frac{3\gamma^2\beta^{\frac{2}{\gamma}}}{\kappa^2}\,, \quad B \equiv -\frac{2\left(\gamma-1\right)}{\gamma} > 0\,. \tag{25}$$

Then instead of (12), we obtain

$$H^2 = \frac{\frac{3}{\kappa^2} \pm \sqrt{\frac{9}{\kappa^4} - 4\alpha A\left(\ln\left(\frac{a_s}{a(t)}\right)\right)^{-B}}}{2\alpha}\,. \tag{26}$$

Then in order that H^2 to be real, we find that there is a maximum $a_{\rm max}$ for $a(t)$,

$$a(t) \leq a_{\rm max} \equiv a_s \mathrm{e}^{-\left(\frac{9}{4A\alpha\kappa^2}\right)^{-\frac{1}{B}}} < a_s\,. \tag{27}$$

Because $a_{\rm max}$ is smaller than a_s, we find that dark universe with the future Type III singularity is transited to the one with Type II singularity due to the account of thermal effects.

C. Thermal radiation for Type II and Type IV singularities

In case of Type II and Type IV singular universes, the Hubble rate H behaves as

$$H \sim H_0 + h_0\left(t_s - t\right)^{-\beta}\,. \tag{28}$$

When $0 > \beta > -1$ the behavior of H corresponds to Type II and when $\beta < -1$ but β is not an integer, to Type IV.

When one considers general matter, the first FRW equation where usual matter and the thermal radiation as in (9) are included, is given by

$$\frac{3}{\kappa^2}H^2 = \rho + \alpha H^4\,. \tag{29}$$

Here ρ is matter energy-density. In case of Type II or Type IV singularity, if $H_0 \neq 0$, near the singularity, the l.h.s. goes to a finite value $\frac{3}{\kappa^2}H^2 \to \frac{3}{\kappa^2}H_0^2$ and the contribution from the thermal radiation in the r.h.s. also becomes finite, $\alpha H^4 \to \alpha H_0^4$. Therefore the thermal radiation does not change the structure of the singularity. If $H_0 = 0$, the r.h.s. behaves as $\left(t_s - t\right)^{-4\beta}$ and the contribution from the thermal radiation behaves as $\left(t_s - t\right)^{-2\beta}$. Because $\beta < 0$, the contribution from the thermal radiation is less dominant and therefore the thermal radiation does not change the structure of the singularity.

D. Little Rip universe with the account of thermal effects

In the qualitatively-different from the above ones, the Little Rip scenario [44–46], the Hubble rate H becomes infinite at the infinite future. A simple example is given by

$$H = H_0 t\,, \tag{30}$$

with positive H_0. Then

$$\rho_{\rm eff} = \frac{3}{\kappa^2}H_0^2 t^2\,, \quad p_{\rm eff} = -\frac{1}{\kappa^2}\left(2H_0 + 3H_0^2 t^2\right)\,. \tag{31}$$

The equation of state is given by

$$p_{\rm eff} = -\rho_{\rm eff} - \frac{2H_0}{\kappa^2},\,. \tag{32}$$

Eq. (30) also shows that the scale factor $a(t)$ is given by

$$a(t) = a_0 \mathrm{e}^{\frac{1}{2}H_0 t^2}\,, \tag{33}$$

with a constant a_0. Then Eq. (31) shows

$$\rho_{\rm eff} = \frac{6H_0}{\kappa^2} \ln\left(\frac{a(t)}{a_0}\right) . \tag{34}$$

When we include the contribution from the thermal radiation, the equation corresponding to (9) or (25) has the following form,

$$\frac{3}{\kappa^2} H^2 = \frac{6H_0}{\kappa^2} \ln\left(\frac{a(t)}{a_0}\right) + \alpha H^4 , \tag{35}$$

and the equation corresponding to (12) or (26) has the following form,

$$H^2 = \frac{3}{2\alpha\kappa^2}\left(1 \pm \sqrt{1 - \frac{2\alpha H_0 \kappa^2}{3} \ln\left(\frac{a(t)}{a_0}\right)}\right) . \tag{36}$$

Again, from Eq. (36) it follows that there is a maximum of the scale factor a,

$$a(t) \leq a_{\rm max} \equiv a_0 {\rm e}^{\frac{3}{2\alpha\kappa^2}} , \tag{37}$$

and therefore the space-time corresponds to Type II singularity. Thus we again see that due to the account of thermal effects, the dark energy which should bring the universe to the Little Rip at the infinite future changes its evolution to Type II singularity. The corresponding transition occurs!

III. FUTURE SINGULARITIES WITH ACCOUNT OF QUANTUM AND THERMAL EFFECTS

In the previous sections, we have shown that any scenario, where the Hubble rate becomes infinite in the finite or infinite future as for Type I (Big Rip) and Type III singularities and in the Little Rip universe scenario, will not be realized if we include the effect from the thermal radiation. The universe will change its evolution to Type II singularity. From other side when the universe approaches to the future singularity, its curvature and other geometrical invariants grow up. As a result, the quantum effects may change the behavior of the future space-time singularity. For example, one can show that quantum effects may change the structure of future singularity, see [22, 35] (see also [47–51]). In this section, we use simple qualitative arguments of Ref. [52] to show the role of quantum effects in conformally-invariant theories to future singularity and compare it with the effect due to thermal radiation.

As is well-known, the conformal anomaly T_A has the following form:

$$T_A = b\left(\mathcal{F} + \frac{2}{3}\Box R\right) + b'\mathcal{G} + b''\Box R . \tag{38}$$

Here $\mathcal{F}$ is the square of the 4D Weyl tensor, and $\mathcal{G}$ is the Gauss-Bonnet invariant, which are given by

$$\mathcal{F} = \frac{1}{3}R^2 - 2R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} , \quad \mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} . \tag{39}$$

In case that matter is conformally-invariant and there appear N scalars, $N_{1/2}$ spinors, N_1 vector fields, N_2 (= 0 or 1) gravitons, and $N_{\rm HD}$ higher-derivative conformal scalars, b and b' have the following forms,

$$b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\rm HD}}{120(4\pi)^2} , \quad b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\rm HD}}{360(4\pi)^2} . \tag{40}$$

As is shown in (40), b is positive and b' is negative for the usual matter. An exception is the higher-derivative conformal scalar. The value of b'' can be always shifted by adding R^2 to the classical action.

If we write the energy density ρ_A and pressure p_A corresponding to the trace anomaly T_A, we find $T_A = -\rho_A + 3p_A$. Then by using the energy conservation law in the FRW universe

$$0 = \frac{d\rho_A}{dt} + 3H\left(\rho_A + p_A\right) , \tag{41}$$

we can delete p_A as

$$T_A = -4\rho_A - \frac{1}{H}\frac{d\rho_A}{dt} , \tag{42}$$

which can be integrated and we find the following expression for ρ_A [5]:

$$\rho_A = -\frac{1}{a^4} \int dt a^4 H T_A \, . \tag{43}$$

By using the above expression and identifying $\rho_{\rm eff} = \rho_A$, one may consider the FRW equation (3).

However, as in [52], for simplicity, we consider the trace of the Einstein equation including the trace anomaly, as follows,

$$R = -\frac{\kappa^2}{2} \left(T_{\rm matter} + T_A \right) \, . \tag{44}$$

Here $T_{\rm matter}$ is the trace of the matter energy-momentum tensor. For the FRW universe (1), $\mathcal{F}$ and $\mathcal{G}$ are given by

$$\mathcal{F} = 0 \, , \quad \mathcal{G} = 24 \left(\dot{H} H^2 + H^4 \right) \, . \tag{45}$$

What we like to show is that if there is a singularity, the trace equation (45) cannot be consistent. Especailly we show that the contribution from the conformal anomaly in the r.h.s. of Eq. (44) is more singular than the scalar curvature in the l.h.s. If the contribution from the matter although the conformal anomaly may give some corrections. Note that rigorous study of the account of quantum effects may be done following Ref. [51] but it requests numerical study depending of particles content of the universe as well as effective dark fluid.

We now assume that H behaves as

$$H \sim H_0 + h_0 \left(t_s - t \right)^{-\beta} \, . \tag{46}$$

When $\beta \geq 1$, the behavior of H corresponds to Type I (Big Rip) singularity, and when $1 \geq \beta > 0$, to Type III, when $0 > \beta > -1$ to Type II, when $\beta < -1$ but β is not an integer, to Type IV singularity. One may also neglect the contribution from matter and put $T_{\rm matter} = 0$.

In case that $\beta > 0$, which corresponds to Type I (Big Rip) and Type III singularity, the first constant term H_0 in (46) seems to be less dominant and we may neglect this term. Then because the scalar curvature R is given by $R = 12H^2 + 6\dot{H}$, when $\beta \geq 1$ (Type I), we find that the scalar curvature R behaves as $R \sim \left(t_s - t\right)^{-2\beta}$ and when $0 < \beta < 1$ (Type III), R behaves as $R \sim \left(t_s - t\right)^{-\beta-1}$. On the other hand, when $-1 < \beta < 0$ (Type II), R behaves as $R \sim \left(t_s - t\right)^{-\beta-1}$. When $\beta < -1$, which corresponds to Type IV singularity if β is not an integer, if $H_0 \neq 0$, R behaves as a constant but if $H_0 = 0$, $R \sim \left(t_s - t\right)^{-\beta-1}$.

We now assume the behavior of the Hubble rate as in (46). Then in case of Type I (Big Rip) case, where $\beta \geq 1$, near the Big Rip singularity, $t \sim t_s$, as seen from (45), the Gauss-Bonnet invariant $\mathcal{G}$ behaves as $\mathcal{G} \sim 24H^4 \sim \left(t_s - t\right)^{-4\beta}$ and therefore $\mathcal{G}$ becomes very large and the contribution from the matter $T_{\rm matter}$ in (44) can be neglected. On the other hand, one finds $\Box R \sim \left(t_s - t\right)^{-2\beta-2}$. Then because $R \sim \left(t_s - t\right)^{-2\beta}$, T_A becomes much larger than R and therefore Eq. (44) cannot be satisfied. This shows that the quantum effects coming from the conformal anomaly also remove Type I (Big Rip) singularity.

In case of Type II singularity, where $-1 < \beta < 0$, we find that $\mathcal{G}$ behaves as $\mathcal{G} \sim 24\dot{H}H^2 \sim \left(t_s - t\right)^{-3\beta-1}$. Because $R \sim \left(t_s - t\right)^{-\beta-1}$, the Gauss-Bonnet term in T_A is less singular and therefore negligible compared with R and the contribution from the matter. Therefore, the Gauss-Bonnet term in T_A does not help to prevent Type II singularity. Note, however, $\Box R$ behaves as $\Box R \sim \left(t_s - t\right)^{-\beta-3}$, which is more singular than the scalar curavature. Then if $2b/3 + b'' \neq 0$, the contribution from T_A becomes mich larger than R near the singularity $t \sim t_s$ and Eq. (44) cannot be satisfied. Therefore if $2b/3 + b'' \neq 0$, even Type II singularity can be also prevented when the quantum effects due conformal anomaly are included.

In case of Type III singularity ($0 < \beta < 1$), the Gauss-Bonnet invariant behaves as $\mathcal{G} \sim 24\dot{H}H^2 \sim \left(t_s - t\right)^{-3\beta-1}$ and $\Box R$ behaves as $\Box R \sim \left(t_s - t\right)^{-\beta-3}$. Because the scalar curvature behaves as $R \sim \left(t_s - t\right)^{-\beta-1}$, both of the terms, $\Box R$ and $\mathcal{G}$, are more singular than the scalar curvature R and Type III singularity is also prevented. Thus, we demonstrated that quantum effects may remove finite-time future singularities. Note that account of quantum gravity effects in specific models also is known to remove the Big Rip singularity [9].

Let us include the thermal effects to above analysis. So far the trace part of the Einstein equation is used. As the radiation is usually conformal, the trace part of the energy-momentum tensor of the radiation should vanish and the thermal radiation does not contribute to the trace equation. We should be, however, more careful in the present situation. The energy-density of the thermal radiation is only determined by the temperature. Therefore, the universe expands and its volume with the thermal radiation increases, the total energy should also be increased if the temperature is not changed or increases as in the case of Type I (Big Rip) or Type III singularity, or the Little Rip cosmology. In other words, say, in the phantom universe, there should exist effectively negative pressure. The energy of the thermal radiation is not conserved because the expansion produces the new thermal radiation. We should note, however, in order that the effective pressure, which includes the effect of the expansion, is consistent with the FRW equations, the energy-density of the thermal radiation and the effective pressure must satisfy the conservation law

$$0 = \frac{d\rho_{\rm t\_rad}}{dt} + 3H\left(\rho_{\rm t\_rad} + p_{\rm t\_rad}\right) . \tag{47}$$

To show the conservation law, we may start from the first FRW equation where usual matter and the thermal radiation as in (9) are included,

$$\frac{3}{\kappa^2}H^2 = \rho + \rho_{\rm t\_rad} , \quad \rho_{\rm t\_rad} = \alpha H^4 . \tag{48}$$

Here ρ is matter energy-density. By considering the derivative of Eq. (48) with respect to time t, we obtain,

$$\frac{6}{\kappa^2}H\dot H = \dot\rho + 4\alpha H^3 \dot H . \tag{49}$$

Then by using the standard conservation law for matter,

$$0 = \dot\rho + 3H\left(\rho + p\right) , \tag{50}$$

with the matter pressure, and combining (48) and (49), we obtain

$$-\frac{1}{\kappa^2}\left(2\dot H + 3H^2\right) = p - \alpha\left(H^4 + \frac{4}{3}H^2\dot H\right) , \tag{51}$$

which is nothing but the second FRW equation and we can identify the effective pressure of the thermal radiation as follows,

$$p_{\rm t\_rad} = -\alpha\left(H^4 + \frac{4}{3}H^2\dot H\right) . \tag{52}$$

Thus effectively, the energy-density and the effective pressure of the thermal radiation satisfy the conservation law (47) or we can find the exact and unique form of the effective pressure in (52) directly by using the conservation law (47) and assuming the form of the energy-density of the radiation in (8).

Then the trace part $T_{\rm t\_rad} = -\rho_{\rm t\_rad} + 3p_{\rm t\_rad}$ of the energy-momentum tensor for the radiation including the effect of the expansion of the universe is given by

$$T_{\rm t\_rad} = -4\alpha\left(H^4 + H^2\dot H\right) . \tag{53}$$

Let us assume the behavior of the Hubble rate H as in (46). Then in case of Type I (Big Rip) case ($\beta \geq 1$), near the singularity, we find $T_{\rm t\_rad} \sim (t_s - t)^{-4\beta}$, whose behavior is not so changed from that of T_A although we need to compare b' with α to see which term is dominant one.

In case of Type II singularity ($-1 < \beta < 0$), we find $T_{\rm t\_rad} \sim (t_s - t)^{-3\beta-1}$. As $R \sim (t_s - t)^{-\beta-1}$, the contribution from $T_{\rm t\_rad}$ is negligible. In case $b'' \neq 0$, which is arbitary and can be put to vanish if we dont add R^2 term, the contribution from $\Box R$ in T_A dominates and the Type II singularity does not occur.

In case of Type III singularity ($0 < \beta < 1$), we find $T_{\rm t\_rad} \sim (t_s - t)^{-3\beta-1}$, whose behavior is not changed from that of the Gauss-Bonnet invariant $\mathcal{G}$ in T_A but weaker than the behavior of $\Box R$. Then if $b'' \neq 0$, the contribution from the thermal radiation is less dominant than that of the conformal anomaly T_A. If $b'' = 0$, the contribution from $T_{\rm t\_rad}$ is not changed from that from T_A and we need to compare b' with α to see which could be dominant, again. Thus, we demonstrated that when quantum effects dominate over thermal effects then future singularities are removed. However, in some cases which depend on the specific features of the theory under consideration the dominant contribution is due to thermal effects. In this case, the most possible universe future is Type II singularity.

IV. SUMMARY

In summary, we studied the singular dark universe future where singularity is caused by the corresponding dark fluid while also thermal effects due to the Hawking radiation on apparent horizon of the FRW universe are included.

It is shown that for dark universe with Type I, III singularities and for the Little Rip universe the transition to Type II singularity at final state occurs. On the same time for Type II and IV singular universe one sees no qualitative effect due to thermal radiation. When in addition to thermal radiation also quantum effects are taken into account the situation is more complicated. Usually, matter quantum effects (at least, for conformally-invariant fields) remove the finite-time future singularity. Together with thermal effects the universe future is defined by which of terms (thermal or quantum) in the effective energy-density is dominant. This depends from the specific features of the universe under consideration (fields content, fluids, coefficient of thermal energy-density, etc.). In particulary, when thermal effects are dominant, then the future universe state is Type II singularity, again.

Some remark is in order. It is known that several million years before the Rip time, there appears some inertial force which may unbound particles producing desintegration of all bound objects at the universe. Let us check the effect of thermal radiation to this inertial force. A test mass m, which is separated from an observer by the distance r, receives an inertial force of tidal force $F_{\rm in}$ when the observer observes the mass, as follows,

$$F_{\rm in} = rm\frac{\ddot a}{a} = rm\left(\dot H + H^2\right) . \tag{54}$$

In case of Type I (Big Rip) or Type III singularity, because H and $\cot H$ become very large near the singulrity and therefore any extended object will be ripped and destroyed. If we take into account the contribution from the thermal radiation, the singularity will reduces to Type II singularity, where although H is finite or vanish, $\dot H$ and therefore the inertial force becomes very large near the singularity. Hence, in this case the bound objects are desintegrated as in the case without thermal effects. In case of Type IV singularity, both of H and H' are finite and therefore the inertial force is also finite. Finally, it may be of interest to study the role of thermal radiation for future singularities in modified gravity theories. This will be done elsewhere.

Acknowledgments

This work is partially supported by MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas "Cosmic Acceleration" No. 15H05890 (S.N.) and the JSPS Grant-in-Aid for Scientific Research (C) No. 18K03615 (S.N.).This work was done partially in the framework of the Russian Government Program of Competitive Growth of the Kazan Federal University.

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