n-EXANGULATED CATEGORIES (I): DEFINITIONS AND FUNDAMENTAL PROPERTIES

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Abstract. For each positive integer n we introduce the notion of n-exangulated categories as higher dimensional analogues of extriangulated categories defined by Nakaoka–Palu. We characterize which n-exangulated categories are n-exact in the sense of Jasso and which are $(n+2)$-angulated in the sense of Geiss–Keller–Oppermann.

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1. Introduction

A fundamental idea in Iyama's higher dimensional Auslander–Reiten theory [I1] is to replace short exact sequences as the basic building blocks for homological

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algebra, by longer exact sequences. A typical setting is to consider an n-cluster tilting subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$. For instance one may take $\mathscr{A}$ to be the category $\mathrm{mod}A$ of finitely generated modules over an n-representation finite algebra A in the sense of [IO] (see [HI1], [HI2] for further examples). Then $\mathrm{mod}A$ has a unique n-cluster tilting subcategory $\mathscr{C}$. In this case there is also an n-cluster tilting subcategory of the bounded derived category $\mathscr{D}^b(\mathrm{mod}A)$, obtained by closing $\mathscr{C}$ under shifts by $\pm n$ and direct sums. To generalize, one may take A to be an n-complete algebra in the sense of [I2] (see [P1] for further examples). Then $\mathrm{mod}A$ has a distinguished exact subcategory that admits an n-cluster tilting subcategory.

To summarize n-cluster tilting subcategories of abelian, exact and triangulated categories play a crucial role in higher dimensional Auslander–Reiten theory and what is sometimes called higher homological algebra. These three settings have all been axiomatized leading to the notions of n-abelian and n-exact categories introduced in [J] as well as the notion of $(n+2)$-angulated categories introduced in [GKO] (see also [BT] for more discussion of the axioms and [BJT] for a more recent class of examples). Setting $n = 1$ recovers the notions of abelian, exact and triangulated categories. Any n-cluster tilting subcategory $\mathscr{C}$ of an abelian or exact category is n-abelian respectively n-exact (see [J, Theorem 3.16] and [J, Theorem 4.14]). Similarly [GKO, Theorem 1] show that if $\mathscr{C}$ is an n-cluster tilting subcategory of a triangulated category closed under shift by n, then it is $(n+2)$-angulated. The condition that $\mathscr{C}$ is closed under shift by n is crucial and no reasonable axiomatization of arbitrary n-cluster tilting subcategories of triangulated categories has to our knowledge been proposed.

The notion of extriangulated categories was recently introduced in [NP] as a common generalization of exact and triangulated categories. The data of such a category is a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, where $\mathscr{C}$ is an additive category, $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is a biadditive functor (modelled after Ext^1) and $\mathfrak{s}$ assigns to each $\delta \in \mathbb{E}(C, A)$ a class of 3-term sequences with end terms A and C such that certain axioms hold. The aim of this paper is to introduce an n-analogue of this notion called n-exangulated categories. Such a category is a similar triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, with the main distinction being that the 3-term sequences mentioned above are replaced by $(n+2)$-term sequences. The precise definition is given in Definition 2.32 (see also Definition 2.22). It is a true analogue in the sense that 1-exangulated categories are the same as extriangulated categories (see Proposition 4.3). As typical examples we have that n-exact and $(n+2)$-angulated categories are n-exangulated (see Proposition 4.34 and Proposition 4.5).

One of the purposes of introducing n-exangulated categories is to provide a common ground for studying the different settings of higher homological algebra. Compared to the classical case $(n = 1)$ many important questions regarding the interplay of n-abelian, n-exact and $(n+2)$-angulated categories remain open. For instance, from any abelian category we obtain a triangulated category by taking its derived category. As far as we know, no satisfactory higher analogue of this procedure has been proposed and in view of [JK] it seems that finding such is non-trivial. It is our hope that by providing the common framework of n-exangulated categories we might contribute to answering some of these questions.

The paper is organized as follows. In Section 2 we introduce n-exangulated categories and related notions. In Section 3 we present basic properties of n-exangulated categories. In Section 4, we show that 1-exangulated categories are the same as extriangulated categories. We also characterize $(n+2)$-angulated categories as n-exangulated categories $(\mathscr{C},\mathbb{E},\mathfrak{s})$ for which $\mathbb{E}=\mathscr{C}(-,\Sigma-)$ for some automorphism Σ of $\mathscr{C}$. Similarly we characterize n-exact categories (defined in a slightly modified way) as n-exangulated categories for which inflations are monomorphisms and deflations are epimorphisms. In Section 5, we introduce a family of examples which are neither n-exact nor $(n+2)$-angulated.

It is natural to ask if there is a reasonable notion of n-cluster tilting subcategories of extriangulated categories and when they are n-exangulated. This question will be addressed in the next article n-Exangulated Categories (II).

2. n-EXANGULATED CATEGORIES

2.1. **$\mathbb{E}$-extensions.** Throughout this paper, let $\mathscr{C}$ be an additive category.

Definition 2.1. Suppose $\mathscr{C}$ is equipped with a biadditive functor $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to Ab$. For any pair of objects $A,C\in\mathscr{C}$, an element $\delta\in\mathbb{E}(C,A)$ is called an $\mathbb{E}$-*extension* or simply an *extension*. We also write such δ as ${}_A\delta_C$ when we indicate A and C.

Remark 2.2. Let ${}_A\delta_C$ be any extension. Since $\mathbb{E}$ is a bifunctor, for any $a\in\mathscr{C}(A,A')$ and $c\in\mathscr{C}(C',C)$, we have extensions

$$\mathbb{E}(C,a)(\delta)\in\mathbb{E}(C,A') \ \text{ and } \ \mathbb{E}(c,A)(\delta)\in\mathbb{E}(C',A).$$

We abbreviately denote them by $a_*\delta$ and $c^*\delta$. In this terminology, we have

$$\mathbb{E}(c,a)(\delta)=c^*a_*\delta=a_*c^*\delta$$

in $\mathbb{E}(C',A')$.

Definition 2.3. Let ${}_A\delta_C, {}_B\rho_D$ be any pair of $\mathbb{E}$-extensions. A *morphism* $(a,c)\colon\delta\to\rho$ of extensions is a pair of morphisms $a\in\mathscr{C}(A,B)$ and $c\in\mathscr{C}(C,D)$ in $\mathscr{C}$, satisfying the equality

$$a_*\delta=c^*\rho.$$

Remark 2.4. Let ${}_A\delta_C$ be any extension. We have the following.

(1) Any morphism $a\in\mathscr{C}(A,B)$ gives rise to a morphism of $\mathbb{E}$-extensions

$$(a,1_C)\colon\delta\to a_*\delta.$$

(2) Any morphism $c\in\mathscr{C}(D,C)$ gives rise to a morphism of $\mathbb{E}$-extensions

$$(1_A,c)\colon c^*\delta\to\delta.$$

Definition 2.5. For any $A,C\in\mathscr{C}$, the zero element ${}_A0_C=0\in\mathbb{E}(C,A)$ is called the *split* $\mathbb{E}$-*extension*.

Definition 2.6. Let ${}_A\delta_C, {}_B\rho_D$ be any pair of $\mathbb{E}$-extensions. Let

$$C\xrightarrow{\iota_C}C\oplus D\xleftarrow{\iota_D}D$$

and

$$A\xleftarrow{p_A}A\oplus B\xrightarrow{p_B}B$$

be coproduct and product in $\mathscr{C}$, respectively. Remark that, by the biadditivity of $\mathbb{E}$, we have a natural isomorphism

$$\mathbb{E}(C\oplus D,A\oplus B)\cong\mathbb{E}(C,A)\oplus\mathbb{E}(C,B)\oplus\mathbb{E}(D,A)\oplus\mathbb{E}(D,B).$$

Let $\delta\oplus\rho\in\mathbb{E}(C\oplus D, A\oplus B)$ be the element corresponding to $(\delta,0,0,\rho)$ through this isomorphism. This is the unique element which satisfies

$$\mathbb{E}(\iota_C,p_A)(\delta\oplus\rho)=\delta \quad , \quad \mathbb{E}(\iota_C,p_B)(\delta\oplus\rho)=0,$$
$$\mathbb{E}(\iota_D,p_A)(\delta\oplus\rho)=0 \quad , \quad \mathbb{E}(\iota_D,p_B)(\delta\oplus\rho)=\rho.$$

If $A=B$ and $C=D$, then the above isomorphism relates the sum $\delta+\rho\in\mathbb{E}(C,A)$ of $\delta,\rho\in\mathbb{E}(C,A)$ coming from the abelian group structure on $\mathbb{E}(C,A)$, to the 'Baer sum', i.e.,

$$\delta+\rho=\mathbb{E}(\Delta_C,\nabla_A)(\delta\oplus\rho),$$

where $\Delta_C=\begin{bmatrix}1\\1\end{bmatrix}\colon C\to C\oplus C$, $\nabla_A=[1\ 1]\colon A\oplus A\to A$.

2.2. **n-exangles.** Let $\mathscr{C}$ be an additive category as before, and let n be any fixed positive integer.

Definition 2.7. Let $\mathbf{C}_{\mathscr{C}}$ be the category of complexes in $\mathscr{C}$. As its full subcategory, define $\mathbf{C}^{n+2}_{\mathscr{C}}$ to be the category of complexes in $\mathscr{C}$ whose components are zero in the degrees outside of $\{0,1,\ldots,n+1\}$. Namely, an object in $\mathbf{C}^{n+2}_{\mathscr{C}}$ is a complex $X^\bullet=\{X^i,d_X^i\}$ of the form

$$X^0\xrightarrow{d_X^0}X^1\xrightarrow{d_X^1}\cdots\xrightarrow{d_X^{n-1}}X^n\xrightarrow{d_X^n}X^{n+1}.$$

We write a morphism $f^\bullet\colon X^\bullet\to Y^\bullet$ simply $f^\bullet=(f^0,f^1,\ldots,f^{n+1})$, only indicating the terms of degrees $0,\ldots,n+1$.

We define the homotopy relation on the morphism sets in the usual way. Thus morphisms $f^\bullet,g^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)$ are *homotopic* if there is a *homotopy*, i.e., a sequence of morphisms $\varphi^\bullet=(\varphi^1,\ldots,\varphi^{n+1})$ of $\varphi^i\in\mathscr{C}(X^i,Y^{i-1})$ satisfying

$$\begin{aligned}g^0-f^0&=\varphi^1\circ d_X^0,\\ g^i-f^i&=d_Y^{i-1}\circ\varphi^i+\varphi^{i+1}\circ d_X^i\quad(1\le i\le n),\\ g^{n+1}-f^{n+1}&=d_Y^n\circ\varphi^{n+1}.\end{aligned}$$

In this case we write as $f^\bullet\sim g^\bullet$, or $f^\bullet\underset{\varphi^\bullet}{\sim}g^\bullet$. We denote the homotopy category by $\mathbf{K}^{n+2}_{\mathscr{C}}$, which is the quotient of $\mathbf{C}^{n+2}_{\mathscr{C}}$ by the ideal of null-homotopic morphisms. If $f^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)$ gives an isomorphism in $\mathbf{K}^{n+2}_{\mathscr{C}}$, we call it a *homotopy equivalence*, as usual. Similarly a *homotopy inverse* of $f^\bullet$ is a morphism $g^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(Y^\bullet,X^\bullet)$ which gives the inverse of $f^\bullet$ in $\mathbf{K}^{n+2}_{\mathscr{C}}$.

Claim 2.8. Assume that $f^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)$ is a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$. For a homotopy inverse $g^\bullet$ of $f^\bullet$, we have the following.

(1) If $X^0=Y^0=A$ and $f^0=1_A$, then $g^\bullet$ can be chosen to satisfy $g^0=1_A$.
(2) Dually, if $X^{n+1}=Y^{n+1}=C$ and $f^{n+1}=1_C$, then $g^\bullet$ can be chosen to satisfy $g^{n+1}=1_C$.
(3) If $f^0=1_A$ and $f^{n+1}=1_C$, then $g^\bullet$ can be chosen to satisfy both $g^0=1_A$ and $g^{n+1}=1_C$.

Proof. Let $h^\bullet$ be any homotopy inverse of $f^\bullet$, and let $f^\bullet\circ h^\bullet\underset{\varphi^\bullet}{\sim}1_{Y^\bullet}$, $h^\bullet\circ f^\bullet\underset{\psi^\bullet}{\sim}1_{X^\bullet}$ be homotopies.

(1) Modifying $h^{\bullet}$ by a homotopy $(\varphi^1, 0, \ldots, 0)$, we obtain a morphism $g^{\bullet}\colon Y^{\bullet} \to X^{\bullet}$ of the form $(1_A, h^1 + d_X^0 \circ \varphi^1, h^2, \ldots, h^{n+1})$. Since $h^{\bullet} \sim g^{\bullet}$, this is also a homotopy inverse of $f^{\bullet}$.

(2) Dually to (1), $g^{\bullet} = (h^0, \ldots, h^{n-1}, h^n + \psi^{n+1} \circ d_Y^n, 1_C)$ gives a homotopy inverse of $f^{\bullet}$ with the desired property.

(3) $g^{\bullet} = (1_A, h^1 + d_X^0 \circ \varphi^1, h^2, \ldots, h^{n-1}, h^n + \psi^{n+1} \circ d_Y^n, 1_C)$ satisfies the desired properties. (If $n = 1$, we put $g^{\bullet} = (1_A, h^1 + d_X^0 \circ \varphi^1 + \psi^2 \circ d_Y^1, 1_C)$.) □

Definition 2.9. Let $\mathscr{C}, \mathbb{E}, n$ be as before. Define the category $\text{Æ} = \text{Æ}^{n+2}_{(\mathscr{C},\mathbb{E})}$ as follows.

(1) An object in $\text{Æ}^{n+2}_{(\mathscr{C},\mathbb{E})}$ is a pair $\langle X^{\bullet}, \delta \rangle$ of $X^{\bullet} \in \mathbf{C}^{n+2}_{\mathscr{C}}$ and $\delta \in \mathbb{E}(X^{n+1}, X^0)$ satisfying

$$(d_X^0)_*\delta = 0 \quad \text{and} \quad (d_X^n)^*\delta = 0.$$

We call such a pair an $\mathbb{E}$-*attached complex of length* $n + 2$. We also denote it by

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \overset{\delta}{\dashrightarrow} .$$

When we emphasize the end-terms $X^0 = A$ and $X^{n+1} = C$, we denote the pair by ${}_A\langle X^{\bullet}, \delta \rangle_C$ or just by ${}_A\langle X^{\bullet}, \delta \rangle$ or $\langle X^{\bullet}, \delta \rangle_C$, depending on our purpose.

(2) For such pairs $\langle X^{\bullet}, \delta \rangle$ and $\langle Y^{\bullet}, \rho \rangle$, a *morphism* $f^{\bullet}\colon \langle X^{\bullet}, \delta \rangle \to \langle Y^{\bullet}, \rho \rangle$ is defined to be a morphism $f^{\bullet} \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet}, Y^{\bullet})$ satisfying

$$(f^0)_*\delta = (f^{n+1})^*\rho.$$

We use the same composition and the identities as in $\mathbf{C}^{n+2}_{\mathscr{C}}$.

Proposition 2.10. *Let $f^{\bullet}\colon \langle X^{\bullet}, \delta \rangle \to \langle Y^{\bullet}, \rho \rangle$ be any morphism in Æ.*

(1) *If a morphism $f'^{\bullet} \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet}, Y^{\bullet})$ satisfies $f^{\bullet} \sim f'^{\bullet}$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$, then $f'^{\bullet}$ also belongs to $\text{Æ}(\langle X^{\bullet}, \delta \rangle, \langle Y^{\bullet}, \rho \rangle)$. Thus we may consider the same homotopy relation $\sim$ in Æ.*

(2) *If $f^{\bullet}$ has a homotopy inverse $g^{\bullet}\colon Y^{\bullet} \to X^{\bullet}$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$, then $g^{\bullet}$ belongs to $\text{Æ}(\langle Y^{\bullet}, \rho \rangle, \langle X^{\bullet}, \delta \rangle)$.*

Proof. (1) Suppose that we have $f^{\bullet} \underset{\varphi^{\bullet}}{\sim} f'^{\bullet}$ as in Definition 2.7. Then we have

$$(f'^0)_*\delta = (f^0 + \varphi^1 \circ d_X^0)_*\delta = (f^0)_*\delta + (\varphi^1)_*(d_X^0)_*\delta = (f^0)_*\delta,$$
$$(f'^{n+1})^*\rho = (f^{n+1} + d_Y^n \circ \varphi^{n+1})^*\rho = (f^{n+1})^*\rho + (\varphi^{n+1})^*(d_Y^n)^*\rho = (f^{n+1})^*\rho$$

since δ and ρ satisfy $(d_X^0)_*\delta = 0$ and $(d_Y^n)^*\rho = 0$. As $f^{\bullet}$ satisfies $(f^0)_*\delta = (f^{n+1})^*\rho$ by the assumption, so does $f'^{\bullet}$.

(2) By assumption, there are homotopies $g^{\bullet} \circ f^{\bullet} \sim 1_{X^{\bullet}}$ and $f^{\bullet} \circ g^{\bullet} \sim 1_{Y^{\bullet}}$. As in the proof of (1), this implies

$$(g^0 \circ f^0)_*\delta = \delta \quad \text{and} \quad (f^{n+1} \circ g^{n+1})^*\rho = \rho.$$

Since $(f^0)_*\delta = (f^{n+1})^*\rho$ by the assumption, it follows that

$$\begin{aligned} (g^{n+1})^*\delta &= (g^{n+1})^*(g^0 \circ f^0)_*\delta = (g^{n+1})^*(g^0)_*(f^0)_*\delta \\ &= (g^{n+1})^*(g^0)_*(f^{n+1})^*\rho = (g^0)_*(f^{n+1} \circ g^{n+1})^*\rho = (g^0)^*\rho, \end{aligned}$$

which means $g^{\bullet} \in \text{Æ}(\langle Y^{\bullet}, \rho \rangle, \langle X^{\bullet}, \delta \rangle)$ by definition. □

Definition 2.11. By Yoneda lemma, any extension $\delta \in \mathbb{E}(C, A)$ induces natural transformations

$$\delta_\sharp \colon \mathscr{C}(-, C) \Rightarrow \mathbb{E}(-, A) \quad \text{and} \quad \delta^\sharp \colon \mathscr{C}(A, -) \Rightarrow \mathbb{E}(C, -).$$

For any $X \in \mathscr{C}$, these $(\delta_\sharp)_X$ and $\delta^\sharp_X$ are given as follows.

(1) $(\delta_\sharp)_X \colon \mathscr{C}(X, C) \to \mathbb{E}(X, A)\ ;\ f \mapsto f^*\delta$.
(2) $\delta^\sharp_X \colon \mathscr{C}(A, X) \to \mathbb{E}(C, X)\ ;\ g \mapsto g_*\delta$.

We abbreviately denote $(\delta_\sharp)_X(f)$ and $\delta^\sharp_X(g)$ by $\delta_\sharp(f)$ and $\delta^\sharp(g)$.

Proposition 2.12. *Let $Q \in \mathscr{C}$ be any object. Then the Hom functor*

$$\mathscr{C}(Q, -) \colon \mathscr{C} \to Ab$$

induces the following functor $\mathfrak{Y}_Q \colon \text{Æ} \to C^{n+3}_{Ab}$. Here Ab denotes the category of abelian groups.

(i) *An object $\langle X^\bullet, \delta\rangle \in \text{Æ}$ is sent to the complex $\mathfrak{Y}_Q(\langle X^\bullet, \delta\rangle)$ defined as*

$$\mathscr{C}(Q, X^0) \xrightarrow{\mathscr{C}(Q, d^0_X)} \cdots \xrightarrow{\mathscr{C}(Q, d^n_X)} \mathscr{C}(Q, X^{n+1}) \xrightarrow{\delta_\sharp} \mathbb{E}(Q, X^0). \tag{2.1}$$

(ii) *A morphism $f^\bullet \in \text{Æ}(\langle X^\bullet, \delta\rangle, \langle Y^\bullet, \rho\rangle)$ is sent to the morphism of complexes*

$$\mathfrak{Y}_Q(f^\bullet) = \big(\mathscr{C}(Q, f^0), \ldots, \mathscr{C}(Q, f^{n+1}), \mathbb{E}(Q, f^0)\big).$$

Similarly, $\mathscr{C}(-, Q) \colon \mathscr{C}^{\mathrm{op}} \to Ab$ induces a functor $\text{Æ}^{\mathrm{op}} \to C^{n+3}_{Ab}$ which sends $\langle X^\bullet, \delta\rangle$ to the complex

$$\mathscr{C}(X^{n+1}, Q) \xrightarrow{\mathscr{C}(d^n_X, Q)} \cdots \xrightarrow{\mathscr{C}(d^0_X, Q)} \mathscr{C}(X^0, Q) \xrightarrow{\delta^\sharp} \mathbb{E}(X^{n+1}, Q). \tag{2.2}$$

Proof. This is straightforward. We remark that for any pair of $X^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ and $\delta \in \mathbb{E}(X^{n+1}, X^0)$, the sequences (2.1), (2.2) are complexes for all $Q \in \mathscr{C}$ if and only if $\langle X^\bullet, \delta\rangle$ belongs to Æ. □

Definition 2.13. An *n-exangle* is a pair $\langle X^\bullet, \delta\rangle$ of $X^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ and $\delta \in \mathbb{E}(X^{n+1}, X^0)$ which satisfies the following conditions.

(1) The following sequence of functors $\mathscr{C}^{\mathrm{op}} \to Ab$ is exact.

$$\mathscr{C}(-, X^0) \xRightarrow{\mathscr{C}(-, d^0_X)} \cdots \xRightarrow{\mathscr{C}(-, d^n_X)} \mathscr{C}(-, X^{n+1}) \xRightarrow{\delta_\sharp} \mathbb{E}(-, X^0) \tag{2.3}$$

(2) The following sequence of functors $\mathscr{C} \to Ab$ is exact.

$$\mathscr{C}(X^{n+1}, -) \xRightarrow{\mathscr{C}(d^n_X, -)} \cdots \xRightarrow{\mathscr{C}(d^0_X, -)} \mathscr{C}(X^0, -) \xRightarrow{\delta^\sharp} \mathbb{E}(X^{n+1}, -) \tag{2.4}$$

In particular any n-exangle is an object in Æ. A *morphism of n-exangles* simply means a morphism in Æ. Thus n-exangles form a full subcategory of Æ.

Remark 2.14. In Æ, a coproduct of objects $\langle X^\bullet, \delta\rangle, \langle Y^\bullet, \rho\rangle$ is given by $\langle X^\bullet \oplus Y^\bullet, \delta \oplus \rho\rangle$, where $X^\bullet \oplus Y^\bullet$ is the direct sum in $\mathbf{C}^{n+2}_{\mathscr{C}}$ and $\delta \oplus \rho$ is the one in Definition 2.6. Remark that $\langle X^\bullet \oplus Y^\bullet, \delta \oplus \rho\rangle$ is an n-exangle if and only if both $\langle X^\bullet, \delta\rangle, \langle Y^\bullet, \rho\rangle$ are n-exangles.

Claim 2.15. For any n-exangle ${}_A\langle X^\bullet, \delta\rangle_C$, the following are equivalent.

(1) $\delta = 0$.
(2) There is $r \in \mathscr{C}(X^1, A)$ satisfying $r \circ d^0_X = 1_A$.
(3) There is $s \in \mathscr{C}(C, X^n)$ satisfying $d^n_X \circ s = 1_C$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows immediately from the exactness of

$$\mathscr{C}(X^1,A)\xrightarrow{-\circ d_X^0}\mathscr{C}(A,A)\xrightarrow{\delta^\sharp}\mathbb{E}(C,A).$$

Similarly for (1) $\Leftrightarrow$ (3). $\square$

Proposition 2.16. *Let $\langle X^\bullet,\delta\rangle,\langle Y^\bullet,\rho\rangle$ be any pair of objects in Æ. Suppose that $f^\bullet\in$ Æ$(\langle X^\bullet,\delta\rangle,\langle Y^\bullet,\rho\rangle)$ is a homotopy equivalence. Then $\langle X^\bullet,\delta\rangle$ is an n-exangle if and only if $\langle Y^\bullet,\rho\rangle$ is.*

Proof. Let $g^\bullet$ be a homotopy inverse of $f^\bullet$, and let $g^\bullet\circ f^\bullet\underset{\varphi^\bullet}{\sim}1_{X^\bullet}$, $f^\bullet\circ g^\bullet\underset{\psi^\bullet}{\sim}1_{Y^\bullet}$ be homotopies. Let $Q\in\mathscr{C}$ be any object.

By Proposition 2.12, we obtain complexes $\mathbb{X}^\bullet=\mathfrak{Y}_Q(\langle X^\bullet,\delta\rangle)$, $\mathbb{Y}^\bullet=\mathfrak{Y}_Q(\langle Y^\bullet,\rho\rangle)$ and morphisms

$$F^\bullet=\mathfrak{Y}_Q(f^\bullet)\colon\mathbb{X}^\bullet\to\mathbb{Y}^\bullet,\quad G^\bullet=\mathfrak{Y}_Q(g^\bullet)\colon\mathbb{Y}^\bullet\to\mathbb{X}^\bullet$$

in C^{n+3}_{Ab}. For the composition $G^\bullet\circ F^\bullet$

$$\begin{array}{ccccccccc}
\mathscr{C}(Q,X^0) & \xrightarrow{\mathscr{C}(Q,d_X^0)} & \mathscr{C}(Q,X^1) & \xrightarrow{\mathscr{C}(Q,d_X^1)} \cdots \xrightarrow{\mathscr{C}(Q,d_X^n)} & \mathscr{C}(Q,X^{n+1}) & \xrightarrow{\delta_\sharp} & \mathbb{E}(Q,X^0)\\
\downarrow{\scriptstyle G^0\circ F^0} & \circlearrowright & \downarrow{\scriptstyle G^1\circ F^1} & \circlearrowright & \downarrow{\scriptstyle G^{n+1}\circ F^{n+1}} & \circlearrowright & \downarrow{\scriptstyle G^{n+2}\circ F^{n+2}}\\
\mathscr{C}(Q,X^0) & \xrightarrow[\mathscr{C}(Q,d_X^0)]{} & \mathscr{C}(Q,X^1) & \xrightarrow[\mathscr{C}(Q,d_X^1)]{} \cdots \xrightarrow[\mathscr{C}(Q,d_X^n)]{} & \mathscr{C}(Q,X^{n+1}) & \xrightarrow[\delta_\sharp]{} & \mathbb{E}(Q,X^0)
\end{array}$$

the sequence of morphisms in Ab

$$\Phi^1=\mathscr{C}(Q,\varphi^1),\dots,\Phi^{n+1}=\mathscr{C}(Q,\varphi^{n+1})$$

satisfies

$$1-G^i\circ F^i=\mathscr{C}(Q,d_X^{i-1})\circ\Phi^i+\Phi^{i+1}\circ\mathscr{C}(Q,d_X^i)\quad(1\le i\le n)$$

and

$$1-G^{n+1}\circ F^{n+1}=\mathscr{C}(Q,d_X^n)\circ\Phi^{n+1}.$$

This shows that $G^\bullet\circ F^\bullet$ induces $H^i(G^\bullet\circ F^\bullet)=1$ on cohomologies for any $1\le i\le n+1$. In the same way, by using $\psi^\bullet$, we can show $H^i(F^\bullet\circ G^\bullet)=1$ for $1\le i\le n+1$. Thus

$$H^i(F^\bullet)\colon H^i(\mathbb{X}^\bullet)\xrightarrow{\cong}H^i(\mathbb{Y}^\bullet)\quad(1\le i\le n+1)$$

are isomorphisms. In particular $\mathbb{X}^\bullet$ is exact if and only if $\mathbb{Y}^\bullet$ is. Similarly for the exactness of (2.4). $\square$

2.3. **The categories $\mathbf{C}^{n+2}_{(A,C)}$ and $\mathbf{K}^{n+2}_{(A,C)}$.** We consider the complexes of length $n+2$ with fixed end-terms, as follows.

Definition 2.17. For any pair of objects $A,C\in\mathscr{C}$, define the subcategory $\mathbf{C}^{n+2}_{(\mathscr{C};A,C)}$ of $\mathbf{C}^{n+2}_{\mathscr{C}}$ as follows. We abbreviately denote $\mathbf{C}^{n+2}_{(\mathscr{C};A,C)}$ by $\mathbf{C}^{n+2}_{(A,C)}$, when $\mathscr{C}$ is clear from the context.

1. An object $X^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}$ belongs to $\mathbf{C}^{n+2}_{(A,C)}$ if it satisfies $X^0=A$ and $X^{n+1}=C$. We also write it as ${}_AX^\bullet_C$ when we emphasize A and C.
2. For any $X^\bullet,Y^\bullet\in\mathbf{C}^{n+2}_{(A,C)}$, the morphism set is defined by

$$\mathbf{C}^{n+2}_{(A,C)}(X^\bullet,Y^\bullet)=\{f^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)\mid f^0=1_A,\ f^{n+1}=1_C\}.$$

This category $\mathbf{C}^{n+2}_{(A,C)}$ is no longer (pre-)additive. However we can take the quotient $\mathbf{C}^{n+2}_{(A,C)}$ by the same homotopy relation $\sim$ as in $\mathbf{C}^{n+2}_{\mathscr{C}}$. Namely, morphisms $f^\bullet, g^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ are *homotopic* if there is a sequence of morphisms $\varphi^\bullet = (\varphi^1, \dots, \varphi^{n+1})$ satisfying

$$\begin{aligned} 0 &= \varphi^1 \circ d_X^0, && (2.5)\\ g^i - f^i &= d_Y^{i-1} \circ \varphi^i + \varphi^{i+1} \circ d_X^i \quad (1 \le i \le n), &&\\ 0 &= d_Y^n \circ \varphi^{n+1}. && (2.6) \end{aligned}$$

We use the same notation $f^\bullet \sim g^\bullet$ and $f^\bullet \underset{\varphi^\bullet}{\sim} g^\bullet$ as before. We denote the resulting category by $\mathbf{K}^{n+2}_{(A,C)}$, which is a subcategory of $\mathbf{K}^{n+2}_{\mathscr{C}}$.

For any morphism $f^\bullet$ in $\mathbf{C}^{n+2}_{(A,C)}$, its image in $\mathbf{K}^{n+2}_{(A,C)}$ will be denoted by $\underline{f^\bullet}$. As the usual terminology, a morphism $f^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(X^\bullet, Y^\bullet)$ is called a *homotopy equivalence* if it induces an isomorphism $\underline{f^\bullet}$ in $\mathbf{K}^{n+2}_{(A,C)}$. Two objects $X^\bullet, Y^\bullet \in \mathbf{C}^{n+2}_{(A,C)}$ are said to be *homotopically equivalent* if there is some homotopy equivalence $X^\bullet \to Y^\bullet$. We denote the homotopy equivalence class of ${}_AX^\bullet_C$ by $[{}_AX^\bullet_C]$ or simply by $[X^\bullet]$.

Remark 2.18. Let $X^\bullet, Y^\bullet \in \mathbf{C}^{n+2}_{(A,C)}$ be any pair of objects. By Claim 2.8 (3), if a morphism $f^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(X^\bullet, Y^\bullet)$ gives a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$, then it is also a homotopy equivalence in $\mathbf{C}^{n+2}_{(A,C)}$.

However in general, a homotopy equivalence $g^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ does not necessarily give rise to a homotopy equivalence in $\mathbf{C}^{n+2}_{(A,C)}$, and thus there can be a difference between homotopy equivalences taken in $\mathbf{C}^{n+2}_{(A,C)}$ and in $\mathbf{C}^{n+2}_{\mathscr{C}}$. To distinguish, we use the notation $[X^\bullet]$ exclusively for the homotopy equivalence class in $\mathbf{C}^{n+2}_{(A,C)}$.

Claim 2.19. Let $f^\bullet \underset{\varphi^\bullet}{\sim} g^\bullet \colon X^\bullet \to Y^\bullet$ be homotopic morphisms in $\mathbf{C}^{n+2}_{\mathscr{C}}$.

(1) If $f^0 = g^0$ and if $\mathscr{C}(X^2, Y^0) \xrightarrow{-\circ d_X^1} \mathscr{C}(X^1, Y^0) \xrightarrow{-\circ d_X^0} \mathscr{C}(X^0, Y^0)$ is exact, then $\varphi^\bullet$ can be modified to satisfy $\varphi^1 = 0$.
(2) Dually, if $f^{n+1} = g^{n+1}$ and if $\mathscr{C}(X^{n+1}, Y^{n-1}) \xrightarrow{d_Y^{n-1}\circ -} \mathscr{C}(X^{n+1}, Y^n) \xrightarrow{d_Y^n \circ -} \mathscr{C}(X^{n+1}, Y^{n+1})$ is exact, then $\varphi^\bullet$ can be modified to satisfy $\varphi^{n+1} = 0$.
(3) If both assumptions of (1),(2) are satisfied and if $n \ge 2$, then $\varphi^\bullet$ can be modified to satisfy $\varphi^1 = 0$ and $\varphi^{n+1} = 0$.

Proof. We only show (1). By $\varphi^1 \circ d_X^0 = g^0 - f^0 = 0$ and the exactness of

$$\mathscr{C}(X^2, Y^0) \xrightarrow{-\circ d_X^1} \mathscr{C}(X^1, Y^0) \xrightarrow{-\circ d_X^0} \mathscr{C}(X^0, Y^0),$$

there is $h \in \mathscr{C}(X^2, Y^0)$ which gives $h \circ d_X^1 = \varphi^1$. Then $(0, \varphi^2 + d_Y^0 \circ h, \varphi^3, \dots, \varphi^{n+1})$ gives the desired homotopy. □

Morphisms in $\mathbf{C}^{n+2}_{(A,C)}$ behave nicely with n-exangles. The following is obvious from the definition.

Remark 2.20. Let ${}_A\delta_C$ be any extension, and let $\langle X^\bullet, \delta\rangle, \langle Y^\bullet, \delta\rangle$ be objects in Æ. Then any morphism $f^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(X^\bullet, Y^\bullet)$ gives a morphism $f^\bullet \colon \langle X^\bullet, \delta\rangle \to \langle Y^\bullet, \delta\rangle$ in Æ.

Proposition 2.21. *Let ${}_A\delta_C$ be any extension, let $\langle X^\bullet,\delta\rangle,\langle Y^\bullet,\delta\rangle$ be n-exangles, and let $f^\bullet\in\mathbf{C}^{n+2}_{(A,C)}(X^\bullet,Y^\bullet)$ be any morphism. If $\mathbf{C}^{n+2}_{(A,C)}(Y^\bullet,X^\bullet)\neq\emptyset$, then $f^\bullet$ is a homotopy equivalence in $\mathbf{C}^{n+2}_{(A,C)}$.*

Proof. By the exactness of $\mathscr{C}(Y^n,X^n)\xrightarrow{-\circ d_X^n}\mathscr{C}(Y^n,C)\xrightarrow{\delta_\sharp}\mathbb{E}(Y^n,A)$ and $(d_Y^n)^*\delta=0$, there is $h\in\mathscr{C}(Y^n,X^n)$ which gives $d_X^n\circ h=d_Y^n$. By assumption, there is some $y^\bullet\in\mathbf{C}^{n+2}_{(A,C)}(Y^\bullet,X^\bullet)$.

Put $\varphi^1=0\in\mathscr{C}(X^1,X^0)$. By the exactness of

$$\mathscr{C}(C,-)\xRightarrow{-\circ d_X^n}\mathscr{C}(X^n,-)\xRightarrow{-\circ d_X^{n-1}}\cdots\xRightarrow{-\circ d_X^0}\mathscr{C}(A,-)\xRightarrow{\delta^\sharp}\mathbb{E}(C,-),$$

we obtain $\varphi^i\in\mathscr{C}(X^i,X^{i-1})$ for $2\le i\le n+1$ satisfying

$$\varphi^{i+1}\circ d_X^i+d_X^{i-1}\circ\varphi^i=1-y^i\circ f^i\quad(1\le i\le n).$$

Then, since

$$\begin{aligned}d_X^n\circ\varphi^{n+1}\circ d_Y^n &= d_X^n\circ\varphi^{n+1}\circ d_X^n\circ h\\ &= d_X^n\circ(1-y^n\circ f^n-d_X^{n-1}\circ\varphi^n)\circ h\\ &= (d_X^n-d_X^n\circ y^n\circ f^n)\circ h\ =\ 0,\end{aligned}$$

the sequence

$$g^\bullet=(y^0,y^1,\ldots,y^{n-1},y^n+\varphi^{n+1}\circ d_Y^n,1_C)$$

gives a morphism $g^\bullet\in\mathbf{C}^{n+2}_{(A,C)}(Y^\bullet,X^\bullet)$. We can easily check that $g^\bullet$ satisfies $g^\bullet\circ f^\bullet\sim 1$ for the homotopy $(\varphi^1,\ldots,\varphi^n,0)$. Thus $f^\bullet$ has a left homotopy inverse $g^\bullet$.

Applying the argument so far to $g^\bullet$ instead of $f^\bullet$, we see that $g^\bullet$ also has a left homotopy inverse $f'^\bullet$, which necessarily satisfies $\underline{f'^\bullet}=\underline{f^\bullet}$. This shows $\underline{g^\bullet}=(\underline{f^\bullet})^{-1}$. □

2.4. **Realization of extensions.**

Definition 2.22. Let $\mathfrak{s}$ be an association which assigns a homotopy equivalence class $\mathfrak{s}(\delta)=[{}_AX^\bullet_C]$ to each extension $\delta={}_A\delta_C$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it satisfies the following condition for any $\mathfrak{s}(\delta)=[X^\bullet]$ and any $\mathfrak{s}(\rho)=[Y^\bullet]$.

(R0) For any morphism of extensions $(a,c)\colon\delta\to\rho$, there exists a morphism $f^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)$ of the form $f^\bullet=(a,f^1,\ldots,f^n,c)$. Such $f^\bullet$ is called a *lift* of (a,c).

In such a case, we abbreviately say that "$X^\bullet$ realizes δ" whenever they satisfy $\mathfrak{s}(\delta)=[X^\bullet]$.

Moreover, a realization $\mathfrak{s}$ of $\mathbb{E}$ is said to be *exact* if it satisfies the following conditions.

(R1) For any $\mathfrak{s}(\delta)=[X^\bullet]$, the pair $\langle X^\bullet,\delta\rangle$ is an n-exangle.

(R2) For any $A\in\mathscr{C}$, the zero element ${}_A0_0=0\in\mathbb{E}(0,A)$ satisfies

$$\mathfrak{s}({}_A0_0)=[A\xrightarrow{1_A}A\to 0\to\cdots\to 0\to 0].$$

Dually, $\mathfrak{s}({}_00_A)=[0\to 0\to\cdots\to 0\to A\xrightarrow{1_A}A]$ holds for any $A\in\mathscr{C}$.

By Proposition 2.16 (and Remark 2.20), the above condition (R1) does not depend on representatives of the class $[X^\bullet]$.

Definition 2.23. Let $\mathfrak{s}$ be an exact realization of $\mathbb{E}$.

(1) An n-exangle $\langle X^\bullet, \delta\rangle$ is called an $\mathfrak{s}$-*distinguished* n-exangle if it satisfies $\mathfrak{s}(\delta) = [X^\bullet]$. We often simply say *distinguished n-exangle* when $\mathfrak{s}$ is clear from the context.
(2) An object $X^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ is called an $\mathfrak{s}$-*conflation* or simply a *conflation* if it realizes some extension $\delta \in \mathbb{E}(X^{n+1}, X^0)$.
(3) A morphism f in $\mathscr{C}$ is called an $\mathfrak{s}$-*inflation* or simply an *inflation* if it admits some conflation $X^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $d^0_X = f$.
(4) A morphism g in $\mathscr{C}$ is called an $\mathfrak{s}$-*deflation* or simply a *deflation* if it admits some conflation $X^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $d^n_X = g$.

Lemma 2.24. Let ${}_A\delta_C$ be any extension, and let $\langle X^\bullet, \delta\rangle, \langle Y^\bullet, \delta\rangle$ be n-exangles. If a morphism $f^\bullet \in \text{Æ}(\langle X^\bullet, \delta\rangle, \langle Y^\bullet, \delta\rangle)$ satisfies $f^{n+1} = 1_C$, then there is a morphism $f'^\bullet$ which is homotopic to $f^\bullet$ and belongs to $\mathbf{C}^{n+2}_{(A,C)}$.

Proof. Since $f^\bullet$ satisfies $(f^0)_*\delta = (f^{n+1})^*\delta = \delta$, there exists $h \in \mathscr{C}(X^1, A)$ satisfying $h \circ d^0_X = 1 - f^0$ by the exactness of

$$\mathscr{C}(X^1, A) \xrightarrow{-\circ d^0_X} \mathscr{C}(A, A) \xrightarrow{\delta^\sharp} \mathbb{E}(C, A).$$

If we modify $f^\bullet$ by a homotopy $\varphi^\bullet = (h, 0, \ldots, 0)$, then the resulting morphism $f'^\bullet$ satisfies the desired properties. □

Proposition 2.25. *Let $\mathfrak{s}$ be an exact realization of $\mathbb{E}$. Suppose that a morphism $f^\bullet \in \text{Æ}({}_A\langle X^\bullet, \delta\rangle_C, {}_B\langle Y^\bullet, \rho\rangle_C)$ satisfies $f^{n+1} = 1_C$ and gives a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$. Then $\langle X^\bullet, \delta\rangle$ is a distinguished n-exangle if and only if $\langle Y^\bullet, \rho\rangle$ is.*

Proof. By Claim 2.8, there is a homotopy inverse $g^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(Y^\bullet, X^\bullet)$ of $f^\bullet$ satisfying $g^{n+1} = 1_C$, which gives a morphism $g^\bullet \colon \langle Y^\bullet, \rho\rangle \to \langle X^\bullet, \delta\rangle$ by Proposition 2.10 (2). Thus it suffices to show the 'if' part, since the statement is symmetric in $\langle X^\bullet, \delta\rangle$ and $\langle Y^\bullet, \rho\rangle$.

Assume that $\langle Y^\bullet, \rho\rangle$ is a distinguished n-exangle, and put $f^0 = a, g^0 = b$ for simplicity. By Proposition 2.16, the pair $\langle X^\bullet, \delta\rangle$ is also an n-exangle. Take $\mathfrak{s}(\delta) = [Z^\bullet]$, to obtain a distinguished n-exangle ${}_A\langle Z^\bullet, \delta\rangle_C$. Since $\langle Y^\bullet, \rho\rangle$ is also a distinguished n-exangle, morphisms $(a, 1_C) \colon \delta \to \rho$ and $(b, 1_C) \colon \rho \to \delta$ have lifts $h^\bullet \colon \langle Z^\bullet, \delta\rangle \to \langle Y^\bullet, \rho\rangle$ and $\ell^\bullet \colon \langle Y^\bullet, \rho\rangle \to \langle Z^\bullet, \delta\rangle$. Composing with $g^\bullet$ and $f^\bullet$, we obtain $g^\bullet \circ h^\bullet \in \text{Æ}(\langle Z^\bullet, \delta\rangle, \langle X^\bullet, \delta\rangle)$ and $\ell^\bullet \circ f^\bullet \in \text{Æ}(\langle X^\bullet, \delta\rangle, \langle Z^\bullet, \delta\rangle)$. Since $g^{n+1} \circ h^{n+1} = 1_C$, it is homotopic to a morphism $k^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(Z^\bullet, X^\bullet)$ by Lemma 2.24. Similarly for $\ell^\bullet \circ f^\bullet$. Then by Proposition 2.21, we have $[X^\bullet] = [Z^\bullet] = \mathfrak{s}(\delta)$, which means that $\langle X^\bullet, \delta\rangle$ is distinguished. □

Corollary 2.26. *Let $\mathfrak{s}$ be an exact realization of $\mathbb{E}$. For any distinguished n-exangle ${}_A\langle X^\bullet, \delta\rangle_C$, i.e.*

$$A \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} \cdots \xrightarrow{d^{n-1}_X} X^n \xrightarrow{d^n_X} C \overset{\delta}{\dashrightarrow},$$

the following holds.

(1) *For any isomorphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$,*

$$A' \xrightarrow{d^0_X \circ a^{-1}} X^1 \xrightarrow{d^1_X} \cdots \xrightarrow{d^{n-1}_X} X^n \xrightarrow{c^{-1} \circ d^n_X} C' \overset{a_*c^*\delta}{\dashrightarrow}$$

is again a distinguished n-exangle.

(2) *If an object $\langle Y^\bullet, \rho\rangle \in \text{Æ}$ is isomorphic in Æ to $\langle X^\bullet, \delta\rangle$, then $\langle Y^\bullet, \rho\rangle$ is also a distinguished n-exangle.*

Proof. (1) We have the following sequence of isomorphisms $\langle X^\bullet, \delta\rangle \xrightarrow{f^\bullet} \langle X'^\bullet, c^*\delta\rangle \xrightarrow{g^\bullet} \langle X''^\bullet, a_*c^*\delta\rangle$ in Æ.

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & C & \overset{\delta}{\dashrightarrow} & \\
\| & \circlearrowright & \| & \circlearrowright & \| & \circlearrowright & & & \| & \circlearrowright & \downarrow{\scriptstyle c^{-1}} & & \\
A & \xrightarrow[d_X^0]{} & X^1 & \xrightarrow[d_X^1]{} & X^2 & \xrightarrow[d_X^2]{} & \cdots & \xrightarrow[d_X^{n-1}]{} & X^n & \xrightarrow[c^{-1}\circ d_X^n]{} & C' & \underset{c^*\delta}{\dashrightarrow} & \\
{\scriptstyle a}\downarrow & \circlearrowright & \| & \circlearrowright & \| & \circlearrowright & & & \| & \circlearrowright & \| & & \\
A' & \xrightarrow[d_X^0\circ a^{-1}]{} & X^1 & \xrightarrow[d_X^1]{} & X^2 & \xrightarrow[d_X^2]{} & \cdots & \xrightarrow[d_X^{n-1}]{} & X^n & \xrightarrow[c^{-1}\circ d_X^n]{} & C' & \underset{a_*c^*\delta}{\dashrightarrow} &
\end{array}$$

Since $f^0 = 1_A$, the middle row becomes a distinguished n-exangle by Proposition 2.25. Then, since $g^{n+1} = 1_{C'}$, the bottom row becomes a distinguished n-exangle by the dual of the same proposition.

(2) Let $h^\bullet = (h^0, h^1, \ldots, h^n)\colon {}_A\langle X^\bullet, \delta\rangle_C \to {}_B\langle Y^\bullet, \rho\rangle_D$ be an isomorphism. By (1), the isomorphism $h^0 \in \mathscr{C}(A, B)$ induces the following distinguished n-exangle.

$$B \xrightarrow{d_X^0\circ(h^0)^{-1}} X^1 \xrightarrow{d_X^1} \cdots \to X^n \xrightarrow{d_X^n} C \overset{(h^0)_*\delta}{\dashrightarrow} \tag{2.7}$$

Since $(1_B, h^1, h^2, \ldots, h^n)$ gives an isomorphism from (2.7) to $\langle Y^\bullet, \rho\rangle$ in Æ, the dual of Proposition 2.25 shows that $\langle Y^\bullet, \rho\rangle$ becomes distinguished. □

2.5. Definition of n-exangulated categories.

Definition 2.27. For a morphism $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ satisfying $f^0 = 1_A$ for some $A = X^0 = Y^0$, its *mapping cone* $M_f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ is defined to be the complex

$$X^1 \xrightarrow{d_{M_f}^0} X^2 \oplus Y^1 \xrightarrow{d_{M_f}^1} X^3 \oplus Y^2 \xrightarrow{d_{M_f}^2} \cdots \xrightarrow{d_{M_f}^{n-1}} X^{n+1} \oplus Y^n \xrightarrow{d_{M_f}^n} Y^{n+1}$$

where

$$\begin{aligned}
d_{M_f}^0 &= \begin{bmatrix} -d_X^1 \\ f^1 \end{bmatrix}, \\
d_{M_f}^i &= \begin{bmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{bmatrix} \quad (1 \le i \le n-1), \\
d_{M_f}^n &= \begin{bmatrix} f^{n+1} & d_Y^n \end{bmatrix}.
\end{aligned}$$

The *mapping cocone* is defined dually, for morphisms $h^\bullet$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $h^{n+1} = 1$.

Proposition 2.28. *Suppose that a diagram in* $\mathbf{C}^{n+2}_{\mathscr{C}}$

$$\begin{array}{ccc}
X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet \\
{\scriptstyle x^\bullet}\downarrow & \overset{\sim}{\varphi^\bullet} & \downarrow{\scriptstyle y^\bullet} \\
W^\bullet & \xrightarrow[g^\bullet]{} & Z^\bullet
\end{array}$$

satisfies the following conditions.

(i) $x^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(X^\bullet, W^\bullet)$, *with* $X^0 = W^0 = A$ *and* $X^{n+1} = W^{n+1} = C$,

(ii) $y^\bullet \in \mathbf{C}^{n+2}_{(A,D)}(Y^\bullet, Z^\bullet)$, *with* $Y^0 = Z^0 = A$ *and* $Y^{n+1} = Z^{n+1} = D$,

(iii) $f^0 = g^0 = 1_A$,

(iv) *$g^\bullet \circ x^\bullet \underset{\varphi^\bullet}{\sim} y^\bullet \circ f^\bullet$ is a homotopy satisfying $\varphi^1 = 0$.*

Then the following holds for the mapping cones $M_f^\bullet$ and $M_g^\bullet$.

(1) *We have a morphism $F^\bullet \in \mathbf{C}_{\mathscr{C}}^{n+2}(M_f^\bullet, M_g^\bullet)$ given by*

$$F^\bullet = \Big(x^1, \begin{bmatrix} x^2 & 0 \\ \varphi^2 & y^1 \end{bmatrix}, \ldots, \begin{bmatrix} x^{n+1} & 0 \\ \varphi^{n+1} & y^n \end{bmatrix}, 1_D\Big).$$

(2) *Assume that $X^\bullet, Y^\bullet, Z^\bullet, W^\bullet$ satisfy the assumption of the exactness in Claim 2.19 (1). If $x^\bullet$ and $y^\bullet$ are homotopy equivalences in $\mathbf{C}_{(A,C)}^{n+2}$ and $\mathbf{C}_{(A,D)}^{n+2}$ respectively, then the above $F^\bullet$ is a homotopy equivalence in $\mathbf{C}_{\mathscr{C}}^{n+2}$.*

Proof. (1) This is straightforward.

(2) Let $w^\bullet \in \mathbf{C}_{(A,C)}^{n+2}(W^\bullet, X^\bullet)$ and $z^\bullet \in \mathbf{C}_{(A,D)}^{n+2}(Z^\bullet, Y^\bullet)$ be homotopy inverses of $x^\bullet$ and $y^\bullet$, with homotopies

$$x^\bullet \circ w^\bullet \underset{\omega^\bullet}{\sim} 1_{W^\bullet} \quad , \quad w^\bullet \circ x^\bullet \underset{\xi^\bullet}{\sim} 1_{X^\bullet},$$
$$y^\bullet \circ z^\bullet \underset{\zeta^\bullet}{\sim} 1_{Z^\bullet} \quad , \quad z^\bullet \circ y^\bullet \underset{\eta^\bullet}{\sim} 1_{Y^\bullet}.$$

As in Claim 2.19, we may assume $\omega^1 = 0, \xi^1 = 0, \eta^1 = 0, \zeta^1 = 0$. Then

$$\psi^i = z^{i-1} \circ g^{i-1} \circ \omega^i - z^{i-1} \circ \varphi^i \circ w^i - \eta^i \circ f^i \circ w^i \quad (1 \le i \le n+1)$$

gives a homotopy $f^\bullet \circ w^\bullet \underset{\psi^\bullet}{\sim} z^\bullet \circ g^\bullet$ satisfying $\psi^1 = 0$. Thus (1) applied to

$$\begin{array}{ccc} W^\bullet & \xrightarrow{g^\bullet} & Z^\bullet \\ {\scriptstyle w^\bullet}\downarrow & \underset{\psi^\bullet}{\sim} & \downarrow{\scriptstyle z^\bullet} \\ X^\bullet & \xrightarrow[f^\bullet]{} & Y^\bullet \end{array}$$

gives a morphism $G^\bullet \in \mathbf{C}_{\mathscr{C}}^{n+2}(M_g^\bullet, M_f^\bullet)$ defined in the same way as $F^\bullet$. We can show that

$$\Phi^\bullet = \Big(\begin{bmatrix} -\xi^2 & 0 \end{bmatrix}, \begin{bmatrix} -\xi^3 & 0 \\ 0 & \eta^2 \end{bmatrix}, \ldots, \begin{bmatrix} -\xi^{n+1} & 0 \\ 0 & \eta^n \end{bmatrix}, \begin{bmatrix} 0 \\ \eta^{n+1} \end{bmatrix} \Big)$$

give a homotopy $G^\bullet \circ F^\bullet \underset{\Phi^\bullet}{\sim} I^\bullet$ where $I^\bullet \in C_{(X^1,D)}^{n+2}(M_f^\bullet, M_f^\bullet)$ is a morphism of the form

$$I^\bullet = \Big(1, \begin{bmatrix} 1 & 0 \\ a^2 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ a^{n+1} & 1 \end{bmatrix}, 1\Big)$$

for some $a^i \in \mathscr{C}(X^i, Y^{i-1})$. Since $I^\bullet$ is an isomorphism, this shows that $F^\bullet$ has a left homotopy inverse.

Similarly, $\psi^\bullet$ induces a homotopy $F^\bullet \circ G^\bullet \sim J^\bullet$ to an isomorphism $J^\bullet$, and $F^\bullet$ also has a right homotopy inverse. Thus $F^\bullet$ is a homotopy equivalence. □

Proposition 2.29. *Let $f^\bullet \colon {}_A\langle X^\bullet, \delta\rangle_C \to {}_A\langle Y^\bullet, \rho\rangle_D$ be a morphism in Æ, satisfying $f^0 = 1_A$. Then $\langle M_f^\bullet, (d_X^0)_*\rho\rangle$ also belongs to Æ.*

Proof. By the definition of $d_{M_f}^0$ and $d_{M_f}^{n+1}$, this follows from

$$(d_X^1)_*(d_X^0)_*\rho = 0, \quad (f^1)_*(d_X^0)_*\rho = (d_Y^0)_*\rho = 0$$

and

$$(f^{n+1})^*(d_X^0)_*\rho = (d_X^0)_*\delta = 0, \quad (d_Y^n)^*(d_X^0)_*\rho = (d_X^0)_*(d_Y^n)^*\rho = 0.$$

□

Corollary 2.30. *Let $f^\bullet, g^\bullet \colon {}_A\langle X^\bullet, \delta\rangle_C \to {}_A\langle Y^\bullet, \rho\rangle_D$ be any pair of morphisms of n-exangles, satisfying $f^0 = g^0 = 1_A$. If $g^\bullet \underset{\varphi^\bullet}{\sim} f^\bullet$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$, then $M_f^\bullet \cong M_g^\bullet$ holds in $\mathbf{C}^{n+2}_{(X^1,D)}$. In particular we have $[M_f^\bullet] = [M_g^\bullet]$.*

Proof. By Claim 2.19 (1), we may modify $\varphi^\bullet$ to satisfy $\varphi^1 = 0$. Applying Proposition 2.28 to

$$\begin{array}{ccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet \\ \| & \underset{\varphi^\bullet}{\sim} & \| \\ X^\bullet & \xrightarrow[g^\bullet]{} & Y^\bullet \end{array},$$

we obtain a homotopy equivalence

$$F^\bullet = \left(1_{X^1}, \begin{bmatrix} 1 & 0 \\ \varphi^2 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ \varphi^{n+1} & 1 \end{bmatrix}, 1_D\right) \colon M_f^\bullet \to M_g^\bullet$$

in $\mathbf{C}^{n+2}_{\mathscr{C}}$, which is indeed an isomorphism. □

Corollary 2.31. *Let $f^\bullet \colon {}_A\langle X^\bullet, \delta\rangle_C \to {}_A\langle Y^\bullet, \rho\rangle_D$ be a morphism of n-exangles, satisfying $f^0 = 1_A$. If $w^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(W^\bullet, X^\bullet)$ and $y^\bullet \in \mathbf{C}^{n+2}_{(A,D)}(Y^\bullet, Z^\bullet)$ are homotopy equivalences in $\mathbf{C}^{n+2}_{(A,C)}$ and $\mathbf{C}^{n+2}_{(A,D)}$ respectively, then the following holds for $g^\bullet = y^\bullet \circ f^\bullet \circ w^\bullet$.*

(1) *If $\langle M_f^\bullet, (d_X^0)_*\rho\rangle$ is an n-exangle, then so is $\langle M_g^\bullet, (d_W^0)_*\rho\rangle$.*
(2) *Moreover, if $\langle M_f^\bullet, (d_X^0)_*\rho\rangle$ is distinguished, so is $\langle M_g^\bullet, (d_W^0)_*\rho\rangle$.*

Proof. By Proposition 2.16 (and Remark 2.20), the pairs $\langle W^\bullet, \delta\rangle$ and $\langle Z^\bullet, \rho\rangle$ are n-exangles. Let $x^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(X^\bullet, W^\bullet)$ be a homotopy inverse of $w^\bullet$, and take a homotopy $w^\bullet \circ x^\bullet \underset{\xi^\bullet}{\sim} 1_{X^\bullet}$. If we define $\varphi^\bullet$ by

$$\varphi^i = y^{i-1} \circ f^{i-1} \circ \xi^i \quad (1 \le i \le n+1),$$

this gives a homotopy $g^\bullet \circ x^\bullet \underset{\varphi^\bullet}{\sim} y^\bullet \circ f^\bullet$. Since $\langle X^\bullet, \delta\rangle$ is an n-exangle, we may assume $\varphi^1 = 0$ by Claim 2.19. Then by Proposition 2.28, we obtain a homotopy equivalence $F^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(M_f^\bullet, M_g^\bullet)$ satisfying $F^0 = x^1$ and $F^{n+1} = 1_D$. By $(x^1)_*(d_X^0)_*\rho = (d_W^0)_*\rho$, this gives a morphism $F^\bullet \in \text{Æ}(\langle M_f^\bullet, (d_X^0)_*\rho\rangle, \langle M_g^\bullet, (d_W^0)_*\rho\rangle)$. Thus (1) follows from Proposition 2.16, and (2) follows from Proposition 2.25. □

Definition 2.32. An *n-exangulated category* is a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ of additive category $\mathscr{C}$, biadditive functor $\mathbb{E} \colon \mathscr{C}^{\text{op}} \times \mathscr{C} \to Ab$, and its exact realization $\mathfrak{s}$, satisfying the following conditions.

(EA1) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be any sequence of morphisms in $\mathscr{C}$. If both f and g are inflations, then so is $g \circ f$. Dually, if f and g are deflations then so is $g \circ f$.

(EA2) For $\rho \in \mathbb{E}(D,A)$ and $c \in \mathscr{C}(C,D)$, let ${}_A\langle X^{\bullet}, c^*\rho\rangle_C$ and ${}_A\langle Y^{\bullet}, \rho\rangle_D$ be distinguished n-exangles. Then $(1_A, c)$ has a *good lift* $f^{\bullet}$, in the sense that its mapping cone gives a distinguished n-exangle $\langle M_f^{\bullet}, (d_X^0)_*\rho\rangle$.

(EA2$^{\mathrm{op}}$) Dual of (EA2).

Remark 2.33. Concerning (EA2), the following holds. Similarly for (EA2$^{\mathrm{op}}$).

(1) By Corollary 2.30, if $g^{\bullet} \sim f^{\bullet} \colon {}_A\langle X^{\bullet}, \delta\rangle_C \to {}_A\langle Y^{\bullet}, \rho\rangle_D$ are lifts of $(1_A, c)$, then $f^{\bullet}$ is a good lift if and only if $g^{\bullet}$ is.

(2) By Corollary 2.31, condition (EA2) is independent from representatives of the classes $[X^{\bullet}]$ and $[Y^{\bullet}]$.

3. Fundamental properties

3.1. **Fundamental properties of n-exangulated categories.** We summarize here some properties of n-exangulated categories, which will be used in the proceeding sections. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category, throughout this section.

Proposition 3.1. *Let ${}_A\delta_C$ be an extension. Suppose that for any $Q \in \mathscr{C}$,*

$$\delta_\sharp \colon \mathscr{C}(Q,C) \to \mathbb{E}(Q,A) \quad \text{and} \quad \delta^\sharp \colon \mathscr{C}(A,Q) \to \mathbb{E}(C,Q)$$

are monomorphic. Then, the following holds for any n-exangle $\langle X^{\bullet}, \delta\rangle$.

(1) $d_X^0 = 0$ *and* $d_X^n = 0$.

(2) $X^{\bullet}$ *is homotopically equivalent in* $\mathbf{C}^{n+2}_{(A,C)}$ *to the object*

$$A \xrightarrow{0} 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \xrightarrow{0} C, \tag{3.1}$$

which will be denoted by $\varnothing^{\bullet} = {}_A\varnothing^{\bullet}_C$ in the rest.

In particular, such δ should satisfy $\mathfrak{s}(\delta) = [\varnothing^{\bullet}]$.

Proof. (1) This immediately follows from $\delta^\sharp(d_X^0) = 0$ and $\delta_\sharp(d_X^n) = 0$.

(2) Remark that the assumption of the monomorphicity of $\delta_\sharp$ and $\delta^\sharp$ is equivalent to that $\langle \varnothing^{\bullet}, \delta\rangle$ is an n-exangle. Since there are morphisms

$$f^{\bullet} = (1_A, 0, \ldots, 0, 1_C) \colon X^{\bullet} \to \varnothing^{\bullet},$$
$$g^{\bullet} = (1_A, 0, \ldots, 0, 1_C) \colon \varnothing^{\bullet} \to X^{\bullet}$$

in $\mathbf{C}^{n+2}_{(A,C)}$ by (1), these are homotopy equivalences by Proposition 2.21. □

Remark 3.2. Let us explain the motivation behind Proposition 3.1 using the case of $(n+2)$-angulated categories. As we will see in Subsection 4.2, any $(n+2)$-angulated category $(\mathscr{C}, \Sigma, \pentagon)$ (triangulated category if $n = 1$) can be regarded as a particular case of an n-exangulated category. In this case, an n-exangle ${}_A\langle X^{\bullet}, \delta\rangle_C$ corresponds to an $(n+2)$-angle

$$A \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} C \xrightarrow{\delta} \Sigma A,$$

from which we can obtain its right rotation

$$\Sigma^{-1}C \xrightarrow{(-1)^n\Sigma^{-1}\delta} A \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} C. \tag{3.2}$$

Then the conditions $d_X^0 = 0$ and $d_X^n = 0$ correspond to that $\Sigma^{-1}\delta$ is an isomorphism, and that (3.2) becomes weakly isomorphic to

$$\Sigma^{-1}C \xrightarrow{(-1)^n\Sigma^{-1}\delta} A \to 0 \to \cdots \to 0 \to C,$$

whose left rotation gives (3.1).

Proposition 3.3. *Let* ${}_A\langle X^{\bullet},\delta\rangle_C$, ${}_B\langle Y^{\bullet},\rho\rangle_D$ *be any pair of objects in* Æ. *Then the following are equivalent.*

(1) $\langle X^{\bullet}\oplus Y^{\bullet},\delta\oplus\rho\rangle$ *is a distinguished* n*-exangle.*
(2) *Both* $\langle X^{\bullet},\delta\rangle$ *and* $\langle Y^{\bullet},\rho\rangle$ *are distinguished* n*-exangles.*

Proof. As in Remark 2.14, $\langle X^{\bullet}\oplus Y^{\bullet},\delta\oplus\rho\rangle$ is an n-exangle if and only if $\langle X^{\bullet},\delta\rangle$ and $\langle Y^{\bullet},\rho\rangle$ are n-exangles.

(1) $\Rightarrow$ (2). Put $\mathfrak{s}(\delta)=[Z^{\bullet}]$, and let us show that $[X^{\bullet}]=[Z^{\bullet}]$ holds in $\mathbf{C}^{n+2}_{(A,C)}$. For simplicity, for any pair of objects $I,J\in\mathscr{C}$, denote the inclusion and projection to the 1st component by

$$j_I=\begin{bmatrix}1\\0\end{bmatrix}\colon I\to I\oplus J\quad\text{and}\quad p_I=[1\ 0]\colon I\oplus J\to I,$$

respectively. Since $(j_A,j_C)\colon\delta\to\delta\oplus\rho$ and $(p_A,p_C)\colon\delta\oplus\rho\to\delta$ are morphisms of extensions, they have lifts $f^{\bullet}\in\mathbf{C}^{n+2}_{\mathscr{C}}(Z^{\bullet},X^{\bullet}\oplus Y^{\bullet})$ and $g^{\bullet}\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet}\oplus Y^{\bullet},Z^{\bullet})$. If we compose them with

$$p^{\bullet}=(p_A,p_{X^1},\ldots,p_C)\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet}\oplus Y^{\bullet},X^{\bullet})$$

and

$$j^{\bullet}=(j_A,j_{X^1},\ldots,j_C)\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet},X^{\bullet}\oplus Y^{\bullet})$$

respectively, we obtain $p^{\bullet}\circ f^{\bullet}\in\mathbf{C}^{n+2}_{(A,C)}(Z^{\bullet},X^{\bullet})$ and $g^{\bullet}\circ j^{\bullet}\in\mathbf{C}^{n+2}_{(A,C)}(X^{\bullet},Z^{\bullet})$. Thus Proposition 2.21 shows $[X^{\bullet}]=[Z^{\bullet}]$. Similarly for $\mathfrak{s}(\rho)=[Y^{\bullet}]$.

(2) $\Rightarrow$ (1). Put $\mathfrak{s}(\delta\oplus\rho)=[W^{\bullet}]$, and let us show $[X^{\bullet}\oplus Y^{\bullet}]=[W^{\bullet}]$. Let $x^{\bullet}\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet},W^{\bullet})$ and $u^{\bullet}\in\mathbf{C}^{n+2}_{\mathscr{C}}(W^{\bullet},X^{\bullet})$ be lifts of (j_A,j_C) and (p_A,p_C), respectively. Similarly, let $y^{\bullet}\in\mathbf{C}^{n+2}_{\mathscr{C}}(Y^{\bullet},W^{\bullet})$ and $v^{\bullet}\in\mathbf{C}^{n+2}_{\mathscr{C}}(W^{\bullet},Y^{\bullet})$ be lifts of $\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$ and $([0\ 1],[0\ 1])$. Then

$$(1,[x^1\ y^1],\ldots,[x^n\ y^n],1)\colon X^{\bullet}\oplus Y^{\bullet}\to W^{\bullet},$$
$$\left(1,\begin{bmatrix}u^1\\v^1\end{bmatrix},\ldots,\begin{bmatrix}u^n\\v^n\end{bmatrix},1\right)\colon W^{\bullet}\to X^{\bullet}\oplus Y^{\bullet}$$

are morphisms in $\mathbf{C}^{n+2}_{(A\oplus B,C\oplus D)}$. Proposition 2.21 shows $[X^{\bullet}\oplus Y^{\bullet}]=[W^{\bullet}]$. $\square$

The following is an analog of [Hu, Lemma 5].

Corollary 3.4. *Suppose that*

$$X^0\oplus A\xrightarrow{d}X^1\oplus A\xrightarrow{[d_X^1\ w]}X^2\xrightarrow{d_X^2}\cdots\xrightarrow{d_X^n}X^{n+1}\overset{\theta}{\dashrightarrow}\tag{3.3}$$

is a distinguished n*-exangle, where* d *is as follows.*

$$d=\begin{bmatrix}x&u\\v&1\end{bmatrix}\in\mathscr{C}(X^0\oplus A,X^1\oplus A)$$

Then for $d_X^0=x-u\circ v$ *and* $p=[1\ 0]\colon X^0\oplus A\to X^0$,

$$X^0\xrightarrow{d_X^0}X^1\xrightarrow{d_X^1}X^2\xrightarrow{d_X^2}\cdots\xrightarrow{d_X^n}X^{n+1}\overset{p_*\theta}{\dashrightarrow}\tag{3.4}$$

becomes a distinguished n*-exangle.*

Proof. For p and $q = [0\ 1]\colon X^0 \oplus A \to A$, put $\delta = p_*\theta$ and $\rho = q_*\theta$. Then θ corresponds to $\begin{bmatrix}\delta\\ \rho\end{bmatrix}$ through the natural isomorphism

$$\mathbb{E}(X^{n+1}, X^0 \oplus A) \cong \mathbb{E}(X^{n+1}, X^0) \oplus \mathbb{E}(X^{n+1}, A), \tag{3.5}$$

and the equality $d_*\theta = 0$ implies $v_*\delta + \rho = 0$. Thus

$$\left(a = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}, b = \begin{bmatrix} 1 & -u \\ 0 & 1 \end{bmatrix}, 1, 1, \ldots, 1\right)$$

gives an isomorphism in Æ from (3.3) to

$$X^0 \oplus A \xrightarrow{d_X^0 \oplus 1_A} X^1 \oplus A \xrightarrow{[d_X^1\ 0]} X^2 \xrightarrow{d_X^2} \cdots \xrightarrow{d_X^n} X^{n+1} \overset{a_*\theta}{\dashrightarrow}, \tag{3.6}$$

with $a_*\theta$ corresponding to $\begin{bmatrix}\delta\\ 0\end{bmatrix}$ through (3.5). Since (3.6) is isomorphic to a coproduct of (3.4) and

$$A \xrightarrow{1_A} A \to 0 \to \cdots \to 0 \overset{0}{\dashrightarrow}$$

in Æ, Corollary 2.26 and Proposition 3.3 shows that (3.4) is also a distinguished n-exangle. □

The following lemma is an analog to parts of the classical [ML, Theorems III.3.2 and III.3.4]. Especially, the treatment of sums of extensions in the proof is quite parallel to that in [ML, Theorem 2.1]. One can also compare with [NP, Proposition 3.3].

Lemma 3.5. For any distinguished n-exangle ${}_A\langle X^\bullet, \delta\rangle_C$, the following holds.

(1) $\mathscr{C}(-, C) \overset{\delta_\sharp}{\Longrightarrow} \mathbb{E}(-, A) \overset{(d_X^0)_*}{\Longrightarrow} \mathbb{E}(-, X^1)$ is exact.
(2) $\mathscr{C}(A, -) \overset{\delta^\sharp}{\Longrightarrow} \mathbb{E}(C, -) \overset{(d_X^n)^*}{\Longrightarrow} \mathbb{E}(X^n, -)$ is exact.

Proof. We only show (1), since (2) can be shown dually. Let us show the exactness of

$$\mathscr{C}(D, C) \xrightarrow{\delta_\sharp} \mathbb{E}(D, A) \xrightarrow{(d_X^0)_*} \mathbb{E}(D, X^1) \tag{3.7}$$

for any $D \in \mathscr{C}$. Suppose that $\theta \in \mathbb{E}(D, A)$ satisfies $(d_X^0)_*\theta = 0$. Put $\mathfrak{s}(\theta) = [Y^\bullet]$, to obtain a distinguished n-exangle ${}_A\langle Y^\bullet, \theta\rangle_D$. By Proposition 3.3, coproduct ${}_{A\oplus A}\langle X^\bullet \oplus Y^\bullet, \delta \oplus \theta\rangle_{C\oplus D}$ is also a distinguished n-exangle.

Let $\nabla_A = [1\ 1]\colon A \oplus A \to A$ be the folding morphism. Put $\mu = (\nabla_A)_*(\delta \oplus \theta)$ and $\mathfrak{s}(\mu) = [Z^\bullet]$, to obtain a distinguished n-exangle ${}_A\langle Z^\bullet, \mu\rangle_{C\oplus D}$. If we write $d_Z^n = \begin{bmatrix}k\\ \ell\end{bmatrix} : Z^n \to C \oplus D$, then $(d_Z^n)^*\mu = 0$ means

$$k^*\delta + \ell^*\theta = 0. \tag{3.8}$$

Since $\begin{bmatrix}1\\ 0\end{bmatrix} : C \to C\oplus D$ satisfies $\begin{bmatrix}1\\ 0\end{bmatrix}^* \mu = (\nabla_A)_* \begin{bmatrix}1\\ 0\end{bmatrix}^* (\delta\oplus\theta) = \delta$, we have a morphism of extensions $\left(1_A, \begin{bmatrix}1\\ 0\end{bmatrix}\right)\colon \delta \to \mu$. By (EA2), it has a good lift $f^\bullet\colon \langle X^\bullet, \delta\rangle \to \langle Z^\bullet, \mu\rangle$, which gives a distinguished n-exangle $\langle M_f^{\bullet}, (d_X^0)_*\mu\rangle$. By definition, the last two terms $M_f^n \xrightarrow{d_{M_f}^n} M_f^{n+1}$ of $M_f^{\bullet}$ is

$$C \oplus Z^n \xrightarrow{\begin{bmatrix} 1 & k \\ 0 & \ell \end{bmatrix}} C \oplus D.$$

Remark that the assumption $(d_X^0)_*\theta = 0$ shows

$$(d_X^0)_*\mu = (d_X^0)_*(\nabla_A)_*(\delta \oplus \theta) = [d_X^0\ d_X^0]_*(\delta \oplus \theta) = 0.$$

Thus by Claim 2.15, morphism $d_{M_f}^n$ has a section $s = \begin{bmatrix} p & q \\ r & t \end{bmatrix} : C \oplus D \to C \oplus Z^n$, and the equality $d_{M_f}^n \circ s = 1$ implies in particular $q + k \circ t = 0$ and $\ell \circ t = 1_D$. Then $q \in \mathscr{C}(D, C)$ satisfies

$$\delta_\sharp(q) = q^*\delta = -(k \circ t)^*\delta = t^*(-k^*\delta) = t^*\ell^*\theta = \theta$$

by (3.8). This shows the exactness of (3.7). □

The following is a consequence of (R0) and (EA2).

Proposition 3.6. *Let ${}_A\langle X^{\bullet}, \delta\rangle_C$ and ${}_B\langle Y^{\bullet}, \rho\rangle_D$ be distinguished n-exangles. Suppose that we are given a commutative square*

$$\begin{array}{ccc} X^0 & \xrightarrow{d_X^0} & X^1 \\ {\scriptstyle a}\downarrow & \circlearrowright & \downarrow{\scriptstyle b} \\ Y^0 & \xrightarrow[d_Y^0]{} & Y^1 \end{array}$$

in $\mathscr{C}$. Then the following holds.

1. *There is a morphism $f^{\bullet} : \langle X^{\bullet}, \delta\rangle \to \langle Y^{\bullet}, \rho\rangle$ which satisfies $f^0 = a$ and $f^1 = b$.*
2. *If $X^0 = Y^0 = A$ and $a = 1_A$ for some $A \in \mathscr{C}$, then the above $f^{\bullet}$ can be taken to give a distinguished n-exangle $\langle M_f^{\bullet}, (d_X^0)_*\rho\rangle$.*

Proof. By Lemma 3.5, $\mathscr{C}(C, D) \xrightarrow{\rho_\sharp} \mathbb{E}(C, B) \xrightarrow{(d_Y^0)_*} \mathbb{E}(C, Y^1)$ is exact. Thus by $(d_Y^0)_*(a_*\delta) = b_*(d_X^0)_*\delta = 0$, there is $c \in \mathscr{C}(C, D)$ satisfying $\rho_\sharp(c) = a_*\delta$. This gives a morphism of extensions $(a, c) : \delta \to \rho$.

By (R0), it has a lift $g^{\bullet} : \langle X^{\bullet}, \delta\rangle \to \langle Y^{\bullet}, \rho\rangle$. Then by the exactness of

$$\mathscr{C}(X^2, Y^1) \xrightarrow{-\circ d_X^1} \mathscr{C}(X^1, Y^1) \xrightarrow{-\circ d_X^0} \mathscr{C}(X^0, Y^1) \tag{3.9}$$

and $(b - g^1) \circ d_X^0 = 0$, there is $m \in \mathscr{C}(X^2, Y^1)$ which gives $m \circ d_X^1 = b - g^1$.

Modifying $g^{\bullet}$ by the homotopy $\varphi^{\bullet} = (0, m, 0, \ldots, 0)$, we obtain a morphism $f^{\bullet} = (a, b, g^2 + d_Y^2 \circ m, g^3, \ldots, g^n, g^{n+1})$, which satisfies the desired condition.

(2) The same construction as (1) works, except for that we take a good lift in the second step. Indeed as above, there is $c \in \mathscr{C}(C, D)$ which gives a morphism $(1_A, c) : \delta \to \rho$. By (EA2), it has a good lift $g^{\bullet} : \langle X^{\bullet}, \delta\rangle \to \langle Y^{\bullet}, \rho\rangle$, which makes $\langle M_g^{\bullet}, (d_X^0)_*\rho\rangle$ a distinguished n-exangle. Then the exactness of (3.9) gives a homotopy $\varphi^{\bullet} = (0, m, 0, \ldots, 0)$ from $g^{\bullet}$ to a morphism $f^{\bullet}$ satisfying $f^0 = 1_A$ and $f^1 = b$. By Corollary 2.30, it follows that $\langle M_g^{\bullet}, (d_X^0)_*\rho\rangle$ is also a distinguished n-exangle. □

3.2. **Relative theory.** In this subsection, we study relative theory for n-exangulated categories and show that n-exangulated structures can be inherited in any relative theory. This is just an n-exangulated analog of the argument for exact categories in [DRSSK, Sections 1.2 and 1.3]. We also remark that subfunctors $\mathbb{F} \subseteq \mathbb{E}$ for extriangulated categories, namely, in the case where $n = 1$ (see Proposition 4.3), are investigated in [ZH].

Definition 3.7. Let $\mathscr{C}$ be a category, and let $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ be a biadditive functor.

(1) A functor $\mathbb{F}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Set$ is called a *subfunctor* of $\mathbb{E}$ if it satisfies the following conditions.
- $\mathbb{F}(C, A)$ is a subset of $\mathbb{E}(C, A)$, for any $A, C \in \mathscr{C}$.
- $\mathbb{F}(c, a) = \mathbb{E}(c, a)|_{\mathbb{F}(C,A)}$ holds, for any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$.

In this case, we write as $\mathbb{F} \subseteq \mathbb{E}$.

(2) A subfunctor $\mathbb{F} \subseteq \mathbb{E}$ is said to be an *additive subfunctor* if $\mathbb{F}(C, A) \subseteq \mathbb{E}(C, A)$ is an abelian subgroup for any $A, C \in \mathscr{C}$. In this case, $\mathbb{F}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ itself becomes a biadditive functor.

Definition 3.8. Let $\mathbb{F} \subseteq \mathbb{E}$ be an additive subfunctor. For a realization $\mathfrak{s}$ of $\mathbb{E}$, define $\mathfrak{s}|_{\mathbb{F}}$ to be the restriction of $\mathfrak{s}$ onto $\mathbb{F}$. Namely, it is defined by $\mathfrak{s}|_{\mathbb{F}}(\delta) = \mathfrak{s}(\delta)$ for any $\mathbb{F}$-extension δ.

Claim 3.9. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category, and let $\mathbb{F} \subseteq \mathbb{E}$ be an additive subfunctor. Then $\mathfrak{s}|_{\mathbb{F}}$ is an exact realization of $\mathbb{F}$. Moreover, the triplet $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ satisfies conditions (EA2) and (EA2$^{\mathrm{op}}$).

Proof. This immediately follows from the definitions of these conditions. $\square$

Thus we may speak of $\mathfrak{s}|_{\mathbb{F}}$-conflations (resp. $\mathfrak{s}|_{\mathbb{F}}$-inflations, $\mathfrak{s}|_{\mathbb{F}}$-deflations) and $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangles as in Definition 2.23. The following condition on $\mathbb{F} \subseteq \mathbb{E}$ gives a necessary and sufficient condition for $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ to be an n-exangulated category, as will be shown in Proposition 3.16.

Definition 3.10. (cf.[DRSSK]) Let $\mathbb{F} \subseteq \mathbb{E}$ be a additive subfunctor.

(1) $\mathbb{F} \subseteq \mathbb{E}$ is *closed on the right* if
$$\mathbb{F}(-, X^0) \overset{(d_X^0)_*}{\Longrightarrow} \mathbb{F}(-, X^1) \overset{(d_X^1)_*}{\Longrightarrow} \mathbb{F}(-, X^2)$$
is exact for any $\mathfrak{s}|_{\mathbb{F}}$-conflation $X^{\bullet}$.

(2) $\mathbb{F} \subseteq \mathbb{E}$ is *closed on the left* if
$$\mathbb{F}(X^{n+1}, -) \overset{(d_X^n)^*}{\Longrightarrow} \mathbb{F}(X^n, -) \overset{(d_X^{n-1})^*}{\Longrightarrow} \mathbb{F}(X^{n-1}, -)$$
is exact for any $\mathfrak{s}|_{\mathbb{F}}$-conflation $X^{\bullet}$.

Proposition 3.11. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be any n-exangulated category. If $\mathfrak{s}|_{\mathbb{F}}$-inflations are closed by composition, then $\mathbb{F} \subseteq \mathbb{E}$ is closed on the right. Dually, if $\mathfrak{s}|_{\mathbb{F}}$-deflations are closed by composition, then $\mathbb{F} \subseteq \mathbb{E}$ is closed on the left.*

Proof. We only show the first statement. Suppose that $\mathfrak{s}|_{\mathbb{F}}$-inflations are closed by composition. Let ${}_A\langle X^{\bullet}, \delta\rangle_C$ be any $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle, and let us show the exactness of
$$\mathbb{F}(F, X^0) \xrightarrow{(d_X^0)_*} \mathbb{F}(F, X^1) \xrightarrow{(d_X^1)_*} \mathbb{F}(F, X^2)$$

for any $F \in \mathscr{C}$. Since $(d_X^1)_* \circ (d_X^0)_* = 0$ follows from $d_X^1 \circ d_X^0 = 0$, it is enough to show $\mathrm{Ker}(d_X^1)_* \subseteq \mathrm{Im}(d_X^0)_*$.

Let $\theta \in \mathrm{Ker}(d_X^1)_*$ be any element, and let ${}_B\langle Y^\bullet, \theta\rangle_F$ be an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle realizing it as follows, where we put $X^1 = B$.

$$B \xrightarrow{d_Y^0} Y^1 \xrightarrow{d_Y^1} Y^2 \to \cdots \to Y^n \xrightarrow{d_Y^n} F \overset{\theta}{\dashrightarrow} .$$

Since d_X^0 and d_Y^0 are $\mathfrak{s}|_{\mathbb{F}}$-inflations, their composition $d_Y^0 \circ d_X^0$ becomes an $\mathfrak{s}|_{\mathbb{F}}$-inflation by assumption. Thus there is some $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle ${}_A\langle Z^\bullet, \tau\rangle_D$ which satisfies $Z^1 = Y^1$ and $d_Z^0 = d_Y^0 \circ d_X^0$ as follows.

$$A \xrightarrow{d_Y^0 \circ d_X^0} Y^1 \xrightarrow{d_Z^1} Z^2 \xrightarrow{d_Z^2} \cdots \to Z^n \xrightarrow{d_Z^n} D \overset{\tau}{\dashrightarrow} .$$

By Proposition 3.6 applied to the following commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{d_X^0} & X^1 \\ \| & \circlearrowright & \downarrow{\scriptstyle d_Y^0} \\ A & \xrightarrow[d_Z^0]{} & Y^1 \end{array}$$

we find a morphism of n-exangles $f^\bullet \colon \langle X^\bullet, \delta\rangle \to \langle Z^\bullet, \tau\rangle$ which satisfies

$$f^0 = 1_A, \quad f^1 = d_Y^0$$

and makes ${}_B\langle M_f^\bullet, (d_X^0)_*\tau\rangle_D$ an $\mathfrak{s}$-distinguished n-exangle. Since $(d_X^0)_*\tau \in \mathbb{F}(D, B)$, this is an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle. Then by Proposition 3.6 applied to the diagram

$$\begin{array}{ccc} B & \xrightarrow{d} & X^2 \oplus Y^1 \\ \| & \circlearrowright & \downarrow{\scriptstyle [0\ 1]} \\ B & \xrightarrow[d_Y^0]{} & Y^1 \end{array}$$

where we put $d = \begin{bmatrix} -d_X^1 \\ d_Y^0 \end{bmatrix} \colon B \to X^2 \oplus Y^1$, we find a morphism of n-exangles $g^\bullet \colon \langle M_f^\bullet, (d_X^0)_*\tau\rangle \to \langle Y^\bullet, \theta\rangle$ which satisfies

$$g^0 = 1_B, \quad g^1 = [0\ 1]$$

and makes $\langle M_g^\bullet, d_*\theta\rangle$ an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle. In particular it satisfies

$$(g^{n+1})^*\theta = (d_X^0)_*\tau. \tag{3.10}$$

By definition, this n-exangle $\langle M_g^\bullet, d_*\theta\rangle$ is of the following form.

$$X^2 \oplus Y^1 \to X^3 \oplus Z^2 \oplus Y^1 \to X^4 \oplus Z^3 \oplus Y^2 \to \cdots \to D \oplus Y^n \xrightarrow{[g^{n+1}\ d_Y^n]} F \overset{d_*\theta}{\dashrightarrow}$$

Then $d_*\theta = 0$ follows from $(d_X^1)_*\theta = 0$ and $(d_Y^0)_*\theta = 0$, which means that $[g^{n+1}\ d_Y^n]$ has a section $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \colon F \to D \oplus Y^n$. If we put $\beta = s_1^*\tau \in \mathbb{F}(F, A)$, then the equalities $g^{n+1} \circ s_1 + d_Y^n \circ s_2 = 1_F$ and (3.10) show

$$\theta = s_1^*(g^{n+1})^*\theta + s_2^*(d_Y^n)^*\theta = s_1^*(d_X^0)_*\tau = (d_X^0)_*\beta,$$

and thus $\theta \in \mathrm{Im}\big(\mathbb{F}(F, X^0) \xrightarrow{(d_X^0)_*} \mathbb{F}(F, X^1)\big)$ holds. $\square$

Corollary 3.12. *For any n-exangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$, the sequences*

$$\mathbb{E}(-,X^0)\xrightarrow{(d_X^0)_*}\mathbb{E}(-,X^1)\xrightarrow{(d_X^1)_*}\mathbb{E}(-,X^2)$$

and

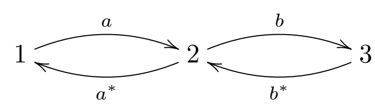
$$\mathbb{E}(X^{n+1},-)\xrightarrow{(d_X^n)^*}\mathbb{E}(X^n,-)\xrightarrow{(d_X^{n-1})^*}\mathbb{E}(X^{n-1},-)$$

are exact for any $\mathfrak{s}$-conflation $X^\bullet$.

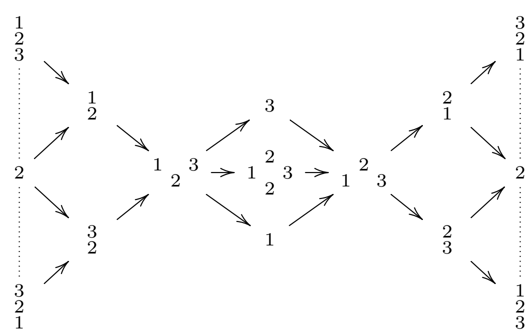
Proof. This immediately follows from Proposition 3.11 applied to $\mathbb{F}=\mathbb{E}$, as $(\mathscr{C},\mathbb{E},\mathfrak{s})$ satisfies condition (EA1). □

In view of the classical case one might expect that the the exact sequences in Lemma 3.5 and Corollary 3.12 are part of longer exact sequences involving $\mathbb{E}(-,X^i)$ for $i>2$, respectively $\mathbb{E}(X^i,-)$ for $i<n-1$. The following example shows that this is not the case in general and so in this sense Lemma 3.5 together with Corollary 3.12 is optimal for n-exangulated categories.

Example 3.13. Let k be field, Q the quiver



and $A=kQ/(aa^*,a^*a-bb^*,b^*b)$, i.e., the preprojective algebra of Dynkin type $\mathbb{A}_3$. The Auslander–Reiten quiver of A is



where each indecomposable is labelled by its Loewy structure and the dotted lines should be identified.

The subcategory $\mathscr{C}=\mathrm{add}(A\oplus \begin{smallmatrix}1\\2\end{smallmatrix}\oplus \begin{smallmatrix}2\end{smallmatrix}\oplus \begin{smallmatrix}2\\1\end{smallmatrix})$ is a 2-cluster tilting subcategory of $\mathrm{mod}A$, and according to [J], it is a 2-abelian category. Hence $\mathscr{C}$ has the structure of a 2-exangulated category, where $\mathbb{E}$ is just Ext_A^2 (see Subsection 4.2 for details). Notice that there is an exact sequence

$$0\to \begin{smallmatrix}2\end{smallmatrix}\to \begin{smallmatrix}1\\2\end{smallmatrix}\to \begin{smallmatrix}2\\1\end{smallmatrix}\to \begin{smallmatrix}2\end{smallmatrix}\to 0,$$

which is a 2-exact sequence in $\mathscr{C}$. However, the following complex

$$\mathrm{Ext}_A^2(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}2\end{smallmatrix})\to\mathrm{Ext}_A^2(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}1\\2\end{smallmatrix})\to\mathrm{Ext}_A^2(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}2\\1\end{smallmatrix})\to\mathrm{Ext}_A^2(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}2\end{smallmatrix})$$

is not exact at $\mathrm{Ext}_A^2(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}2\\1\end{smallmatrix})$. Indeed, it is isomorphic to $0\to 0\to k\to 0$.

Lemma 3.14. Assume that an additive subfunctor $\mathbb{F}\subseteq\mathbb{E}$ is closed on the right. Let $\theta\in\mathbb{E}(C,A)$ be any $\mathbb{E}$-extension. If it satisfies $x_*\theta\in\mathbb{F}(C,B)$ for some $\mathfrak{s}|_{\mathbb{F}}$-inflation $x\in\mathscr{C}(A,B)$, then $\theta\in\mathbb{F}(C,A)$ follows.

Proof. Take an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle ${}_A\langle X^{\bullet}, \delta\rangle$ satisfying $d_X^0 = x$. We have the following commutative diagram, whose bottom row is exact by Lemma 3.5 and Corollary 3.12.

$$\begin{array}{ccccccc} & & \mathbb{F}(C,A) & \xrightarrow{x_*} & \mathbb{F}(C,B) & \xrightarrow{(d_X^1)_*} & \mathbb{F}(C,X^2) \\ & \overset{\delta_\sharp}{\nearrow} \ \circlearrowright & \downarrow & \circlearrowright & \downarrow & \circlearrowright & \downarrow \\ \mathscr{C}(C,X^{n+1}) & \xrightarrow[\delta_\sharp]{} & \mathbb{E}(C,A) & \xrightarrow[x_*]{} & \mathbb{E}(C,B) & \xrightarrow[(d_X^1)_*]{} & \mathbb{E}(C,X^2) \end{array}$$

By assumption, the upper row is exact at $\mathbb{F}(C,B)$. Thus there exists some $\nu \in \mathbb{F}(C,A)$ satisfying $x_*\nu = x_*\theta$. Then by $x_*(\theta - \nu) = 0$, there exists $f \in \mathscr{C}(C, X^{n+1})$ which gives $\theta - \nu = \delta_\sharp(f)$. Thus it follows $\theta = \nu + \delta_\sharp f \in \mathbb{F}(C,A)$. □

Lemma 3.15. For any additive subfunctor $\mathbb{F} \subseteq \mathbb{E}$, the following are equivalent.

(1) $\mathbb{F}$ is closed on the right.
(2) $\mathbb{F}$ is closed on the left.

Thus in the following, we simply say $\mathbb{F} \subseteq \mathbb{E}$ is *closed*, if either of the conditions are satisfied.

Proof. We only show (1) ⇒ (2). Let $\langle X^{\bullet}, \delta\rangle_C$ be any $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle, and let us show the exactness of $\mathbb{F}(C,A) \overset{(d_X^n)^*}{\Longrightarrow} \mathbb{F}(X^n, A) \overset{(d_X^{n-1})^*}{\Longrightarrow} \mathbb{F}(X^{n-1}, A)$ for any $A \in \mathscr{C}$.

Take any element $\theta \in \mathbb{F}(X^n, A)$ satisfying $(d_X^{n-1})^*\theta = 0$. By the exactness of

$$\mathbb{E}(C,A) \overset{(d_X^n)^*}{\Longrightarrow} \mathbb{E}(X^n, A) \overset{(d_X^{n-1})^*}{\Longrightarrow} \mathbb{E}(X^{n-1}, A),$$

there exists $\nu \in \mathbb{E}(C,A)$ satisfying $(d_X^n)^*\nu = \theta$. It suffices to show $\nu \in \mathbb{F}(C,A)$. Realize θ by an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle ${}_A\langle Y^{\bullet}, \theta\rangle_{X^n}$, and ν by an $\mathfrak{s}$-distinguished n-exangle ${}_A\langle Z^{\bullet}, \nu\rangle_C$. Take a good lift $f^{\bullet}$ of $(1_A, d_X^n)\colon \theta \to \nu$, to obtain $\mathfrak{s}$-distinguished n-exangle $\langle M_f^{\bullet}, (d_Y^0)_*\nu\rangle$ as follows.

$$Y^1 \to Y^2 \oplus Z^1 \to \cdots \to Y^n \oplus Z^{n-1} \to X^n \oplus Z^n \xrightarrow{[d_X^n \ d_Z^n]} C \overset{(d_Y^0)_*\nu}{\dashrightarrow}$$

By the dual of Proposition 3.6 applied to the following diagram,

$$\begin{array}{ccc} X^n & \xrightarrow{d_X^n} & C \\ {\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\downarrow & \circlearrowright & \| \\ X^n \oplus Z^n & \xrightarrow[{[d_X^n \ d_Z^n]}]{} & C \end{array}$$

we obtain a morphism of n-exangles $g^{\bullet}\colon \langle X^{\bullet}, \delta\rangle_C \to \langle M_f^{\bullet}, (d_Y^0)_*\nu\rangle$ satisfying $g^{n+1} = 1_C$. In particular we have $(d_Y^0)_*\nu = (g^0)_*\delta \in \mathbb{F}(Y^1, C)$. Since d_Y^0 is an $\mathfrak{s}|_{\mathbb{F}}$-inflation, Lemma 3.14 shows $\nu \in \mathbb{F}(C,A)$. □

Proposition 3.16. *For any additive subfunctor $\mathbb{F} \subseteq \mathbb{E}$, the following are equivalent.*

(1) $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ *is n-exangulated.*
(2) $\mathfrak{s}|_{\mathbb{F}}$*-inflations are closed under composition.*
(3) $\mathfrak{s}|_{\mathbb{F}}$*-deflations are closed under composition.*
(4) $\mathbb{F} \subseteq \mathbb{E}$ *is closed.*

Proof. (1) holds if and only if both (2) and (3) hold, by Claim 3.9. Proposition 3.11 means that (2) or (3) implies (4). Since (4) is self-dual by Lemma 3.15, it remains to show (4) ⇒ (2).

Let ${}_A\langle X^\bullet, \delta\rangle_C, {}_B\langle Y^\bullet, \rho\rangle_F$ be any pair of $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exanlges with $X^1 = B$. Under the assumption of (4), let us show that $d_Y^0 \circ d_X^0$ becomes an $\mathfrak{s}|_{\mathbb{F}}$-inflation. Since $d_Y^0 \circ d_X^0$ is an $\mathfrak{s}$-inflation, there is an $\mathfrak{s}$-distinguished n-exangle ${}_A\langle Z^\bullet, \tau\rangle_D$ satisfying $d_Z^0 = d_Y^0 \circ d_X^0$. Similarly as in the proof of Proposition 3.11, we obtain by Proposition 3.6 a morphism of n-exangles $f^\bullet \colon \langle X^\bullet, \delta\rangle \to \langle Z^\bullet, \tau\rangle$ which satisfies $f^0 = 1_A, f^1 = d_Y^0$ and makes $\langle M_f^\bullet, (d_X^0)_*\tau\rangle$ an $\mathfrak{s}$-distinguished n-exangle as follows.

$$B \xrightarrow{\left[\begin{smallmatrix} -d_X^1 \\ d_Y^0 \end{smallmatrix}\right]} X^2 \oplus Y^1 \to \cdots \to C \oplus Z^n \to D \overset{(d_X^0)_*\tau}{\dashrightarrow} .$$

Applying Proposition 3.6 to the following diagram,

$$\begin{array}{ccc} B & \xrightarrow{\left[\begin{smallmatrix} -d_X^1 \\ d_Y^0 \end{smallmatrix}\right]} & X^2 \oplus Y^1 \\ \| & \circlearrowright & \downarrow{\scriptstyle [0\ 1]} \\ B & \xrightarrow[d_Y^0]{} & Y^1 \end{array}$$

we obtain a morphism of n-exangles $g^\bullet \colon \langle M_f^\bullet, (d_X^0)_*\tau\rangle \to \langle Y^\bullet, \rho\rangle$ satisfying $g^0 = 1_B$ and $g^1 = [0\ 1]$. In particular we have $(d_X^0)_*\tau = (g^{n+1})^*\rho \in \mathbb{F}(D, B)$. Since d_X^0 is an $\mathfrak{s}|_{\mathbb{F}}$-inflation, Lemma 3.14 shows $\tau \in \mathbb{F}(D, A)$. □

Corollary 3.17. *Let $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of additive subfunctors of $\mathbb{E}$. If each $\mathbb{F}_\lambda \subseteq \mathbb{E}$ is closed, then so is their intersection $\bigcap_{\lambda\in\Lambda} \mathbb{F}_\lambda \subseteq \mathbb{E}$.*

Proof. It can be easily confirmed that the intersection satisfies condition (2) in Proposition 3.16. □

Definition 3.18. Let $\mathcal{I} \subseteq \mathscr{C}$ be a full subcategory. Define subfunctors $\mathbb{E}_{\mathcal{I}}$ and $\mathbb{E}^{\mathcal{I}}$ of $\mathbb{E}$ by

$$\begin{aligned} \mathbb{E}_{\mathcal{I}}(C, A) &= \{\delta \in \mathbb{E}(C, A) \mid (\delta_\sharp)_I = 0 \text{ for any } I \in \mathcal{I}\}, \\ \mathbb{E}^{\mathcal{I}}(C, A) &= \{\delta \in \mathbb{E}(C, A) \mid \delta_I^\sharp = 0 \text{ for any } I \in \mathcal{I}\}. \end{aligned}$$

Proposition 3.19. *For any full subcategory $\mathcal{I} \subseteq \mathscr{C}$, these $\mathbb{E}_{\mathcal{I}}$ and $\mathbb{E}^{\mathcal{I}}$ are closed subfunctors of $\mathbb{E}$.*

Proof. We only show for the statement for $\mathbb{E}_{\mathcal{I}}$. To show that $\mathbb{E}_{\mathcal{I}}$ is a subfunctor of $\mathbb{E}$, it suffices to show

$$a_*c^*(\mathbb{E}_{\mathcal{I}}(C, A)) \subseteq \mathbb{E}_{\mathcal{I}}(C', A')$$

for any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$. Let $\delta \in \mathbb{E}_{\mathcal{I}}(C, A)$ be any element. Then $a_*c^*\delta \in \mathbb{E}(C', A')$ satisfies

$$(a_*c^*\delta)_\sharp(f') = f'^*a_*c^*\delta = a_*(\delta_\sharp(c \circ f')) = 0$$

for any $I \in \mathcal{I}$ and any $f' \in \mathscr{C}(I, C')$. Thus $((a_*c^*\delta)_\sharp)_I = 0$ holds for any $I \in \mathcal{I}$, which means $a_*c^*\delta \in \mathbb{E}_{\mathcal{I}}(C', A')$. Thus $\mathbb{E}_{\mathcal{I}} \subseteq \mathbb{E}$ is a subfunctor. Moreover, since

$$0_\sharp = 0 \quad \text{and} \quad (\delta - \delta')_\sharp = \delta_\sharp - \delta'_\sharp$$

holds for $0 \in \mathbb{E}(C, A)$ and any $\delta, \delta' \in \mathbb{E}(C, A)$, we see that $\mathbb{E}_{\mathcal{I}} \subseteq \mathbb{E}$ is additive.

For any $\mathfrak{s}|_{\mathbb{E}_{\mathcal{I}}}$-distinguished n-exangle $\langle X^{\bullet},\delta\rangle$, let us show the exactness of

$$\mathbb{E}_{\mathcal{I}}(-,X^0)\overset{(d_X^0)_*}{\Longrightarrow}\mathbb{E}_{\mathcal{I}}(-,X^1)\overset{(d_X^1)_*}{\Longrightarrow}\mathbb{E}_{\mathcal{I}}(-,X^2).$$

Remark that $\mathbb{E}(-,X^0)\overset{(d_X^0)_*}{\Longrightarrow}\mathbb{E}(-,X^1)\overset{(d_X^1)_*}{\Longrightarrow}\mathbb{E}(-,X^2)$ is exact. Thus for any $C\in\mathscr{C}$, if $\theta\in\mathbb{E}_{\mathcal{I}}(C,X^1)$ satisfies $(d_X^1)_*\theta=0$, then there is $\nu\in\mathbb{E}(C,X^0)$ which gives $(d_X^0)_*\nu=\theta$. It is enough to show $\nu_\sharp(f)=0$ for any $I\in\mathcal{I}$ and any $f\in\mathscr{C}(I,C)$. Since $0\to\mathbb{E}(I,X^0)\overset{(d_X^0)_*}{\longrightarrow}\mathbb{E}(I,X^1)$ is exact, this follows from the equation

$$(d_X^0)_*(\nu_\sharp(f))=(d_X^0)_*f^*\nu=f^*\theta=\theta_\sharp(f)=0.$$

□

4. Typical cases

4.1. **Extriangulated categories.** In this subsection, we consider the case $n=1$. Let $\mathscr{C}$ be an additive category, and let $\mathbb{E}\colon\mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to Ab$ be a biadditive functor.

Lemma 4.1. For any $A,C\in\mathscr{C}$, let $X^{\bullet},Y^{\bullet}\in\mathbf{C}^3_{(A,C)}$ be any pair of objects. Assume that

$$\mathscr{C}(C,-)\overset{\mathscr{C}(d_X^1,-)}{\Longrightarrow}\mathscr{C}(X^1,-)\overset{\mathscr{C}(d_X^0,-)}{\Longrightarrow}\mathbb{E}(A,-),\tag{4.1}$$

$$\mathscr{C}(-,A)\overset{\mathscr{C}(-,d_X^0)}{\Longrightarrow}\mathscr{C}(-,X^1)\overset{\mathscr{C}(-,d_X^1)}{\Longrightarrow}\mathbb{E}(-,C),\tag{4.2}$$

are exact, and similarly for $Y^{\bullet}$. Then for any morphism $f^{\bullet}=(1_A,f^1,1_C)\in\mathbf{C}^3_{(A,C)}(X^{\bullet},Y^{\bullet})$, the following are equivalent.

(1) $f^{\bullet}$ is a homotopy equivalence in $\mathbf{C}^3_{(A,C)}$.
(2) $f^{\bullet}$ is an isomorphism in $\mathbf{C}^3_{(A,C)}$.
(3) f^1 is an isomorphism in $\mathscr{C}$.

Thus $X^{\bullet},Y^{\bullet}$ are homotopically equivalent in $\mathbf{C}^3_{(A,C)}$ if and only if they are equivalent in the sense of [NP, Definition 2.7].

Proof. (2) ⇔ (3) is obvious. (2) ⇒ (1) is also trivial. Let us show that (1) implies (3). Suppose that $f^{\bullet}$ has a homotopy inverse $g^{\bullet}\in\mathbf{C}^3_{(A,C)}(Y^{\bullet},X^{\bullet})$.

Let $g^{\bullet}\circ f^{\bullet}\underset{\varphi^{\bullet}}{\sim}1_{X^{\bullet}}$ be a homotopy. By Claim 2.19, we may assume that $\varphi^{\bullet}$ is of the form $\varphi^{\bullet}=(0,\varphi^2)$. Then we have $\varphi^2\circ d_X^1=1-g^1\circ f^1$. Thus $x=g^1+\varphi^2\circ d_Y^1$ satisfies

$$x\circ f^1=g^1\circ f^1+\varphi^2\circ d_X^1=1,$$

and gives a left inverse of f^1 in $\mathscr{C}$. Similarly we can show that f^1 has a right inverse, which means it is an isomorphism. □

Lemma 4.2. Assume that $(\mathscr{C},\mathbb{E},\mathfrak{s})$ is a 1-exangulated category, and suppose we are given distinguished 1-exangles $A\overset{f}{\longrightarrow}B\overset{f'}{\longrightarrow}D\overset{\delta}{\dashrightarrow}$ and $B\overset{g}{\longrightarrow}C\overset{g'}{\longrightarrow}F\overset{\delta'}{\dashrightarrow}$. Remark that $h=g\circ f$ is an $\mathfrak{s}$-inflation by (EA1), and thus there is also some distinguished 1-exangle $A\overset{h}{\longrightarrow}C\overset{h'}{\longrightarrow}E\overset{\delta''}{\dashrightarrow}$.

Then, there exist $d\in\mathscr{C}(D,E)$ and $e\in\mathscr{C}(E,F)$ which satisfy the following conditions.

(i) $D\overset{d}{\longrightarrow}E\overset{e}{\longrightarrow}F\overset{f'_*\delta'}{\dashrightarrow}$ is a distinguished 1-exangle.
(ii) $d^*\delta''=\delta$.

(iii) $f_*\delta'' = e^*\delta'$.

Proof. This is an analog of [Hu, 3.5]. By Proposition 3.6 (2), there is $d \in \mathscr{C}(D,E)$ which satisfies $d \circ f' = h' \circ g$, $d^*\delta'' = \delta$ and makes

$$B \xrightarrow{u} D \oplus C \xrightarrow{[d\ h']} E \overset{f_*\delta''}{\dashrightarrow}$$

a distinguished 1-exangle for $u = \begin{bmatrix} -f' \\ g \end{bmatrix}$. Again by the same proposition applied to the following,

$$\begin{array}{ccccccc} B & \xrightarrow{u} & D \oplus C & \xrightarrow{[d\ h']} & E & \overset{f_*\delta''}{\dashrightarrow} & \\ \| & \circlearrowright & \downarrow{\scriptstyle [0\ 1]} & & & & \\ B & \xrightarrow[g]{} & C & \xrightarrow[g']{} & F & \underset{\delta'}{\dashrightarrow} & \end{array}$$

we obtain $e \in \mathscr{C}(E,F)$ which satisfies $e \circ [d\ h'] = g' \circ [0\ 1]$, $e^*\delta' = f_*\delta''$ and makes

$$D \oplus C \xrightarrow{v} E \oplus C \xrightarrow{[e\ g']} F \overset{u_*\delta'}{\dashrightarrow} \tag{4.3}$$

a distinguished 1-exangle for $v = \begin{bmatrix} -d & -h' \\ 0 & 1 \end{bmatrix}$. Thus Corollary 3.4 shows that

$$D \xrightarrow{-d} E \xrightarrow{e} F \overset{-f'_*\delta'}{\dashrightarrow}$$

is a distinguished 1-exangle. This is isomorphic to $D \xrightarrow{d} E \xrightarrow{e} F \overset{f'_*\delta'}{\dashrightarrow}$, and thus Corollary 2.26 can be applied. □

Proposition 4.3. *Let $\mathscr{C}$ and $\mathbb{E}$ be as before. Then, a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is a 1-exangulated category if and only if it is an extriangulated category.*

Proof. First, suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is a 1-exangulated category, and show it is an extriangulated category. By duality, let us just confirm conditions (ET1),(ET2),(ET3),(ET4) in [NP, Definition 2.12].

(ET1) is already assumed. By Lemma 4.1, the homotopy equivalence class $\mathfrak{s}(\delta) = [X^\bullet]$ is equal to the equivalence class of $X^\bullet$ in the sense of [NP, Definition 2.7] for any extension δ. Thus (ET2) follows from (R0),(R2) and Proposition 3.3. (ET3) is shown in Proposition 3.6. Lemma 4.2 shows (ET4).

Conversely, suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category. By [NP, Proposition 3.3], sequences (4.1) and (4.2) are exact for any $X^\bullet$ realizing an extension δ. Thus the equivalence class of $X^\bullet$ in the sense of [NP] is equal to the homotopy equivalence class of $X^\bullet$ in $\mathbf{C}^3_{(A,C)}$, by Lemma 4.2. Similarly as above, let us just confirm conditions (R0),(R1),(R2) and (EA1),(EA2).

(R0),(R2) follow from (ET2). (R1) is shown in [NP, Proposition 3.3]. (EA1) follows from (ET4), as stated in [NP, Remark 2.16]. (EA2) follows from the dual of [LN, Proposition 1.20]. □

4.2. **$(n+2)$-angulated categories.** In this subsection, we consider the case where the additive category $\mathscr{C}$ is equipped with an automorphism $\Sigma\colon \mathscr{C} \xrightarrow{\cong} \mathscr{C}$. Then Σ gives a biadditive functor $\mathbb{E}_\Sigma = \mathscr{C}(-, \Sigma-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$, defined by the following.

(i) For any $A, C \in \mathscr{C}$, $\mathbb{E}_\Sigma(C,A) = \mathscr{C}(C, \Sigma A)$.
(ii) For any $a \in \mathscr{C}(A,A')$ and $c \in \mathscr{C}(C',C)$, the map $\mathbb{E}_\Sigma(c,a)\colon \mathscr{C}(C,\Sigma A) \to \mathscr{C}(C', \Sigma A')$ sends $\delta \in \mathscr{C}(C,\Sigma A)$ to $c^*a_*\delta = (\Sigma a) \circ \delta \circ c$.

The aim of this subsection is to show the equivalence of the following (I) and (II).

(I) To give a class of $(n+2)$-Σ-sequences $\pentagon$ which makes $(\mathscr{C},\Sigma,\pentagon)$ an $(n+2)$-angulated category in the sense of [GKO].

(II) To give an exact realization $\mathfrak{s}$ of $\mathbb{E}_\Sigma$ which makes $(\mathscr{C},\mathbb{E}_\Sigma,\mathfrak{s})$ an n-exangulated category.

First let us show that (I) implies (II). Let $(\mathscr{C},\Sigma,\pentagon)$ be an $(n+2)$-angulated category as in [GKO]. We assume that Σ is an automorphism as above. For each $\delta\in\mathbb{E}_\Sigma(C,A)$, complete it into an $(n+2)$-angle

$$A\xrightarrow{d_X^0}X^1\xrightarrow{d_X^1}X^2\xrightarrow{d_X^2}\cdots\xrightarrow{d_X^{n-1}}X^n\xrightarrow{d_X^n}C\xrightarrow{\delta}\Sigma A$$

by (F1) (c) and (F2) in [GKO]. Then define $\mathfrak{s}_{\pentagon}(\delta)=[X^\bullet]$ by using $X^\bullet\in\mathbf{C}^{n+2}_{(A,C)}$ given by

$$X^0\xrightarrow{d_X^0}X^1\xrightarrow{d_X^1}X^2\xrightarrow{d_X^2}\cdots\xrightarrow{d_X^{n-1}}X^n\xrightarrow{d_X^n}X^{n+1}\quad(X^0=A,\ X^{n+1}=C).$$

Lemma 4.4. For each ${}_A\delta_C$, the above $\mathfrak{s}_{\pentagon}(\delta)=[X^\bullet]$ is well-defined.

Proof. Let $A\xrightarrow{d_Y^0}Y^1\xrightarrow{d_Y^1}Y^2\xrightarrow{d_Y^2}\cdots\xrightarrow{d_Y^{n-1}}Y^n\xrightarrow{d_Y^n}C\xrightarrow{\delta}\Sigma A$ be another choice of $(n+2)$-angle, and let $Y^\bullet$ be the corresponding object in $\mathbf{C}^{n+2}_{(A,C)}$ given by

$$Y^0\xrightarrow{d_Y^0}Y^1\xrightarrow{d_Y^1}Y^2\xrightarrow{d_Y^2}\cdots\xrightarrow{d_Y^{n-1}}Y^n\xrightarrow{d_Y^n}Y^{n+1}\quad(Y^0=A,\ Y^{n+1}=C).$$

Let us show $[X^\bullet]=[Y^\bullet]$. By (F2),(F3) in [GKO], there is a morphism $f^\bullet\in\mathbf{C}^{n+2}_{(A,C)}(X^\bullet,Y^\bullet)$ as follows.

$$\begin{array}{ccccccccccc}
A & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & C\\
\| & \circlearrowright & \downarrow{\scriptstyle f^1} & \circlearrowright & \downarrow{\scriptstyle f^2} & \circlearrowright & & & \downarrow{\scriptstyle f^n} & \circlearrowright & \|\\
A & \xrightarrow[d_Y^0]{} & Y^1 & \xrightarrow[d_Y^1]{} & Y^2 & \xrightarrow[d_Y^2]{} & \cdots & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & C
\end{array}$$

Similarly, there is $g^\bullet\in\mathbf{C}^{n+2}_{(A,C)}(Y^\bullet,X^\bullet)$. Remark that by [GKO, Proposition 2.5] and its dual, sequences (2.3) and (2.4) are exact for $X^\bullet$ and $Y^\bullet$, which means $\langle X^\bullet,\delta\rangle$ and $\langle Y^\bullet,\delta\rangle$ are n-exangles. Thus $f^\bullet$ is an homotopy equivalence by Proposition 2.21. $\square$

Proposition 4.5. *With the above definition, $(\mathscr{C},\mathbb{E}_\Sigma,\mathfrak{s}_{\pentagon})$ becomes an n-exangulated category.*

Proof. Let us confirm the conditions. (R0) follows from (F2) and (F3). (R1) follows from [GKO, Proposition 2.5] and its dual. (R2) follows from (F1)(b) and (F2). (EA1) becomes trivial, since any morphism is both inflation and deflation by (F2).

Let us show (EA2). The following argument has been given in (the dual of) [BT, Lemma 4.1]. Let $c\in\mathscr{C}(C,D)$ be a morphism, and let ${}_A\langle X^\bullet,\delta=c^*\rho\rangle_C,{}_A\langle Y^\bullet,\rho\rangle_D$ be n-exangles. By the definition of $\mathfrak{s}_{\pentagon}$ and Remark 2.33 (2), we may assume that they correspond to $(n+2)$-angles

$$A\xrightarrow{d_X^0}X^1\xrightarrow{d_X^1}\cdots\xrightarrow{d_X^{n-1}}X^n\xrightarrow{d_X^n}C\xrightarrow{\delta}\Sigma A,$$
$$A\xrightarrow{d_Y^0}Y^1\xrightarrow{d_Y^1}\cdots\xrightarrow{d_Y^{n-1}}Y^n\xrightarrow{d_Y^n}D\xrightarrow{\rho}\Sigma A.$$

By (F2), we can 'rotate' them to obtain $(n+2)$-angles

$$\Sigma^{-1}C \xrightarrow{(-1)^n\Sigma^{-1}\delta} A \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} C,$$
$$\Sigma^{-1}D \xrightarrow{(-1)^n\Sigma^{-1}\rho} A \xrightarrow{d_Y^0} Y^1 \xrightarrow{d_Y^1} \cdots \xrightarrow{d_Y^{n-1}} Y^n \xrightarrow{d_Y^n} D.$$

By (F4), we obtain a morphism of $(n+2)$-Σ-sequences

$$\begin{array}{ccccccccccc} \Sigma^{-1}C & \xrightarrow{(-1)^n\Sigma^{-1}\delta} & A & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & C \\ {\scriptstyle \Sigma^{-1}c}\downarrow & \circlearrowright & \Vert & \circlearrowright & \downarrow{\scriptstyle f^1} & \circlearrowright & & & \downarrow{\scriptstyle f^n} & \circlearrowright & \downarrow{\scriptstyle c} \\ \Sigma^{-1}D & \xrightarrow[(-1)^n\Sigma^{-1}\rho]{} & A & \xrightarrow[d_Y^0]{} & Y^1 & \xrightarrow[d_Y^1]{} & \cdots & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & D \end{array}$$

which gives an $(n+2)$-angle

(4.4)
$$A\oplus\Sigma^{-1}D \xrightarrow{d^0} X^1\oplus A \xrightarrow{d^1} X^2\oplus Y^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^n\oplus Y^{n-1} \xrightarrow{d^n} C\oplus Y^n \xrightarrow{d^{n+1}} \Sigma A\oplus D$$

where

$$d^0 = \begin{bmatrix} -d_X^0 & 0 \\ 1 & (-1)^n\Sigma^{-1}\rho \end{bmatrix}, \quad d^i = \begin{bmatrix} -d_X^i & 0 \\ f^i & d_Y^{i-1} \end{bmatrix} \quad (1 \le i \le n),$$
$$d^{n+1} = \begin{bmatrix} (-1)^{n+1}\delta & 0 \\ c & d_Y^n \end{bmatrix}.$$

Then the sequence of isomorphisms in $\mathscr{C}$

$$\left(\begin{bmatrix} 1 & (-1)^n\Sigma^{-1}\rho \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & d_X^0 \end{bmatrix}, 1, 1, \ldots, 1, \begin{bmatrix} 1 & (-1)^n\rho \\ 0 & 1 \end{bmatrix} \right)$$

gives an isomorphism of $(n+2)$-sequences from (4.4) to

(4.5)
$$A\oplus\Sigma^{-1}D \xrightarrow{e^0} A\oplus X^1 \xrightarrow{e^1} X^2\oplus Y^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^n\oplus Y^{n-1} \xrightarrow{d^n} C\oplus Y^n \xrightarrow{e^{n+1}} \Sigma A\oplus D,$$

with

$$e^0 = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^n d_X^0 \circ \Sigma^{-1}\rho \end{bmatrix}, \quad e^1 = \begin{bmatrix} 0 & -d_X^1 \\ 0 & f^1 \end{bmatrix}, \quad e^{n+1} = \begin{bmatrix} 0 & 0 \\ c & d_Y^n \end{bmatrix}$$

and the same $d^2, \ldots, d^n$. Thus (4.5) belongs to $\pentagon$. Since this is equal to the direct sum of

$$A \xrightarrow{1_A} A \to 0 \to \cdots \to 0 \to \Sigma A \quad \text{and}$$

(4.6)
$$\Sigma^{-1}D \xrightarrow{q^0} X^1 \xrightarrow{q^1} X^2 \oplus Y^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^n \oplus Y^{n-1} \xrightarrow{d^n} C \oplus Y^n \xrightarrow{q^{n+1}} D$$

with

$$q^0 = (-1)^n d_X^0 \circ \Sigma^{-1}\rho, \quad q^1 = \begin{bmatrix} -d_X^1 \\ f^1 \end{bmatrix}, \quad q^{n+1} = [c \ d_Y^n],$$

we see that (4.6) also belongs to $\pentagon$ by (F1)(a). Rotating it by (F2), we obtain an $(n+2)$-angle

$$X^1 \xrightarrow{q^1} X^2 \oplus Y^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^n \oplus Y^{n-1} \xrightarrow{d^n} C \oplus Y^n \xrightarrow{q^{n+1}} D \xrightarrow{(\Sigma d_X^0)\circ\rho} \Sigma X^1.$$

By the definition of $\mathfrak{s}_{\pentagon}$, this shows that $f^\bullet = (1_A, f^1, \ldots, f^n, c) \colon \langle X^\bullet, \delta \rangle \to \langle Y^\bullet, \rho \rangle$ gives a distinguished n-exangle

$$X^1 \xrightarrow{q^1} X^2 \oplus Y^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^n \oplus Y^{n-1} \xrightarrow{d^n} C \oplus Y^n \xrightarrow{q^{n+1}} D \overset{(d_X^0)_*\rho}{\dashrightarrow},$$

that is what we wanted to show. $\square$

Conversely, let us show that (II) implies (I). Suppose we are given an exact realization of $\mathbb{E}_\Sigma$ which makes $(\mathscr{C}, \mathbb{E}_\Sigma, \mathfrak{s})$ an n-exangulated category. Remark that any object in Æ

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \overset{\delta}{\dashrightarrow}$$

can be naturally regarded as an $(n+2)$-Σ-*sequence*

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{\delta} \Sigma X^0 \tag{4.7}$$

in the sense of [GKO, Definition 2.1].

Remark 4.6. The above correspondence gives a fully faithful functor from Æ to the category of $(n+2)$-Σ-sequences. In this way, we may identify Æ with the full subcategory of the category of $(n+2)$-Σ-sequences, consisting of (4.7) satisfying $d_X^{i+1} \circ d_X^i = 0$ $(0 \le i \le n-1)$, $\delta \circ d_X^n = 0$ and $(\Sigma d_X^0) \circ \delta = 0$. This subcategory is closed by isomorphisms, and by taking finite direct sums and summands.

Lemma 4.7. For any $A \in \mathscr{C}$, let us denote $1_{\Sigma A} \in \mathscr{C}(\Sigma A, \Sigma A)$ by $\iota = {}_A\iota_{\Sigma A} \in \mathbb{E}_\Sigma(\Sigma A, A)$ when we regard it as an extension. For this extension, we have $\mathfrak{s}(\iota) = [\varnothing^\bullet]$. Namely,

$$A \xrightarrow{0} 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \xrightarrow{0} \Sigma A \overset{\iota}{\dashrightarrow}$$

is a distinguished n-exangle.

Proof. This immediately follows from Proposition 3.1 applied to $\delta = \iota$. $\square$

Proposition 4.8. *Define $\pentagon_{\mathfrak{s}}$ to be the class of $(n+2)$-Σ-sequences obtained as* (4.7) *from distinguished n-exangles. Then $(\mathscr{C}, \mathbb{E}_\Sigma, \pentagon_{\mathfrak{s}})$ becomes an $(n+2)$-angulated category.*

Proof. By Corollary 2.26 and Remark 4.6, the class $\pentagon_{\mathfrak{s}}$ is closed by isomorphisms of $(n+2)$-Σ-sequences. Thus we do not have to take any isomorphism closure. Let us confirm conditions (F1),...,(F4) in [GKO]. (F1)(a) follows from Proposition 3.3 and Remark 4.6. (F1)(b) follows from (R2).

(F2) Let ${}_A\langle X^\bullet, \delta\rangle_C$ be any distinguished n-exangle. It suffices to show that we can rotate $\langle X^\bullet, \delta\rangle$ in both directions to obtain distinguished n-exangles. As in Lemma 4.7, the pair ${}_A\langle \varnothing^\bullet, \iota\rangle_{\Sigma A}$ is also a distinguished n-exangle. By (EA2), the morphism $(1_A, \delta)\colon \delta \to \iota$ has a good lift $f^\bullet\colon \langle X^\bullet, \delta\rangle \to \langle \varnothing^\bullet, \iota\rangle$, which should be as follows without any other possibility.

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & C & \overset{\delta}{\dashrightarrow} & \\
\Vert & \circlearrowright & \downarrow{\scriptstyle 0} & \circlearrowright & \downarrow{\scriptstyle 0} & \circlearrowright & & & \downarrow{\scriptstyle 0} & \circlearrowright & \downarrow{\scriptstyle \delta} & & \\
A & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & \cdots & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & \Sigma A & \underset{\iota}{\dashrightarrow} &
\end{array}$$

Its mapping cone induces a distinguished n-exangle

$$X^1 \xrightarrow{-d_X^1} X^2 \xrightarrow{-d_X^2} \cdots \to X^{n-1} \xrightarrow{-d_X^{n-1}} X^n \xrightarrow{-d_X^n} C \xrightarrow{\delta} \Sigma A \overset{(d_X^0)_*\iota}{\dashrightarrow} . \tag{4.8}$$

Remark that we have $(d_X^0)_*\iota = (\Sigma d_X^0) \circ 1_{\Sigma A} = \Sigma d_X^0$ by definition. Since

$$\big((-1)^n, (-1)^{n-1}, \ldots, 1, -1, 1_C, 1_{\Sigma A}\big)$$

gives an isomorphism from (4.8) to

$$X^1 \xrightarrow{d_X^1} X^2 \xrightarrow{d_X^2} \cdots \to X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} C \xrightarrow{\delta} \Sigma A \overset{(-1)^n \Sigma d_X^0}{\dashrightarrow}, \tag{4.9}$$

this (4.9) becomes a distinguished n-exangle by Corollary 2.26. Rotation to the opposite direction can be performed in a dual manner.

(F1)(c) Any morphism $f \in \mathscr{C}(A, B)$ can be regarded as an extension $f \in \mathbb{E}_\Sigma(A, \Sigma^{-1}B)$, and then there exists some distinguished n-exangle

$$\Sigma^{-1}B \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \to X^n \xrightarrow{d_X^n} A \overset{f}{\dashrightarrow} .$$

Applying (F2) repeatedly, we obtain a distinguished n-exangle of the form

$$A \xrightarrow{f} B \to \Sigma X^1 \to \cdots \to \Sigma X^n \dashrightarrow .$$

(F3) This follows from (F2) and (R0), or from Proposition 3.6.

(F4) The same argument on the axioms of triangulated category [Hu] works. Suppose that we are given distinguished n-exangles $\langle X^\bullet, \delta\rangle, \langle Y^\bullet, \rho\rangle$ and a commutative square

$$\begin{array}{ccc} X^0 & \xrightarrow{d_X^0} & X^1 \\ {\scriptstyle f^0}\downarrow & \circlearrowright & \downarrow{\scriptstyle f^1} \\ Y^0 & \xrightarrow[d_Y^0]{} & Y^1 \end{array} \tag{4.10}$$

in $\mathscr{C}$. Let us construct a morphism $f^\bullet = (f^0, f^1, f^2, \ldots, f^{n+1})\colon \langle X^\bullet, \delta\rangle \to \langle Y^\bullet, \rho\rangle$ to fulfill the requirement of (F4). Remark that

$$Y^0 \xrightarrow{1_{Y^0}} Y^0 \to 0 \to \cdots \to 0 \overset{0}{\dashrightarrow},$$
$$X^0 \to 0 \to \cdots \to 0 \to \Sigma X^0 \overset{\iota}{\dashrightarrow}$$

are distinguished n-exangles by (R2) and Lemma 4.7. Taking coproducts with $\langle X^\bullet, \delta\rangle$ and $\langle Y^\bullet, \rho\rangle$, we obtain distinguished n-exangles

$$X^0 \oplus Y^0 \xrightarrow{d_X^0 \oplus 1} X^1 \oplus Y^0 \xrightarrow{[d_X^1\ 0]} X^2 \xrightarrow{d_X^2} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \overset{\mu}{\dashrightarrow}, \tag{4.11}$$

$$X^0 \oplus Y^0 \xrightarrow{[0\ d_Y^0]} Y^1 \xrightarrow{d_Y^1} Y^2 \xrightarrow{d_Y^2} \cdots \xrightarrow{d_Y^{n-1}} Y^n \xrightarrow{\left[\begin{smallmatrix} 0 \\ d_Y^n \end{smallmatrix}\right]} \Sigma X^0 \oplus Y^{n+1} \overset{\nu}{\dashrightarrow} \tag{4.12}$$

by Proposition 3.3, where $\mu \in \mathbb{E}_\Sigma(X^{n+1}, X^0 \oplus Y^0)$ and $\nu \in \mathbb{E}_\Sigma(\Sigma X^0 \oplus Y^{n+1}, X^0 \oplus Y^0)$ correspond to

$$\begin{bmatrix} \delta \\ 0 \end{bmatrix} \in \mathscr{C}(X^{n+1}, \Sigma X^0 \oplus \Sigma Y^0),$$
$$\begin{bmatrix} 1_{\Sigma X^0} & 0 \\ 0 & \rho \end{bmatrix} \in \mathscr{C}(\Sigma X^0 \oplus Y^{n+1}, \Sigma X^0 \oplus \Sigma Y^0)$$

respectively, through the natural isomorphism

$$\Sigma(X^0 \oplus Y^0) \cong \Sigma X^0 \oplus \Sigma Y^0. \tag{4.13}$$

Using the automorphism

$$a = \begin{bmatrix} -1 & 0 \\ f^0 & 1 \end{bmatrix} : X^0 \oplus Y^0 \xrightarrow{\cong} X^0 \oplus Y^0,$$

we can modify (4.12) to obtain a distinguished n-exangle

$$X^0 \oplus Y^0 \xrightarrow{[d_Y^0 \circ f^0 \ d_Y^0]} Y^1 \xrightarrow{d_Y^1} Y^2 \xrightarrow{d_Y^2} \cdots \xrightarrow{d_Y^{n-1}} Y^n \xrightarrow{\begin{bmatrix} 0 \\ d_Y^n \end{bmatrix}} \Sigma X^0 \oplus Y^{n+1} \overset{a_*\nu}{\dashrightarrow} \tag{4.14}$$

by Corollary 2.26 (1), where $a_*\nu$ corresponds to

$$\begin{bmatrix} -1 & 0 \\ \Sigma f^0 & \rho \end{bmatrix} \in \mathscr{C}(\Sigma X^0 \oplus Y^{n+1}, \Sigma X^0 \oplus \Sigma Y^0)$$

through (4.13). By Proposition 3.6 (2) applied to the following commutative square,

$$\begin{array}{ccc} X^0 \oplus Y^0 & \xrightarrow{d_X^0 \oplus 1} & X^1 \oplus Y^0 \\ \| & \circlearrowright & \downarrow{\scriptstyle [f^1 \ d_Y^0]} \\ X^0 \oplus Y^0 & \xrightarrow[{[d_Y^0 \circ f^0 \ d_Y^0]}]{} & Y^1 \end{array}$$

we obtain a morphism

$$g^\bullet = \left(1, [f^1 \ d_Y^0], f^2, \ldots, f^n, \begin{bmatrix} x \\ y \end{bmatrix}\right)$$

from (4.11) to (4.14), which makes $\langle M_g^\bullet, (d_X^0 \oplus 1)_* a_*\nu \rangle$ a distinguished n-exangle. If we put $f^{n+1} = y$, then the equalities

$$\begin{bmatrix} x \\ y \end{bmatrix} \circ d_X^n = \begin{bmatrix} 0 \\ d_Y^n \end{bmatrix} \circ f^n \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix}^* a_*\nu = \mu$$

imply $x = -\delta$ and $f^{n+1} \circ d_X^n = d_Y^n \circ f^n$, and

$$(\Sigma f^0) \circ \delta = \rho \circ f^{n+1},$$

which means $(f^0)_*\delta = (f^{n+1})^*\rho$. Thus $f^\bullet = (f^0, f^1, f^2, \ldots, f^{n+1}) \colon \langle X^\bullet, \delta \rangle \to \langle Y^\bullet, \rho \rangle$ is a morphism. Moreover, the obtained distinguished n-exangle $\langle M_g^\bullet, (d_X^0 \oplus 1)_* a_*\nu \rangle$ is of the form

$$X^1 \oplus Y^0 \xrightarrow{d_{M_g}^0} X^2 \oplus Y^1 \xrightarrow{d_{M_g}^1} \cdots \xrightarrow{d_{M_g}^{n-1}} X^{n+1} \oplus Y^n \xrightarrow{d_{M_g}^n} \Sigma X^0 \oplus Y^{n+1} \overset{\tau}{\dashrightarrow}$$

where

$$d_{M_g}^i = \begin{bmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{bmatrix} \ (0 \le i \le n-1), \quad d_{M_g}^n = \begin{bmatrix} -\delta & 0 \\ f^{n+1} & d_Y^n \end{bmatrix},$$

and $\tau = (d_X^0 \oplus 1)_* a_*\nu$ corresponds to

$$\begin{bmatrix} -\Sigma d_X^0 & 0 \\ \Sigma f^0 & \rho \end{bmatrix} \in \mathscr{C}(\Sigma X^0 \oplus Y^{n+1}, \Sigma X^1 \oplus \Sigma Y^0)$$

through the isomorphism $\Sigma(X^1 \oplus Y^0) \cong \Sigma X^1 \oplus \Sigma Y^0$. This yields the desired $(n+2)$-angle. □

4.3. **n-exact categories.** In this subsection, we will see the relation with the notion of an *n-exact category* introduced in [J]. We briefly recall its definition and related notions from [J].

Definition 4.9. (cf. [J, Definitions 2.2 and 2.4]) Let $\mathscr{C}$ be an additive category, and let $X^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ be any object.

(1) $X^\bullet$ is called an *n-kernel sequence* if the following sequence of functors $\mathscr{C}^{\mathrm{op}} \to Ab$ is exact.

$$0 \Rightarrow \mathscr{C}(-,X^0) \overset{\mathscr{C}(-,d_X^0)}{\Longrightarrow} \mathscr{C}(-,X^1) \overset{\mathscr{C}(-,d_X^1)}{\Longrightarrow} \cdots \overset{\mathscr{C}(-,d_X^n)}{\Longrightarrow} \mathscr{C}(-,X^{n+1})$$

In particular d_X^0 is a monomorphism in $\mathscr{C}$.

(2) $X^\bullet$ is called an *n-cokernel sequence* if the following sequence of functors $\mathscr{C} \to Ab$ is exact.

$$0 \Rightarrow \mathscr{C}(X^{n+1},-) \overset{\mathscr{C}(d_X^n,-)}{\Longrightarrow} \mathscr{C}(X^n,-) \overset{\mathscr{C}(d_X^{n-1},-)}{\Longrightarrow} \cdots \overset{\mathscr{C}(d_X^0,-)}{\Longrightarrow} \mathscr{C}(X^0,-)$$

In particular d_X^n is an epimorphism in $\mathscr{C}$.

(3) $X^\bullet$ is called an *n-exact sequence* if it is both n-kernel and n-cokernel sequence.

Remark that n-kernel (respectively, n-cokernel, or n-exact) sequences are closed by homotopy equivalences in $\mathbf{C}^{n+2}_{\mathscr{C}}$.

The following can be shown easily.

Proposition 4.10. *Let $\mathscr{C}$ be an additive category, and let $X^\bullet, Y^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ be any pair of n-exact sequences.*

(1) *Let $k \in \{0,\ldots,n\}$ be any integer. For any commutative square*

$$\begin{array}{ccc} X^k & \xrightarrow{d_X^k} & X^{k+1} \\ {\scriptstyle a}\downarrow & \circlearrowright & \downarrow{\scriptstyle b} \\ Y^k & \xrightarrow[d_Y^k]{} & Y^{k+1} \end{array}$$

in $\mathscr{C}$, there exists $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)$ satisfying $f^k = a$ and $f^{k+1} = b$. Moreover, such $f^\bullet$ is unique up to homotopy. Especially, if both a,b are isomorphisms, then $f^\bullet$ becomes a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$.

(2) *Let $a \in \mathscr{C}(X^0,Y^0), c \in \mathscr{C}(X^{n+1},Y^{n+1})$ be any pair of morphisms. If there exists $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)$ satisfying $f^0 = a$ and $f^{n+1} = c$, then such $f^\bullet$ is unique up to homotopy in $\mathbf{C}^{n+2}_{\mathscr{C}}$.*

Proof. This is straightforward. (See [J, Proposition 2.7] for (1), and [J, Comparison Lemma 2.1] for (2).) □

In particular, the following holds in $\mathbf{C}^{n+2}_{(A,C)}$.

Corollary 4.11. *Let $\mathscr{C}$ be an additive category, let $A, C \in \mathscr{C}$ be any pair of objects. For any pair of n-exact sequences $X^\bullet, Y^\bullet \in \mathbf{C}^{n+2}_{(A,C)}$, we have*

$$|\mathbf{K}^{n+2}_{(A,C)}(X^\bullet,Y^\bullet)| \le 1.$$

Thus if $\mathbf{C}^{n+2}_{(A,C)}(X^\bullet,Y^\bullet) \neq \emptyset$ and $\mathbf{C}^{n+2}_{(A,C)}(Y^\bullet,X^\bullet) \neq \emptyset$, then $X^\bullet$ and $Y^\bullet$ are homotopically equivalent in $\mathbf{C}^{n+2}_{(A,C)}$.

Proof. This is an immediate consequence of Proposition 4.10 (2). $\square$

Definition 4.12. Let $\mathscr{C}$ be an additive category, and let $A, C \in \mathscr{C}$ be any pair of objects. Denote the class of all homotopy equivalence classes of n-exact sequences in $\mathbf{C}^{n+2}_{(A,C)}$ by $\Lambda^{n+2}_{(A,C)}$. This is a subclass of $\mathrm{Ob}(\mathbf{K}^{n+2}_{(A,C)})/\cong$.

For $[X^\bullet], [Y^\bullet] \in \Lambda^{n+2}_{(A,C)}$, we write $[X^\bullet] \le [Y^\bullet]$ if $\mathbf{C}^{n+2}_{(A,C)}(X^\bullet, Y^\bullet) \neq \emptyset$. By Corollary 4.11, this relation makes $\Lambda^{n+2}_{(A,C)}$ a poset (provided it forms a set).

Corollary 4.13. *For any n-exact sequence $X^\bullet \in \mathbf{C}^{n+2}_{(A,C)}$, the following are equivalent.*

(1) $[X^\bullet]$ *is isolated, in the sense that*

$$[X^\bullet] \le [X'^\bullet] \ \ \textit{or} \ \ [X'^\bullet] \le [X^\bullet] \quad \Rightarrow \quad [X^\bullet] = [X'^\bullet].$$

holds in $\Lambda^{n+2}_{(A,C)}$.

(2) $X^\bullet$ *satisfies the following* (I1) *and* (I2) *for any n-exact sequence $Y^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$.*
  - (I1) *If there is $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ in which $f^0 = a$ and $f^{n+1} = c$ are isomorphisms, then $f^\bullet$ is a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$.*
  - (I2) *Dually, if there is $g^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(Y^\bullet, X^\bullet)$ in which g^0 and g^{n+1} are isomorphisms, then $g^\bullet$ is a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$.*

Proof. Assume that $[X^\bullet]$ is isolated in $\Lambda^{n+2}_{(A,C)}$, and let us show (I1). Suppose that $Y^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ is an n-exact sequence, and let $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ be a morphism in which $f^0 = a, f^{n+1} = c$ are isomorphisms. Then

$$A \xrightarrow{d_Y^0 \circ a} Y^1 \xrightarrow{d_Y^1} Y^2 \to \cdots \xrightarrow{d_Y^{n-1}} Y^n \xrightarrow{c^{-1} \circ d_Y^n} C \tag{4.15}$$

is an n-exact sequence in $\mathbf{C}^{n+2}_{(A,C)}$, with an isomorphism $(a, 1, \ldots, 1, c)$ to $Y^\bullet$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$. Since $(1, f^1, \ldots, f^n, 1)$ gives a morphism from $X^\bullet$ to (4.15) in $\mathbf{C}^{n+2}_{(A,C)}$, it becomes a homotopy equivalence by (1). As their composition, $f^\bullet$ gives a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$. Similarly for (I2).

Conversely, assume that $X^\bullet$ satisfies (I1), and suppose $[X^\bullet] \le [X'^\bullet]$ holds for some n-exact sequence $X'^\bullet \in \mathbf{C}^{n+2}_{(A,C)}$. Then there is a morphism $f^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(X^\bullet, X'^\bullet)$, which becomes a homotopy equivalence in $\mathbf{C}^{n+2}_{(A,C)}$ by (I1) and Remark 2.18. This means $[X^\bullet] = [X'^\bullet]$. Similarly, (I2) shows $[X'^\bullet] \le [X^\bullet] \Rightarrow [X^\bullet] = [X'^\bullet]$. $\square$

Definition 4.14. ([J, (dual of) Definition 2.11]) Let $Y^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ be any object. A commutative diagram in $\mathscr{C}$

$$\begin{array}{ccccccccc} X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\ {\scriptstyle f^1}\downarrow & \circlearrowright & {\scriptstyle f^2}\downarrow & \circlearrowright & & & {\scriptstyle f^n}\downarrow & \circlearrowright & \downarrow{\scriptstyle f^{n+1}} \\ Y^1 & \xrightarrow[d_Y^1]{} & Y^2 & \xrightarrow[d_Y^2]{} & \cdots & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & Y^{n+1} \end{array} \tag{4.16}$$

is called an *n-pullback diagram* if

$$X^1 \xrightarrow{d^0} X^2 \oplus Y^1 \xrightarrow{d^1} X^3 \oplus Y^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^{n+1} \oplus Y^n \xrightarrow{d^n} Y^{n+1} \tag{4.17}$$

is an n-kernel sequence, where d^i are defined by

$$\begin{aligned} d^0 &= \begin{bmatrix} -d_X^1 \\ f^1 \end{bmatrix}, \\ d^i &= \begin{bmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{bmatrix} \quad (1 \le i \le n-1), \\ d^n &= \begin{bmatrix} f^{n+1} & d_Y^n \end{bmatrix}. \end{aligned}$$

An *n-pushout diagram* is defined dually.

Remark 4.15. Let (4.16) be an n-pullback diagram. If we put $X^0 = Y^0$, then the exactness of

$$0 \to \mathscr{C}(X^0, X^1) \xrightarrow{\mathscr{C}(X^0, d^0)} \mathscr{C}(X^0, X^2 \oplus Y^1) \xrightarrow{\mathscr{C}(X^0, d^1)} \mathscr{C}(X^0, X^3 \oplus Y^2)$$

gives a unique morphism $d_X^0 \in \mathscr{C}(X^0, X^1)$ satisfying $f^1 \circ d_X^0 = d_Y^0$ and $d_X^1 \circ d_X^0 = 0$. Then the sequence

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1}$$

gives an object $X^\bullet \in \mathbf{C}_{\mathscr{C}}^{n+2}$, and $f^\bullet = (1, f^1, \ldots, f^{n+1}) \in \mathbf{C}_{\mathscr{C}}^{n+2}(X^\bullet, Y^\bullet)$ becomes a morphism. Sequence (4.17) is nothing but the mapping cone $M_f^\bullet$ (in Definition 2.27) of this morphism $f^\bullet$.

Definition 4.16. Let $Y^\bullet \in \mathbf{C}_{(A,C)}^{n+2}$ be any n-exact sequence, and denote its homotopy equivalence class in $\mathbf{C}_{(A,C)}^{n+2}$ by $[Y^\bullet]$, as before. Let $c \in \mathscr{C}(C', C)$ be any morphism.

If there exists an n-exact sequence $X^\bullet \in \mathbf{C}_{(A,C')}^{n+2}$ equipped with a morphism

$$f^\bullet = (1_A, f^1, \ldots, f^n, f^{n+1} = c) \in \mathbf{C}_{\mathscr{C}}^{n+2}(X^\bullet, Y^\bullet)$$

which makes (4.16) an n-pullback diagram, then we define $c^*[Y^\bullet]$ to be

$$c^*[Y^\bullet] = [X^\bullet]. \tag{4.18}$$

Dually, for a morphism $a \in \mathscr{C}(A, A')$, the class $a_*[Y^\bullet]$ is defined by using an n-pushout diagram when it exists. Well-definedness of this definition will be shown in Proposition 4.18.

Lemma 4.17. Let $f^\bullet = (1_A, f^1, \ldots, f^n, f^{n+1} = c) \in \mathbf{C}_{\mathscr{C}}^{n+2}({}_AX_{C'}^\bullet, {}_AY_C^\bullet)$ be any morphism, which makes (4.16) an n-pullback diagram. Then for any morphism

$$g^\bullet = (a, g^1, \ldots, g^n, c) \in \mathbf{C}_{\mathscr{C}}^{n+2}({}_{A'}Z_{C'}^\bullet, {}_AY_C^\bullet),$$

there exists a morphism $h^\bullet = (a, h^1, \ldots, h^n, 1) \in \mathbf{C}_{\mathscr{C}}^{n+2}(Z^\bullet, X^\bullet)$ and a homotopy $\varphi^\bullet = (0, \varphi^2, \varphi^3, \ldots, \varphi^n, 0)$ which gives $g^\bullet \underset{\varphi^\bullet}{\sim} f^\bullet \circ h^\bullet$. Moreover, such $h^\bullet$ is unique up to homotopy.

Proof. This is shown in a straightforward way, only using the fact that $M_f^\bullet$ is an n-kernel sequence (cf. dual of [J, Proposition 2.13] and Remark 4.15). □

Proposition 4.18. *For any n-exact sequence ${}_AY_C^\bullet \in \mathbf{C}_{(A,C)}^{n+2}$ and any $c \in \mathscr{C}(C', C)$, the class $c^*[Y^\bullet]$ in (4.18) is unique if it exists, only depending on the homotopy equivalence class $[Y^\bullet]$.*

Proof. Suppose that there are $X^{\bullet} \in \mathbf{C}^{n+2}_{(A,C')}$ and $f^{\bullet} \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet}, Y^{\bullet})$ with the required properties in Definition 4.16 to give $c^*[Y^{\bullet}] = [X^{\bullet}]$. Lemma 4.17 shows that $[X^{\bullet}]$ is unique for $Y^{\bullet}$. Let us show that it only depends on the class $[Y^{\bullet}]$. Assume that $Y^{\bullet}$ and $Y'^{\bullet}$ are homotopically equivalent in $\mathbf{C}^{n+2}_{(A,C)}$. Then $Y'^{\bullet}$ is also an n-exact sequence. Take a homotopy equivalence $y^{\bullet} \in \mathbf{C}^{n+2}_{(A,C)}(Y^{\bullet}, Y'^{\bullet})$ in $\mathbf{C}^{n+2}_{(A,C)}$, and put $f'^{\bullet} = y^{\bullet} \circ f^{\bullet}$. Proposition 2.28 applied to

$$\begin{array}{ccc} X^{\bullet} & \xrightarrow{f^{\bullet}} & Y^{\bullet} \\ \| & \circlearrowright & \downarrow{\scriptstyle y^{\bullet}} \\ X^{\bullet} & \xrightarrow[f'^{\bullet}]{} & Y'^{\bullet} \end{array}$$

gives a homotopy equivalence between $M^{\bullet}_f$ and $M^{\bullet}_{f'}$. In particular $M^{\bullet}_{f'}$ also becomes an n-kernel sequence. Thus $X^{\bullet} \in \mathbf{C}^{n+2}_{(A,C')}$ and $f'^{\bullet} \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet}, Y'^{\bullet})$ satisfy the required properties to give $c^*[Y'^{\bullet}] = [X^{\bullet}]$. This shows $c^*[Y'^{\bullet}] = [X^{\bullet}] = c^*[Y^{\bullet}]$. □

The following is the definition of an n-exact category in [J]. Later we will rephrase it in Definition 4.21 (see Proposition 4.23).

Definition 4.19. ([J, Definition 4.2]) Let $\mathscr{C}$ be an additive category, and let $\mathcal{X}$ be a class of n-exact sequences in $\mathscr{C}$. The pair $(\mathscr{C}, \mathcal{X})$ is called an *n-exact category* if it satisfies the following closedness (EC) and conditions (E0),(E1),…,(E2$^{\mathrm{op}}$).

In the following, a morphism $a \in \mathscr{C}(A,B)$ is called an *$\mathcal{X}$-admissible monomorphism* (respectively, an *$\mathcal{X}$-admissible epimorphism*) if there is some $X^{\bullet} \in \mathcal{X}$ of the form $A \xrightarrow{a} B \to X^2 \to \cdots \to X^{n+1}$ (resp. $X^0 \to \cdots \to X^{n-1} \to A \xrightarrow{a} B$).

- (EC) The following holds for any morphism $f^{\bullet} \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^{\bullet}, Y^{\bullet})$ between n-exact sequences $X^{\bullet}, Y^{\bullet}$.
  - (i) If f^k and f^{k+1} are isomorphisms for some $k \in \{0, \dots, n\}$, then $X^{\bullet} \in \mathcal{X}$ holds if and only if $Y^{\bullet} \in \mathcal{X}$.
  - (ii) If f^0 and f^{n+1} are isomorphisms, then $X^{\bullet} \in \mathcal{X}$ holds if and only if $Y^{\bullet} \in \mathcal{X}$.
- (E0) The sequence ${}_0\varnothing^{\bullet}_0 \in \mathbf{C}^{n+2}_{\mathscr{C}}$ (see Proposition 3.1)

$$0 \to 0 \to \cdots \to 0$$

  belongs to $\mathcal{X}$.
- (E1) $\mathcal{X}$-admissible monomorphisms are closed by composition.
- (E1$^{\mathrm{op}}$) Dually, $\mathcal{X}$-admissible epimorphisms are closed by composition.
- (E2) For any $X^{\bullet} \in \mathcal{X}$, any $Y^0 \in \mathscr{C}$ and any $f^0 \in \mathscr{C}(X^0, Y^0)$, there is an n-pushout diagram in $\mathscr{C}$ as follows, such that d^0_Y is an $\mathcal{X}$-admissible monomorphism.

$$\begin{array}{ccccccccc} X^0 & \xrightarrow{d^0_X} & X^1 & \xrightarrow{d^1_X} & X^2 & \xrightarrow{d^2_X} & \cdots & \xrightarrow{d^{n-1}_X} & X^n \\ {\scriptstyle f^0}\downarrow & \circlearrowright & \downarrow{\scriptstyle f^1} & \circlearrowright & \downarrow{\scriptstyle f^2} & & \circlearrowright & & \downarrow{\scriptstyle f^n} \\ Y^0 & \xrightarrow[d^0_Y]{} & Y^1 & \xrightarrow[d^1_Y]{} & Y^2 & \xrightarrow[d^2_Y]{} & \cdots & \xrightarrow[d^{n-1}_Y]{} & Y^n \end{array}$$

(E2$^{\mathrm{op}}$) Dually, for any $Y^\bullet \in \mathcal{X}$, any $X^{n+1} \in \mathscr{C}$ and any $f^{n+1} \in \mathscr{C}(X^{n+1}, Y^{n+1})$, there is an n-pullback diagram in $\mathscr{C}$ as in (4.16), such that d_X^n is an $\mathcal{X}$-admissible epimorphism.

The following is a consequence of being an n-exact category, shown in [J].

Fact 4.20. ([J, dual of Proposition 4.8 (iv) ⇒ (ii)].) Let $(\mathscr{C}, \mathcal{X})$ be an n-exact category. For any ${}_AX^\bullet, {}_AY^\bullet \in \mathcal{X}$, if $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ satisfies $f^0 = 1_A$, then $M_f^\bullet \in \mathcal{X}$ holds. Dually for $f^\bullet$ satisfying $f^{n+1} = 1$.

In order to rephrase the definition of an n-exact category, let us consider the following conditions.

Definition 4.21. Let $\mathscr{C}$ be an additive category, and let $\mathcal{X}$ be a class of n-exact sequences in $\mathscr{C}$. Define conditions (EC′),(E2′),(E2′$^{\mathrm{op}}$) and (EI) as follows.

(EC′) For any $A, C \in \mathscr{C}$,

$$\{X \in \mathcal{X} \mid X^0 = A, X^{n+1} = C\} \subseteq \mathrm{Ob}(\mathbf{C}^{n+2}_{(A,C)})$$

is closed by homotopy equivalences in $\mathbf{C}^{n+2}_{(A,C)}$.

(E2′) The dual of the following (E2′$^{\mathrm{op}}$).

(E2′$^{\mathrm{op}}$)
- (i) For any $c \in \mathscr{C}(C', C)$ and any ${}_AY_C^\bullet \in \mathcal{X}$, there exists ${}_AX_{C'}^\bullet \in \mathcal{X}$ equipped with a morphism $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ satisfying $f^0 = 1_A$ and $f^{n+1} = c$.
- (ii) For any ${}_AX_{C'}^\bullet, {}_AY_C^\bullet \in \mathcal{X}$ and any $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ satisfying $f^0 = 1_A$, we have $M_f^\bullet \in \mathcal{X}$.

(EI) $[X^\bullet] \in \Lambda^{n+2}_{(A,C)}$ is isolated, for any ${}_AX_C^\bullet \in \mathcal{X}$.

In Proposition 4.23, we will see that conditions (EC),(E2),(E2$^{\mathrm{op}}$) in Definition 4.19 can be replaced by the above conditions. First let us show the following.

Lemma 4.22. Let $\mathscr{C}$ be an additive category, and let $\mathcal{X}$ be a class of n-exact sequences in $\mathscr{C}$. If $(\mathscr{C}, \mathcal{X})$ satisfies (EC′),(E2′),(E2′$^{\mathrm{op}}$), then the following holds.

(1) Let $A, C \in \mathscr{C}$ be any pair of objects. If ${}_AX_C^\bullet, {}_AY_C^\bullet \in \mathcal{X}$, then any $f^\bullet \in \mathbf{C}^{n+2}_{(A,C)}(X^\bullet, Y^\bullet)$ is a homotopy equivalence in $\mathbf{C}^{n+2}_{(A,C)}$.

(2) For any $c \in \mathscr{C}(C', C)$, the class $\mathcal{X}$ is closed by c^*. Namely, for any ${}_AY_C^\bullet \in \mathcal{X}$, there exists ${}_AX_{C'}^\bullet \in \mathcal{X}$ which gives $c^*[Y^\bullet] = [X^\bullet]$. Dually, $\mathcal{X}$ is closed by a_* for any morphism a in $\mathscr{C}$.

(3) For any ${}_AX_C^\bullet \in \mathcal{X}$, any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, we have $a_*(c^*[X^\bullet]) = c^*(a_*[X^\bullet])$ in $\Lambda^{n+2}_{(A',C')}$.

(4) $\mathcal{X}$ is closed by homotopy equivalences in $\mathbf{C}^{n+2}_{\mathscr{C}}$.

Proof. (1) By (E2′$^{\mathrm{op}}$), we have $M_f^\bullet \in \mathcal{X}$. In particular $M_f^\bullet$ is n-exact. Since $d^n_{M_f} \colon C \oplus Y^n \to C$ has a section $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \colon C \to C \oplus Y^n$, we can construct a homotopy

$$0 \underset{\varphi^\bullet}{\sim} 1_{M_f^\bullet} \colon M_f^\bullet \to M_f^\bullet$$

satisfying $\varphi^{n+1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (cf. dual of [J, Proposition 2.6].) If we write φ^k as

$$\varphi^1 = [p^2 \ q^1] \colon X^2 \oplus Y^1 \to X^1,$$

$$\varphi^k=\begin{bmatrix} p^{k+1} & q^k \\ r^{k+1} & s^k \end{bmatrix}\colon X^{k+1}\oplus Y^k\to X^k\oplus Y^{k-1}\quad (2\le k\le n),$$

then by definition they satisfy

$$\varphi^1\circ d^0_{M_f}=1\ \text{ and }\ d^{k-1}_{M_f}\circ\varphi^k+\varphi^{k+1}\circ d^k_{M_f}=1\quad(1\le k\le n).$$

In particular we have

$$d^0_{M_f}\circ q^1+\begin{bmatrix} q^2\\ s^2\end{bmatrix}\circ d^1_Y=\begin{bmatrix}0\\1\end{bmatrix},$$
$$q^{k+1}\circ d^k_Y=d^k_X\circ q^k\quad(1\le k\le n-1),$$
$$d^n_Y=d^n_X\circ q^n.$$

Then the monomorphicity of $d^0_{M_f}$ and the equality

$$d^0_{M_f}\circ q^1\circ d^0_Y=\begin{bmatrix}0\\1\end{bmatrix}\circ d^0_Y-\begin{bmatrix} q^2\\ s^2\end{bmatrix}\circ d^1_Y\circ d^0_Y=d^0_{M_f}\circ d^0_X$$

shows $q^1\circ d^0_Y=d^0_X$. Thus $q^\bullet=(1,q^1,\ldots,q^n,1)$ gives a morphism $q^\bullet\in\mathbf{C}^{n+2}_{(A,C)}(Y^\bullet,X^\bullet)$. Corollary 4.11 shows that $f^\bullet$ is a homotopy equivalence in $\mathbf{C}^{n+2}_{(A,C)}$.

(2) This follows immediately from (E2$'^{\mathrm{op}}$) and the definition of $c^*[Y^\bullet]$. Dually for the closedness by a_*.

(3) By (2), there are ${}_{A'}Y^\bullet_{C'},{}_{A'}Z^\bullet_{C'}\in\mathcal{X}$ which give

$$a_*(c^*[X^\bullet])=[Y^\bullet]\quad\text{and}\quad c^*(a_*[X^\bullet])=[Z^\bullet].$$

By Lemma 4.17 and its dual, we find a morphism $f^\bullet\in\mathbf{C}^{n+2}_{(A',C')}(Y^\bullet,Z^\bullet)$. By (1) this becomes a homotopy equivalence in $\mathbf{C}^{n+2}_{(A',C')}$, and thus $[Y^\bullet]=[Z^\bullet]$ holds.

(4) Let $f^\bullet=(a,f^1,\ldots,f^n,c)\in\mathbf{C}^{n+2}_{\mathscr{C}}({}_AX^\bullet_C,{}_BY^\bullet_D)$ be a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$, with a homotopy inverse $g^\bullet=(b,g^1,\ldots,g^n,d)$. Assume $X^\bullet\in\mathcal{X}$, and let us show $Y^\bullet\in\mathcal{X}$. Existence of a homotopy equivalence implies that $Y^\bullet$ is also an n-exact sequence. By (2), there are ${}_BU^\bullet_C,{}_AV^\bullet_D\in\mathcal{X}$ which give

$$a_*[X^\bullet]=[U^\bullet]\quad\text{and}\quad d^*[X^\bullet]=[V^\bullet].$$

By (2) and (3), there is ${}_BZ^\bullet_D\in\mathcal{X}$ which gives $[Z^\bullet]=d^*[U^\bullet]=a_*[V^\bullet]$. Remark that there are morphisms $u^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,U^\bullet)$ and $v^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(V^\bullet,X^\bullet)$ satisfying $u^0=a,u^{n+1}=1_C$ and $v^0=1_A,v^{n+1}=d$. Applying Lemma 4.17 to $u^\bullet\circ g^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(Y^\bullet,U^\bullet)$, we obtain some $y^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(Y^\bullet,Z^\bullet)$ satisfying $y^0=a\circ b$ and $y^{n+1}=1_D$. Since $f^\bullet\circ g^\bullet\sim 1_{Y^\bullet}$ by assumption, there is $\varphi^1\in\mathscr{C}(Y^1,B)$ which gives $\varphi^1\circ d^0_Y=1_B-a\circ b$. Modifying $y^\bullet$, we obtain a morphism

$$(1_B,y^1+d^0_Z\circ\varphi^1,y^2,\ldots,y^n,1_D)\in\mathbf{C}^{n+2}_{(B,D)}(Y^\bullet,Z^\bullet).$$

Similarly, the dual of Lemma 4.17 applied to $f^\bullet\circ v^\bullet$ gives $z^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(Z^\bullet,Y^\bullet)$ satisfying $z^0=1_B$ and $z^{n+1}=c\circ d$, and thus we obtain

$$(1_B,z^1,\ldots,z^{n-1},z^n+\varphi^{n+1}\circ d^n_Z,1_D)\in\mathbf{C}^{n+2}_{(B,D)}(Z^\bullet,Y^\bullet).$$

By Corollary 4.11, it follows $[Y^\bullet]=[Z^\bullet]$. Thus (EC$'$) shows $Y^\bullet\in\mathcal{X}$. □

Proposition 4.23. *Let $\mathscr{C}$ be an additive category, and let $\mathcal{X}$ be a class of n-exact sequences in $\mathscr{C}$. Assume that $(\mathscr{C},\mathcal{X})$ satisfies* (E0),(E1),(E1$^{\mathrm{op}}$). *Then the following are equivalent.*

(1) *$(\mathscr{C},\mathcal{X})$ is an n-exact category.*

(2) $(\mathscr{C},\mathcal{X})$ *satisfies conditions* (EC′),(E2′),(E2′$^{\rm op}$) *and* (EI) *in Definition 4.21.*

Proof. (1) ⇒ (2). (EC′) is a particular case of (EC)(ii). (E2′$^{\rm op}$)(i) follows from (E2$^{\rm op}$) and Remark 4.15. (E2′$^{\rm op}$)(ii) is given in Fact 4.20. Dually for (E2′). Thus we can apply Lemma 4.22 to $(\mathscr{C},\mathcal{X})$. Then (EI) follows from (EC)(ii).

(2) ⇒ (1). Let $(\mathscr{C},\mathcal{X})$ be as in (2). Let us confirm conditions in Definition 4.19. (EC)(i) follows from Proposition 4.10 (1) and Lemma 4.22 (4). To show (EC)(ii), let $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)$ be a morphism between n-exact sequences $X^\bullet,Y^\bullet$ in which f^0 and f^{n+1} are isomorphisms. If one of $X^\bullet,Y^\bullet$ belongs to $\mathcal{X}$, then (EI) implies that $f^\bullet$ is a homotopy equivalence in $\mathbf{C}^{n+2}_{\mathscr{C}}$, by Corollary 4.13. Thus the other also belongs to $\mathcal{X}$ by Lemma 4.22 (4). (E2$^{\rm op}$) follows from (E2′$^{\rm op}$). Dually for (E2). □

We proceed to show that each n-exact category is n-exangulated. By the equivalence shown in Proposition 4.23, we may use conditions (EC′),(E2′),(E2′$^{\rm op}$) and consequently Lemma 4.22. Indeed, we can avoid using (EI) (see Remark 4.35). We begin by defining the bifunctor $\mathbb{E}$ similarly to the usual Yoneda extension functor. The procedure follows very closely the classical case of exact categories (see e.g. [FS]). This is also shown in [L, Section 5] in the case of n-abelian categories[1].

Definition 4.24. Let $(\mathscr{C},\mathcal{X})$ be an n-exact category. For $A,C\in\mathscr{C}$, let $\mathbb{E}(C,A)$ be the subclass of $\Lambda^{n+2}_{(A,C)}$ consisting of all $[X^\bullet]$ such that $X^\bullet\in\mathcal{X}$. This is well-defined by (EC′). From now on we assume that $\mathbb{E}(C,A)$ is a set for all $A,C\in\mathscr{C}$. We consider the assignment $(C,A)\mapsto\mathbb{E}(C,A)$ as a functor

$$\mathbb{E}\colon \mathscr{C}^{\rm op}\times\mathscr{C}\to \mathit{Set}$$

by defining $\mathbb{E}(c,a)[X^\bullet]=a_*(c^*[X^\bullet])$ for all $(c,a)\in\mathscr{C}(C',C)\times\mathscr{C}(A,A')$ and ${}_AX^\bullet_C\in\mathcal{X}$. That $\mathbb{E}$ is well-defined is shown in Lemma 4.26.

Remark 4.25. To compute the functor $\mathbb{E}$, it is useful to note that

$$c^*[X^\bullet]=[Y^\bullet]$$

holds for ${}_AX^\bullet,{}_AY^\bullet\in\mathcal{X}$ if and only if there is $f^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(Y^\bullet,X^\bullet)$ such that $f^0=1_A$ and $f^{n+1}=c$. This follows from (E2′$^{\rm op}$) (or alternatively from [J, Proposition 4.8]). Dually,

$$a_*[X^\bullet]=[Y^\bullet]$$

holds for $X^\bullet_C,Y^\bullet_C\in\mathcal{X}$ if and only if there is $f^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,Y^\bullet)$ such that $f^0=a$ and $f^{n+1}=1_C$.

Lemma 4.26. The functor

$$\mathbb{E}\colon \mathscr{C}^{\rm op}\times\mathscr{C}\to \mathit{Set}$$

in Definition 4.24 is well-defined.

Proof. First note that $\mathbb{E}(c,a)\colon\mathbb{E}(C,A)\to\mathbb{E}(C',A')$ is a well-defined map by Proposition 4.18 and Lemma 4.22 (2).

Considering the identity morphism on ${}_AX^\bullet_C\in\mathcal{X}$ and Remark 4.25 we find $(1_C)^*[X^\bullet]=[X^\bullet]$ and $(1_A)_*[X^\bullet]=[X^\bullet]$ and so $\mathbb{E}(1_C,1_A)=1_{\mathbb{E}(C,A)}$.

Now let ${}_AX^\bullet,{}_AY^\bullet,{}_AZ^\bullet\in\mathcal{X}$ and suppose that $c^*[X^\bullet]=[Y^\bullet]$ and $d^*[Y^\bullet]=[Z^\bullet]$. By Remark 4.25 there are $f^\bullet\colon Y^\bullet\to X^\bullet$ and $g^\bullet\colon Z^\bullet\to Y^\bullet$, with $f^0=1_A$, $f^{n+1}=c$,

[1]The authors wishes to thank the referee for introducing them [L].

$g^0 = 1_A$, $g^{n+1} = d$. By considering $f^\bullet \circ g^\bullet \colon Z^\bullet \to X^\bullet$ we find that $(cd)^* = d^* c^*$. Similarly $(ab)_* = a_* b_*$. By Lemma 4.22 (3) it follows that

$$\mathbb{E}((d,b) \circ (c,a)) = \mathbb{E}(cd, ba) = (cd)^*(ba)_* = d^* c^* b_* a_* = d^* b_* c^* a_* = \mathbb{E}(d,b) \circ \mathbb{E}(c,a).$$

□

Next we want to endow $\mathbb{E}(C,A)$ with the structure of an abelian group. As for exact categories this is done using the Baer sum.

Remark 4.27. In an n-exact category $(\mathscr{C}, \mathcal{X})$,

$$X^\bullet, Y^\bullet \in \mathcal{X} \ \Rightarrow \ X^\bullet \oplus Y^\bullet \in \mathcal{X}$$

holds. This has been shown in [J, Proposition 4.6]. We also remark that if $(\mathscr{C}, \mathcal{X})$ satisfies (EC′),(E0),(E1),(E1$^{\mathrm{op}}$),(E2′),(E2′$^{\mathrm{op}}$), then the same proof as in [J, Lemma 4.5, Proposition 4.6] works, because of Lemma 4.22.

Definition 4.28. Let ${}_AX^\bullet_C, {}_AY^\bullet_C \in \mathcal{X}$. As in Remark 4.27, the direct sum $X^\bullet \oplus Y^\bullet \in \mathcal{X}$. Moreover, $[X^\bullet \oplus Y^\bullet]$ only depends on $[X^\bullet]$ and $[Y^\bullet]$ so we may define

$$[X^\bullet] \oplus [Y^\bullet] = [X^\bullet \oplus Y^\bullet].$$

Finally define the Baer sum of $[X^\bullet]$ and $[Y^\bullet]$ to be

$$[X^\bullet] + [Y^\bullet] = (\Delta_C)^*(\nabla_A)_*([X^\bullet] \oplus [Y^\bullet]) \in \mathbb{E}(C,A).$$

Remark 4.29. By Lemma 4.22 (3), we also have

$$[X^\bullet] + [Y^\bullet] = (\nabla_A)_*(\Delta_C)^*([X^\bullet] \oplus [Y^\bullet]).$$

Using this together with $X^\bullet \oplus (Y^\bullet \oplus Z^\bullet) = (X^\bullet \oplus Y^\bullet) \oplus Z^\bullet$ one easily checks that

$$([X^\bullet] + [Y^\bullet]) + [Z^\bullet] = [X^\bullet] + ([Y^\bullet] + [Z^\bullet]).$$

To show that $\mathbb{E}(C,A)$ with the Baer sum is an abelian group, we will use the following result.

Lemma 4.30. Let ${}_AX^\bullet_C, {}_{A'}Y^\bullet_{C'} \in \mathcal{X}$ and $f^\bullet \colon X^\bullet \to Y^\bullet$ with $f^0 = a$, $f^{n+1} = c$. Then

$$a_*[X^\bullet] = c^*[Y^\bullet].$$

Proof. By Lemma 4.17 and Remark 4.25, there are ${}_{A'}Z^\bullet_C \in \mathcal{X}$ and morphisms $g^\bullet \colon X^\bullet \to Z^\bullet$, $h^\bullet \colon Z^\bullet \to Y^\bullet$ satisfying $g^0 = a$, $g^{n+1} = 1_C$, $h^0 = 1_{A'}$ and $h^{n+1} = c$ (see also [J, Proposition 4.9]). Hence $a_*[X^\bullet] = [Z^\bullet] = c^*[Y^\bullet]$. □

The following lemma is analogous to [FS, Proposition 6.10] and has a similar proof, which we include for the sake of completeness.

Lemma 4.31. Let ${}_AX^\bullet_C, {}_AY^\bullet_C \in \mathcal{X}$ and $a, b \in \mathscr{C}(A, A')$. Then the following statements hold.

(1) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_* ([X^\bullet] \oplus [Y^\bullet]) = a_*[X^\bullet] \oplus b_*[Y^\bullet]$
(2) $(a+b)_*[X^\bullet] = a_*[X^\bullet] + b_*[X^\bullet]$
(3) $a_*([X^\bullet] + [Y^\bullet]) = a_*[X^\bullet] + a_*[Y^\bullet]$

Proof. (1) Write $a_*[X^\bullet] = [Z^\bullet]$ and $b_*[Y^\bullet] = [W^\bullet]$. Then there are $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Z^\bullet)$ and $g^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(Y^\bullet, W^\bullet)$ such that $f^0 = a$, $f^{n+1} = 1_C$, $g^0 = b$ and $g^{n+1} = 1_C$ by Remark 4.25. Considering $h^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet \oplus Y^\bullet, Z^\bullet \oplus W^\bullet)$ given by

$$h^k = \begin{bmatrix} f^k & 0 \\ 0 & g^k \end{bmatrix}$$

we find that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_* ([X^\bullet] \oplus [Y^\bullet]) = [Z^\bullet] \oplus [W^\bullet] = a_*[X^\bullet] \oplus b_*[Y^\bullet].$$

(2) Consider $\Delta^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, X^\bullet \oplus X^\bullet)$ defined by $\Delta^k = \Delta_{X^k}$. By Lemma 4.30, we get $(\Delta_A)_*[X^\bullet] = (\Delta_C)^*([X^\bullet] \oplus [X^\bullet])$. Now by (1)

$$\begin{aligned}
(a+b)_*[X^\bullet] &= \left(\nabla_{A'} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Delta_A\right)_* [X^\bullet] = (\nabla_{A'})_* \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_* (\Delta_A)_*[X^\bullet] \\
&= (\nabla_{A'})_* \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_* (\Delta_C)^*([X^\bullet] \oplus [X^\bullet]) \\
&= (\nabla_{A'})_* (\Delta_C)^* \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_* ([X^\bullet] \oplus [X^\bullet]) \\
&= (\nabla_{A'})_* (\Delta_C)^* (a_*[X^\bullet] \oplus b_*[X^\bullet]) \\
&= a_*[X^\bullet] + b_*[X^\bullet].
\end{aligned}$$

(3) Using (1) we compute

$$\begin{aligned}
a_*([X^\bullet] + [Y^\bullet]) &= a_*(\nabla_A)_*(\Delta_C)^*([X^\bullet] \oplus [Y^\bullet]) \\
&= (\nabla_{A'})_* \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}_* (\Delta_C)^*([X^\bullet] \oplus [Y^\bullet]) \\
&= (\nabla_{A'})_* (\Delta_C)^* \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}_* ([X^\bullet] \oplus [Y^\bullet]) \\
&= (\nabla_{A'})_* (\Delta_C)^* (a_*[X^\bullet] \oplus a_*[Y^\bullet]) \\
&= a_*[X^\bullet] + a_*[Y^\bullet].
\end{aligned}$$

□

Proposition 4.32. ([L, Section 5] *for the n-abelian case.*) *For all $C, A \in \mathscr{C}$ the Baer sum defines the structure of an abelian group on $\mathbb{E}(C, A)$. This enhances the functor $\mathbb{E}$ in Definition 4.24 to a biadditive functor*

$$\mathbb{E} \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab.$$

Proof. Since this is well-known for $n = 1$, we assume $n \geq 2$. Let ${}_AX^\bullet_C, {}_AY^\bullet_C \in \mathcal{X}$ and consider the canonical isomorphism $t \colon X^\bullet \oplus Y^\bullet \to Y^\bullet \oplus X^\bullet$. By Lemma 4.30 we get

$$(t^0)_*([X^\bullet] \oplus [Y^\bullet]) = (t^{n+1})^*([Y^\bullet] \oplus [X^\bullet])$$

Now

$$\begin{aligned}[X^\bullet]+[Y^\bullet] &= (\Delta_C)^*(\nabla_A)_*([X^\bullet]\oplus[Y^\bullet]) = (\Delta_C)^*(\nabla_A t^0)_*([X^\bullet]\oplus[Y^\bullet])\\ &= (\Delta_C)^*(\nabla_A)_*(t^0)_*([X^\bullet]\oplus[Y^\bullet]) = (\Delta_C)^*(\nabla_A)_*(t^{n+1})^*([Y^\bullet]\oplus[X^\bullet])\\ &= (\Delta_C)^*(t^{n+1})^*(\nabla_A)_*([Y^\bullet]\oplus[X^\bullet])\\ &= (t^{n+1}\Delta_C)^*(\nabla_A)_*([Y^\bullet]\oplus[X^\bullet])\\ &= (\Delta_C)^*(\nabla_A)_*([Y^\bullet]\oplus[X^\bullet]) = [Y^\bullet]+[X^\bullet].\end{aligned}$$

Together with Remark 4.29 this shows that $\mathbb{E}(C,A)$ is an abelian semigroup.

Let ${}_AX^\bullet_C\in\mathcal{X}$ and $N^\bullet$ be the complex

$$A\xrightarrow{1_A}A\to 0\to\cdots\to 0\to C\xrightarrow{1_C}C \tag{4.19}$$

It follows from (E0) and Lemma 4.22 (4) (or alternatively from [J, Remark 4.7]) that $N^\bullet\in\mathcal{X}$. By considering $f^\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet,N^\bullet)$ defined by $f^{n+1}=1_C$, $f^n=d_X^n$ and $f^k=0$ for $k<n$, we find that $0_*[X^\bullet]=[N^\bullet]$. By Lemma 4.31, we have $[N^\bullet]+[X^\bullet]=0_*[X^\bullet]+1_*[X^\bullet]=[X^\bullet]$, and so $[N^\bullet]$ is the neutral element in $\mathbb{E}(C,A)$. Similarly, $(-1)_*[X^\bullet]$ is the inverse of $[X^\bullet]$. Hence $\mathbb{E}(C,A)$ is an abelian group. From Lemma 4.31 and its dual it follows that

$$\mathbb{E}\colon\mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to Ab$$

is well-defined and biadditive. □

Remark 4.33. As in the above proof, the element $0\in\mathbb{E}(C,A)$ is given by the sequence (4.19) if $n\geq 2$. If $n=1$, it is given by $A\xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}A\oplus C\xrightarrow{[0\ 1]}C$.

Proposition 4.34. *Let $(\mathscr{C},\mathcal{X})$ be an n-exact category such that $\mathbb{E}(C,A)$ is a set for all $A,C\in\mathscr{C}$. For all $\delta\in\mathbb{E}(C,A)$, set $\mathfrak{s}(\delta)=[X^\bullet]$, where $\delta=[X^\bullet]$. Then $(\mathscr{C},\mathbb{E},\mathfrak{s})$ is n-exangulated.*

Proof. Similarly as before, we only deal with the case $n\geq 2$. As for the case $n=1$, a similar proof to the one below works, if we take Remark 4.33 into account. The case $n=1$ also follows from Proposition 4.3 and [NP, Example 2.13].

By Proposition 4.32 we know that $\mathbb{E}\colon\mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to Ab$ is a biadditive functor. So it remains to check the conditions (R0),(R1),(R2) and also (EA1),(EA2),(EA2)$^{\mathrm{op}}$.

(R0) Let $(a,c):{}_A\delta_C\to{}_{A'}\rho_{C'}$ be a morphism of extensions where $\delta=[X^\bullet]$ and $\rho=[Y^\bullet]$. Then $a_*[X^\bullet]=c^*[Y^\bullet]=[Z^\bullet]$ for some ${}_{A'}Z^\bullet_C\in\mathcal{X}$ and so there are morphisms $f^\bullet\colon X^\bullet\to Z^\bullet$, $g^\bullet\colon Z^\bullet\to Y^\bullet$ satisfying $f^0=a$, $f^{n+1}=1_C$, $g^0=1_{A'}$ and $g^{n+1}=c$. The composition $g^\bullet\circ f^\bullet\colon X^\bullet\to Y^\bullet$ is a lift of (a,c).

(R1) Let $X^\bullet\in\mathcal{X}$ and $\delta=[X^\bullet]$. We need to check that $\langle X^\bullet,\delta\rangle$ is an n-exangle. Since $X^\bullet$ is n-exact it is enough to check that

$$\mathscr{C}(Y,X^n)\xrightarrow{\mathscr{C}(Y,d_X^n)}\mathscr{C}(Y,X^{n+1})\xrightarrow{\delta_\sharp}\mathbb{E}(Y,X^0)$$

and

$$\mathscr{C}(X^1,Y)\xrightarrow{\mathscr{C}(d_X^0,Y)}\mathscr{C}(X^0,Y)\xrightarrow{\delta^\sharp}\mathbb{E}(X^{n+1},Y)$$

are exact for all $Y\in\mathscr{C}$. We only check the first case as the second is similar.

Let $f\colon Y \to X^{n+1}$. Then $\delta_\sharp(f) = f^*[X^\bullet]$ is zero if and only if there is a commutative diagram of the form

$$\begin{array}{ccccccccccccc}
X^0 & = & X^0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & Y & = & Y \\
\| & \circlearrowright & \downarrow & \circlearrowright & \downarrow & & & & \downarrow & \circlearrowright & \downarrow & \circlearrowright & \downarrow{\scriptstyle f} \\
X^0 & \xrightarrow[d_X^0]{} & X^1 & \xrightarrow[d_X^1]{} & X^2 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \xrightarrow[d_X^{n-1}]{} & X^n & \xrightarrow[d_X^n]{} & X^{n+1}
\end{array}.$$

Evidently, this is equivalent to $f = d_X^n \circ g$ for some $g\colon Y \to X^n$, i.e., f is in the image of $\mathscr{C}(Y, d_X^n)$.

(R2) immediately follows from the description of $0 \in \mathbb{E}(0, A)$ and $0 \in \mathbb{E}(A, 0)$.

(EA1) follows from (E1) and (E1$^{\mathrm{op}}$).

(EA2) Let ${}_AY^\bullet_C \in \mathcal{X}$ and $c \in \mathscr{C}(C', C)$. Let ${}_AX^\bullet_{C'} \in \mathcal{X}$ such that $c^*[Y^\bullet] = [X^\bullet]$. Then there is $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ such that by $f^0 = 1_A$, $f^{n+1} = c$. We claim that this is a good lift of $(1_A, c)$. First of all $M_f^\bullet \in \mathcal{X}$ by (E2$'^{\mathrm{op}}$). Next, existence of the morphism $g^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(Y^\bullet, M_f^\bullet)$ given by $g^0 = d_X^0$, $g^{n+1} = 1_C$ and

$$g^k = \begin{bmatrix} 0 \\ 1_{Y^k} \end{bmatrix}$$

for all other k shows $(d_X^0)_*[Y^\bullet] = [M_f^\bullet]$. The dual case (EA2$^{\mathrm{op}}$) is similar. □

Remark 4.35. Every n-exangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ coming from an n-exact category $(\mathscr{C}, \mathcal{X})$ as in Proposition 4.34 satisfies the condition that all inflations are monomorphisms and all deflations are epimorphisms. In fact, the arguments so far show that if $(\mathscr{C}, \mathcal{X})$ satisfies conditions (EC$'$),(E0),(E1),(E1$^{\mathrm{op}}$),(E2$'$),(E2$'^{\mathrm{op}}$), then it gives an n-exangulated category of this type. Next we will show the converse of this (Proposition 4.37).

Lemma 4.36. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. Assume that any $\mathfrak{s}$-inflation is monomorphic, and any $\mathfrak{s}$-deflation is epimorphic in $\mathscr{C}$. Note that this is equivalent to assuming that any $\mathfrak{s}$-conflation is n-exact. If we denote the class of all $\mathfrak{s}$-conflations by $\mathcal{X}$, then we have the following.

(1) For any n-exangle ${}_A\langle Y^\bullet, \delta\rangle_C$ and any $c \in \mathscr{C}(C', C)$, if we put $\mathfrak{s}(c^*\delta) = [X^\bullet]$, then any lift $f^\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ of $(1_A, c)\colon \delta \to c^*\delta$ satisfies $M_f^\bullet \in \mathcal{X}$. In particular,

$$\begin{array}{ccccccccc}
X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
{\scriptstyle f^1}\downarrow & \circlearrowright & {\scriptstyle f^2}\downarrow & \circlearrowright & & & {\scriptstyle f^n}\downarrow & \circlearrowright & \downarrow{\scriptstyle f^{n+1}} \\
Y^1 & \xrightarrow[d_Y^1]{} & Y^2 & \xrightarrow[d_Y^2]{} & \cdots & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & Y^{n+1}
\end{array}$$

becomes an n-pullback diagram in $\mathscr{C}$.

(2) For any pair of distinguished n-exangles ${}_A\langle X^\bullet, \delta\rangle_C$, ${}_B\langle Y^\bullet, \rho\rangle_D$, we have

$$\mathbf{C}^{n+2}_{\mathscr{C}}(X^\bullet, Y^\bullet) = \text{Æ}(\langle X^\bullet, \delta\rangle, \langle Y^\bullet, \rho\rangle).$$

(3) If $\delta, \delta' \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(\delta) = \mathfrak{s}(\delta')$, then $\delta = \delta'$ holds. Thus for any $A, C \in \mathscr{C}$, the realization $\mathfrak{s}$ gives the following bijective correspondence.

$$\mathbb{E}(C, A) \xrightarrow{\text{bij.}} \frac{\{X^\bullet \in \mathcal{X} \mid X^0 = A, X^{n+1} = C\}}{(\text{homotopy equivalence in } \mathbf{C}^{n+2}_{(A,C)})}.$$

Proof. (1) By (EA2), there is a good lift $g^{\bullet}$ of $(1_A, c)$, which makes $\langle M_g^{\bullet}, (d_X^0)_*\delta\rangle$ a distinguished n-exangle by definition. Since $f^{\bullet} \sim g^{\bullet}$ holds by Proposition 4.10 (2), it follows that $f^{\bullet}$ is also a good lift by Remark 2.33 (1), and thus $M_f^{\bullet} \in \mathcal{X}$.

(2) It suffices to show $\mathbf{C}_{\mathscr{C}}^{n+2}(X^{\bullet}, Y^{\bullet}) \subseteq \text{Æ}(\langle X^{\bullet}, \delta\rangle, \langle Y^{\bullet}, \rho\rangle)$. Let $f^{\bullet} \in \mathbf{C}_{\mathscr{C}}^{n+2}(X^{\bullet}, Y^{\bullet})$ be any morphism. By Proposition 3.6, there is some $g^{\bullet} \in \text{Æ}(\langle X^{\bullet}, \delta\rangle, \langle Y^{\bullet}, \rho\rangle)$ satisfying $g^0 = f^0$ and $g^1 = f^1$. Since $X^{\bullet}$ and $Y^{\bullet}$ are n-exact sequences, Proposition 4.10 (1) shows $f^{\bullet} \sim g^{\bullet}$. Thus Proposition 2.10 shows $f^{\bullet} \in \text{Æ}(\langle X^{\bullet}, \delta\rangle, \langle Y^{\bullet}, \rho\rangle)$.

(3) This immediately follows from (2). Indeed, $1_{X^{\bullet}} \in \mathbf{C}_{\mathscr{C}}^{n+2}(X^{\bullet}, X^{\bullet})$ should give a morphism $1_{X^{\bullet}} \in \text{Æ}(\langle X^{\bullet}, \delta\rangle, \langle X^{\bullet}, \delta'\rangle)$, which in particular satisfies $\delta = \delta'$. □

Proposition 4.37. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category, in which any $\mathfrak{s}$-inflation is monomorphic and any $\mathfrak{s}$-deflation is epimorphic. Let $\mathcal{X}$ be the class of all $\mathfrak{s}$-conflations, as in Lemma 4.36. Then, the following holds.*

1. *The pair $(\mathscr{C}, \mathcal{X})$ satisfies conditions* (EC$'$),(E0),(E1),(E1$^{\mathrm{op}}$),(E2$'$),(E2$'^{\mathrm{op}}$).
2. *Moreover, if $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies the following conditions* (a),(b) *for any pair of morphisms $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$, then $(\mathscr{C}, \mathcal{X})$ also satisfies* (EI)*, and thus becomes an n-exact category in the sense of* [J] *by Proposition 4.23.*
   (a) *If $b \circ a$ is an $\mathfrak{s}$-inflation, then so is a.*
   (b) *If $b \circ a$ is an $\mathfrak{s}$-deflation, then so is b.*

Proof. (1) (EC$'$) is obvious from the definition of $\mathcal{X}$. (E0) follows from (R2). (E1) and (E1$^{\mathrm{op}}$) follow from (EA1). (E2$'^{\mathrm{op}}$)(i) follows from the functoriality of $\mathbb{E}$ and (R0). (E2$'^{\mathrm{op}}$)(ii) follows from Lemma 4.36 (1),(2). Dually for (E2$'$).

(2) By Corollary 4.13, it suffices to show that any $X^{\bullet} \in \mathcal{X}$ satisfies (I1) and (I2) in Corollary 4.13. Since (I2) is dual to (I1), we only show that (b) implies (I1). For ${}_AX_C^{\bullet} \in \mathcal{X}$, let $f^{\bullet} \in \mathbf{C}_{\mathscr{C}}^{n+2}(X^{\bullet}, Y^{\bullet})$ be any morphism to an n-exact sequence $Y^{\bullet}$, in which f^0 and f^{n+1} are isomorphisms in $\mathscr{C}$. Modifying $X^{\bullet}$ using isomorphisms f^0 and f^{n+1} by Corollary 2.26, we may assume $f^0 = 1_A$ and $f^{n+1} = 1_C$ from the beginning.

By the equality $d_Y^n \circ f^n = d_X^n$, condition (b) implies that d_Y^n is an $\mathfrak{s}$-deflation. Thus there is $Z^{\bullet} \in \mathcal{X}$ of the form

$$Z^0 \xrightarrow{d_Z^0} Z^1 \xrightarrow{d_Z^1} \cdots \xrightarrow{d_Z^{n-2}} Z^{n-1} \xrightarrow{d_Z^{n-1}} Y^n \xrightarrow{d_Y^n} Y^{n+1}.$$

Since both $Y^{\bullet}$ and $Z^{\bullet}$ are n-exact sequences, the commutative square in $\mathscr{C}$

$$\begin{array}{ccccccccccc} Z^0 & \xrightarrow{d_Z^0} & Z^1 & \xrightarrow{d_Z^1} & \cdots & \xrightarrow{d_Z^{n-2}} & Z^{n-1} & \xrightarrow{d_Z^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} \\ & & & & & & & & \| & \circlearrowright & \| \\ Y^0 & \xrightarrow[d_Y^0]{} & Y^1 & \xrightarrow[d_Y^1]{} & \cdots & \xrightarrow[d_Y^{n-2}]{} & Y^{n-1} & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & Y^{n+1} \end{array}$$

can be completed into a homotopy equivalence $Z^{\bullet} \to Y^{\bullet}$ by Proposition 4.10 (1). Thus $Z^{\bullet} \in \mathcal{X}$ implies $Y^{\bullet} \in \mathcal{X}$ by Lemma 4.22 (4). □

Remark 4.38. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category, and let $\mathbb{F} \subseteq \mathbb{E}$ be a closed subfunctor. Trivially, if any $\mathfrak{s}$-inflation is monomorphic (respectively, if any $\mathfrak{s}$-deflation is epimorphic), then so is any $\mathfrak{s}|_{\mathbb{F}}$-inflation (resp. $\mathfrak{s}|_{\mathbb{F}}$-deflation).

Let $\mathcal{X}_{\mathfrak{s}}$ and $\mathcal{X}_{\mathfrak{s}|_{\mathbb{F}}}$ be the classes of all $\mathfrak{s}$-conflations and $\mathfrak{s}|_{\mathbb{F}}$-conflations respectively, as in Lemma 4.36. If $(\mathscr{C}, \mathcal{X}_{\mathfrak{s}})$ moreover satisfies condition (EI), then so does

$(\mathscr{C}, \mathcal{X}_{\mathfrak{s}|_{\mathbb{F}}})$. By the arguments so far, this means that any relative theory for an n-exact category induces an n-exact category.

Remark 4.39. Let $(\mathscr{C}, \Sigma, \pentagon)$ be an $(n+2)$-angulated category, and regard it as an n-exangulated category through Proposition 4.5. Then by Proposition 3.16, any closed subfunctor $\mathbb{F} \subseteq \mathbb{E}_\Sigma$ gives an n-exangulated category $(\mathscr{C}, \mathbb{F}, \mathfrak{s}_{\pentagon}|_{\mathbb{F}})$, which is not n-exact unless $\mathbb{F} = 0$. Indeed, if d_X^0 is monomorphic in $\mathscr{C}$ for $\delta \in \mathbb{F}(C, A)$ with $\mathfrak{s}_{\pentagon}(\delta) = [X^{\bullet}]$, then d_X^0 should be a split monomorphism, which implies $\delta = 0$.

It is not $(n+2)$-angulated either, in general. Especially for the closed subfunctor $\mathbb{F} = \mathbb{E}_\Sigma^{\mathcal{I}}$ associated with a full subcategory $\mathcal{I} \subseteq \mathscr{C}$ as in Definition 3.18, the resulting n-exangulated category is not $(n+2)$-angulated unless $\mathcal{I} = 0$. In fact if $\mathcal{I} \neq 0$, any object $0 \neq I \in \mathcal{I}$ satisfies $\mathbb{E}_\Sigma^{\mathcal{I}}(C, I) = 0$ for any $C \in \mathscr{C}$, which cannot happen in an $(n+2)$-angulated category. Similarly for $(\mathbb{E}_\Sigma)_{\mathcal{I}}$.

5. Examples

In this section we construct a family of examples of n-exangulated categories using relative theory as introduced in Subsection 3.2. Since we will start from an $(n+2)$-angulated category, the resulting relative versions will not be $(n+2)$-angulated nor n-exact by Remark 4.39.

We start by considering a finite dimensional algebra A over a field k given by a quiver with relations (see [ASS] for details on such algebras). More precisely, let A be the path algebra of the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5 \quad \text{with relation} \quad abcd = 0.$$

Next we explain how to get a 4-angulated category $\mathscr{C}$ from A. In fact $\mathscr{C}$ will be one instance of a family of $(n+2)$-angulated categories that is discussed in detail in [F, Section 7], where many of the facts used below can be found. In the interest of brevity, we proceed without precise references to [F] for the most part.

The category mod$A$ of finitely generated right A-modules has a unique 2-cluster tilting subcategory $\mathscr{M}$ consisting of all modules that can be written as a direct sum of a projective and an injective module. There are 5 indecomposable projective A-modules and 5 indecomposable injective A-modules. Among these there are 2 indecomposable A-modules that are both projective and injective. Hence $\mathscr{M}$ has 8 indecomposable objects. In fact, one may label these indecomposables as C_1, C_2, C_3, C_4, C_5, C_6, C_7 and C_8 in a unique way such that

$$\dim_k \operatorname{Hom}_A(C_i, C_j) = \begin{cases} 1 & \text{if } 0 \leq j - i \leq 3, \\ 0 & \text{else.} \end{cases}$$

Note that C_i is projective for $1 \leq i \leq 5$ and injective for $4 \leq i \leq 8$. Now consider the bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}A)$ and let $\Sigma = [2]$ be the twofold suspension of $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}A)$. Since the global dimension of A is 2 it follows from [GKO] that

$$\mathscr{C} := \operatorname{add}\{\Sigma^m C_i \mid 1 \leq i \leq 8,\ m \in \mathbb{Z}\}$$

is 4-angulated. By Proposition 4.8 we obtain a 2-exangulated category $(\mathscr{C}, \mathbb{E}_\Sigma, \mathfrak{s}_{\pentagon})$.

We denote $\Sigma^m C_i$ by C_{8m+i} so that the indecomposables in $\mathscr{C}$ are precisely $\{C_i \mid i \in \mathbb{Z}\}$ and satisfy

$$\dim_k \mathscr{C}(C_i, C_j) = \begin{cases} 1 & \text{if } 0 \leq j - i \leq 3, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad \Sigma C_i = C_{i+8}.$$

It is useful to note that, if $i \leq k \leq j$, then any morphism $C_i \to C_j$ factors through C_k.

Since $\mathbb{E}_\Sigma(C_i, C_j) = \mathscr{C}(C_i, \Sigma C_j)$ we get

$$\dim_k \mathbb{E}_\Sigma(C_i, C_j) = \begin{cases} 1 & \text{if } 5 \leq i - j \leq 8, \\ 0 & \text{else.} \end{cases}$$

Using [F, Remark 7.3] one can show that each non-zero $\delta \in \mathbb{E}_\Sigma(C_i, C_j)$ gives rise to a 4-angle of the form

$$C_j \longrightarrow C_{i-4} \longrightarrow C_{j+4} \longrightarrow C_i \overset{\delta}{\longrightarrow} \Sigma C_j.$$

Hence

$$\mathfrak{s}_{\pentagon}(\delta) = [C_j \longrightarrow C_{i-4} \longrightarrow C_{j+4} \longrightarrow C_i].$$

Now let us consider a relative version of $(\mathscr{C}, \mathbb{E}_\Sigma, \mathfrak{s}_{\pentagon})$. Fix $t \in \mathbb{Z}$ and set $\mathcal{I}_t = \{C_t\}$. Further set $\mathbb{E}_t = \mathbb{E}_\Sigma^{\mathcal{I}_t}$ and $\mathfrak{s}_t = (\mathfrak{s}_{\pentagon})|_{\mathbb{E}}$. Then $(\mathscr{C}, \mathbb{E}_t, \mathfrak{s}_t)$ is 2-exangulated.

To calculate $\mathbb{E}_t(C_i, C_j)$ we need to consider $\delta^\sharp_{C_t} \colon \mathscr{C}(C_j, C_t) \to \mathbb{E}_\Sigma(C_i, C_t)$ for $\delta \in \mathbb{E}_\Sigma(C_i, C_j)$. If $\delta = 0$, then $\delta^\sharp_{C_t} = 0$, so assume that $\delta \neq 0$. Then we claim that $\delta^\sharp_{C_t} = 0$ if and only if $\mathscr{C}(C_j, C_t) = 0$ or $\mathbb{E}_\Sigma(C_i, C_t) = 0$. Clearly, the 'if' part holds. On the other, if $\mathscr{C}(C_j, C_t) \neq 0$, then $j \leq t$, and since $\delta \neq 0$, we get $i - j \leq 8$ so that $i \leq j + 8 \leq t + 8$. As noted above it follows that any morphism $g \colon C_i \to C_{t+8}$ factors through C_{j+8} and as $\dim_k \mathscr{C}(C_i, C_j + 8) = 1$, we get that g even factors through $\delta \in \mathscr{C}(C_i, C_j + 8)$. Hence $\delta^\sharp_{C_t}$ is surjective and non-zero if $\mathbb{E}_\Sigma(C_i, C_t) \neq 0$.

To simplify our notation we let $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ so that $\mathscr{C}(C_j, C_t) = 0$ if and only if $j \notin [t-3, t]$. Similarly, $\mathbb{E}_\Sigma(C_i, C_t) = 0$ if and only if $i \notin [5 + t, 8 + t]$. Hence

$$\dim_k \mathbb{E}_t(C_i, C_j) = \begin{cases} 1 & \text{if } 5 \leq i - j \leq 8 \text{ and } t \notin [j, i-5] \\ 0 & \text{else.} \end{cases}$$

Since $\mathfrak{s}_t$ is just induced from $\mathfrak{s}_{\pentagon}$ this gives a substantial control over the 2-exangulated category $(\mathscr{C}, \mathbb{E}_t, \mathfrak{s}_t)$.

To generalize we may choose any subset $T \subseteq \mathbb{Z}$ and obtain a 4-angulated category $(\mathscr{C}, \mathbb{E}_T, \mathfrak{s}_T)$, by setting $\mathcal{I}_T = \{C_t \mid t \in T\}$, $\mathbb{E}_T = \mathbb{E}_\Sigma^{\mathcal{I}_T}$ and $\mathfrak{s}_T = (\mathfrak{s}_{\pentagon})|_{\mathbb{E}_T}$. As above we find that

$$\dim_k \mathbb{E}_T(C_i, C_j) = \begin{cases} 1 & \text{if } 5 \leq i - j \leq 8 \text{ and } T \cap [j, i-5] = \emptyset \\ 0 & \text{else.} \end{cases}$$

We note that $(\mathscr{C}, \mathbb{E}_T, \mathfrak{s}_T)$ is 4-angulated if and only if $T = \emptyset$ and that $(\mathscr{C}, \mathbb{E}_T, \mathfrak{s}_T)$ is 2-exact if and only if $\mathbb{E}_T = 0$.

Many similar examples can be constructed. See for instance the next article n-Exangulated Categories (II).

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