Moment extract method for drift kinetic simulation of magnetized plasma

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Abstract

A novel time integration method of the drift kinetic equation is proposed, where low order fluid moment equations including the lowest order terms for the dispersive Alfvén wave propagation are extracted from the electron drift kinetic equations. The moment extract method enables ones to implement an implicit solver for wave propagation by means of the operator splitting, while the kinetic part is solved explicitly. Numerical tests of the new scheme have verified stable and accurate time-integration of the drift kinetic equation and enables to take a large time step size when the electron beta value is smaller than the electron-to-ion mass ratio.

Keywords: Vlasov simulation, drift kinetic equation, dispersive Alfvén wave

1. Introduction

Comprehension of turbulent transport mechanism in high temperature plasmas is one of the crucial issues in magnetic fusion research. The anomalous plasma transport is driven by drift wave turbulence of which time scales are longer than that of gyromotion of particles while the wavelengths are of the order of the ion Larmor radius. Therefore, numerical simulations based on the gyrokinetic theory [1, 2, 3], where the gyrophase dependent fluctuations are averaged out, are widely employed for the turbulent transport analysis [4].

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The ion temperature gradient (ITG) mode turbulence has widely been investigated as a main cause for the ion heat transport in high temperature magnetized plasma. Conventional analyses of the ITG turbulence often assume the adiabatic electron response because of fast motions of electrons with smaller mass. This simplification demands less computational costs than that for the gyrokinetic or drift kinetic description of electrons. However, it is found that the ITG simulations with kinetic electrons result in a higher heat transport level than that with adiabatic electrons [5, 6, 7] because response to the potential fluctuation is reduced in case with trapped electrons. Therefore, for quantitative analyses of turbulent transport, kinetic response of electrons should be taken into account even in simulations of turbulence in the ion gyroradius scale.

In gyrokinetic Vlasov simulation codes with kinetic electrons, the local flux tube model [8] is often employed such as GKV [9], GENE [10], GKW [11], and GS2 [12], where the simulation domain is given by a long thin magnetic flux tube, and turbulent fluctuations with strong anisotropy can be handled with less computational costs. Several types of time integration schemes are employed in the flux tube codes. The fourth-order explicit Runge-Kutta (RK) schemes are used in GKV, GENE, and GKW codes, where the time step size Δt is restricted by the Courant-Friedrichs-Lewy (CFL) condition. The CFL limitation to Δt becomes severer in non-axisymmetric systems, because magnetic ripples along a field line in a helical geometry demand shorter grid spacing, leading to smaller time step size. Furthermore, if kinetic electrons are included, Δt may also be limited not only by the parallel advection term, but also by the propagation of dispersive Alfvén waves (DAW). In the low β and long wave length limit, the latter condition determines the upper limit of Δt for the numerical stability. On the other hand, GS2 code applies an implicit method to linear terms with the second order accuracy for space and time. While the implicit scheme is free from the CFL condition, generally speaking, a lot of computational costs are necessary for implicitly solving the kinetic equations in multi-dimensional phase space.

To relax the numerical difficulty in solving the gyrokinetic equation with

kinetic electrons, we propose a new time-integration scheme named the moment extract (ME) method, where the kinetic equation is divided into three equations, that is, the zeroth and the first order moment equations, and the remnant kinetic equation responsible for the second or higher order moments of the distribution function. This technique allows us to decouple the main terms for the wave propagation from the kinetic equation, and to apply an implicit solver only to the wave propagation terms in the fluid equations for relaxation of the CFL condition. It leads to reduction of the matrix size inverted for time-integration in comparison to that required for the full implicit kinetic simulation, and works efficiently when the electron beta value is smaller than the electron-to-ion mass ratio ($\beta_e < m_e/m_i$).

Separation of fluid moment equations from the kinetic one has been pursued in drift kinetic or gyrokinetic simulations One also finds similar ideas in a broader context of multi-scale phenomena. Some recent examples are a moment guided Monte Carlo method [16], a conservation-moment-based implicit lattice Boltzmann method [17], a unified gas-kinetic scheme for unsteady flow [18], and a fully implicit moment particle-in-cell method for Vlasov-Ampére system [19]. More generally, it would be regarded as one of high-order/low-order algorithms [20]. In this paper, we discuss the ME method applied to a Vlasov-type solver for the drift kinetic equations and demonstrate the numerical stability for Δt beyond the CFL condition for the dispersive Alfvén wave.

This paper is organized as follows. In Section 2, equations used in the ME method are described. Numerical methods used in the present study are given in Sections 3 and 4. The detailed analyses of numerical errors and stability are shown in Section 5. Concluding remarks are summarized in the last section.

2. Moment extract method

In this paper, we focus on electron motions given by the drift kinetic equations in the limit of small electron Larmor radius compared to the perpendicular wavelength characterized by the ion Larmor radius. If the Fourier transforma-

tion is applied to the perpendicular coordinates in the linear regime, the phase space dimension is reduced to two, that is the field aligned coordinate z and the parallel velocity coordinate $v_\parallel$. The key issue that the DAW propagation restricts the time step size is still retained even in the reduced model. In the limit of uniform and constant external magnetic field and the background plasma density, the linearized drift kinetic equation (DKE) for the perturbed electron gyrocenter distribution function $\delta f_e\left(z, v_\parallel, t\right)$ from the Maxwellian, the quasi-neutrality condition and the Ampere's law are given by

$$\frac{\partial \delta f_e}{\partial t} + v_\parallel \frac{\partial \delta f_e}{\partial z} = -\frac{q_e}{T_{e0}} v_\parallel F_{eM} \left(\frac{\partial \phi}{\partial z} + \frac{\partial A_\parallel}{\partial t} \right), \tag{1}$$

$$\frac{{q_i}^2 n_{i0}}{T_{i0}} k_\perp^2 \rho_i^2 \phi = q_e n_e \tag{2}$$

and

$$k_\perp^2 A_\parallel = \mu_0 q_e n_{e0} U_{e\parallel} \ . \tag{3}$$

Here, q_a, n_{a0}, T_{a0}, and ρ_a are charge, equilibrium density, equilibrium temperature and Larmor radius of particle species a with the thermal speed v_{ta}, respectively. Subscripts $a = e$ and $a = i$ mean electrons and ions. Here, the charge neutrality $q_e n_{e0} + q_i n_{i0} = 0$ is assumed. The perturbed electrostatic potential and parallel component of vector potential are denoted by ϕ and $A_\parallel$, respectively. Also, n_e and $U_{e\parallel}$ represents fluctuations of the electron density and parallel electron flow velocity. The above set of equations are the same as those discussed in Ref. [21]. The equilibrium distribution function is given by the Maxwellian distribution function

$$F_{eM} = \frac{n_{e0}}{\sqrt{2\pi} v_{te}} \exp\left(-\frac{v_\parallel^2}{2{v_{te}}^2} \right), \tag{4}$$

where $v_{te} = \sqrt{T_{e0}/m_e}$. The above equations describe the parallel electron motion and the DAW propagation. Effect of the ion polarization is taken into account through the term on the left hand side of Eq.(2). If we assume a sinusoidal perturbation in proportion to $\exp[i(k_\parallel z - \omega t)]$, the dispersion relation is written as

$$k_\perp^2 \rho_s^2 = \left[1 + \zeta Z(\zeta) \right] \left(\frac{2{v_{te}}^2}{{v_A}^2} \zeta^2 - 1 \right), \tag{5}$$

where $Z(\zeta)$ is the plasma dispersion function [22] and $\zeta = \omega/\sqrt{2}k_\parallel v_{te}$. The electron beta value and the Larmor radius for the ion accoustic speed $c_s = \sqrt{T_{e0}/m_i}$ are denoted by $\beta_e = \mu_0 n_{e0} T_{e0}/B^2$ and $\rho_s = m_i c_s/q_i B$, respectively. The Alfvén velocity is defined as $v_A \equiv c_s/\sqrt{\beta_e}$. In the limit of $k_\parallel v_{te} \gg \omega$, Eq.(5) reduces to

$$\omega = k_\parallel v_A \sqrt{1 + k_\perp^2 \rho_s^2} = k_\parallel c_s \sqrt{\left(1 + k_\perp^2 \rho_s^2\right)/\beta_e} \ . \tag{6}$$

Equation (6) indicates that the DAW frequency becomes lower with increase of β_e while fixing c_s and ρ_s. It is noteworthy that c_s and ρ_s represent characteristic speed and length of the ITG mode. Thus, in the time integration of Eq.(1) by an explicit scheme, the time step size Δt is restricted by the parallel electron motion such that $C_v = v_{\parallel max}\Delta t/\Delta z < 1$ when $v_A < v_{te}$ and $k_\perp \rho_s \ll 1$, where $v_{\parallel max}$ means the maximum value of the parallel velocity. On the other hand, in the cold electron limit $k_\parallel v_{te} \ll \omega$ and $\beta_e \to 0$, Eq.(5) reduces to

$$\omega \equiv \omega_H = \frac{k_\parallel v_{te}}{k_\perp \rho_s} \ . \tag{7}$$

In the small perpendicular wavenumber $k_\perp$ and the zero β_e limits, the time step size Δt is restricted by the DAW propagation so that $C_w = \omega_H \Delta t/k_\parallel \Delta z < 1$. The sever limitation degrades efficiency of the gyrokinetic simulation of drift waves with kinetic electrons.

In order to relax the CFL condition in the low β_e regime, we have developed the ME method. The basic idea is summarized below. Taking the zeroth and the first order moments of Eq.(1), the fluid moment equations of electrons are given by

$$\frac{\partial n_e}{\partial t} = -n_{e0}\frac{\partial U_{e\parallel}}{\partial z} \tag{8}$$

$$n_{e0} m_e \frac{\partial U_{e\parallel}}{\partial t} = -q_e n_{e0}\left(\frac{\partial \phi}{\partial z} + \frac{\partial A_\parallel}{\partial t}\right) - \frac{\partial}{\partial z}\left(n_e T_{e0} + n_{e0} T_{e\parallel}\right) \ , \tag{9}$$

where perturbations of the density n_e, the parallel flow $U_{e\parallel}$, and the parallel temperature $T_{e\parallel}$ are represented in terms of the zeroth, the first, and the second order moments of δf_e, that is,

$$n_e = \int \delta f_e dv_\parallel \tag{10}$$

$$n_{e0}U_{e\parallel} = \int v_\parallel \delta f_e dv_\parallel \tag{11}$$

and

$$n_{e0}T_{e\parallel} + n_e T_{e0} = m_e \int {v_\parallel}^2 \delta f_e dv_\parallel \ , \tag{12}$$

respectively. Here, we also divide the perturbed distribution function δf_e into three components

$$\delta f_e \equiv \frac{n_e}{n_{e0}} F_{eM} + \frac{m_e}{T_{e0}} v_\parallel U_{e\parallel} F_{eM} + h_e \ . \tag{13}$$

It is found that coefficients of the first and the second terms of Eq.(13) are given by the zeroth and the first order moments of δf_e, respectively. Thus, for consistency, the zeroth and the first order moments of h_e should vanish. Namely, Eq.(13) corresponds to the Hermite polynomial expansion of δf_e up to the first order. From Eqs. (12) and (13), one finds that the parallel temperature $T_{e\parallel}$ can be represented as

$$n_{e0}T_{e\parallel} = m_e \int {v_\parallel}^2 h_e dv_\parallel \ . \tag{14}$$

Substituting Eqs. (8), (9), and (13) into Eq.(1), the drift kinetic equation is rewritten in terms of the remnant distribution function h_e,

$$\frac{\partial h_e}{\partial t} + v_\parallel \frac{\partial h_e}{\partial z} = \left(1 - \frac{m_e {v_\parallel}^2}{T_{e0}}\right) F_{eM} \frac{\partial U_{e\parallel}}{\partial z} + \frac{v_\parallel}{T_{e0}} F_{eM} \frac{\partial T_{e\parallel}}{\partial z} \ . \tag{15}$$

It is noteworthy that the dispersive Alfvén wave propagation in the limit of $\beta_e \to 0$ is described by the low-order moment fluid equations and the quasi-neutrality. The above set of equations are similar to those by Zocco et al. [15], where the low order moment equations are separated from the drift kinetic equation by means of the Hermite polynomial expansion.

If we naively apply the explicit time integration to Eqs.(8), (9) and (15), the time step size Δt is still restricted by the wave propagation in the low-β_e regime. The time step restriction due to the DAW propagation can be relaxed partly by the fluid-kinetic hybrid electron model which is also useful to avoid the "cancellation" problem in solving the parallel electric field [13, 14] (while the "cancellation" is not a serious issue in the present study as well as

in flux tube gyrokinetic Vlasov codes, such as GKV). In the present method, we employ a kind of implicit-explicit (IMEX) scheme (see, for example, Ref. [23]), where only the terms responsible for the wave propagation are implicitly time-integrated while others are explicitly computed. This approach enables an efficient time integration with a larger Δt than the CFL limit in the low β_e regime of $\beta_e < m_e/m_i$.

Physical meaning and related numerical requirements contained in Eqs. (2), (3), (8), and (9) become clearer by means of the Elsässer variables. When $q_i = e$ and $q_e = -e$, we can rewrite Eqs.(8)-(9) by using Eqs.(2) and (3) as

$$\frac{\partial R}{\partial t} = -v_p \frac{\partial R}{\partial z} - \frac{n_{e0}}{M} \frac{\partial T_{e\parallel}}{\partial z} \tag{16}$$

$$\frac{\partial L}{\partial t} = v_p \frac{\partial L}{\partial z} - \frac{n_{e0}}{M} \frac{\partial T_{e\parallel}}{\partial z} \ . \tag{17}$$

In Eqs.(16) and (17), the Elsässer variables R and L, the advection velocity of the DAW v_p, and an effective mass M are, respectively, defined by

$$R = n_{e0} U_{e\parallel} + v_p n_e, \tag{18}$$

$$L = n_{e0} U_{e\parallel} - v_p n_e, \tag{19}$$

and

$$v_p = \sqrt{\frac{T_{e0}}{M} \frac{1 + k_\perp^2 \rho_s^2}{k_\perp^2 \rho_s^2}} = c_s \sqrt{\frac{1 + k_\perp^2 \rho_s^2}{\frac{m_e}{m_i} k_\perp^2 \rho_s^2 + \beta_e}} = v_A \sqrt{\frac{1 + k_\perp^2 \rho_s^2}{1 + k_\perp^2 \delta_e^2}} \ , \tag{20}$$

where

$$M = m_e + m_i \frac{\beta_e}{k_\perp^2 \rho_s^2} \ , \tag{21}$$

and $\delta_e = c/\omega_p$ means the electron skin depth. Here, R and L represent the DAW propagating in the positive and negative z-directions, respectively, with modification by the parallel temperature gradient. Also, one should note that $v_p = v_{te}$ for $\beta_e = m_e/m_i$. Since Eqs. (16) and (17) are one-dimensional advection equations for R and L, application of an IMEX scheme is straightforward and efficient. Restriction of the time step size due to the dispersive Alfvén wave

propagation is now rewritten as $v_p \Delta t/\Delta z < 1$. In the limit of $\beta_e \to 0$ and low $k_\perp \rho_s$, it reduces to $\omega_H \Delta t < k_\parallel \Delta z$ (or equivalently, $v_{te}\Delta t/k_\perp \rho_s < \Delta z$), which corresponds to the CFL condition of $C_w < 1$. Therefore, when $\omega_H > k_\parallel v_{\parallel max}$, the DAW propagation limits the time step size of an explicit scheme. In the following, we focus on the moment equations with the Elsässer variables, Eqs. (16) and (17), since it is simpler and more efficient than those with n_e and $U_{e\parallel}$, that is, Eqs. (8) and (9). Indeed, a structure of the electron fluid equations becomes clearer in Eqs. (16) and (17). Also, decoupling the two equations of fluid moments reduces the matrix dimension for the implicit solver by half. However, the Elsässer variables are useful only in cases where one can analytically derive the advection speed, v_p. In addition, the use of R and L is not mandatory in the ME method, while separation of n_e, $U_{e\parallel}$, and h_e is essential. Actually, we can construct the ME scheme for Eqs. (8) and (9), while we need to handle the implicit matrix equation of $2N \times 2N$ instead of $N \times N$ where N means the number of grid points in the z direction.

In the following, we employ the gyrokinetic (or gyrofluid) units with the equilibrium (parallel) scale length L, the ion acoustic gyroradius ρ_s, the equilibrium electron density n_{e0} and temperature T_{e0}, and the electron thermal velocity v_{te}. For example, the perturbed electron density is normalized as $\overline{n}_e = n_e L/n_{e0}\rho_s$. Thus, a set of the normalized equations for the ME method is summarized as

$$\frac{\partial \overline{h}_e}{\partial \overline{t}} = -\overline{v}_\parallel \frac{\partial}{\partial \overline{z}} \left(\overline{h}_e - \overline{F}_{eM} \overline{T}_{e\parallel} \right) + \left(1 - \overline{v}_\parallel^2 \right) \overline{F}_{eM} \frac{\partial \overline{U}_{e\parallel}}{\partial \overline{z}} \tag{22}$$

$$\frac{\partial \overline{R}}{\partial \overline{t}} = -\overline{v}_p \frac{\partial \overline{R}}{\partial \overline{z}} - \frac{1}{\overline{M}} \frac{\partial \overline{T}_{e\parallel}}{\partial \overline{z}} \tag{23}$$

$$\frac{\partial \overline{L}}{\partial \overline{t}} = \overline{v}_p \frac{\partial \overline{L}}{\partial \overline{z}} - \frac{1}{\overline{M}} \frac{\partial \overline{T}_{e\parallel}}{\partial \overline{z}} \tag{24}$$

$$\overline{R} = \overline{U}_{e\parallel} + \overline{v}_p \, \overline{n}_e \tag{25}$$

$$\overline{L} = \overline{U}_{e\parallel} - \overline{v}_p \, \overline{n}_e \tag{26}$$

$$\overline{T}_{e\parallel} = \int \overline{v}_\parallel{}^2 \overline{h}_e d\overline{v}_\parallel \tag{27}$$

$$\overline{v}_p = \sqrt{\frac{1}{\overline{M}} \frac{1+\overline{k}_\perp^2}{\overline{k}_\perp^2}} \tag{28}$$

and

$$\overline{M} = \left(1 + \overline{m}_i \frac{\beta_e}{\overline{k}_\perp^2}\right). \tag{29}$$

where $\overline{m}_i = m_i/m_e$. Hereafter, we drop the overbar from the normalized quantities for simplicity.

3. Time integration scheme

In the above, we have seen how to separate the terms responsible for the DAW propagation from the drift kinetic equation. In order to reduce the computational costs in the low β_e regime ($\beta_e < m_e/m_i$), we employ an IMEX time integration scheme, where the wave propagation (stiff) terms are time-integrated implicitly and the other terms are solved explicitly. Equations (22)-(24) can be expressed by

$$\frac{\partial}{\partial t}\begin{pmatrix} R \\ L \end{pmatrix} = S_{adv}\begin{pmatrix} -R \\ L \end{pmatrix} + N_T(h_e), \tag{30}$$

$$\frac{\partial h_e}{\partial t} = N_{adv}(h_e) + N_{div}\,(R+L) \tag{31}$$

where operators N_{adv}, N_{div}, S_{adv}, and N_T are defined by

$$S_{adv} = v_p \frac{\partial}{\partial z}, \tag{32}$$

$$N_T = -\frac{1}{M}\frac{\partial}{\partial z}\int dv_\parallel v_\parallel{}^2, \tag{33}$$

$$N_{adv} = v_\parallel \left(-\frac{\partial}{\partial z} + F_{eM}\frac{\partial}{\partial z}\int dv_\parallel v_\parallel{}^2\right), \tag{34}$$

$$N_{div} = \frac{1}{2}\left(1 - {v_\parallel}^2\right) F_{eM} \frac{\partial}{\partial z}, \tag{35}$$

respectively. S_{adv} is an operator for the stiff terms which describe the wave propagation, and the others are supposed to be non-stiff. The formal solution of Eqs. (30) and (31) is given by the product of matrices and vectors such that

$$\begin{bmatrix} R\left(t+\Delta t\right) \\ L\left(t+\Delta t\right) \\ h_e\left(t+\Delta t\right) \end{bmatrix} = \exp\left\{\Delta t\left(\mathcal{S}+\mathcal{N}\right)\right\} \begin{bmatrix} R\left(t\right) \\ L\left(t\right) \\ h_e\left(t\right) \end{bmatrix}, \tag{36}$$

where $\mathcal{S}$ and $\mathcal{N}$ consist of the matrix elements, and denote the operators for stiff and non-stiff terms, respectively. According to the Strang's symmetric product for operators [24], the operator in Eq.(36) can be expanded as

$$\exp\left\{\Delta t\left(\mathcal{S}+\mathcal{N}\right)\right\} = \exp\left(\frac{\Delta t}{2}\mathcal{S}\right)\exp\left(\Delta t\mathcal{N}\right)\exp\left(\frac{\Delta t}{2}\mathcal{S}\right) + O\left(\Delta t^2\right) \ , \tag{37}$$

where

$$\exp\left(\frac{\Delta t}{2}\mathcal{S}\right) = \begin{bmatrix} -\frac{\Delta t}{2}S_{adv} & 0 & 0 \\ 0 & \frac{\Delta t}{2}S_{adv} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{38}$$

and

$$\exp\left(\Delta t\mathcal{N}\right) = \begin{bmatrix} 0 & 0 & \Delta t N_T \\ 0 & 0 & \Delta t N_T \\ \Delta t N_{div} & \Delta t N_{div} & N_{adv}\Delta t \end{bmatrix}. \tag{39}$$

Thus, the time integration is carried out in three steps. The first and the last steps with $\mathcal{S}$ are solved implicitly, while the middle step with $\mathcal{N}$ is solved explicitly. Although calculation costs per a time step may increase, the total cost is expected to decrease by taking a large Δt. Here, we apply the second order midpoint implicit rule to $\mathcal{S}$, and the fourth order Runge-Kutta-Gill (RKG) scheme is used for $\mathcal{N}$. A higher order scheme is preferable for time-integration of the non-stiff terms as it is inserted in the innermost step of the 2nd order splitting scheme with the time-reversal symmetry so that the symmetry of the midpoint implicit rule is approximately preserved. The fourth order RK method is chosen, since it is used in several flux tube gyrokinetic codes. For the z derivative in

these operators, the fourth order central finite difference is employed. In total, the time integration scheme has the second-order temporal and the fourth order spatial accuracy.

4. Correction to the equilibrium distribution

According to Eq.(13), the zeroth and first order moments of h_e should vanish. The conservation holds if the even- (zeroth and second) and odd- (first) order moments of the Maxwellian F_M are equal to 1 and 0, respectively. However, discretization in the finite velocity space causes numerical errors in computation of the zeroth and second order moments of F_M. Even if the numerical errors of low order moments of F_M are around of $10^{-7} \sim 10^{-5}$, accumulation of the secular error during the time integration may influence the simulation result. In order to minimize the error, we introduce a correction to the equilibrium distribution function $F_{Mc} = F_M + \tilde{F}$, to satisfy $\int F_{Mc} dv_\parallel = 1$ and $\int {v_\parallel}^2 F_{Mc} dv_\parallel = 1$ within the round-off error level. A similar requirement also motivates to find discretization scheme that preserves the analytical equilibrium for a Rosenbluth-Fokker-Planck collision operator [25].

To find $\tilde{F}$, let us define the even order moments of the discretized Maxwellian $[F_M]_j$ by means of the trapezoidal rule,

$$E_0 = \sum_j [F_M]_j \, \Delta v \tag{40}$$

$$E_1 = \sum_j \left[{v_\parallel}^2 F_M\right]_j \Delta v \tag{41}$$

and

$$E_2 = \sum_j \left[{v_\parallel}^4 F_M\right]_j \Delta v \ , \tag{42}$$

where $[f]_j \equiv f(v_\parallel = v_{\parallel j} = j\Delta v)$ for an integer j. Here, we consider that $\tilde{F}$ is given by a solution of a variational problem in a similar manner that one finds in derivation of the Maxwellian distribution with the maximum entropy under constraint of particle and energy conservation. A variational approach has also

been pursued for implementation of the Landau collision operator for kinetic plasma [26].

The target functional for the variational formulation is given by a discretized version of the quadratic integral of the correction $\tilde{F}$,

$$\mathcal{E} = \int dv \frac{\tilde{F}^2}{2F_M} \ . \tag{43}$$

For a small perturbation from the Maxwellian, $|\tilde{F}| \ll F_M$, it is straightforward to find that $\mathcal{E}$ corresponds to the second order deviation of the entropy $\int dv(-F_{Mc} \ln F_{Mc})$ from the equilibrium state,

$$-\int dv F_{Mc} \ln F_{Mc} \approx -\int dv F_M \ln F_M - \int dv (1 + \ln F_M)\tilde{F} - \int dv \frac{\tilde{F}^2}{2F_M} \ . \tag{44}$$

The first order deviation from $\int dv(-F_M \ln F_M)$, that is, the second term on the right hand side of Eq. (44), plays no role in the following variational formulation as it can be absorbed into the Lagrangian multipliers as found below. A discretized form of $\mathcal{E}$ is written as

$$\mathcal{E} = \sum_j \frac{\left[\tilde{F}\right]_j^2}{2\left[F_M\right]_j} \Delta v \tag{45}$$

which is minimized under constraints of

$$E_0 + \sum_j \left[\tilde{F}\right]_j \Delta v = 1 \tag{46}$$

and

$$E_1 + \sum_j v_{\|j}^2 \left[\tilde{F}\right]_j \Delta v = 1. \tag{47}$$

By means of the Lagrange multipliers λ_1 and λ_2, we consider to minimize S,

$$\begin{aligned} S\left(\left[\tilde{F}\right]_j\right) = \sum_j \frac{\left[\tilde{F}\right]_j^2}{2\left[F_M\right]_j} \Delta v - \lambda_1 \left[\sum_j \left[\tilde{F}\right]_j \Delta v - (1 - E_0)\right] \\ - \lambda_2 \left[\sum_j v_{\|j}^2 \left[\tilde{F}\right]_j \Delta v - (1 - E_1)\right] . \end{aligned} \tag{48}$$

The variational of S for $\left[\tilde{F}\right]_j$ are represented by

$$\delta S = \sum_j \left(\frac{\left[\tilde{F}\right]_j}{[F_M]_j} - \lambda_1 - {v_{\|j}}^2 \lambda_2 \right) \Delta v \left[\delta \tilde{F}\right]_j \tag{49}$$

where $\left[\delta \tilde{F}\right]_j$ means an arbitrary perturbation of $\left[\tilde{F}\right]_j$ for each j. Requiring $\delta S = 0$, a group of terms inside of the round brackets for each j in Eq.(49) should cancel as $\left[\delta \tilde{F}\right]_j$ is arbitrary. Thus, $\left[\tilde{F}\right]_j$ can be obtained as

$$\left[\tilde{F}\right]_j = \left(\lambda_1 + {v_{\|j}}^2 \lambda_2\right) [F_M]_j \tag{50}$$

for each j. Substituting Eq. (50) into (46) and (47), one obtains expressions for the Lagrange multipliers, λ_1 and λ_2. Then,

$$\left[\tilde{F}\right]_j = \left(\frac{(E_2 - E_1) + {v_{\|j}}^2 (E_0 - E_1)}{E_2 E_0 - {E_1}^2} - 1 \right) [F_M]_j \ . \tag{51}$$

Thus, the numerical (or corrected) Maxwellian is obtained as

$$[F_{Mc}]_j = \frac{(E_2 - E_1) + {v_{\|j}}^2 (E_0 - E_1)}{E_2 E_0 - {E_1}^2} [F_M]_j \ . \tag{52}$$

Equation (52) confirms that the constraints of $\sum_j [F_{Mc}]_j \Delta v = 1$ and $\sum_j v_{\|j}^2 [F_{Mc}]_j \Delta v = 1$ are satisfied within the round-off error level. According to the definitions of E_0, E_1 and E_2, the numerical values are estimated as $E_0 = 1 + O(\varepsilon)$, $E_1 = 1 + O(\varepsilon)$, and $E_2 = 3 + O(\varepsilon)$, where ε denotes magnitudes of numerical errors in computation of the integrals E_0, E_1, and E_2. [It is known, or is easily confirmed that the integral errors of the trapezoidal rule applied to the Maxwellian distribution or its low order moments, such as E_0, E_1, and E_2, are quite small, that is, $\varepsilon \lesssim 10^{-3}$, if one employs moderate numbers of velocity grid points (for example, only 9 grids for $0 \le v \le 5v_{te}$)]. Thus, one finds $E_2 - E_1 \approx 2 + O(\varepsilon)$, $E_2 E_0 - E_1 \approx 2 + O(\varepsilon)$, and $E_0 - E_1 \approx O(\varepsilon)$. From (51), thus, $\left[\tilde{F}\right]_j \approx \left[O(\varepsilon)(1/2)({v_{\|j}}^2 - 1) + O(\varepsilon^2)\right] [F_M]_j$. Indeed, the maximum amplitude of $\tilde{F}$ for the present parameters is found to be negligible, that is, $\tilde{F} = -2.5 \times 10^{-6}$ at $v_\| = 0$ for $v_{\|max} = 5v_{te}$ and $\Delta v = (5/128)v_{te}$.

Table 1 shows errors in computation of the zeroth and the second order moments of F_M and F_{Mc}, where $v_{\|max} = 5v_{te}$, $\Delta v = (5/128)v_{te}$, and the

	F_M	F_{Mc}
1 - (0th order moment)	5.2×10^{-7}	2.0×10^{-16}
1 - (2nd order moment)	1.4×10^{-5}	$< 1.0 \times 10^{-16}$

Table 1: Errors in computation of the zeroth and the second order moments of original (F_M) and corrected Maxwellian (F_{Mc}).

numerical integration is carried out by means of the trapezoidal rule. The numerical accuracy is largely improved by means of the equilibrium distribution F_{Mc} of which errors in moment calculations are in the round-off error level of 10^{-16}. Numerical results applied to the ME method will be discussed in the next section in terms of the conservation of the zeroth and the first order moments of h_e.

5. Numerical results

Before testing the ME method, we survey the advection speed of the DAW, v_p, in Eq.(20) for $m_i = 1836.15\ m_e$ on the β_e - $k_\perp \rho_s$ space. Figure 1 shows contours of $\log_{10}(v_p/v_{te})$. The parameter range where the v_p exceeds $v_{\parallel max}$ is found in the low-β_e (electrostatic) regime of $\beta_e < 10^{-5}$ for $v_{\parallel max} = 5v_{te}$. Especially, for $\beta_e < 10^{-7}$ and $k_\perp \rho_s < 10^{-2}$, v_p becomes more than about 20 times of $v_{\parallel max}$. Therefore, it is expected that the ME method with the IMEX time integration scheme can largely reduce the computational costs in case with long wave length modes of $k_\perp \rho_s < 10^{-2}$ in the low-β_e electrostatic limit.

In the following, we evaluate numerical properties of the ME method applied to propagation of DAWs. We calculate the time development of R, L and h_e for a given initial condition, $\delta f_e(z, v_\parallel, t = 0) = 0.01 F_{Mc} \cos z$. Thus, $n_e(z, t = 0) = 0.01 \cos z$ and $U_{e\parallel}(z, t = 0) = h_e(z, v_\parallel, t = 0) = 0$. Figure 2 shows snapshots of the real part of h_e in the $z - v_\parallel$ space at $t = 0, 0.001, 0.015$, and $0.198\ L/c_s$ with $k_\perp \rho_s = 0.01$ and $\Delta t = 5.0 \times 10^{-6}\ L/c_s$, where $-\pi \leq z < \pi$, and $-v_{\parallel max} \leq v_\parallel \leq v_{\parallel max}$. Throughout the paper, we set $v_{\parallel max} = 5v_{te}$. Grid spacing in the z and $v_\parallel$ directions are $\Delta z = \pi/N_z$ and $\Delta v = 5v_{\parallel max}/N_v$,

Figure 1: Contour plot of the phase velocity v_p/v_{te} in $k_\perp \rho_s$ - β_e space, where the contour levels are taken logarithmically, that is, $\log_{10}(v_p/v_{te})$.

respectively, with $N_z = 32$, and $N_v = 128$. Starting from the initial condition, h_e develops through Eq. (22). It is found in Figs. 2 (b), (c) and (d) that the distribution function is elongated due to the parallel motion with different $v_\parallel$, and then, an oscillation arises in the velocity space with a time-dependent scale length in the $v_\parallel$ space characterized by the ballistic motion of electrons.

Let us check conservation of the zeroth and the first order moments of h_e normalized by the averaged values of n_e and $U_{e\parallel}$,

$$h_{em0} = \sqrt{\frac{\sum_i \sum_j |h_{e\,i,j}|^2 \Delta v}{\sum_i |n_{e\,i}|^2}} \tag{53}$$

and

$$h_{em1} = \sqrt{\frac{\sum_i \sum_j \left|v_{\parallel j} h_{e\,i,j}\right|^2 \Delta v}{\sum_i \left|U_{e\parallel\,i}\right|^2}} \tag{54}$$

where subscripts i and j represent grid points in the spatial and velocity coordinates, respectively. The long-time evolutions of the zeroth and first order moments of h_e are also compared with those of the uncorrected Maxwellian case. Figure 3 demonstrates that, for the uncorrected Maxwellian case, the normal-

Figure 2: Phase space structures of $Re(h_e)$ at (a) $t = 0(L/c_s)$, (b) $t = 0.001(L/c_s)$, (c) $t = 0.015(L/c_s)$, and (d) $t = 0.198(L/c_s)$.

ized zeroth order moment grows to the order of 10^{-2}, causing a non-negligible error in calculation of the density perturbation. On the other hand, using the corrected equilibrium distribution, F_{Mc}, the error is reduced to $\sim 10^{-10}$ at $t = 10\,(L/c_s)$. Conservation of the first order moments are also improved by use of F_{Mc}. The result confirms that separation of the zeroth and the first order moments from h_e is well maintained through out the numerical time-integration. The correction to the Maxwellian is, thus, essential to the ME method.

As discussed in section 3, accuracy of the present scheme is second order in time, because of the second-order midpoint implicit rule and the operator splitting. In order to confirm the orders of numerical accuracy in space and time, we define a measure of numerical errors, L_E

$$L_E = \sqrt{\frac{\sum_i \sum_j \left| f_{i,j} - f^{\rm ref}_{i,j} \right|^2}{\sum_i \sum_j \left| f^{\rm ref}_{i,j} \right|^2}}, \ . \tag{55}$$

where subscripts i and j represent grid points in the spatial and velocity coordinates, respectively. Figures 4 (a) and (b) show L_E at $t = 0.1364\,(2\pi/\omega)$ for $\beta_e = 0$ and 0.01, respectively. In these runs, other parameters are set to $N_v = 128$, and $k_\perp \rho_s = 0.01$. The initial condition is the same as those used in

Figure 3: Time evolutions of (a) the zeroth and (b) the first order moments of h_e. Result with the corrected Maxwellian F_{Mc} is compared with that of the uncorrected Maxwellian F_M case.

the above. The reference value, $f_{i,j}^{\rm ref}$, is obtained from a higher resolution run with $N_z = 256$ and $\Delta t = 1.0 \times 10^{-8}$. As shown in Fig.1, the cases for $\beta_e = 0$ and 0.01 correspond to the wave-propagation and the advection velocity dominant cases, respectively. It is found that L_E is in proportion to $(\Delta t)^2$ for large values of Δt. In other words, L_E is governed by errors due to the second order time integration scheme. In contrast, the norm of errors is independent of the time step size Δt for small values of Δt, where L_E is in proportion to $(\Delta z)^4$. The results shown in Figs. 4(a) and (b) confirm the expected spatial and temporal accuracy of the scheme.

The dispersion relation of the DAWs obtained by simulations with the ME

Figure 4: A measure of the numerical error, L_E, is plotted as a function of Δt with $t = 0.1364\,(2\pi/\omega)\,(L/c_s)$, $\delta f_e = 0.01 F_{Mc} \cos z$, $N_v = 128$, $N_z = 32, 64$, and 128 for (a) $\beta_e = 0$ and (b) $\beta_e = 0.01$.

method is given in Figs. 5(a) and (b) for $\beta_e = 0$ and 0.01, respectively. Here, we set $N_z = 32$, $N_v = 128$, and employ the same initial condition as mentioned above. In the case with $\beta_e = 0$, the time step size is set to $\Delta t = 4.0 \times 10^{-5}\ L/c_s$, where the Courant numbers for the advection (C_v) and for the DAW propagation (C_w) are 0.087 and $0.0175/k_\perp \rho_s$, respectively. On the other hand, the simulation for $\beta_e = 0.01$ is carried out with $\Delta t = 1.0 \times 10^{-4}\ L/c_s$, which corresponds to $C_v = 0.218$ and $C_w = 0.218\sqrt{1 - 17.3615/\left(18.3615 + k_\perp^2 \rho_s^2\right)}$. For the both cases, one finds that frequencies and growth rates numerically obtained agree well with the analytical results shown by the solid curves. It confirms that

Figure 5: Dispersion relation of the dispersive Alfvén waves for (a) $\beta_e = 0$ and $\beta_e = 0.01$ obtained by the moment extract method. The real frequency ω and the damping rate γ are compared with the analytical results shown by solid curves.

the linear propagation and the Landau damping of the DAW can be correctly simulated by means of the ME method without unphysical growth nor decay due to numerical instability.

Figure 6 shows dependence of the real and imaginary parts of frequencies on the Courant number for the DAW propagation C_w with $k_\perp \rho_s = 1.0 \times 10^{-2}$, $\beta_e = 0$ and $N_z = 32$. In this calculation, $C_w = 1$ corresponds to $\Delta t = 2.29 \times 10^{-5}\ L/c_s$. According to the definitions of C_v and C_w, we estimate $C_v \sim 0.049 C_w$. It is noteworthy that the time-integration can be successfully carried out even with much larger values of C_w than unity unless the parallel advection

Figure 6: Observed wave frequency (ω) and damping rate (γ) for $\beta_e = 0$ with $k_\perp \rho_s = 10^{-2}$ compared with the analytical estimate. Here, the vertical axis is scaled with factors of 10^{-3} and 10^3 for ω and γ, respectively.

term of electron motion limits the time step size. Figure 6 also shows that the wave frequency is underestimated for the large values of C_w, while the time integration is stably carried out. (Note that the damping rate γ of the DAW is quite small in the present parameter regime of $k_\perp \rho_s = 1.0 \times 10^{-2}$ and $\beta_e = 0$, and that the scale difference of 10^6 between ω_r and γ is applied to the vertical axis of Fig. 6.) The ME method with the IMEX integrator, thus, enables ones to carry out time integration of the drift kinetic equation with much larger Δt than the CFL limit for the conventional explicit scheme in the low β_e regime.

We have also compared computational time per a single time step between the moment extract method with the IMEX integrator (combination of the midpoint implicit rule and the fourth order RKG method) and the conventional drift kinetic simulation solving Eqs. (1)-(3) by means of the RKG method. The used grid numbers are $(N_z, N_v) = (32, 128)$, respectively, and we run the code $2,000$ steps with $\Delta t = 10^{-6}$ and $\beta_e = 0$. The computational speeds are nearly comparable (or the IMEX case was even slightly faster about 12 %). It means that increase of the computational cost due to the implicit solver for the fluid moment equations does not much influence the total performance, while cost reduction in the moment calculation is an advantage. In the conventional

approach, one must compute the charge and current density by means of the velocity space integral of δf_e while one needs only the temperature perturbation computed from the second order moment of h_e. However, the computational cost for the explicit scheme should be reduced if we employ a second order explicit scheme. Nevertheless, the main advantage of the ME scheme is the stability for a larger time step size than the CFL condition as shown in Fig. 6 in the case of $\beta_e < m_e/m_i$. In nonlinear drift wave turbulence simulations, the most expensive computations (or internode communications as well) are devoted to nonlinear terms (such as the $E \times B$ convection) with the spectral method. Computational time used for the parallel advection is typically minor. Thus, if we could take a larger time step size by means of the ME method, it should be a figure of merit in reducing the total computational time, even with a slight increase of computational costs per a single time step.

6. Concluding remarks

In this paper, we have discussed the moment extract (ME) method, where the low order moment equations of electrons are separated from the drift kinetic equation (DKE) while the remnant part of the distribution function is fully taken into account. The ME approach is analytically equivalent to the original DKE, but has an advantage in the numerical time integration. As the wave propagation is governed by the low order moment (fluid) equations, the time integration can be efficiently carried out by decoupling the dispersive Alfvén wave propagation and the thermal motion of electrons. Application of the implicit solver to the low order moment equations realizes stable time integration of DKE with larger Δt than that for the explicit scheme in the low-β_e regime of $\beta_e < m_e/m_i$. Also, the matrix size inverted for time-integration is smaller than that of the full implicit scheme. It is expected that numerical simulations of plasma instabilities such as the electrostatic ITG instability will benefit from the present method because one does not need to take into account of the limitation of Δt due to the dispersive Alfvén wave propagation.

For $\beta_e > m_e/m_i$, where the electron thermal velocity exceeds the DAW advection speed, the CFL condition for the explicit time integration is given by the parallel electron motion not by the DAWs propagation. In the case, we may apply the semi-Lagrangian scheme (or the splitting scheme) [27, 28, 29] to Eq. (15) with the ME approach, which should be useful to relax the limitation of the time-step size due to the parallel electron motion.

For application of the present scheme to the drift wave instabilities and turbulence , we need to extend the code so as to include effects of gyrokinetic ions, the toroidal geometry, the finite collisionality, and the magnetic, diamagnetic and $E \times B$ drifts. Time scales of drift waves with the perpendicular wavelengths characterized by the ion acoustic or thermal gyroradius, such as the ITG modes, are longer than that of the DAWs, when the ion acoustic or thermal speed is slower than the Alfvén speed or the phase velocity of DAWs. As the time scales of drift waves are given by the parallel and perpendicular motion of ion gyrocenters or the precession drift motion of electrons, stiffness of the governing equations is brought by the parallel propagation of DAWs (or by the parallel electron motion if $\beta_e > m_e/m_i$ as discussed above). Thus, the perpendicular drift terms, including the $E \times B$ nonlinear term, in the toroidal gyrokinetic equation are regarded as non-stiff terms and may be handled by means of explicit integrators. Extensions to turbulence or toroidal systems remain for future works.

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