RESEARCH REPORTS

ON THE EQUATION OF LONGITUDINAL VIBRATION OF A CYLINDER WITH MODERATE THICKNESS

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1. Preliminaries and Notations

The equation of longitudinal vibration of a thin rod, the cross-sectional form of which is freely chosen, is expressed by:

$$\rho\frac{\partial^2}{\partial t^2}g = E\frac{\partial^2}{\partial z^2}g,$$

where g is longitudinal displacement. The notations used here are given at the end of this paragraph. Taking into account the lateral motion of the material points of the bar, the equation is modified into;

$$\rho\frac{\partial^2}{\partial t^2}g - K^2\sigma^2\rho\frac{\partial^4}{\partial t^2\partial z^2}g = E\frac{\partial^2}{\partial z^2}g,$$

where K is the radius of gyration of a cross-section about the central line.

In this paper, begining with the equations of elasticity, we are going to deduce the equation of the longitudinal vibration of a circular cylinder of radius a, in which case K^2 corresponds to $a^2/2$, taking into account the higher order deformations of the cross-sectional plane. The radius of the bar being small, the approximative procedures can be generally performed successively. The detailed calculations are carried out up to the terms of order K^4.

Notations

r, ϑ, z: cylindrical coordinates,

u_r, u_ϑ, u_z: components of displacement,

ρ: density,

t: time,

λ, μ: Lamé's constants.

We shall write in order to simplify denotations as follows:

$$(m, n) = m\lambda + n\mu,$$

$$(l, m, n) = l\lambda^2 + m\lambda\mu + n\mu^2, \text{ etc.}$$

$$E = \mu(3, 2)/(1, 1): \text{ Young's modulus},$$

$$\sigma = \lambda/\{2(1, 1)\}: \text{ Poisson's ratio}.$$

We also use the following abbreviations.

$$[\alpha,\ \beta]=\alpha\rho\frac{\partial^2}{\partial t^2}+\beta\frac{\partial^2}{\partial z^2},$$

$$[\alpha,\ \beta,\ \gamma]=\alpha\rho^2\frac{\partial^4}{\partial t^4}+\beta\rho\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2}+\gamma\frac{\partial^4}{\partial z^4},\quad \text{etc.}$$

2. Fundamental Equations and Longitudinal Vibration of a Circular Cylinder

The equations of motion, when no body force exists, are written as;

$$\left.\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2}&=(1,\ 2)\frac{\partial\Theta}{\partial r}-\frac{2\mu}{r}\frac{\partial\tilde{\omega}_z}{\partial\vartheta}+2\mu\frac{\partial\tilde{\omega}_\vartheta}{\partial z},\\
\rho\frac{\partial^2 u_\vartheta}{\partial t^2}&=(1,\ 2)\frac{1}{r}\frac{\partial\Theta}{\partial\vartheta}-2\mu\frac{\partial\tilde{\omega}_r}{\partial z}+2\mu\frac{\partial\tilde{\omega}_z}{\partial r},\\
\rho\frac{\partial^2 u_z}{\partial t^2}&=(1,\ 2)\frac{\partial\Theta}{\partial z}-\frac{2\mu}{r}\frac{\partial}{\partial r}(r\tilde{\omega}_\vartheta)+\frac{2\mu}{r}\frac{\partial\tilde{\omega}_r}{\partial\vartheta},
\end{aligned}\right\}\tag{1}$$

where

$$\left.\begin{aligned}
2\tilde{\omega}_r&=\frac{1}{r}\frac{\partial u_z}{\partial\vartheta}-\frac{\partial u_\vartheta}{\partial z},\\
2\tilde{\omega}_\vartheta&=\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r},\\
2\tilde{\omega}_z&=\frac{1}{r}\left(\frac{\partial(ru_\vartheta)}{\partial r}-\frac{\partial u_r}{\partial\vartheta}\right),
\end{aligned}\right\}\tag{2}$$

and

$$\Theta=\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}+\frac{\partial u_z}{\partial z}.\tag{3}$$

Let the cylinder occupy the region: $r\leqq a,\ -\infty<z<+\infty$, and let us take no surface traction acting on the peripheral surface. Considering the symmetric properties of the displacement of the longitudinal vibration, which is finite at the central line $r=0$, we put

$$u_r=\sum_{n=0}^{\infty}f_n r^{2n+1},$$

$$u_z=\sum_{n=0}^{\infty}g_n r^{2n},$$

$$u_\vartheta=0,\qquad \frac{\partial}{\partial\vartheta}=0.$$

Then, from (2) and (3) we obtain:

$$\tilde{\omega}_r=\tilde{\omega}_z=0,$$

$$\tilde{\omega}_\vartheta=\sum_{n=0}^{\infty}\left\{\frac{1}{2}\frac{\partial f_n}{\partial z}-(n+1)g_{n+1}\right\}r^{2n+1},$$

$$\Theta=\sum_{n=0}^{\infty}\left\{2(n+1)f_n+\frac{\partial g_n}{\partial z}\right\}r^{2n}.$$

The equations of motion (1) become:

$$\left.\begin{aligned}\sum_{n=0}^{\infty}\rho\frac{\partial^2 f_n}{\partial t^2}r^{2n+1}=\sum_{n=0}^{\infty}\Big[\,2(1,\ 2)\ (n+1)\left\{2(n+2)f_{n+1}+\frac{\partial g_{n+1}}{\partial z}\right\}\\ +\mu\left\{\frac{\partial^2 f_n}{\partial z^2}-2(n+1)\ \frac{\partial g_{n+1}}{\partial z}\right\}\Big]r^{2n+1},\\ \sum_{n=0}^{\infty}\rho\frac{\partial^2 g_n}{\partial t^2}r^{2n}=\sum_{n=0}^{\infty}\Big[\ (1,\ 2)\left\{2(n+1)\ \frac{\partial f_n}{\partial z}+\frac{\partial^2 g_n}{\partial z^2}\right\}\\ -2\,\mu(n+1)\left\{\frac{\partial f_n}{\partial z}-2(n+1)\,g_{n+1}\right\}\Big]r^{2n}.\end{aligned}\right\}\qquad(4)$$

Equating the coefficients of the same powers of r in (4), we obtain a system of equations as follows:

$$\left.\begin{aligned}\rho\frac{\partial^2 f_n}{\partial t^2}&=\mu\frac{\partial^2 f_n}{\partial z^2}+4\,(1,\ 2)\,(n+1)\,(n+2)f_{n+1}+2\,(1,\ 1)\,(n+1)\frac{\partial g_{n+1}}{\partial z},\\ \rho\frac{\partial^2 g_n}{\partial t^2}&=(1,\ 2)\frac{\partial^2 g_n}{\partial z^2}+4\,\mu(n+1)^2 g_{n+1}+2(1,\ 1)\,(n+1)\frac{\partial f_n}{\partial z}.\\ &\qquad (n=0,\ 1,\ 2,\ 3,\ \ldots\ .)\end{aligned}\right\}\qquad(5)$$

The boundary conditions, that the cylinder is free from surface traction, are:

$$r=a\,;\qquad \lambda\Theta+2\,\mu\frac{\partial u_r}{\partial r}=0,$$

$$\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}=0,$$

i.e.

$$\left.\begin{aligned}0&=\sum_{n=0}^{\infty}\left\{2(n+1,\ 2n+1)f_n+\lambda\frac{\partial}{\partial z}g_n\right\}a^{2n},\\ 0&=\sum_{n=0}^{\infty}\left\{\frac{\partial f_n}{\partial z}+2\,(n+1)g_{n+1}\right\}a^{2n}.\end{aligned}\right\}\qquad(6)$$

3. Approximative Calculations for Moderately Thick Bar

From (5) we obtain:

$$\left.\begin{aligned}g_{n+1}&=\frac{1}{4\,\mu(n+1)^2}[1,\ -(1,\ 2)]g_n-\frac{(1,\ 1)}{2\,\mu(n+1)}\ \frac{\partial}{\partial z}f_n,\\ f_{n+1}&=\frac{1}{4\,(1,\ 2)\,(n+1)\,(n+2)}[1,\ -\mu]f_n-\frac{(1,\ 1)}{2\,(1,\ 2)\,(n+2)}\ \frac{\partial}{\partial z}g_{n+1}\\ &=\frac{1}{4\,(1,\ 2)\,(n+1)\,(n+2)}\left[1,\ \frac{\lambda(1,\ 2)}{\mu}\right]f_n\\ &\quad-\frac{(1,\ 1)}{8\,\mu(1,\ 2)\,(n+1)^2\,(n+2)}[1,\ -(1,\ 2)]\frac{\partial}{\partial z}g_n.\end{aligned}\right\}\qquad(7)$$

Introducing a new variable φ:

$$\varphi=f_0+\frac{\lambda}{2(1,\ 1)}\ \frac{\partial g_0}{\partial z},$$

and taking operators G_n, $\overline{G}_n$, F_n and $\overline{F}_n$, which are successively defined as;

$$\left.\begin{aligned} g_n &= G_n g_0 + \overline{G}_n \varphi, \\ f_n &= F_n g_0 + \overline{F}_n \varphi, \qquad (n = 1,\ 2,\ 3,\ \ldots\ .) \end{aligned}\right\} \tag{8}$$

we obtain, from (6) and (7),

$$0 = \varphi + \sum_{i=1}^{\infty} (C_i g_0 + \overline{C}_i \varphi) a^{2i}, \tag{9}$$

together with

$$\left.\begin{aligned} C_i &= \frac{(i+1,\ 2i+1)}{(1,\ 1)} F_i + \frac{\lambda}{2(1,\ 1)} \frac{\partial}{\partial z} G_i, \\ \overline{C}_i &= \frac{(i+1,\ 2i+1)}{(1,\ 1)} \overline{F}_i + \frac{\lambda}{2(1,\ 1)} \frac{\partial}{\partial z} \overline{G}_i, \end{aligned}\right\} \tag{10}$$

and

$$0 = [1,\ -E] g_0 + H\varphi + \sum_{i=1}^{\infty} (D_i g_0 + \overline{D}_i \varphi) a^{2i}, \tag{11}$$

together with

$$\left.\begin{aligned} D_i &= 2\mu \frac{\partial}{\partial z} F_i + 4\mu(i+1) G_{i+1}, \\ \overline{D}_i &= 2\mu \frac{\partial}{\partial z} \overline{F}_i + 4\mu(i+1) \overline{G}_{i+1}, \end{aligned}\right\} \tag{12}$$

where

$$H = -2\lambda \frac{\partial}{\partial z}.$$

Some of the explicit expressions of the operators in (8), (10) and (12), are as follows:

$$G_1 = \frac{1}{4\mu}[1,\ -2\mu], \qquad \overline{G}_1 = -\frac{(1,\ 1)}{2\mu} \frac{\partial}{\partial z},$$

$$G_2 = \frac{1}{2^6 \mu^2}\left[1,\ -\frac{\mu(3,\ 7)}{(1,\ 2)},\ 3\mu^2\right], \qquad \overline{G}_2 = \frac{-(1,\ 1)}{2^5 \mu^2}\left[\frac{(1,\ 3)}{(1,\ 2)},\ -2\mu\right]\frac{\partial}{\partial z},$$

$$G_3 = \frac{1}{3^2 \cdot 2^8 \mu^3}\left[1,\ -\frac{\mu(4,\ 17,\ 19)}{(1,\ 2)^2},\ 3\mu^2 \frac{(2,\ 5)}{(1,\ 2)},\ -4\mu^3\right],$$

$$F_1 = \frac{1}{2^4 \mu(1,\ 1)}\left[\frac{-(1,\ 3,\ 1)}{(1,\ 2)},\ \mu(2,\ 1)\right]\frac{\partial}{\partial z},$$

$$\overline{F}_1 = \frac{1}{2^3(1,\ 2)}\left[1,\ \frac{\lambda(1,\ 2)}{\mu}\right],$$

$$F_2 = \frac{-1}{3 \cdot 2^7 \mu^2 (1,\ 1)(1,\ 2)}\left[\frac{(1,\ 5,\ 8,\ 3)}{(1,\ 2)},\ -\mu(3,\ 10,\ 5),\ \mu^2(1,\ 2)(3,\ 2)\right]\frac{\partial}{\partial z},$$

$$\overline{F}_2 = \frac{1}{3 \cdot 2^6 (1,\ 2)}\left[\frac{1}{(1,\ 2)},\ \frac{(1,\ 4,\ 1)}{\mu^2},\ -\frac{(1,\ 2)(2,\ 1)}{\mu}\right],$$

$$C_1 = \frac{-1}{2^4 (1,\ 1)^2}\left[\frac{(3,\ 7,\ 3)}{(1,\ 2)},\ -\mu(4,\ 3)\right]\frac{\partial}{\partial z},$$

$$\overline{C}_1 = \frac{1}{2^3 (1,\ 1)}\left[\frac{(2,\ 3)}{(1,\ 2)},\ \lambda\right],$$

$$C_2 = \frac{-1}{3 \cdot 2^7 \mu (1,\ 1)^2 (1,\ 2)}\left[\frac{(5,\ 25,\ 37,\ 15)}{(1,\ 2)},\ -\mu(15,\ 44,\ 25),\ 2\mu^2(1,\ 2)(6,\ 5)\right]\frac{\partial}{\partial z},$$

$$D_1 = \frac{1}{2^3 \mu (1,\ 1)}\left[(1,\ 1),\ -\frac{\mu(4,\ 13,\ 8)}{(1,\ 2)},\ \mu^2(5,\ 4)\right],$$

$$\overline{D}_1 = -\frac{1}{4\mu}\left[\frac{(1,4,2)}{(1,2)},\ -\mu(3,2)\right]\frac{\partial}{\partial z},$$

$$D_2 = \frac{1}{3\cdot 2^6\mu^2(1,1)(1,2)}\Big[(1,1)(1,2),\ \frac{-\mu(5,26,44,22)}{(1,2)},$$

$$\mu^2(9,31,20),\ -\mu^3(1,2)\ (7,6)\Big].$$

Approximative Procedure

I. *The zero-th order approximation*

From (9) and (11), neglecting the terms of $O(a^2)$, we obtain:

$$\varphi = 0 \qquad \text{and} \qquad 0 = [1,\ -E]g_0, \tag{13}$$

resulting the usual equation for a thin rod or bar.

II. *The first order approximation*

Inserting (13) into the terms of $O(a^2)$ of (9) and (11), and neglecting the quantities of $O(a^4)$, we get

$$C_1 = C_1' = \frac{-\lambda\mu(5,12,6)}{2^4(1,1)^3(1,2)}\frac{\partial^3}{\partial z^3},$$

$$D_1 = D_1' = \frac{\lambda^2\mu}{2^2(1,1)(1,2)}\frac{\partial^4}{\partial z^4},$$

$$\left.\begin{aligned}&\varphi = -a^2C_1g_0 = -a^2C_1'g_0,\\ &0 = [1,\ -E]g_0 + a^2(D_1' - HC_1')g_0,\end{aligned}\right\} \tag{14}$$

where the operators with a prime ′, represent those in which $\rho\frac{\partial^2}{\partial t^2}$ is replaced by $E\frac{\partial^2}{\partial z^2}$.

The detailed calculation of (14) leads to the following:

$$0 = [1,\ -E]g_0 - \frac{a^2}{2}\sigma^2E\frac{\partial^4}{\partial z^4}g_0. \tag{15}$$

The equation (15) coincides with the one obtained by taking into account the inertia of the lateral motion in the cross-sectional plane of the bar.

III. *The second order approximation*

Inserting (13) and (14) respectively into the terms of $O(a^4)$ and $O(a^2)$ of (9) and (11), we have:

$$C_1 = C_1' + a^2C_1'', \qquad \overline{C}_1 = \overline{C}_1', \qquad C_2 = C_2',$$

$$D_1 = D_1' + a^2D_1'', \qquad \overline{D}_1 = \overline{D}_1', \qquad D_2 = D_2',$$

$$\left.\begin{aligned}&\varphi = -a^2C_1'g_0 + a^4Cg_0,\\ &0 = [1,\ -E]g_0 + a^2(D_1' - HC_1')g_0 + a^4(HC + D_1'' + D_2' - \overline{D}_1'C_1')g_0,\end{aligned}\right\} \tag{16}$$

with $$C = \overline{C}_1'C_1' - C_1'' - C_2',$$

where the double primed operators are those in the terms of $O(a^2)$ in C_1 and D_1.

After some complicated calculations, we obtain:

$$\overline{C}_1' = \frac{(1,\ 9,\ 15,\ 6)}{2^3(1,\ 1)^2(1,\ 2)} \frac{\partial^2}{\partial z^2},$$

$$C_1'' = \frac{-\lambda^2\mu(3,\ 2)(3,\ 7,\ 3)}{2^7(1,\ 1)^5(1,\ 2)} \frac{\partial^5}{\partial z^5},$$

$$C_2' = \frac{-\lambda\mu(6,\ 35,\ 65,\ 45,\ 10)}{3\cdot 2^6(1,\ 1)^4(1,\ 2)^2} \frac{\partial^5}{\partial z^5},$$

$$\overline{D}_1' = \frac{-\lambda\mu(3,\ 2)}{2^2(1,\ 1)(1,\ 2)} \frac{\partial^3}{\partial z^3},$$

$$D_1'' = \frac{\lambda^3\mu(2,\ 3)(3,\ 2)}{2^6(1,\ 1)^4(1,\ 2)} \frac{\partial^6}{\partial z^6},$$

$$D_2' = \frac{\lambda^2\mu(1,\ 6,\ 7,\ 2)}{3\cdot 2^5(1,\ 1)^3(1,\ 2)^2} \frac{\partial^6}{\partial z^6},$$

$$C = \frac{-\lambda\mu}{3\cdot 2^6(1,\ 1)^5(1,\ 2)^2}(-12,\ -23,\ 68,\ 208,\ 170,\ 44)\frac{\partial^5}{\partial z^5},$$

and

$$HC + D_1'' + D_2' - \overline{D}_1' C_1' = \frac{\lambda^2\mu}{3\cdot 2^5(1,\ 1)^5(1,\ 2)}(-2,\ 13,\ 56,\ 52,\ 14)\frac{\partial^6}{\partial z^6}.$$

The equation (16) becomes:

$$0 = [1,\ -E]g_0 - K^2\sigma^2 E\frac{\partial^4}{\partial z^4}g_0 + K^4\sigma^2\frac{\mu(-2,\ 13,\ 56,\ 52,\ 14)}{3\cdot 2(1,\ 1)^3(1,\ 2)}\frac{\partial^6}{\partial z^6}g_0, \tag{17}$$

where $K = a/\sqrt{2}$ is the radius of gyration of a cross-section about the central line.

In the second term of (17), replacing $E\frac{\partial^2}{\partial z^2}$ by $\left(\rho\frac{\partial^2}{\partial t^2} - K^2\sigma^2 E\frac{\partial^4}{\partial z^4}\right)$, we obtain:

$$0 = \rho\frac{\partial^2}{\partial t^2}g_0 - E\frac{\partial^2}{\partial z^2}g_0 - K^2\sigma^2\rho\frac{\partial^4}{\partial t^2\partial z^2}g_0 + K^4\sigma^2\frac{(5,\ 50,\ 124,\ 104,\ 28)}{3\cdot 2^2(1,\ 1)^2(1,\ 2)(3,\ 2)}\rho\frac{\partial^6}{\partial t^2\partial^4 z}g_0. \tag{18}$$

Putting

$$g_0 = g_0 e^{i(\omega t - \gamma z)},$$

into (17) or (18), we have the velocity V of propagation of the wave:

$$V = \frac{\omega}{\gamma} = \sqrt{\frac{E}{\rho}}\cdot\left\{1 - \frac{1}{2}K^2\sigma^2\gamma^2 - K^4\sigma^2\frac{(-7,\ 128,\ 460,\ 416,\ 112)}{3\cdot 2^5(1,\ 1)^2(1,\ 2)(3,\ 2)}\gamma^4\right\}.$$

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