RESEARCH REPORTS

ON THE NUMBER OF TYPES OF SYMMETRIC BOOLEAN OUTPUT MATRICES

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1. Introduction

The operation performed by a combinational switching network can be described by a Boolean output matrix[1] (*BO-matrix*), whose rows and columns correspond to the terminals 1, 2, . . . , m, say, and ij-entry is a Boolean function f_{ij} of the input variables $x_1, \ldots, x_n$, say, representing the states of connection between the terminals i and j. The diagonal entries are all 1, because every terminal can be considered to be always connected to itself. If all the contact elements of the network are bilateral, we have $f_{ij}=f_{ji}$, i.e., the matrix is symmetric. But if some of the contact elements are unilateral, the matrix is not generally symmetric.

There exist many different real networks which perform a given operation described by a BO-matrix, and it is a very important but still open problem to select the simplest or at least economical ones from among them. However, we will not take care of this aspect of problem. On the contrary, we will not differentiate networks with the same BO-matrix and even more. That is: If a BO-matrix $\mathbf{F}'$ can be obtained from a given BO-matrix $\mathbf{F}$ by permuting its rows and columns, and (or) by permuting the input variables, and (or) by complementing some of them, we will consider that the networks corresponding to $\mathbf{F}$ and $\mathbf{F}'$ are all physically the same.

It will be convenient to define two BO-matrices in the above relation to each other to be of the same type. Then there exist as many different physical networks as types of BO-matrices. The main purpose of this paper consists in the enumeration of the types of BO-matrices in the above mentioned sense. Precisely, we will consider symmetric BO-matrices only, and the treatment of the general case will be put off to later occasions.

2. Boolean Output Matrix

BO-matrix can be characterized as an $m\times m$ matrix $\mathbf{F}=(f_{ij})$ over a Boolean algebra generated by a finite set of elements $x_1, \ldots, x_n$, satisfying the conditions

$$f_{ii}=1 \qquad i=1, \ldots, m \tag{2.1}$$

and

$$\mathbf{F}^2=\mathbf{F}. \tag{2.2}$$

Expanding a BO-matrix F by 2^n fundamental products p_k $(k=0, 1, \ldots, 2^n-1)$ formed from $x_1, \ldots, x_n$, we have

$$\mathbf{F}=\sum_{k=0}^{2^n-1} p_k \mathbf{A}_k, \tag{2.3}$$

where $\mathbf{A}_k$ are $m \times m$ matrices over a two-element Boolean algebra, i.e., their entries are either 1 or 0.

It can easily be seen that the properties of F, i.e., (2.1) and (2.2), induce the same on each $\mathbf{A}_k$. Conversely, if each $\mathbf{A}_k$ satisfies these properties, then any matrix of the form of (2.3) is a BO-matrix. If, in addition, F is symmetric, then each $\mathbf{A}_k$ is also symmetric and vice versa.

Since there holds a one-to-one correspondence between an $m \times m$ matrix over a two-element Boolean algebra and a dyadic relation defined in a finite set of m elements, we may call it a relation matrix or *R-matrix* for short. A symmetric $m \times m$ R-matrix satisfying the conditions (2.1) and (2.2) corresponds to a reflexive, symmetric, and transitive relation defined in a finite set of m elements, and consequently to a partition of the set. Therefore we may call it a partition matrix or ***P-matrix*** for short. Thus a symmetric BO-matrix is equivalent to a sequence of 2^n P-matrices. When we denote the number of possible $m \times m$ P-matrices by $\Phi(m)$, the total number of possible symmetric BO-matrices is given by $(\Phi(m))^{2^n}$. As for $\Phi(m)$, the following generating function is well known

$$e^{e^t-1}=\sum_{m=0}^{\infty} \frac{\Phi(m)}{m!} t^m. \tag{2.4}$$

3. Computing Principle

A permutation of the rows and columns $1, 2, \ldots, m$, and (or) a permutation of the input variables $x_1, \ldots, x_n$, and (or) a complementation of some of them, will induce a permutation of $(\Phi(m))^{2^n}$ BO-matrices among each other.

R-matrices of order $(\Phi(m))^{2^n}$ describing these permutations of BO-matrices constitute a representation $\mathfrak{D}$ of the group $\mathfrak{S}_m \times \mathfrak{O}_n$, where $\mathfrak{S}_m$ is the symmetric group of m-th order, and $\mathfrak{O}_n$ is the hyperoctahedral group of n-th order. It can be observed that this representation is reducible and contains as many identity representations as the types of symmetric BO-matrices. Denoting the number of types of symmetric BO-matrices by $N_{m,n}$, we obtain, from the theorem of group representation,

$$N_{m,n}=\frac{1}{m!\,2^n n!} \sum_{c} n_c \chi_c, \tag{3.1}$$

where the summation is taken over all the conjugate classes C of the group $\mathfrak{S}_m \times \mathfrak{O}_n$, n_c is the number of elements of class C, and χ_c is the character of the class C in the representation $\mathfrak{D}$.

A conjugate class of $\mathfrak{S}_m$ is given by a symbol $(\mu_1, \ldots, \mu_m)$ or (μ) for short, specifying the cycle structure of the class. The number of elements of the class $(\mu_1, \ldots, \mu_m)$ is given by

$$h_\mu=\frac{m!}{\prod_{j=1}^{m} \mu_j!\, j^{\mu_j}}, \tag{3.2}$$

and the number of classes is equal to that of partitions of m.

A conjugate class of $\mathfrak{O}_n$ is given by a symbol $(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ or $(\alpha ; \beta)$ for short, where $(\alpha_1, \ldots, \alpha_n)$ specifies the cycle structure of permutations of $x_1, \ldots, x_n$, and $(\beta_1, \ldots, \beta_n)$ specifies the complementation structure. Precisely β_j $(j=1, \ldots, n)$ is the number of e-cycles—the cycles in which even number of the variables are subjected to complementation—of length j.

The number of elements of the class $(\alpha ; \beta)$ is given by

$$h_{\alpha,\beta} = n! \prod_{i=1}^{n} \frac{2^{(i-1)\alpha_i}}{\beta_i! \, (\alpha_i - \beta_i)! \, i^{\alpha_i}} \tag{3.3}$$

and the number of classes of $\mathfrak{O}_n$ is given by

$$\sum_{\sum_i i\alpha_i = n} \prod_{i=1}^{n} (\alpha_i + 1). \tag{3.4}$$

As a conjugate class of a product group is a direct product of conjugate classes of factor groups, (3.1) can be rewritten as

$$N_{m,n} = \frac{1}{m! \, 2^n n!} \sum_{\mu ; \alpha, \beta} h_\mu h_{\alpha,\beta} \chi_{\mu ; \alpha, \beta}. \tag{3.1'}$$

Now, we can interprete $\chi_{\mu \; \alpha,\beta}$ as the number of symmetric BO-matrices which are invariant under the operation of the class $(\mu ; \alpha, \beta)$ of $\mathfrak{S}_m \times \mathfrak{O}_n$. Let ab be an element of the class $(\mu ; \alpha, \beta)$ where $a \in \mathfrak{S}_m$ and $b \in \mathfrak{O}_n$. The cycle structure of the permutation of 2^n foundamental products p_k induced by b, is uniquely determined by the class of b, and does not depend on the choice of b. Let $(p_{k_1}, \ldots, p_{k_r})$ be a typical cycle of the permutation of p_k. Since a transforms $\mathbf{A}_k$ into $a\mathbf{A}_k$, a necessary and sufficient condition that a symmetric BO-matrix $\mathbf{F} = \sum_{k=1}^{2^n-1} p_k \mathbf{A}_k$ may be invariant under the operation of ab, can be given by

$$a\mathbf{A}_{k_1} = \mathbf{A}_{k_2}, \ a\mathbf{A}_{k_2} = \mathbf{A}_{k_3}, \ \ldots, \ a\mathbf{A}_{k_r} = \mathbf{A}_{k_1} \tag{3.5}$$

for each cycle. (3.5) implies $a^r \mathbf{A}_{k_i} = \mathbf{A}_{k_i}$ $(i=1, \ldots, r)$, i.e., $\mathbf{A}_{k_i}$ are invariant under the operation of a^r. Let $\mathbf{A}^{(j)}$ $(j=1, \ldots, \xi_{\mu^r})$ be all the P-matrices invariant under the operation of a^r, then putting $\mathbf{A}_{k_i} = a^{i-1}\mathbf{A}^{(j)}$ $(i=1, \ldots, r ; j=1, \ldots, \xi_{\mu^r})$, we can obtain ξ_{μ^r} solutions for this cycle. Clearly ξ_{μ^r} depends only on the class (μ), and the length r of the cycles of the permutation of p_k. Thus, $\chi_{\mu ; \alpha,\beta}$ can be given by

$$\chi_{\mu ; \alpha,\beta} = \prod_{r=1}^{2^n} \xi_{\mu^r}^{\nu_r}, \tag{3.6}$$

where $(\nu_1, \ldots, \nu_{2^n})$, or simply (ν) for short, is the cycle structure of the permutation of p_k induced by the class $(\alpha ; \beta)$. Suppose, now, that two classes $(\alpha ; \beta)$ and $(\alpha' ; \beta')$ induce the permutations of p_k with the same cycle structure (ν). Then the above reasoning shows that $\chi_{\mu ; \alpha,\beta} = \chi_{\mu ; \alpha',\beta'}$. Thus, summing up all $h_{\alpha,\beta}$ for the classes $(\alpha ; \beta)$ which induces the same cycle structure (ν), we can get

$$N_{m,n} = \frac{1}{m! \, 2^n n!} \sum_{\mu,\nu} h_\mu h_\nu \chi_{\mu,\nu}, \tag{3.7}$$

where h_ν is the number of elements of $\mathfrak{O}_n$ which induce the cycle structure (ν), and $\chi_{\mu,\nu}$ is given by (3.6).

A method to inquire after the cycle structure (ν) induced by a class $(\alpha ; \beta)$, has been worked out by Slepian.[2] Here we will not try to restate the detail of his method. We will only show the results for $n = 1, 2, 3, 4, 5$, obtained by his method in the following table, where k_ν is the number of cycles, and the ordering of (ν) is somewhat arbitrary.

TABLE 1. The Cycle Structures of the Permutations of the 2^n Fundamental Products Induced by the Elements of $\mathfrak{O}_n$

$n=1,\ 2^n=2,\ 2^n n!=2$

ν	ν_1	ν_2	k_ν	h_ν
1	2	0	2	1
2	0	1	1	1
				$\sum h_\nu=2$

$n=2,\ 2^n=4,\ 2^n n!=8$

ν	ν_1	ν_2	ν_4	k_ν	h_ν
1	4	0	0	4	1
2	2	1	0	3	2
3	0	2	0	2	3
4	0	0	1	1	2
					$\sum h_\nu=8$

$n=3,\ 2^n=8,\ 2^n n!=48$

ν	ν_1	ν_2	ν_3	ν_4	ν_6	k_ν	h_ν
1	8	0	0	0	0	8	1
2	4	2	0	0	0	6	6
3	2	0	2	0	0	4	8
4	0	4	0	0	0	4	13
5	0	1	0	0	1	2	8
6	0	0	0	2	0	2	12
							$\sum h_\nu=48$

$n=4,\ 2^n=16,\ 2^n n!=384$

ν	ν_1	ν_2	ν_3	ν_4	ν_6	ν_8	k_ν	h_ν
1	16	0	0	0	0	0	16	1
2	8	4	0	0	0	0	12	12
3	4	6	0	0	0	0	10	12
4	4	0	4	0	0	0	8	32
5	0	8	0	0	0	0	8	51
6	2	1	0	3	0	0	6	48
7	0	2	0	0	2	0	4	96
8	0	0	0	4	0	0	4	84
9	0	0	0	0	0	2	2	48
								$\sum h_\nu=384$

$n=5,\ 2^n=32,\ 2^n n!=3{,}840$

ν	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_8	ν_{10}	ν_{12}	k_ν	h_ν
1	32	0	0	0	0	0	0	0	0	32	1
2	16	8	0	0	0	0	0	0	0	24	20
3	8	12	0	0	0	0	0	0	0	20	60
4	8	0	8	0	0	0	0	0	0	16	80
5	0	16	0	0	0	0	0	0	0	16	231
6	4	2	4	0	0	2	0	0	0	12	160
7	4	2	0	6	0	0	0	0	0	12	240
8	0	4	0	6	0	0	0	0	0	10	240
9	2	0	0	0	6	0	0	0	0	8	384
10	0	4	0	0	0	4	0	0	0	8	720
11	0	0	0	8	0	0	0	0	0	8	520
12	0	1	0	0	0	0	0	3	0	4	384
13	0	0	0	2	0	0	0	0	2	4	320
14	0	0	0	0	0	0	4	0	0	4	480
											$\sum h_\nu=3{,}840$

In order to be able to compute $N_{m,n}$ by (3.7), there remains only one task, i.e., that of computing ξ_{μ^r}. Since ξ_{μ^r} can be given by $\xi_{\mu'}$ for a certain μ', it will be sufficient to compute ξ_μ for each class (μ) of $\mathfrak{S}_m$.

4. The Number of P-Matrices Invariant under Permutations of Rows and Columns with a Given Cycle Structure

Let $(\mu) = (\mu_1, \ldots, \mu_m)$ be a given cycle structure. As a typical permutation p with this structure, we take $(1)(2) \ldots (\mu_1)(\mu_1+1,\ \mu_1+2) \ldots (\mu_1+2\mu_2-1,\ \mu_1+2\mu_2) \ldots\ldots$ If we classify the rows and columns of an R-matrix $\mathbf{A} = (a_{ij})$ according to these cycles, $\mathbf{A}$ will be partitioned into blocks. First, we will seek for a necessary and sufficient condition, so that an R-matrix may be invariant under the permutation, p. Consider any block $B=(b_{ij})$ where $i=1, \ldots, h$ and $j=1, \ldots, k$. Let us define that b_{ij} and $b_{i'j'}$ are equivalent, if and only if $j-i \equiv j'-i'$ (mod. g), where g is the G.C.M. of h and k. Then a classification among b_{ij} will occur. We call the classes of b_{ij}, thus formed, *strings*. Thus B will be divided into g strings $S_0, S_1, \ldots, S_{g-1}$, where S_a is the string consisting of all b_{ij} such that $j-i \in a$ (mod. g). If it is necessary to specify the block to which a string belongs, we will write like $S_a(B)$.

Now, when p is applied to A, a_{ij} will permuted within each block, and it can be seen that the latter permutation is the product of cyclic permutations, of the form $(b_{st}, b_{s+1\,t+1}, \ldots, b_{s-1\,t-1})$ within each string where the two suffixes i and j of b_{ij} should be considered modulo h and k respectively.

Therefore A is invariant under p, if and only if the each string of the each block consists exclusively of 1 or 0, or symbolically $S=1$ or $S=0$. We call an R-matrix with this property an *S-matrix*, or more exactly an S-matrix with respect to p. Hereafter we will consider only S-matrices, so we need no longer take account of invariance under p.

Let us proceed to the consideration of an S-matrix A which is a P-matrix at the same time. At first, we will inquire after the condition that a diagonal block D of A should be a part of P-matrix, or, we may say, a solution. Let k be the order of D, then D is divided into k strings $S_0, S_1, \ldots, S_{k-1}$, where S_0 is the main diagonal and the others are sub-diagonals, and S_a and S_{k-a} make a transpose pair.

Lemma 1. The number of $k \times k$ reflexive and transitive R-matrices which are invariant under the cyclic permutation $(1, 2, \ldots, k)$ of the rows and columns, is equal to the number of divisors of k inclusive of 1 and k. Any R-matrix with these properties is always symmetric i.e., it is a P-matrix.

Proof. The *unit matrix* $S_0=1$, $S_i=0$, $(i \not\equiv 0 \pmod{k})$, is clearly a solution satisfying the conditions of the lemma. If, in a solution, $S_i=1$ for a certain i, $(i \not\equiv 0 \pmod{k})$, then transitivity implies $S_{ni}=1$ for $n=1, 2, \ldots, k$. Suppose that i is prime to k, then all the strings should be 1. This is the case of the *universal matrix*. Evidently the universal matrix is a solution satisfying the conditions of the lemma. Suppose, on the other hand, that i is not prime to k and let d be their G.C.M., then $S_0, S_d, \ldots, S_{k-d}$ should be all 1. Conversely, it can be readily proved that the R-matrix with the above k/d strings put to 1, and the other strings put to 0, is a solution satisfying the conditions of the lemma.

Since the G.C.M. of any i and k is a divisor of k, and conversely any divisor d of k is the G.C.M. of d and k, we can see that there exist as many solutions as divisors of k, whereby we consider the universal matrix and the unit matrix correspond to $d=1$ and $d=k$ respectively. Incidentally, the value of d thus associated to each solution, coincides with the number of components constituting the partition, corresponding to the solution, of a set of k elements.

From the above lemma, the number of solutions for a $k \times k$ diagonal block is equal to the number of divisors of its order k, and to each solution, there corresponds a divisor of k, and vice versa. We call the divisor of k associated to a solution for a $k \times k$ diagonal block, the *index* of the solution. Next, we take two diagonal blocks $D(k \times k)$ and $D'(k' \times k')$. Assume that D and D' are the solutions with the indices d and d' respectively. Then how many solutions are there for the pair of non-diagonal blocks $\mathbf{B}(k \times k')$ and $\mathbf{B}^T(k' \times k)$ related to $\mathbf{D}$ and $\mathbf{D}'$?

Lemma 2. The number of solutions for the pair of non-diagonal blocks $\mathbf{B}(k \times k')$ and $\mathbf{B}^T(k' \times k)$ related to two diagonal blocks $\mathbf{D}(k \times k)$ and $\mathbf{D}'(k' \times k')$ with the indices d and d' respectively, is 1 or $d+1$ according as $d \neq d'$ or $d = d'$ respectively.

Proof. Zero matrices for $\mathbf{B}$ and $\mathbf{B}^T$ are clearly a solution, whether $d \neq d'$ or $d = d'$. In order to find solutions other than the trivial zero matrix, we assume $\mathbf{S}_c(\mathbf{B}) = \mathbf{1}$, where c is considered modulo g, the G.C.M. of k and k'. From the symmetry, we have $\mathbf{S}_{-c}(\mathbf{B}^T) = \mathbf{1}$.

First, assume $d \neq d'$. We may take $d < d'$ without loss of generality. Now, the transitivity applied to $\mathbf{S}_d(\mathbf{D}) = \mathbf{1}$ and $\mathbf{S}_c(\mathbf{B}) = \mathbf{1}$, gives $\mathbf{S}_{c+d}(\mathbf{B}) = \mathbf{1}$ and again applied to $\mathbf{S}_{-c}(\mathbf{B}^T) = \mathbf{1}$ and $\mathbf{S}_{c+d}(\mathbf{B}) = \mathbf{1}$, gives $\mathbf{S}_d(\mathbf{D}') = \mathbf{1}$. But this is a contradiction, because the index of $\mathbf{D}'$ is d'. Consequently, there is no solution other than zero matrix in this case.

Next, assume $d = d'$, then d must be a divisor of g. Applying the transitivity to $\mathbf{S}_{sd}(\mathbf{D}) = \mathbf{1}$, $(s = 0, 1, \ldots, k/d - 1)$, and $\mathbf{S}_c(\mathbf{B}) = \mathbf{1}$, we obtain $\mathbf{S}_{c+sd}(\mathbf{B}) = \mathbf{1}$. Considering $c + sd$, $(s = 0, 1, \ldots, k/d - 1)$ modulo g, and taking account of the fact that d is a divisor of g, we see that the results, obtained above, coincide with $\mathbf{S}_{c+td}(\mathbf{B}) = \mathbf{1}$, $(t = 0, 1, \ldots, g/d - 1)$, as a whole.

It can be observed that the application of the transitivity to $\mathbf{S}_{sd}(\mathbf{D}) = \mathbf{1}$ and $\mathbf{S}_{c+td}(\mathbf{B}) = \mathbf{1}$ gives just as much as $\mathbf{S}_{c+td}(\mathbf{B}) = \mathbf{1}$, $(t = 0, 1, \ldots, g/d - 1)$. In a similar manner we obtain $\mathbf{S}_{-c+t'd}(\mathbf{B}^T) = \mathbf{1}$, $(t' = 0, 1, \ldots, g/d - 1)$, from $\mathbf{S}_{s'd}(\mathbf{D}') = \mathbf{1}$, $(s' = 0, 1, \ldots, k'/d - 1)$, and $\mathbf{S}_{-c}(\mathbf{B}^T) = \mathbf{1}$. Since the application of the transitivity to $\mathbf{S}_{c+td}(\mathbf{B}) = \mathbf{1}$ and $\mathbf{S}_{-c+t'd}(\mathbf{B}^T) = \mathbf{1}$ brings out $\mathbf{S}_{sd}(\mathbf{D}) = \mathbf{1}$ and $\mathbf{S}_{s'd}(\mathbf{D}') = \mathbf{1}$, and no more, and this arrangement is clearly symmetric, we conclude that this is a solution. Taking $c = 0, 1, \ldots, d-1$, d different solutions are obtained. If there exists any other solution, it must be of the following form,

$$\mathbf{S}_{c_r+td}(\mathbf{B}) = \mathbf{1}, \qquad \mathbf{S}_{-c_r+t'd}(\mathbf{B}^T) = \mathbf{1},$$
$$(r = 1, \ldots, l\,;\ t,\ t' = 0, 1, \ldots, g/d - 1)$$

where $c_r \not\equiv c_s \pmod{d}$, $(r \neq s,\ r, s = 1, \ldots, l)$.

But the transitivity applied to $\mathbf{S}_{c_1}(\mathbf{B}) = \mathbf{1}$, and $\mathbf{S}_{-c_2}(\mathbf{B}^T) = \mathbf{1}$, for instance, gives $\mathbf{S}_{c_1-c_2}(\mathbf{D}) = \mathbf{1}$, which is a contradiction.

Therefore, there is no non-trivial solution other than those given above. We call the d values of c the *indices* of solutions for non-diagonal blocks $\mathbf{B}$ and $\mathbf{B}^T$. Incidentally, we can interprete these non-trivial solutions as follows. That is: Each of these solutions corresponds to a one-to-one mapping from a set of elements $1, 2, \ldots, d$, with the cyclic order $(1, 2, \ldots, d)$ onto itself, preserving this cyclic order, whereby the index corresponds to the amount of shift in this mapping.

Now we proceed to the next step. When the indices of the solutions for all the diagonal blocks are given, how many solutions are there for the whole matrix? From the lemma 2, the non-diagonal blocks related to two diagonal blocks with different indices must be zero matrices. Therefore, the solutions for the groups of blocks related to the diagonal blocks with the same indices can be determined independently to each other, and the number of solutions for the entire matrix are given as the product of the numbers of solutions for all the different groups of blocks. Let a group consist of q diagonal blocks $D_1, \ldots, D_q$ with the same index d, and the related non-diagonal blocks B_{ij} $(ij=1, \ldots, q)$, where apparently B_{ij} is the block formed from the rows of D_i and the columns of D_j, and B_{ij} and B_{ji} form a transpose pair.

We first consider the solutions in which every non-diagonal blocks is non-trivial. Let us call these solutions as *full solutions.*

Lemma 3. The number of full solutions for a group of blocks related ot q diagonal blocks with the same index d is d^{q-1}.

Proof. As every non-diagonal block must be a non-trivial solution in the sense of the Lemma 2, there corresponds to each non-diagonal block B_{ij}, an index of the solution c_{ij}, where of course $c_{ji}=-c_{ij}$ from the symmetry. Because the transitivity is assured between the diagonal and the related non-diagonal blocks, or between the transpose pair of non-diagonal blocks by the fact that the each B_{ij} is a solution, it is sufficient that the transitivity should be established between two non-diagonal blocks B_{ij} and B_{jk} for $i \neq j$, $j \neq k$, and $i \neq k$ so that the whole group may be a solution.

Let B_{ij}, B_{jk} and B_{ik} be $h \times h'$, $h' \times h''$, and $h \times h''$ respectively, and g, g' and g'' be the G.C.M.'s of, h and h', h' and h'', and h and h'' respectively. When we apply the transitivity to $S_{c_{ij}+td}(B_{ij})=1$, $(t=0, 1, \ldots, g/d-1)$ and $S_{c_{jk}+t'd}(B_{jk})=1$, $(t'=0, 1, \ldots, g'/d)$, we obtain $S_c(B_{ik})=1$, where

$$c \equiv c_{ij}+c_{jk}+(t+t')\,d+ng+n'g' \pmod{g''},$$
$$(n=0, 1, \ldots, h/g-1\ ;\ n'=0, 1, \ldots, h''/g'-1).$$

We can rewrite the result obtained above, as

$$S_{c_{ij}+c_{jk}+t''d}(B_{ik})=1, \qquad (t''=0, 1, \ldots, g''/d-1).$$

Accordingly, if $c_{ij}+c_{jk}=c_{ik}$ for all the combinations of i, j and k $(i \neq j,\ j \neq k,\ i \neq k)$, this group is a solution. Then, how and in how many ways can we arrange c_{ij} to get a solution?

This is easy. That is: To begin with, we choose indices for $q-1$ blocks, B_{12}, $B_{13}, \ldots, B_{1q}$, for instance. This can be done independently. Therefore the number of choices of indices for these blocks is equal to d^{q-1}. For each choice, we determine the remaining c_{ij} using

$$c_{j1}=-c_{1j} \qquad (j=2, \ldots, q)$$

and

$$c_{ij}=c_{i1}-c_{j1} \qquad (i, j=2, \ldots, q\ ;\ i \neq j).$$

These arrangements of c_{ij} are readily proved to be consistent as the solutions. This completes the proof.

Next we consider general solutions for the group of blocks, i.e., solutions in which some of the non-diagonal blocks can be zero matrices. If there exist some zero blocks, it can be inferred that the diagonal blocks should be classified, in such a manner that any non-diagonal block related to two diagonal blocks belonging to different classes is zero matrix, and the sub-group of blocks related to any one class forms a full solution.

Lemma 4. The number of solutions, $\Phi(q, d)$, for a group of blocks consisting of q diagonal blocks with the same index d, and the related non-diagonal blocks can be computed by means of the following generating function

$$e^{\frac{1}{d}(e^t-1)}=\sum_{q=0}^{\infty}\frac{\Phi(q,d)}{d^q q!}t^q.$$

Proof. Let $(\lambda_1, \ldots, \lambda_q)$ be a structure of a classification of diagonal blocks. As the full solutions corresponding to different classes can be determined independently, the number of solutions for any one classification of the above structure can be given by

$$\prod_{i=1}^{q} d^{(i-1)\lambda_i}.$$

This can be rewritten as $d^{q-\sum_{i=1}^{q}\lambda_i}$ because of $\sum_{i=1}^{q} i\lambda_i = q$. As there are

$$\frac{q!}{\prod_{i=1}^{q}(i!)^{\lambda_i}\lambda_i!}.$$

of classifications with the same structure, the number of solutions for this structure, is

$$\frac{q!\,d^{q-\Sigma\lambda_i}}{\prod_{i=1}^{q}(i!)^{\lambda_i}\lambda_i!}.$$

Summing these quantities for all the structures, we obtain

$$\frac{\Phi(q,d)}{d^q q!}=\sum\frac{d^{-\Sigma\lambda_i}}{\prod_{i=1}^{q}(i!)^{\lambda_i}\lambda_i!}.$$

Putting $d=1$, we have

$$\frac{\Phi(q)}{q!}=\sum\frac{1}{\prod_{i=1}^{q}(i!)^{\lambda_i}\lambda_i!},$$

because of $\Phi(q,1)=\Phi(q)$.

Using the same reasoning as that which leads to the generation function of $\Phi(q)$, and considering that $\sum_{i=1}^{q}\lambda_i$ is the number of classes, we can conclude that

$$e^{\frac{1}{d}(e^t-1)}=\sum_{q=0}^{\infty}\frac{\Phi(q,d)}{d^q q!}t^q.$$

From the Lemma 4, we can derive the following results for $q=1, 2, 3, 4, 5, 6$

$$\begin{aligned}
\Phi(1, d) &= 1, \\
\Phi(2, d) &= 1+d, \\
\Phi(3, d) &= 1+3d+d^2, \\
\Phi(4, d) &= 1+6d+7d^2+d^3, \\
\Phi(5, d) &= 1+10d+25d^2+15d^3+d^4, \\
\Phi(6, d) &= 1+15d+65d^2+90d^3+31d^4+d^5.
\end{aligned}$$

Now, we are in a position to answer the final question.

Theorem 1. Let $(\rho_1, \ldots, \rho_L)$ be the alternative symbol for the given cycle structure $(\mu_1, \ldots, \mu_m)$, i.e., ρ_i is the length of i-th cycle and L is the number of cycles. Let d_i be any divisor of ρ_i. Suppose that rearranging the sequence $(d_1, \ldots, d_L)$, we find σ_1 of d_1, σ_2 of $d_2, \ldots, \sigma_L$ of d_L. Then the number of P-matrix ξ_μ in question is given by

$$\xi_\mu = \sum \prod_{j=1}^{L} \Phi(\sigma_j, d_j),$$

where the summation is taken over all the possible sequences $(d_1, \ldots, d_L)$.

The proof is immediate. The following table shows the result obtained by the theorem for $m=1, 2, 3, 4, 5, 6$.

TABLE 2

$m=2$, $m!=2$

μ	μ_1	μ_2	h_μ	ξ_μ
1	2	0	1	2
2	0	1	1	2

$\sum h_\mu=2$

$m=3$, $m!=6$

μ	μ_1	μ_2	μ_3	h_μ	ξ_μ
1	3	0	0	1	5
2	1	1	0	3	3
3	0	0	1	2	2

$\sum h_\mu=6$

$m=4$, $m!=24$

μ	μ_1	μ_2	μ_3	μ_4	h_μ	ξ_μ
1	4	0	0	0	1	15
2	2	1	0	0	6	7
3	0	2	0	0	3	7
4	1	0	1	0	8	3
5	0	0	0	1	6	3

$\sum h_\mu=24$

$m=5$, $m!=120$

μ	μ_1	μ_2	μ_3	μ_4	μ_5	h_μ	ξ_μ
1	5	0	0	0	0	1	52
2	3	1	0	0	0	10	20
3	1	2	0	0	0	15	12
4	2	0	1	0	0	20	7
5	0	1	1	0	0	20	5
6	1	0	0	1	0	30	4
7	0	0	0	0	1	24	2

$\sum h_\mu=120$

$m=6$, $m!=720$

μ	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	h_μ	ξ_μ
1	6	0	0	0	0	0	1	203
2	4	1	0	0	0	0	15	67
3	2	2	0	0	0	0	45	31
4	3	0	1	0	0	0	40	20
5	2	0	0	1	0	0	90	9
6	1	1	1	0	0	0	120	10
7	0	3	0	0	0	0	15	31
8	1	0	0	0	1	0	144	3
9	0	1	0	1	0	0	90	9
10	0	0	2	0	0	0	40	8
11	0	0	0	0	0	1	120	4

$\sum h_\mu=720$

5. Conclusion

The number of types of symmetric Boolean output matrices was investigated. This work is a generalization of Slepian's[2] work, containing his work as a special case for $m=2$, on the one hand. And it would be a generalization of Davis'[3] work, containing his work as a special case for $n=0$, on the other hand, if we had treated general Boolean matrix instead of symmetric Boolean output matrix. The reason why we did not treat general Boolean output matrix, is the same as that why Davis did not treat reflexive and transitive relations. The following table shows the results obtained by the method of this paper, for some combinations of moderate values of m and n. The figures for $m=2$ reproduce Slepian's results, and those for $n=0$ is nothing but the numbers of partitions of m.

TABLE 3

$n\rightarrow$ / $m\downarrow$	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	3	6	22	402	1,228,158	400,507,806,843,728
3	3	7	38	864	67,731,281	1,010,601,961,371,087,726	*
4	5	21	536	2,361,211	712,923,958,172,867	*	*
5	7	54	10,919	9,353,218,050	*	*	*
6	11	167	372,341	*	*	*	*

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