

THE BOLTZMANN-MAXWELL EQUATION AS THE FUNDAMENTAL EQUATION OF GASDYNAMICS

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1. Introduction

Boltzmann proposed his equation rather intuitively. Subsequently several authors reconsidered the same equation as based on Liouville's equation of a system of particles. We now know that B-M Eq.* is rightly applied to those cases in which we may reasonably expect certain predominant distribution functions with small fluctuations even though the states under consideration deviate considerably from thermal equilibrium¹⁾²⁾³⁾⁴⁾. When we consider gas dynamics as a boundary or initial value problem of B-M Eq., we must remember that in the process of reducing the exact dynamical expression of systems of particles to B-M Eq., we have performed operations of degeneration. We should note that, even though the values of a set of gross variables of finite number of two given states are similar, it is possible that the two distribution functions of molecules are entirely different.

One method of approach to obtaining the molecular distribution function by solving B-M Eq. was given by Enskog and Chapman and has been developed by Chapman, Cowling and Burnett¹⁾. They expand the function in a series, the first term being the Maxwell distribution function based on local quantities. The coefficients of the higher terms are taken as functions of derivatives of local state variables, temperature and pressure. Grad⁵⁾ proposed another method. In Grad's method the molecular distribution function is expanded in terms of Hermite polynomials. Although the first term of his series is the same Maxwell function as that of Enskog and Chapman, the coefficients of higher terms are given in terms not only of temperature and pressure but also of other variables, moments. We note that the first term is the Maxwell function in both Enskog's and Grad's methods.

The present author⁶⁾ suggested the possibility of a solution of B-M Eq., the first term being a function different from Maxwell's, viz.,

$$f_0 = \exp(\alpha + \sum \beta_{\xi} c_{\xi} + \sum \sum \gamma_{\xi\eta} c_{\xi} c_{\eta})$$

where c denotes the velocity of a particle while α , β , γ denote functions of space and time. This function f_0 expresses the most probable distribution under the condition that density, mass velocity and stress are given.

These three trials suggest that there is the possibility of more than one solution which simultaneously satisfies B-M Eq. The discrepancy comes from the difference in choice of the first term of the polynomials in terms of which the function is expanded, and in choice of independent gross variables. Enskog and Grad assume that the states under consideration do not deviate too far from

* Hereafter B-M Eq. is the abbreviation of the Boltzmann-Maxwell equation.

thermal equilibrium. On the other hand, the present author assumes from the beginning that the states under consideration are located near another equilibrium different from thermal equilibrium. As is well known, in slip flow we can easily see an example of equilibrium different from thermal equilibrium which is expressed by a function other than any of the above three functions.⁷⁾ From among the various possible solutions, we can select one only after comparing each with experimental results. It is essential to make the proper choice of one solution among many which are equally possible in the sense that B-M Eq. and given boundary conditions of gross variables are satisfied. This means that B-M Eq. is one of the necessary principles on which our problem is based. We do not yet know exactly the other principles. B-M Eq. alone does not answer the question as to which one of the approximate solutions of B-M Eq. satisfying a certain given set of boundary conditions given in terms of gross variables of finite number, is proper to the cases under consideration. We seem now to be searching for these principles by trial and error.⁸⁾

The essential points in obtaining solutions of B-M Eq. are

1) selection of most essential gross variables which are necessary to present states under consideration

2) the most probable distribution in the zeroth approximation under a given condition given in terms of a certain set of values of the essential gross variables and external restriction.

From these viewpoints, the problem facing us may be subdivided as follows:

- a) shock waves sufficiently apart from solid bodies
- b) flows adjacent to surfaces of solid bodies
- c) flows under magneto-hydrodynamical forces
- d) miscellaneous.

In shock waves, the states deviate from thermal equilibrium, gases being subjected to no external force, while in flows adjacent to surfaces of solid bodies the distribution function of gases under consideration is conveniently given by the sum of two functions which are entirely different⁷⁾. One presents the molecules impinging upon solid surface, and the other presents reflecting molecules. Here the latter is a Maxwell-type function corresponding to the temperature of solid surfaces and is independent of the mode of distribution of impinging molecules. That the distribution of molecules adjacent to surface is regulated in this way and that the state deviates from thermal equilibrium, may be certain. If we may expect that a certain predominant probable distribution mode of molecules even under the condition that molecules are under restriction of solid bodies, and that fluctuations are small, then the distribution functions are to be subject to B-M Eq. except for exact surfaces of solid bodies.

On the other hand, in flows under magneto-hydrodynamical force, the condition is distinguishable from the others because molecules are always influenced by external forces.

After having considered the general feature of solutions of B-M Eq. mentioned above, we now reconsider flows adjacent to surfaces of solid bodies in more detail.

2. Steady Flows

In a slip flow, the conception of which was initiated and developed by Max-

well⁸⁾, Milikan⁹⁾ and Epstein¹⁰⁾, we now consider⁷⁾ that the distribution function is conveniently given by the sum of two clearly different functions, one for the impinging molecules and the other for the reflecting molecules*. The function for reflecting molecules is almost independent of the function for impinging molecules and is a Maxwell's function corresponding to the temperature of wall surfaces. This sum is a function of velocities of molecules and some parameters which are functions of space and time. The boundary conditions of the parameters included in this function are given in terms of the values of gross variables on the wall surface. This sum, the parameters being functions of space and time, should be subject to B-M Eq. The difference of the two functions which, being large, is distinguishable on the wall surface, gradually disappears as we go away from the wall, say the length of mean free path. The domain of slip is the domain where the difference of two functions is distinguishable.

As Tsien¹¹⁾ proposes, in boundary layer flows λ/δ gives a characteristic number of flow where δ is the boundary layer thickness and λ is the mean free path of molecules**. As for flows between two parallel flat plates in different states*** from each other, we have λ/D as the characteristic number where D is the distance between the two considered parallel flat plates. These numbers are important measures which show the degree of irregularity in molecule distributions directly caused by wall surfaces. As Tsien calculated, for extremely small Reynolds' number

$$\frac{\lambda}{\delta} \sim \frac{M}{Re(l)} \frac{l}{\delta} \sim \frac{M}{Re(l)}, \quad Re(l) < 1. \quad (1)$$

For large Reynolds' number, it is well-known that

$$\frac{\lambda}{\delta} \sim \frac{M}{Re(l)} \frac{l}{\delta} \sim \frac{M}{\sqrt{Re(l)}}, \quad Re(l) > 1. \quad (2)$$

Here l is the distance between the leading edge and the place under consideration. Another fact to be considered is: How much does the state in a boundary layer flow deviate from thermal equilibrium? As Nonweiler¹²⁾ says, there is no slip when impinging molecules distribute in the same way as the molecules of reflection. The measure of the deviation is p_{xy}/p_{xx} where p_{xy} is the stress component of x - y plane, x being the direction of flow and y being the direction perpendicular to the wall surface. Now we consider this measure in detail:

$$\frac{p_{xy}}{p_{xx}} = \frac{\mu \frac{U}{\delta}}{p} = \frac{\gamma \mu \frac{U}{\delta \rho}}{\gamma \frac{p}{\rho}} = \frac{U^2 \gamma \nu}{a^2 U \delta} = \gamma \frac{M^2 l}{Re \delta}. \quad (3)$$

* We must note that there may possibly be other functions which satisfy boundary conditions and which are at least continuously connected to the state of the flow outside the boundary layer by being subjected to B-M Eq. We distinguish one from the possible others only by examining the values of variables at places other than the boundary mentioned above.

** We may conveniently consider λ/L when the entire dimension of solid body L is to be compared with λ .

*** i.e., two flat plates moving in opposite directions parallel to each other or having different temperatures, etc. See Appendix.

Here U is the velocity of flow outside the boundary layer, μ is the coefficient of viscosity and a is the velocity of sound. When M is larger than unit, we see clearly that

$$\frac{\dot{p}_{xy}}{\dot{p}_{xx}} > \frac{\lambda}{\delta} \quad (4)$$

by comparing (1) and (2) with (3). This relation means that the deviation of state from thermal equilibrium is large at the same time when the slip effect is large.

In cases of flow between two parallel flat plates moving in opposite directions, we have

$$\frac{\lambda}{D} = 1.88\sqrt{\tau} M/R_e(D)$$

and

$$\frac{\dot{p}_{xy}}{\dot{p}_{xx}} \doteq \frac{\mu \frac{U}{D}}{\dot{p}_{xx}} = \tau M^2/R_e(D).$$

Namely

$$\frac{\dot{p}_{xy}}{\dot{p}_{xx}} > \frac{\lambda}{D} \quad \text{when} \quad M > 1. \quad (5)$$

Based on these facts, at the same time that we have to consider "slip", we must take into consideration that the deviation of state of flow from thermal equilibrium is not small. The distribution functions of impinging molecules are of modes considerably different from the Maxwell type. This means that Navier-Stokes' equation may not be properly used for flows where slip is effectively considered, even in the domain more than the length of mean free path apart from a wall.*

3. Rayleigh's Problem

When we deal with Rayleigh's problem of transition phenomena, we have to take into consideration another measure besides λ/δ and \dot{p}_{xy}/\dot{p} , namely, t/τ . Here t is the time interval between the instant of time under consideration and the initial time when the plate under consideration begins to move in a certain constant velocity, and τ is the mean free time of molecules. Assume that we observe the distribution of molecules at a certain point in boundary layer, say λ away from wall. Until the value of t/τ attains a certain magnitude, there may be no large change in the distribution of molecules and the distribution function continues to be Maxwell's if the initial state is of thermal equilibrium. As time passes and t increases to a sufficiently large value as compared with τ , the mode of distribution will change to the final steady state which we considered in the last section. Consequently the essential point of Rayleigh's problem is to consider the process of the change of state from thermal equilibrium to that of steady flow. The time period required for the entire procedure being denoted by T , T is of the order of l/U where l is the distance between the leading edge of plate and the point under consideration, and U is the velocity of the flow. Then considering

* As for heat transfer, see Appendix.

$$T = l/U, \quad \frac{\lambda}{\tau} = \bar{c} = a \sqrt{\frac{8}{\pi \bar{\gamma}}}, \quad \lambda = 1.88 \sqrt{\bar{\gamma}} \nu / a,$$

we have

$$T/\tau = \frac{0.85}{\bar{\gamma}} \frac{Re(l)}{M^2}. \quad (6)$$

When $T \gg \tau$ or $\frac{M^2}{Re(l)} \ll 1$, then we may treat the process of transition as continuous. In the sense of both time and space, it is not necessary to consider "slip". On the other hand, when $T \leq \tau$ or $\frac{M^2}{Re(l)} \geq 1$, we have to take into consideration "slip" not only in the sense of space but also in the sense of time. In this case, according to condition (4), the final steady states deviate considerably from thermal equilibrium.

4. Conclusion

Under the assumption that there is a predominant probable distribution mode of molecules with small fluctuations, we can apply B-M Eq. as the fundamental equation of gasdynamics. In solving B-M Eq. we have to presume a proper probable mode of distribution in each case. Especially when the state under consideration deviates considerably from thermal equilibrium, we have to presume the mode of distribution of molecules according to the following items:

- 1) what are the gross-variables?
- 2) what is the effect of given external force on the mode of molecular distribution?
- 3) how far the state under consideration is away from external restriction (for example, from solid body), in the sense of both time and space?

We should note that, even though the values of a set of gross variables of finite number of two given states are similar, the distributions of molecules can possibly be different.

References

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Appendix. Heat Transfer

In the case when there is heat transfer between two parallel flat plates, which have no relative velocity, we consider another dimensionless number which shows the degree of deviation of state from thermal equilibrium. That is given by

$$\frac{H}{\rho c_v \bar{\theta} \bar{c}},$$

where

$$H = K \frac{\theta_2 - \theta_1}{D} = K \frac{\Delta\theta}{D}$$

and

- θ_1, θ_2 : temperatures of two flat plates
- D : distance between two plates
- K : heat conductivity
- $\bar{\theta}$: mean temperature of gas between two plates
- \bar{c} : mean velocity of molecules.

As we know that

$$K \doteq k \mu c_v,$$

$$k = \begin{cases} 2.5 & \text{for monatomic gases,} \\ 1.9 & \text{for diatomic gases,} \end{cases}$$

$$\bar{c} \doteq a \sqrt{\frac{8}{\pi \gamma}},$$

we obtain the following result:

$$\frac{H}{\rho c_v \bar{\theta} \bar{c}} = 0.63 k \sqrt{\gamma} \frac{1}{Re(D, a)} \frac{\Delta\theta}{\bar{\theta}}, \quad Re(D, a) = \frac{Da}{\nu}.$$

This is a measure of deviation of state from thermal equilibrium. On the other hand

$$\frac{\lambda}{D} = 1.88 \sqrt{\gamma} \frac{1}{Re(D, a)}.$$