Special Mathematics Lecture

Differential equations and dynamical systems

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Differential equations and dynamical systems

I Introduction through examples

We start by presenting examples and by introducing the main ideas and some keywords. More precise definitions and a systematic approach will come later.

I.1 Newton's equation

usual 3 dimensional space

Consider
$$\begin{aligned} x : \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto x(t) \end{aligned}$$

time

the description of the position of a particle at time t. We set $v := \dot{x} \equiv x'$

derivative of the function x

for the velocity of the particle,

and $a := \dot{v} = v' = \ddot{x} \equiv x''$ second derivative of the function x

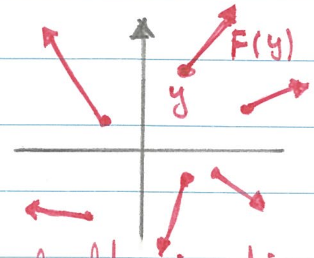
for the acceleration of the particle. Note that $v : \mathbb{R} \to \mathbb{R}^3$ and

time

that $a : \mathbb{R} \to \mathbb{R}^3$.

time

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be an <u>external</u> <u>force</u> (a vector field). It associates to any point of $\mathbb{R}^3$ a vector in $\mathbb{R}^3$.



a vector field in dim. 2

Then, Newton's second law of motion states

mass of the particle, $m > 0$

$$\underline{\left| m\ddot{x}(t) = F(x(t)) \right.} \qquad \forall t \in \mathbb{R}$$

for all

second order differential equation

In fact, it corresponds to a system of 3 equations

$$\begin{cases} m\,\ddot{x}_1(t) = F_1(x(t)) \\ m\,\ddot{x}_2(t) = F_2(x(t)) \\ m\,\ddot{x}_3(t) = F_3(x(t)) \end{cases}$$

with $x = (x_1, x_2, x_3)$ and $F = (F_1, F_2, F_3)$

If $F(y) = (0, 0, -mg) \equiv \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$

a constant

small abuse of notation

then one gets
$$\begin{cases} m\,\ddot{x}_1(t) = 0 \\ m\,\ddot{x}_2(t) = 0 \\ m\,\ddot{x}_3(t) = -mg. \end{cases} \qquad \circledast$$

observe that the right hand side is independent of t, in this example

In this case, a solution is

$$x(t) = \underbrace{x_0}_{\in \mathbb{R}^3} + \underbrace{v_0}_{\in \mathbb{R}^3} t - \frac{1}{2} g t^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

initial position initial velocity.

Thus, a solution for ⊛ can be found explicitly, but initial (or boundary) conditions are necessary for a unique (?) solution.

I.2 The exponential function

Consider $x : \mathbb{R} \to \mathbb{R}$, $t \mapsto x(t)$ and the equation

$$\boxed{\dot{x}(t) \equiv x'(t) = a\, x(t)} \quad \forall\, t \in \mathbb{R}, \text{ and } a \in \mathbb{R}.$$

first order differential equation

Then, a solution is $x(t) = \underbrace{x_0}_{\in \mathbb{R}} e^{at}$ initial condition

and one shows in calculus I that this is the only solution satisfying $x(0) = x_0$.

(usually, we consider $x_0 = 1$ only)

__Remark__: The choice of an initial condition at $t=0$ is arbitrary, we could fix $x(t_0) = x_{t_0}$ and then the problem reads

$$\begin{cases} \dot{x}(t) = a\, x(t) & \forall\, t \in \mathbb{R} \\ x(t_0) = x_{t_0} \end{cases}$$

and the unique solution is $x(t) = x_{t_0}\, e^{a(t-t_0)}$.

a detailed proof, please?

I.3 The logistic equation

For $a \in \mathbb{R}$, $N > 0$, consider $x : \mathbb{R} \to \mathbb{R}$ satisfying

$$\dot{x}(t) = a\, x(t) \left(1 - \frac{x(t)}{N}\right) \quad \forall t \in \mathbb{R}$$

similar to previous case when $\frac{x(t)}{N}$ small but quite different when $\frac{x(t)}{N} \simeq 1$.

By multiplying both side by $\frac{1}{N}$ and by setting $x := \frac{x}{N}$ (rescaling) one gets

$$\boxed{\dot{x}(t) = a\, x(t)\,(1 - x(t))} \quad (*) \quad \forall t \in \mathbb{R}$$

first order non linear differential equation

A solution is given by

$$x(t) = \frac{x_0\, e^{at}}{1 - x_0 + x_0 e^{at}} \qquad \forall t \in \mathbb{R}.$$

a detailed proof, please ?

Observe that if $x_0 = 0$, $x(t) = 0 \ \forall t$ is a solution of $(*)$, and if $x_0 = 1$, $x(t) = 1 \ \forall t$ is also a solution of $(*)$. These solutions are called equilibrium solution.

Remark : If $a > 0$, $x_0 > 0$, then $\lim_{t \to \infty} x(t) = 1$, but if $x_0 < 0$ and $a > 0$, then $\exists t > 0$ such that $1 - x_0 + x_0 e^{at} = 0$

$\Leftrightarrow e^{at} = 1 - \frac{1}{x_0} > 1$. Thus, the function $x(t)$ is not well defined for $t = \frac{1}{a} \ln\left(1 - \frac{1}{x_0}\right)$. ⚠

I.4 Logistic equation + constant term

Fix $a = 1$, $h \geq 0$ and the equation

$$\boxed{\dot{x} = x(1-x) - h} \quad (*)$$

t dependence not written

first order non linear differential equation depending on a parameter (h)

Observe that
$$\begin{aligned} & x(1-x) - h \\ &= -(x^2 - x + h) \\ &= -\left(x - \tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4} - h\right). \quad (\square) \end{aligned}$$

If $h = \frac{1}{4}$, $x(t) = \frac{1}{2}$ $\forall t$ is a solution of $(*)$.

If $h > \frac{1}{4}$, then $(\square) < 0$ $\forall x$

$\Rightarrow$ no equilibrium solution to $(*)$.

there exist(s)

If $h \in [0, \frac{1}{4})$, then $\exists$ 2 solutions of $(\square) = 0$, and thus 2 equilibrium solutions.

Thus, depending on h, there exist 0, 1, or 2 equilibrium solutions. $\rightsquigarrow$ bifurcation

to be defined!

<u>In summary</u> : we have heard about :

- the order of a differential equation
- linear or non linear
- initial / boundary conditions
- unicity of the solution ?
- equilibrium solution
- asymptotic ($\lim_{t \to \infty} x(t)$)
- dependence on a parameter
- bifurcation ...

II First order linear systems

This chapter presents the natural n-dimensional generalization of eq. $\dot{x} = ax$ seen in Section I.2. Namely, let $x : \mathbb{R} \to \mathbb{R}^n$ and let $A \in M_n(\mathbb{R})$ (set of all $n \times n$ real matrices).

Then, our first aim is to study

$$\boxed{\dot{x}(t) = A x(t)} \qquad \forall\, t \in \mathbb{R}.$$

first order linear system of differential equations

II.1 Fundamental theorem for linear systems

For $x \in \mathbb{R}^n$, recall that $x = (x_1, x_2, \dots, x_n)$ and

$$\|x\| := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\sum_{j=1}^{n} x_j^2}$$

the Euclidean norm on $\mathbb{R}^n$

For $A \in M_n(\mathbb{R})$, we set

This is one way to measure the size of a matrix

$$\|A\| := \max_{\substack{x \in \mathbb{R}^n \\ \|x\| \leq 1}} \|Ax\|$$

it is the same notation, but it should not be misleading.

Then the following properties hold : if and only if

1) $\|A\| \geq 0$ and $\|A\| = 0$ iff $A = 0$ ← 0-matrix

2) $\|\lambda A\| = |\lambda| \, \|A\| \qquad \forall \lambda \in \mathbb{R}$

3) $\|A + B\| \leq \|A\| + \|B\|, \qquad B \in M_n(\mathbb{R})$

These 3 properties imply that $\|\cdot\|$ is a norm on $M_n(\mathbb{R})$

4) $\|Ax\| \leq \|A\| \, \|x\| \qquad \forall x \in \mathbb{R}^n$

5) $\|AB\| \leq \|A\| \, \|B\|$

6) $\|A^k\| \leq \|A\|^k$.

Someone for proving these properties ?

<u>Lemma II.1</u>: For any $A \in M_n(\mathbb{R})$ and $t \in \mathbb{R}$, the expression $\underline{e^{tA}} := \sum_{k=0}^{\infty} \frac{1}{k!} (tA)^k$ is well defined and satisfies :

$$\|e^{tA}\| \leq e^{|t| \|A\|}$$

The proof is easy from the properties above

Lemma II.2 For any $s,t \in \mathbb{R}$, $A \in M_n(\mathbb{R})$, one has $e^{sA} e^{tA} = e^{(s+t)A}$. The map $\mathbb{R} \ni t \mapsto e^{tA} \in M_n(\mathbb{R})$ is differentiable with $\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$.

Somebody for a proof ?

(it is necessary to define precisely the meaning of differentiable.)

⚠ $e^{t(A+B)} \neq e^{tA} e^{tB}$ in general, except if $AB = BA$. A proof ?

Theorem II.3: For any $A \in M_n(\mathbb{R})$ and $x_0 \in \mathbb{R}^n$, the system $\begin{cases} \dot{x}(t) = A\,x(t) \\ x(0) = x_0 \end{cases}$ has a unique solution given by

$$x(t) = e^{tA} x_0 .$$

Proof quite similar to calculus I.

Since the solution is given by the previous Theorem, we could stop here, but this result does not give any information on the behavior of $x(t) = e^{tA} x_0$ for $t \to +\infty$?

II.2 Linearity + decomposition

An important result for linear system :

Theorem II.4 If $x(t)$ satisfies $\dot{x} = Ax$ and $y(t)$ satisfies $\dot{y} = Ay$, then $x(t)+y(t)$ satisfies

$$(x+y)' \equiv \dot{x} + \dot{y} = A(x+y).$$

this result is in fact a definition of linear systems : Superposition

Proof : Let $x(t) = e^{tA} x_0$, $y(t) = e^{tA} y_0$,

then $(x(t)+y(t))' = A e^{tA} x_0 + A e^{tA} y_0 =$

$$= A(e^{tA} x_0 + e^{tA} y_0) = A(x(t)+y(t)).$$

■

Example : Let $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, then $A^k = \begin{pmatrix} 2^k & 0 \\ 0 & (-1)^k \end{pmatrix}$

and $e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} t^k 2^k & 0 \\ 0 & t^k(-1)^k \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}$.

Thus, if $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $x(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \xrightarrow[t\to\infty]{} \begin{pmatrix} \infty \\ 0 \end{pmatrix}$

while if $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $x(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix} \xrightarrow{t\to\infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The asymptotic behaviors are quite different, depending on the initial condition x_0.

We shall come back to this

More generally one has :

Proposition II.5 If $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} \equiv \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then $e^{tA} = \operatorname{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_n})$. If $B^{-1}AB = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ for some invertible matrix B, then $B^{-1} e^{tA} B = \operatorname{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_n})$.

change of basis

Somebody for the proof ?

⚠ It is possible that $\lambda_j \in \mathbb{C}$! and ...
$B \in M_n(\mathbb{C})$

⚠ Not all matrices can be diagonalized. For example there is no B such that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = B^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B .$$

Thus, Proposition II.5 applies to some matrices A, but not all. The general decomposition result is the following:

Theorem II.6 (Jordan canonical form)

Let $A \in M_n(\mathbb{R})$, $\exists B$ an invertible $n \times n$ matrix such that

$$B^{-1} A B = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_m \end{pmatrix}$$

with each block J_j of the form

$$J_j = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \lambda & \ddots \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

and λ a solution of the characteristic polynomial of A, namely

$$\det(A - z) = \prod_{j=1}^{\ell} (\lambda_j - z)^{a_j} \quad *$$

⚠ $\lambda_j \neq \lambda_k$ in $*$, but some λ in the Jordan canonical form can be the same.

The values λ_j are called the <u>eigenvalues</u> of A, and a_j is called the <u>algebraic multiplicity</u> of λ_j.

Based on this decomposition one infers the following form for e^{tA}:

<u>Theorem II.7</u> With the previous notation one has
$$B^{-1} e^{tA} B = \begin{pmatrix} e^{tJ_1} & & & 0 \\ & e^{tJ_2} & & \\ & & \ddots & \\ 0 & & & e^{tJ_m} \end{pmatrix}$$

If $J_j = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ a $k \times k$ matrix, then
$$e^{tJ_j} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2 & \cdots & t^{k-1}/(k-1)! \\ & 1 & t & & \vdots \\ & & 1 & \ddots & \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}.$$

somebody for this proof?

<u>Remark:</u> If $A \in M_n(\mathbb{R})$, there exists a real version of the Jordan canonical form, only with real entries (but it is more complicated and not useful).

II.3 Asymptotic evolution (for $t \to \infty$)

Recall that $A \in M_n(\mathbb{R})$ and for some invertible matrix B one has

$$B^{-1} e^{tA} B = \begin{pmatrix} e^{tJ_1} & & & 0 \\ & \boxed{e^{tJ_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{e^{tJ_m}} \end{pmatrix}$$

with

$$e^{tJ_j} = e^{\lambda_j t} \begin{pmatrix} 1 & t & t^2/2! & \cdots & t^{k_j-1}/(k_j-1)! \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & t \\ 0 & & & & 1 \end{pmatrix}.$$

$$\in M_{k_j}(\mathbb{R})$$

λ_j is one eigenvalue of A, and $k_j \in \mathbb{N}$ depends on j. Thus, for any $x \in \mathbb{R}^n$, we have

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \quad \text{with} \quad x_j \in \mathbb{R}^{k_j}$$

m, not n!

and

$$B^{-1} e^{tA} B x = \begin{pmatrix} e^{tJ_1} x_1 \\ e^{tJ_2} x_2 \\ \vdots \\ e^{tJ_m} x_m \end{pmatrix}.$$

Lemma II. 8 Let $\lambda = Re(\lambda) + i\,Im(\lambda)$

1) If $Re(\lambda) < 0$, then $e^{tJ_j} x_j \xrightarrow{t \to \infty} 0 \in \mathbb{R}^{k_j}$

2) If $Re(\lambda) > 0$, then $\| e^{tJ_j} x_j \| \xrightarrow{t \to \infty} \infty$ if $x_j \neq 0$

3) If $Re(\lambda) = 0$, then $\| e^{tJ_j} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \| = 1 \quad \forall t$

while $\| e^{tJ_j} \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix} \| \xrightarrow{t \to \infty} \infty$ if $x_j \neq 0$.

The proof of this Lemma is straightforward ♡

Remark: If $\lambda_j \in \mathbb{C}$ is an eigenvalue of A, then $\bar{\lambda}_j$ is also an eigenvalue of A. It implies that in case 3, at least two independent vectors remain bounded under the evolution.

stable vector space generated by these vectors

Def: Let $E^s := \text{span}\left\{ \begin{pmatrix} 0 \\ \vdots \\ x_j \\ \vdots \\ 0 \end{pmatrix} \;\middle|\; Re(\lambda_j) < 0 \right\}$

unstable

$E^u := \text{span}\left\{ \begin{pmatrix} 0 \\ \vdots \\ x_j \\ \vdots \\ 0 \end{pmatrix} \;\middle|\; Re(\lambda_j) > 0 \right\}$

$E^c := \text{span}\left\{ \begin{pmatrix} 0 \\ \vdots \\ x_j \\ \vdots \\ 0 \end{pmatrix} \;\middle|\; Re(\lambda_j) = 0 \right\}$

center or critical!

Clearly $E^s \oplus E^u \oplus E^c = \mathbb{R}^n$ and each of these subspaces is <u>invariant</u> under $B^{-1} e^{tA} B$.

"a vector inside this subspace stays inside this subspace under the evolution"

⚠ In E^c, some vectors remain of bounded norm under the evolution, while others have a norm growing polynomially (and not exponentially).

<u>Remark</u> : The solution $x(t) = 0 \ \forall t$ is always an equilibrium solution of the equation $\dot{x}(t) = Ax(t)$.

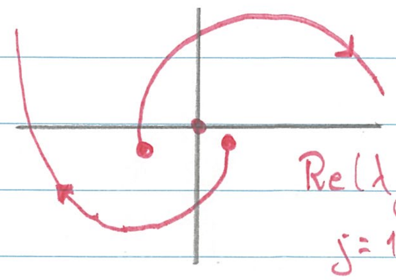
We say that 0 is a <u>sink</u> if $\mathrm{Re}(\lambda_j) < 0 \ \forall j$ or a <u>source</u> if $\mathrm{Re}(\lambda_j) > 0 \ \forall j$. If $\mathrm{Re}(\lambda_j) < 0$ $\forall j$, we also say that the linear system is <u>asymptotically stable</u>, while it is <u>stable</u> if $\mathrm{Re}(\lambda_j) \leq 0 \ \forall j$ and no element $x \in E^c$ satisfies $\lim\limits_{t \to \infty} \| e^{tJ_j} x \| = \infty$.

This last condition is about the geometric multiplicity of λ_j.

Phase portraits (in $\mathbb{R}^2$) (look in the reference books for nicer pictures).

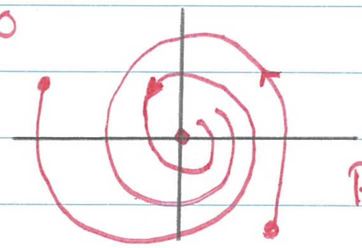
given $x_0 \in \mathbb{R}^2$, we look at

$x(t) := e^{tA} x_0$; arrows are for $t \to \infty$.

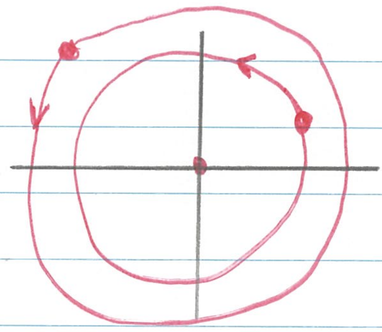


$\mathrm{Re}(\lambda_j) > 0$
$j = 1, 2$

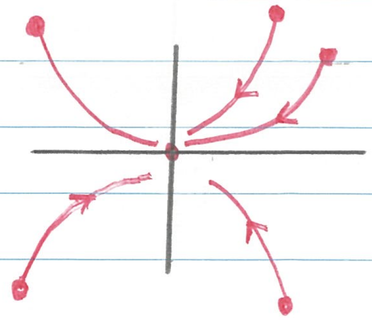
$\mathrm{Im}(\lambda_j) \neq 0$
$j = 1, 2$



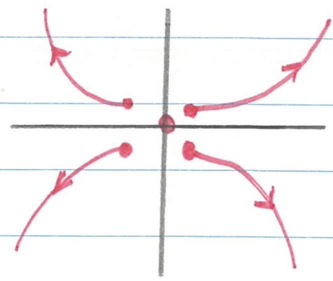
$\mathrm{Re}(\lambda_j) < 0$
$j = 1, 2$



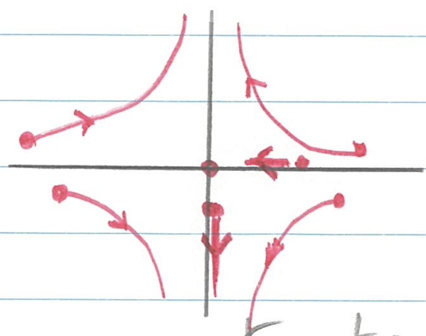
$\mathrm{Re}(\lambda_j) = 0$
$\mathrm{Im}(\lambda_j) \neq 0$



$\lambda_j < 0$
$j = 1, 2$



$\lambda_j > 0$ $j = 1, 2$



$\lambda_1 < 0$
$\lambda_2 > 0$

It is clearly more interesting in higher dimension.

II.4 n^{th} order linear systems

Let $x : \mathbb{R} \to \mathbb{R}$ and consider the n^{th} order linear system :

n-1 derivative

$$\boxed{x^{(n)} + c_{n-1} x^{(n-1)} + \dots + c_1 x^{(1)} + c_0 x = 0} \quad (*)$$

with $c_j \in \mathbb{R}$ or $\mathbb{C}$.

The initial condition are $x(0) = x_0$, $x^{(1)}(0) = x_1, \dots$ $x^{(n-1)}(0) = x_{n-1}$.

Then by setting $A := \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ 0 & & & \ddots & 1 \\ -c_0 & -c_1 & \dots & & -c_{n-1} \end{pmatrix} \in M_n(\mathbb{R})$ or $M_n(\mathbb{C})$

we can consider $y : \mathbb{R} \to \mathbb{R}^n$ with

$y_j(t) := x^{(j-1)}(t)$, $y_j(0) := x^{(j-1)}(0) = x_{j-1}$

and the equation $\dot{y}(t) = A\,y(t)$ is equivalent to the equation $(*)$. Thus, all results of Sections II.1 – II.3 apply, and this problem can be solved with these results.

In fact, due to the special form of A, the solution is even more explicit:

Theorem II.9

Let λ_j be the factorization of the polynomial

$$z^n + c_{n-1} z^{n-1} + c_{n-2} z^{n-2} \ldots + c_1 z + c_0 = \prod_{j=1}^{\ell} (z - \lambda_j)^{a_j},$$

Then the functions $x_{j,k}(t) := t^k \exp(\lambda_j t)$,

with $0 \leq k < a_j$, are linearly independent solution of ⊛, and any solution of ⊛ is a linear combination of these functions.

Can one check this statement?

II.5 Inhomogeneous equation

some regularity!

For $x : \mathbb{R} \to \mathbb{R}^n$, $A \in M_n(\mathbb{R})$ and $g : \mathbb{R} \to \mathbb{R}^n$ consider the inhomogeneous equation

$$\dot{x}(t) = A\, x(t) + g(t) \quad \forall t \in \mathbb{R}$$

with initial condition $x(0) = x_0$.

Then the solution of this equation is

$$x(t) := e^{tA} x_0 + \int_0^t e^{(t-s)A} g(s)\, ds. \qquad (*)$$

Can one prove it?

sometimes called Duhamel's formula

Note that the term with the integral is not easy to compute. Quite often, and in particular when g has a special form, one can "guess" the form of the second term (ansatz).

<u>Example:</u> If $g(t) = p(t)\, e^{\beta t}$ with $p : \mathbb{R} \to \mathbb{R}^n$ and p_j a polynomial, $j = 1, \dots, n$. Then the second term in $(*)$ can be chosen as

$q(t)\, e^{\beta t}$ with $q : \mathbb{R} \to \mathbb{R}^n$ with q_j polynomial.

why?

Exercise : look at RLC circuit in [T], p 77-79.

II.6 General first order systems

We consider $x : I \to \mathbb{R}^n$ (an interval) and the equation

$$\dot{x}(t) = A(t)\, x(t) \quad \forall t \in I, \qquad x(0) = x_0$$

time dependent $n \times n$ matrix, continuous in t.

It turns out that there exists a unique solution of this system (see Chapter III).

⚠ The solution is not given by $e^{\int_{t_0}^{t} A(s)\,ds}\, x_0$, except if $A(s)A(t) = A(t)A(s)$ $\forall s,t$ Why ?

In fact, there does not exist a simple formula for $x(t)$, but we can still say something about the solution.

1° Observe that if $x(t)$, $y(t)$ satisfy $\dot{x}(t) = A(t)\,x(t)$ and $\dot{y}(t) = A(t)\,y(t)$, then

$$(x(t)+y(t))' = A(t)\,(x(t)+y(t))$$

superposition principle still holds.

$\Rightarrow$ The set of solutions is a vector space.

2° Since $x_{t_0} = \sum_{j=1}^{n} x_{t_0,j} \, E_j$ canonical basis of $\mathbb{R}^n$

we can look for solution $\phi(\cdot, t_0, E_j) : I \to \mathbb{R}^n$

satisfying $\phi(t_0, t_0, E_j) = E_j$. Then, the solution

$\phi(\cdot, t_0, x_{t_0})$ is obtained by

$$\phi(\cdot, t_0, x_{t_0}) = \sum_{j=1}^{n} \phi(\cdot, t_0, E_j) \, x_j(t_0) \equiv \Pi(\cdot, t_0) \, x_{t_0}$$

$n \times n$ matrix

with $\Pi(\cdot, t_0) := \big(\phi(\cdot, t_0, E_1), \phi(\cdot, t_0, E_2), \dots, \phi(\cdot, t_0, E_n) \big)$

matrix defined with the solutions as columns.

called <u>the principal matrix solution at t_0</u>.

Then the initial equation is equivalent to

$$\dot{\Pi}(t, t_0) = A(t) \, \Pi(t, t_0) \quad , \quad \Pi(t_0, t_0) = \mathbb{1}$$

derivative of each entry of the matrix.

Given the initial condition x_{t_0}, the unique solution is

$x(t) = \Pi(t, t_0) \, x_{t_0}$.

3° Observe that $\Pi(t, t_1)\,\Pi(t_1, t_0) = \Pi(t, t_0)$.

since both sides satisfy $\dot{M}(t) = A(t)\,M(t)$

$\Big(\big(\Pi(t,t_1)\,\Pi(t_1,t_0)\big)' = \dot{\Pi}(t,t_1)\,\Pi(t_1,t_0) = A(t)\,\Pi(t,t_1)\,\Pi(t_1,t_0)$

+ for $t = t_1$: $\underbrace{\Pi(t_1,t_1)}_{=\mathbb{1}}\,\Pi(t_1,t_0) = \Pi(t_1,t_0)$, they are equal.

In particular, for $t = t_0$ one gets

$$\Pi(t_0, t_1)\,\Pi(t_1, t_0) = \Pi(t_0, t_0) = \mathbb{1}$$

$$\Rightarrow \boxed{\Pi(t, t_0)^{-1} = \Pi(t_0, t)}.$$

4° Let $\phi_1, \dots, \phi_n$ be n solutions of the initial equation, and satisfying $\det\big(\phi_1(t), \dots, \phi_n(t)\big) \neq 0$ for some $t \in I$. Set $U := (\phi_1, \dots, \phi_n) \in M_{n\times n}(\mathbb{R})$.

One has:

<u>Lemma II.10</u> : 1) If $\det(U(t)) \neq 0$ for some $t \in I$, then $\det(U(t)) \neq 0 \ \forall\, t \in I$.

2) The relation $\Pi(t, t_0) = U(t)\,U(t_0)^{-1}$ holds, for all $t \in I$.

See [T], Lemma 3.11

The first statement of the lemma says that it is sufficient to check the linear independence for one t, and then it holds for all t. The second statement follows by the uniqueness of the solution of the differential equation with a given boundary condition, since for $t=t_0$

$$\Pi(t_0, t_0) = \mathbb{1} = U(t_0)\, U(t_0)^{-1}.$$

If we consider the inhomogeneous equation

$$\dot{x}(t) = A(t)\, x(t) + g(t) \quad \forall\, t \in I, \quad x(t_0) = x_{t_0}$$

with $A \in C(I, M_n(\mathbb{R}))$, $g \in C(I, \mathbb{R}^n)$,

then its solution is given by

$$x(t) = \Pi(t, t_0)\, x_{t_0} + \int_{t_0}^{t} \Pi(t,s)\, g(s)\, ds$$

Somebody to check it ?

solution of the homogeneous ($= g(t) = 0$) system

As already mentioned, there is no general solution for $\Pi(\cdot, t_0)$, but if one solution is known, the problem can be reduced (reduction of order, d'Alembert). Suppose $\phi_1 : I \to \mathbb{R}^n$ is a solution and that $(\phi_1(t))_1 \neq 0$ (first component different from 0).

Set $X(t) := (\phi_1(t), E_2, \dots, E_n) \in M_n(\mathbb{R})$, and $y(t) := X(t)^{-1} x(t)$. Then

$$\dot{y}(t) = X(t)^{-1}\dot{x}(t) - X(t)^{-1}\dot{X}(t)X(t)^{-1}x(t)$$

$$= X(t)^{-1}\left(A(t)x(t) - \dot{X}(t)X^{-1}(t)x(t)\right)$$

$$= X(t)^{-1}\left(A(t)X(t) - \dot{X}(t)\right)y(t)$$

$$A(t)X(t) - \dot{X}(t) = A(t)X(t) - (\dot{\phi}_1, 0, 0, \dots 0)$$

$$= A(t)X(t) - A(t)(\phi_1(t), 0, 0, \dots 0)$$

$$= A(t)(\phi_1 - \phi_1, E_2, E_3, \dots E_n).$$

$$\Rightarrow \dot{y}(t) = A(t)\underline{(0, E_2, \dots, E_n)}\,y(t) = A(t)\begin{pmatrix} 0 \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

no $y_1(t)$!

$\Rightarrow$ equation for $y_2(t), \dots, y_n(t)$.

We get a system of dimension $n-1$, to be solved. Then y_1 can be obtained by one integration. See Example p 84-85 of [T].

Remark: By the trick developed in Section II.4 and by the content of this section, equations of the form $x^{(n)} + q_{n-1}(t)\,x^{(n-1)} + \dots + q_1(t)\,\dot{x} + q_0(t)\,x = 0$ (homogeneous)

or $x^{(n)} + q_{n-1}(t)\,x^{(n-1)} + \dots + q_1(t)\,\dot{x} + q_0(t)\,x = g(t)$

(inhomogeneous)

can be treated similarly.

A few examples in Section 3.5 of [T] ?

II. 7 Periodic linear systems

Consider again $\dot{x}(t) = A(t)\,x(t)$ ⊛ with

$A(t+T) = A(t)$ for some $T>0$ and all t.

↑ periodic condition

Thus, if $x(\cdot)$ is a solution of ⊛, $x(\cdot + T)$ is also a solution.

Lemma II.11 : If $A(t+T) = A(t)$, then

$$\Pi(t+T, t_0+T) = \Pi(t, t_0), \quad \forall t.$$

Proof : Since $\frac{d}{dt}\Pi(t+T, t_0+T) = A(t+T)\,\Pi(t+T, t_0+T)$

$= A(t)\,\Pi(t+T, t_0+T)$ and $\Pi(t_0+T, t_0+T) = \mathbb{1}$,

it follows by uniqueness that $\Pi(t+T, t_0+T) = \Pi(t, t_0)$. $\square$

Def The <u>monodromy matrix</u> is defined by

$$M(t_0) := \Pi(t_0+T, t_0).$$

Clearly, $M(t_0+T) = M(t_0)$, but $M(t_0) \neq \mathbb{1}$

($M(t_0) = \mathbb{1}$ would mean that after one period, the system would come back to its initial condition).

Also, one has

see p 24

$$\Pi(t_0+\ell T, t_0) = \Pi(t_0+\ell T, t_0+(\ell-1)T)\,\Pi(t_0+(\ell-1)T, t_0)$$

$$= M(t_0+(\ell-1)T)\,\Pi(t_0+(\ell-1)T, t_0)$$

$$= M(t_0)\,\Pi(t_0+(\ell-1)T, t_0) = \dots = M(t_0)^\ell\,\Pi(t_0, t_0) = M(t_0)^\ell.$$

$\Rightarrow$ $\Pi(t, t_0)$ exhibits an exponential behavior, but what happens if $t \neq \ell T$?

not unique

Theorem II.12 (Floquet) $\exists\ Q(t_0) \in M_n(\mathbb{C})$ s.t. $\Pi(t, t_0) = P(t, t_0)\, e^{(t-t_0)Q(t_0)}$, with

$P(t+T, t_0) = P(t, t_0)$ and $P(t_0, t_0) = \mathbb{1}$.

$Q(t_0)$ is defined as matrix logarithm, see [T] p 108
Somebody for this definition ?

Observe that $M(t_0) = \Pi(t_0+T, t_0) = \underbrace{P(t_0+T, t_0)}_{=\mathbb{1}}\, e^{(t_0+T-t_0)Q(t_0)} = e^{TQ(t_0)}$.

The eigenvalues of $M(t_0)$ are called the Floquet multipliers, and the eigenvalues of $Q(t_0)$ are called the Floquet exponent. They are related by

$$\rho_j = e^{T\mu_j}$$

μ_j: Floquet exponent

ρ_j: Floquet multiplier

From p 17 one infers: A periodic system is asymptotically stable if $\mathrm{Re}(\mu_j) < 0$ $\forall j$, and stable if $\mathrm{Re}(\mu_j) \leq 0$ with geometric mult = alg. mult. if $\mathrm{Re}(\mu_j) = 0$.

III Initial value problems

III.1 Fixed point theorem

Let X be a real vector space

addition of its elements + multiplication by real constants are available

Examples: $\mathbb{R}^n$, but also $M_n(\mathbb{R})$, $C([a,b])$

continuous functions

$C^p([a,b])$, $L^2(\mathbb{R}^n)$ ...

p times differentiable functions

my favourite Hilbert space!

A norm on X is a function $\|\cdot\| : X \to [0,\infty)$ satisfying for $x, y \in X$:

i) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.

ii) $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall \lambda \in \mathbb{R}$

iii) $\|x+y\| \leq \|x\| + \|y\|$ triangle inequality

Then we also get $\big| \|x\| - \|y\| \big| \leq \|x-y\|$.

(inverse triangle inequality)

Somebody for the proof?

A vector space with a norm is called a <u>normed vector space</u>.

<u>Question</u>: Consider $c_0(\mathbb{Z}) := \{(a_n)_{n\in\mathbb{Z}}$ with $a_n \to 0$ as $|n| \to \infty\}$, and $c_c(\mathbb{Z}) := \{(a_n)_{n\in\mathbb{Z}}$ with $a_n \neq 0$ only for finitely many $n\}$.

Clearly $c_c(\mathbb{Z}) \subset c_0(\mathbb{Z})$, but what is a fundamental difference between these 2 normed vector spaces? By the way, the norm on these spaces is: $\|(a_n)_{n\in\mathbb{Z}}\| = \sup_{n\in\mathbb{Z}} |a_n| \equiv \|(a_n)_{n\in\mathbb{Z}}\|_\infty$ ← we call it the sup norm.

<u>Answer</u>: a converging sequence in $c_0(\mathbb{Z})$ will always converge in $c_0(\mathbb{Z})$, but a "converging" sequence in $c_c(\mathbb{Z})$ will not always converge in $c_c(\mathbb{Z})$ but only in $c_0(\mathbb{Z})$. This is similar to the convergence in $\mathbb{R}$ or in $\mathbb{Q}$.

⚠ We are dealing with a sequence (index N) of sequences (index n)

Example : For any $N \in \mathbb{N}$, consider the sequence $(a_n^N)_{n\in\mathbb{Z}}$ defined by $a_n^N := \frac{1}{|n|+1}$ if $n \leq N$, and $a_n^N = 0$ if $n > N$. Clearly $(a_n^N)_{n\in\mathbb{Z}} \in c_c(\mathbb{Z})$ but $\lim_{N\to\infty} (a_n^N)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$ but does not belong to $c_c(\mathbb{Z})$.

Definition : Let X be a normed vector space. A Cauchy sequence in X is a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ s.t. for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ with $\|x_n - x_m\| \leq \varepsilon$ $\forall n, m \geq N$.

for a given ε, all elements with $n \geq N$ are at a distance at most ε. There is an accumulation!

Definition : A converging sequence in X is a sequence $(x_n)_{n\in\mathbb{N}} \in X$ s.t. $\exists x_\infty \in X$ satisfying : $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ with $\|x_n - x_\infty\| \leq \varepsilon$ $\forall n \geq N$.

Exercise: Show that any converging sequence in X is Cauchy, but not every Cauchy sequence is converging in X.

Definition♡ : A normed space is complete ($\equiv$ a Banach space) if every Cauchy sequence converge in X.

$I = [a, b]$

Examples: $\mathbb{R}^n$, $M_n(\mathbb{R})$, $c_0(\mathbb{Z})$, $C(I)$, $C_0(\mathbb{R})$ are Banach spaces, but $\mathbb{Q}$, $C_c(\mathbb{Z})$, $C_c(\mathbb{R})$ are not Banach spaces.

Let X be a normed space and $C \subset X$ a subset. We shall consider a function $K : C \to C$.

Def: A fixed point for K is an element $x \in C$ satisfying $K(x) = x$.

<u>Def</u> : $K : C \to C$ is called a <u>contraction</u>

if $\exists\ \theta \in [0,1)$ with

$$\|K(x) - K(y)\| \leq \theta \|x - y\|, \quad \forall\, x, y \in C$$

↑ K decreases the distance.

<u>Theorem III.1</u> (Contraction principle) [T], thm 2.1

Let X be a Banach space, $C \subset X$ a non-empty (closed) subset, and $K : C \to C$ a contraction. Then K has a unique fixed point $\bar{x} \in C$ and it satisfies

$$\|K^n(x) - \bar{x}\| \leq \frac{\theta^n}{1-\theta} \|K(x) - x\| \qquad \forall\, x \in C.$$

$= \underbrace{K(K(K\ldots(K}_{n \text{ times}}(x))\ldots))$

Somebody for the proof which is not difficult.

<u>Remark</u> : If C is closed, the limit of any Cauchy sequence in C will also be in C.

III. 2 Existence and uniqueness

We consider the equation

$$\circledast \quad \begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_{t_0} \end{cases} \Bigg\} \text{ Initial value problem (IVP)}$$

for some $f \in C(U; \mathbb{R}^n)$ and $U \subset \mathbb{R}^{n+1}$, open, with $(t_0, x_0) \in U$.

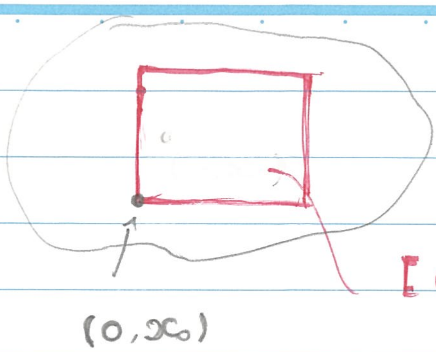
same function

Observe that $\circledast \iff x(t) = x_{t_0} + \int_{t_0}^{t} f(s, x(s))\, ds$. $\odot$

Thus, if we set $[K(x)](t) := x_{t_0} + \int_{t_0}^{t} f(s, x(s))\, ds$, then $\odot$ corresponds to $x = K(x)$, a fixed point for K. Thus, in order to apply Thm III.1, one has to guess who are X, C and check if K is a contraction.

For simplicity, we fix $t_0 = 0$ and consider $t \geq 0$. We also choose $X := C([0,T]; \mathbb{R}^n)$ for some $T > 0$. Then $C([0,T]; \mathbb{R}^n)$ is a Banach space.

↑ fixed later



U, open

closure

$[0,T] \times \overline{B_\delta(x_0)} =: V$

ball centered at x_0 and of radius δ

Since U is open we can choose a cylinder

$[0,T] \times B_\delta(x_0) =: V \subset U$. If we set

$$M := \max_{(t,x)\in V} \| f(t,x) \|$$

Then one has for any fixed t ($x(s)$ is a curve in V)

$$\| [K(x)](t) - x_0 \| = \| \int_0^t f(s, x(s))\, ds \|$$

$$\leq \int_0^t \| f(s, x(s)) \| ds \leq Mt,$$

which means that $t \leq T_0 := \min\{\frac{\delta}{M}, T\}$, $K(x)$ still belongs to V.

Now we can fix $X = C([0,T_0]; \mathbb{R}^n)$, with norm $\|x\|_\infty = \max_{t\in[0,T_0]} \|x(t)\|$ (Euclidean norm)

and $C := \{ x \in X \mid \| x - x_0 \|_\infty \leq \delta \}$

(x_0 considered here as a constant function (indep. of t))

What about contraction? namely

$$\| K(x) - K(y) \|_\infty \overset{?}{\leq} \theta \| x - y \|_\infty \qquad ?$$

$$\|$$

$$\| \int_0^t f(s, x(s))\, ds - \int_0^t f(s, y(s))\, ds \|_\infty$$

$$= \| \int_0^t \left(f(s, x(s)) - f(s, y(s)) \right) ds \|_\infty$$

$$= \sup_{t \in [0, T_0]} \| \int_0^t \left(f(s, x(s)) - f(s, y(s)) \right) ds \|$$

$$\leq \sup_{t \in [0, T_0]} \int_0^t \| f(s, x(s)) - f(s, y(s)) \| \, ds \quad \circledast$$

Suppose that $\displaystyle \sup_{\substack{(s,x),(s,y) \in \\ [a,b] \times \overline{B_\varepsilon(z)} \subset U}} \frac{\| f(s,x) - f(s,y) \|}{\| x - y \|} =: L(a,b,\varepsilon,z) < \infty$ constant

any such cylinder

called a local Lipschitz condition

Then $\circledast$ $\displaystyle \leq \sup_{t \in [0,T_0]} L(0,t,\delta,x_0) \int_0^t \| x(s) - y(s) \| \, ds$

$$\leq \sup_{t \in [0,T_0]} L(0,t,\delta,x_0)\, t \, \| x - y \|_\infty$$

$$\leq L \, T_0 \, \| x - y \|_\infty$$

By choosing T_0 such that $L T_0 < 1$, one gets:

Theorem III.2 (Picard - Lindelöf)

Let $f \in C(U; \mathbb{R}^n)$, with U open in $\mathbb{R}^{n+1}$, let $(0, x_0) \in U$ and suppose that f is locally Lipschitz, in the sense mentioned before. Then there exists an interval $I \subset \mathbb{R}$ with $0 \in I$, and a unique $x \in C^1(I; \mathbb{R}^n)$ such that $\dot{x}(t) = f(t, x(t))$ and $x(0) = x_0$.

Remark 1: If $f \in C^k(U; \mathbb{R}^n)$ for any $k \in \mathbb{N}^*$, then the Lipshitz condition is satisfied, and $x \in C^{k+1}(I; \mathbb{R}^n)$.

Remark 2: Various extensions of this theorem exist, with weaker conditions $\rightsquigarrow$ proofs become more technical, but it is also an interesting topic, and useful for some aplications.

III.3 Miscellaneous

Dependence on initial conditions:

In the framework of Thm III.2: if $(0, x_0)$ and $(0, y_0)$ belong to U, if $\dot{x}(t) = f(t, x(t))$ with $x(0) = x_0$ and $\dot{y}(t) = f(t, y(t))$ with $y(0) = y_0$ then $\|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{Lt}$ exponential growth of distance

for $t > 0$, and $L := \sup_{(s,x),(s,y) \in V} \frac{\|f(s,x) - f(s,y)\|}{\|x - y\|}$, with V containing the curves $t \mapsto x(t)$ and $t \mapsto y(t)$.

2° More generally, if $t_0 \leq s_0 \leq t$, and if $\dot{x}(t) = f(t, x(t))$ with $x(t_0) = x_{t_0}$ and $\dot{y}(t) = f(t, y(t))$ with $y(s_0) = y_{s_0}$, then

$$\|x(t) - y(t)\| \leq \|x_{t_0} - y_{s_0}\| e^{L(t-t_0)} + M\left(|t_0 - s_0| e^{L(t-s_0)}\right).$$

$M = \max_{(t,x) \in V} \|f(t,x)\|$

In this case, the initial conditions are at different times.

Extension : The existence and uniqueness of the solution proved in Thm III.2 is very local in time. However, it can often be extended, and one gets:

If the the IVP of p 35 has a unique local solution, then there exists a unique maximal solution defined on a maximal time interval. ~~in~~ one could be a little bit more precise ...

If the solution is defined for all $t \in \mathbb{R}$, one speaks about a global solution.

In particular if $\| f(t,x) \| \leq C_T + L_T \|x\|$ for all $(t,x) \in [-T,T] \times \mathbb{R}^n$, then the solution is global (see [T] Thm 2.17).

Constants depending on T, but not on x.

IV Dynamical systems

IV.1 Main definitions

Def: Let G be a semi-group (additive notation) with identity 0, and let M be a non-empty set. A dynamical system is a triple (G, M, ϕ) with

$$\phi : W \longrightarrow M, \quad (g, x) \longmapsto \phi(g, x)$$

with $W \subset G \times M$, satisfying

1) $\{0\} \times M \subset W$, $\phi(0, x) = x$

2) $\phi(g_2, \phi(g_1, x)) = \phi(g_1 + g_2, x)$ whenever it is well defined! Namely, if we set

$I_x := \{ g \in G \mid (g, x) \in W \}$ then it is necessary to have $g_1, g_1 + g_2 \in I_x$ and $g_2 \in I_{\phi(g_1, x)}$.

The "simplest" example is provided by the iterated map:

For any $T : M \to M$, we set $W := \mathbb{Z}_+ \times M$ and

$$\phi(n, x) := \underbrace{T(T(T(\ldots(T}_{n \text{ times}}(x))\ldots))), \quad \phi(0, x) = x.$$

If $G = \mathbb{Z}_+$ or $\mathbb{Z}$ we speak about a discrete dynamical system, while if $G = \mathbb{R}_+$ or $\mathbb{R}$, we speak about a continuous dynamical system. If G is a group one speaks about invertible dynamical systems.

Our leading example of a continuous dyn. sys. is given by the flow of an autonomous differential equation:

Consider $M \subset \mathbb{R}^n$, open, $f \in C^k(M; \mathbb{R}^n)$ for some $k \geq 1$ and the equation

f does not depend explicitly on t
$\iff$ autonomous

$$\dot{x}(t) = f(x(t)) \quad (*) \quad , \quad x(0) = x_0 \in M.$$

By Thm III.2 and its extension (on t), there exists a unique maximal solution defined on $I_{x_0} \subset \mathbb{R}$.

maximal interval containing 0

Since x_0 is arbitrary in M, we shall just call it x and $x(t) =: \phi(t, x)$ with $\phi(0, x) = x$.

In other words $\phi(\cdot, x) : I_x \to \mathbb{R}^n$ is the unique maximal solution of ⊛ with $\phi(0,x) = x \in M$.

The map $\phi(\cdot, x) : I_x \ni t \longmapsto \phi(t,x) \equiv \underbrace{\phi_t(x) \equiv \phi_x(t)}_{\text{different notations}} \in M$

is called an <u>integral curve</u>, or a <u>trajectory</u>.

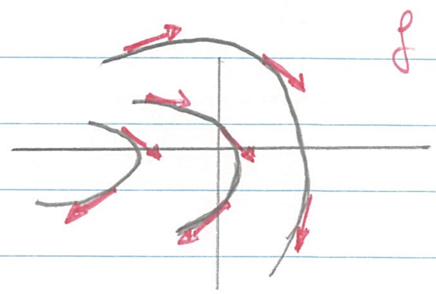
We then set $W := \bigcup_{x \in M} I_x \times \{x\} \subset \mathbb{R} \times M$,

and $\phi : W \ni (t,x) \longmapsto \phi(t,x) \in M$.

The map $\phi \in C^k(W; M)$ (if $f \in C^k(M; \mathbb{R}^n)$) is called <u>the flow</u> associated with ⊛. $(\mathbb{R}, M, \phi)$ is a dynamical system.

<u>Remark</u> Since f is a vector field, and from the equation $\dot{x}(t) = f(x(t))$, it follows that the trajectories are tangent to the vector fields.

In other words, the flow follows the vector field.



IV. 2 Orbits and invariant sets

We continue with the notations just introduced, but the following concepts hold for general dynamical system.

Def: For any $x \in M$, the orbit of x is defined by $\gamma(x) := \{\phi(t,x) \mid t \in I_x\} \subset M$.

Observe that if $y \in \gamma(x)$, then $\gamma(y) = \gamma(x)$.
Thus, different orbits are disjoint ($\equiv$ empty intersection).
We can introduce the forward and backward orbit by $\gamma_\pm(x) := \phi(t,x) \mid t \in I_x, t \gtrless 0 \}$.

Definition: x is a fixed $\equiv$ stationary $\equiv$ equilibrium point if $\gamma(x) = \{x\}$. An orbit $\gamma(x)$ is periodic if $\exists\, T > 0$ s.t. $\phi(t+T,x) = \phi(t,x)$. Then $\gamma(t+T,y) = \gamma(t,y)$ for all $y \in \gamma(x)$.

Periodic orbits are also called <u>closed</u> orbits. If an orbit is not reduced to a point or to a periodic orbit is called a <u>non-closed</u> orbit. If $I_x = \mathbb{R}$, we say that this orbit is <u>complete</u>. If $I_x = \mathbb{R}$ for all $x \in M$, then ϕ is <u>globally defined</u>, namely $W = \mathbb{R} \times M$.

<u>Definition</u>: A set $U \subset M$ is called <u>$\pm$-invariant</u> if $\gamma_{\pm}(x) \subset U$ $\forall x \in U$, and <u>invariant</u> if $\gamma(x) \subset U$ $\forall x \in U$.

<u>Lemma IV.1</u>: If U, V are $\pm$-invariant or invariant, then $U \cap V$, $U \cup V$, $U \setminus V$, and $\bar{U}$ are $\pm$-invariant or invariant.

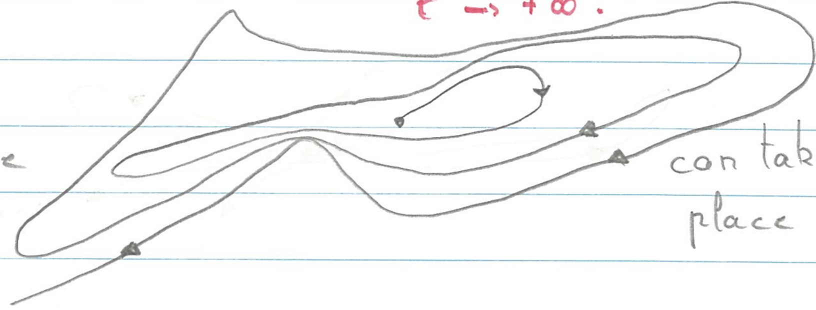
closure in M (= the set and its boundary)

<u>Remark</u>: If $\gamma(x) \subset U \subset M$ with U bounded and closed, then $I_x = \mathbb{R}$ and this orbit is complete.

If $\gamma(x)$ is complete, what is the long-time behavior of $t \mapsto \phi(t, x)$?

can be $t \to -\infty$ or $t \to +\infty$.

⚠ situations like



can take place

<u>Definition</u> The <u>$\omega_\pm$-limit set</u> of the orbit $\gamma(x)$ (or simply of x) is the set of $y \in M$ such that there exists $(t_n)_{n \in \mathbb{N}}$, $t_n \to \pm\infty$ and $\lim_{n \to \infty} \phi(t_n, x) = y$.

In other words, $\phi(\cdot, x)$ is approaching y on a sequence of time going to $\pm\infty$.

We denote this set by $\omega_\pm(x)$ but as already mentioned it depends only on the orbit of x, namely $\omega_\pm(x) = \omega_\pm(y)$ if $y \in \gamma(x)$.

<u>Lemma IV.2</u> $\omega_\pm(x)$ is closed and invariant.

Somebody for the proof ?

Remark : If $\gamma_\pm(x)$ is contained in a bounded and closed set, then $\omega_\pm(x)$ is non-empty and connected (≡ in one piece). In addition,

$$\lim_{t\to\pm\infty} \underbrace{d(\phi(t,x),\ \omega_\pm(x))}_{=\text{ distance between } \phi(t,x) \text{ and } \omega_\pm(x)} = 0 .$$

IV. 3 Stability (towards the future : $t\to\infty$)

Recall that we study the flow generated by the differential equation $\dot{x} = f(x(t))$ with $f \in C^k(M,\mathbb{R}^n)$ for some $k\geq 1$ and $M \subset \mathbb{R}^n$.
↑ autonomous

A fixed point satisfies $\dot{x} = 0$, or equivalently $\gamma(x) = \{x\}$.

Definition : A fixed point x is stable (Lyapunov) if for any $B_\varepsilon(x)$ (ball of center x and radius ε), $\exists\, B_\delta(x) \subset B_\varepsilon(x)$ with $\phi(t,y) \in B_\varepsilon(x)$ for any $t\geq 0$ and any $y \in B_\delta(x)$.

all trajectories starting in $B_\delta(x)$ stay in $B_\varepsilon(x)$.

<u>Definition</u> : A fixed point x is <u>asymptotically stable</u> if it is Lyapunov stable and $\exists\, B_\varepsilon(x)$ with, $\forall y \in B_\varepsilon(x)$,

$$\lim_{t \to \infty} \| \phi(t,y) - x \| = 0 . \quad (*)$$

⚠ $(*)$ does not imply Lyapunov stability why?

<u>Definition</u> : A fixed point x is <u>exponentially stable</u> if $\exists\, \alpha, C, \delta > 0$ such that $\forall y \in B_\delta(x)$,

$$\| \phi(t,y) - x \| \leq C\, e^{-\alpha t} \| y - x \| .$$

⚠ this implies Lyapunov stability and $(*)$ why?

Observe that the notions of stability and asymptotic stability correspond to the ones introduced in Section II.3 for linear systems ($f(x) = Ax$).

In fact, the following relation holds:

Theorem IV.3 : Let f be C^1 and let x be a fixed point. If $J_f(x) \equiv f'(x)$ ($\equiv$ Jacobian matrix) has all its eigenvalues with a strictly negative real part, then x is exponentially stable.

This is a linear approximation theorem.

Exercise : Study examples p 199 of [T] + equations 6.31 – 6.33 of [T], p 200 about bifurcation.

One tool for studying stability :

Definition : Let x_0 be a fixed point. A Lyapunov function for x_0 is a continuous function $L : B_\varepsilon(x_0) \to [0, \infty)$ for some $\varepsilon > 0$, which satisfies $L(x_0) = 0$ and

$L(\phi(t_1, x)) \geq L(\phi(t_2, x))$ for any $t_2 > t_1$ and

$\phi(t_1, x), \phi(t_2, x) \in B_\varepsilon(x_0) \setminus \{x_0\}$.

decay along the evolution

strict Lyapunov function if strict inequality

The following result then holds, see [T] p 201-202.

<u>Theorem IV.4</u>: If x_0 is a fixed point and if a Lyapunov function for x_0 exists, then x_0 is Lyapunov stable. If the function is strict, then x_0 is asymptotically stable.

use monotone convergence theorem + k-level sets

More generally one has:

If $L : U \to \mathbb{R}$ is continuous and bounded from below ($\equiv L(x) \geq a$ for a fixed $a \in \mathbb{R}$ and all $x \in U$) and if $\gamma_+(x) \subset U$ for some $x \in U$ with

$$L(\phi(t_1, x)) \geq L(\phi(t_2, x)) \quad \forall \; t_2 \geq t_1 \geq 0,$$

then L is constant on $\omega_+(x) \cap U$.

In particular, if L is differentiable,

$$\frac{d}{dt} L(\phi(t,x)) = [\nabla L](\phi(t,x)) \cdot \dot{\phi}(t,x) = [\nabla L](\phi(t,x)) \cdot f(\phi(t,x))$$

$$\leq 0.$$

The expression $\nabla L \cdot f$ is called the Lie derivative of L along f, and if $[\nabla L](\phi(t,x)) \cdot f(\phi(t,x)) = 0$ for any $x \in U$, $t \geq 0$, L is called a constant of motion. In particular ω_+-limit sets are contained in the 0-level set of $\nabla L \cdot f$

$\equiv$ where the function vanishes.

V Examples in 2D

V.1 : 1D Newton's equation

a 2D example

Consider $f : \mathbb{R} \to \mathbb{R}$ and the equation $\ddot{x}(t) = f(x(t))$. This can be recast in a first order differential equation in $\mathbb{R}^2$:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ f(x(t)) \end{pmatrix} \quad (*)$$

If f is $C^1(\mathbb{R})$, and for $\begin{pmatrix} x(t_0) \\ \dot{x}(t_0) \end{pmatrix} = \begin{pmatrix} x_{t_0} \\ \dot{x}_{t_0} \end{pmatrix}$, it has a unique local solution, by Thm III.2. The set $\mathbb{R}^2 \ni \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$ is called the phase space, and $\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$ a phase point.

Set $T(\dot{x}) := \frac{\dot{x}^2}{2}$ and $U(x) = -\int_{x_0}^{x} f(\xi)\, d\xi$

fixed, but arbitrary

Then, the function $E(x, \dot{x}) := T(\dot{x}) + U(x)$ is a constant of motion since if $(x(t), \dot{x}(t)) \equiv \phi(t, x)$ is a solution of $(*)$, then

$$\frac{d}{dt} E(x(t), \dot{x}(t)) = \dot{x}(t)\, \ddot{x}(t) - f(x(t))\, \dot{x}(t) =$$
$$= \dot{x}(t)\left(\ddot{x}(t) - f(x(t))\right) = 0.$$

In this setting E is called the <u>energy</u>, T the <u>kinetic energy</u>, and U the <u>potential energy</u>.

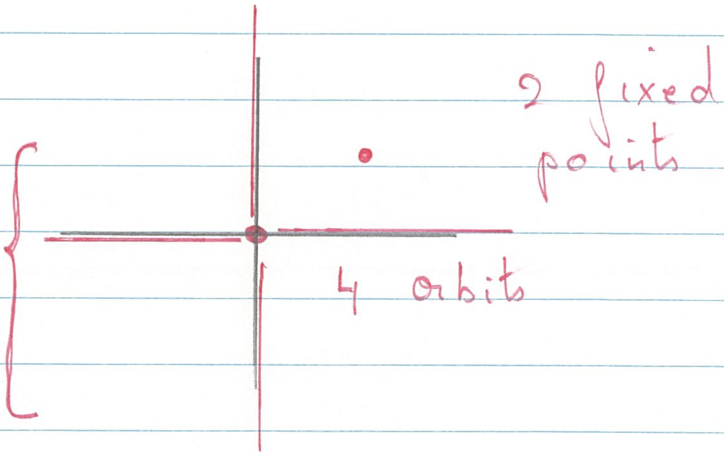
<u>Exercises</u>: Work on the 1D examples and problems in [T] p 204 - 208.

<u>V.2 Lotka - Volterra equation</u> (or predator - prey equation)

In $\mathbb{R}^2$, consider $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} (1-y)x \\ \alpha(x-1)y \end{pmatrix}$ $\alpha > 0$.

Observe that $(0,0)$ and $(1,1)$ are fixed points, and that $\phi(t,(0,y)) = (0, y e^{-\alpha t})$ $\forall y \in \mathbb{R}$ and $\phi(t,(x,0)) = (x e^t, 0)$, $\forall x \in \mathbb{R}$, are solutions.

4 distinct regions $\Leftarrow$ which do not interact.



We concentrate on $(0,\infty)\times(0,\infty)$. If we suppose that $y = y(x)$, one has

chain rule — derivative of an inverse

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\left(\frac{dt}{dx}\right) = \frac{dy}{dt}\cdot\left(\frac{dx}{dt}\right)^{-1} = \alpha\,\frac{(x-1)\,y}{(1-y)\,x}$$

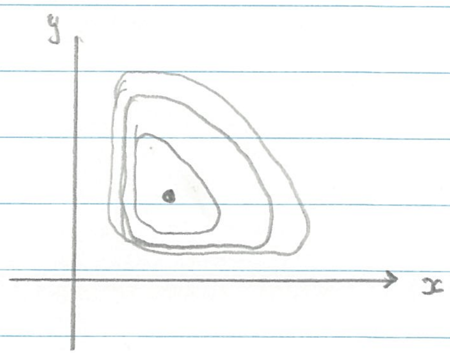
$\rightsquigarrow$ separation of variables

Somebody for the details ?

$f(y) + \alpha f(x) = \text{cst}$ with $f(x) = x - 1 - \ln(x)$

Since $\lim\limits_{x\searrow 0} f(x) = \infty = \lim\limits_{x\nearrow\infty} f(x)$, and f has a minimum at 1, we infers that the orbits are closed. In fact

closed and periodic orbits.



For further extensions, see [T] p 210 - 215

A volunteer ?

V.3 Liénard's equation

Consider $f : \mathbb{R}\to\mathbb{R}$ with $f(0) = 0$ and the equation : $\begin{cases} \dot{x} = y - f(x) \\ \dot{y} = -x \end{cases}$ ⊛. Observe that $(0,0)$ is a fixed point.

<u>1°</u> If $x f(x) > 0$ on $B_\varepsilon((0,0))$ for some $\varepsilon > 0$,

then we can consider $R(x,y) = \frac{1}{2}(x^2+y^2)$

$\frac{1}{2}$ square of distance to $(0,0)$

and if (x,y) satisfies ⊛ one has

$$\frac{d}{dt} R(x(t), y(t)) = x(t)\dot{x}(t) + y(t)\dot{y}(t)$$

$$= x(t)y(t) - x(t) f(x(t)) - y(t)x(t)$$

$$= -x(t) f(x(t)) < 0$$

for $(x(t), y(t)) \in B_\varepsilon((0,0))$.

Then R is strict Lyapunov function and by Thm <u>IV</u>.4, $(0,0)$ is asymptotically stable.

<u>2°</u> If $x f(x) < 0$, $(0,0)$ is repelling, since

$$\frac{d}{dt} R(x(t), y(t)) > 0.$$

the distance to $(0,0)$ increases.

In fact, if f satisfies some properties, one can say more

<u>Thm V.1</u> : Suppose that $f \in C^1(\mathbb{R})$ satisfies

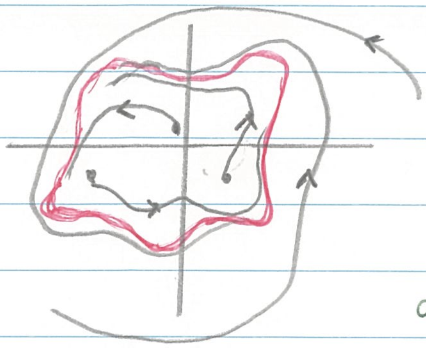
i) f is odd ($\Rightarrow f(0) = 0$)

ii) $f(x) < 0, \forall x \in (0, \alpha)$, $f(\alpha) = 0$

iii) f is increasing on $[\alpha, \infty)$ and

$$\lim_{x \to \infty} f(x) = \infty.$$

Then $\exists!$ periodic orbit enclosing $(0,0)$, and every orbits converge to it (except $(0,0)$)



The proof of this theorem is not trivial and rather long.

<u>Exercise</u> : Show that the van der Pol's equation $\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$, $\mu > 0$, is equivalent to the Liénard's equation with $f(x) = \mu\left(\frac{x^3}{3} - x\right)$.

V.4 About $\omega_\pm$-limit sets in 2D

So far we have seen fixed points and periodic orbits, as $\omega_\pm$-limit sets. In fact, a "complete" description exists in 2D. What makes 2D special is the Jordan curve theorem:

Every non-intersecting continuous closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ in 2 connected disjoint sets.

Then one has:

Thm V.2 (Generalized Poincaré-Bendixson theorem)

Let $M \subset \mathbb{R}^2$ be open and $f \in C^1(M;\mathbb{R}^2)$. For $x \in M$, suppose that $\gamma_\pm(x)$ is contained in a bounded and closed subset of M. (⇒ $\omega_\pm(x)$ is non-empty and connected, see p 47), and suppose that $\omega_\pm(x)$ contains only finitely many fixed points (a counterexample?)

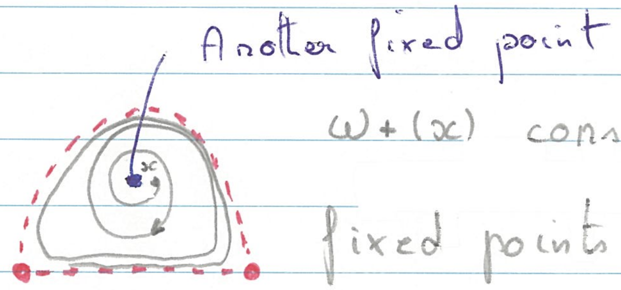
Then one of the following situation holds:

i) $\omega_\pm(x)$ is a fixed point

ii) $\omega_\pm(x)$ is a periodic orbit

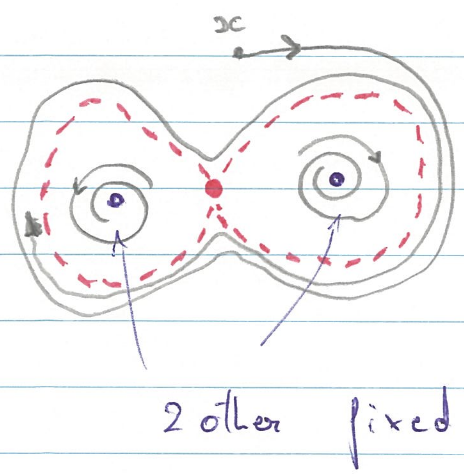
iii) $\omega_\pm(x)$ consists in finitely many fixed points $\{x_j\}$ and non-closed orbits $\gamma(y)$ with $\omega_\pm(y) \in \{x_j\}$.

Examples: 1)



$\omega_+(x)$ consists in 2 fixed points and 2 orbits connecting them.

2)



$\omega_+(x)$ consists in 1 fixed point and 2 orbits satisfying iii)

Exercises: study the examples p 224 and 225 of [T].

VI Dynamical systems in higher dimensions

VI.1 Attracting sets (we assume ϕ globally defined)

Recall that for a dynamical system $(\mathbb{R}, M, \phi)$ and for $x \in M$, $\gamma_\pm(x) := \{\phi(t,x) \in M \mid t \gtrless 0\}$

More generally, for $X \subset M$, we set

$$\gamma_\pm(X) := \bigcup_{x \in X} \gamma_\pm(x) = \{\phi(t,x) \mid x \in X,\ t \gtrless 0\}$$

Clearly, $\gamma_\pm(X)$ are $\pm$-invariant, and so are the closures $\overline{\gamma_\pm(X)}$. For their limit sets, we define

$$\omega_\pm(X) := \{y \in M \mid \exists (t_n, x_n) \text{ with } t_n \to \pm\infty, \text{ and}$$
$$\lim_{n\to\infty} \phi(t_n, x_n) = y\}.$$

Clearly $\bigcup_{x \in X} \omega_\pm(x) \subset \omega_\pm(X)$, but in general the second set is bigger (we can choose different x_n).

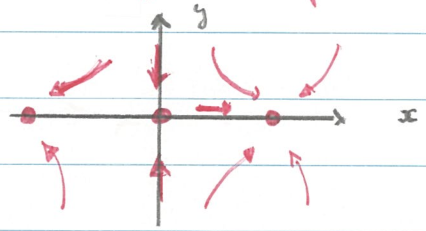
Note that $\omega_\pm(X)$ is closed and $\pm$-invariant, and that $\omega_\pm(X) = \bigcap_{t \gtrless 0} \{\phi(t,x) \mid x \in \overline{\gamma_\pm(X)}\}$.

not so simple, see Lemma 8.1 of [T].

Example : Consider $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} x(1-x^2)\\ -y\end{pmatrix}$ in $\mathbb{R}^2$.

$x = \pm 1$ are stable fixed point for the first equation, $x=0$ is unstable. $y(t) = y_0\, e^{-t}$, and thus $y=0$ is a stable fixed point for the second equation.

(Recall that stable is defined for $t \to +\infty$)



Then $\omega_+ (B_r(0)) = [-1, 1] \times \{0\}$ $\forall\, r > 1$

but $\displaystyle\bigcup_{x \in B_r(0)} \omega_+(x) = \{(-1,0), (0,0), (0,1)\}$.

The set $\omega_+ (B_2(0))$ contains $(-1,0) \times \{0\}$ and $(0,1) \times \{0\}$ because $x=0$ is an unstable fixed point: points near $x=0$ move with an arbitrary small "speed".

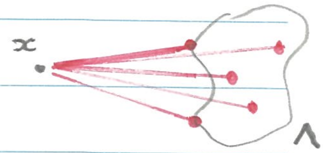
Question : what to do with $\omega_+(X)$ and how to choose X?

already used on page 47

For a set $\Lambda \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, recall that

Euclidean norm

$$d(x, \Lambda) = \inf \{ \| x - y \| \mid y \in \Lambda \}$$



distance between x and the set Λ

Note that the distance between 2 sets is given by

$$d(\Lambda_1, \Lambda_2) = \inf \{ \| x - y \| \mid x \in \Lambda_1, y \in \Lambda_2 \}.$$

<u>Definition</u> : For an invariant set $\Lambda \subset M$, we define

$$W^{\pm}(\Lambda) := \{ x \in M \mid \lim_{t \to \pm\infty} d(\phi(t, x), \Lambda) = 0 \},$$

$W^{+}(\Lambda) =$ <u>stable set</u> , $W^{-}(\Lambda) =$ <u>unstable set</u> .

set of points approaching Λ under the evolution

set of points moving away from Λ under the evolution

One easily observes that $W^{\pm}(\Lambda)$ are invariant.

<u>Definition</u> : If $W^{+}(\Lambda) \supset \Lambda$ is open, then we say that Λ is <u>attracting</u>, and call $W^{+}(\Lambda)$ its <u>domain</u>, or its <u>basin of attraction</u>.

Attracting sets can be detected by looking at <u>trapping regions</u>, namely an open, connected and bounded subset $E \subset M$ satisfying $\phi(t, \bar{E}) \subset E$ for all $t > 0$. Then

$\Lambda := \omega_+(E)$ is non-empty, invariant, closed, bounded, connected, and attracting. Proof not difficult.

However, as seen on the example p 60, a trapping region and the corresponding set Λ are not always so meaningful: for this example $B_r(0)$ is a trapping region, for any $r > 1$, and $\Lambda = [-1, 1] \times \{0\}$, while only $(-1, 0)$ and $(1, 0)$ are really attractive. In fact we have

<u>Lemma VI. 1:</u> For any traping region E, one has $W^-(x) \subset \omega_+(E)$, for every fixed point $x \in \omega_+(E)$. the unstable set

[T] Lemma 8.6

Clearly, one would like to exclude $W^-(x)$ from an attracting set.

<u>Definition</u>: A closed and invariant set Λ is called <u>topologically transitive</u> if for any open subsets $U, V \subset \Lambda$ one has $\phi(t, U) \cap V \neq \emptyset$ for some $t \in \mathbb{R}$.

any 2 open subsets can not stay disjoint under the evolution.

Note that a closed invariant set Λ is topologically transitive if it contains a dense orbit.

<u>Definition</u>: A closed, invariant, attracting and topologically transitive subset of M is called an <u>attractor</u>.

it can not be split into smaller parts

with $W^+((\pm 1, 0)) = \{(x,y) \in \mathbb{R}^2 \mid x \gtrless 0\}$

In the previous examples, only $(\pm 1, 0)$ are attractors, $(-1,0) \times \{0\}$, $(0,1) \times \{0\}$, $(0,0)$ are not.

VI.2 The Lorenz attractor

So far, we have seen fixed point attractors, or periodic orbit's attractors (see Thm V.1 p56). More complicated situations also take place.

Consider the system:

$$\begin{cases} \dot{x} = -\sigma(x-y) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases} \qquad , \quad \sigma, r, b > 0$$

The system is invariant under the transformation $(x,y,z) \longmapsto (-x,-y,z)$, and a solution is $(0,0,z_0 e^{-bz})$, leaving the z-axis invariant.

Also, if $r<1$, $(0,0,0)$ is an exponentially stable fixed point, since the Jacobian matrix $J(0)$ $= \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$ has eigenvalues $-b<0$ and

$$-\frac{1}{2}\left(1+\sigma \pm \sqrt{(1+\sigma)^2 + 4(r-1)\sigma}\right) < 0,$$ see Thm IV.3 p 49.

In fact, if we set $L(x,y,z) = r x^2 + \sigma y^2 + \sigma z^2$,
then one has for the derivative along the evolution:

$$\frac{d}{dt} L(x(t), y(t), z(t)) = -2\sigma\left[\left(r(x-y)^2 + (1-r)y^2 + bz^2\right)\right](t) < 0.$$

Then, by a slight adaptation of Thm IV.4 p 50, one gets that that __all__ trajectories converge to 0, as long as $r < 1$.

For $r > 1$, 2 new fixed points are

$\left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$. In this case, 0 is no more stable, and the 2 new fixed points are asymptotically stable for $1 < r < cst(b,\sigma)$.

Let us still define a trapping region: we set
$L(x,y,z) := r x^2 + \sigma y^2 + \sigma(z-2r)^2$. Then

$$\frac{d}{dt} L(x(t), y(t), z(t)) = -2\sigma\left(r x^2 + y^2 + b(z-r)^2 - br^2\right),$$ and

let $E := \{(x,y,z) \in \mathbb{R}^3 \mid \dot{L}(x,y,z) \geq 0\}$.

Observe that E is an ellipsoid. We also set

$M := \max_{(x,y,z)\in E} L(x,y,z)$, and $E_1 := \{(x,y,z) \mid L(x,y,z) < M+1\}$.

Then $E \subset E_1$ (strict inclusion). In addition,

for $(x,y,z) \in \mathbb{R}^3 \setminus E_1$ one has $\dot{L}(x,y,z) \leq -\delta < 0$

for some $\delta > 0$ $\Rightarrow$ $\phi(t,(x,y,z))$ must enter

E_1 (because the "distance" to $(0,0,2r)$ is decreasing)

$\Rightarrow$ E_1 is a trapping region.

Then, $\Lambda := \omega_+(E_1)$ is an attracting set,

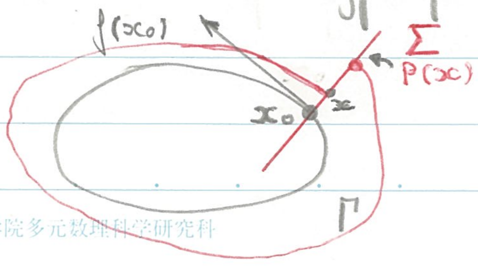
the <u>strange attractor</u> of the Lorentz system.

Its basin of attraction is $\mathbb{R}^3$.

VI.3 The Poincaré map

The behavior near a fixed point can be studied with the Jacobian matrix of f, and such an analysis is performed extensively in [T, chapter 9]. The Poincaré map is used to study the behavior near a periodic orbit. For this part we refer to [P], Section 3.4 and 3.5.

We still consider an autonomous system. Let $M \subset \mathbb{R}^n$, and $f \in C^1(M, \mathbb{R}^n)$. The underlying differential equation is $\dot{x}(t) = f(x(t))$, and let $\phi : W \to \mathbb{R}^n$ be the resulting flow. Suppose that $\Gamma \subset M$ is a periodic orbit, with $\phi(t+T, x) = \phi(t, x)$ $\forall x \in \Gamma$, and let $x_0 \in \Gamma$. Let Σ be the hyperplane perpendicular to Γ at x_0.



For any $x \in \Sigma$, with x close to x_0, we set $P(x) :=$ the first intersection of $\phi(t,x)$ with Σ for $t > 0$.

__Definition__ : The map $x \to P(x)$ is called the __Poincaré map.__

Note that Σ could be any hypersurface not tangent to Γ at x_0, it is not necessarily a hyperplane.

__Theorem VI.2__ : Let $f \in C^1(M, \mathbb{R}^n)$, $\Gamma := \gamma(x_0)$ a T-periodic orbit inside M, and let Σ be the hyperplane $\Sigma = \{ x \in \mathbb{R}^n \mid (x - x_0) \cdot f(x_0) = 0 \}$.

Then $\exists\, \delta > 0$ and $\tau \in C^1(B_\delta(x_0), \mathbb{R}_+)$ with $\tau(x_0) = T$ and such that $\phi(\tau(x), x) \in \Sigma$, $\forall\, x \in B_\delta(x_0)$.

The proof uses the implicit function theorem. Somebody for it?

In the framework of the previous theorem, one can define $\mathbf{P}: B_\delta(x_0) \cap \Sigma \to \Sigma$ with

$$\mathbf{P}(x) = \phi(\tau(x), x), \quad \forall x \in \underbrace{B_\delta(x_0) \cap \Sigma}_{=: V_\delta}.$$

It follows also from the previous thm that $\mathbf{P}: V_\delta \to \Sigma$ is C^1. By translating x_0 to 0 and by observing that $\Sigma \simeq \mathbb{R}^{n-1}$, one obtains that $\mathbf{P}: B_\delta(0) \cap \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, and its Jacobian matrix becomes a $(n-1)\times(n-1)$ matrix, denoted by $J_{\mathbf{P}}(0)$ or $D\mathbf{P}(0)$.

Recall from Section II.7 about periodic linear systems that the matrix solution of the equation $\underline{\dot{\Pi}(t,0) = A(t)\,\Pi(t,0)}$, with $A(t+T) = A(t)$, takes the form (for $t_0 = 0$): $\Pi(t,0) = P(t,0)\, e^{tQ(0)}$

$\equiv P(t)\, e^{tQ}$ with $P(t+T) = P(t)$, $P(0) = \mathbb{1}$ and $Q \in M_n(\mathbb{C})$

see p.29.

Now, if ϕ denotes the flow of the initial equation with T-periodic orbit $\gamma(x_0)$, let us define $H(t,x) := \mathcal{J}_\phi(t,x) \equiv [D\phi](t,x)$ ⊛ the (space) derivative of ϕ.

it is a matrix $(n \times n)$

From $\dot{\phi}(t,x) = f(\phi(t,x))$ one infers that

$$\frac{d}{dt} H(t,x) = [Df](\phi(t,x))\, H(t,x).$$

derivative of a composition

If we fix $x = x_0$, and observe that

$$\phi(t+T, x_0) = \phi(t, x_0) \Rightarrow Df(\phi(t+T,x_0)) = Df(\phi(t,x_0)),$$

we get the matrix equation $\underline{\dot{H}(t,x_0) = A(t)\, H(t,x_0)}$, with $A(t) := Df(\phi(t,x_0))$, and $H(0,x_0) = \mathbb{1}$ (from ⊛). Thus, we infer from Section II. 7 that

$$H(t,x_0) = P(t)\, e^{tQ} \quad \text{with} \quad P(t+T) = P(t) \text{ and}$$

$P(0) = \mathbb{1}$. Also $H(T,x_0) = e^{TQ}$.

its eigenvalues are called Floquet exponents, or characteristic exponents.

its eigenvalues are called Floquet multipliers, or characteristic multipliers.

Let us now link the Poincaré map with the above construction. For simplicity we shift x_0 to 0.

<u>Theorem VI.3</u> : Let $f \in C^1(M, \mathbb{R}^n)$, ϕ the flow associated with $\dot{x} = f(x)$, and $\gamma(0)$ a T-periodic orbit. Let P be the associated Poincaré map at 0, and $H(T,0) = e^{TQ}$ constructed as above. Then, one Floquet multiplier is 1, while the others are eigenvalues of the $(n-1)\times(n-1)$ matrix $DP(0)$.

<u>Idea for the proof</u> : Somebody for the details ?

1) $\dot{\phi}(0,0) = f(0)$ satisfies $H(T,0)f(0) = f(0)$

$\Rightarrow$ 1 is an eigenvalue of $H(T,0)$.

2) Consider $h : B_\delta(0) \to \mathbb{R}^n$, $h(x) = \phi(J(x), x)$.

space derivative

Then $Dh(x) = \underbrace{\dot{\phi}(J(x), x)}_{\in M_{n\times 1}} \underbrace{[DJ](x)}_{\in M_{1\times n}} + D\phi(J(x), x)$

$= \dot{\phi}(J(x), x)[DJ](x) + H(J(x), x)$.

For $x = 0$ one gets

$$Dh(0) = \dot{\phi}(T,0)[DJ](0) + H(T,0)$$

$$= f(0)\,[DJ](0) + H(T,0). \quad \circledast$$

If we choose a basis of $\mathbb{R}^n$ with the first $n-1$ coordinates in Σ, and the last one along $f(0)$, then $\circledast$ reads

$$\begin{pmatrix} & 0 & \\ \frac{\partial J}{\partial x_1} & \cdots & \frac{\partial J}{\partial x_n} \end{pmatrix}(0) + \begin{pmatrix} H(T,0)|_\Sigma & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\ 0 \;\cdots\; 0 & 1 \end{pmatrix}$$

↑ corresponding to the eigenvalue 1

Since $P = h(0)|_\Sigma$, one deduces that

$$DP(0) = H(T,0)|_\Sigma. \qquad \square$$

Observe that $H(T,0)$ corresponds also to the monodromy matrix $M(0)$ introduced on p 28. It implies that the eigenvalues of $DP(0)$ plus the eigenvalue 1 correspond to the eigenvalues of $M(0)$. Then, the periodic orbit is asymptotically stable if all eigenvalues of $DP(0)$ lie inside the unit disk in the complex plane. Indeed, a Floquet multiplier inside the unit disk correspond to a Floquet exponent with negative real part. In addition, these eigenvalues do not depend on the initial choice of x_0 inside the periodic orbit (and on the hyperplane Σ). If some eigenvalues are inside the unit disk, while others are outside, one can define a stable and an unstable set, see [T. p 321 - 324] or [P. p 225 - 231] for further details.

VI. 4 Miscellaneous

1) For the linear differential equation $\dot{x} = Ax$, the stable, unstable, and center (or critical) subspaces have been introduced on p16 and the first 2 subspaces correspond for the fixed point 0 to the stable set $W^+(\{0\})$ and $W^-(\{0\})$ if the center subspace is trivial. This situation correspond to the absence of eigenvalues of A with real part 0, and is called <u>hyperbolic</u>. Hyperbolic systems are much more stable under perturbations compared to non-hyperbolic systems (with at least one eigenvalue with real part 0).

A fixed point of a general dynamical

system is called hyperbolic if the correspon-ding Jacobian matrix is hyperbolic.

2) Let x_1, x_2 be fixed points of a dynamical system. Let $\gamma(x)$ be an orbit satisfying $\omega_-(x) = x_1$, $\omega_+(x) = x_2$. If $x_1 \neq x_2$, one speaks about a <u>heteroclinic</u> orbit, while if $x_1 = x_2$ one speaks about a <u>homoclinic</u> orbit.

3) The next statement is about the linear aproxi-mation near a fixed point.

<u>Theorem</u> VI.4 (Hartman-Grobman) Let f be $C^1(M,\mathbb{R}^n)$ and assume that 0 is a hyperbolic fixed point. Let ϕ be the flow and set $A = Df(0) \equiv \mathcal{J}_f(0)$ for the Jacobian matrix. Then $\exists\, \varphi$ a locally bijective,

continuous, and with inverse continuous, function near 0 satisfying $\varphi(x) = x + h(x)$ with h bounded, such that

$$\varphi \circ e^{tA} = \phi(t, \cdot) \circ \varphi = \phi(t, \varphi(\cdot))$$

in a small ball around 0.

In other words, the flow around 0 is a continuously deformed version of the linear flow.

VII Discrete dynamical systems

VII.1 General framework

no difference with the continuous setting

So far, only the continuous setting has been illustrated, let's move to the discrete setting, and see that there is no difference in the general theory. We consider now $G = \mathbb{Z}$ or $\mathbb{Z}_+$, and let M be a subset of $\mathbb{R}^n$ or a subset of a metric space (see later). Let $f : W \to M$ with $W \subset G \times M$ and define the evolution equation by

$$x(m+1) = f(m, x(m)), \quad \forall m \in G.$$

<u>Remark</u>: Two special instances of this framework are the <u>iterated map</u>: $f : M \to M$ and

$$x(m+1) = f(x(m)) = f^m(x(0)) \qquad (*)$$

(this corresponds to the autonomous system)

and the linear situation:

$x(m+1) = A(m)\,x(m)$ with $A(m) \in M_{n\times n}(\mathbb{R})$.

If $A(m)$ is independent of m, it is a linear autonomous system (it is also an iterated map).

VII. 2 Iterated maps

For $f : M \to M$ a fixed point of ⊛ satisfies $x = f(x)$, and a n-periodic point satisfies $f^n(x) = x$. If x is a n-periodic point, we assume that $f^m(x) \neq x$, $\forall\, m \in \{1, \dots n-1\}$ and observe that $\{x, f(x), f^2(x), \dots, f^{n-1}(x)\}$ are all n-periodic fixed points. This is called a n-periodic orbit, and corresponds to a periodic orbit in the continuous case.

We also define the forward orbit $\gamma_+(x)$ of x by

$\gamma_+(x) = \{ f^n(x) \mid n \in \mathbb{Z}_+ \}$. Clearly, such an orbit is $+$-invariant, namely $f(\gamma_+(x)) \subset \gamma_+(x)$.

<u>Remark:</u> It seems that the notion of $(\pm)$-invariance is meaningful only in the autonomous case, and this holds also for the continuous setting.

A point $x \in M$ <u>converges to a n-periodic point</u> ($n \in \mathbb{N}$) if $\lim\limits_{\substack{k \to \infty \\ k \in \mathbb{N}}} f^{nk}(x)$ exists and belongs to M. More generally, if Λ is an $+$-invariant set, we define

$$W^+(\Lambda) := \{ x \in M \mid \lim_{m \to \infty} d(f^m(x), \Lambda) = 0 \}$$

and call it the <u>stable set of Λ</u>. If $W^+(\Lambda)$ is open and $\Lambda \subsetneq W^+(\Lambda)$, we say that Λ is <u>attracting</u>.

As in the continuous case, the behavior near a n-periodic point x is determined by the Jacobian of f^n, whenever $f \in C^1(M, M)$, namely the eigenvalues of $Df^n(x) \equiv J_{f^n}(x)$ dictate the dynamics near x: attracting if all eigenvalues are inside the unit disk (of $\mathbb{C}$), and repelling if outside of the unit disk.

Observe that

$$Df^n(x) = Df(f^{n-1}(x)) \; Df(f^{n-2}(x)) \cdots Df(f(x)) \, Df(x).$$

The fixed point x is also called <u>hyperbolic</u> if the Jacobian matrix has no eigenvalue on the unit circle, and a Hartman-Grobman theorem also holds in this framework.

Note that it can be aplied to f^n, if necessary.

In a vague sense, it is similar to the Poincaré map!

<u>Theorem VII.1</u> Let $f \in C^1(M,M)$ and suppose that 0 is a hyperbolic fixed point, with f bijective near 0, with f^{-1} also C^1 near 0. Then, $\exists\, \varphi$ a locally continuous and bijective function (with inverse also continuous) s.t.

$$\varphi(Ax) = f(\varphi(x))$$

for $A := Df(0)$ and all x close to 0.

VII.3 The linear situation

The linear system $x(m+1) = A(m)\,x(m)$ can be easily solved: $x(m+1) = \prod_{j=0}^{m} A(j)\, x(0)$. In this setting the <u>principal matrix solution</u> $\Pi(m, m_0) := \prod_{j=m_0}^{m-1} A(j)$ satisfies

product sign

$$\Pi(m+1, m_0) = A(m)\, \Pi(m, m_0)$$

with $\Pi(m_0, m_0) = \mathbb{1}$. For the inhomogeneous equation

$$x(m+1) = A(m)\,x(m) + g(m) \quad , \quad x(m_0) = x_{m_0}$$

the solution is given by

$$x(m) = \Pi(m, m_0)\, x_{m_0} + \sum_{j=m_0}^{m-1} \Pi(m,j)\, g(j)$$

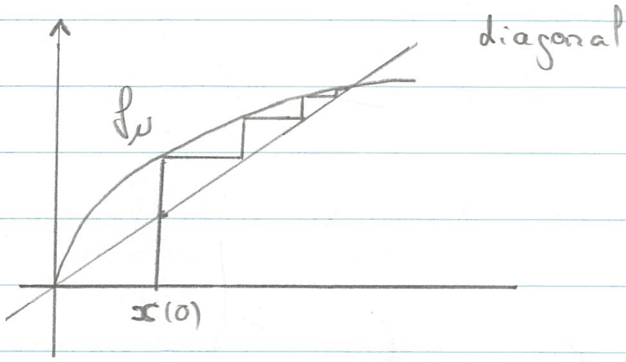
Somebody for the proof ?

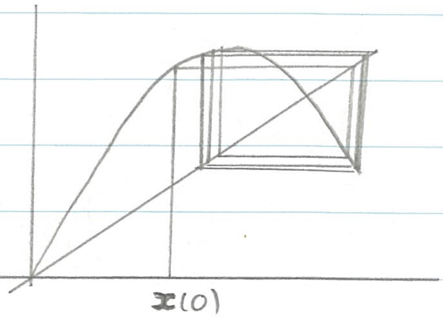
VII. 4 The logistic equation

For $\mu \in [0,4]$, consider the function $f_\mu : [0,1] \to [0,1]$ defined by $f_\mu(x) = \mu x(1-x)$. The logistic equation corresponds to the equation

$$x(m+1) = f_\mu(x(m)) = \mu\, x(m)\,(1 - x(m)) .$$

For any fixed μ and any $x(0) \in [0,1]$, the iterated values $f_\mu^m(x(0))$ can be obtained by a cobweb plot ≡ Verhulst diagram :





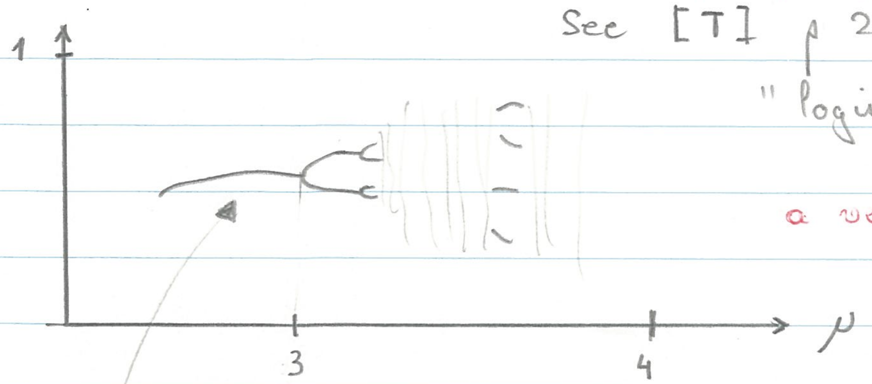
different evolutions can take place, depending on $x(0)$ and on ν.

See [T] ~ p. 282 and 293 for mathematica codes.

In the first plot, $f_\nu^m(x(0))$ converges to the unique solution of $f_\nu(x) = x$, while in the second plot there exist 2 solutions of the equation $f_\nu^2(x) = x$, and these solutions generate a 2-periodic orbit.

Note that the existence of n-periodic orbits depends on the value of ν, as discussed below.

A summary of the asymptotic evolutions can be obtained by ploting the "asymptotic" values of $f_\nu^m(x(0))$ for various values of ν, and $x(0)$ fixed.



See [T] p 294, or Wikipedia "logistic map"

a very complicated and famous picture

$\{ f_\mu^m(x(0)) \mid m \in (200, 300) \}$ for a fixed initial condition $x(0) = 0.4$.

Let us now briefly describe the behavior of the logistic equation for "small" μ. Somebody for the details?

For $\mu \in [0,1]$, only fixed point 0, attracting. For $\mu \in (1, 3]$, a new fixed point $p := 1 - \frac{1}{\mu}$, with $W^+(0) = \{0, 1\}$ and $W^+(p) = (0,1)$.

stable set

For $\mu \in (3, 1+\sqrt{6})$, there exist two 2-periodic points $p_\pm := \frac{1}{2\mu}\left(1 + \mu \pm \sqrt{(\mu+1)(\mu-3)}\right)$, solutions of $f_\mu^2(x) = x$. p is no more attracting, but $p_\pm$ generate an attracting 2-periodic orbit.

The bifurcation at $\mu = 3$ is called a <u>period doubling</u>. Let us emphasize that the fixed point p still exists, but is unstable. Bifurcation theory is a huge subject, see for example [P], let us just scratch the surface.

<u>VII. 5</u> <u>Bifurcation theory</u>

Very roughly, a bifurcation occurs when a small change in a parameter leads to a qualitative change in the behavior of the system. This idea applies to continuous time dynamical systems or to discrete time dynamical systems, <u>in the autonomous case</u>. Let's "try" to be more precise:

continuous

<u>Definition</u> Let $M \subset \mathbb{R}^d$, $f_c \in C^1(M, \mathbb{R}^d)$, and

discrete

$f_d \in C^1(M, M)$. f_c or f_d are called <u>structurally stable</u> if $\exists\, \varepsilon > 0$ such that for all $g_c \in C^1(M, \mathbb{R}^d)$ or $g_d \in C^1(M, M)$ satisfying for $* = c$ or d:

$$\| f_* - g_* \|_1 := \sup_{x \in M} \| f_*(x) - g_*(x) \| + \sup_{x \in M} \| Df_*(x) - Dg_*(x) \| < \varepsilon$$

norm in $\mathbb{R}^n$ norm in $M_n(\mathbb{R})$

there exists $\varphi : M \to M$ bijective and bi-continuous mapping the trajectories of $\dot{x}(t) = f_c(x(t))$ onto the trajectories of $\dot{x}(t) = g_c(x(t))$, or of $x(m+1) = f_d(x(m))$ onto the ones of $x(m+1) = g_d(x(m))$.

In other words, f_* is structurally stable if any small modification of f_* preserves the trajectories.

Based on this notion, bifurcations can be defined:

<u>Definition</u>: Consider $f_\mu \in C^1(M, \mathbb{R}^n)$ or $f_\mu \in C^1(M,M)$ for $\mu \in I$. Then $\mu_0 \in I$ is a <u>bifurcation value</u> if f_{μ_0} is not structurally stable.

There exist plenty of types of bifurcations, often nicely illustrated (see for example [PI]). The simplest situations are the bifurcation at a fixed point or around a periodic orbit. The continuous time periodic case can be studied with the Poincaré map. The fixed point case, or the discrete time periodic case, can be studied directly with the Jacobian matrix.

For a continuous time system, a fixed point x_0 satisfying $0 = f_{\mu_0}(x_0)$ can be a bifurcation for μ_0 if one eigenvalue of $Df_\mu(x)$ crosses $\{ z \in \mathbb{C} \mid Re(z) = 0 \}$ at (x_0, μ_0).

For a discrete time system, a fixed point x_0 satisfying $x_0 = f_{\mu_0}(x_0)$ can be a bifurcation for μ_0 if one eigenvalue of $Df_\mu(x)$ crosses $\{ z \in \mathbb{C} \mid |z| = 1 \}$ at (x_0, μ_0).

(Observe that the above conditions correspond to two conditions: 1) a fixed point at x_0, 2) a specific behavior of the eigenvalues of $Df_{\mu_0}(x_0)$).

Depending on the number of eigenvalues involved in the crossing, the bifurcations can be of very different natures. [P] studies this in detail, we shall only look at the discrete time logistic map.

VII. 6 Back to the logistic equation

Let's look at the period doubling bifurcation (also called pitchfork bifurcation) already mentioned in Section VII. 4. The following theorem (based on the implicit function theorem) plays a "crucial" role (see p 335 of [HSD])

Theorem VII. 2 (Bifurcation criterion)

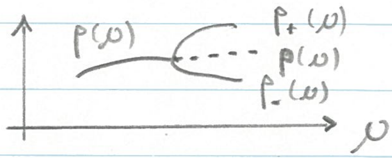
Let $f_\cdot \in C^1(\underset{\mu}{\mathbb{R}} \times \underset{x}{\mathbb{R}}, \mathbb{R})$ (cont. diff. in the variable and in the parameter). Suppose that $\exists (\mu_0, x_0)$ s.t. $f_{\mu_0}(x_0) = x_0$ and $f'_{\mu_0}(x_0) \neq 1$. Then $\exists$ an interval $J \ni \mu_0$ and an interval $I \ni x_0$ and a continuous function $p : J \to I$ s.t. $p(\mu_0) = x_0$, and $f_\mu(p(\mu)) = p(\mu)$ for all $\mu \in J$. In addition f_μ has no other fixed point in I.

$\ni \mu$ $\ni x$

Consider $f_\cdot \in C^1(\mathcal{J} \times \mathbb{R}, \mathbb{R})$, and suppose that $f_\mu(x) = x$ has a single solution $p(\mu)$, $\forall \mu \in \mathcal{J}$.

By the previous theorem, we infer that $f'_\mu(p(\mu)) \neq 1$ "in general", but there is no implication ⚠.

Let also $\mu_0 \in \mathcal{J}$, and assume that $f^2_\mu(x) = x$ ⊛ has 2 addition solutions (denoted by $p_-(\mu)$, $p_+(\mu)$) for $\mu > \mu_0$, and no additional solution for $\mu \leq \mu_0$.



From the previous theorem applied to f^2_μ one gets

$$f^2_{\mu_0}(x)'\Big|_{p(\mu_0)} = f'_{\mu_0}\big(\underbrace{f_{\mu_0}(p(\mu_0))}_{=\,p(\mu_0)}\big)\, f'_{\mu_0}(p(\mu_0)) = f_{\mu_0}(p(\mu_0))^2 = 1$$

because f^2_μ has a change in the number of solutions of ⊛. As consequence, we call a <u>period doubling bifurcation</u> a value μ_0 verifying the above framework and $f_{\mu_0}(p(\mu_0)) = -1$.

<u>Observations</u>: 1) The fact that $f'_{\mu_0}(p(\mu_0)) = -1$ corresponds to the general idea that $Df_\mu(x)$ should have an eigenvalue crossing the unit circle.

2) The existence of an unstable fixed point $p(\mu)$ even for $\mu > \mu_0$ is not an accident, see below.

3) The same process and result can then be applied to f_μ^2 and f_μ^4, and iteratively for $f_\mu^{2^n}, f_\mu^{2^{n+1}}$.

Observe now that all natural numbers can be written uniquely as $2^m(2n+1)$ for $n, m \in \mathbb{Z}_+$. Let's give them a new order:

$3 > 5 > 7 > \ldots > 2\cdot 3 > 2\cdot 5 > 2\cdot 7 > \ldots \quad > 2^m\cdot 3 > 2^m\cdot 5 >$

$> 2^m\cdot 7 > \ldots > \quad \ldots > 2^4 > 2^3 > 2^2 > 2 > 1$

Observe that each natural number appears only once, but most of them are preceded by an infinite number of them!

Theorem VII. 3 (Sarkovskii's theorem)

If $f : M \subset \mathbb{R} \to M$ is continuous and has an orbit of period m, then it also has orbits with prime period n for all n satisfying $m \triangleright n$.

meaning that $f^m(x) = x$ is not satisfied for any $m < n$, if $f^n(x) = x$ is satisfied.

Corollary : If f has an orbit of period 3, it also has orbits of prime period n for all numbers $n \in \mathbb{N}$.

Remark : For the logistic equation, f_μ has an orbit of period 3 for $\mu = 4$, or $\mu = 1 + 2\sqrt{2}$. For checking this, one has to solve

$$f_\mu^3(x) = x .$$

VII . 7 Chaos

There exists several definitions of chaos, depending on the context. The definition provided below holds for continuous and discrete time dynamical systems, for autonomous systems.

Recall that the definition of topologically transitive has been introduced for a flow in Section VI.1 (p 63). In the discrete time setting, let $f: M \to M$ generating an iterated map. Then f is <u>topologically transitive</u> if

non empty, open

$\forall\, U, V \subset M$, $\exists\, n \in \mathbb{N}$ s.t. $f^n(U) \cap V \neq \emptyset$.

Again, f is topologically transitive if M contains a dense orbit.

Another notion which is often associated with chaos (not defined yet) is the <u>sensitive dependence on initial conditions</u>, namely $\exists\, \delta > 0$ such that $\forall\, x \in M$ and $\forall\, \varepsilon > 0$, there exists $y \in M$ and $n \in \mathbb{N}$ with $d(x, y) < \varepsilon$ but $d(f^n(x), f^n(y)) > \delta$.

distance, for example $\| x - y \|$ in $\mathbb{R}^n$

⚠ This condition is not sufficient for defining a chaotic behavior: The map $f : (0, \infty) \longrightarrow (0, \infty)$, $f(x) = 2x$ satisfies the above property, but one wouldn't say that it is a chaotic map.

<u>Definition</u> : A dynamical system is said to be <u>chaotic</u> if :

1) It is topologically transitive,

2) It has dense periodic orbits,

3) It has a sensitive dependence on initial conditions.

<u>Lemma VII.4</u> : In the discrete time setting, if $f: M \to M$ is continuous, then

1) + 2) $\Rightarrow$ 3).

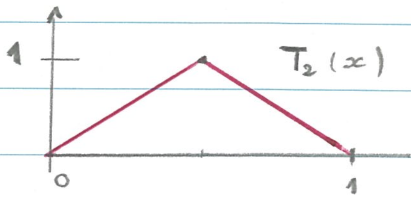
<u>Remark</u>: Sometimes, only 1) and 3) are assumed, but it is a weaker definition.

Our next aim is to show that the logistic equation with $\mu = 4$ is chaotic. For this, we shall use another system: the tent map.

Consider $T_\mu : [0,1] \to \mathbb{R}$, $T_\mu(x) := \frac{\mu}{2}(1 - |2x - 1|)$

for $\mu \in \mathbb{R}$. In particular, we consider

$$T_2(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$



and $T_2 : [0,1] \to [0,1]$.

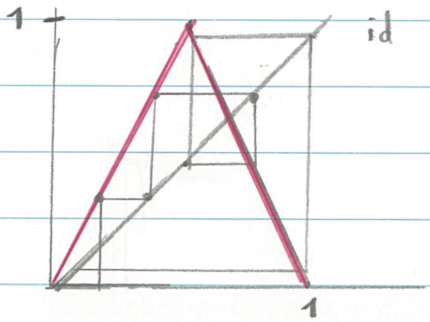
T_2 defines an iterated map on $M = [0,1]$

Then one has:

$$T_2^n(x) = \begin{cases} 2^n x - 2j & \text{if } \frac{2j}{2^n} \leq x \leq \frac{2j+1}{2^n} \\ 2(j+1) - 2^n x & \text{if } \frac{2j+1}{2^n} \leq x \leq \frac{2j+2}{2^n} \end{cases}$$

for $j = 0, 1, \ldots 2^{n-1} - 1$

Somebody for the proof? By induction over n?



Then the interval $\left[\frac{2j}{2^n}, \frac{2j+1}{2^n}\right]$ is mapped onto $[0,1]$, and the point $x = \frac{2j}{2^n - 1} \in$ and solves $T_2^n(x) = x$. It is thus a n-periodic point.

From this, we infer that periodic points are dense in $[0,1]$.

We can choose very small intervals, they always contain some periodic point.

To show that the system is topologically transitive, take any U, V open on non-empty. Then $\exists\, j, n$ such that $\left[\frac{2j}{2^n}, \frac{2j+1}{2^n}\right] \subset U$, and since $T_2^n\left(\left[\frac{2j}{2^n}, \frac{2j+1}{2^n}\right]\right) = [0,1]$, $T_2^n(U) \cap V \neq \emptyset$.

Property 3) of chaotic systems follows from Lemma VII.4. Thus, T_2 is chaotic.

<u>Exercise</u>: Set $\varphi(x) := \sin\left(\frac{\pi x}{2}\right)^2$ for $x \in [0,1]$. Check that $\varphi \circ T_2 = f_4 \circ \varphi$.

Since $\varphi : [0,1] \to [0,1]$ is bijective and bi-continuous, the chaotic properties of T_2 are transferred to f_4 (they are stable under <u>homeomorphisms</u>). We also say that T_2 and f_4 are <u>topologically equivalent</u>.

VII. 8 Cantor sets and strange attractors

Consider the tent map T_μ for $\mu > 2$. In this case, one has to consider

$$T_\mu : \mathbb{R} \to \mathbb{R}, \quad T_\mu(x) = \frac{\mu}{2}\left(1 - |2x - 1|\right).$$

One easily observes that $T_\mu^n(x) \xrightarrow{n\to\infty} -\infty$ for $x \notin [0,1]$. In fact, set $\Lambda_1 := [0, \frac{1}{\mu}] \cup [1-\frac{1}{\mu}, 1]$, and observe that $T_\mu^n(\mathbb{R} \setminus \Lambda_1) \xrightarrow{n\to\infty} -\infty$, since $T_\mu^1(\mathbb{R}\setminus\Lambda_1) \subset \mathbb{R}\setminus[0,1]$. Similarly, if

$$\Lambda_2 := [0, \tfrac{1}{\mu^2}] \cup [\tfrac{1}{\mu} - \tfrac{1}{\mu^2}, \tfrac{1}{\mu}] \cup [1-\tfrac{1}{\mu}, 1-\tfrac{1}{\mu}+\tfrac{1}{\mu^2}] \cup [1-\tfrac{1}{\mu^2}, 1],$$

then $T_\mu^2(\mathbb{R}\setminus\Lambda_2) \subset \mathbb{R}\setminus[0,1]$. More generally set $\Lambda_n := (\frac{1}{\mu}\Lambda_{n-1}) \cup (1 - \frac{1}{\mu}\Lambda_{n-1})$. Then $T_\mu^n(\mathbb{R}\setminus\Lambda_n) \subset \mathbb{R}\setminus[0,1]$. Observe that Λ_n consists of 2^n intervals of length μ^{-n}. Finally, set

$$\Lambda := \bigcap_{n\in\mathbb{N}} \Lambda_n \subset [0,1].$$

By construction, $T_\mu(\Lambda) = \Lambda$, and

$T_\mu^n(x) \xrightarrow{n \to \infty} -\infty$ for any $x \in \mathbb{R} \setminus \Lambda$.

Clearly $\Lambda \neq \emptyset$ since $T_\mu^{-n}(\{0,1\}) \in \Lambda$ $\forall n$ ⊛.

Also, Λ does not contain any interval,

and therefore its Lebesgue measure is 0.

Any set which does not contain any interval

is called <u>totally disconnected</u>. Λ is also

<u>perfect</u>: any point $x \in \Lambda$ is an accumulation

point, or in other words is the limit of a

sequence of points of Λ. This property can

also be deduced from ⊛.

Sets which are closed, totally discon-

nected and perfect are called <u>Cantor sets</u>.

⚠ the most general definition is slightly more general!

Clearly, Λ is a Cantor set.

As in the continuous time setting, an invariant set Λ ($\equiv f(\Lambda) = \Lambda$) is called <u>attracting</u> if $\exists\, U$ open with $\Lambda \subsetneq U$ such that

$$d(f^n(x), \Lambda) \xrightarrow{n \to \infty} 0 \quad \text{for all } x \in U.$$

Λ is <u>repelling</u> if $\exists\, U$ open with $\Lambda \subsetneq U$ such that for all $x \in U \setminus \Lambda$, $\exists\, n_x$ with $f^n(x) \notin U$ $\forall\, n \geq n_x$. In these situations, Λ is also called an <u>attractor</u> or a <u>repeller</u>.

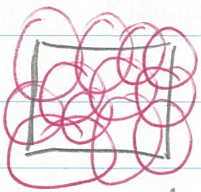
<u>Def</u>: An attractor or a repeller Λ is <u>strange</u> if the underlying dynamical system (Λ, f) is chaotic and Λ is <u>fractal</u>.

the restriction of the dynamical system to Λ

⚠ Strange non chaotic attractors exist, and fractals are not uniquely defined...

A possible definition for a fractal is a set with a Hausdorff dimension which is not an integer.

For any set $U \subset \mathbb{R}^n$, let us define the diameter of U by $D(U) := \sup_{x,y \in U} \|x-y\|$. Recall also that for any $\Lambda \subset \mathbb{R}^n$, a cover of Λ is a collection $\{U_j\}_j$ with $U_j \subset \mathbb{R}^n$ and $\Lambda \subset \bigcup_j U_j$



: Λ and a cover

We suppose that covers are countable (at most $\mathbb{N}$ elements).

For any $\alpha \geq 0$ and $\delta > 0$ we set

$$h^\alpha_\delta(\Lambda) := \inf\Big\{ \sum_j D(U_j)^\alpha \;\Big|\; \{U_j\} \text{ cover of } \Lambda,\ D(U_j) \leq \delta \Big\}$$

$$\in [0, \infty]$$

Observe that $h^\alpha_\delta(\Lambda)$ increases as δ decreases, since less possible covering are available.

As a consequence, one can set

$$h^\alpha(\Lambda) := \lim_{\delta \searrow 0} h^\alpha_\delta(\Lambda) \equiv \sup_{\delta > 0} h^\alpha_\delta(\Lambda) \in [0, \infty]$$

h^α has the properties that $h^\alpha(U) \leq h^\alpha(V)$ if $U \subset V$, and $h^\alpha(\bigcup_j U_j) \leq \sum_j h^\alpha(U_j)$, with an equality for any countable union of disjoint (Borel) sets

Let us now observe that if $\delta < 1$ and $D(U_j) < \delta$ then $D(U_j)^\beta \leq D(U_j)^\alpha$ if $\beta > \alpha$

$\Rightarrow$ $h^\beta(\Lambda) \leq h^\alpha(\Lambda)$ if $\beta > \alpha$. In addition, from

$$D(U_j)^\beta = D(U_j)^{\beta-\alpha} D(U_j)^\alpha$$
$$\leq \delta^{\beta-\alpha} D(U_j)^\alpha$$

one infers that $\sum_j D(U_j)^\beta \leq \delta^{\beta-\alpha} \sum_j D(U_j)^\alpha$.

Thus $h^\beta_\delta(\Lambda) \leq \delta^{\beta-\alpha} h^\alpha_\delta(\Lambda) \leq \delta^{\beta-\alpha} h^\alpha(\Lambda)$.

$\delta < 1$ $\qquad$ by def.

As a consequence, if $h^\alpha(\Lambda) < \infty$, then

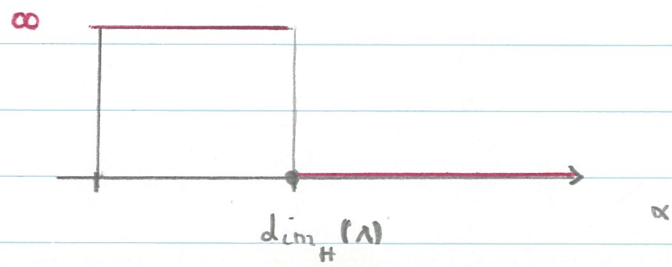
$$h^\beta(\Lambda) = \lim_{\delta \searrow 0} h^\beta_\delta(\Lambda) \leq \lim_{\delta \searrow 0} \delta^{\beta-\alpha} h^\alpha(\Lambda) = 0$$

for any $\beta > \alpha$. We can then set

$$\underline{\dim_H(\Lambda)} := \inf\{\beta \geq 0 \mid h^\beta(\Lambda) = 0\}.$$

In fact, this quantity is also equal to

$$\sup\{\alpha \geq 0 \mid h^\alpha(\Lambda) = \infty\}.$$



<u>Remark</u>: $\dim_H(\Lambda)$ is usually difficult to compute. But $\dim_H(\{x_1, \dots, x_N\}) = 0$, $\dim_H(U) \leq n$ for $U \subset \mathbb{R}^n$, $\dim_H(\Lambda) = \frac{\log(2)}{\log(\mu)}$ for the strange repeller of the tent map for $\mu > 2$.

For the Lorentz attractor of Section VI.2 with $\sigma = 10$, $r = 28$ and $b = 8/3$, one has

$\dim_H(\Lambda) \simeq 2,0627160$.

What a journey!