On the analytic behaviour of higher derivatives of Hardy's Z-function, and a certain discrete moment of the first derivative of Dirichlet L-functions
(ハーディの Z-関数の高階導関数の解析的挙動, 及びディリクレ L-関数の一階導関数のある離散的平均値について)

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Abstract

In this thesis, we give some results on certain meromorphic functions related to the Riemann ζ-function. One of those is associated with the k-th derivative of Hardy's Z-function and denoted by $Z_k(s)$. Hardy's Z-function is a real-valued function whose zeros coincide with those of the Riemann ζ-function on the critical line. Thus Hardy's Z-function is an important tool to study the distribution of the zeros of the Riemann ζ-function. To investigate the analytic behaviour of a function, it is helpful to consider its derivative. Edwards and Mozer obtained the result that the zeros of the first derivative of Hardy's Z-function are interlaced with those of Hardy's Z-function. After that, Anderson, Matsumoto, Tanigawa, and Matsuoka progressed the knowledge on the distribution of the zeros of Hardy's Z-function and its higher derivatives. In their proof, the meromorphic function $Z_k(s)$ plays an important role. We present miscellaneous results on the completed $Z_k(s)$ which is a natural generalization of the Riemann ξ-function. Next, we consider a discrete mean value of higher derivatives of Hardy's Z-function. This study is motivated by Matsuoka's result.

We also give a result on a discrete mean value of the first derivative of Dirichlet L-functions. The result is a generalization of Fujii's result on the Riemann ζ-function. His study is inspired by Shanks' conjecture and his result implies that the conjecture is true. The conjecture states that the first derivative of the Riemann ζ-function at the zeros of the Riemann ζ-function is positive and real in the mean. Therefore, the results are also related to the problem on the simplicity of the zeros of the function, which is an important problem to study some arithmetic functions.

Contents

Preface

In 1859, B. Riemann published a paper on the distribution of prime numbers and conjectured that the *Riemann ζ-function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

has non-real zeros only on the critical line $\Re s = 1/2$. This conjecture is called the *Riemann Hypothesis*. Since then, the analytic behaviour of the Riemann ζ-function has been an attractive research theme. In 1914, G. H. Hardy proved that there are infinitely many zeros of $\zeta(s)$ on the critical line. This is one of the most important achievements towards the Riemann Hypothesis (RH). He studied an integral

$$\int_0^\infty \frac{\xi(\frac{1}{2}+it)}{t^2+\frac{1}{4}} t^{2n} \cosh \frac{\pi t}{4} dt,$$

where $\xi(s)$ is called the *Riemann ξ-function* derived from the Riemann ζ-function. His argument is simplified by considering the real-valued function $Z(t)$, named *Hardy's Z-function* now. Since Hardy's Z-function is real and continuous, the sign change indicates the existence of its zero. Moreover, the zeros of Hardy's Z-function coincide with those of the Riemann ζ-function on the critical line. From these properties, comparing $\int Z(t)dt$ with $\int |Z(t)|dt$, we can show that Hardy's statement is true.

In this thesis, we consider the analytic behaviour of higher derivatives of Hardy's Z-function. Our motivation is originated from Matsuoka's theorem.

Theorem (K. Matsuoka). *If the RH is true, then for any non-negative integer k there exists a $T = T(k) > 0$ such that for $t > T$ the function $Z^{(k+1)}(t)$ has exactly one zero between consecutive zeros of $Z^{(k)}(t)$.*

When we study higher derivatives of Hardy's Z-function, a meromorphic function $Z_k(s)$ associated with the k-th derivative of Hardy's Z-function plays an important role. In Chapter 1, we give the definitions of $Z_k(s)$ and $\xi_k(s)$ and a concise survey of higher derivatives of

Hardy's Z-function. In Chapter 2, we discuss some of basic properties of $Z_k(s)$. Chapter 3 is devoted to the study of an entire function $\xi_k(s)$, which can be regarded as a natural generalization of the Riemann ξ-function. First we prove the functional equation for $\xi_k(s)$.

THEOREM. *For all $s \in \mathbb{C}$,*

$$\xi_k(s) = (-1)^k \xi_k(1-s).$$

Next we show that $\xi_k(s)$ can be represented as a product over its zeros.

THEOREM. *For $k \geq 0$, there are constants a_k and B_k such that*

$$\xi_k(s) = e^{A_k + B_k s} \prod_{\rho_k} \left(1 - \frac{s}{\rho_k}\right) e^{\frac{s}{\rho_k}}$$

for all s. Here the product is extended over all zeros ρ_k of $\xi_k(s)$.

We can determine constants e^{A_k} and B_k in the above theorem.

THEOREM. *In the previous theorem, for $k \geq 0$ we have*

$$e^{A_k} = \xi_k(0) = \frac{(-1)^k (2k-1)!!}{(4\sqrt{\pi})^k}$$

and

$$B_k = -\frac{2k(k-1)}{2k-1} \log 2 - \frac{1}{2(2k-1)} \log 4\pi + \frac{\gamma}{2(2k-1)} - 1,$$

where γ is the Euler constant.

In a similar manner, we can obtain special values of $\xi_k(s)$ at integer points.

THEOREM. *Let $k \geq 0$. We have*

$$\xi_k(1) = \frac{(2k-1)!!}{(4\sqrt{\pi})^k},$$

and for $n \geq 1$,

$$\begin{aligned} &\xi_k(2n+1) \\ &= (-1)^{kn+1} \pi^{-\frac{2n+k}{2}} \frac{(2n+1)!2n}{4^n \cdot n!} \left(\frac{4^{n-1}(n!)^2}{(2n)!}\right)^k (2k-3)!!\zeta(2n+1) \end{aligned}$$

and

$$\xi_k(2n) = (-1)^{kn} \pi^{-\frac{2n+k}{2}} 2n(2n-1)(n-1)! \left(\frac{(2n)!}{4^n n!(n-1)!}\right)^k Z_k(2n).$$

Finally, we give an alternative proof of Mozer's formula.

Proposition. *If the RH is true, for any non-negative integer k, we have*
$$\frac{d}{dt}\frac{Z^{(k+1)}(t)}{Z^{(k)}(t)} = -\sum_{\gamma_k}\frac{1}{(t-\gamma_k)^2} + O_k(t^{-1}),$$
where γ_k are zeros of $Z^{(k)}(t)$.

This formula was proved by Matsuoka [**28**]. He considered a contour integral to prove this formula. However, we can avoid calculating the integral by using the factorization of $\xi_k(s)$.

Chapter 4 is dedicated to a discrete moment of higher derivatives of Hardy's Z-function. We prove the following theorem.

Theorem. *Let j and k be fixed non-negative integers and $L = \log(T/2\pi)$. If the RH is true, then as $T \to \infty$,*
$$\begin{aligned}
&\sum_{0<\gamma_k\le T}\left|Z^{(j)}(\gamma_k)\right|^2 \\
&= \delta_{0,k}\frac{T}{2^{2j+1}(2j+1)\pi}L^{2j+2} - \frac{(k+1)\{1+(-1)^j\}}{2^{2j+1}(j+1)^2}\frac{T}{2\pi}L^{2j+2} \\
&\quad + \sum_{u=1}^{j}\frac{1}{2j+1-u}\frac{j!}{(j-u)!}(-1)^{-u}\sum_{g=1}^{k}\frac{1}{\theta_g^{u+1}}\frac{T}{2^{2j+1}\pi}L^{2j+2} \\
&\quad + (-1)^{j+1}\sum_{g=1}^{k}\frac{(j!)^2}{\theta_g^{2j+2}}\frac{T}{2^{2j+2}\pi}L^{2j+2} \\
&\quad + (-1)^j(j!)^2\sum_{g=1}^{k}\frac{\left(\frac{T}{2\pi}\right)^{z_g-1}}{\theta_g^{2j+2}}\left(\sum_{\mu=0}^{j}\frac{\theta_g^{\mu}}{\mu!}\right)^2\frac{T}{2^{2j+2}\pi}L^{2j+2} \\
&\quad + O_{j,k}\left(TL^{2j+1}\right),
\end{aligned}$$
where $\delta_{0,k}$ is Kronecker's delta, z_g $(g = 1, 2, \cdots, k)$ are the zeros of $\mathscr{Z}_k(s,T) := (L/2 + d/ds)^k\zeta(s)$ with $z_g = 1 - 2\theta_g/L + O(L^{-2})$, and θ_g satisfies $\sum_{\mu=0}^{k}\theta_g^{\mu}/\mu! = 0$. When $j = 0$ or $k = 0$, we consider the sums on the right-hand side as the empty sums.

This study is inspired by Matsuoka's result, and our motivation is to research the distribution of the zeros of higher derivatives of Hardy's Z-function. From Matsuoka's result, we can guess the existence of the deviation of the distribution of the zeros of the k-th derivatives of Hardy's Z-function depending on the parity of k.

Another main theme of this thesis is a moment of the first derivative of Dirichlet L-functions. Dirichlet L-functions are some generalization of the Riemann ζ-function, thus the functions have some properties similar to those of the Riemann ζ-function. In this thesis, we consider one of such properties. In Chapter 1, we present the definition of Dirichlet L-functions and related concepts and explain some previous researches and the motivation of our study. In Chapter 5, we give the author's result on a moment of the first derivative of Dirichlet L-functions. The main theorem in Chapter 5 is

THEOREM. *Let c_1 be a positive constant. Let $\chi \pmod q$ be a primitive character. Then, uniformly for $q \le \exp(c_1\sqrt{\log T})$, we have*

$$\sum_{0<\gamma_\chi\le T} L'(\rho_\chi,\chi) = \frac{1}{4\pi}T\left(\log\frac{qT}{2\pi}\right)^2 + a_1\frac{T}{2\pi}\log\frac{qT}{2\pi} + a_2\frac{T}{2\pi} + a_3 + O\left(T\exp\left(-c\sqrt{\log T}\right)\right),$$

where the implicit constant is absolute, c is a positive absolute constant depends on c_1 and

$$a_1 = \sum_{p|q}\frac{\log p}{p-1} + \gamma_0 - 1,$$

$$a_2 = \frac{1}{2}\left(\sum_{p|q}\frac{\log p}{p-1}\right)^2 + (\gamma_0-1)\sum_{p|q}\frac{\log p}{p-1} - \frac{3}{2}\sum_{p|q}p\left(\frac{\log p}{p-1}\right)^2 + 1 - \gamma_0 - \gamma_0^2 + 3\gamma_1$$

with the Stieltjes constants γ_0, γ_1 and

$$a_3 = \frac{\omega\chi(-1)\tau(\overline{\chi})\tau(\overline{\omega}\chi)}{q\varphi(q)}\frac{L'(\beta,\omega)}{\beta}\left(\frac{qT}{2\pi}\right)^\beta$$

when $L(s,\omega)$ with a quadratic character $\omega \pmod q$ has an exceptional zero β, otherwise $a_3 = 0$.

Assuming the GRH, we can replace the error term by $(qT)^{\frac{1}{2}+\varepsilon}$ uniformly for $q \ll T^{1-\varepsilon}$.

This implies that the derivative coefficient of each Dirichlet L-function at its zero is positive real in mean. A. Fujii proved a similar formula and that the same statement is true in the case of the Riemann ζ-function. In other words, our result is a generalization of Fujii's result. However, the term a_3 does not appear in the case of the Riemann ζ-function and it is not a simple problem to estimate the error term as good as Fujii's result under the GRH.

CHAPTER 1

Introduction

1.1. Hardy's Z-function

Hardy's Z-function is derived from the Riemann ζ-function. Thus we give the definition and some properties of the Riemann ζ-function in the beginning. In the followings, we will always assume that $s = \sigma + it \in \mathbb{C}$ with $\sigma, t \in \mathbb{R}$.

Definition 1.1.1. *The Riemann ζ-function is defined by*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1.1}$$

for $\sigma > 1$.

This function is defined in the half complex plane $\sigma > 1$, but the functional equation implies that the Riemann ζ-function is an analytic function in $\mathbb{C} \setminus \{1\}$.

Proposition 1.1.1 (The functional equation, [**30**, p.329]). *The Riemann ζ-function can be continued meromorphically to the whole plane, and has the functional equation*

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{1.1.2}$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s). \tag{1.1.3}$$

The Riemann ζ-function has a simple pole at $s = 1$, and from (1.1.2) we can see that the function has simple zeros at negative even integers (called "trivial zeros"). Here we define an entire function that has only non-trivial zeros.

Definition 1.1.2.

$$\xi(s) := s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{1.1.4}$$

This function is called the Riemann ξ-function. The zeros of this function coincide precisely with those of the Riemann ζ-function in the critical strip $0 < \sigma < 1$ and the trivial zeros are cancelled due to $\Gamma(s/2)$. Therefore the RH can be stated as follows, Conjecture 1.1.1. *All the zeros of the Riemann ξ-function lie on the critical line $\sigma = 1/2$.*

The functional equation (1.1.2) implies that we have

$$\xi(s) = \xi(1-s). \tag{1.1.5}$$

To study the distribution of the zeros of the Riemann ζ-function, Hardy's Z-function plays an important role.

Definition 1.1.3. *Hardy's Z-function is defined by*

$$Z(t) = \chi\left(\frac{1}{2}+it\right)^{-\frac{1}{2}} \zeta\left(\frac{1}{2}+it\right). \tag{1.1.6}$$

By the functional equation (1.1.2), we have

$$\chi(s)\chi(1-s) = 1. \tag{1.1.7}$$

This implies that $\overline{Z(t)}$, the complex conjugate of $Z(t)$, equals to $Z(t)$ for real t, namely, Hardy's Z-function is a real-valued function. When we put

$$\theta(t) := \arg\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) - \frac{t}{2}\log\pi$$

we see that $\chi(1/2+it) = e^{-2i\theta(t)}$. Thus we can rewrite (1.1.6) as follows:

$$Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right). \tag{1.1.8}$$

From this form, we can immediately see that

$$|Z(t)| = \left|\zeta\left(\frac{1}{2}+it\right)\right|,$$

and hence the zeros of the Riemann ζ-function on the critical line correspond to those of Hardy's Z-function. One approach to understanding the distribution of the zeros of the Riemann zeta function is therefore to investigate the change of signs of Hardy's Z-function. In fact, some mathematicians computed the locations of the zeros of the Riemann ζ-function on the critical line in such a way. They calculate the Riemann-Siegel formula

$$Z(t) = 2\sum_{n\le\sqrt{t/2\pi}} n^{-1/2}\cos(\theta(t) - t\log n) + R(t), \tag{1.1.9}$$

where $R(t)$ is the error term (e.g. [**40**], [**26**]).

1.2. Higher derivatives of $Z(t)$

To investigate the analytic behaviour of Hardy's Z-function, it is natural to study its derivatives.

In 1985, Conrey and Ghosh [**5**] introduced a meromorphic function

$$Z_1(s) := \zeta'(s) - \frac{1}{2}\omega(s)\zeta(s),$$

where

$$\omega(s) = \frac{\chi'}{\chi}(s) = \log 2\pi - \frac{\Gamma'}{\Gamma}(s) + \frac{\pi}{2}\tan\frac{\pi s}{2}.$$

This function is derived from the first derivative of Hardy's Z-function, i.e. this function satisfies

$$Z'(t) = ie^{i\theta(t)}Z_1\left(\frac{1}{2}+it\right).$$

Thus

$$|Z'(t)| = \left|Z_1\left(\frac{1}{2}+it\right)\right|.$$

Moreover, we have the functional equation

$$Z_1(s) = -\chi(s)Z_1(1-s).$$

From the above, the meromorphic function $Z_1(s)$ can be regarded as a counterpart of the Riemann ζ-function.

Conrey and Ghosh [**5**] suggested the existence of a meromorphic function $Z_k(s)$ which satisfies the above properties without constructing it, for $k \geq 0$.

By the definition (1.1.6), we obtain for $k \geq 0$

$$Z^{(k)}(t) = \frac{d^k}{dt^k}\left\{\chi\left(\frac{1}{2}+it\right)^{-\frac{1}{2}}\zeta\left(\frac{1}{2}+it\right)\right\}. \tag{1.2.1}$$

From this fact, in 1990, Yıldırım [**44**] defined

$$Z_k(s) := (\chi(s))^{\frac{1}{2}}\frac{d^k}{ds^k}\left((\chi(s))^{-\frac{1}{2}}\zeta(s)\right). \tag{1.2.2}$$

This function is a generalization of $Z_1(s)$. This function satisfies

$$|Z^{(k)}(t)| = \left|Z_k\left(\frac{1}{2}+t\right)\right|$$

and

$$Z_k(s) = (-1)^k\chi(s)Z_k(1-s).$$

On the other hand, Matsuoka [**28**] gave another definition of $Z_k(s)$:

Definition 1.2.1. *Let $Z_0(s) = \zeta(s)$, and for $k \geq 1$, we define $Z_k(s)$ as*

$$Z_k(s) = Z'_{k-1}(s) - \frac{1}{2}\omega(s)Z_{k-1}(s). \tag{1.2.3}$$

In fact, we can show that Matsuoka's definition is the same as Yıldırım's, but Matsuoka's definition is more convenient. Therefore, in this thesis, we define $Z_k(s)$ by Matsuoka's way.

For $Z_k(s)$, we can obtain a counterpart of the Riemann ξ-function. We denote that function by $\xi_k(s)$ and I define it as follows:

Definition 1.2.2. *For $k \geq 0$, we define*

$$\xi_k(s) := \pi^{-\frac{s}{2}} s(s-1) \frac{Z_k(s)}{\Gamma(\frac{s}{2})^{k-1}\Gamma(\frac{1-s}{2})^k}. \tag{1.2.4}$$

This $\xi_k(s)$ is introduced by the author [**23**]. When $k = 0$, $\xi_k(s)$ coincides with the Riemann ξ-function. In Chapter 3, we give various results on $\xi_k(s)$. Since those facts on $\xi_k(s)$ are valid for $k = 0$, we may expect that $\xi_k(s)$ is a natural generalization of the Riemann ξ-function. For entire functions, the concept of the order is defined and the function $\xi_k(s)$ is an entire function of order 1. Thus we can factorize this function by Hadamard's factorization theorem. Moreover, we can represent some constants which appear in the factorized form explicitly. We show these facts in Section 3.3. It is in Section 3.4 that we give the special values of $\xi_k(s)$ at positive integers. As we see in Section 3.1, $\xi_k(s)$ has a functional equation. It implies that we can obtain the special values at all integers.

Remark 1.2.1. There is another variant of a meromorphic function associated with higher derivatives of Hardy's Z-function. In 1986 (a year later the paper of Conrey and Ghosh [**5**] is published), Anderson [**1**] introduced the meromorphic function

$$\eta(s) = \zeta(s) - \frac{2}{\omega(s)}\zeta'(s),$$

to study the distribution of zeros of the first derivative of Hardy's Z-function. Later, in 1999, Matsumoto and Tanigawa [**27**] considered $\eta_k(s)$, a generalization of $\eta(s)$. They defined $\eta_1(s) = \eta(s)$ and for $k \geq 2$,

$$\eta_{k+1}(s) = \lambda(s)\eta_k(s) + \eta'_k(s),$$

where

$$\lambda(s) = \frac{\omega'}{\omega}(s) - \frac{1}{2}\omega(s).$$

In fact, we can see that

$$Z_k(s) = -\frac{\omega(s)}{2}\eta_k(s)$$

for $k \geq 1$. Therefore, $\eta_k(s)$ is the essentially same as $Z_k(s)$. However, it seems that Yıldırım's definition and Matsuoka's are not well-known. For example, Das and Pujahari [**7**] gave a result on the distribution of the zeros of $\eta_k(s)$. They used $\eta_k(s)$ and did not refer to Yıldırım's and Matsuoka's definitions.

1.3. The zeros of higher derivatives of $Z(t)$

One of the main themes of the researches on higher derivatives of Hardy's Z-function has been the existence of its zeros in a short interval. Various authors studied that by using an approximate functional equation (like the Riemann-Siegel formula) for higher derivatives of Hardy's Z-function (see Karatsuba [**20**], Ivić [**16**], A. A. Lavrik [**24**]).

In this thesis, we treat another theme, a relation between the zeros of the j-th derivative of Hardy's Z-function and those of the k-th derivative of the function.

In 1974, Edwards [**8**] and Mozer [**31**] independently proved

Theorem 1.3.1 (H. M. Edwards and J. Mozer, 1974). *If the RH is true, then there exists a $t_0 > 0$ such that for $t > t_0$ the function $Z'(t)$ has exactly one zero between consecutive zeros of $Z(t)$.*

This theorem implies that under the RH, for $t > t_0$, there is no positive local minimum and no negative local maximum of the Hardy's Z-function (see Fig.1). Thus, if we could find those, the RH would be disproved.

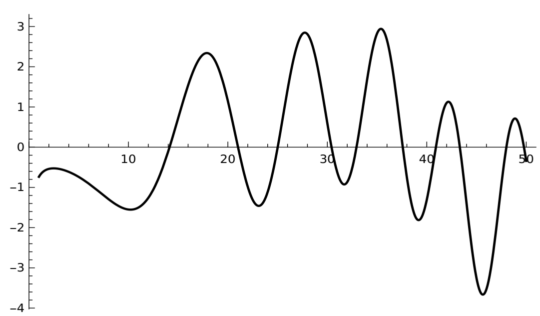

Fig. 1. The graph of $Z(t)$ for $0 \leq t \leq 50$

This study is inspired by the Lehmer phenomenon. The Lehmer phenomenon is a phenomenon that some consecutive zeros of Hardy's Z-function are unusually close to each other. Lehmer [**26**] found this phenomenon around $t = 7005$ (see Fig.2 and Fig.3) by the numerical calculation. It is an unsolved problem whether the phenomenon does happen infinitely many times.

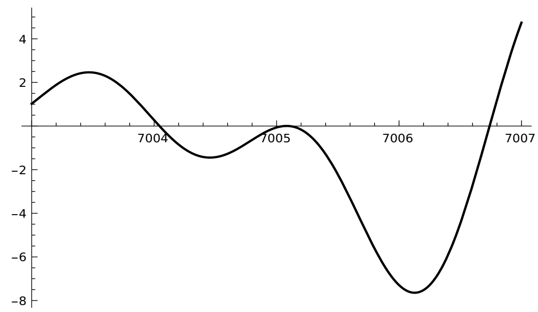

FIG. 2. The graph of $Z(t)$ around $t = 7005$

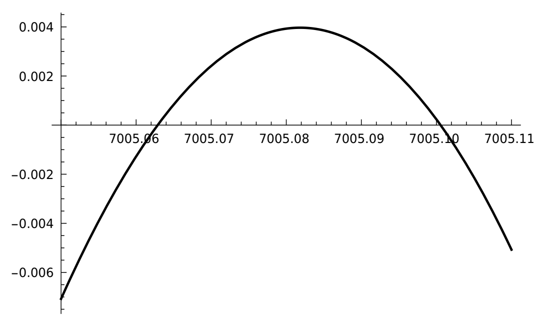

FIG. 3. Enlarged view of Fig. 2 ($7005.05 \leq t \leq 7005.11$)

Later, Anderson [**1**] established a milestone.

THEOREM 1.3.2 (R. J. Anderson, 1986). *If the RH is true, then there exists a $t_0 > 0$ such that for $t > t_0$ the function $Z''(t)$ has exactly one zero between consecutive zeros of $Z'(t)$.*

To prove this theorem, he showed the following statement:

THEOREM 1.3.3 (R. J. Anderson, 1986). *Let*

$$R := \{s = \sigma + it \in \mathbb{C} \mid 0 < t < T, -7 < \sigma < 8\}$$

and denote the number of zeros of $Z_1(s)$ in R by $N(T; Z_1)$. Then

$$N(T; Z_1) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Moreover, on the RH, all non-real zeros of $Z_1(s)$ are on the critical line.

His method of proof is ingenious and applied to a general case by Matsuoka [**28**], Matsumoto and Tanigawa [**27**].

Matsumoto and Tanigawa [**27**] generalized Theorem 1.3.3.

THEOREM 1.3.4 (K. Matsumoto and Y. Tanigawa, 1999). *Let k be any positive integer and*

$$R_k := \{s = \sigma + it \in \mathbb{C} \mid 0 < t < T, -2m + 1 < \sigma < 2m\},$$

where $m = m(k)$ is sufficiently large positive integer. And denote the number of zeros of $Z_k(s)$ in R_k by $N(T; Z_k)$. Then

$$N(T; Z_k) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O_k(\log T).$$

Moreover, on the RH, the zeros of $Z_k(s)$ in $-2m+1 < \sigma < 2m$ are on the critical line except for finitely many exceptions.

REMARK 1.3.1. As we mentioned in the previous section, Anderson, Matsumoto and Tanigawa did not use $Z_k(s)$ but $\eta_k(s)$.

Matsuoka [**28**] extended Theorem 1.3.2.

THEOREM 1.3.5 (K. Matsuoka, 2012). *If the RH is true, then for any non-negative integer k there exists a $T = T(k) > 0$ such that for $t > T$ the function $Z^{(k+1)}(t)$ has exactly one zero between consecutive zeros of $Z^{(k)}(t)$.*

We show Matsuoka's theorem, or more precisely, Mozer's formula by a different way from Matsuoka's proof. The factorization of $\xi_k(s)$ leads to this different way. This method is a generalization of the method by which Edwards [**8**] and Mozer [**31**] proved Theorem 1.3.1.

Finally, we note Hall's result [**12**] on a relation between the number of zeros of $Z_1(s)$ and those of $Z_0(s) = \zeta(s)$. Let

$$H(s) := \frac{\zeta'}{\zeta}(s) - \frac{1}{2}\omega(s)$$

and put

$$A(T) := \frac{1}{\pi} \arg H\left(\frac{1}{2} + iT\right),$$

defined by continuous variation along the line segments $[8, 8+iT]$, $[8+iT, 1/2+iT]$ ($\arg H(8) := 0$). Then we have the following theorem.

THEOREM 1.3.6 (R. R. Hall, 2004). *Suppose that T is not the ordinate of a zero of $\zeta(s)$ or of $H(s)$. Then we have, for sufficiently large T,*

$$N(T; Z_1) = N(T; Z_0) + A(T) + \frac{3}{2}.$$

Assuming the RH, $A(T)$ can be replaced by $-\mathrm{sgn}(Z'/Z)(T)/2$.

This is a more sophisticated theorem than Theorem 1.3.3. For general k, it is not known whether a similar statement holds or not.

1.4. The moments of higher derivatives of $Z(t)$

There are various type of problems on the moments of higher derivatives of Hardy's Z-function. One of those is the second continuous moment of the function. Hall [**11**] obtained the following theorem:

THEOREM 1.4.1 (R. R. Hall, 1999). *For each $k = 0, 1, 2, \ldots,$ and any sufficiently large T, we have*

$$\int_0^T Z^{(k)}(t)^2 dt = \frac{1}{4^k(2k+1)} T P_{2k+1}\left(\log \frac{T}{2\pi}\right) + O\left(T^{\frac{3}{4}} \log^{2k+\frac{1}{2}} T\right), \tag{1.4.1}$$

where $P_{2k+1}(x)$ is the monic polynomial of degree $2k+1$ given by

$$P_{2k+1}(x) = W_{2k+1}(x) + (4k+2) \sum_{h=0}^{2k} \binom{2k}{h} (-2)^h c_h W_{2k-h}(x),$$

in which

$$W_g(v) = \frac{1}{e^v} \int_0^{e^v} \log^g u du, \quad \zeta(s) = \frac{1}{s-1} + \sum_{h=0}^{\infty} \frac{(-1)^h c_h}{h!} (s-1)^h.$$

This is the first result on the moment of higher derivatives of Hardy's Z-function. He referred to Ingham's result [**15**]:

$$\begin{aligned} &\int_0^T \left| \zeta^{(k)}\left(\frac{1}{2} + it\right) \right|^2 dt \\ &= \frac{1}{2k+1} T \tilde{P}_{2k+1}\left(\log \frac{T}{2\pi}\right) + O_k\left(T^{\frac{1}{2}} \log^{2k+2} T\right), \end{aligned} \tag{1.4.2}$$

where $\tilde{P}_{2k+1}(x)$ is a certain polynomial of degree $2k+1$ in x and conjectured that an error term in (1.4.1) can be at least as good as that in (1.4.2). In 2020, Minamide and Tanigawa [**29**] solved his conjecture:

THEOREM 1.4.2 (M. Minamide and Y. Tanigawa, 2020). *For each $k = 0, 1, 2, \ldots,$ and any sufficiently large T, we have*

$$\int_0^T Z^{(k)}(t)^2 dt = \frac{1}{4^k(2k+1)} T P_{2k+1}\left(\log \frac{T}{2\pi}\right) + O(T^{\frac{1}{2}} \log^{2k+1} T), \tag{1.4.3}$$

where $P_{2k+1}(x)$ is the monic polynomial of degree $2k+1$.

In this thesis, we consider a discrete second moment of the higher derivative of Hardy's Z-function, but, the second continuous moment (1.4.3) is important for our argument. Moreover, our problem is related to the distribution of zeros of the function which we referred to in the previous section.

In 1984, Gonek [**10**] published an important paper on a discrete mean of the higher derivative of the Riemann ζ-function. The most important case of his result is the case of the first derivative of the Riemann ζ-function under the RH, and the result can be expressed by Hardy's Z-function as follows.

THEOREM 1.4.3 (S. M. Gonek, 1984). *If the RH is true, then for any sufficiently large T,*

$$\sum_{0<\gamma\le T} |Z'(\gamma)|^2 = \frac{T}{24\pi}\left(\log\frac{T}{2\pi}\right)^4 + O(T(\log T)^3), \tag{1.4.4}$$

where the summation is over the zeros of Hardy's Z-function counted with the multiplicity.

His method of the proof has become a basic method to study discrete mean problems nowadays.

A year after Gonek's result, Conrey and Ghosh [**5**] showed that

THEOREM 1.4.4 (J. B. Conrey and A. Ghosh, 1985). *Assume the RH is true, and let $\gamma \le \gamma^+$ be successive ordinates of zeros of Hardy's Z-function. Then*

$$\sum_{0<\gamma\le T} \max_{\gamma<t\le\gamma^+} |Z(t)|^2 = \frac{e^2-5}{4\pi} T \left(\log\frac{T}{2\pi}\right)^2 + O(T\log T)$$

as $T \to \infty$.

Theorem 1.3.1 implies that the result of Conrey and Ghosh gives an asymptotic formula for the mean square of the extremal value of Hardy's Z-function.

Later, Yıldırım [**44**] generalized their result:

THEOREM 1.4.5 (C. Y. Yıldırım, 1990). *Assume the RH and let k be a fixed natural number. Let γ_k run through the zeros of the k-th derivative of Hardy's Z-function. Then as $T \to \infty$*

$$\sum_{\gamma_k\le T} |Z(\gamma_k)|^2 \sim \begin{cases} \frac{TL^2}{2\pi}(1+\frac{1}{k}+O(\frac{\log k}{k^2})) & (k \text{ odd and } k>1) \\ \frac{TL^2}{2\pi}(1-\frac{3}{k}+O(\frac{\log k}{k^2})) & (k \text{ even}), \end{cases}$$

where $L = \log(T/2\pi)$.

To prove this, Yıldırım calculated the integral

$$\frac{1}{2\pi i}\int_C \frac{Z_k'}{Z_k}(s)\zeta(s)\zeta(1-s)ds,$$

where C is an appropriate integral path. To treat $Z_k'/Z_k(s)$, he introduced the function $\mathscr{Z}_k(s,T)$ defined by

$$\mathscr{Z}_k(s,T) := \left(\frac{L}{2} + \frac{d}{ds}\right)^k \zeta(s). \tag{1.4.5}$$

In Chapter 4, we give a more general theorem. We consider a second moment of the j-th derivative of Hardy's Z-function. However, our motivation is not just to generalize Yıldırım's result. Matsuoka's result suggests that there is the deviation of the distribution of the zeros of the k-th derivative of Hardy's Z-function depending on the parity of k. Indeed, Yıldırım's result supports the existence of the deviation. Our main theorem is the result of an attempt to show the existence of the deviation for more general cases in the mean.

1.5. Dirichlet L-functions

Dirichlet L-functions are meromorphic functions defined by Dirichlet series with Dirichlet characters.

We first define Dirichlet character.

Definition 1.5.1. *Let q be a positive integer. A Dirichlet character χ modulo q is an arithmetic function satisfying;*

(i) $\chi(mn) = \chi(m)\chi(n)$ *for all* $m, n \in \mathbb{Z}$,

(ii) $\chi(n+q) = \chi(n)$ *for all* $n \in \mathbb{Z}$,

(iii) *If* $(n,q) > 1$, *then* $\chi(n) = 0$.

Here (n,q) denotes the greatest common divisor of n and q. When $\chi(n) = 1$ for all $n \in \mathbb{Z}$ with $(n,q) = 1$, then we call that character the principal character, and it is denoted by $\chi_0(n)$, otherwise we say that it is a non-principal character.

Now we can define Dirichlet L-functions.

Definition 1.5.2. *The Dirichlet L-function attached to χ is defined by*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $\sigma > 1$.

For $\chi \pmod 1$ we can see that $L(s,\chi) = \zeta(s)$.

Dirichlet L-functions have the Euler product expansions.

PROPOSITION 1.5.1. *For $\sigma > 1$, we have*

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product runs through all primes.

This implies that $L(s,\chi_0) = \zeta(s)\prod_{p|q}(1-p^{-s})$.

All Dirichlet L-functions can be analytically continued to $\mathbb{C}$ except possibly for a pole at $s=1$. When $\chi = \chi_0$, the Dirichlet L-function has only one pole at $s=1$ and it is simple. When χ is non-principal, the Dirichlet L-function is an entire function.

Let χ be a Dirichlet character modulo q. It is said that d is a quasiperiod of χ if $\chi(m) = \chi(n)$ whenever $m \equiv n \pmod{d}$ and $(mn, q) = 1$. On the quasiperiod, the following fact is known:

LEMMA 1.5.1 ([**30**, p. 282]). *The least quasiperiod of χ is a divisor of q.*

The least quasiperiod of χ is called the conductor of χ. Suppose that $d \mid q$ and that $\chi^\star$ is a Dirichlet character modulo d satisfying

$$\chi(n) = \begin{cases} \chi^\star(n) & (n,q)=1, \\ 0 & \text{otherwise.} \end{cases}$$

In this situation, we say that $\chi^\star$ induces χ. Now we can define primitive characters.

DEFINITION 1.5.3. *A Dirichlet character χ modulo q is said to be primitive when q is the conductor of χ.*

When χ is not primitive, then we call χ imprimitive character. It is known that any imprimitive character is induced by a unique primitive character. We summarize the above as follows.

THEOREM 1.5.1 ([**30**, p. 283]). *Let χ be a Dirichlet character modulo q and let d be the conductor of χ. Then $d \mid q$, and there is a unique primitive character $\chi^\star$ modulo d which induces χ.*

Here we introduce the Gauss sum. For a Dirichlet character $\chi \pmod{q}$ the Gauss sum is defined by

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) \exp\left(2\pi i \frac{a}{q}\right).$$

When $\chi \pmod{q}$ is a primitive character, we have $|\tau(\chi)| = \sqrt{q}$.

When $\chi \pmod{q}$ is induced by a primitive character $\chi^\star$ modulo d, for some $d \mid q$, the Euler product leads to

$$L(s,\chi) = L(s,\chi^\star) \prod_{p \mid q} \left(1 - \frac{\chi^\star(p)}{p^s}\right).$$

This implies that it is sufficient to study the Dirichlet L-functions attached to primitive characters.

The Dirichlet L-functions also satisfy the functional equation.

Proposition 1.5.2 ([**30**, p. 333]). *Let χ be a primitive character modulo q. Setting*

$$\kappa = \frac{1-\chi(-1)}{2}$$

and

$$\varepsilon(\chi) = \frac{\tau(\chi)}{i^\kappa \sqrt{q}},$$

we have

$$L(s,\chi) = \Delta(s,\chi) L(1-s,\overline{\chi}), \tag{1.5.1}$$

where

$$\Delta(s,\chi) = \varepsilon(\chi) 2^s \pi^{s-1} q^{\frac{1}{2}-s} \Gamma(1-s) \sin\frac{\pi}{2}(s+\kappa).$$

We note that $\Delta(s,\chi)$ is a meromorphic function with only real zeros and poles satisfying the functional equation

$$\Delta(s,\chi)\Delta(1-s,\overline{\chi}) = 1.$$

Finally, we note a generalization of the RH.

Conjecture 1.5.1. *All zeros of every Dirichlet L-function in the strip $0 < \sigma < 1$ lie on the critical line $\sigma = 1/2$.*

This is called the Generalized Riemann Hypothesis (GRH).

1.6. The discrete moments and some conjectures

Let $\rho = \beta + i\gamma$ be the non-trivial zeros of the Riemann ζ-function. Reviewing the tables of numerical computations by Haselgrove and Miller [**13**], Shanks [**36**] found a tendency and gave the following conjecture on the non-trivial zeros of the Riemann ζ-function.

Conjecture 1.6.1. *$\zeta'(1/2 + i\gamma)$ is positive real in the mean.*

More precisely, he conjectured that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \pi^{-1} \arg \zeta'\left(\frac{1}{2} + i\gamma_n\right) = 0.$$

Concerning this, Fujii [**9**] proved the following asymptotic formula:

THEOREM 1.6.1 (A. Fujii, 1994).

$$\sum_{0<\gamma\leq T}\zeta'(\rho)=\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^2+(c_0-1)\frac{T}{2\pi}\log\frac{T}{2\pi}$$
$$+(1-c_0-\gamma_0^2+3c_1)\frac{T}{2\pi}+O(T\exp(-C\sqrt{\log T})),$$

where c_0 and c_1 are the Stieltjes constants. Assuming the RH, we can replace the error term by $T^{\frac{1}{2}}(\log T)^{\frac{7}{2}}$.

This implies that Shanks' conjecture is true.

REMARK 1.6.1. Later, Trudgian [**39**] proved that more sophisticated statement is true. He showed that, for any $\alpha>0$,

$$\lim_{N\to\infty}\frac{1}{N^\alpha}\sum_{n=1}^{N}\pi^{-1}\arg\zeta'\left(\frac{1}{2}+i\gamma_n\right)=0.$$

Conjecture 1.5.1 is solved. However, the study of the discrete moment which Fujii considered has another motivation. We can easily see that there are infinitely many simple zeros of the Riemann ζ-function by Fujii's result. Indeed, there is the following conjecture.

CONJECTURE 1.6.2. *The Riemann ζ-function has only simple zeros.*

This is called the Simple Zero Conjecture. In 2020, Pratt, Robles, Zaharescu and Zeindler [**35**] unconditionally proved that the proportion of simple zeros of the Riemann ζ-function is at least 40.75%. On the other hand, it is shown that the proportion is at least 70.37% under the RH by Bui and Heath-Brown [**3**]. This conjecture is extended to Dirichlet L-functions. Wu [**41**] unconditionally proved that for any Dirichlet character, more than 40.74% of the zeros of Dirichlet L-functions are simple. Later, he [**42**] considered the proportion of simple zeros on a certain average over the family of all Dirichlet L-functions and showed that the proportion is at least 60.26% unconditionally. In the same paper, it is also shown that the proportion can be improved to more than 66.43% under the GRH. It should also be noted that some much stronger results have been obtained for the family of Dirichlet L-functions. Assuming the GRH, Özlük [**32**] treated a certain average of the q-analogue of the pair correlation function and proved that the proportion of simple zeros on that average is more than 91.66%. Under the GRH, it is shown that more than 93.50% of the zeros of Dirichlet L-functions with primitive Dirichlet characters are simple in the above sense by Chirre, Gonçalves and de Laat [**4**].

In Chapter 6, we consider a generalization of Fujii's result to Dirichlet L-functions with primitive Dirichlet characters. However, the result does not depend on the characters and we also use some properties on Dirichlet L-functions with imprimitive characters in the proof.

CHAPTER 2

A meromorphic function $Z_k(s)$ and its basic properties

In this chapter, we introduce a meromorphic function $Z_k(s)$ associated with the k-th derivative of Hardy's Z-function and some of its basic properties.

2.1. A meromorphic function $Z_k(s)$

As we noted, we use Matsuoka's definition.

DEFINITION 2.1.1. *Let $Z_0(s) = \zeta(s)$, and for $k \geq 1$, we define $Z_k(s)$ as*

$$Z_k(s) = Z'_{k-1}(s) - \frac{1}{2}\omega(s)Z_{k-1}(s). \tag{2.1.1}$$

By the relation $\chi(1/2+it) = e^{-2i\theta(t)}$, we see that

$$\omega\left(\frac{1}{2}+it\right) = -2\theta'(t). \tag{2.1.2}$$

PROPOSITION 2.1.1 (Proposition 2.1 in [**28**]). *For any non-negative k, we have*

$$Z^{(k)}(t) = i^k e^{i\theta(t)} Z_k\left(\frac{1}{2}+it\right). \tag{2.1.3}$$

PROOF. The case $k=0$ is the definition of $Z(t)$. If we assume that the equation is true for k, then

$$Z^{(k+1)}(t) = i^{k+1}e^{i\theta(t)}\left(Z'_k\left(\frac{1}{2}+it\right) + \theta'(t)Z_k\left(\frac{1}{2}+it\right)\right).$$

By the definition (2.1.1) and (2.1.2), we find that the equation (2.1.3) is true for $k+1$. $\square$

This leads to

$$|Z^{(k)}(t)| = \left|Z_k\left(\frac{1}{2}+it\right)\right|.$$

The function $Z_k(s)$ satisfies a functional equation like as the Riemann ζ-function.

PROPOSITION 2.1.2 (The Functional Equation, Lemma 2 in [**44**] or Proposition 2.2 in [**28**]). *For any non-negative k, we have*

$$Z_k(s) = (-1)^k \chi(s) Z_k(1-s). \tag{2.1.4}$$

PROOF. The case $k = 0$ is the functional equation for the Riemann ζ-function. If we assume that the equation is true for k, then by the definition,

$$\begin{aligned}
\chi(s) Z_{k+1}(1-s) &= \chi(s)\left(Z_k'(1-s) - \frac{1}{2}\omega(1-s) Z_k(1-s)\right) \\
&= \chi'(s) Z_k(1-s) - (-1)^k Z_k'(s) - \frac{(-1)^k}{2}\omega(s) Z_k(s) \\
&= (-1)^{k+1} Z_k'(s) + (-1)^k \omega(s) Z_k(s) - \frac{(-1)^k}{2}\omega(s) Z_k(s) \\
&= (-1)^{k+1}\left(Z_k'(s) - \frac{1}{2}\omega(s) Z_k(s)\right) \\
&= (-1)^{k+1} Z_{k+1}(s).
\end{aligned}$$

The proof is completed. □

2.2. THE FUNCTION $f_k(s)$

Let $f_0(s) = 1$, and define $f_k(s)$ for $k \geq 1$ by

$$f_{k+1}(s) = f_k'(s) - \frac{1}{2}\omega(s) f_k(s) \quad (k \geq 1). \tag{2.2.1}$$

The following proposition is inspired by Proposition 3 in [**27**].

PROPOSITION 2.2.1. *For any non-negative k, we have*

$$Z_k(s) = \sum_{j=0}^{k} \binom{k}{j} f_{k-j}(s) \zeta^{(j)}(s). \tag{2.2.2}$$

PROOF. The case $k = 0$ is clear. We assume that this is valid for k. By the definition,

$$\begin{aligned}
Z_{k+1}(s) &= Z_k'(s) - \frac{1}{2}\omega(s)Z_k(s) \\
&= \sum_{j=0}^{k} \binom{k}{j} f_{k-j}'(s)\zeta^{(j)}(s) + \sum_{j=0}^{k} \binom{k}{j} f_{k-j}(s)\zeta^{(j+1)}(s) \\
&\quad - \frac{1}{2}\omega(s) \sum_{j=0}^{k} \binom{k}{j} f_{k-j}(s)\zeta^{(j)}(s) \\
&= \sum_{j=0}^{k} \binom{k}{j} f_{k+1-j}(s)\zeta^{(j)}(s) + \sum_{j=0}^{k} \binom{k}{j} f_{k-j}(s)\zeta^{(j+1)}(s) \\
&= \sum_{j=0}^{k+1} \left\{ \binom{k}{j} + \binom{k}{j-1} \right\} f_{k+1-j}(s)\zeta^{(j)}(s) \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j} f_{k-j}(s)\zeta^{(j)}(s).
\end{aligned}$$

Here, to obtain the last equality, we use the relation

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}.$$

□

The function $f_k(s)$ can be expressed explicitly as below but the expression is complicated.

PROPOSITION 2.2.2. *For $k \geq 1$, we have*

$$f_k(s) = k! \sum_{a_1+2a_2+\cdots+ka_k=k} \left(-\frac{1}{2}\right)^{a_1+\cdots+a_k} \prod_{l=1}^{k} \frac{1}{a_l!} \left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{a_l}.$$

We can guess this expression by (1.2.2) and Faà di Bruno's formula (see [**33**], [**18**]).

PROOF. We will prove this statement by induction on k. When $k = 1$, this formula is trivial. We assume that this is true for $k = j$.

By the definition, we have

$$\begin{aligned}
&f_{j+1}(s)\\
&= f_j'(s) - \frac{1}{2}\omega(s) f_j(s)\\
&= j! \sum_{a_1+2a_2+\cdots+ja_j=j} \left(-\frac{1}{2}\right)^{a_1+\cdots+a_j} \sum_{\substack{m=1\\ a_m\geq 1}}^{j} \prod_{\substack{l=1\\ l\neq m}}^{k} \frac{1}{a_l!}\left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{a_l}\\
&\quad \times \frac{1}{(a_m-1)!}\left(\frac{1}{m!}\right)^{a_m} \omega^{(m-1)}(s)^{a_m-1}\omega^{(m)}(s)\\
&\quad - \frac{1}{2}\omega(s) j! \sum_{a_1+2a_2+\cdots+ja_j=j} \left(-\frac{1}{2}\right)^{a_1+\cdots+a_j} \prod_{l=1}^{j} \frac{1}{a_l!}\left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{a_l}.
\end{aligned}$$

On the first term, let $b_m = a_m - 1, b_{m+1} = a_{m+1}+1 \quad (1 \leq m \leq j-1)$ and $b_l = a_l \quad (l \neq m, m+1)$. Then we can see that the first term is

$$\begin{aligned}
&j! \sum_{b_1+2b_2+\cdots+jb_j=j+1} \left(-\frac{1}{2}\right)^{b_1+\cdots+b_j}\\
&\quad \times \sum_{m=1}^{j-1}(m+1)b_{m+1} \prod_{l=1}^{j} \frac{1}{b_l!}\left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{b_l}\\
&\quad + (j+1)! \sum_{\substack{b_1+2b_2+\cdots+(j+1)b_{j+1}=j+1\\ b_{j+1}=1}} \left(-\frac{1}{2}\right)^{b_1+\cdots+b_{j+1}} \prod_{l=1}^{j+1} \frac{1}{b_l!}\left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{b_l}\\
&= (j+1)! \sum_{b_1+2b_2+\cdots+(j+1)b_{j+1}=j+1} \left(-\frac{1}{2}\right)^{b_1+\cdots+b_{j+1}} \prod_{l=1}^{j+1} \frac{1}{b_l!}\left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{b_l}\\
&\quad - j! \sum_{b_1+2b_2+\cdots+(j+1)b_{j+1}=j+1} \left(-\frac{1}{2}\right)^{b_1+\cdots+b_{j+1}} b_1 \prod_{l=1}^{j+1} \frac{1}{b_l!}\left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{b_l}
\end{aligned}$$

As for the second term, when we put $c_1 = a_1 + 1$ and $c_l = a_l \quad (l \geq 2)$ then

$$j! \sum_{c_1+2c_2+\cdots+jc_j=j+1} \left(-\frac{1}{2}\right)^{c_1+\cdots+c_j} c_1 \prod_{l=1}^{j} \frac{1}{c_l!}\left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{c_l}.$$

Therefore

$$f_{j+1}(s) = (j+1)! \sum_{a_1+2a_2+\cdots+(j+1)a_{j+1}=j+1} \left(-\frac{1}{2}\right)^{a_1+\cdots+a_{j+1}} \prod_{l=1}^{j+1} \frac{1}{a_l!} \left(\frac{\omega^{(l-1)}(s)}{l!}\right)^{a_l}$$

This completes the proof. □

2.3. The poles and zeros of $f_k(s)$ and $Z_k(s)$

We investigate the poles of $f_k(s)$ and $Z_k(s)$.

Lemma 2.3.1 (Lemma 1 in [**27**]). *The poles of $\omega(s)$ are all simple, and located at positive odd integers with residue -1 and at non-positive even integers with residue 1.*

Proof. Since $\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)$, the zeros of $\chi(s)$ are non-positive even integers and the poles are positive odd integers. Therefore we obtain the lemma. □

Lemma 2.3.2 (Lemma 4.2 in [**28**]). *For $k \geq 0$, the function $f_k(s)$ has poles of order k which are located only at $\dots, -4, -2, 0, 1, 3, 5, \dots$.*

Proof. The case $k = 1$ is the previous lemma. We assume that the lemma is valid for $k \geq 1$. Let a be a pole of $f_k(s)$.Then by Laurent expansion with centre a, we have

$$f_k(s) = \frac{c_k}{(s-a)^k} + \cdots ,$$

where c_k does not vanish. By the definition and the previous lemma, we have

$$f_{k+1}(s) = \frac{-kc_k \pm \frac{c_k}{2}}{(s-a)^{k+1}} + \cdots .$$

Since $-kc_k \pm c_k/2 \neq 0$, the lemma is true for $k+1$. This completes the lemma. □

This lemma and Proposition 2.2.1 immediately lead to the following lemma.

Lemma 2.3.3 (Lemma 4.4 in [**28**]). *For $k \geq 0$, the function $Z_k(s)$ has poles of order k located at $0, 3, 5, 7, \dots$, that of order $k+1$ located at 1, and those of order $k-1$ located at $-2, -4, -6, \dots$.*

We understand that "poles of order -1" means zeros of order 1.

Finally, we present some results of Matsumoto and Tanigawa on the zeros of $Z_k(s)$.

PROPOSITION 2.3.1 (see the proof of Theorem 2 in [**27**]). *If the RH is true, then the number of zeros of* $Z_k(s)$ *in* $\{s \mid 1-2m \leq \sigma \leq 2m, \sigma \neq 1/2\}$ *is* $O_k(1)$.

They proved that the difference of the number of zeros of $Z^{(k)}(t)$ and that of $Z_k(s)$ is $O_k(1)$. From this fact, they obtained the following approximate formula:

PROPOSITION 2.3.2 (Theorem 2 in [**27**]). *Let* $N_k(T)$ *be the number of the zero of* $Z^{(k)}(t)$ *in the interval* $(0, T)$. *If the RH is true, then for any* $k \geq 1$,

$$N_k(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O_k(\log T).$$

CHAPTER 3

An entire function $\xi_k(s)$ associated with the higher derivative of $Z(t)$

This chapter is dedicated to the function $\xi_k(s)$. This function can be regarded as a natural generalization of the Riemann ξ-function because $\xi_k(s)$ has similar properties to the Riemann ξ-function, the functional equation, the factorization, and the special values at positive integers. In the final section, we give an alternative proof of Matsuoka's theorem and our method of the proof is a generalization of Edwards and Mozer's. This chapter is based on the author's paper [**23**]

3.1. The functional equation

In the beginning, we note that $\xi_k(s)$ is entire. By Lemma 2.3.3, we can determine the poles of $Z_k(s)$, and those poles are cancelled by Γ-functions in the definition of $\xi_k(s)$ (1.2.4).

The functional equation of $Z_k(s)$ leads to the following theorem.

Theorem 3.1.1. *For all s, we have*

$$\xi_k(s) = (-1)^k \xi_k(1-s). \tag{3.1.1}$$

Proof. We can transform $\chi(s)$ to

$$\chi(s) = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\pi^{s-\frac{1}{2}}.$$

Thus

$$\begin{aligned}
\xi_k(s) &= \pi^{-\frac{s}{2}} s(s-1) \frac{Z_k(s)}{\Gamma(\frac{s}{2})^{k-1}\Gamma(\frac{1-s}{2})^k} \\
&= \pi^{-\frac{s}{2}} s(s-1) \frac{(-1)^k \chi(s) Z_k(1-s)}{\Gamma(\frac{s}{2})^{k-1}\Gamma(\frac{1-s}{2})^k} \\
&= (-1)^k \pi^{-\frac{1-s}{2}} s(s-1) \frac{Z_k(1-s)}{\Gamma(\frac{1-s}{2})^{k-1}\Gamma(\frac{s}{2})^k} \\
&= (-1)^k \xi_k(1-s).
\end{aligned}$$

This completes the proof. ∎

3.2. Some estimates on $Z_k(s)$

In this section, we prove some estimates on $Z_k(s)$. Proposition 2.2.1 implies that we need the estimates of $f_k(s)$ and $\zeta^{(k)}(s)$ for any non-negative integer k.

First we consider $f_k(s)$. Stirling's formula implies that

Lemma 3.2.1. *For $\sigma \geq 1/4$, we have*

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s}(1 + O(|s|^{-1})),$$
$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1}),$$

and

$$\frac{d^k}{ds^k}\frac{\Gamma'}{\Gamma}(s) = O_k(|s|^{-k}) \quad (k \geq 1).$$

Define the set $\mathscr{D}$ by removing all small circles whose centres are odd positive integers and even non-positive integers with radii depending on k from the complex plane.

Lemma 3.2.2 (Lemma 3.1 in [**28**]). *In the region $\{s \in \mathscr{D} | \sigma > 1/4\}$, we have*

$$\tan\frac{\pi s}{2} = \begin{cases} i + O(e^{-2t}) & (t \geq 0), \\ -i + O(e^{2t}) & (t \leq 0), \end{cases} \tag{3.2.1}$$

and

$$\frac{d^k}{ds^k}\tan\frac{\pi s}{2} = O_k(e^{-2|t|}) \quad (k \geq 1). \tag{3.2.2}$$

By this lemma and Lemma 3.2.1, we have

Lemma 3.2.3 (Lemma 3.2 in [**28**]). *For $s \in \mathscr{D}$, we have*

$$\omega(s) = -\log|s| + O(1), \tag{3.2.3}$$

and

$$\omega^{(k)}(s) = O_k(1) \quad (k \geq 1). \tag{3.2.4}$$

Here we use the fact that $\omega(s) = \omega(1-s)$.

Therefore we can see that $f_k(s) \ll_k (\log(|s|+2))^k$ for $s \in \mathscr{D}$ by Proposition 2.2.2.

By Lemma 2.3.2,

$$\left(\cos\frac{\pi s}{2}\right)^k f_k(s)$$

has no pole in the half plane $\sigma \geq 1/2$.

From the above, for $\sigma \geq 1/2$ there is a constant $C_1 = C_1(k)$ such that

$$\left(\cos\frac{\pi s}{2}\right)^k f_k(s) \ll_k e^{C_1|s|}. \tag{3.2.5}$$

Next we consider $\zeta^{(k)}(s)$. Let $k=0$. It is known that

$$\zeta(s) = \frac{1}{s-1} - s\int_1^\infty \frac{x^2-[x]-\frac{1}{2}}{x^{s+1}}dx + \frac{1}{2}.$$

Therefore we can see that when s is not close to 1,

$$\zeta(s) \ll_\eta |s| \quad (\sigma > \eta)$$

for any $\eta > 0$. Therefore, by Cauchy's integral formula applied to a circle which has centre s and radius $1/\log(|s|+2)$,

$$\zeta^{(k)}(s) \ll_{\eta'} |s| \log^k(|s|+2) \quad (\sigma > \eta')$$

for $k \geq 1$ and any $\eta' = \eta'(k) > 0$ unless s is close to 1. Thus for all $\sigma > \eta'$, there is a constant $C_2 = C_2(k)$ such that

$$\left(\cos\frac{\pi s}{2}\right)^k (s-1)\zeta^{(k)}(s) \ll_k e^{C_2|s|}. \tag{3.2.6}$$

3.3. The factorization of $\xi_k(s)$

By the estimates shown in the previous section, we can prove the following theorem:

THEOREM 3.3.1. *For $k \geq 0$, there are constants a_k and B_k such that*

$$\xi_k(s) = e^{A_k + B_k s} \prod_{\rho_k} \left(1 - \frac{s}{\rho_k}\right) e^{\frac{s}{\rho_k}} \tag{3.3.1}$$

for all s. Here the product is extended over all zeros ρ_k of $\xi_k(s)$.

We note that it is possible that some of the ρ_k's are real. Actually, Anderson [**1**] proved the existence of real zeros for $k=1$. This fact is reproduced by Hall [**12**].

PROOF. It is sufficient to prove that the order of $\xi_k(s)$ is 1, for then Hadamard's factorization theorem can be applied.

Let $\sigma \geq 1/2$. By the formulas

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \quad \text{and} \quad \Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2s}\Gamma(2s),$$

we see that

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = 2^{s-1}\sqrt{\pi}\frac{\Gamma(\frac{s}{2})^2}{\Gamma(s)\cos\frac{\pi s}{2}}.$$

Then, by the definition of $\xi_k(s)$ (1.2.4) and Proposition 2.2.1,

$$\xi_k(s) = 2^{-k(s-1)}\pi^{-\frac{s+k}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)\left(\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\right)^k\left(\cos\frac{\pi s}{2}\right)^k \times \sum_{j=0}^{k}\binom{k}{j}f_{k-j}(s)\zeta^{(j)}(s).$$

Lemma 3.2.1 implies that

$$\xi_k(s) = 2^{-\frac{s+k}{2}+1}\pi^{-\frac{s-1}{2}-k}s^{\frac{s+k+1}{2}}e^{-\frac{s}{2}} \times \sum_{j=0}^{k}\binom{k}{j}\left(\cos\frac{\pi s}{2}\right)^k f_{k-j}(s)(s-1)\zeta^{(j)}(s)(1+O_k(|s|^{-1})). \tag{3.3.2}$$

Finally, by the estimates in the previous section (3.2.6) and (3.2.5), we have

$$\xi_k(s) \ll_k \exp(C|s|\log(|s|+2)),$$

where the constant C depends only on k. The functional equation of $\xi_k(s)$ (3.1.1) implies that this estimate is valid for $\sigma \leq 1/2$ too. Thus the order of $\xi_k(s)$ is at most 1. By (3.3.2), we have $\log\xi_k(\sigma) \sim (\sigma\log\sigma)/2$ for $\sigma \to \infty$. This completes the proof. $\square$

We can determine constants e^{A_k} and B_k in (3.3.1).

THEOREM 3.3.2. *In the previous theorem, for $k \geq 0$ we have*

$$e^{A_k} = \xi_k(0) = \frac{(-1)^k(2k-1)!!}{(4\sqrt{\pi})^k}$$

and

$$B_k = -\frac{2k(k-1)}{2k-1}\log 2 - \frac{1}{2(2k-1)}\log 4\pi + \frac{\gamma}{2(2k-1)} - 1,$$

where γ is the Euler constant.

Here, we note that

$$(2k-1)!! = \prod_{l=1}^{k}(2l-1) = (2k-1)(2k-3)\cdots 3\cdot 2\cdot 1,$$

and we define $(-1)!! = 1$ and $(-3)!! = -1$.

PROOF. The logarithmic derivative leads to

$$\frac{\xi_k'}{\xi_k}(0) = B_k,$$

by the previous theorem. Thus, by the definition of $\xi_k(s)$ (1.2.4),

$$B_k = -\frac{1}{2}\log\pi - 1 + \frac{k}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) + \lim_{s\to 0}\left(\frac{1}{s} - \frac{k-1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{Z_k'}{Z_k}(s)\right).$$

Since

$$\frac{1}{s} + \frac{\Gamma'}{\Gamma}(s) = \frac{\Gamma'}{\Gamma}(s+1),$$

we have

$$\begin{aligned}\frac{1}{s} - \frac{k-1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{Z_k'}{Z_k}(s) &= \frac{k}{s} + \frac{Z_k'}{Z_k}(s) - \frac{k-1}{2}\left(\frac{2}{s} + \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\right)\\ &= \frac{k}{s} + \frac{Z_k'}{Z_k}(s) - \frac{k-1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right).\end{aligned}$$

We consider the logarithmic derivative of $Z_k(s)$. We recall $Z_k(s)$ has a pole with order k at $s = 0$. Then we can express

$$Z_k(s) = \sum_{l=-k}^{\infty} a_{k,l}s^l.$$

Then we obtain

$$\begin{aligned}\frac{k}{s} + \frac{Z_k'}{Z_k}(s) &= \frac{k}{s} + \frac{-ka_{k,-k}s^{-k-1} - (k-1)a_{k,-k+1}s^{-k} + O(|s|^{-k+1})}{a_{k,-k}s^{-k} + a_{k,-k+1}s^{-k+1} + O(|s|^{-k+2})}\\ &= \frac{ka_{k,-k}s^{-k-1} + ka_{k,-k+1}s^{-k} + O(|s|^{-k+1})}{a_{k,-k}s^{-k} + a_{k,-k+1}s^{-k+1} + O(|s|^{-k+2})}\\ &\quad + \frac{-ka_{k,-k}s^{-k-1} - (k-1)a_{k,-k+1}s^{-k} + O(|s|^{-k+1})}{a_{k,-k}s^{-k} + a_{k,-k+1}s^{-k+1} + O(|s|^{-k+2})}\\ &= \frac{a_{k,-k+1} + O(|s|)}{a_{k,-k} + a_{k,-k+1}s + O(|s|^2)}.\end{aligned}$$

Therefore

$$\lim_{s\to 0}\left(\frac{k}{s} + \frac{Z_k'}{Z_k}(s)\right) = \frac{a_{k,-k+1}}{a_{k,-k}}.$$

By the definition (2.1.1) and Lemma 2.3.1, we can see that for $k \geq 1$

$$a_{k,-k} = -(k-1)a_{k-1,-k+1} - \frac{1}{2}a_{k-1,-k+1} = \left(-k + \frac{1}{2}\right)a_{k-1,-k+1},$$

(see the argument in the proof of Lemma 2.3.2) and

$$\begin{aligned}\frac{a_{k,-k}}{a_{0,0}} &= \frac{a_{k,-k}}{a_{k-1,-k+1}} \cdot \frac{a_{k-1,-k+1}}{a_{k-2,-k+2}} \cdots \frac{a_{1,-1}}{a_{0,0}}\\ &= \left(-k + \frac{1}{2}\right)\left(-k + \frac{3}{2}\right)\cdots\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^k (2k-1)!!.\end{aligned}$$

Since $a_{0,0} = \zeta(0) = -1/2$, we have

$$a_{k,-k} = \left(-\frac{1}{2}\right)^{k+1} (2k-1)!!.$$

On the other hand, by the definition (2.1.1) and Lemma 2.3.1, we can obtain

$$a_{k,-k+1} = -\frac{1}{2}(2k-3)a_{k-1,-k+2} - \frac{C_0}{2}a_{k-1,-k+1},$$

where C_0 is the constant term of the Laurent expansion of $\omega(s)$ around 0 and it is $\log 2\pi + \gamma$. When we put

$$a'_k = \frac{a_{k,-k+1}}{-\frac{C_0}{2}a_{k-1,-k+1}}$$

we have

$$a'_k = a'_{k-1} + 1 = a'_1 + k - 1.$$

Thus we can see that

$$a_{k,-k+1} = \left(\frac{\gamma}{\log 2\pi + \gamma} + k - 1\right)(\log 2\pi + \gamma)\left(-\frac{1}{2}\right)^{k+1} \frac{(2k-1)!!}{2k-1}$$

by the fact that $a_{1,0} = (\zeta'(0) - \zeta(0)\log 2\pi - \zeta(0)\gamma)/2 = \gamma/4$.

Hence we have

$$\begin{aligned}
B_k &= -\frac{1}{2}\log \pi - 1 + \frac{k}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) - \frac{k-1}{2}\frac{\Gamma'}{\Gamma}(1) \\
&\quad + \left(\frac{\gamma}{\log 2\pi + \gamma} + k - 1\right)\frac{\log 2\pi + \gamma}{2k-1} \\
&= -\frac{2k(k-1)}{2k-1}\log 2 - \frac{1}{2(2k-1)}\log 4\pi + \frac{\gamma}{2(2k-1)} - 1.
\end{aligned}$$

Here, to show the last equality, we use

$$\frac{\Gamma'}{\Gamma}(1) = -\gamma \quad \text{and} \quad \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) = -\gamma - 2\log 2.$$

Lastly, we calculate $\xi_k(0) = e^{A_k}$. It is known that

$$\Gamma(s+1) = s\Gamma(s).$$

Hence we obtain

$$\begin{aligned}
\xi_k(0) &= -\Gamma\left(\frac{1}{2}\right)^{-k} \lim_{s\to 0} \frac{sZ_k(s)}{\Gamma\left(\frac{s}{2}\right)^{k-1}} \\
&= -\Gamma\left(\frac{1}{2}\right)^{-k} \lim_{s\to 0} \left(\frac{s}{2}\right)^{k-1} \frac{sZ_k(s)}{\Gamma\left(\frac{s}{2}+1\right)^{k-1}} \\
&= -\frac{2^{-k+1}}{\Gamma(\frac{1}{2})^k} \lim_{s\to 0} s^k Z_k(s) = -2^{-k+1}\pi^{-\frac{k}{2}} a_{k,-k} = \frac{(-1)^k(2k-1)!!}{(4\sqrt{\pi})^k}.
\end{aligned}$$

□

3.4. The special values of $\xi_k(s)$ at integer points

In this section, we give an explicit expression of special values of $\xi_k(s)$ at positive integer points.

THEOREM 3.4.1. *Let $k \geq 0$. We have*

$$\xi_k(1) = \frac{(2k-1)!!}{(4\sqrt{\pi})^k},$$

and for $n \geq 1$,

$$\begin{aligned}
&\xi_k(2n+1) \\
&= (-1)^{kn+1}\pi^{-\frac{2n+k}{2}} \frac{(2n+1)!2n}{4^n \cdot n!} \left(\frac{4^{n-1}(n!)^2}{(2n)!}\right)^k (2k-3)!!\zeta(2n+1)
\end{aligned}$$

and

$$\xi_k(2n) = (-1)^{kn}\pi^{-\frac{2n+k}{2}} 2n(2n-1)(n-1)! \left(\frac{(2n)!}{4^n n!(n-1)!}\right)^k Z_k(2n).$$

Before starting the proof, we note that Proposition 2.2.1 and 2.2.2 imply

$$\begin{aligned}
Z_k(2n) = \sum_{j=0}^{k} \frac{k!}{j!}\zeta^{(j)}(2n) &\sum_{a_1+2a_2+\cdots+(k-j)a_{k-j}=k-j} \left(-\frac{1}{2}\right)^{a_1+\cdots+a_{k-j}} \\
&\times \prod_{l=1}^{k-j} \frac{1}{a_l!}\left(\frac{\omega^{(l-1)}(2n)}{l!}\right)^{a_l}.
\end{aligned}$$

On the $\omega^{(l-1)}(2n)$, since

$$\left.\frac{d^{l-1}}{ds^{l-1}} \tan\frac{\pi s}{2}\right|_{s=2n} = \begin{cases} \dfrac{(-1)^{m-1}B_{2m}(4^m-1)\pi^{2m-1}}{2m} & l = 2m, \\ 0 & l = 2m+1, \end{cases}$$

and for $l \geq 2$

$$\frac{d^{l-1}}{ds^{l-1}} \frac{\Gamma'}{\Gamma}(s)\Big|_{s=2n} = (-1)^l (l-1)! \left(\zeta(l) - \sum_{j=1}^{2n-1} \frac{1}{j^l} \right),$$

we have

$$\omega^{(l-1)}(2n) = \begin{cases} (-1)^m \dfrac{B_{2m}(2-4^m)\pi^{2m}}{4m} + (2m-1)! \displaystyle\sum_{j=1}^{2n-1} \frac{1}{j^{2m}} & l = 2m, \\ -(2m)! \left(\displaystyle\sum_{j=1}^{2n-1} \frac{1}{j^{2m+1}} - \zeta(2m+1) \right) & l = 2m+1, \end{cases}$$

where B_{2m} is the $2m$-th Bernoulli number and therefore $\omega^{(l-1)}(2n) > 0$. When $l = 1$, we can obtain

$$\omega(2n) = \log 2\pi + \gamma - \sum_{j=1}^{2n-1} \frac{1}{j}$$

because

$$\frac{\Gamma'}{\Gamma}(n) = -\gamma + \sum_{j=1}^{n-1} \frac{1}{j}.$$

PROOF OF THEOREM 3.4.1. The case of $s = 1$ is trivial.

Let $n \geq 1$. First we consider the case of $s = 2n$. When $s = 2n$ we have

$$\begin{aligned} \xi_k(2n) &= \pi^{-n} 2n(2n-1) \frac{Z_k(2n)}{\Gamma(n)^{k-1}\Gamma(\frac{1}{2}-n)^k} \\ &= (-1)^{kn} \pi^{-\frac{2n+k}{2}} 2n(2n-1)(n-1)! \left(\frac{(2n)!}{4^n n!(n-1)!} \right)^k Z_k(2n). \end{aligned}$$

Next we treat the case of $s = 2n+1$. In this case, we see that

$$\begin{aligned}
\xi_k(2n+1) &= \pi^{-\frac{2n+1}{2}}(2n+1)2n\frac{1}{\Gamma(\frac{2n+1}{2})^{k-1}}\lim_{s\to 2n+1}\frac{Z_k(s)}{\Gamma(\frac{1-s}{2})^k} \\
&= \pi^{-\frac{2n+1}{2}}(2n+1)2n\frac{(4^n n!)^{k-1}}{((2n)!)^{k-1}\pi^{\frac{k-1}{2}}} \\
&\quad \times \lim_{s\to 2n+1}\frac{\prod_{m=0}^{n-1}(2m+1-s)^k}{2^{kn+k}\Gamma(\frac{2n+3-s}{2})^k}(2n+1-s)^k Z_k(s) \\
&= (-1)^{kn+k}\pi^{-\frac{2n+k}{2}}(2n+1)2n\frac{(n!)^k(4^n n!)^{k-1}}{2^k((2n)!)^{k-1}} \\
&\quad \times \lim_{s\to 2n+1}(s-2n-1)^k Z_k(s).
\end{aligned}$$

We consider the Laurent expansion at $s = 2n+1$ i.e.

$$Z_k(s) = \sum_{l=-k}^{\infty} b_{k,l}(s-2n-1)^l.$$

By the definition (2.1.1) and Lemma 2.3.1, we can see that

$$b_{k,-k} = -(k-1)b_{k-1,-k+1} + \frac{1}{2}b_{k-1,-k+1} = -\left(k-\frac{3}{2}\right)b_{k-1,-k+1}.$$

Thus we have

$$\begin{aligned}
\frac{b_{k,-k}}{b_{0,0}} &= \frac{b_{k,-k}}{b_{k-1,-k+1}}\cdot \cdots \cdot \frac{b_{1,-1}}{b_{0,0}} \\
&= \left(-k+\frac{3}{2}\right)\cdots\left(-\frac{1}{2}\right)\frac{1}{2} = -\left(-\frac{1}{2}\right)^k(2k-3)!!.
\end{aligned}$$

Since $b_{0,0} = \zeta(2n+1)$, we obtain

$$b_{k,-k} = -\left(-\frac{1}{2}\right)^k(2k-3)!!\zeta(2n+1).$$

This leads to

$$\begin{aligned}
&\xi_k(2n+1) \\
&= (-1)^{kn+1}\pi^{-\frac{2n+k}{2}}\frac{(2n+1)!2n}{4^n\cdot n!}\left(\frac{4^{n-1}(n!)^2}{(2n)!}\right)^k(2k-3)!!\zeta(2n+1).
\end{aligned}$$

$\square$

3.5. Necessary Additional statements

To show Matsuoka's theorem (Theorem 3.6.1), we need some additional statements.

By Proposition 2.2.2 and Lemma 3.2.3,

$$f_k(s) = \left(-\frac{\omega(s)}{2}\right)^k (1 + O_k((\log|s|)^{-2})) \tag{3.5.1}$$

for $s \in \mathscr{D}(2m) = \{s \in \mathscr{D} \mid \sigma \geq 2m\}$ with sufficiently large integer $m = m(k)$. This implies $\Re f_k(s) > 0$ in this region. Thus, by Lemma 2.3.2 the argument principle, we have

Lemma 3.5.1 (Lemma 4.3 in [**28**]). *For $s \in \{s \mid \sigma < 1-2m\} \cup \{s \mid \sigma > 2m\}$, the zeros of $f_k(s)$ are in small circles centred negative even integers and positive odd integers, and the number of those is k in each circles.*

By Proposition 2.2.1, (3.5.1) and the facts that

$$\zeta(s) = 1 + O(2^{-\sigma}), \quad \zeta^{(k)} = O_k(2^{-\sigma}),$$

we see that

$$\begin{aligned} Z_k(s) &= f_k(s)\left\{\zeta(s) + \sum_{j=1}^{k} \binom{k}{j} \frac{f_{k-j}(s)}{f_k(s)} \zeta^{(j)}(s)\right\} \\ &= f_k(s)\{1 + O(2^{-\sigma})\} \end{aligned}$$

for $s \in \mathscr{D}(2m) = \{s \in \mathscr{D} \mid \sigma \geq 2m\}$ and $k \geq 1$.

Hence, by (3.5.1)

$$Z_k(s) = \left(-\frac{\omega(s)}{2}\right)^k \{1 + O_k((\log|s|)^{-2})\} \quad (k \geq 1) \tag{3.5.2}$$

for $s \in \mathscr{D}(2m)$. This leads to $\Re Z_k(s) > 0$ in this region. Hence, by Lemma 2.3.3 the same argument as in Lemma 3.5.1, we obtain

Lemma 3.5.2 (Lemma 4.5 in [**28**]). *For $s \in \{s \mid \sigma < 1-2m\} \cup \{s \mid \sigma > 2m\}$, the zeros of $Z_k(s)$ are all located in small circles centred negative even integers and positive odd integers, and the number of those is k in each circles.*

3.6. An Alternative Proof of Matsuoka's result

In this section, we give an alternative proof of Matsuoka's result:

Theorem 3.6.1 (Theorem 1.1 in [**28**]). *If the RH is true, then for any non-negative integer k there exists a $T = T(k) > 0$ such that,* *for $t \geq T$, $Z^{(k+1)}(t)$ has exactly one zero between consecutive zeros of $Z^{(k)}(t)$.*

More precisely, we prove Mozer's formula

$$\frac{d}{dt}\frac{Z^{(k+1)}}{Z^{(k)}}(t) = -\sum_{\gamma_k}\frac{1}{(t-\gamma_k)^2} + O_k(t^{-1}) \tag{3.6.1}$$

in a different way from Matsuoka's proof. Here the sum is taken over zeros of $Z^{(k)}(t)$. If we can prove this formula, then we see that

$$\begin{aligned}\frac{d}{dt}\frac{Z^{(k+1)}}{Z^{(k)}}(t) &< -\sum_{0<\gamma_k<t}\frac{1}{(t-\gamma_k)^2} + At^{-1}\\ &< t^{-1}(A - N_k(t)t^{-1}).\end{aligned}$$

By Proposition 2.3.2, this is negative for large t. Matsuoka's proof is inspired by Anderson's method ([**1**]), which implies the above theorem for $k = 1$. Matsuoka considered the integral

$$\int_C \frac{G_k'}{G_k}(w)\frac{s}{w(s-w)}dw,$$

where C is an appropriate rectangle and

$$G_k(w) = h(w)\frac{Z_k(w)}{f_k(w)}$$

with $h(w) = \pi^{-s/2}\Gamma(s/2)$. However, in view of Theorem 3.3.1, we can prove the formula more easily.

Proof of (3.6.1). By the definition of $\xi_k(s)$ and (2.1.3), we have

$$\xi_k\left(\frac{1}{2}+it\right) = -i^{-k}\pi^{-\frac{1}{4}}\left(\frac{1}{4}+t^2\right)\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|^{-2k+1} Z^{(k)}(t).$$

Hence when we put

$$g_k(t) = i^{-k}\pi^{-\frac{1}{4}}\left(\frac{1}{4}+t^2\right)\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|^{-2k+1}$$

then, by the logarithmic derivative with respect to t, we can obtain

$$i\frac{\xi_k'}{\xi_k}\left(\frac{1}{2}+it\right) = \frac{g_k'}{g_k}(t) + \frac{Z^{(k+1)}}{Z^{(k)}}(t). \tag{3.6.2}$$

As for the function $g_k'/g_k(t)$, we can see that

$$\frac{g_k'}{g_k}(t) = -(2k-1)\frac{d}{dt}\Re\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) + \frac{2t}{\frac{1}{2}+t^2}$$

and hence

$$\frac{d}{dt}\frac{g_k'}{g_k}(t) \ll_k t^{-1}$$

by Lemma 3.2.1.

On the other hand, Theorem 3.3.1 implies that

$$\frac{\xi_k'}{\xi_k}\left(\frac{1}{2}+it\right) = B_k + \sum_{\rho_k}\left(\frac{1}{\frac{1}{2}+it-\rho_k}+\frac{1}{\rho_k}\right).$$

Therefore we have

$$\begin{aligned}
\frac{d}{dt}\frac{\xi_k'}{\xi_k}\left(\frac{1}{2}+it\right) &= \sum_{\rho_k}\frac{-i}{(\frac{1}{2}+it-\rho_k)^2} \\
&= i\sum_{\gamma_k}\frac{1}{(t-\gamma_k)^2} + \sum_{\substack{\rho_k \\ \beta_k\neq\frac{1}{2}}}\frac{-i}{(\frac{1}{2}+it-\rho_k)^2} \\
&= i\sum_{\gamma_k}\frac{1}{(t-\gamma_k)^2} \\
&\quad + \sum_{\substack{\rho_k \\ \beta_k<1-2m,\ 2m<\beta_k}}\frac{-i}{(\frac{1}{2}+it-\rho_k)^2} + O(t^{-2}) \\
&= i\sum_{\gamma_k}\frac{1}{(t-\gamma_k)^2} + O(t^{-1})
\end{aligned}$$

by Proposition 2.3.1 and Lemma 3.5.2. To show the last equality, we use the same argument as Matsuoka's (see [**28**, p.15]):

$$\sum_{\substack{\rho_k \\ \beta_k<1-2m,\ 2m<\beta_k}}\frac{1}{(\frac{1}{2}+it-\rho_k)^2} \ll_k \sum_{n=m}^{\infty}\frac{k}{(t+2n+1)^2} \ll_k \int_{2m+1}^{\infty}\frac{dx}{(t+x)^2}.$$

Thus we have

$$\frac{d}{dt}\frac{Z^{(k+1)}}{Z^{(k)}}(t) = -\sum_{\gamma_k}\frac{1}{(t-\gamma_k)^2} + O_k(t^{-1}).$$

□

CHAPTER 4

A discrete moment of the higher derivative of $Z(t)$

This chapter is based on the author's paper [**22**]. In this chapter, we will prove the following theorem.

THEOREM 4.0.1. *Let j and k be fixed non-negative integers. If the RH is true, then as $T \to \infty$,*

$$\begin{aligned}
&\sum_{0<\gamma_k\le T} \left|Z^{(j)}(\gamma_k)\right|^2 \\
&= \delta_{0,k} \frac{T}{2^{2j+1}(2j+1)\pi} \left(\log \frac{T}{2\pi}\right)^{2j+2} \\
&- \frac{(k+1)\{1+(-1)^j\}}{2^{2j+1}(j+1)^2} \frac{T}{2\pi} \left(\log \frac{T}{2\pi}\right)^{2j+2} \\
&\quad + \sum_{u=1}^{j} \frac{1}{2j+1-u} \frac{j!}{(j-u)!} (-1)^{-u} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \frac{T}{2^{2j+1}\pi} \left(\log \frac{T}{2\pi}\right)^{2j+2} \\
&\quad + (-1)^{j+1} \sum_{g=1}^{k} \frac{(j!)^2}{\theta_g^{2j+2}} \frac{T}{2^{2j+2}\pi} \left(\log \frac{T}{2\pi}\right)^{2j+2} \\
&\quad + (-1)^j (j!)^2 \sum_{g=1}^{k} \frac{\left(\frac{T}{2\pi}\right)^{z_g-1}}{\theta_g^{2j+2}} \left(\sum_{\mu=0}^{j} \frac{\theta_g^\mu}{\mu!}\right)^2 \frac{T}{2^{2j+2}\pi} \left(\log \frac{T}{2\pi}\right)^{2j+2} \\
&\quad + O_{j,k}\left(T(\log T)^{2j+1}\right),
\end{aligned}$$

where $\delta_{0,k}$ is Kronecker's delta, z_g $(g = 1, 2, \cdots, k)$ are the zeros of $\mathscr{Z}_k(s,T)$ with $z_g = 1 - 2\theta_g/L + O(L^{-2})$, and θ_g satisfies $\sum_{\mu=0}^{k} \theta_g^\mu/\mu! = 0$. When $j = 0$ or $k = 0$, we consider the sums on the right-hand side as the empty sums.

At the last main term, since $L = \log(T/2\pi)$, we see that

$$\left(\frac{T}{2\pi}\right)^{z_g-1} = e^{-2\theta_g + O\left(\frac{1}{L}\right)}.$$

Therefore we can write the approximate formula in the form $C_{j,k} T L^{2j+2}$.

REMARK 4.0.1. Matsuoka [**28**] proved that the zeros of $Z^{(k+1)}(t)$ are interlaced with those of $Z^{(k)}(t)$ for sufficiently large t. Therefore our sum contains the mean square of the extremal value of $|Z^{(j)}(t)|$.

REMARK 4.0.2. When $k = 2$, it is not clear whether the coefficient of Yıldırım's asymptotic formula is positive or negative, hence his result does not give precise information, and our main theorem too. This is because we have no exact information on the location of zeros of $\mathscr{Z}_2(s, T)$ near $s = 1$. In general, it is difficult to confirm even $C_{j,k} \geq 0$. However, we can verify $C_{k,k} = 0$ because it is known that

$$\sum_{g=1}^{k} \frac{1}{\theta_g^{2k+2}} = \frac{(-1)^{k+1} + 1}{k!(k+1)!}$$

and

$$\sum_{g=1}^{k} \frac{1}{\theta_g^{u}} = \begin{cases} -1 & (u = 1), \\ 0 & (2 \leq u \leq k), \\ \frac{1}{k!} & (u = k+1) \end{cases}$$

(see [**43**]).

4.1. PRELIMINARY LEMMAS

First we present the following lemma on the distance between the real part of the zeros of $Z_k(s)$ and the critical line.

LEMMA 4.1.1 (Lemma 4 in [**44**]). *Assuming the RH, the zeros of $Z_k(s)$ which are not on $\sigma = 1/2$ are within a distance $1/9$ from the line $\sigma = 1/2$.*

From Proposition 2.3.2, we see that there exists a sequence of positive numbers $\{T_r\}_{r=1}^{\infty}(T_r \to \infty \; as \; r \to \infty)$ such that if $Z_k(\beta_k + i\gamma_k) = 0$ then $|\gamma_k - T_r|^{-1} = O_k(\log T_r)$. Moreover, Proposition 2.3.1 says that for sufficiently $T_0 = T_0(k)$, all zeros of $Z_k(s)$ for $t > T_0$ is on the critical line. When we take T, we understand that it is $> T_0$ and in $\{T_r\}_{r=1}^{\infty}$ hereafter.

$\mathscr{Z}_k(s, T)$ has important properties for our purpose.

LEMMA 4.1.2 (Lemma 5 in [**44**]). *Assuming the RH, we have*

$$\frac{Z_k'}{Z_k}(s) - \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s, T) \ll \frac{U}{T}$$

for $\sigma \geq 5/8$ and $T \leq t \leq T + U \leq 2T$.

LEMMA 4.1.3 (Lemma 6 in [**44**]). *We assume RH and let $k \geq 1$. At $s = 1$ $\mathscr{Z}_k(s, T)$ has a pole of order $k+1$. There are k zeros of $\mathscr{Z}_k(s, T)$*

located at $z_g = 1 - 2\theta_g/L + O_k(1/L^2)$ $(g = 1, \dots, k)$, *where* θ_g*'s are the roots of* $\sum_{\mu=0}^{k} \theta^\mu/\mu! = 0$. *There are no other zeros or poles of* $\mathscr{Z}_k(s,T)$ *with* $5/8 \le \sigma \le 2$. *Thus we have*

$$\frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T) = \frac{-(k+1)}{s-1} + \sum_{g=1}^{k} \frac{1}{s - z_g} + W(s,T)$$

where $W(s,T)$ *is regular for* $5/8 \le \sigma \le 9/8$.

Lemma 4.1.4. *For* $\sigma \ge 9/8$, *there is an absolutely convergent Dirichlet series such that*

$$\frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T) = \sum_{m=1}^{\infty} \frac{a_k(m)}{m^s} + O(T^{-1})$$

where, as $T \to \infty$, $a_k(m) = a_k(m,L) \ll_\varepsilon T^\varepsilon$ *for any* $\varepsilon > 0$ *and* $m \ll T$.

Proof. This result has been proved in [**6**] $\square$

Under the RH, we can obtain

$$\frac{\zeta'}{\zeta}(s) \ll ((\log(|t|+2))^{2-2\sigma} + 1) \min\left(\frac{1}{|\sigma - 1|}, \log\log(|t|+2)\right)$$

uniformly for $1/2 + 1/\log\log(|t|+2) \le \sigma \le 3/2, |t| \ge 1$ (see [**30**], p.435). We can see that

$$\frac{\zeta^{(\mu+1)}}{\zeta}(s) = \frac{d}{ds}\frac{\zeta^{(\mu)}}{\zeta}(s) + \frac{\zeta^{(\mu)}}{\zeta}(s)\frac{\zeta'}{\zeta}(s).$$

Hence, inductively applying Cauchy's integral theorem in a disk of radius $(\log(|t|+2))^{-1}$ around s, we have

$$\frac{\zeta^{(\mu+1)}}{\zeta}(s) \ll_\mu ((\log(|t|+2))^{\mu+2-2\sigma} + (\log(|t|+2))^{\mu}) \log\log(|t|+2)$$

uniformly for $5/8 \le \sigma \le 9/8, |t| \ge 2$. Therefore

$$
\begin{aligned}
\frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T) &= \left(\sum_{\mu=0}^{k}\binom{k}{\mu}(\frac{L}{2})^{k-\mu}\zeta^{(\mu+1)}(s)\right)\left(\sum_{\mu=0}^{k}\binom{k}{\mu}(\frac{L}{2})^{k-\mu}\zeta^{(\mu)}(s)\right)^{-1}\\
&= \left(\sum_{\mu=0}^{k}\binom{k}{\mu}(\frac{L}{2})^{-\mu}\frac{\zeta^{(\mu+1)}}{\zeta}(s)\right)\\
&\quad\times\left(1+\sum_{\mu=1}^{k}\binom{k}{\mu}(\frac{L}{2})^{-\mu}\frac{\zeta^{(\mu)}}{\zeta}(s)\right)^{-1}\\
&= \sum_{\mu=0}^{k}\binom{k}{\mu}\left(\frac{L}{2}\right)^{-\mu}\frac{\zeta^{(\mu+1)}}{\zeta}(s)(1+o(1))\\
&\ll_{k,\varepsilon} |t|^{\varepsilon}
\end{aligned}
$$

uniformly for $5/8 \le \sigma \le 9/8, |t| \ge 2$.

As in the paper of Yıldırım [**44**], we apply the following lemma by Gonek [**10**] :

Lemma 4.1.5 (Lemma 5 in [**10**]). *Let $a > 1$ be fixed and let m be a non-negative integer. Let the Dirichlet series $\sum_{n=1}^{\infty} b_n n^{-a-it}$ be absolutely convergent with a sequence of complex number $\{b_n\}_{n=1}^{\infty}$. Then for any sufficiently large T,*

$$
\begin{aligned}
&\frac{1}{2\pi}\int_1^T\left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^m dt\\
&= \sum_{1\le n\le \frac{T}{2\pi}} b_n(\log n)^m + O\left(T^{a-\frac{1}{2}}(\log T)^m\right).
\end{aligned}
$$

Finally, we introduce some fundamental lemmas. Stirling's formula implies

Lemma 4.1.6. *For $-1 < \sigma < 2$ and $t \ge 1$, we have*

$$\chi(1-s) = e^{-\frac{\pi i}{4}}\left(\frac{t}{2\pi}\right)^{\sigma-\frac{1}{2}}\exp\left(it\log\frac{t}{2\pi e}\right)\left(1+O\left(\frac{1}{t}\right)\right), \quad (4.1.1)$$

$$\frac{\chi'}{\chi}(s) = -\log\frac{t}{2\pi} + O\left(\frac{1}{t}\right), \quad (4.1.2)$$

and

$$\left(\frac{\chi'}{\chi}\right)^{(k)}(s) = O_k\left(\frac{1}{t}\right). \quad (4.1.3)$$

Then, by the definition of $f_k(s)$ (2.2.1), for $-1 < \sigma < 2$ and $t \geq 1$

$$f_k(s) = \left(\frac{1}{2}\log\frac{t}{2\pi}\right)^k + O_k\left(t^{-1}(\log t)^{k-1}\right). \tag{4.1.4}$$

If the RH is true, then the Lindelöf Hypothesis

$$\zeta\left(\frac{1}{2}+it\right) \ll_\varepsilon |t|^\varepsilon \quad (|t| \geq 1)$$

is also true. Therefore we can obtain the following estimates.

LEMMA 4.1.7. *If the RH is true, then for $\mu = 0, 1, 2, \dots$ and $|t| \geq 1$,*

$$\zeta^{(\mu)}(s) \ll_{\mu,\varepsilon} \begin{cases} 1 & 1 < \sigma, \\ |t|^\varepsilon & \frac{1}{2} \leq \sigma \leq 1, \\ |t|^{\frac{1}{2}-\sigma+\varepsilon} & -1 < \sigma < \frac{1}{2}. \end{cases}$$

When $\mu = 0$, these estimates are well-known. For $\mu \geq 1$, we can obtain this estimates, using Cauchy's theorem in a disk of radius $(\log(|t|+2))^{-1}$ around s.

By (2.2.2), this lemma leads to

$$Z_k(s) \ll_{k,\varepsilon} \begin{cases} |t|^\varepsilon & \frac{1}{2} \leq \sigma < 2, \\ |t|^{\frac{1}{2}-\sigma+\varepsilon} & -1 < \sigma < \frac{1}{2} \end{cases}$$

for $|t| \geq 1$.

Now we can show that

$$\frac{Z_k'}{Z_k}(\sigma + iT) = O_k((\log T)^2)$$

uniformly for $-1 \leq \sigma \leq 2$ by applying the following lemma

LEMMA 4.1.8 (Lemma α in [**38**]). *If $f(s)$ is regular, and*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M \quad (M > 1)$$

in the circle $|s - s_0| \leq r$, then

$$\left|\frac{f'(s)}{f(s)} - \sum_\rho \frac{1}{s-\rho}\right| < \frac{AM}{r} \quad \left(|s - s_0| \leq \frac{r}{4}\right),$$

where ρ runs over the zeros of $f(s)$ such that $|\rho - s_0| \leq r/2$ and A is an absolute positive constant.

We use this lemma with $f(s) = Z_k(s), r = 12$ and $s_0 = 2 + iT$. The estimate of $Z_k(s)$ implies that we can take $M = \log T$ in this lemma. Hence we have

$$\frac{Z_k'}{Z_k}(\sigma + iT) = \sum_{\substack{\rho \\ |\rho-(2+iT)|\le 6}} \frac{1}{s-\rho} + O_k(\log T).$$

By the way of taking T and Proposition 2.3.2, we see that

$$\begin{aligned}\sum_{\substack{\rho \\ |\rho-(2+iT)|\le 6}} \frac{1}{s-\rho} &\ll_k \sum_{\substack{\rho \\ |\rho-(2+iT)|\le 6}} \log T \\ &\ll_k (\log T)^2.\end{aligned}$$

4.2. The proof of Theorem

Our proof is inspired by the proof of Yıldırım [**44**]. As we mentioned before, we consider sufficiently large T in $\{T_r\}_{r=1}^{\infty}$. This restriction will be removed at the end of the proof. Now by Proposition 2.3.1, $Z_k(s)$ has at most $O_k(1)$ zeros off the critical line up to T. At such a zero, by Lemma 4.1.1,

$$|Z_j(\rho_k)|^2 \ll_{j,\varepsilon} |\Im\rho_k|^{\frac{2}{9}+\varepsilon},$$

whence

$$\sum_{\substack{0<\Im\rho_k\le T \\ \Re\rho_k\ne\frac{1}{2}}} |Z_j(\rho_k)|^2 \ll_{j,k,\varepsilon} T^{\frac{2}{9}+\varepsilon},$$

where ρ_k is the zeros of $Z_k(s)$. Therefore, by (2.1.3) and (2.1.4),

$$\begin{aligned}\sum_{0<\gamma_k\le T} \left|Z^{(j)}(\gamma_k)\right|^2 &= \sum_{0<\gamma_k\le T} \left|Z_j\left(\frac{1}{2}+i\gamma_k\right)\right|^2 \\ &= \sum_{\substack{\rho_k \\ 0<\Im\rho_k\le T}} |Z_j(\rho_k)|^2 + O_{j,k,\varepsilon}\left(T^{\frac{2}{9}+\varepsilon}\right) \\ &= M(T) + O_{j,k,\varepsilon}\left(T^{\frac{2}{9}+\varepsilon}\right),\end{aligned}$$

say. For convenience, we consider a sum over a shorter range. Let

$$U = T^{\frac{3}{4}}$$

and let R be the positively oriented rectangular path with vertices $c+iT$, $c+i(T+U)$, $1-c+i(T+U)$ and $1-c+iT$, where $c = 5/8$. Then we need to consider

$$M(T+U) - M(T) = \frac{1}{2\pi i}\int_R \frac{Z_k'}{Z_k}(s) Z_j(s) Z_j(1-s)ds.$$

On the horizontal line, since

$$\frac{Z_k'}{Z_k}(s) \ll_{k,\varepsilon} T^{\varepsilon} \quad \text{and} \quad Z_k(s)Z_k(1-s) \ll_{k,\varepsilon} T^{c-\frac{1}{2}+\varepsilon},$$

we can see that

$$\int_{1-c+iT}^{c+iT} \frac{Z_k'}{Z_k}(s)Z_j(s)Z_j(1-s)ds \ll_{k,\varepsilon} T^{c-\frac{1}{2}+\varepsilon}.$$

Thus

$$\begin{aligned}
&\frac{1}{2\pi i}\int_R \frac{Z_k'}{Z_k}(s)Z_j(s)Z_j(1-s)ds \\
&= \frac{1}{2\pi i}\int_{c+iT}^{c+i(T+U)} \frac{Z_k'}{Z_k}(s)Z_j(s)Z_j(1-s)ds \\
&\quad + \frac{1}{2\pi i}\int_{1-c+i(T+U)}^{1-c+iT} \frac{Z_k'}{Z_k}(s)Z_j(s)Z_j(1-s)ds + O_{j,k,\varepsilon}\left(T^{c-\frac{1}{2}+\varepsilon}\right) \\
&= I_1 + I_2 + O_{j,k,\varepsilon}\left(T^{c-\frac{1}{2}+\varepsilon}\right),
\end{aligned}$$

say. On the integral I_2,

$$\begin{aligned}
I_2 &= -\frac{1}{2\pi i}\int_{1-c+iT}^{1-c+i(T+U)} \frac{Z_k'}{Z_k}(s)Z_j(s)Z_j(1-s)ds \\
&= -\frac{1}{2\pi i}\int_{1-c+iT}^{1-c+i(T+U)} \left(\frac{\chi'}{\chi}(s) - \frac{Z_k'}{Z_k}(1-s)\right) Z_j(s)Z_j(1-s)ds \\
&= -\frac{1}{2\pi i}\int_{1-c+iT}^{1-c+i(T+U)} \frac{\chi'}{\chi}(s)Z_j(s)Z_j(1-s)ds \\
&\quad + \frac{1}{2\pi i}\int_{1-c+iT}^{1-c+i(T+U)} \frac{Z_k'}{Z_k}(1-s)Z_j(s)Z_j(1-s)ds.
\end{aligned}$$

When we replace s by $1-s$, the second integral is

$$-\frac{1}{2\pi i}\int_{c-iT}^{c-i(T+U)} \frac{Z_k'}{Z_k}(s)Z_j(s)Z_j(1-s)ds = \overline{I_1}.$$

Now we see that

$$\begin{aligned}
&M(T+U) - M(T) \\
&= -\frac{1}{2\pi i}\int_{1-c+iT}^{1-c+i(T+U)} \frac{\chi'}{\chi}(s)Z_j(s)Z_j(1-s)ds \\
&\quad + 2\Re I_1 + O_{j,k,\varepsilon}\left(T^{c-\frac{1}{2}+\varepsilon}\right).
\end{aligned}$$

We divide the following argument into 5 steps;

Step 1:
Calculate the integral

$$-\frac{1}{2\pi i}\int_{1-c+iT}^{1-c+i(T+U)}\frac{\chi'}{\chi}(s)Z_j(s)Z_j(1-s)ds,$$

Step 2:
Transform the integral I_1 to certain sums of arithmetic functions,

Step 3:
To derive some approximate formula for those sums by Perron's formula,

Step 4:
Express I_1 with that formula and simplify the coefficients,

Step 5:
Concluding.

Step 1. By Cauchy's integral theorem, the integral is equal to

$$-\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(T+U)}\frac{\chi'}{\chi}(s)Z_j(s)Z_j(1-s)ds+O_{j,\varepsilon}\left(T^{c-\frac{1}{2}+\varepsilon}\right).$$

From (4.1.2) and Lemma 4.1.7 we see that the above integral is

$$\frac{1}{2\pi}\int_{T}^{T+U}\log\frac{t}{2\pi}Z^{(j)}(t)^2dt+O_{j,\varepsilon}\left(T^{\varepsilon}\right). \tag{4.2.1}$$

Here we put

$$Y_j(t)=\int_1^t Z^{(j)}(x)^2dx.$$

Using integration by parts and the result of Minamide and Tanigawa (1.4.3), we can show that the integral in (4.2.1) is equal to

$$
\begin{aligned}
&\frac{1}{2\pi}\log\frac{T+U}{2\pi}Y_j(T+U)-\frac{1}{2\pi}\log\frac{T}{2\pi}Y_j(T)-\frac{1}{2\pi}\int_T^{T+U}t^{-1}Y_j(t)dt\\
&=\frac{T+U}{2\cdot 4^j(2j+1)\pi}P_{2j+1}\left(\log\frac{T+U}{2\pi}\right)\log\frac{T+U}{2\pi}\\
&\quad-\frac{T}{2\cdot 4^j(2j+1)\pi}P_{2j+1}\left(\log\frac{T}{2\pi}\right)\log\frac{T}{2\pi}\\
&\quad-\frac{1}{2\cdot 4^j(2j+1)\pi}\int_T^{T+U}\left\{P_{2j+1}\left(\log\frac{t}{2\pi}\right)+O\left(t^{-\frac12}\log^{2j+1}t\right)\right\}dt\\
&\quad+O_j(T^{\frac12}\log^{2j+1}T)\\
&=\frac{U}{2\cdot 4^j(2j+1)\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}+O_j\left(U(\log T)^{2j+1}\right),
\end{aligned}
$$

because

$$
\log\frac{T+U}{2\pi}=\log\frac{T}{2\pi}\left(1+O\left(\frac{U}{T\log T}\right)\right).
$$

Step 2. We calculate I_1. By the functional equation (2.1.4) and Lemma 4.1.2, we have

$$
\begin{aligned}
I_1&=\frac{(-1)^j}{2\pi i}\int_{c+iT}^{c+i(T+U)}\frac{Z_k'}{Z_k}(s)Z_j(s)^2\chi(1-s)ds\\
&=\frac{(-1)^j}{2\pi i}\int_{c+iT}^{c+i(T+U)}\frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)Z_j(s)^2\chi(1-s)ds+O_{j,k,\varepsilon}\left(U^2T^{c-\frac32+\varepsilon}\right).
\end{aligned}
$$

The representation of (2.2.2) and the approximation of $f_k(s)$ (4.1.4) imply that the above is

$$
\begin{aligned}
&=\frac{(-1)^j}{2\pi i}\int_{c+iT}^{c+i(T+U)}\frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\left(\sum_{\mu=0}^{j}\binom{j}{\mu}f_{j-\mu}(s)\zeta^{(\mu)}(s)\right)^2\chi(1-s)ds\\
&\quad+O_{j,k,\varepsilon}\left(U^2T^{c-\frac32+\varepsilon}\right)\\
&=\frac{(-1)^j}{2\pi i}\int_{c+iT}^{c+i(T+U)}\frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\\
&\quad\times\left(\sum_{\mu=0}^{j}\binom{j}{\mu}\left(\frac12\log\frac{t}{2\pi}\right)^{j-\mu}\zeta^{(\mu)}(s)\right)^2\chi(1-s)ds\\
&\quad+O_{j,k,\varepsilon}\left(U^2T^{c-\frac32+\varepsilon}\right)+O_{j,k,\varepsilon}\left(T^{c-\frac12+\varepsilon}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^j}{2\pi i} \int_{b+iT}^{b+i(T+U)} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T) \\
&\quad \times \left(\sum_{\mu=0}^{j} \binom{j}{\mu} \left(\frac{1}{2} \log \frac{t}{2\pi} \right)^{j-\mu} \zeta^{(\mu)}(s) \right)^2 \chi(1-s) ds \\
&\quad + O_{j,k,\varepsilon}\left(U^2 T^{c-\frac{3}{2}+\varepsilon} \right) + O_{j,k,\varepsilon}\left(T^{b-\frac{1}{2}+\varepsilon} \right),
\end{aligned}
$$

where $b = 9/8$. To show the last equality, we use Cauchy's integral theorem. We note that

$$
\begin{aligned}
&\left(\sum_{\mu=0}^{j} \binom{j}{\mu} \left(\frac{1}{2} \log \frac{t}{2\pi} \right)^{j-\mu} \zeta^{(\mu)}(s) \right)^2 \\
&= \sum_{\mu=0}^{j} \sum_{\nu=0}^{j} \binom{j}{\mu} \binom{j}{\nu} \left(\frac{1}{2} \log \frac{t}{2\pi} \right)^{2j-\mu-\nu} \zeta^{(\mu)}(s) \zeta^{(\nu)}(s).
\end{aligned}
\tag{4.2.2}
$$

Therefore, by Lemma 4.1.5, our problem is reduced to consider

$$
\begin{aligned}
&\frac{1}{2\pi i} \int_{b+iT}^{b+i(T+U)} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T) \zeta^{(\mu)}(s) \zeta^{(\nu)}(s) \chi(1-s) \left(\log \frac{t}{2\pi} \right)^{2j-\mu-\nu} ds \\
&= \frac{1}{2\pi i} \int_{b+iT}^{b+i(T+U)} \sum_{m=1}^{\infty} \frac{a_k(m)}{m^s} \zeta^{(\mu)}(s) \zeta^{(\nu)}(s) \chi(1-s) \left(\log \frac{t}{2\pi} \right)^{2j-\mu-\nu} ds \\
&\quad + O_{\mu,\nu,k,\varepsilon}\left(T^{b-\frac{1}{2}+\varepsilon} \right) \\
&= \sum_{\frac{T}{2\pi} \le mn \le \frac{T+U}{2\pi}} a_k(m) D_{\mu\nu}(n) (\log mn)^{2j-\mu-\nu} + O_{\mu,\nu,k,\varepsilon}\left(T^{b-\frac{1}{2}+\varepsilon} \right),
\end{aligned}
\tag{4.2.3}
$$

where $D_{\mu\nu}(n)$ satisfies

$$
\zeta^{(\mu)}(s) \zeta^{(\nu)}(s) = \sum_{n=1}^{\infty} \frac{D_{\mu\nu}(n)}{n^s}
$$

for $\sigma > 1$. If we can calculate the sum

$$
\sum_{mn \le x} a_k(m) D_{\mu\nu}(n),
$$

then by partial summation we are able to compute the sum on the right-hand side in (4.2.3).

Step 3. By Perron's formula,

$$\sum_{mn\le x} a_k(m)D_{\mu\nu}(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT}\sum_{m=1}^{\infty}\frac{a_k(m)}{m^s}\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s}ds + O(x^\varepsilon) + R,$$

where R is the error term appearing in Perron's formula (see [**30**, p.140]) which satisfies that

$$R \ll \sum_{\substack{x/2<mn<2x\\ mn\ne x}} |a_k(m)D_{\mu\nu}(n)| \min\left(1, \frac{x}{T|x-mn|}\right) + \frac{(4x)^b}{T}\sum_{mn=1}^{\infty}\frac{|a_k(m)D_{\mu\nu}(n)|}{(mn)^b}.$$

On the first term, we see that

$$\begin{aligned}
&\sum_{\substack{x/2<mn<2x\\ n\ne x}} |a_k(m)D_{\mu\nu}(n)| \min\left(1, \frac{x}{T|x-mn|}\right)\\
&\ll \frac{x}{T}\sum_{x/2<mn<x-1}\left|\frac{a_k(m)D_{\mu\nu}(n)}{x-mn}\right| + \sum_{x-1\le mn\le x+1}|a_k(m)D_{\mu\nu}(n)|\\
&\quad + \frac{x}{T}\sum_{x+1<mn<2x}\left|\frac{a_k(m)D_{\mu\nu}(n)}{x-mn}\right|\\
&= \frac{x}{T}\sum_{x/2<l<x-1}\sum_{l=mn}\left|\frac{a_k(m)D_{\mu\nu}(n)}{x-l}\right| + \sum_{x-1\le l\le x+1}\sum_{l=mn}|a_k(m)D_{\mu\nu}(n)|\\
&\quad + \frac{x}{T}\sum_{x+1<l<2x}\sum_{l=mn}\left|\frac{a_k(m)D_{\mu\nu}(n)}{x-l}\right|\\
&\ll_{\mu,\nu} \frac{x^{1+\varepsilon}}{T}\sum_{x/2<l<x-1}\sum_{l=mn}\frac{1}{x-l} + x^\varepsilon\sum_{x-1\le l\le x+1}\sum_{l=mn}1\\
&\quad + \frac{x^{1+\varepsilon}}{T}\sum_{x+1<l<2x}\sum_{l=mn}\frac{1}{l-x}\\
&= \frac{x^{1+\varepsilon}}{T}\sum_{x/2<l<x-1}\frac{d(l)}{x-l} + x^\varepsilon\sum_{x-1\le l\le x+1}d(l) + \frac{x^{1+\varepsilon}}{T}\sum_{x+1<l<2x}\frac{d(l)}{l-x}\\
&\ll_\varepsilon \frac{x^{1+\varepsilon}}{T}\sum_{x/2<l<x-1}\frac{1}{x-l} + x^\varepsilon + \frac{x^{1+\varepsilon}}{T}\sum_{x+1<l<2x}\frac{1}{l-x}
\end{aligned}$$

$$\ll \frac{x^{1+\varepsilon}}{T} \sum_{1<l<x} \frac{1}{l} + x^{\varepsilon} \ll \frac{x^{1+\varepsilon}}{T} + x^{\varepsilon}.$$

Therefore we obtain

$$R \ll_{\mu,\nu,\varepsilon} \frac{x^b}{T} + x^{\varepsilon}.$$

By using Lemmas 4.1.3, 4.1.4 and the residue theorem, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{b-iT}^{b+iT} \sum_{m=1}^{\infty} \frac{a_k(m)}{m^s} \zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} ds \\
&= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} ds + O_{\mu,\nu,\varepsilon}(x^b T^{-1+\varepsilon}) \\
&= \operatorname*{Res}_{s=1} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} + \sum_{g=1}^{k} \operatorname*{Res}_{s=z_g} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} \\
&\quad + \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} ds + O_{\mu,\nu,\varepsilon}(x^b T^{-1+\varepsilon}) \\
&= \operatorname*{Res}_{s=1} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} + \sum_{g=1}^{k} \operatorname*{Res}_{s=z_g} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} \\
&\quad + O_{\mu,\nu,\varepsilon}(x^c T^{\varepsilon} + x^b T^{-1+\varepsilon}).
\end{aligned}$$

To calculate the residues, we note that

$$\frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T) = \frac{-(k+1)}{s-1} + \sum_{g=1}^{k} \frac{1}{s-z_g} + W(s,T),$$

$$\frac{x^s}{s} = x\sum_{l=0}^{\infty} \left(\sum_{r=0}^{l} \frac{(-1)^r}{(l-r)!} (\log y)^{l-r} \right) (s-1)^l$$

and

$$\zeta^{(\mu)}(s) = \frac{(-1)^{\mu}\mu!}{(s-1)^{\mu+1}} + \sum_{n=\mu}^{\infty} \frac{n!}{(n-\mu)!} c_n (s-1)^{n-\mu},$$

where c_n is the n-th Stieltjes constant.

On the residue at $s = z_g$, we have

$$\begin{aligned}
&\sum_{g=1}^{k} \operatorname*{Res}_{s=z_g} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} \\
&= \sum_{g=1}^{k} \zeta^{(\mu)}(z_g)\zeta^{(\nu)}(z_g)\frac{x^{z_g}}{z_g} \\
&= \sum_{g=1}^{k} \frac{x^{z_g}}{z_g}\left\{\frac{(-1)^{\mu+\nu}\mu!\nu!}{(z_g-1)^{\mu+\nu+2}} + (-1)^{\mu}\mu!\sum_{n=\mu}^{\infty}\frac{n!}{(n-\mu)!}c_n(z_g-1)^{n-\mu-\nu-1}\right. \\
&\qquad +(-1)^{\mu}\mu!\sum_{m=\nu}^{\infty}\frac{m!}{(m-\nu)!}c_m(z_g-1)^{m-\mu-\nu-1} \\
&\qquad \left. +\sum_{n=\mu}^{\infty}\sum_{m=\nu}^{\infty}\frac{n!m!c_nc_m}{(n-\mu)!(m-\nu)!}(z_g-1)^{m+n-\mu-\nu}\right\},
\end{aligned}$$

because

$$\begin{aligned}
\zeta^{(\mu)}(s)\zeta^{(\nu)}(s) &= \frac{(-1)^{\mu+\nu}\mu!\nu!}{(s-1)^{\mu+\nu+2}} + (-1)^{\nu}\nu!\sum_{n=\mu}^{\infty}\frac{n!}{(n-\mu)!}c_n(s-1)^{n-\mu-\nu-1} \\
&\quad + (-1)^{\mu}\mu!\sum_{m=\nu}^{\infty}\frac{m!}{(m-\nu)!}c_m(s-1)^{m-\mu-\nu-1} \\
&\quad + \sum_{n=\mu}^{\infty}\sum_{m=\nu}^{\infty}\frac{n!m!c_nc_m}{(n-\mu)!(m-\nu)!}(s-1)^{m+n-\mu-\nu}.
\end{aligned}$$

Next we consider the residue at $s = 1$. We see that

$$\begin{aligned}
&\operatorname*{Res}_{s=1} \frac{\mathscr{Z}_k'}{\mathscr{Z}_k}(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} \\
&= -\operatorname*{Res}_{s=1}\frac{k+1}{s-1}\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} + \operatorname*{Res}_{s=1}\sum_{g=1}^{k}\frac{1}{s-z_g}\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} \\
&\quad + \operatorname*{Res}_{s=1} W(s,T)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s} = R_1 + R_2 + R_3,
\end{aligned}$$

say. Since

$$\begin{aligned}
&\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s}\\
&= (-1)^{\mu+\nu}\mu!\nu!x\sum_{l=0}^{\infty}\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)(s-1)^{l-\mu-\nu-2}\\
&\quad + (-1)^{\nu}\nu!x\sum_{n=\mu}^{\infty}\sum_{l=0}^{\infty}\frac{n!}{(n-\mu)!}c_n\\
&\quad \times\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)(s-1)^{l+n-\mu-\nu-1}\\
&\quad + (-1)^{\mu}\mu!x\sum_{m=\nu}^{\infty}\sum_{l=0}^{\infty}\frac{m!}{(m-\nu)!}c_m\\
&\quad \times\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)(s-1)^{l+m-\mu-\nu-1}\\
&\quad + x\sum_{n=\mu}^{\infty}\sum_{m=\nu}^{\infty}\sum_{l=0}^{\infty}\frac{n!m!c_nc_m}{(n-\mu)!(m-\nu)!}\\
&\quad \times\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)(s-1)^{l+m+n-\mu-\nu},
\end{aligned}$$

we have

$$\begin{aligned}
R_1 &= (-1)^{\mu+\nu+1}(k+1)\mu!\nu!x\sum_{r=0}^{\mu+\nu+2}\frac{(-1)^r}{(\mu+\nu+2-r)!}(\log x)^{\mu+\nu+2-r}\\
&\quad + (-1)^{\nu+1}(k+1)\nu!x\\
&\quad \times\sum_{n=\mu}^{\mu+\nu+1}\sum_{l=0}^{\mu+\nu+1-n}\frac{n!}{(n-\mu)!}c_n\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\\
&\quad + (-1)^{\mu+1}(k+1)\mu!x\\
&\quad \times\sum_{m=\nu}^{\mu+\nu+1}\sum_{l=0}^{\mu+\nu+1-m}\frac{m!}{(m-\nu)!}c_m\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\\
&\quad - (k+1)\mu!\nu!c_\mu c_\nu x.
\end{aligned}$$

We emphasise that the largest term is

$$(k+1)\frac{(-1)^{\mu+\nu+1}\mu!\nu!}{(\mu+\nu+2)!}x(\log x)^{\mu+\nu+2}.$$

As for R_2,

$$
\begin{aligned}
&\frac{1}{s-z_g}\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{x^s}{s}\\
&= (-1)^{\mu+\nu}\mu!\nu!x\sum_{\lambda=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^\lambda}{(1-z_g)^{\lambda+1}}\\
&\quad\times\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)(s-1)^{\lambda+l-\mu-\nu-2}\\
&\quad+(-1)^{\mu}\mu!x\sum_{\lambda=0}^{\infty}\sum_{n=\mu}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^\lambda n!}{(n-\mu)!(1-z_g)^{\lambda+1}}c_n\\
&\quad\times\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)(s-1)^{\lambda+l+n-\mu-\nu-1}\\
&\quad+(-1)^{\nu}\nu!x\sum_{\lambda=0}^{\infty}\sum_{m=\nu}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^\lambda m!}{(m-\nu)!(1-z_g)^{\lambda+1}}c_m\\
&\quad\times\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)(s-1)^{\lambda+l+m-\mu-\nu-1}\\
&\quad+x\sum_{\lambda=0}^{\infty}\sum_{n=\mu}^{\infty}\sum_{m=\nu}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^\lambda n!m!c_nc_m}{(n-\mu)!(m-\nu)!(1-z_g)^{\lambda+1}}\\
&\quad\times\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)(s-1)^{\lambda+l+m+n-\mu-\nu},
\end{aligned}
$$

because

$$
\frac{1}{s-z_g}=\sum_{\lambda=0}^{\infty}\frac{(-1)^\lambda}{(1-z_g)^{\lambda+1}}(s-1)^\lambda.
$$

Thus we have

$$
\begin{aligned}
R_2 &= (-1)^{\mu+\nu}\mu!\nu!x\sum_{g=1}^{k}\sum_{\substack{\lambda+l=\mu+\nu+1\\0\le\lambda,l}}\frac{(-1)^\lambda}{(1-z_g)^{\lambda+1}}\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)\\
&\quad+(-1)^{\mu}\mu!x\\
&\quad\times\sum_{g=1}^{k}\sum_{\substack{\lambda+l+n=\mu+\nu\\0\le\lambda,l\\\mu\le n}}\frac{(-1)^\lambda n!c_n}{(n-\mu)!(1-z_g)^{\lambda+1}}\left(\sum_{r=0}^{l}\frac{(-1)^r}{(l-r)!}(\log x)^{l-r}\right)\\
&\quad+(-1)^{\nu}\nu!x
\end{aligned}
$$

$$\times \sum_{g=1}^{k} \sum_{\substack{\lambda+l+m=\mu+\nu \\ 0\le\lambda,l \\ \nu\le m}} \frac{(-1)^{\lambda} m! c_m}{(m-\nu)!(1-z_g)^{\lambda+1}} \left(\sum_{r=0}^{l} \frac{(-1)^r}{(l-r)!} (\log x)^{l-r} \right).$$

The main term in our final result will come from the first sum with $r = 0$, namely,

$$\begin{aligned}
&(-1)^{\mu+\nu}\mu!\nu!x \sum_{g=1}^{k} \sum_{\substack{\lambda+l=\mu+\nu+1 \\ 0\le\lambda,l}} \frac{(-1)^{\lambda}}{(1-z_g)^{\lambda+1}} \frac{(\log x)^l}{l!} \\
&= (-1)^{\mu+\nu+1}\mu!\nu!x \sum_{g=1}^{k} \sum_{\lambda=0}^{\mu+\nu+1} \frac{1}{(\mu+\nu+1-\lambda)!} \frac{(\log x)^{\mu+\nu+1-\lambda}}{(z_g-1)^{\lambda+1}}.
\end{aligned}$$

Since we can see that

$$W(s,T) = \sum_{\lambda_1=0}^{\infty} \frac{W^{(\lambda_1)}(1,T)}{\lambda_1!} (s-1)^{\lambda_1},$$

in a similar manner,

$$\begin{aligned}
&R_3 \\
&= (-1)^{\mu+\nu}\mu!\nu!x \sum_{\substack{\lambda_1+l=\mu+\nu+1 \\ 0\le\lambda_1,l}} \frac{W^{(\lambda_1)}(1,T)}{\lambda_1!} \left(\sum_{r=0}^{l} \frac{(-1)^r}{(l-r)!} (\log x)^{l-r} \right) \\
&\quad + (-1)^{\mu}\mu!x \sum_{\substack{\lambda_1+l+n=\mu+\nu \\ 0\le\lambda_1,l \\ \mu\le n}} \frac{W^{(\lambda_1)}(1,T) n! c_n}{(n-\mu)!\lambda_1!} \left(\sum_{r=0}^{l} \frac{(-1)^r}{(l-r)!} (\log x)^{l-r} \right) \\
&\quad + (-1)^{\nu}\nu!x \sum_{\substack{\lambda_1+l+m=\mu+\nu \\ 0\le\lambda_1,l \\ \nu\le m}} \frac{W^{(\lambda_1)}(1,T) m! c_m}{(m-\nu)!\lambda_1!} \left(\sum_{r=0}^{l} \frac{(-1)^r}{(l-r)!} (\log x)^{l-r} \right).
\end{aligned}$$

We note that the order of R_3 is at least $x(\log x)^{\mu+\nu+1}$.

From the above computations, we obtain

$$\begin{aligned}
&\sum_{mn\le x} a_k(m)D_{\mu\nu}(n)\\
&= (-1)^{\mu+\nu+1}\frac{\mu!\nu!}{(\mu+\nu+2)!}(k+1)x(\log x)^{\mu+\nu+2}\\
&\quad + (-1)^{\mu+\nu+1}\mu!\nu! x\sum_{g=1}^{k}\sum_{\lambda=0}^{\mu+\nu+1}\frac{1}{(\mu+\nu+1-\lambda)!}\frac{(\log x)^{\mu+\nu+1-\lambda}}{(z_g-1)^{\lambda+1}}\\
&\quad + \sum_{g=1}^{k}\zeta^{(\mu)}(z_g)\zeta^{(\nu)}(z_g)\frac{x^{z_g}}{z_g} + x\sum_{\lambda=1}^{\mu+\nu+1}C'_{\mu,\nu}(\lambda)(\log x)^{\mu+\nu+2-\lambda}\\
&\quad + x\sum_{\lambda_1=1}^{\mu+\nu}\sum_{g=1}^{k}\sum_{\lambda=0}^{\mu+\nu+1-\lambda_1}C''_{\mu,\nu}(\lambda,\lambda_1)\frac{(\log x)^{\mu+\nu+1-\lambda_1-\lambda}}{(z_g-1)^{\lambda+1}}\\
&\quad + O_{\mu,\nu,k,\varepsilon}\left((x^c+x^bT^{-1})T^{\varepsilon}(\log x)^{\mu+\nu+1}\right),
\end{aligned}$$

where $C'_{\mu,\nu}(\lambda)$ and $C''_{\mu,\nu}(\lambda_1)$ are some constants.

This leads to

$$\begin{aligned}
&\sum_{\frac{T}{2\pi}\le mn\le\frac{T+U}{2\pi}} a_k(m)D_{\mu\nu}(n)(\log mn)^{2j-\mu-\nu}\\
&= (-1)^{\mu+\nu+1}\frac{\mu!\nu!}{(\mu+\nu+2)!}(k+1)\frac{U}{2\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad + (-1)^{\mu+\nu+1}\mu!\nu!\frac{U}{2\pi}\sum_{g=1}^{k}\sum_{\lambda=0}^{\mu+\nu+1}\frac{1}{(\mu+\nu+1-\lambda)!}\frac{(\log\frac{T}{2\pi})^{2j+1-\lambda}}{(z_g-1)^{\lambda+1}}\\
&\quad + (-1)^{\mu+\nu}\mu!\nu!\frac{U}{2\pi}\left(\log\frac{T}{2\pi}\right)^{2j-\mu-\nu}\sum_{g=1}^{k}\frac{\left(\frac{T}{2\pi}\right)^{z_g-1}}{(z_g-1)^{\mu+\nu+2}}\\
&\quad + O_{\mu,\nu,k}\left(U(\log T)^{2j+1}\right).
\end{aligned}$$

To deduce the last main term, we used that

$$\begin{aligned}
(T+U)^{z_g}-T^{z_g} &= T^{z_g}\left(\left(1+\frac{U}{T}\right)^{z_g}-1\right)\\
&= z_gUT^{z_g-1}+O_k\left(U^2|T^{z_g-2}|\right)\\
&= UT^{z_g-1}+O_k(U(\log T)^{-1}),
\end{aligned}$$

$$\frac{1}{z_g}=\frac{1}{1-\frac{2}{L}\theta_g+O(L^{-2})}=1+O_k\left(L^{-1}\right)$$

and

$$\frac{1}{(z_g-1)^\lambda}=\frac{1}{(-2\nu_g L^{-1}+O_k(L^{-2}))^\lambda}=\left(-2\nu_g L^{-1}\right)^{-\lambda}+O_k\left(L^{\lambda-1}\right)$$

for positive integer λ, because

$$z_g=1-\frac{2}{L}\theta_g+O_k\left(L^{-2}\right),$$

where $L=\log(T/2\pi)$.

Step 4. From the previous steps, recalling (4.2.2), we obtain

$$\begin{aligned}
I_1 &= (-1)^{j+1}(k+1)\frac{U}{2\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad\times\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\frac{\mu!\nu!}{(\mu+\nu+2)!}\left(-\frac{1}{2}\right)^{2j-\mu-\nu}\\
&\quad+(-1)^{j+1}\frac{U}{2\pi}\left(\log\frac{T}{2\pi}\right)^{2j+1}\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\mu!\nu!\left(-\frac{1}{2}\right)^{2j-\mu-\nu}\\
&\quad\times\sum_{g=1}^{k}\frac{1}{z_g-1}\sum_{\lambda=0}^{\mu+\nu+1}\frac{1}{(\mu+\nu+1-\lambda)!}\frac{(\log\frac{T}{2\pi})^{-\lambda}}{(z_g-1)^\lambda}\\
&\quad+(-1)^{j}\frac{U}{2\pi}\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\mu!\nu!\left(-\frac{1}{2}\right)^{2j-\mu-\nu}\left(\log\frac{T}{2\pi}\right)^{2j-\mu-\nu}\\
&\quad\times\sum_{g=1}^{k}\frac{\left(\frac{T}{2\pi}\right)^{z_g-1}}{(z_g-1)^{\mu+\nu+2}}+O_{j,k}\left(U(\log T)^{2j+1}\right)\\
&= (-1)^{j+1}(k+1)\frac{U}{2\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad\times\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\frac{\mu!\nu!}{(\mu+\nu+2)!}\left(-\frac{1}{2}\right)^{2j-\mu-\nu}\\
&\quad+(-1)^{j+1}\frac{U}{2\pi}\left(\frac{1}{2}\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad\times\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\mu!\nu!\sum_{g=1}^{k}\frac{1}{\theta_g^{\mu+\nu+2}}\sum_{\lambda=0}^{\mu+\nu+1}\frac{(-2\theta_g)^\lambda}{\lambda!}\\
&\quad+(-1)^{j}\frac{U}{2\pi}\left(\frac{1}{2}\log\frac{T}{2\pi}\right)^{2j+2}\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\mu!\nu!\sum_{g=1}^{k}\frac{\left(\frac{T}{2\pi}\right)^{z_g-1}}{\theta_g^{\mu+\nu+2}}
\end{aligned}$$

$$+ O_{j,k}\left(U(\log T)^{2j+1}\right).$$

As for the first term,

$$\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\frac{\mu!\nu!}{(\mu+\nu+2)!}\left(-\frac{1}{2}\right)^{2j-\mu-\nu}$$
$$= (j!)^2\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\frac{1}{(j-\mu)!(j-\nu)!(\mu+\nu+2)!}\left(-\frac{1}{2}\right)^{2j-\mu-\nu}$$
$$= (j!)^2\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\frac{1}{\mu!\nu!(2j+2-\mu-\nu)!}\left(-\frac{1}{2}\right)^{\mu+\nu}$$
$$= \frac{(j!)^2}{(2j+2)!}\sum_{\mu=0}^{j}\binom{2j+2}{\mu}\left(-\frac{1}{2}\right)^{\mu}\sum_{\nu=0}^{j}\binom{2j+2-\mu}{\nu}\left(-\frac{1}{2}\right)^{\nu}.$$

Here we note that

$$0 = \left(1-\frac{1}{2}-\frac{1}{2}\right)^{2j+2} = \sum_{0\le\mu+\nu\le 2j+2}\frac{(2j+2)!}{\mu!\nu!(2j+2-\mu-\nu)!}\left(-\frac{1}{2}\right)^{\mu+\nu}$$
$$= \sum_{\mu=0}^{2j+2}\binom{2j+2}{\mu}\left(-\frac{1}{2}\right)^{\mu}\sum_{\nu=0}^{2j+2-\mu}\binom{2j+2-\mu}{\nu}\left(-\frac{1}{2}\right)^{\nu}.$$

Thus we have

$$\sum_{\mu=0}^{j}\binom{2j+2}{\mu}\left(-\frac{1}{2}\right)^{\mu}\sum_{\nu=0}^{j}\binom{2j+2-\mu}{\nu}\left(-\frac{1}{2}\right)^{\nu}$$
$$= \left(1-\frac{1}{2}-\frac{1}{2}\right)^{2j+2}$$
$$\quad - 2\sum_{\mu=j+1}^{2j+2}\binom{2j+2}{\mu}\left(-\frac{1}{2}\right)^{\mu}\sum_{\nu=0}^{2j+2-\mu}\binom{2j+2-\mu}{\nu}\left(-\frac{1}{2}\right)^{\nu}$$
$$\quad + \binom{2j+2}{j+1}\left(-\frac{1}{2}\right)^{2j+2}$$
$$= -2\sum_{\mu=j+1}^{2j+2}\binom{2j+2}{\mu}\left(-\frac{1}{2}\right)^{\mu}\left(\frac{1}{2}\right)^{2j+2-\mu} + \binom{2j+2}{j+1}\left(-\frac{1}{2}\right)^{2j+2}$$
$$= -\frac{1}{2^{2j+2}}\left(2\sum_{\mu=j+1}^{2j+2}\binom{2j+2}{\mu}(-1)^{\mu} - \binom{2j+2}{j+1}\right)$$

$$= \frac{1}{2^{2j+2}}\left(2\sum_{\mu=0}^{j}\binom{2j+2}{\mu}(-1)^{\mu}+\binom{2j+2}{j+1}\right).$$

The sum is the coefficient of x^j in $(1-x)^{2j+2}(1-x)^{-1}$. Thus we can see that

$$\begin{aligned}
&\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\frac{\mu!\nu!}{(\mu+\nu+2)!}\left(-\frac{1}{2}\right)^{2j-\mu-\nu}\\
&= \frac{(j!)^2}{2^{2j+2}(2j+2)!}\left(2\binom{2j+1}{j}(-1)^j+\binom{2j+2}{j+1}\right)\\
&= \frac{1+(-1)^j}{2^{2j+2}(j+1)^2}.
\end{aligned}$$

On the second term, putting $u=\mu+\nu+1-\lambda$ and dividing the sum to four parts according as the conditions $u=0$, $1\le u\le j$ with $0\le i\le u-1$, $1\le u\le j$ with $u\le i\le j$ and $j+1\le u\le 2j+1$, we have

$$\begin{aligned}
&\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\mu!\nu!\sum_{g=1}^{k}\frac{1}{\theta_g^{\mu+\nu+2}}\sum_{\lambda=0}^{\mu+\nu+1}\frac{(-2\theta_g)^{\lambda}}{\lambda!}\\
&=\sum_{g=1}^{k}\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\sum_{u=0}^{\mu+\nu+1}\binom{j}{\mu}\binom{j}{\nu}\mu!\nu!\frac{1}{\theta_g^{\mu+\nu+2}}\frac{(-2\theta_g)^{\mu+\nu+1-u}}{(\mu+\nu+1-u)!}\\
&=\sum_{g=1}^{k}\frac{1}{\theta_g}\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\mu!\nu!\frac{(-2)^{\mu+\nu+1}}{(\mu+\nu+1)!}\\
&\quad+\sum_{g=1}^{k}\sum_{u=1}^{j}\frac{1}{\theta_g^{u+1}}\sum_{\mu=0}^{u-1}\sum_{\nu=u-1-i}^{j}\binom{j}{i}\binom{j}{h}\mu!\nu!\frac{(-2)^{\mu+\nu+1-u}}{(\mu+\nu+1-u)!}\\
&\quad+\sum_{g=1}^{k}\sum_{u=1}^{j}\frac{1}{\theta_g^{u+1}}\sum_{\mu=u}^{j}\sum_{\nu=0}^{j}\binom{j}{\mu}\binom{j}{\nu}\mu!\nu!\frac{(-2)^{\mu+\nu+1-u}}{(\mu+\nu+1-u)!}\\
&\quad+\sum_{g=1}^{k}\sum_{u=j+1}^{2j+1}\frac{1}{\theta_g^{u+1}}\sum_{\mu=u-1-j}^{j}\sum_{\nu=u-1-\mu}^{j}\binom{j}{\mu}\binom{j}{\mu}\mu!\nu!\frac{(-2)^{\mu+\nu+1-u}}{(\mu+\nu+1-u)!}\\
&=S_1+S_2+S_3+S_4,
\end{aligned}$$

say.

To calculate these sums, we prepare a lemma on combinatorics.

LEMMA 4.2.1. *For non-negative integers j and u ($j \geq u$),*

$$\sum_{\mu=0}^{j-u} \binom{2j+1-u}{\mu} \sum_{\nu=0}^{j} \binom{2j+1-u-\mu}{\nu} (-2)^{2j+1-u-\mu-\nu}$$
$$= (-1)^{j+1} \binom{2j-u}{j} \{1 + (-1)^{-u}\}.$$

PROOF. Since

$$(1+1-2)^{2j+1-u}$$
$$= \sum_{\mu=0}^{2j+1-u} \binom{2j+1-u}{\mu} \sum_{\nu=0}^{2j+1-u-\mu} \binom{2j+1-u-\mu}{\nu} (-2)^{2j+1-u-\mu-\nu},$$

we have

$$\sum_{\mu=0}^{j-u} \binom{2j+1-u}{\mu} \sum_{\nu=0}^{j} \binom{2j+1-u-\mu}{\nu} (-2)^{2j+1-u-\mu-\nu}$$
$$= (1+1-2)^{2j+1-u}$$
$$- \sum_{\mu=j-u+1}^{2j+1-u} \binom{2j+1-u}{\mu} \sum_{\nu=0}^{2j+1-u-\mu} \binom{2j+1-u-\mu}{\nu} (-2)^{2j+1-u-\mu-\nu}$$
$$- \sum_{\nu=j+1}^{2j+1-u} \binom{2j+1-u}{\nu} \sum_{\mu=0}^{2j+1-u-\nu} \binom{2j+1-u-\nu}{\mu} (-2)^{2j+1-u-\nu-\mu}$$
$$= - \sum_{\mu=j-u+1}^{2j+1-u} \binom{2j+1-u}{\mu} (-1)^{2j+1-u-\mu}$$
$$- \sum_{\nu=j+1}^{2j+1-u} \binom{2j+1-u}{\nu} (-1)^{2j+1-u-\nu}$$
$$= - \sum_{\mu=0}^{j} \binom{2j+1-u}{\mu} (-1)^{\mu} - \sum_{\nu=0}^{j-u} \binom{2j+1-u}{\nu} (-1)^{\nu}.$$

These sums are coefficients of x^j and x^{j-u} in $(1-x)^{2j+1-u}(1-x)^{-1}$ and are therefore equal to the coefficient of x^j and x^{j-u} in $(1-x)^{2j-u}$.

Thus we obtain

$$\begin{aligned}
&\sum_{\mu=0}^{j-u}\binom{2j+1-u}{\mu}\sum_{\nu=0}^{j}\binom{2j+1-u-\mu}{\nu}(-2)^{2j+1-u-\mu-\nu}\\
&=(-1)^{j+1}\binom{2j-u}{j}+(-1)^{j+1-u}\binom{2j-u}{j-u}\\
&=(-1)^{j+1}\binom{2j-u}{j}\{1+(-1)^{-u}\}.
\end{aligned}$$

$\square$

By Lemma 4.2.1 with $u=0$, when $k\neq 0$,

$$\begin{aligned}
S_1&=(j!)^2\sum_{g=1}^{k}\frac{1}{\theta_g}\sum_{\mu=0}^{j}\sum_{\nu=0}^{j}\frac{(-2)^{2j+1-i-h}}{\mu!\nu!(2j+1-\mu-\nu)!}\\
&=\frac{(j!)^2}{(2j+1)!}\sum_{g=1}^{k}\frac{1}{\theta_g}\sum_{\mu=0}^{j}\binom{2j+1}{\mu}\sum_{\nu=0}^{j}\binom{2j+1-\mu}{\nu}(-2)^{2j+1-\mu-\nu}\\
&=(-1)^{j+1}2\frac{(j!)^2}{(2j+1)!}\binom{2j}{j}\sum_{g=1}^{k}\frac{1}{\theta_g}\\
&=(-1)^{j+1}\frac{2}{2j+1}\sum_{g=1}^{k}\frac{1}{\theta_g}=(-1)^{j}\frac{2}{2j+1}.
\end{aligned}$$

At the last equality, we use the fact that

$$\sum_{g=1}^{k}\frac{1}{\theta_g}=-1.$$

This can be obtained by the Newton-Girard formulas. We note that if $k=0$, then $S_1=0$.

On S_2, recalling the proof of Lemma 4.2.1, we see that

$$\begin{aligned}
&S_2\\
&=\sum_{u=1}^{j}(j!)^2\sum_{g=1}^{k}\frac{1}{\theta_g^{u+1}}\sum_{\mu=0}^{u-1}\frac{1}{(j-\mu)!}\sum_{\nu=0}^{j+\mu+1-u}\frac{(-2)^{\nu}}{\nu!(j+\mu+1-u-\nu)!}\\
&=\sum_{u=1}^{j}\frac{(j!)^2}{(2j+1-u)!}\sum_{g=1}^{k}\frac{1}{\theta_g^{u+1}}\\
&\quad\times\sum_{\mu=0}^{u-1}\binom{2j+1-u}{\mu+j+1-u}\sum_{\nu=0}^{\mu+j+1-u}\binom{\mu+j+1-u}{\nu}(-2)^{\nu}
\end{aligned}$$

$$= \sum_{u=1}^{j} \frac{(j!)^2}{(2j+1-u)!} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \sum_{\mu=0}^{u-1} \binom{2j+1-u}{\mu+j+1-u} (-1)^{\mu+j+1-u}$$

$$= \sum_{u=1}^{j} \frac{(j!)^2}{(2j+1-u)!} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \sum_{\mu=j+1-u}^{j} \binom{2j+1-u}{\mu} (-1)^{\mu}$$

$$= \sum_{u=1}^{j} \frac{(j!)^2}{(2j+1-u)!} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \left\{ (-1)^j \binom{2j-u}{j} - (-1)^{j-u} \binom{2j-u}{j-u} \right\}$$

$$= (-1)^j \sum_{u=1}^{j} \frac{j!}{2j+1-u} \frac{1}{(j-u)!} \{1-(-1)^{-u}\} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}}.$$

By Lemma 4.2.1,

$$\begin{aligned} S_3 &= \sum_{u=1}^{j} (j!)^2 \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \\ &\quad \times \sum_{\mu=u}^{j} \sum_{\nu=0}^{j} \frac{1}{(j-\mu)!(j+1-u+\mu-\nu)!} \frac{(-2)^{j+1-u+\mu-\nu}}{\nu!} \\ &= \sum_{u=1}^{j} (j!)^2 \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \sum_{\mu=0}^{j-u} \frac{1}{\mu!} \sum_{\nu=0}^{j} \frac{(-2)^{2j+1-u-\mu-\nu}}{\nu!(2j+1-u-\mu-\nu)!} \\ &= \sum_{u=1}^{j} \frac{(j!)^2}{(2j+1-u)!} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \\ &\quad \times \sum_{\mu=0}^{j-u} \binom{2j+1-u}{\mu} \sum_{\nu=0}^{j} \binom{2j+1-u-\mu}{\nu} (-2)^{2j+1-u-\mu-\nu} \\ &= (-1)^{j+1} \sum_{u=1}^{j} \frac{1}{2j+1-u} \frac{j!}{(j-u)!} \{1+(-1)^{-u}\} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}}. \end{aligned}$$

Since

$$\sum_{\mu=0}^{2j+1-u} \binom{2j+1-u}{\mu} (-1)^{\mu} = \begin{cases} 1 & u = 2j+1, \\ 0 & otherwise, \end{cases}$$

S_4

$$= \sum_{u=j+1}^{2j+1} (j!)^2 \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \sum_{\mu=u-1-j}^{j} \frac{1}{(j-\mu)!} \sum_{\nu=0}^{j+\mu+1-u} \frac{(-2)^{\nu}}{\nu!(j+\mu+1-u-\nu)!}$$

$$
\begin{aligned}
&= \sum_{u=j+1}^{2j+1} (j!)^2 \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \\
&\quad \times \sum_{\mu=u-1-j}^{j} \frac{1}{(j-\mu)!(j+\mu+1-u)!} \sum_{\nu=0}^{j+\mu+1-u} \binom{j+\mu+1-u}{\nu} (-2)^{\nu} \\
&= \sum_{u=j+1}^{2j+1} \frac{(j!)^2}{(2j+1-u)!} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \sum_{\mu=0}^{2j+1-u} \binom{2j+1-u}{\mu} (-1)^{\mu} = \sum_{g=1}^{k} \frac{(j!)^2}{\theta_g^{2j+2}}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&S_1 + S_2 + S_3 + S_4 \\
&= (-1)^j \frac{2}{2j+1} \\
&\quad + (-1)^{j+1} 2 \sum_{u=1}^{j} \frac{1}{2j+1-u} \frac{j!}{(j-u)!} (-1)^{-u} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} + \sum_{g=1}^{k} \frac{(j!)^2}{\theta_g^{2j+2}}.
\end{aligned}
$$

On the third term, we see that

$$
\sum_{\mu=0}^{j} \sum_{\nu=0}^{j} \binom{j}{\mu} \binom{j}{\nu} \mu! \nu! \sum_{g=1}^{k} \frac{\left(\frac{T}{2\pi}\right)^{z_g - 1}}{\theta_g^{\mu+\nu+2}} = (j!)^2 \sum_{g=1}^{k} \frac{\left(\frac{T}{2\pi}\right)^{z_g - 1}}{\theta_g^{2j+2}} \left(\sum_{\mu=0}^{j} \frac{\theta_g^{\mu}}{\mu!} \right)^2 .
$$

Therefore, when $k \neq 0$,

$$
\begin{aligned}
I_1 &= -\frac{(k+1)\{1+(-1)^j\}}{2^{2j+2}(j+1)^2} \frac{U}{2\pi} \left(\log \frac{T}{2\pi} \right)^{2j+2} \\
&\quad - \frac{U}{2^{2j+2}(2j+1)\pi} \left(\log \frac{T}{2\pi} \right)^{2j+2} \\
&\quad + \sum_{u=1}^{j} \frac{1}{2j+1-u} \frac{j!}{(j-u)!} (-1)^{-u} \sum_{g=1}^{k} \frac{1}{\theta_g^{u+1}} \frac{U}{2^{2j+2}\pi} \left(\log \frac{T}{2\pi} \right)^{2j+2} \\
&\quad - (-1)^j \sum_{g=1}^{k} \frac{(j!)^2}{\theta_g^{2j+2}} \frac{U}{2^{2j+3}\pi} \left(\log \frac{T}{2\pi} \right)^{2j+2} \\
&\quad + (-1)^j (j!)^2 \sum_{g=1}^{k} \frac{\left(\frac{T}{2\pi}\right)^{z_g - 1}}{\theta_g^{2j+2}} \left(\sum_{\mu=0}^{j} \frac{\theta_g^{\mu}}{\mu!} \right)^2 \frac{U}{2^{2j+3}\pi} \left(\log \frac{T}{2\pi} \right)^{2j+2} \\
&\quad + O_{j,k}\left(U (\log T)^{2j+1} \right).
\end{aligned}
$$

If $k = 0$, then these main terms vanish except for the first.

Step 5. Finally, we obtain

$$
\begin{aligned}
&M(T+U)-M(T)\\
&=\delta_{0,k}\frac{U}{2^{2j+1}(2j+1)\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad-\frac{(k+1)\{1+(-1)^j\}}{2^{2j+1}(j+1)^2}\frac{U}{2\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad+\sum_{u=1}^{j}\frac{1}{2j+1-u}\frac{j!}{(j-u)!}(-1)^{-u}\sum_{g=1}^{k}\frac{1}{\theta_g^{u+1}}\frac{U}{2^{2j+1}\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad+(-1)^{j+1}\sum_{g=1}^{k}\frac{(j!)^2}{\theta_g^{2j+2}}\frac{U}{2^{2j+2}\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad+(-1)^j(j!)^2\sum_{g=1}^{k}\frac{\left(\frac{T}{2\pi}\right)^{z_g-1}}{\theta_g^{2j+2}}\left(\sum_{\mu=0}^{j}\frac{\theta_g^{\mu}}{\mu!}\right)^2\frac{U}{2^{2j+2}\pi}\left(\log\frac{T}{2\pi}\right)^{2j+2}\\
&\quad+O_{j,k}\left(U(\log T)^{2j+1}\right).
\end{aligned}
$$

This completes the proof for the special T which are chosen at the beginning of the proof.

To complete the proof, we take away the condition on T. When T increases continuously in bounded interval, the number of relevant $|Z^{(j)}(\gamma_k)|^2$ is at most $O_k(\log T)$ and the order is $O_j(T^{\varepsilon})$. Thus it is smaller than the error in our main theorem that the contribution of these terms. Thus the formula is true for all $T>T_0$.

CHAPTER 5

A discrete moment of $L'(s,\chi)$

This chapter is based on the author's paper [**21**]. Let $\rho_\chi = \beta_\chi + i\gamma_\chi$ be the non-trivial zeros of a Dirichlet L-function $L(s,\chi)$. Our purpose of this chapter is to prove the following theorem.

Theorem 5.0.1. *Let c_1 be a positive constant. Let $\chi \pmod q$ be a primitive character. Then, uniformly for $q \le \exp(c_1\sqrt{\log T})$, we have*

$$\sum_{0<\gamma_\chi\le T} L'(\rho_\chi,\chi) = \frac{1}{4\pi}T\left(\log\frac{qT}{2\pi}\right)^2 + a_1\frac{T}{2\pi}\log\frac{qT}{2\pi} + a_2\frac{T}{2\pi} + a_3$$
$$+ O\left(T\exp\left(-c\sqrt{\log T}\right)\right),$$

where the implicit constant is absolute, c is a positive absolute constant depends on c_1 and

$$a_1 = \sum_{p|q}\frac{\log p}{p-1} + \gamma_0 - 1,$$

$$a_2 = \frac{1}{2}\left(\sum_{p|q}\frac{\log p}{p-1}\right)^2 + (\gamma_0-1)\sum_{p|q}\frac{\log p}{p-1}$$
$$-\frac{3}{2}\sum_{p|q}p\left(\frac{\log p}{p-1}\right)^2 + 1 - \gamma_0 - \gamma_0^2 + 3\gamma_1$$

with the Stieltjes constants γ_0, γ_1 and

$$a_3 = \frac{\omega\chi(-1)\tau(\overline{\chi})\tau(\overline{\omega}\chi)}{q\varphi(q)}\frac{L'(\beta,\omega)}{\beta}\left(\frac{qT}{2\pi}\right)^\beta$$

when $L(s,\omega)$ with a quadratic character $\omega \pmod q$ has an exceptional zero β, otherwise $a_3 = 0$.

Assuming the GRH, we can replace the error term by $(qT)^{\frac{1}{2}+\varepsilon}$ uniformly for $q \ll T^{1-\varepsilon}$.

REMARK 5.0.1. Let q be a prime power. If we could obtain the estimate

$$\sum_{\gamma_\chi \le T} |L'(\rho_\chi, \chi)|^2 \ll T(\log qT)^4, \tag{5.0.1}$$

where the implicit constant is absolute, we could replace the error term by $\sqrt{qT}(\log qT)^{\frac{7}{2}}$ under the GRH. We will give the details in the last section. In view of Gonek's formula (5.4.1), the above estimate (5.0.1) may be plausible.

When $q = 1$, the above theorem implies Fujii's Theorem 1 in [**9**]. Our proof is a generalization of his method. However, it is not easy to obtain his Theorem 2 in [**9**] and we give a weaker statement. Kaptan, Karabulut and Yıldırım [**19**] considered more general cases and gave the asymptotic formula, that is for $\mu \ge 1$ and $q \le (\log T)^A$ with any fixed $A > 0$

$$\sum_{0 \le \gamma_\chi \le T} L^{(\mu)}(\rho_\chi, \chi) = \frac{(-1)^{\mu+1}}{\mu+1} \frac{T}{2\pi} \left(\log \frac{qT}{2\pi}\right)^{\mu+1} + O(T(\log T)^{\mu+\varepsilon})$$

for any fixed $\varepsilon > 0$. Our result is the case $\mu = 1$ in their paper and gives a more sophisticated formula. Jakhlouti and Mazhouda [**17**] considered the sum

$$\sum_{\substack{\rho_{a,\chi} \\ 0 < \gamma_{a,\chi} \le T}} L'(\rho_{a,\chi}, \chi) X^{\rho_{a,\chi}},$$

where $\rho_{a,\chi} = \beta_{a,\chi} + i\gamma_{a,\chi}$ are the zeros of $L(s, \chi) - a$ for any fixed complex number a and X is a fixed positive number. They also fixed χ throughout their paper. Hence our main theorem treats a special case of their sum, but our result gives a more precise form because we do not fix χ.

5.1. Preliminary lemmas

By Stirling's formula, we can show that

LEMMA 5.1.1. *For $-1 \le \sigma \le 2$ and $t \ge 1$, we have*

$$\Delta(1-s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} e^{-\frac{\pi i}{4}} \left(\frac{qt}{2\pi}\right)^{\sigma - \frac{1}{2}} \exp\left(it \log \frac{qt}{2\pi e}\right)\left(1 + O\left(\frac{1}{t}\right)\right) \tag{5.1.1}$$

and

$$\frac{\Delta'}{\Delta}(s, \chi) = -\log \frac{qt}{2\pi} + O\left(\frac{1}{t}\right). \tag{5.1.2}$$

A theorem from [**14**] and an application of the Phragmen-Lindelöf principle yields the estimates;

$$L(s,\chi) \ll (q(|t|+2))^{\frac{3}{16}+\varepsilon} \quad \text{for} \quad \frac{1}{2} \le \sigma \le 1 + \frac{1}{\log qT}, \tag{5.1.3}$$

and

$$L(s,\chi) \ll (q(|t|+2))^{\frac{1}{2}} \log q(|t|+2) \quad \text{for} \quad -\frac{1}{\log qT} \le \sigma < \frac{1}{2} \tag{5.1.4}$$

uniformly in $|t| \ll T$ for any non-principal character $\chi \pmod q$. When we assume the GRH, the bound of (5.1.3) can be replaced by $(q(|t|+2))^{\varepsilon}$. For the principal character, we need the restriction $|s-1| \gg 1$ in (5.1.3). For the logarithmic derivative, it is known that for $q \ge 1$ and $\chi \pmod q$

$$\frac{L'}{L}(s,\chi) = \sum_{|t-\gamma_\chi|\le 1} \frac{1}{s-\rho_\chi} + O(\log q(|t|+2)) \quad \text{for} \quad -1 \le \sigma \le 2,\ |t| \ge 1 \tag{5.1.5}$$

(see [**34**, p. 225]). For $q \ge 1$, $\chi \pmod q$ and $t \ge 0$ we have (see [**34**, p. 220])

$$\begin{aligned} N(t+1,\chi) - N(t,\chi) &:= \#\{\rho_\chi = \beta_\chi + i\gamma_\chi : t < \gamma_\chi \le t+1\} \\ &\ll \log q(t+2). \end{aligned} \tag{5.1.6}$$

Hence for any $T_0 \ge 0$, there exists a $t = t(\chi),\ t \in (T_0, T_0+1]$, such that

$$\min_{\gamma_\chi} |t-\gamma_\chi| \gg \frac{1}{\log q(t+2)}. \tag{5.1.7}$$

By the expression (5.1.5), it follows that for $q \ge 1$, $\chi \pmod q$ and t satisfying (5.1.7)

$$\frac{L'}{L}(\sigma+it,\chi) \ll (\log q(|t|+2))^2 \quad \text{for} \quad -1 \le \sigma \le 2 \tag{5.1.8}$$

uniformly. This estimate is valid for $|s-\rho_\chi| \gg (\log(q(|t|+2)))^{-1}$ though t is not satisfying (5.1.7).

We will apply the following approximate functional equation for $L(s,\chi)$.

LEMMA 5.1.2 (A. F. Lavrik [**25**]). *We let* $0 \leq \sigma \leq 1$, $2\pi xy = t$, $x \geq 1$ *and* $y \geq 1$. *Then for* $t > 0$, *we get*

$$L(s,\chi) = \sum_{n\leq x} \frac{\chi(n)}{n^s} + \Delta(s,\chi) \sum_{n\leq y} \frac{\overline{\chi}(n)}{n^{1-s}} + O\left(\sqrt{q}\left(y^{-\sigma} + x^{\sigma-1}(qt)^{\frac{1}{2}-\sigma}\right)\log 2t\right).$$

On the other hand, for $t > t_0 > 0$ and $\sigma > 1$, using partial summation, we get

$$L(s,\chi) = \sum_{n\leq qt} \frac{\chi(n)}{n^s} + O\left(\frac{q|s|}{\sigma}(qt)^{-\sigma}\right). \tag{5.1.9}$$

We will use the following modified Gonek's lemma ([**10**, Lemma 5]).

LEMMA 5.1.3. *Let* $\{b_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers such that* $b_n \ll n^{\varepsilon}$ *for any* $\varepsilon > 0$. *Let* $a > 1$ *and let* m *be a non-negative integer. Then for any sufficiently large* T,

$$\begin{aligned}&\frac{1}{2\pi}\int_1^T \left(\sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}}\right) \Delta(1-a-it,\chi) \left(\log \frac{qt}{2\pi}\right)^m dt \\ &= \frac{\tau(\chi)}{q} \sum_{1\leq n\leq qT/2\pi} b_n e\left(-\frac{n}{q}\right)(\log n)^m \\ &\quad + O\left(\left|\sum_{n=1}^{\infty}\frac{b_n}{n^a}\right|(qT)^{a-1/2}(\log qT)^m\right).\end{aligned}$$

This is provided implicitly by Steuding in [**37**].

5.2. The proof of Theorem in unconditional

In this section, we prove the claim of the unconditional part of Theorem 5.0.1. Let $(\log 2q)^{-1} \ll b \leq 1$ and $T \geq 2$ be such that

$$\min_{\gamma_\chi} |b - \gamma_\chi| \gg \frac{1}{\log 2q} \quad \text{and} \quad \min_{\gamma_\chi} |T - \gamma_\chi| \gg \frac{1}{\log qT}.$$

We prove the theorem under this situation. At the end of the proof, we remove this restriction. Let $a = 1 + (\log qT)^{-1}$ and define the contour C as the positively oriented rectangular path with vertices $a+ib, a+iT, 1-a+iT$ and $1-a+ib$. By the residue theorem, our sum can be written as a contour integral

$$\sum_{0<\gamma_\chi\leq T} L'(\rho_\chi,\chi) = \frac{1}{2\pi i}\int_C \frac{L'}{L}(s,\chi)L'(s,\chi)ds + E,$$

where E consists of the terms $L'(\rho_\chi,\chi)$ with $0<\gamma_\chi<b$.

For zeros $\rho_\chi=\beta_\chi+i\gamma_\chi$ with $0<\gamma_\chi<b$ we have

$$L'(\rho_\chi,\chi)\ll q^{\frac{1}{2}}(\log 2q)^2$$

by (5.1.3), (5.1.4) and the Cauchy's integral formula applied to the circle with centre ρ_χ and radius $(\log 2q)^{-1}$. Therefore, by (5.1.6), we have

$$E=\sum_{0<\gamma_\chi<b} L'(\rho_\chi,\chi)\ll q^{\frac{1}{2}}(\log 2q)^2\sum_{0<\gamma_\chi<b}1\ll q^{\frac{1}{2}}(\log 2q)^3.$$

Next we consider the contour integral

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C\frac{L'}{L}(s,\chi)L'(s,\chi)ds\\
&=\frac{1}{2\pi i}\left\{\int_{a+ib}^{a+iT}+\int_{1-a+iT}^{1-a+ib}+\int_{a+iT}^{1-a+iT}+\int_{1-a+ib}^{a+ib}\right\}\frac{L'}{L}(s,\chi)L'(s,\chi)ds\\
&=I_1+I_2+I_3+I_4,
\end{aligned}$$

say.

By the Laurent expansion of the Riemann ζ-function, it is easily seen that

$$\begin{aligned}
I_1&=\frac{1}{2\pi}\int_b^T\frac{L'}{L}(a+it,\chi)L'(a+it,\chi)dt\\
&=\frac{1}{2\pi}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{\chi(m)\Lambda(m)\chi(n)\log n}{(mn)^a}\int_b^T\frac{dt}{(mn)^{it}}\\
&\ll\left|\frac{\zeta'}{\zeta}(a)\right||\zeta'(a)|\ll(\log qT)^3,
\end{aligned}$$

where $\Lambda(m)$ is the von-Mangoldt function. To estimate the integral on the horizontal line, we will show the following lemma.

LEMMA 5.2.1. *Let χ be a primitive character, then*

$$\int_{1-a}^{a}L'(\sigma+iT,\chi)d\sigma\ll\sqrt{qT}\log qT.$$

PROOF. Let

$$\delta=\frac{1}{\log qT}.$$

Then $L(w,\chi)$ is analytic on the disk $|s-w|\le\delta$, for $s=\sigma+iT$ with $1-a\le\sigma\le a$. Therefore, by Cauchy's integral formula,

$$\begin{aligned}L'(s,\chi)&=\frac{1}{2\pi i}\int_{|s-w|=\delta}\frac{L(w,\chi)}{(s-w)^2}dw\\&\ll\log qT\int_0^{2\pi}|L(s+\delta e^{i\theta},\chi)|d\theta.\end{aligned}$$

Thus it suffices to prove that

$$\int_{1-a}^{a}\int_0^{2\pi}|L(s+\delta e^{i\theta},\chi)|d\theta d\sigma=\int_0^{2\pi}\int_{1-a}^{a}|L(s+\delta e^{i\theta},\chi)|d\sigma d\theta\ll\sqrt{qT}.$$

From the functional equation and, for $1-a\le\sigma\le 1/2$, we have

$$\begin{aligned}&\int_{1-a}^{\frac{1}{2}}|L(s+\delta e^{i\theta},\chi)|d\sigma\\&=\int_{1-a}^{\frac{1}{2}}|\Delta(s+\delta e^{i\theta},\chi)L(1-s-\delta e^{i\theta},\overline{\chi})|d\sigma\\&=\int_{\frac{1}{2}}^{a}\left|\Delta(1-\sigma+iT+\delta e^{i\theta},\chi)L(\sigma-iT-\delta e^{i\theta},\overline{\chi})\right|d\sigma.\end{aligned}$$

On the second equality, we change the variable σ to $1-\sigma$. Since

$$\begin{aligned}&\Delta(1-(\sigma-iT-\delta e^{i\theta}),\chi)\\&=\overline{\overline{\Delta(1-(\sigma+iT-\delta e^{-i\theta}),\overline{\chi})}}\\&=\frac{\tau(\overline{\chi})}{\sqrt{q}}e^{\frac{\pi i}{4}}\left(\frac{qT}{2\pi}\right)^{\sigma-\delta\cos\theta-\frac{1}{2}}\exp\left(iT\log\frac{qT}{2\pi e}\right)\left(1+O\left(\frac{1}{T}\right)\right)\end{aligned}$$

by Lemma 5.1.1, the integral can be bounded by

$$\int_{\frac{1}{2}}^{a}(qT)^{\sigma-\delta\cos\theta-\frac{1}{2}}\left|L(\sigma+iT-\delta e^{-i\theta},\chi)\right|d\sigma.$$

Therefore we obtain

$$
\begin{aligned}
&\int_{1-a}^{a} |L(s+\delta e^{i\theta},\chi)| d\sigma \\
&\ll \int_{\frac{1}{2}}^{a} |L(\sigma+iT+\delta e^{i\theta},\chi)| d\sigma \\
&\quad + \int_{\frac{1}{2}}^{a} (qT)^{\sigma-\delta\cos\theta-\frac{1}{2}} \left|L(\sigma+iT-\delta e^{-i\theta},\chi)\right| d\sigma \\
&\ll \int_{\frac{1}{2}}^{a} (qT)^{\sigma-\frac{1}{2}} \left|L(\sigma+iT\pm\delta e^{\pm i\theta},\chi)\right| d\sigma.
\end{aligned}
$$

On the last inequality, we use the facts that

$$(qT)^{\delta} = e$$

with $\delta = (\log qT)^{-1}$. This integral is

$$
\begin{aligned}
&= \left\{ \int_{\frac{1}{2}}^{1} + \int_{1}^{a} \right\} (qT)^{\sigma-\frac{1}{2}} \left|L(\sigma+iT\pm\delta e^{\pm i\theta},\chi)\right| d\sigma \\
&= S_1 + S_2,
\end{aligned}
$$

say. Using Lemma 5.1.2, we have

$$
\begin{aligned}
S_1 &\ll (qT)^{-\frac{1}{2}} \sum_{n\ll\sqrt{qT}} n^{\delta} \int_{\frac{1}{2}}^{1} \left(\frac{qT}{n}\right)^{\sigma} d\sigma + \sum_{n\ll\sqrt{qT}} n^{\delta-1} \int_{\frac{1}{2}}^{1} n^{\sigma} d\sigma \\
&\quad + \sqrt{q}\log 2T \int_{\frac{1}{2}}^{1} (qT)^{\frac{\sigma+\delta-1}{2}} d\sigma \ll \sqrt{qT}.
\end{aligned}
$$

On the other hand, by (5.1.9), we get

$$
\begin{aligned}
S_2 &\ll (qT)^{-\frac{1}{2}} \sum_{n\le\frac{qT}{2}} n^{\delta} \int_{1}^{a} \left(\frac{qT}{n}\right)^{\sigma} d\sigma + \sqrt{qT}\int_{1}^{a} \frac{d\sigma}{\sigma} \\
&\ll \sqrt{qT}.
\end{aligned}
$$

Hence we complete the proof. □

By (5.1.8) and the above lemma, we get

$$
\begin{aligned}
I_3 + I_4 &\ll (\log qT)^2 \int_{1-a}^{a} |L'(\sigma+iT,\chi)| d\sigma \\
&\ll \sqrt{qT}(\log qT)^3.
\end{aligned}
$$

Now we consider I_2. By the functional equation, we have

$$
\begin{aligned}
&\frac{L'}{L}(1-a+it,\chi)L'(1-a+it,\chi)\\
&=\left(\frac{\Delta'}{\Delta}(1-a+it,\chi)-\frac{L'}{L}(a-it,\overline{\chi})\right)\\
&\quad\times\left(\Delta'(1-a+it,\chi)L(a-it,\overline{\chi})-\Delta(1-a+it,\chi)L'(a-it,\overline{\chi})\right)\\
&=\frac{\Delta'}{\Delta}(1-a+it,\chi)\Delta'(1-a+it,\chi)L(a-it,\overline{\chi})\\
&\quad-2\Delta'(1-a+it,\chi)L'(a-it,\overline{\chi})\\
&\quad+\Delta(1-a+it,\chi)\frac{L'}{L}(a-it,\overline{\chi})L'(a-it,\overline{\chi}).
\end{aligned}
$$

Thus we can divide I_2 into the following three integrals:

$$
\begin{aligned}
I_2&=\frac{1}{2\pi}\int_T^b\frac{L'}{L}(1-a+it,\chi)L'(1-a+it,\chi)dt\\
&=\frac{1}{\pi}\int_b^T\Delta'(1-a+it,\chi)L'(a-it,\overline{\chi})dt\\
&\quad-\frac{1}{2\pi}\int_b^T\frac{\Delta'}{\Delta}(1-a+it,\chi)\Delta'(1-a+it,\chi)L(a-it,\overline{\chi})dt\\
&\quad-\frac{1}{2\pi}\int_b^T\Delta(1-a+it,\chi)\frac{L'}{L}(a-it,\overline{\chi})L'(a-it,\overline{\chi})dt\\
&=J_1+J_2+J_3,
\end{aligned}
$$

say. We take complex conjugates of J_i $(i=1,2,3)$ to apply Lemma 5.1.3. Then we have

$$
\begin{aligned}
\overline{J_1}&=\overline{\frac{1}{\pi}\int_b^T\Delta'(1-a+it,\chi)L'(a-it,\overline{\chi})dt}\\
&=\frac{1}{\pi}\int_b^T\Delta'(1-a-it,\overline{\chi})L'(a+it,\chi)dt\\
&=-\frac{1}{\pi}\int_b^T L'(a+it,\chi)\Delta(1-a-it,\overline{\chi})\log\frac{qt}{2\pi}dt\\
&\quad+O\left(\sum_{n=1}^{\infty}\frac{\log n}{n^a}\int_b^T\frac{(qt)^{a-\frac{1}{2}}}{t}dt\right)\\
&=\frac{1}{\pi}\int_b^T\sum_{n=1}^{\infty}\frac{\chi(n)\log n}{n^{a+it}}\Delta(1-a-it,\overline{\chi})\log\frac{qt}{2\pi}dt
\end{aligned}
$$

$$+ O\left((qT)^{a-\frac{1}{2}}(\log qT)^2\right)$$
$$= 2\frac{\tau(\overline{\chi})}{q} \sum_{1\le n\le qT/2\pi} \chi(n)e\left(-\frac{n}{q}\right)(\log n)^2 + O\left((qT)^{a-\frac{1}{2}}(\log qT)^3\right).$$

On the third equality, we use the approximation (5.1.2). For convenience, we put $x = qT/2\pi$. By partial summation, the above sum can be calculated as

$$\begin{aligned}
&\sum_{1\le n\le x} \chi(n)e\left(-\frac{n}{q}\right)(\log n)^2 \\
&= (\log x)^2 \sum_{m=1}^{q} \chi(m)e\left(-\frac{m}{q}\right) \sum_{\substack{n\le x\\ n\equiv m \bmod q}} 1 \\
&\quad - 2\int_1^x \left(\sum_{m=1}^{q} \chi(m)e\left(-\frac{m}{q}\right) \sum_{\substack{n\le y\\ n\equiv m \bmod q}} 1\right)\frac{\log y}{y}dy \\
&= \left(\frac{x}{q}\chi(-1)\tau(\chi) + O(\sqrt{q})\right)(\log x)^2 \\
&\quad - 2\int_1^x \left(\frac{y}{q}\chi(-1)\tau(\chi) + O(\sqrt{q})\right)\frac{\log y}{y}dy \\
&= \frac{\chi(-1)\tau(\chi)}{q}\left(x(\log x)^2 - 2\int_1^x \log y dy\right) \\
&\quad + O\left(\sqrt{q}(\log x)^2 + \sqrt{q}\int_1^x \frac{\log y}{y}dy\right) \\
&= \frac{\chi(-1)\tau(\chi)}{q}\left(x(\log x)^2 - 2x\log x + 2x\right) + O\left(\sqrt{q}(\log x)^2\right),
\end{aligned}$$

and we can see that

$$\frac{\chi(-1)\tau(\chi)\tau(\overline{\chi})}{q^2} = \frac{\overline{\tau(\chi)}\tau(\chi)}{q^2} = \frac{q}{q^2} = \frac{1}{q}.$$

Therefore we obtain

$$J_1 = 2\left(\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}\right)^2 - \frac{T}{\pi}\log\frac{qT}{2\pi} + \frac{T}{\pi}\right) + O\left((qT)^{a-\frac{1}{2}}(\log qT)^3\right).$$

Next we consider J_2. We have, by (5.1.2) again,

$$
\begin{aligned}
\overline{J_2} &= \overline{-\frac{1}{2\pi}\int_b^T \frac{\Delta'}{\Delta}(1-a+it,\chi)\Delta'(1-a+it,\chi)L(a-it,\overline{\chi})dt} \\
&= -\frac{1}{2\pi}\int_b^T L(a+it,\chi)\frac{\Delta'}{\Delta}(1-a-it,\overline{\chi})\Delta'(1-a-it,\overline{\chi})dt \\
&= -\frac{1}{2\pi}\int_b^T \sum_{n=1}^{\infty}\frac{\chi(n)}{n^{a+it}}\Delta(1-a-it,\overline{\chi})\left(\log\frac{qt}{2\pi}\right)^2 dt \\
&\quad + O\left(\sum_{n=1}^{\infty}\frac{1}{n^a}\int_b^T (qt)^{a-\frac{1}{2}}\frac{\log qt}{t}dt\right) \\
&= -\frac{\tau(\overline{\chi})}{q}\sum_{1\le n\le qT/2\pi}\chi(n)e\left(-\frac{n}{q}\right)(\log n)^2 + O\left((qT)^{a-\frac{1}{2}}(\log qT)^3\right).
\end{aligned}
$$

This sum is the same as the previous one. Hence we get

$$
J_2 = -\left(\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}\right)^2 - \frac{T}{\pi}\log\frac{qT}{2\pi} + \frac{T}{\pi}\right) + O\left((qT)^{a-\frac{1}{2}}(\log qT)^3\right).
$$

Finally, we calculate J_3. We have

$$
\begin{aligned}
\overline{J_3} &= \overline{-\frac{1}{2\pi}\int_b^T \Delta(1-a+it,\chi)\frac{L'}{L}(a-it,\overline{\chi})L'(a-it,\overline{\chi})dt} \\
&= -\frac{1}{2\pi}\int_b^T \frac{L'}{L}(a+it,\chi)L'(a+it,\chi)\Delta(1-a-it,\overline{\chi})dt \\
&= -\frac{1}{2\pi}\int_b^T \left(\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^{a+it}}\right)\left(\sum_{n=1}^{\infty}\frac{\chi(n)\log n}{n^{a+it}}\right)\Delta(1-a-it,\overline{\chi})dt \\
&= -\frac{\tau(\overline{\chi})}{q}\sum_{1\le mn\le qT/2\pi}\chi(mn)e\left(-\frac{mn}{q}\right)\Lambda(m)\log n \\
&\quad + O\left((qT)^{a-\frac{1}{2}}(\log qT)^3\right).
\end{aligned}
$$

By the orthogonality of Dirichlet characters, we see that

$$\begin{aligned}
&\sum_{mn\le x}\chi(m)\chi(n)e\left(-\frac{mn}{q}\right)\Lambda(m)\log n\\
&=\sum_{a=1}^{q}\chi(a)\sum_{b=1}^{q}\chi(b)e\left(-\frac{ab}{q}\right)\sum_{\substack{mn\le x\\ m\equiv a \bmod q\\ n\equiv b \bmod q}}\Lambda(m)\log n\\
&=\frac{1}{\varphi(q)^2}\sum_{\substack{\psi \bmod q\\ \psi' \bmod q}}\sum_{a=1}^{q}\overline{\psi}(a)\chi(a)\sum_{b=1}^{q}\overline{\psi'}(b)\chi(b)e\left(-\frac{ab}{q}\right)\\
&\quad\times\sum_{mn\le x}\psi(m)\psi'(n)\Lambda(m)\log n.
\end{aligned}$$

We will divide the sum into four parts, according to the following conditions:

(i) $\psi=\psi_0,\ \psi'=\psi_0'$,
(ii) $\psi=\psi_0,\ \psi'\neq\psi_0'$,
(iii) $\psi\neq\psi_0,\ \psi'=\psi_0'$,
(iv) $\psi\neq\psi_0,\ \psi'\neq\psi_0'$,

where $\psi_0=\psi_0'$ is the principal character modulo q. Before discussing further, we will remind some facts on the sum of Dirichlet characters (see [**2**, Sec. 8]). We define $G(n,\chi)$ as

$$G(n,\chi):=\sum_{a=1}^{q}\chi(a)e\left(\frac{an}{q}\right).$$

If a Dirichlet character $\chi \pmod q$ is primitive, then we have

$$G(a,\chi)=\overline{\chi}(a)\tau(\chi).$$

Now we consider the above four parts.
(i) In this case, we have

$$\begin{aligned}
&\frac{1}{\varphi(q)^2}\sum_{a=1}^{q}\chi(a)\sum_{b=1}^{q}\chi(b)e\left(-\frac{ab}{q}\right)\sum_{mn\le x}\psi_0(m)\psi_0(n)\Lambda(m)\log n\\
&=\frac{1}{\varphi(q)^2}\sum_{a=1}^{q}\chi(a)G(-a,\chi)\sum_{mn\le x}\psi_0(m)\psi_0(n)\Lambda(m)\log n\\
&=\frac{\chi(-1)\tau(\chi)}{\varphi(q)}\sum_{mn\le x}\psi_0(m)\psi_0(n)\Lambda(m)\log n.
\end{aligned}$$

By Perron's formula we get

$$\begin{aligned}
&\sum_{mn\le x}\psi_0(m)\Lambda(m)\psi_0(n)\log n\\
&=\frac{1}{2\pi i}\int_{a-iU}^{a+iU}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds+R,
\end{aligned}$$

where R is the error term appearing in Perron's formula (see [**30**, p.140]) and satisfies that

$$\begin{aligned}
R \ll &\sum_{\substack{\frac{x}{2}<mn<2x\\ mn\ne x}}|\Lambda(m)\log n|\min\left(1,\frac{x}{U|x-mn|}\right)\\
&+\frac{(4x)^a}{U}\sum_{mn=1}^{\infty}\frac{|\Lambda(m)\log n|}{(mn)^a}.
\end{aligned}$$

We will choose an appropriate U later. The first term of the error term R can be estimated as follows;

$$\begin{aligned}
&\frac{x}{U}\sum_{\frac{x}{2}<mn<x-1}\frac{\Lambda(m)\log n}{x-mn}+\sum_{x-1\le mn\le x+1}\Lambda(m)\log n\\
&+\frac{x}{U}\sum_{x+1<mn<2x}\frac{\Lambda(m)\log n}{mn-x}\\
\ll &\frac{x}{U}\log x\sum_{m<x-1}\frac{\Lambda(m)}{m}\sum_{\frac{x}{2m}<n<\frac{x-1}{m}}\frac{1}{\frac{x}{m}-n}\\
&+(\log x)^2\sum_{x-1\le l\le x+1}\sum_{l=mn}1\\
&+\frac{x}{U}\log x\sum_{m<2x}\frac{\Lambda(m)}{m}\sum_{\frac{x+1}{m}<n<\frac{2x}{m}}\frac{1}{n-\frac{x}{m}}\\
\ll &\frac{x}{U}(\log x)^2\sum_{m<2x}\frac{\Lambda(m)}{m}+(\log x)^2\sum_{x-1\le l\le x+1}d(l)\\
\ll &\frac{x}{U}(\log x)^3+x^{\varepsilon},
\end{aligned}$$

where $d(l)$ is the divisor function. On the last estimates, we use

$$\sum_{m\le x}\frac{\Lambda(m)}{m}=\log x+O(1)$$

and

$$d(x)\ll x^{\varepsilon}.$$

The second is

$$\ll \frac{(4x)^a}{U}\sum_{mn=1}^{\infty}\frac{|\Lambda(m)\log n|}{(mn)^a} \ll \frac{x^a}{U}(\log qT)^3.$$

Therefore

$$R \ll \frac{x}{U}(\log x)^3 + x^{\varepsilon}.$$

Since $L(s,\psi_0) = \zeta(s)\prod_{p|q}(1-p^{-s})$, there is an absolute constant $C>0$ such that

$$L(s,\psi_0) \neq 0 \quad \text{for} \quad \sigma \geq 1 - \frac{C}{\log(|t|+2)}$$

(see [**30**, p.172]). With regard to this zero-free region for $L(s,\psi_0)$, let $a' = 1 - C/\log U$ and $U = \exp\left(4c_1\sqrt{\log qT}\right)$. By the residue theorem, the integral is

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{a-iU}^{a+iU}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds \\
&= \operatorname*{Res}_{s=1}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s} \\
&\quad + \frac{1}{2\pi i}\left\{\int_{a+iU}^{a'+iU} + \int_{a'+iU}^{a'-iU} + \int_{a'-iU}^{a-iU}\right\}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds.
\end{aligned}$$

By an argument similar to the proof of Lemma 5.2.1, we can see that the integral on the horizontal line can be estimated as

$$\begin{aligned}
\int_{a\pm iU}^{a'\pm iU}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds &\ll \frac{(\log qU)^3}{U}x^a(qU)^{\frac{3}{16}+\varepsilon}(a-a') \\
&\ll xU^{-\frac{1}{2}} = x\exp\left(-2c_1\sqrt{\log x}\right),
\end{aligned}$$

noting the condition $q \leq \exp\left(c_1\sqrt{\log T}\right) \leq \exp\left(4c_1\sqrt{\log qT}\right) = U$ and (5.1.8). Since $L'/L(s,\psi_0) \ll |s-1|^{-1}$ and $L'(s,\psi_0) \ll |s-1|^{-2}$ in the neighbourhood around $s=1$, the integral on the vertical line can be bounded by

$$\begin{aligned}
&\ll x^{a'}(qU)^{\frac{3}{16}+\varepsilon}(\log qU)^3\int_{-U}^{U}\frac{dt}{1+|t|} + x^{a'}(\log qU)^3\int_{-1}^{1}\frac{dt}{|a'+it|} \\
&\ll x^{a'}(qU)^{\frac{3}{16}+\varepsilon}(\log U)^4 \\
&\ll x^{a'}U^{\frac{1}{2}} = x\exp\left(\left(2c_1 - \frac{C}{4c_1}\right)\sqrt{\log x}\right).
\end{aligned}$$

When we put $c_1 = \sqrt{C}/4$, we obtain that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{a-iU}^{a+iU}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds\\&=\operatorname*{Res}_{s=1}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}+O\left(x\exp\left(-\frac{\sqrt{C}}{2}\sqrt{\log x}\right)\right).\end{aligned}$$

Note that

$$\begin{aligned}&\operatorname*{Res}_{s=1}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}\\&=\frac{1}{2!}\lim_{s\to 1}\frac{d^2}{ds^2}(s-1)^3\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}.\end{aligned}$$

To calculate this residue, we observe that

$$\begin{aligned}L'(s,\psi_0)&=\zeta'(s)\prod_{p|q}(1-p^{-s})+\zeta(s)\left(\prod_{p|q}(1-p^{-s})\right)'\\&=\left(-\frac{1}{(s-1)^2}+\sum_{k=1}^{\infty}\gamma_k k(s-1)^{k-1}\right)\prod_{p|q}(1-p^{-s})\\&\quad+\left(\frac{1}{s-1}+\sum_{k=0}^{\infty}\gamma_k(s-1)^k\right)\left(\prod_{p|q}(1-p^{-s})\sum_{p|q}\frac{\log p}{p^s-1}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{L'}{L}(s,\psi_0)&=\frac{\zeta'}{\zeta}(s)+\sum_{p|q}\frac{\log p}{p^s-1}\\&=-\frac{1}{s-1}+\sum_{k=0}^{\infty}\eta_k(s-1)^k+\sum_{p|q}\frac{\log p}{p^s-1},\end{aligned}$$

where γ_k is the k-th Stieltjes constant and can be defined by the limit

$$\gamma_k=\lim_{n\to\infty}\left\{\left(\sum_{m=1}^{n}\frac{(\log m)^k}{m}\right)-\frac{(\log n)^{k+1}}{k+1}\right\},$$

and η_k can be represented by the sum

$$\eta_k=(-1)^k\left\{\frac{k+1}{k!}\gamma_k+\sum_{n=0}^{k-1}\frac{(-1)^{n-1}}{(k-n-1)!}\eta_n\gamma_{k-n-1}\right\}.$$

Hence we get

$$
\begin{aligned}
&\operatorname*{Res}_{s=1}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}\\
&=\frac{1}{2!}\lim_{s\to1}\frac{d^2}{ds^2}\prod_{p|q}\left(1-\frac{1}{p^s}\right)\frac{x^s}{s}\\
&\quad-\frac{2}{2!}\lim_{s\to1}\frac{d}{ds}\prod_{p|q}(1-p^{-s})\left(\sum_{p|q}\frac{\log p}{p^s-1}+\eta_0+\sum_{p|q}\frac{\log p}{p^s-1}\right)\frac{x^s}{s}\\
&\quad-\frac{2}{2!}\lim_{s\to1}\prod_{p|q}\left(1-\frac{1}{p^s}\right)\left\{\gamma_1+\gamma_0\sum_{p|q}\frac{\log p}{p^s-1}+\eta_1\right.\\
&\quad\left.-\sum_{p|q}\frac{\log p}{p^s-1}\left(\eta_0+\sum_{p|q}\frac{\log p}{p^s-1}\right)\right\}\frac{x^s}{s}\\
&=\frac{\varphi(q)}{q}x\\
&\quad\times\left\{\frac{1}{2}(\log x)^2-\left(\sum_{p|q}\frac{\log p}{p-1}+\gamma_0+1\right)\log x-\frac{1}{2}\left(\sum_{p|q}\frac{\log p}{p-1}\right)^2\right.\\
&\quad\left.+\frac{3}{2}\sum_{p|q}p\left(\frac{\log p}{p-1}\right)^2+(1-\gamma_0)\sum_{p|q}\frac{\log p}{p-1}+\gamma_0^2+\gamma_0-3\gamma_1+1\right\}.
\end{aligned}
$$

Therefore we can see that

$$
\begin{aligned}
&\frac{\chi(-1)\tau(\chi)}{\varphi(q)}\sum_{mn\le x}\psi_0(m)\psi_0(n)\Lambda(m)\log n\\
&=\frac{\tau(\overline{\chi})}{q}x\\
&\quad\times\left\{\frac{1}{2}(\log x)^2-\left(\sum_{p|q}\frac{\log p}{p-1}+\gamma_0+1\right)\log x-\frac{1}{2}\left(\sum_{p|q}\frac{\log p}{p-1}\right)^2\right.\\
&\quad\left.+\frac{3}{2}\sum_{p|q}p\left(\frac{\log p}{p-1}\right)^2+(1-\gamma_0)\sum_{p|q}\frac{\log p}{p-1}+\gamma_0^2+\gamma_0-3\gamma_1+1\right\}\\
&\quad+O\left(x\exp\left(-\frac{\sqrt{C}}{2}\sqrt{\log x}\right)\right).
\end{aligned}
$$

Here we note that $\tau(\chi)/\varphi(q) \ll 1$.

(ii) In the same way, we obtain

$$\begin{aligned}
&\frac{1}{\varphi(q)^2} \sum_{\substack{\psi' \bmod q \\ \psi' \neq \psi'_0}} \sum_{b=1}^{q} \overline{\psi'}(b)\chi(b) \sum_{a=1}^{q} \chi(a) e\left(-\frac{ab}{q}\right) \\
&\times \sum_{mn \le x} \psi_0(m)\psi'(n)\Lambda(m)\log n \\
=& \frac{\chi(-1)\tau(\chi)}{\varphi(q)^2} \sum_{\psi' \neq \psi'_0} \sum_{b=1}^{q} \overline{\psi'}(b) \sum_{mn \le x} \psi_0(m)\psi'(n)\Lambda(m)\log n.
\end{aligned}$$

The sum of $\overline{\psi'}$ is 0. Hence we see that the sum in this case vanishes.

(iii) This case is the same as the case (ii).

(iv)

$$\begin{aligned}
&\frac{1}{\varphi(q)^2} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \sum_{a=1}^{q} \overline{\psi}(a)\chi(a) \sum_{b=1}^{q} \overline{\psi'}(b)\chi(b) e\left(-\frac{ab}{q}\right) \\
&\times \sum_{mn \le x} \psi(m)\psi'(n)\Lambda(m)\log n \\
=& \frac{1}{\varphi(q)^2} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \sum_{a=1}^{q} \overline{\psi}(a)\chi(a)\psi'(-a)\overline{\chi}(-a)\tau(\overline{\psi'}\chi) \\
&\times \sum_{mn \le x} \psi(m)\psi'(n)\Lambda(m)\log n \\
=& \frac{\chi(-1)}{\varphi(q)^2} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \sum_{a=1}^{q} \overline{\psi}(a)\psi'(-a)\tau(\overline{\psi'}\chi) \sum_{mn \le x} \psi(m)\psi'(n)\Lambda(m)\log n \\
=& \frac{\chi(-1)}{\varphi(q)^2} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \psi'(-1)\tau(\overline{\psi'}\chi) \sum_{a=1}^{q} \overline{\psi}(a)\psi'(a) \sum_{mn \le x} \psi(m)\psi'(n)\Lambda(m)\log n \\
=& \frac{\chi(-1)}{\varphi(q)} \sum_{\psi \neq \psi_0} \psi(-1)\tau(\overline{\psi}\chi) \sum_{mn \le x} \psi(m)\psi(n)\Lambda(m)\log n.
\end{aligned}$$

To show the last equality, we use the fact that the sum over a does not equal to 0 if and only if $\psi = \psi'$.

In this case, we know the fact that there is an absolute constant $C' > 0$ such that

$$L(s,\chi) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{C'}{\log q(|t|+2)}$$

unless χ is a quadratic character, in which case $L(s,\chi)$ has at most one, necessarily real, zero $\beta < 1$ (see [**30**, p. 360]). By the same argument as in the case (i), when we put $c_1 = \sqrt{C'}/4$ we have

$$\begin{aligned}&\sum_{mn\le x} \psi(m)\Lambda(m)\psi(n)\log n \\ &= -L'(\beta,\psi)\frac{x^\beta}{\beta} + O\left(x\exp\left(-\frac{\sqrt{C'}}{2}\sqrt{\log x}\right)\right)\end{aligned}$$

when $L(s,\psi)$ with a quadratic character ω has an exceptional zero β. If there is no exceptional zero, then the first term vanishes. Hence when $L(s,\omega)$ has an exceptional zero β we have

$$\begin{aligned}&\frac{\chi(-1)}{\varphi(q)}\sum_{\psi\neq\psi_0}\psi(-1)\tau(\overline{\psi}\chi)\sum_{mn\le x}\psi(m)\psi(n)\Lambda(m)\log n \\ &= -\frac{\chi(-1)}{\varphi(q)}\omega(-1)\tau(\overline{\omega}\chi)L'(\beta,\omega)\frac{x^\beta}{\beta} + O\left(\sqrt{q}x\exp\left(-\frac{\sqrt{C'}}{2}\sqrt{\log x}\right)\right),\end{aligned}$$

otherwise the main term does not appear.

From the above, when we put $c_1 = \min\{\sqrt{C}/4, \sqrt{C'}/4\}$ and $c = c_1/2$, we have

$$\begin{aligned}&J_3 \\ &= -\frac{T}{2\pi}\left\{\frac{1}{2}\left(\log\frac{qT}{2\pi}\right)^2 - \left(\sum_{p|q}\frac{\log p}{p-1} + \gamma_0 + 1\right)\log\frac{qT}{2\pi}\right. \\ &\quad -\frac{1}{2}\left(\sum_{p|q}\frac{\log p}{p-1}\right)^2 + \frac{3}{2}\sum_{p|q}p\left(\frac{\log p}{p-1}\right) \\ &\quad \left.+(1-\gamma_0)\sum_{p|q}\frac{\log p}{p-1} + \gamma_0^2 + \gamma_0 + \gamma_1 + 1\right\} \\ &\quad + \frac{\omega\chi(-1)\tau(\overline{\chi})\tau(\overline{\omega}\chi)}{q\varphi(q)}\frac{L'(\beta,\omega)}{\beta}\left(\frac{qT}{2\pi}\right)^\beta + O\left(T\exp\left(-c\sqrt{\log T}\right)\right).\end{aligned}$$

We note that $\tau(\chi)\sqrt{q}/q \ll 1$.

To complete the proof, we take away the condition on T. When T increases continuously in $|T-\gamma_\chi| \ll (\log qT)^{-1}$, the number of relevant $L'(\rho_\chi,\chi)$ is at most $O(\log qT)$ and the order of each term is $O((qT)^{\frac{3}{16}+\varepsilon})$. Thus the contribution of these terms is smaller than the error in our main theorem. Therefore the proof in the unconditional case is completed.

5.3. The conditional estimate

In this section, we assume the GRH. We choose $a' = 1/2+(\log qT)^{-1}$ and $U = qT$. In the case (i), by Cauchy's theorem,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{a-iU}^{a+iU} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds \\
&= \operatorname*{Res}_{s=1}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s} \\
&\quad + \frac{1}{2\pi i}\left\{\int_{a+iU}^{a'+iU} + \int_{a'+iU}^{a'-iU} + \int_{a'-iU}^{a-iU}\right\}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds,
\end{aligned}$$

The integral on the horizontal line is

$$\int_{a'\pm iU}^{a\pm iU} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds \ll \frac{x^a}{U}(qU)^\varepsilon(\log qU)^3 \ll (qT)^\varepsilon.$$

As for the vertical line, we note that

$$\frac{L'}{L}(s,\psi_0) = \frac{\zeta'}{\zeta}(s) + \sum_{p|q}\frac{\log p}{p^s-1} \ll \log 2q$$

for $s = a'+it$ and $0 \le |t| \le 1$. Thus we have

$$\begin{aligned}
&\int_{a'-iU}^{a'+iU} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds \\
&= i\int_{-U}^{U} \frac{L'}{L}(a'+it,\psi_0)L'(a'+it,\psi_0)\frac{x^{a'+it}}{a'+it}dt \\
&\ll x^{a'}(\log qU)^3\int_1^U \frac{(qt)^\varepsilon}{t}dt + x^{a'}(\log 2q)^2\int_{-1}^1 \frac{q^\varepsilon}{a'}dt \\
&\ll (qT)^{\frac12+\varepsilon}.
\end{aligned}$$

Concerning the case (iv), we can see that

$$\sum_{nm\le x}\psi(m)\Lambda(m)\psi(n)\log n \ll (qT)^{\frac12+\varepsilon}$$

by a similar argument. Therefore we can replace the error term in our theorem by $(qT)^{\frac{1}{2}+\varepsilon}$.

5.4. The Details of Remark 5.0.1

We consider the case when q is a prime power. Let $q = p^\alpha$, $a' = -(\log qT)^{-1} = 1 - a$ and $U = qT$. In the case (i), by the residue theorem

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{a-iU}^{a+iU}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds\\
&= \operatorname*{Res}_{s=1}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s} + \operatorname*{Res}_{s=0}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}\\
&\quad + \sum_{\substack{\rho\neq 0\\ |\Im\rho|\le U}} L'(\rho,\psi_0)\frac{x^\rho}{\rho}\\
&\quad + \frac{1}{2\pi i}\left\{\int_{a+iU}^{a'+iU} + \int_{a'+iU}^{a'-iU} + \int_{a'-iU}^{a-iU}\right\}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds,
\end{aligned}$$

where ρ runs over the zeros of $L(s,\psi_0)$. With regard to the residue at $s=0$, we can see that

$$\frac{L'}{L}(s,\psi_0) = \frac{\zeta'}{\zeta}(s) + \frac{\log p}{p^s-1} \quad (q = p^\alpha)$$

and

$$\frac{\log p}{p^s-1} = \frac{1}{s}\cdot\frac{s\log p}{e^{s\log p}-1} = \frac{1}{s}\sum_{n=0}^{\infty}\frac{B_n}{n!}(s\log p)^n,$$

where B_n is the n-th Bernoulli number, and hence we have

$$\begin{aligned}
&\operatorname*{Res}_{s=0}\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}\\
&= \lim_{s\to 0}\frac{d}{ds}s\frac{L'}{L}(s,\psi_0)L'(s,\psi_0)x^s\\
&= \lim_{s\to 0}\frac{d}{ds}s\left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s}\sum_{n=0}^{\infty}\frac{B_n}{n!}(s\log p)^n\right)L'(s,\psi_0)x^s\\
&= L''(0,\psi_0) + \left(\frac{\zeta'}{\zeta}(0) + B_1\log p + \log x\right)L'(0,\psi_0)\\
&= 3\zeta'(0)\log p - \frac{3}{2}\zeta(0)(\log p)^2 + \zeta(0)\log x \ll (\log qT)^2.
\end{aligned}$$

The integral on the horizontal line is

$$\begin{aligned}
&\int_{1-a\pm iU}^{a\pm iU} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds \\
&\ll \left\{ \int_{\frac{1}{2}\pm iU}^{a\pm iU} + \int_{1-a\pm iU}^{\frac{1}{2}\pm iU} \right\} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds \\
&\ll \frac{x^a}{U}(qU)^{\varepsilon}(\log qU)^3 + \frac{x^{\frac{1}{2}}}{U}(qU)^{\frac{1}{2}}(\log qU)^4 \\
&\ll (qU)^{\varepsilon}(\log qU)^3 + \sqrt{q}(\log qU)^4.
\end{aligned}$$

On the integral along the vertical line, since $|s-\rho_{\psi_0}| \gg 1$, by (5.1.5), we can see that

$$\frac{L'}{L}(s,\psi_0) \ll \log q(|t|+2).$$

Therefore we have

$$\begin{aligned}
&\int_{1-a-iU}^{1-a+iU} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds \\
&= i\int_{-U}^{U} \frac{L'}{L}(1-a+it,\psi_0)L'(1-a+it,\psi_0)\frac{x^{1-a+it}}{1-a+it}dt \\
&\ll (\log qU)^2 \left| \int_{-U}^{U} \zeta(1-a+it)\frac{dt}{1-a+it} \right| \\
&\ll (\log qU)^2 \left(\log U \int_1^U t^{-\frac{1}{2}}dt + \int_{-1}^{1} \frac{dt}{|1-a+it|} \right) \\
&\ll \sqrt{U}(\log qU)^3.
\end{aligned}$$

Here we use the well-known estimate

$$\zeta(s) \ll (|t|+2)^{\frac{1}{2}}\log(|t|+2) \quad \text{for} \quad -\frac{1}{\log T} \le \sigma < \frac{1}{2}.$$

The sum over ρ consists of two sums as

$$\begin{aligned}
\sum_{\substack{\rho\neq 0 \\ |\Im\rho|\le U}} L'(\rho,\psi_0)\frac{x^\rho}{\rho} &= \sum_{|\gamma|\le U} L'\left(\frac{1}{2}+i\gamma,\psi_0\right)\frac{x^{\frac{1}{2}+i\gamma}}{\frac{1}{2}+i\gamma} \\
&\qquad + \sum_{\substack{\left|\frac{2\pi k}{\log p}\right|\le U \\ k\neq 0}} L'\left(\frac{2\pi ik}{\log p},\psi_0\right)\frac{x^{\frac{2\pi k}{\log p}i}\log p}{2\pi ik} \\
&= S_1 + S_2,
\end{aligned}$$

say. Since

$$L'\left(\frac{1}{2}+i\gamma,\psi_0\right)=\zeta'\left(\frac{1}{2}+i\gamma\right)\left(1-p^{-\frac{1}{2}-i\gamma}\right),$$

we have

$$\begin{aligned}S_1&\ll x^{\frac{1}{2}}\sum_{\gamma\le U}\frac{\left|L'\left(\frac{1}{2}+i\gamma,\psi_0\right)\right|}{\gamma}\\&\ll x^{\frac{1}{2}}\left(\sum_{\gamma\le U}\frac{\left|\zeta'\left(\frac{1}{2}+i\gamma\right)\right|^2}{\gamma}\right)^{\frac{1}{2}}\left(\sum_{\gamma\le U}\frac{1}{\gamma}\right)^{\frac{1}{2}}\\&\ll x^{\frac{1}{2}}(\log U)^{\frac{7}{2}}\end{aligned}$$

by partial summation and the fact that

$$\sum_{0<\gamma\le T}\left|\zeta'\left(\frac{1}{2}+i\gamma\right)\right|^2\asymp T(\log T)^4 \tag{5.4.1}$$

proved by Gonek [**10**].

On the other hand, since

$$L'\left(\frac{2\pi ik}{\log p},\psi_0\right)=\zeta\left(\frac{2\pi ik}{\log p}\right)\log p,$$

we see that

$$S_2\ll(\log p)^2\sum_{\frac{2\pi k}{\log p}\le U}\frac{\left|\zeta\left(\frac{2\pi ik}{\log p}\right)\right|}{2\pi k}\ll\sqrt{U}\log U(\log q)^2$$

by the estimate

$$\zeta(s)\ll(|t|+2)^{\frac{1}{2}}\log(|t|+2)\quad\text{for}\quad -\frac{1}{\log T}\le\sigma<\frac{1}{2}$$

again. Therefore we can see that

$$\frac{\chi(-1)\tau(\chi)}{\varphi(q)} \sum_{mn\le x} \psi_0(m)\psi_0(n)\Lambda(m)\log n$$
$$= \frac{\tau(\overline{\chi})}{q} x \left\{ \frac{1}{2}(\log x)^2 - \left(\sum_{p|q} \frac{\log p}{p-1} + \gamma_0 + 1 \right) \log x \right.$$
$$- \frac{1}{2}\left(\sum_{p|q} \frac{\log p}{p-1} \right)^2 + \frac{3}{2} \sum_{p|q} p \left(\frac{\log p}{p-1} \right)^2$$
$$\left. +(1-\gamma_0) \sum_{p|q} \frac{\log p}{p-1} + \gamma_0^2 + \gamma_0 - 3\gamma_1 + 1 \right\} + O\left(x^{\frac{1}{2}} (\log U)^{\frac{7}{2}} \right).$$

As for the case (iv), we need to deal with the Dirichlet L-functions with primitive and also imprimitive characters. However, it is sufficient to consider these with only primitive characters, for we put $q = p^\alpha$. For primitive characters, the integral on the vertical line can be estimated as

$$\int_{1-a-iU}^{1-a+iU} \frac{L'}{L}(s,\psi) L'(s,\psi) \frac{x^s}{s} ds$$
$$= \int_{1-a-iU}^{1-a+iU} \Delta(s,\psi) \left\{ \left(\frac{\Delta'}{\Delta}(s,\psi) \right)^2 L(1-s,\overline{\psi}) \right.$$
$$\left. -2\frac{\Delta'}{\Delta}(s,\psi) L'(1-s,\overline{\psi}) + \frac{L'}{L}(1-s,\overline{\psi}) L'(1-s,\overline{\psi}) \right\} \frac{x^s}{s} ds$$
$$\ll q^{a-\frac{1}{2}} \left| \int_{-U}^{U} \left(t^{a-\frac{1}{2}} \exp\left(it \log \frac{2\pi e}{qt} \right) + O(t^{a-\frac{3}{2}}) \right) \right.$$
$$\left. \times \left((\log qU)^2 L(a-it,\psi) + \frac{L'}{L}(a-it,\psi) L'(a-it,\psi) \right) \frac{x^{1-a+it}}{1-a+it} dt \right|$$
$$\ll x^{1-a} q^{a-\frac{1}{2}}$$
$$\times \left((\log U)^2 \sum_{n=1}^{\infty} \frac{1}{n^a} \left| \int_1^U \left(t^{a-\frac{3}{2}} \exp\left(it \log \frac{2\pi exn}{qt} \right) + O(t^{a-\frac{5}{2}}) \right) dt \right| \right.$$
$$+ \sum_{m=2}^{\infty} \frac{\Lambda(m)}{m^a} \sum_{n=1}^{\infty} \frac{\log n}{n^a}$$
$$\left. \times \left| \int_1^U \left(t^{a-\frac{3}{2}} \exp\left(it \log \frac{2\pi exmn}{qt} \right) + O(t^{a-\frac{5}{2}}) \right) dt \right| \right)$$

$$+ O\left(q^{a-\frac{1}{2}}(\log U)^3\right).$$

Since

$$\frac{d^2}{dt^2}\left(t\log\frac{2\pi exn}{qt}\right) = -t^{-1},$$

by the second derivative test,

$$\begin{aligned}
&\int_1^U t^{a-\frac{3}{2}}\exp\left(it\log\frac{2\pi exn}{qt}\right)dt \\
&\ll \sum_{l\le[\log U]+1}\int_{\frac{U}{2^l}}^{\frac{U}{2^{l-1}}} t^{a-\frac{3}{2}}\exp\left(it\log\frac{2\pi exn}{qt}\right)dt \\
&\ll \sum_{l\le[\log U]+1} 1 \ll \log U.
\end{aligned}$$

Therefore we obtain

$$\int_{1-a-iU}^{1-a+iU}\frac{L'}{L}(s,\psi)L'(s,\psi)\frac{x^s}{s}ds \ll q^{a-\frac{1}{2}}(\log U)^4.$$

On the sum S_1, we assume the estimate (5.0.1). By partial summation and this assumption, we have

$$\begin{aligned}
S_1 &\ll x^{\frac{1}{2}}\sum_{0<\gamma_\psi\le U}\frac{\left|L'\left(\frac{1}{2}+i\gamma_\psi,\psi\right)\right|}{\gamma_\psi} \\
&\ll x^{\frac{1}{2}}\left(\sum_{0<\gamma_\psi\le U}\frac{\left|L'\left(\frac{1}{2}+i\gamma_\psi,\psi\right)\right|^2}{\gamma_\psi}\right)^{\frac{1}{2}}\left(\sum_{0<\gamma_\psi\le U}\frac{1}{\gamma_\psi}\right)^{\frac{1}{2}} \\
&\ll x^{\frac{1}{2}}(\log U)^{\frac{7}{2}}.
\end{aligned}$$

On the other hand, the counterpart of the sum S_2 does not appear. When $\psi \pmod q$ is induced by $\psi^\star \pmod d$ with $d \mid q$, we see that

$$L(s,\psi) = L(s,\psi^\star)\prod_{\substack{p\mid q\\ p\nmid d}}\left(1-\frac{\psi^\star(p)}{p^s}\right).$$

However we assume that $q = p^\alpha$. Thus the product on the right-hand side is 1. Hence there are no zeros on the imaginary axis.

Therefore we can replace the estimate of the error term by

$$\sqrt{qT}(\log qT)^{\frac{7}{2}}.$$

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