Attainability of a stationary Navier-Stokes flow around a moving rigid body

(運動する剛体周りの定常 Navier-Stokes 流の attainability)

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Contents

Chapter 1

Introduction

We consider the large time behavior of a viscous incompressible flow around a moving rigid body $\mathscr{O} \subset \mathbb{R}^n$ ($n \geq 3$). Specifically, if the rigid body $\mathscr{O} \subset \mathbb{R}^3$ translates with a prescribed constant velocity, we then expect from the physical point of view that solutions to the Navier-Stokes equation possess anisotropic decay structure at spatial infinity. In a series of his celebrated papers, see for instance [17–20], Finn succeeded in constructing a stationary solution, termed by him physically reasonable solution, that exhibits a paraboloidal wake region behind the body, and developed the theory of stationary Navier-Stokes problem in exterior domains. For even better understanding, in [18, Section 6], he raised a question relating to the convergence of nonstationary solutions to stationary solutions. This is well known as Finn's starting problem; to be precise, suppose both a rigid body and the fluid filling the outside of the body are initially at rest and the body starts to translate with a velocity which gradually increases and is maintained after a certain finite time, does a nonstationary flow then converge to a stationary solution corresponding to a terminal velocity of the body as time goes to infinity? If the answer is affirmative, the stationary solution is said to be attainable by following Heywood [29, Section 6], who first studied Finn's starting problem but gave a partial answer merely in a special situation that the net force exerted by the fluid on the body is identically zero, yielding the square summability of stationary solutions. Since stationary solutions do not belong to L^2 space in general, energy methods are far from enough to analyze the starting problem and we do need L^q framework. Thus the problem had remained open until Kobayashi and Shibata [39] developed the L^q theory of the linearized problem, which is called the Oseen problem. By making use of estimates established in [39] via Kato's approach [37], Finn's starting problem was affirmatively solved by Galdi, Heywood and Shibata [27] for small terminal velocity of the body. Nevertheless, convergence rates deduced by them were the same as those in stability analysis (Shibata [45]), and they can be improved as we will clarify in this thesis. The concept of attainability is somewhat similar to, however, different from stability. Analysis of attainability is more difficult because the equation is non-autonomous and because one has to deal with several delicate terms. Since the other rigid motion is rotation, it should be even more challenging to study Finn's starting problem above in which translation is replaced by rotation. This problem was proposed by Hishida [31], but it has remained open since there seems to be no chance to avoid the non-autonomous character unlike the translational case. If a stationary solution is attainable, then it is interesting to compare convergence rates with those in the translational regime. This issue is closely related to asymptotic structure of stationary solutions at infinity.

The objective of this thesis is two-fold. The first one is to develop further analysis of Finn's starting problem with translation to derive new convergence rates which improve [27].

Moreover, we extend this result to the case of higher dimensions. Such generalization is never obvious because our knowledge about stationary solutions in higher dimensions is quite less than in 3D case. Therefore, analysis of the stationary problem must be an important step. The second objective is to prove attainability of a stationary solution around a rigid body rotating from rest.

In Chapter 3, we discuss the first objective. Let us introduce the mathematical formulation of Finn's starting problem. Suppose that a compact set $\mathscr{O}$, with non-empty interior, is translating with a velocity $-\psi(t)ae_1$, where $a>0$, $e_1=(1,0,\cdots,0)^\top$ and ψ is a function on $\mathbb{R}$ describing the transition of the translational velocity in such a way that

$$\psi\in C^1(\mathbb{R};\mathbb{R}),\quad |\psi(t)|\le 1\quad \text{for } t\in\mathbb{R},\quad \psi(t)=0\quad \text{for } t\le 0,\quad \psi(t)=1\quad \text{for } t\ge 1. \tag{1.1}$$

Here and hereafter, $(\cdot)^\top$ denotes the transpose. We take the frame attached to the body, then the motion of the fluid filling $D=\mathbb{R}^n\setminus\mathscr{O}$ $(n\ge 3)$, which is assumed to be an exterior domain with smooth boundary ∂D, and being started from rest obeys

$$\begin{cases} \dfrac{\partial u}{\partial t}+u\cdot\nabla u = \Delta u-\psi(t)a\dfrac{\partial u}{\partial x_1}-\nabla p, & x\in D,\ t>0,\\ \nabla\cdot u = 0, & x\in D, t\ge 0,\\ u|_{\partial D} = -\psi(t)ae_1, & t>0,\\ u\to 0 & \text{as } |x|\to\infty,\\ u(x,0) = 0, & x\in D. \end{cases} \tag{1.2}$$

Here, $u=(u_1(x,t),\cdots,u_n(x,t))^\top$ and $p=p(x,t)$ denote unknown velocity and pressure of the fluid, respectively. It should be emphasized that the fluid is assumed to be initially at rest since we focus our attention on attainability of stationary solutions as limits of nonstationary fluid motions induced only by the motion of the body. Since $\psi(t)=1$ for $t\ge 1$, the large time behavior of solutions is related to the stationary problem

$$\begin{cases} u_s\cdot\nabla u_s = \Delta u_s-a\dfrac{\partial u_s}{\partial x_1}-\nabla p_s, & x\in D,\\ \nabla\cdot u_s = 0, & x\in D,\\ u_s|_{\partial D} = -ae_1, & \\ u_s\to 0 & \text{as } |x|\to\infty. \end{cases} \tag{1.3}$$

We seek a nonstationary solution to (1.2), which tends to a stationary solution to (1.3) as $t\to\infty$. Furthermore, we derive new convergence rate, that is determined by the summability of the stationary solution at spatial infinity. We thus first construct a small stationary solution possessing the optimal summability at infinity. As already mentioned above, if $n=3$, we then have a solution with anisotropic decay structure due to Finn, but this thesis gives much shorter proof of the existence theorem for stationary solutions in all dimension $n\ge 3$ by focusing on summability at infinity rather than pointwise estimates. Note that the summability of the Oseen fundamental solution

$$\mathbf{E}\in L^q(\{x\in\mathbb{R}^n\mid |x|>1\}),\quad q>\frac{n+1}{n-1},\qquad \nabla\mathbf{E}\in L^r(\{x\in\mathbb{R}^n\mid |x|>1\}),\quad r>\frac{n+1}{n}, \tag{1.4}$$

see Galdi [26, Section VII], would be optimal summability of stationary solutions at infinity because their leading profile is expected to be given by the fundamental solution $\mathbf{E}$ as long as the net force does not vanish, see for instance Farwig [12] and Galdi [26] when $n = 3$. For the proof, we find a certain closed ball N so that the map $\Psi : N \ni v \mapsto u \in N$, which provides the solution to the problem

$$\begin{cases} \Delta u - a\dfrac{\partial u}{\partial x_1} = \nabla p + v \cdot \nabla v, & x \in D, \\ \nabla \cdot u = 0, & x \in D, \\ u|_{\partial D} = -ae_1, & \\ u \to 0 & \text{as } |x| \to \infty, \end{cases}$$

is well-defined and contractive. To do this, we rely on the L^q theory of the Oseen system developed by Galdi [26], however, the proof is never straightforward and we in fact need a device to capture the optimal summability of stationary solutions at infinity.

Let us proceed to the starting problem. We prove the attainability of the stationary solution obtained above by making use of L^q-L^r estimates of the Oseen semigroup [10, 39]. Since the fluid is initially at rest and the stationary solution u_s appears in the forcing term of the equation for perturbation, we expect that the convergence rate is determined by the summability of u_s. In fact, we derive the following convergence properties

$$\|u(t) - u_s\|_{L^q(D)} = O(t^{-\frac{1}{2}+\frac{n}{2q}-\frac{\rho_1}{2}}), \quad n \le q \le \infty, \tag{1.5}$$

$$\|\nabla u(t) - \nabla u_s\|_{L^n(D)} = O(t^{-\frac{1}{2}-\frac{\rho_1}{2}}) \tag{1.6}$$

as $t \to \infty$, where $u_s \in L^{n/(1+\rho_1)}(D)$ with some $\rho_1 > 0$. Our result should be compared to $\|u(t) - u_s\|_{L^q(D)} = O(t^{-1/2+n/(2q)})$ $(n \le q)$ with $n = 3$, that was proved by Galdi, Heywood and Shibata [27]. Therefore, our convergence rates are the improvement of those derived by [27] and seem to be sharp since we can take ρ_1 so that $n/(1+\rho_1)$ is close to $(n+1)/(n-1)$, see (1.4). For the proof of (1.5)–(1.6), the key step is to derive the L^n convergence, that is,

$$\|u(t) - u_s\|_{L^n(D)} = O(t^{-\frac{\rho_1}{2}}) \tag{1.7}$$

as $t \to \infty$. To this end, we first observe slower convergence $\|u(t) - u_s\|_{L^n(D)} = O(t^{-\rho/2})$ with some $\rho \in (0, 1)$. Then estimates of other norms are improved. With them at hand, we repeat improvements of the L^n estimate step by step to find (1.7). In this procedure, L^{q_0} convergence of the solution is needed especially for $n \ge 4$, where $q_0 < n$ is appropriately chosen.

In Chapter 4, we are aiming at attainability of a stationary solution around a rigid body rotating from rest. Suppose that $\mathscr{O} \subset \mathbb{R}^3$ is rotating with an angular velocity $\psi(t)\omega_0$, where $\psi(t)$ is given by (1.1) and $\omega_0 = (0, 0, a)^\top$. Then the fluid motion which occupies the exterior domain $D = \mathbb{R}^3 \setminus \mathscr{O}$ with smooth boundary ∂D and is started from rest obeys

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = \Delta u + (\psi(t)\omega_0 \times x) \cdot \nabla u - \psi(t)\omega_0 \times u - \nabla p, & x \in D, t > 0, \\ \nabla \cdot u = 0, & x \in D, t \ge 0, \\ u|_{\partial D} = \psi(t)\omega_0 \times x, & t \ge 0, \\ u \to 0 & \text{as } |x| \to \infty, \\ u(x, 0) = 0, & x \in D. \end{cases} \tag{1.8}$$

The purpose of Chapter 4 is to show that (1.8) admits a global solution which tends to a solution for the stationary problem

$$\begin{cases} u_s\cdot\nabla u_s = \Delta u_s + (\omega_0\times x)\cdot\nabla u_s - \omega_0\times u_s - \nabla p_s, & x\in D,\\ \nabla\cdot u_s = 0, \qquad x\in D,\\ u_s|_{\partial D} = \omega_0\times x,\\ u_s\to 0 \qquad \text{as } |x|\to\infty \end{cases} \tag{1.9}$$

provided the terminal angular velocity ω_0 of the body is small enough. In [25], Galdi first constructed a unique stationary solution which satisfies

$$|u_s(x)|\le \frac{C|\omega_0|}{|x|},\quad |\nabla u_s(x)|+|p_s(x)|\le \frac{C|\omega_0|}{|x|^2} \tag{1.10}$$

if $|\omega_0|$ is small. Farwig and Hishida [13] also captured the scale-critical decay rate (1.10) in terms of weak Lebesgue spaces even for the external force being in a Lorentz-Sobolev space of order (-1). The leading term of stationary solutions which decay like $O(1/|x|)$ was first studied by Farwig and Hishida [14]. In fact, they proved that the leading term at infinity is a specific Landau solution being symmetric about the axis of rotation.

Let us mention some difficulties caused by rotation and how to overcome them in this thesis. In the translational case (Chapter 3), the key tool is L^q-L^r estimates of the Oseen semigroup. In the rotational case as well, Hishida and Shibata [36] established L^q-L^r estimates of the semigroup generated by the Stokes operator with the additional term $(\omega_0\times x)\cdot\nabla-\omega_0\times$. If we try to solve the starting problem by using the semigroup as in the translational case, we have to treat the term $(\psi(t)-1)(\omega_0\times x)\cdot\nabla v$ which is no longer subordinate to the semigroup mentioned above on account of the unbounded coefficient $\omega_0\times x$, where $v=u-\psi(t)u_s$. To overcome this difficulty, we make use of the evolution operator $\{T(t,s)\}_{t\ge s\ge 0}$ on the solenoidal space $L^q_\sigma(D)$ $(1<q<\infty)$, which is the solution operator to the initial value problem for the non-autonomous linearized system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + (\psi(t)\omega_0\times x)\cdot\nabla u - \psi(t)\omega_0\times u - \nabla p, & x\in D, t>s,\\ \nabla\cdot u = 0, \quad x\in D, t\ge s,\\ u|_{\partial D} = 0, \quad t>s,\\ u\to 0 \quad \text{as } |x|\to\infty,\\ u(x,s) = f, \quad x\in D. \end{cases}$$

The evolution operator $\{T(t,s)\}_{t\ge s\ge 0}$ was successfully constructed by Hansel and Rhandi [28], and they also derived the L^q-L^r smoothing rate near the initial time. It is remarkable that they constructed the evolution operator in their own way without using any general theory since the corresponding semigroup is not analytic, see Hishida [30] and Farwig and Neustupa [15]. Recently, Hishida [33,34] developed the L^q-L^r decay estimates of this evolution operator. By those estimates together with the framework of Lorentz spaces developed by Yamazaki [48], we solve the weak formulation of the integral equation which perturbation from the stationary solution obeys. Why we employ the Lorentz space is that it is hopeless to analyze the integral equation in the Lebesgue space because of the scale-critical rate (1.10) of stationary solution. Once we get a solution of the weak formulation, the same procedure as in Kozono and Yamazaki [40] implies that the solution actually satisfies the integral equation. In order to derive the L^∞ convergence rate, we considerably simplify the argument of Koba [38], who first discussed such convergence with less sharp rate in the context of stability analysis (within the translational regime) under the condition $u_s(x) = O(1/|x|)$. Our idea is to employ merely L^1-L^r estimates of the adjoint evolution operator $T(t,s)^*$.

This thesis is organized as follows. In Chapter 2, we introduce some function spaces, which are used throughout this thesis. Chapter 3 is devoted to studies of the Navier-Stokes flow past a rigid body. We first construct a stationary solution past the body translating with a small constant velocity and then discuss the attainability of the stationary solution obtained above. In Chapter 4, we first collect some results on stationary solutions as well as the evolution operator mentioned above and then prove the attainability of a stationary solution around a rigid body rotating from rest.

Chapter 2

Function spaces

In this chapter, we introduce some notation. We begin with some function spaces. Let $D \subset \mathbb{R}^3$ be an exterior domain with smooth boundary. By $C_0^\infty(D)$, we denote the set of all C^∞ functions with compact support in D. For $1 \le q \le \infty$ and nonnegative integer m, $L^q(D)$ and $W^{m,q}(D)$ denote the standard Lebesgue and Sobolev spaces, respectively. We write the L^q norm as $\|\cdot\|_q$. The completion of $C_0^\infty(D)$ in $W^{m,q}(D)$ is denoted by $W_0^{m,q}(D)$. Let $1 < q < \infty$ and $1 \le r \le \infty$. Then the Lorentz spaces $L^{q,r}(D)$ are defined by

$$L^{q,r}(D) = \{f : \text{Lebesgue measurable function} \mid \|f\|_{q,r}^* < \infty\},$$

where

$$\|f\|_{q,r}^* = \begin{cases} \left(\displaystyle\int_0^\infty \left(t\mu(\{x \in D \mid |f(x)| > t\})^{\frac{1}{q}} \right)^r \frac{dt}{t} \right)^{\frac{1}{r}} & 1 \le r < \infty, \\ \displaystyle\sup_{t>0} t\mu(\{x \in D \mid |f(x)| > t\})^{\frac{1}{q}} & r = \infty \end{cases}$$

and $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^3$. The space $L^{q,r}(D)$ is a quasi-normed space and it is even a Banach space equipped with norm $\|\cdot\|_{q,r}$ equivalent to $\|\cdot\|_{q,r}^*$. The real interpolation functor is denoted by $(\cdot,\cdot)_{\theta,r}$, then we have

$$L^{q,r}(D) = (L^{q_0}(D), L^{q_1}(D))_{\theta,r}\,,$$

where $1 \le q_0 < q < q_1 \le \infty$ and $0 < \theta < 1$ satisfy $1/q = (1-\theta)/q_0 + \theta/q_1$, while $1 \le r \le \infty$, see Bergh-Löfström [3]. We note that if $1 \le r < \infty$, the dual of the space $L^{q,r}(D)$ is $L^{q/(q-1),r/(r-1)}(D)$. It is well known that if $1 \le r < \infty$, the space $C_0^\infty(D)$ is dense in $L^{q,r}(D)$, while the space $C_0^\infty(D)$ is not dense in $L^{q,\infty}(D)$.

We next introduce some solenoidal function spaces. Let $C_{0,\sigma}^\infty(D)$ be the set of all C_0^∞-vector fields f which satisfy $\operatorname{div} f = 0$ in D. For $1 < q < \infty$, $L_\sigma^q(D)$ denotes the completion of $C_{0,\sigma}^\infty(D)$ in $L^q(D)$. For every $1 < q < \infty$, we have the following Helmholtz decomposition:

$$L^q(D) = L_\sigma^q(D) \oplus \{\nabla p \in L^q(D) \mid p \in L_{\mathrm{loc}}^q(\overline{D})\},$$

see Fujiwara and Morimoto [22], Miyakawa [44], and Simader and Sohr [47]. Let P_q denote the Fujita-Kato projection from $L^q(D)$ onto $L_\sigma^q(D)$ associated with the decomposition. We remark that the adjoint operator of P_q coincides with $P_{q/(q-1)}$. We simply write $P = P_q$. By real interpolation, it is possible to extend P to a bounded operator on $L^{q,r}(D)$. We then define the solenoidal Lorentz spaces $L_\sigma^{q,r}(D)$ by

$$L_\sigma^{q,r}(D) = PL^{q,r}(D) = (L_\sigma^{q_0}(D), L_\sigma^{q_1}(D))_{\theta,r}\,,$$

where $1 < q_0 < q < q_1 < \infty$ and $0 < \theta < 1$ satisfy $1/q = (1-\theta)/q_0 + \theta/q_1$, while $1 \le r \le \infty$, see Borchers and Miyakawa [5]. We then have the duality relation $L^{q,r}_\sigma(D)^* = L^{q/(q-1),r/(r-1)}_\sigma(D)$ for $1 < q < \infty$ and $1 \le r < \infty$. We denote various constants by C and they may change from line to line. The constant dependent on $A, B, \cdots$ is denoted by $C(A, B, \cdots)$. Finally, if there is no confusion, we use the same symbols for denoting spaces of scalar-valued functions and those of vector-valued ones.

Chapter 3

Existence of a stationary Navier-Stokes flow past a rigid body and Finn's starting problem in higher dimensions

3.1 Introduction and main results

We consider a viscous incompressible flow past a rigid body $\mathscr{O} \subset \mathbb{R}^n\,(n \geq 3)$. We suppose that $\mathscr{O}$ is translating with a velocity $-\psi(t)ae_1$, where $a > 0$, $e_1 = (1, 0, \cdots, 0)^\top$ and ψ is a function on $\mathbb{R}$ describing the transition of the translational velocity in such a way that

$$\psi \in C^1(\mathbb{R}; \mathbb{R}), \quad |\psi(t)| \leq 1 \quad \text{for } t \in \mathbb{R}, \quad \psi(t) = 0 \quad \text{for } t \leq 0, \quad \psi(t) = 1 \quad \text{for } t \geq 1. \tag{3.1}$$

Here and hereafter, $(\cdot)^\top$ denotes the transpose. We take the frame attached to the body, then the fluid motion which occupies the exterior domain $D = \mathbb{R}^n \setminus \mathscr{O}$ with C^2 boundary ∂D and is started from rest obeys

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = \Delta u - \psi(t) a \dfrac{\partial u}{\partial x_1} - \nabla p, & x \in D,\ t > 0, \\ \nabla \cdot u = 0, & x \in D, t \geq 0, \\ u|_{\partial D} = -\psi(t) a e_1, & t > 0, \\ u \to 0 & \text{as } |x| \to \infty, \\ u(x, 0) = 0, & x \in D. \end{cases} \tag{3.2}$$

Here, $u = (u_1(x,t), \cdots, u_n(x,t))^\top$ and $p = p(x,t)$ denote unknown velocity and pressure of the fluid, respectively. Since $\psi(t) = 1$ for $t \geq 1$, the large time behavior of solutions is related to the stationary problem

$$\begin{cases} u_s \cdot \nabla u_s = \Delta u_s - a \dfrac{\partial u_s}{\partial x_1} - \nabla p_s, & x \in D, \\ \nabla \cdot u_s = 0, & x \in D, \\ u_s|_{\partial D} = -a e_1, & \\ u_s \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.3}$$

When $n = 3$, the pioneering work due to Leray [41] provided the existence theorem for weak solution to problem (3.3), what is called D-solution, having finite Dirichlet integral without smallness assumption on data. From the physical point of view, solutions of (3.3) should reflect the anisotropic decay structure caused by the translation, but his solution had little information about the behavior at large distances. To fill this gap, Finn [17]–[20] introduced the class of solutions with pointwise decay property

$$u_s(x) = O(|x|^{-\frac{1}{2}-\varepsilon}) \qquad \text{as } |x| \to \infty \tag{3.4}$$

for some $\varepsilon > 0$ and proved that if a is small enough, (3.3) admits a unique solution satisfying (3.4) and exhibiting paraboloidal wake region behind the body $\mathscr{O}$. He called the Navier-Stokes flows satisfying (3.4) physically reasonable solutions. It is remarkable that D-solutions become physically reasonable solutions no matter how large a would be, see Babenko [2], Galdi [24] and Farwig and Sohr [16]. Galdi developed the L^q-theory of the linearized system, that we call the Oseen system, to prove that every D-solution has the same summability as the one of the Oseen fundamental solution without any smallness assumption, see [26, Theorem X.6.4]. It is not straightforward to generalize his result to the case of higher dimensions and it remains open whether the same result holds true for $n \geq 4$. We also refer to Farwig [11] who gave another outlook on Finn's results by using anisotropically weighted Sobolev spaces, and to Shibata [45] who developed the estimates of physically reasonable solutions and then proved their stability in the L^3 framework when a is small. There is less literature concerning the problem (3.3) for the case $n \geq 4$. When $n \geq 3$, Shibata and Yamazaki [46] constructed a solution u_s, which is uniformly bounded with respect to small $a \geq 0$ in the Lorentz space $L^{n,\infty}$, and investigated the behavior of u_s as $a \to 0$. If, in particular, $n \geq 4$ and if $a \geq 0$ is sufficiently small, they also derived

$$u_s \in L^{\frac{n}{1+\rho_1}}(D) \cap L^{\frac{n}{1-\rho_2}}(D), \quad \nabla u_s \in L^{\frac{n}{2+\rho_1}}(D) \cap L^{\frac{n}{2-\rho_2}}(D) \tag{3.5}$$

for some $0 < \rho_1, \rho_2 < 1$, see [46, Remark 4.2].

Let us turn to the initial value problem. Finn [18] conjectured that (3.2) admits a solution which tends to a physically reasonable solution as $t \to \infty$ if a is small enough. This is called Finn's starting problem. It was first studied by Heywood [29], in which a stationary solution is said to be attainable if the fluid motion converges to this solution as $t \to \infty$. Later on, by using Kato's approach [37] (see also Fujita and Kato [21]) together with the L^q-L^r estimates for the Oseen initial value problem established by Kobayashi and Shibata [39], Finn's starting problem was affirmatively solved by Galdi, Heywood and Shibata [27]. After that, Hishida and Maremonti [35] constructed a sort of weak solution u that enjoys

$$\|u(t) - u_s\|_\infty = O(t^{-\frac{1}{2}}) \quad \text{as } t \to \infty \tag{3.6}$$

if a is small, but $u(\cdot, 0) \in L^3(D)$ can be large. Although we concentrate ourselves on attainability in this thesis, stability of stationary solutions was also studied by Shibata [45], Enomoto and Shibata [10] and Koba [38] in the L^q framework. Those work except [10] studied the three-dimensional exterior problem, while [10] showed the stability of a stationary solution satisfying (3.5) for some $0 < \rho_1, \rho_2 < 1$ in n-dimensional exterior domains with $n \geq 3$. Stability of physically reasonable solutions in 2D is much more involved for several reasons and it has been recently proved by Maekawa [43].

The aim of this chapter is two-fold. The first one is to construct a small stationary solution possessing the optimal summability at spatial infinity, which is the same as that of the Oseen fundamental solution $\mathbf{E}$:

$$\mathbf{E} \in L^q(\{x \in \mathbb{R}^n \mid |x| > 1\}), \quad q > \frac{n+1}{n-1}, \qquad \nabla \mathbf{E} \in L^r(\{x \in \mathbb{R}^n \mid |x| > 1\}), \quad r > \frac{n+1}{n}, \tag{3.7}$$

see Galdi [26, Chapter VII]. As already mentioned above, this result is well known in three-dimensional case even for large $a > 0$, but it is not found in the literature for higher dimensional case $n \geq 4$. Our theorem covers the three-dimensional case as well and the proof is considerably shorter than the one given by authors mentioned above since we focus our interest only on summability at infinity rather than anisotropic pointwise estimates. The second aim is to give an affirmative answer to the starting problem as long as a is small enough, that is, to show the attainability of the stationary solution obtained above. The result extends Galdi, Heywood and Shibata [27] to the case of higher dimensions. Even for the three-dimensional case, our theorem not only recovers [27] but also provides better decay properties, for instance,

$$\|u(t) - u_s\|_\infty = O(t^{-\frac{1}{2}-\frac{\rho}{2}}) \quad \text{as } t \to \infty \tag{3.8}$$

for some $\rho > 0$, that should be compared with (3.6). This is because the fluid is initially at rest and because the three-dimensional stationary solution u_s belongs to $L^q(D)$ with $q < 3$; to be precise, since q can be close to 2, one can take ρ close to $1/2$ in (3.8). Due to the L^q-L^r estimates of the Oseen semigroup established by Kobayashi and Shibata [39], Enomoto and Shibata [9, 10], see Proposition 3.3.1, this decay rate is sharp in view of presence of u_s, see (3.19) below, in forcing terms of the equation (3.18) for the perturbation. Our result can be also compared with the starting problem in which translation is replaced by rotation of the body $\mathscr{O} \subset \mathbb{R}^3$. Under the circumstance of rotational case, the optimal spatial decay of stationary solutions observed in general is the scale-critical rate $O(|x|^{-1})$, so that they cannot belong to $L^q(D)$ with $q \leq 3 = n$, and therefore, we have no chance to deduce (3.8). Another remark is that, in comparison with stability theorem due to [10] for $n \geq 3$, more properties of stationary solutions are needed to establish the attainability theorem. Therefore, those properties must be deduced in constructing a solution of (3.3).

Let us state the first main theorem on the existence and summability of stationary solutions.

Theorem 3.1.1. *Let $n \geq 3$. For every $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying*

$$\frac{n+1}{n-1} < \alpha_1 \leq n+1 \leq \alpha_2 < \frac{n(n+1)}{2}, \qquad \frac{n+1}{n} < \beta_1 \leq \frac{n+1}{2} \leq \beta_2 < \frac{n(n+1)}{n+2}, \tag{3.9}$$

there exists a constant $\delta = \delta(\alpha_1, \alpha_2, \beta_1, \beta_2, n, D) \in (0,1)$ such that if

$$0 < a^{\frac{n-2}{n+1}} < \delta,$$

problem (3.3) admits a unique solution u_s along with

$$\|u_s\|_{\alpha_1} + \|u_s\|_{\alpha_2} \leq C a^{\frac{n-1}{n+1}}, \quad \|\nabla u_s\|_{\beta_1} + \|\nabla u_s\|_{\beta_2} \leq C a^{\frac{n}{n+1}}, \tag{3.10}$$

where $C > 0$ is independent of a.

The upper bounds of α_2 and β_2 come from (3.30) with $q < n/2$ in Proposition 3.2.1 on the L^q-theory of the Oseen system, whereas the lower bounds of α_1 and β_1 are just (3.7).

For the proof of Theorem 3.1.1, we define a certain closed ball N and a contraction map $\Psi : N \ni v \mapsto u \in N$ which provides the solution to the problem

$$\begin{cases} \Delta u - a\dfrac{\partial u}{\partial x_1} = \nabla p + v\cdot\nabla v, & x\in D,\\ \nabla\cdot u = 0, & x\in D,\\ u|_{\partial D} = -ae_1, & \\ u\to 0 & \text{as } |x|\to\infty. \end{cases} \tag{3.11}$$

In doing so, we rely on the L^q-theory of the Oseen system developed by Galdi [26, Theorem VII.7.1], see Proposition 3.2.1, which gives us sharp summability estimates of solutions at infinity together with explicit dependence on $a>0$. As long as we only use Proposition 3.2.1, the only space in which estimates of Ψ are closed is

$$\{u\in L^{n+1}(D) \mid \nabla u\in L^{\frac{n+1}{2}}(D)\}.$$

From this, we can capture neither the optimal summability at infinity nor regularity required in the study of the starting problem. We thus use at least two spaces $L^{\alpha_i}(D)$ $(i=1,2)$ for u and $L^{\beta_i}(D)$ $(i=1,2)$ for ∇u, and intend to find a solution within a closed ball N of

$$\{u\in L^{\alpha_1}(D)\cap L^{\alpha_2}(D) \mid \nabla u\in L^{\beta_1}(D)\cap L^{\beta_2}(D)\}. \tag{3.12}$$

However, it is not possible to apply Proposition 3.2.1 to $f = v\cdot\nabla v$ with

$$v\in L^{\alpha_1}(D), \qquad \nabla v\in L^{\beta_1}(D) \tag{3.13}$$

or

$$v\in L^{\alpha_2}(D), \qquad \nabla v\in L^{\beta_2}(D) \tag{3.14}$$

if α_1 and β_1 are simultaneously close to $(n+1)/(n-1)$ and $(n+1)/n$, or if α_2 and β_2 are simultaneously close to $n(n+1)/2$ and $n(n+1)/(n+2)$, because the relation

$$\frac{2}{n} < \frac{1}{\alpha_2}+\frac{1}{\beta_2} < \frac{1}{\alpha_1}+\frac{1}{\beta_1} < 1$$

required in the linear theory, see Proposition 3.2.1, is not satisfied. In order to overcome this difficulty, given $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ satisfying (3.9), we choose auxiliary exponents (q_1,q_2,r_1,r_2) fulfilling

$$\alpha_1\le q_1\le q_2\le\alpha_2, \quad \beta_1\le r_1\le r_2\le\beta_2, \quad \frac{2}{n}<\frac{1}{q_i}+\frac{1}{r_i}<1, \quad i=1,2$$

such that the application of Proposition 3.2.1 to $f = v\cdot\nabla v$ with $v\in L^{q_1}(D)$ and $\nabla v\in L^{r_1}(D)$ (resp. $v\in L^{q_2}(D)$ and $\nabla v\in L^{r_2}(D)$) recovers (3.13) (resp. (3.14)) with u.

Another possibility to prove Theorem 3.1.1 is combining Proposition 3.2.1 with the Sobolev inequality. We then get a solution $(u_s,p_s)\in X_q(n)$ for all $q\in(1,\infty)$ with $n/3\le q\le(n+1)/3$, where $X_q(n)$ is defined in Proposition 3.2.1. The restriction $n/3\le q\le(n+1)/3$ is removed by applying a bootstrap argument to decrease the lower bound to 1 and to increase the upper bound to $n/2$. As compared with this way, in our proof, we do not any use a bootstrap argument and directly construct a solution possessing the optimal summability at infinity as well as regularity required in the study of the starting problem.

Let us proceed to the starting problem. To study the attainability of the stationary solution u_s of class (3.12) with $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying (3.9), it is convenient to set

$$\alpha_1 = \frac{n}{1+\rho_1}, \qquad \alpha_2 = \frac{n}{1-\rho_2}, \qquad \beta_1 = \frac{n}{2+\rho_3}, \qquad \beta_2 = \frac{n}{2-\rho_4} \tag{3.15}$$

with $(\rho_1, \rho_2, \rho_3, \rho_4)$ satisfying

$$0 < \rho_1 < \frac{n^2-2n-1}{n+1}, \quad \frac{1}{n+1} \le \rho_2 < \frac{n-1}{n+1}, \quad 0 < \rho_3 < \frac{n^2-2n-2}{n+1}, \quad \frac{2}{n+1} \le \rho_4 < \frac{n}{n+1} \tag{3.16}$$

and we need the additional condition

$$\rho_2 + \rho_4 > 1. \tag{3.17}$$

We note that the set of those parameters is nonvoid. It is reasonable to look for a solution to (3.2) of the form

$$u(x,t) = v(x,t) + \psi(t)u_s, \quad p(x,t) = \phi(x,t) + \psi(t)p_s.$$

Then the perturbation (v, ϕ) satisfies the following initial boundary value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v - a\dfrac{\partial v}{\partial x_1} - v\cdot\nabla v - \psi(t)v\cdot\nabla u_s - \psi(t)u_s\cdot\nabla v + (1-\psi(t))a\dfrac{\partial v}{\partial x_1} \\ \qquad\qquad + h_1(x,t) + h_2(x,t) - \nabla\phi, \quad x\in D,\ t>0, \\ \nabla\cdot v = 0, \quad x\in D,\ t\ge 0, \\ v|_{\partial D} = 0, \quad t>0, \\ v \to 0 \quad \text{as } |x|\to\infty, \\ v(x,0) = 0, \quad x\in D, \end{cases} \tag{3.18}$$

where

$$h_1(x,t) = -\psi'(t)u_s, \tag{3.19}$$

$$h_2(x,t) = \psi(t)\big(1-\psi(t)\big)\Big(u_s\cdot\nabla u_s + a\frac{\partial u_s}{\partial x_1}\Big). \tag{3.20}$$

In what follows, we study the problem (3.18) instead of (3.2). In fact, if we obtain the solution v of (3.18) which converges to 0 as $t\to\infty$, the solution u of (3.2) converges to u_s as $t\to\infty$. Problem (3.18) is converted into

$$v(t) = \int_0^t e^{-(t-\tau)A_a} P\Big[-v\cdot\nabla v - \psi(\tau)v\cdot\nabla u_s - \psi(\tau)u_s\cdot\nabla v + \big(1-\psi(\tau)\big)a\frac{\partial v}{\partial x_1} + h_1(\tau) + h_2(\tau)\Big]d\tau \tag{3.21}$$

by using the Oseen semigroup e^{-tA_a} (see Section 3.3).

We are now in a position to give the second main theorem on attainability of stationary solutions.

Theorem 3.1.2. *Let $n \geq 3$ and let ψ be a function on $\mathbb{R}$ satisfying* (3.1). *We set $M = \max_{t\in\mathbb{R}} |\psi'(t)|$. Suppose that ρ_1, ρ_2, ρ_3 and ρ_4 satisfy* (3.16)–(3.17) *and let δ be the constant in Theorem 3.1.1 with* (3.15). *Then there exists a constant $\varepsilon = \varepsilon(n, D) \in (0, \delta]$ such that if*

$$0 < (M + 1)a^{\frac{n-2}{n+1}} < \varepsilon,$$

(3.21) *admits a unique solution v within the class*

$$Y_0 := \{v \in BC([0,\infty); L^n_\sigma(D)) \mid t^{\frac{1}{2}}v \in BC((0,\infty); L^\infty(D)), t^{\frac{1}{2}}\nabla v \in BC((0,\infty); L^n(D)), \lim_{t\to 0} t^{\frac{1}{2}}\big(\|v(t)\|_\infty + \|\nabla v(t)\|_n\big) = 0\}. \tag{3.22}$$

Moreover, we have the following.

1. *(sharp decay) Let $n = 3$. Then there exists a constant $\varepsilon_* = \varepsilon_*(D) \in (0, \varepsilon]$ such that if $0 < (M + 1)a^{1/4} < \varepsilon_*$, the solution v enjoys decay properties*

$$\|v(t)\|_q = O(t^{-\frac{1}{2}+\frac{3}{2q}-\frac{\rho_1}{2}}), \qquad 3 \leq \forall q \leq \infty, \tag{3.23}$$

$$\|\nabla v(t)\|_3 = O(t^{-\frac{1}{2}-\frac{\rho_1}{2}}) \tag{3.24}$$

as $t \to \infty$.

Let $n \geq 4$ and suppose that $\rho_3 > 1$ and $1 < \rho_1 \leq 1 + \rho_3$ in addition to (3.16) *(the set of those parameters is nonvoid when $n \geq 4$). Then there exists a constant $\varepsilon_* = \varepsilon_*(n, D) \in (0, \varepsilon]$ such that if $0 < (M + 1)a^{(n-2)/(n+1)} < \varepsilon_*$, the solution v enjoys*

$$\|v(t)\|_q = O(t^{-\frac{1}{2}+\frac{n}{2q}-\frac{\rho_1}{2}}), \qquad n \leq \forall q \leq \infty, \tag{3.25}$$

$$\|\nabla v(t)\|_n = O(t^{-\frac{1}{2}-\frac{\rho_1}{2}}) \tag{3.26}$$

as $t \to \infty$.

2. *(Uniqueness) There exists a constant $\hat{\varepsilon} = \hat{\varepsilon}(n, D) \in (0, \varepsilon]$ such that if $0 < (M + 1)a^{(n-2)/(n+1)} < \hat{\varepsilon}$, the solution v obtained above is unique even within the class*

$$Y := \{v \in BC([0,\infty); L^n_\sigma(D)) \mid t^{\frac{1}{2}}v \in BC((0,\infty); L^\infty(D)), t^{\frac{1}{2}}\nabla v \in BC((0,\infty); L^n(D))\}. \tag{3.27}$$

For the sharp decay properties (3.23)–(3.26), the key step is to prove the L^n-decay of the solution, that is,

$$\|v(t)\|_n = O(t^{-\frac{\rho_1}{2}}) \tag{3.28}$$

as $t \to \infty$. Once we have (3.28), the other decay properties can be derived by the similar argument to [10]. Note that the condition $\rho_1 \leq 1 + \rho_3$ is always fulfilled and thus redundant for $n = 3$ since $\rho_1 < 1/2$ and $\rho_3 < 1/4$. On the other hand, it is enough for $n \geq 4$ to consider the case $\rho_1, \rho_3 > 1$. To prove (3.28), we first derive slower decay

$$\|v(t)\|_n = O(t^{-\frac{\rho}{2}})$$

with some $\rho \in (0, 1)$ by making use of $u_s \in L^{n/(1+\rho_1)}(D)$ and $\nabla u_s \in L^{n/(2+\rho_3)}(D)$, see Lemma 3.3.6 in Section 3. When $n = 3$, one can take $\rho := \min\{\rho_1, \rho_3\}$, yielding better decay properties of the other norms of the solution. With them at hand, we repeat improvement of the estimate of $\|v(t)\|_n$ step by step to find (3.28). However, this procedure does not work for $n \geq 4$ because of $\rho_1 > 1$. In order to get around the difficulty, our idea is to deduce the L^{q_0}-decay of the solution with some $q_0 < n$, that is appropriately chosen, see Lemma 3.3.8. We are then able to repeat improvement of estimates of several terms to arrive at (3.28), where the argument is more involved than the three-dimensional case above. Finally, to prove the uniqueness within Y, we employ the idea developed by Brezis [7], which shows that the solution $v \in Y$ necessarily satisfies the behavior as $t \to 0$ in (3.22).

In the next section we introduce the L^q-theory of the Oseen system and then prove Theorem 3.1.1. The final section is devoted to the proof of Theorem 3.1.2.

3.2 Proof of Theorem 3.1.1

In order to prove Theorem 3.1.1, we first recall the result on the Oseen boundary value problem due to Galdi [26, Theorem VII.7.1], see also Galdi [23] for the first proof of this result.

Proposition 3.2.1. *Let $n \geq 3$ and let $D \subset \mathbb{R}^n$ be an exterior domain with C^2 boundary. Suppose $a > 0$ and $1 < q < (n+1)/2$. Given $f \in L^q(D)$ and $u_* \in W^{2-1/q,q}(\partial D)$, problem*

$$\begin{cases} \Delta u - a\dfrac{\partial u}{\partial x_1} = \nabla p + f, & x \in D, \\ \nabla \cdot u = 0, & x \in D, \\ u|_{\partial D} = u_*, & \\ u \to 0 & \text{as } |x| \to \infty \end{cases} \tag{3.29}$$

admits a unique (up to an additive constant for p) solution (u,p) within the class

$$X_q(n) := \Big\{ (u,p) \in L^1_{\mathrm{loc}}(D) \,\Big|\, u \in L^{s_2}(D), \nabla u \in L^{s_1}(D),\ \nabla^2 u \in L^q(D), \\ \frac{\partial u}{\partial x_1} \in L^q(D),\ \nabla p \in L^q(D) \Big\},$$

where

$$\frac{1}{s_1} = \frac{1}{q} - \frac{1}{n+1}, \quad \frac{1}{s_2} = \frac{1}{q} - \frac{2}{n+1}. \tag{3.30}$$

Here, by $W^{2-1/q,q}(\partial D)$ we denote the trace space on ∂D from the Sobolev space $W^{2,q}(D)$ (see, for instance, [1] and [26]).

If, in particular, $a \in (0,1]$ and $q < n/2$, then the solution (u,p) obtained above satisfies

$$a^{\frac{2}{n+1}}\|u\|_{s_2} + a\left\|\frac{\partial u}{\partial x_1}\right\|_q + a^{\frac{1}{n+1}}\|\nabla u\|_{s_1} + \|\nabla^2 u\|_q + \|\nabla p\|_q \leq C\big(\|f\|_q + \|u_*\|_{W^{2-\frac{1}{q},q}(\partial D)}\big)$$

with a constant $C > 0$ dependent on q, n and D, however, independent of a.

For later use, we prepare the following lemma. The proof is essentially same as the one of Young's inequality for convolution, thus we omit it.

Lemma 3.2.2. *Let R_0, $d > 0$. Assume that $1 \le q, s \le \infty$ and $1/q + 1/s \ge 1$. Suppose $u \in L^q(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B_d := \{x \in \mathbb{R}^n \mid |x| < d\}$ and $\rho \in L^s(\mathbb{R}^n \setminus B_{R_0})$. Then for all $R \ge R_0 + d$, $\rho * u$ is well-defined as an element of $L^r(\mathbb{R}^n \setminus B_R)$ together with*

$$\|\rho * u\|_{L^r(\mathbb{R}^n\setminus B_R)} \le \|\rho\|_{L^s(\mathbb{R}^n\setminus B_{R_0})}\|u\|_{L^q(B_d)},$$

where $*$ denotes the convolution and $1/r := 1/q + 1/s - 1$.

When the external force f is taken from $L^{q_1}(D) \cap L^{q_2}(D)$ with $1 < q_1, q_2 < (n+1)/2$ and $q_1 \ne q_2$, we can apply Proposition 3.2.1 to $f \in L^{q_i}(D)$ $(i = 1, 2)$. The following tells us that the corresponding solutions coincide with each other.

Lemma 3.2.3. *Suppose $n \ge 3$, $1 < q_1, q_2 < (n+1)/2$ and $f \in L^{q_1}(D) \cap L^{q_2}(D)$. Let (u_i, p_i) be a unique solution obtained in Proposition 3.2.1 with $f \in L^{q_i}(D)$ and $u_* = -ae_1$. Then $u_1 = u_2$.*

Proof. We first show that $u_1 - u_2$ behaves like the Oseen fundamental solution $\mathbf{E}$ at large distances. We fix $R_0 > 0$ satisfying $\mathbb{R}^n \setminus D \subset B_{R_0}$. Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a cut-off function such that $\zeta(x) = 0$ for $|x| \le R_0$, $\zeta(x) = 1$ for $|x| \ge R_0 + 1$, and set

$$\begin{aligned} u(x) &:= u_1(x) - u_2(x), \qquad & p(x) &:= p_1(x) - p_2(x),\\ v(x) &:= \zeta(x)u(x) - \mathbb{B}[u \cdot \nabla\zeta], \qquad & \pi(x) &:= \zeta(x)p(x). \end{aligned}$$

Here, $\mathbb{B}$ is the Bogovskiĭ operator defined on the domain $B_{R_0+1} \setminus B_{R_0}$, see Bogovskiĭ [4], Borchers and Sohr [6] and Galdi [26]. Then we have

$$-\Delta v + a\frac{\partial v}{\partial x_1} + \nabla\pi = g(x), \quad \nabla\cdot v = 0 \quad \text{in } \mathscr{S}'(\mathbb{R}^n), \tag{3.31}$$

where $\mathscr{S}'(\mathbb{R}^n)$ is the set of tempered distributions on $\mathbb{R}^n$ and

$$g(x) = -(\Delta\zeta)u - 2(\nabla\zeta\cdot\nabla)u + a\frac{\partial\zeta}{\partial x_1}u + p\nabla\zeta + \Big(\Delta - a\frac{\partial}{\partial x_1}\Big)\mathbb{B}[u\cdot\nabla\zeta].$$

For (3.31) with $g = 0$, we have $\operatorname{supp}\hat{v} \subset \{0\}$ and $\operatorname{supp}\hat{\pi} \subset \{0\}$, where $\hat{(\cdot)}$ denotes the Fourier transform. We thus find

$$v(x) = \int_{\mathbb{R}^n} \mathbf{E}(x-y)g(y)\,dy + P(x), \qquad \pi(x) = C(n)\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}\cdot g(y)\,dy + Q(x)$$

with some polynomials $P(x)$, $Q(x)$ and some constant $C(n)$. In view of $v \in L^{(1/q_1 - 2/(n+1))^{-1}}(\mathbb{R}^n)$ $+L^{(1/q_2-2/(n+1))^{-1}}(\mathbb{R}^n)$ and $\nabla\pi \in L^{q_1}(\mathbb{R}^n) + L^{q_2}(\mathbb{R}^n)$, we have $P(x) = 0$ and $Q(x) = \overline{p}$. Here, $\overline{p}$ is some constant. Then Lemma 3.2.2 with

$$\rho = \mathbf{E},\ \nabla\mathbf{E},\ \frac{x-y}{|x-y|^n},$$

$u = g$, $d = R_0 + 1$, $q = 1$ and $r = s$ leads us to

$$u \in L^q(\mathbb{R}^n \setminus B_{2R_0+1}), \qquad \nabla u \in L^r(\mathbb{R}^n \setminus B_{2R_0+1}), \qquad p - \overline{p} \in L^s(\mathbb{R}^n \setminus B_{2R_0+1}) \tag{3.32}$$

for all $q > (n+1)/(n-1)$, $r > (n+1)/n$ and $s > n/(n-1)$, see (3.7).

Let $\varphi \in C^\infty[0,\infty)$ be a cut-off function such that $\varphi(t)=1$ for $t\le 1$, $\varphi(t)=0$ for $t\ge 2$, and set $\varphi_R(x)=\varphi(|x|/R)$ for $R\ge 2R_0+1$, $x\in\mathbb{R}^n$. We note that there exists a constant $C>0$ independent of R such that

$$\|\nabla\varphi_R\|_n \le C. \tag{3.33}$$

It follows from

$$-\Delta u + a\frac{\partial u}{\partial x_1} + \nabla p = 0, \quad \nabla\cdot u = 0 \quad \text{in } D, \quad u|_{\partial D}=0$$

that

$$\begin{aligned} 0 &= \int_D \Big\{ -\Delta u + a\frac{\partial u}{\partial x_1} + \nabla(p-\overline{p})\Big\}\cdot(\varphi_R u)\,dx \\ &= \int_D |\nabla u|^2\varphi_R\,dx + \int_{R\le|x|\le 2R}\Big\{(\nabla u\cdot\nabla\varphi_R)u - \frac{a}{2}\frac{\partial\varphi_R}{\partial x_1}|u|^2 - (p-\overline{p})\nabla\varphi_R\cdot u\Big\}\,dx. \end{aligned} \tag{3.34}$$

Since we can see

$$|\nabla u||u|,\ |u|^2,\ (p-\overline{p})|u| \in L^{n/(n-1)}\big(\mathbb{R}^n\setminus B_{2R_0+1}\big)$$

from (3.32), letting $R\to\infty$ in (3.34) yields $\|\nabla u\|_2^2=0$ because of (3.33). From this together with $u|_{\partial D}=0$, we conclude $u_1=u_2$. □

Proof of Theorem 3.1.1. Let $n\ge 3$ and let $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ satisfy (3.9). We first choose parameters (q_1,q_2,r_1,r_2) satisfying

$$\frac{n+1}{n-1} < \alpha_1 \le q_1 \le n+1 \le q_2 \le \alpha_2 < \frac{n(n+1)}{2}, \tag{3.35}$$

$$\frac{n+1}{n} < \beta_1 \le r_1 \le \frac{n+1}{2} \le r_2 \le \beta_2 < \frac{n(n+1)}{n+2}, \tag{3.36}$$

$$\max\left\{\frac{1}{\alpha_1}+\frac{2}{n+1},\ \frac{1}{\beta_1}+\frac{1}{n+1}\right\} \le \frac{1}{q_1}+\frac{1}{r_1} < 1, \tag{3.37}$$

$$\frac{2}{n} < \frac{1}{q_2}+\frac{1}{r_2} \le \min\left\{\frac{1}{\alpha_2}+\frac{2}{n+1},\ \frac{1}{\beta_2}+\frac{1}{n+1}\right\}. \tag{3.38}$$

It is actually possible to choose those parameters. In fact, we put

$$\alpha_1 = \frac{n+1}{n-1-\gamma_1}, \quad \alpha_2 = \frac{n(n+1)}{2+\gamma_2}, \quad \beta_1 = \frac{n+1}{n-\eta_1}, \quad \beta_2 = \frac{n(n+1)}{n+2+\eta_2}$$

with arbitrarily small $\gamma_i,\eta_i\in(0,n-2]$ and look for (q_1,q_2,r_1,r_2) of the form

$$q_1 = \frac{n+1}{n-1-\tilde{\gamma}_1}, \quad q_2 = \frac{n(n+1)}{2+\tilde{\gamma}_2}, \quad r_1 = \frac{n+1}{n-\tilde{\eta}_1}, \quad r_2 = \frac{n(n+1)}{n+2+\tilde{\eta}_2}.$$

Then the conditions (3.35)–(3.38) are accomplished by

$$\begin{aligned} &n-2 < \tilde{\gamma}_1+\tilde{\eta}_1 \le n-2+\min\{\gamma_1,\eta_1\}, \qquad n-2 < \tilde{\gamma}_2+\tilde{\eta}_2 \le n-2+\min\{\gamma_2,\eta_2\}, \\ &\gamma_i \le \tilde{\gamma}_i, \qquad \eta_i \le \tilde{\eta}_i, \quad i=1,2. \end{aligned}$$

For each $i = 1, 2$, the set of $(\tilde{\gamma}_i, \tilde{\eta}_i)$ with those conditions is nonvoid for given γ_i and η_i; for instance, we may take $\tilde{\gamma}_i = \gamma_i$, $\tilde{\eta}_i = n-2$ when $\gamma_i \le \eta_i$ and take $\tilde{\gamma}_i = n-2$, $\tilde{\eta}_i = \eta_i$ when $\gamma_i \ge \eta_i$.

To obtain a small solution, we use the contraction mapping principle. We define

$$B := \{u \in L^{\alpha_1}(D) \cap L^{\alpha_2}(D) \mid \nabla u \in L^{\beta_1}(D) \cap L^{\beta_2}(D)\}$$

which is a Banach space endowed with the norm

$$\|u\|_B := \sum_{i=1}^{2} (a^{\frac{2}{n+1}} \|u\|_{\alpha_i} + a^{\frac{1}{n+1}} \|\nabla u\|_{\beta_i}).$$

Given $v \in B$, which satisfies

$$v \cdot \nabla v \in \bigcap_{i=1}^{2} L^{\kappa_i}(D), \qquad \frac{1}{\kappa_i} = \frac{1}{q_i} + \frac{1}{r_i}, \quad 1 < \kappa_i < \frac{n}{2}$$

for $i = 1, 2$, we can employ Proposition 3.2.1 with $f = v \cdot \nabla v$, $q = \kappa_i$ $(i = 1, 2)$ and $u_* = -ae_1$. Then, due to Lemma 3.2.3, the problem (3.11) admits a unique solution (u, p) such that

$$\begin{aligned} a^{\frac{2}{n+1}} \|u\|_{\mu_i} + a \left\| \frac{\partial u}{\partial x_1} \right\|_{\kappa_i} + a^{\frac{1}{n+1}} \|\nabla u\|_{\lambda_i} + \|\nabla^2 u\|_{\kappa_i} + \|\nabla p\|_{\kappa_i} \\ \le C'(\|v \cdot \nabla v\|_{\kappa_i} + a) \le C'(\|v\|_{q_i} \|\nabla v\|_{r_i} + a) \le C'(a^{-\frac{3}{n+1}} \|v\|_B^2 + a) \end{aligned}$$

for $i = 1, 2$. Here, $1/\lambda_i = 1/\kappa_i - 1/(n+1)$, $1/\mu_i = 1/\kappa_i - 2/(n+1)$. Furthermore, because the conditions (3.37) and (3.38) ensure $\mu_1 \le \alpha_1 \le \alpha_2 \le \mu_2$ and $\lambda_1 \le \beta_1 \le \beta_2 \le \lambda_2$, we find $u \in B$ with

$$\|u\|_B \le 4C'(a^{-\frac{3}{n+1}} \|v\|_B^2 + a).$$

Hence, we assume

$$a^{\frac{n-2}{n+1}} < \frac{1}{64C'^2} =: \delta \tag{3.39}$$

and set

$$N_a := \{u \in B \mid \|u\|_B \le 8C'a\}$$

to see that the map $\Psi : N_a \ni v \mapsto u \in N_a$ is well-defined. Moreover, for $v_i \in N_a$ $(i = 1, 2)$, set $u_i = \Psi(v_i)$ and let p_i be the pressure associated with u_i. Then we have

$$\begin{cases} \Delta(u_1 - u_2) - a \dfrac{\partial}{\partial x_1}(u_1 - u_2) = \nabla(p_1 - p_2) + (v_1 - v_2) \cdot \nabla v_1 + v_2 \cdot \nabla(v_2 - v_1), & x \in D, \\ \nabla \cdot (u_1 - u_2) = 0, \quad x \in D, \\ (u_1 - u_2)|_{\partial D} = 0, \\ u_1 - u_2 \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

By applying Proposition 3.2.1 again, we find

$$\|u_1 - u_2\|_B \le 4C'a^{-\frac{3}{n+1}} (\|v_1\|_B + \|v_2\|_B) \|v_1 - v_2\|_B \le 64C'^2 a^{\frac{n-2}{n+1}} \|v_1 - v_2\|_B$$

and the map Ψ is contractive on account of (3.39). The proof is complete. □

3.3 Proof of Theorem 3.1.2

In this section, we prove Theorem 3.1.2. We define the operator $A_a : L^q_\sigma(D) \to L^q_\sigma(D)$ $(a > 0, 1 < q < \infty)$ by

$$\mathscr{D}(A_a) = W^{2,q}(D) \cap W^{1,q}_0(D) \cap L^q_\sigma(D), \quad A_a u = -P\left[\Delta u - a\frac{\partial u}{\partial x_1}\right].$$

It is well known that $-A_a$ generates an analytic C_0-semigroup e^{-tA_a} called the Oseen semigroup in $L^q_\sigma(D)$, see Miyakawa [44, Theorem 4.2], Enomoto and Shibata [9, Theorem 4.4]. The following L^q-L^r estimates of e^{-tA_a}, which play an important role in the proof of Theorem 3.1.2, were established by Kobayashi and Shibata [39] in the three-dimensional case and further developed by Enomoto and Shibata [9,10] for $n \geq 3$. We also note that L^q-L^r estimates in the two-dimensional case were first established by Hishida [32], and recently Maekawa [42] derived those estimates uniformly in small $a > 0$ as a significant improvement of [32].

Proposition 3.3.1 ([9,10,39]). *Let $n \geq 3$, $\sigma_0 > 0$ and assume $|a| \leq \sigma_0$.*

1. *Let $1 < q \leq r \leq \infty$ $(q \neq \infty)$. Then we have*

$$\|e^{-tA_a} f\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_q \tag{3.40}$$

   for $t > 0$ and $f \in L^q_\sigma(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.

2. *Let $1 < q \leq r \leq n$. Then we have*

$$\|\nabla e^{-tA_a} f\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|f\|_q \tag{3.41}$$

   for $t > 0$ and $f \in L^q_\sigma(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.

3. *Let $n/(n-1) \leq q \leq r \leq \infty$ $(q \neq \infty)$. Then we have*

$$\|e^{-tA_a} P\nabla \cdot F\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|F\|_q \tag{3.42}$$

   for $t > 0$ and $F \in L^q(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.

The proof of the assertion 3 is simply based on duality argument together with semigroup property especially for the case $r = \infty$.

We also prepare the following lemma, which plays a role to prove the uniqueness within Y defined by (3.27).

Lemma 3.3.2. *Let $n \geq 3$ and $a > 0$. For each precompact set $K \subset L^n_\sigma(D)$, we have*

$$\lim_{t\to 0}\sup_{f\in K} t^{\frac{1}{2}}\left(\|e^{-tA_a} f\|_\infty + \|\nabla e^{-tA_a} f\|_n\right) = 0. \tag{3.43}$$

Proof. By applying Proposition 3.3.1 and approximating $f \in L^n_\sigma(D)$ by a sequence in $C^\infty_{0,\sigma}(D)$, we have

$$\lim_{t\to 0} t^{\frac{1}{2}}\left(\|e^{-tA_a} f\|_\infty + \|\nabla e^{-tA_a} f\|_n\right) = 0 \tag{3.44}$$

for all $f \in L^n_\sigma(D)$. Given $\eta > 0$, let $f_1, \cdots, f_m \in K$ fulfill $K \subset \bigcup_{j=1}^m B(f_j;\eta)$, where $B(f_j;\eta) := \{g \in L^n_\sigma(D) \mid \|g - f_j\|_n < \eta\}$. For each $f \in K$, we choose $f_i \in K$ such that $f \in B(f_i;\eta)$. Then it follows from (4.24) that

$$\begin{aligned} t^{\frac{1}{2}}\|e^{-tA_a}f\|_\infty &\le t^{\frac{1}{2}}\|e^{-tA_a}f_i\|_\infty + t^{\frac{1}{2}}\|e^{-tA_a}(f-f_i)\|_\infty \\ &\le t^{\frac{1}{2}}\|e^{-tA_a}f_i\|_\infty + C\|f-f_i\|_n \le \sum_{j=1}^m t^{\frac{1}{2}}\|e^{-tA_a}f_j\|_\infty + C\eta. \end{aligned}$$

Since the right-hand side is independent of $f \in K$ and since η is arbitrary, (3.44) yields

$$\lim_{t\to 0}\sup_{f\in K} t^{\frac{1}{2}}\|e^{-tA_a}f\|_\infty = 0.$$

We can discuss the L^n norm of the first derivative similarly and thus conclude (3.43). □

We recall a function space Y_0 defined by (3.22), which is a Banach space equipped with norm $\|\cdot\|_Y = \|\cdot\|_{Y,\infty}$, where

$$\begin{aligned} &\|v\|_{Y,t} := [v]_{n,t} + [v]_{\infty,t} + [\nabla v]_{n,t}, \\ &[v]_{q,t} := \sup_{0<\tau<t} \tau^{\frac{1}{2}-\frac{n}{2q}}\|v(\tau)\|_q, \quad q = n, \infty; \qquad [\nabla v]_{n,t} := \sup_{0<\tau<t} \tau^{\frac{1}{2}}\|\nabla v(\tau)\|_n \end{aligned}$$

for $t \in (0,\infty]$. Construction of the solution is based on the following.

Lemma 3.3.3. *Suppose $0 < a^{(n-2)/(n+1)} < \delta$, where δ is a constant in Theorem 3.1.1 with* (3.15)–(3.17). *Let ψ be a function on $\mathbb{R}$ satisfying* (3.1) *and set $M = \max_{t\in\mathbb{R}} |\psi'(t)|$. Suppose that u_s is the stationary solution obtained in Theorem 3.1.1. For $u, v \in Y_0$, we set*

$$\begin{aligned} &G_1(u,v)(t) = \int_0^t e^{-(t-\tau)A_a}P[u\cdot\nabla v](\tau)\,d\tau, \quad G_2(v)(t) = \int_0^t e^{-(t-\tau)A_a}P[\psi(\tau)v\cdot\nabla u_s]\,d\tau, \\ &G_3(v)(t) = \int_0^t e^{-(t-\tau)A_a}P[\psi(\tau)u_s\cdot\nabla v]\,d\tau, \\ &G_4(v)(t) = \int_0^t e^{-(t-\tau)A_a}P\left[(1-\psi(\tau))a\frac{\partial v}{\partial x_1}(\tau)\right]d\tau, \\ &H_1(t) = \int_0^t e^{-(t-\tau)A_a}Ph_1(\tau)\,d\tau, \quad H_2(t) = \int_0^t e^{-(t-\tau)A_a}Ph_2(\tau)\,d\tau, \end{aligned}$$

where h_1 and h_2 are defined by (3.19) *and* (3.20)*, respectively. Then we have $G_1(u,v), G_i(v), H_j \in Y_0$ $(i = 2,3,4, j = 1,2)$ along with*

$$\|G_1(u,v)\|_{Y,t} \le C[u]_{n,t}^{\frac{1}{2}}[u]_{\infty,t}^{\frac{1}{2}}[\nabla v]_{n,t}, \tag{3.45}$$
$$\|G_2(v)\|_{Y,t} \le C\big(\|\nabla u_s\|_{\frac{n}{2+\rho_3}} + \|\nabla u_s\|_{\frac{n}{2}} + \|\nabla u_s\|_{\frac{n}{2-\rho_4}}\big)[v]_{\infty,t}, \tag{3.46}$$
$$\|G_3(v)\|_{Y,t} \le C\big(\|u_s\|_{\frac{n}{1+\rho_1}} + \|u_s\|_n + \|u_s\|_{\frac{n}{1-\rho_2}}\big)[\nabla v]_{n,t}, \tag{3.47}$$
$$\|G_4(v)\|_{Y,t} \le Ca[\nabla v]_{n,t}, \tag{3.48}$$
$$\|H_1\|_{Y,t} \le CM\|u_s\|_n, \tag{3.49}$$
$$\|H_2\|_{Y,t} \le C\big(\|u_s\|_{\frac{n}{1-\rho_2}}\|\nabla u_s\|_{\frac{n}{2-\rho_4}} + a\|\nabla u_s\|_{\frac{n}{2-\rho_4}}\big) \tag{3.50}$$

for all $t \in (0, \infty]$ and

$$\lim_{t \to 0} \|H_j(t)\|_{Y,t} = 0 \tag{3.51}$$

for $j = 1, 2$. Here, C is a positive constant independent of u, v, ψ, a and t.

Proof. The continuity of those functions in t is deduced by use of properties of analytic semigroups together with Proposition 3.3.1 in the same way as in Fujita and Kato [21]. Since L^∞ estimate is always the same as L^n estimate of the first derivative, the estimate of $[\cdot]_{\infty,t}$ may be omitted. Although (3.45)–(3.47) are discussed in Enomoto and Shibata [10, Lemma 3.1.] we briefly give the proof for completeness. We find that $u \in Y_0$ satisfies $u(t) \in L^{2n}(D)$ and

$$\|u(t)\|_{2n} \le t^{-\frac{1}{4}} [u]_{n,t}^{\frac{1}{2}} [u]_{\infty,t}^{\frac{1}{2}}$$

for all $t > 0$, which together with Proposition 3.3.1 implies

$$\int_0^t \|e^{-(t-\tau)A_a} P[u \cdot \nabla v](\tau)\|_n \, d\tau \le C \int_0^t (t-\tau)^{-\frac{1}{4}} \|u(\tau)\|_{2n} \|\nabla v(\tau)\|_n \, d\tau \le C [u]_{n,t}^{\frac{1}{2}} [u]_{\infty,t}^{\frac{1}{2}} [\nabla v]_{n,t}$$

and

$$\begin{aligned} \int_0^t \|\nabla e^{-(t-\tau)A_a} P[u \cdot \nabla v](\tau)\|_n \, d\tau &\le C \int_0^t (t-\tau)^{-\frac{3}{4}} \|u(\tau)\|_{2n} \|\nabla v(\tau)\|_n \, d\tau \\ &\le C t^{-\frac{1}{2}} [u]_{n,t}^{\frac{1}{2}} [u]_{\infty,t}^{\frac{1}{2}} [\nabla v]_{n,t}. \end{aligned}$$

We thus conclude (3.45). It follows from Proposition 3.3.1 that

$$\int_0^t \|e^{-(t-\tau)A_a} P[\psi(\tau) v \cdot \nabla u_s]\|_n \, d\tau \le C \int_0^t (t-\tau)^{-\frac{1}{2}} \|v(\tau)\|_\infty \|\nabla u_s\|_{\frac{n}{2}} \, d\tau \le C [v]_{\infty,t} \|\nabla u_s\|_{\frac{n}{2}} \tag{3.52}$$

and that

$$\begin{aligned} \int_0^t \|\nabla e^{-(t-\tau)A_a} P[\psi(\tau) v \cdot \nabla u_s]\|_n \, d\tau &\le C \int_0^t (t-\tau)^{-1+\frac{\rho_4}{2}} \|v(\tau)\|_\infty \|\nabla u_s\|_{\frac{n}{2-\rho_4}} \, d\tau \\ &\le C t^{-\frac{1}{2}+\frac{\rho_4}{2}} [v]_{\infty,t} \|\nabla u_s\|_{\frac{n}{2-\rho_4}} \end{aligned} \tag{3.53}$$

for $t > 0$. Furthermore, for $t \ge 2$, we split the integral into

$$\int_0^t \|\nabla e^{-(t-\tau)A_a} P[\psi(\tau) v \cdot \nabla u_s]\|_n \, d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t \tag{3.54}$$

as in [8] and [10]. By applying (3.41), we have

$$\int_0^{\frac{t}{2}} \le C \int_0^{\frac{t}{2}} (t-\tau)^{-1} \|v(\tau)\|_\infty \|\nabla u_s\|_{\frac{n}{2}} \, d\tau \le C t^{-\frac{1}{2}} [v]_{\infty,t} \|\nabla u_s\|_{\frac{n}{2}}, \tag{3.55}$$

$$\int_{\frac{t}{2}}^{t-1} \le C \int_{\frac{t}{2}}^{t-1} (t-\tau)^{-1-\frac{\rho_3}{2}} \|v(\tau)\|_\infty \|\nabla u_s\|_{\frac{n}{2+\rho_3}} \, d\tau \le C t^{-\frac{1}{2}} [v]_{\infty,t} \|\nabla u_s\|_{\frac{n}{2+\rho_3}}, \tag{3.56}$$

$$\int_{t-1}^t \le C \int_{t-1}^t (t-\tau)^{-1+\frac{\rho_4}{2}} \|v(\tau)\|_\infty \|\nabla u_s\|_{\frac{n}{2-\rho_4}} \, d\tau \le C t^{-\frac{1}{2}} [v]_{\infty,t} \|\nabla u_s\|_{\frac{n}{2-\rho_4}}. \tag{3.57}$$

Combining (3.52)–(3.57) yields (3.46). By the same manner, we obtain (3.47). We use Proposition 3.3.1 to find

$$\begin{aligned}\int_0^t \left\|\nabla^k e^{-(t-\tau)A_a} P\left[\big(1-\psi(\tau)\big)a\frac{\partial v}{\partial x_1}\right]\right\|_n d\tau &\le Ca\int_0^{\min\{1,t\}} (t-\tau)^{-\frac{k}{2}}\|\nabla v(\tau)\|_n\, d\tau \\ &\le Ca[\nabla v]_{n,t}\int_0^{\min\{1,t\}} (t-\tau)^{-\frac{k}{2}}\tau^{-\frac{1}{2}}\, d\tau\end{aligned}$$

for $k=0,1$, which lead us to (3.48). We see (3.49) from

$$\int_0^t \left\|\nabla^k e^{-(t-\tau)A_a} P[\psi'(\tau)u_s]\right\|_n d\tau \le CM\|u_s\|_n \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{k}{2}}\, d\tau \tag{3.58}$$

for $k=0,1$ and (3.50) from

$$\begin{aligned}\int_0^t &\left\|\nabla^k e^{-(t-\tau)A_a} P\left[\psi(\tau)(1-\psi(\tau))\Big(u_s\cdot\nabla u_s + a\frac{\partial u_s}{\partial x_1}\Big)\right]\right\|_n d\tau \\ &\le C\|u_s\|_{\frac{n}{1-\rho_2}}\|\nabla u_s\|_{\frac{n}{2-\rho_4}}\int_0^{\min\{1,t\}} (t-\tau)^{\frac{\rho_2+\rho_4}{2}-1-\frac{k}{2}}\, d\tau \\ &\quad + Ca\|\nabla u_s\|_{\frac{n}{2-\rho_4}}\int_0^{\min\{1,t\}} (t-\tau)^{-\frac{1}{2}+\frac{\rho_4}{2}-\frac{k}{2}}\, d\tau\end{aligned} \tag{3.59}$$

for $k=0,1$, where the condition (3.17) is used. The behavior of $G_1(u,v)(t)$ and $G_i(v)(t)$ as well as the one of $H_j(t)$, see (3.51), as $t\to 0$ follows from (3.45)–(3.48) and (3.58)–(3.59) with $t<1$, so that $G_1(u,v), G_i(v), H_j \in Y_0$ and $\|G_1(u,v)(t)\|_n + \|G_i(v)(t)\|_n + \|H_j(t)\|_n \to 0$ as $t\to 0$. The proof is complete. □

Let us construct a solution of (3.21) by applying Lemma 3.3.3.

Proposition 3.3.4. *Let δ be the constant in Theorem 3.1.1 with* (3.15)–(3.17). *Let ψ be a function on $\mathbb{R}$ satisfying* (3.1) *and set $M = \max_{t\in\mathbb{R}}|\psi'(t)|$. Then there exists a constant $\varepsilon=\varepsilon(n,D)\in(0,\delta]$ such that if $0<(M+1)a^{(n-2)/(n+1)}<\varepsilon$,* (3.21) *admits a solution $v\in Y_0$ with*

$$\|v\|_Y \le C(M+1)a^{\frac{n-2}{n+1}} \tag{3.60}$$

and

$$\lim_{t\to 0}\|v(t)\|_n = 0. \tag{3.61}$$

Proof. We set

$$\begin{aligned}v_0(t) &= 0,\\ v_{m+1}(t) &= \int_0^t e^{-(t-\tau)A_a}P\Big[-v_m\cdot\nabla v_m - \psi(\tau)v_m\cdot\nabla u_s - \psi(\tau)u_s\cdot\nabla v_m + (1-\psi(\tau))a\frac{\partial v_m}{\partial x_1} \\ &\qquad + h_1(\tau)+h_2(\tau)\Big]d\tau\end{aligned} \tag{3.62}$$

for $m \geq 0$. It follows from Theorem 3.1.1, Lemma 3.3.3 and $a \in (0,1)$ that $v_m \in Y_0$ together with

$$\|v_m\|_{Y,t} \leq \|G_1(v_{m-1}, v_{m-1})\|_{Y,t} + \sum_{i=2}^{4} \|G_i(v_{m-1})\|_{Y,t} + \|H_1\|_{Y,t} + \|H_2\|_{Y,t}, \tag{3.63}$$

$$\|v_m\|_Y \leq C_1\|v_{m-1}\|_Y^2 + C_2 a^{\frac{n-2}{n+1}}\|v_{m-1}\|_Y + C_3(M+1)a^{\frac{n-2}{n+1}}, \tag{3.64}$$

$$\|v_{m+1} - v_m\|_Y \leq \{C_1(\|v_m\|_Y + \|v_{m-1}\|_Y) + C_2 a^{\frac{n-2}{n+1}}\}\|v_m - v_{m-1}\|_Y$$

for all $m \geq 1$. Hence, if we assume

$$(M+1)a^{\frac{n-2}{n+1}} < \min\left\{\delta, \frac{1}{2C_2}, \frac{1}{16C_1C_3}\right\} =: \varepsilon, \tag{3.65}$$

it holds that

$$\|v_m\|_Y \leq \frac{1 - C_2 a^{\frac{n-2}{n+1}} - \sqrt{(1 - C_2 a^{\frac{n-2}{n+1}})^2 - 4C_1C_3(M+1)a^{\frac{n-2}{n+1}}}}{2C_1} \leq 4C_3(M+1)a^{\frac{n-2}{n+1}}, \tag{3.66}$$

$$\|v_{m+1} - v_m\|_Y \leq \{8C_1C_3(M+1)a^{\frac{n-2}{n+1}} + C_2 a^{\frac{n-2}{n+1}}\}\|v_m - v_{m-1}\|_Y$$

for all $m \geq 1$ and that

$$8C_1C_3(M+1)a^{\frac{n-2}{n+1}} + C_2 a^{\frac{n-2}{n+1}} < 1.$$

Therefore, we obtain a solution $v \in Y_0$ satisfying (3.60) with $C = 4C_3$. Moreover, by letting $m \to \infty$ in (3.63) and by using (3.45)–(3.48) and (3.51), we have (3.61), which completes the proof. □

Remark 3.3.5. Let $b \in L^n_\sigma(D)$. By the same procedure, we can also construct a solution $T(t)b := v(t) \in Y_0$ for the integral equation

$$v(t) = e^{-tA_a}b + \int_0^t e^{-(t-\tau)A_a} P\Big[- v \cdot \nabla v - \psi(\tau)v \cdot \nabla u_s - \psi(\tau)u_s \cdot \nabla v + (1-\psi(\tau))a\frac{\partial v}{\partial x_1} + h_1(\tau) + h_2(\tau)\Big] d\tau \tag{3.67}$$

whenever

$$\|b\|_n + (M+1)a^{\frac{n-2}{n+1}} < \min\left\{\delta, \frac{1}{2C_2}, \frac{1}{16C_1C_0}, \frac{1}{16C_1C_3}\right\}$$

is satisfied. Here, the constant C_0 is determined by the following three estimates:

$$\|e^{-tA_a}b\|_q \leq C_0 t^{-\frac{1}{2}+\frac{3}{2q}}\|b\|_n, \qquad q = n, \infty; \qquad \|\nabla e^{-tA_a}b\|_n \leq C_0 t^{-\frac{1}{2}}\|b\|_n.$$

Moreover, we find that the solution $T(t)b$ is estimated by

$$\|T(\cdot)b\|_Y \leq 4\big(C_0\|b\|_n + C_3(M+1)a^{\frac{n-2}{n+1}}\big).$$

This will be used in the proof of uniqueness of solutions within Y, see (3.27).

We further derive sharp decay properties of the solution $v(t)$ obtained above. To this end, the first step is the following. In what follows, for simplicity of notation, we write

$$G_1(t) = G_1(v,v)(t), \quad G_i(t) = G_i(v)(t)$$

for $i = 2, 3, 4$, which are defined in Lemma 3.3.3.

Lemma 3.3.6. *Let ε be the constant in Proposition 3.3.4. Given $\rho \in (0,1)$ satisfying $\rho \le \min\{\rho_1, \rho_3\}$, there exists a constant $\varepsilon' = \varepsilon'(\rho, n, D) \in (0, \varepsilon]$ such that if $0 < (M+1)a^{(n-2)/(n+1)} < \varepsilon'$, then the solution $v(t)$ obtained in Proposition 3.3.4 satisfies*

$$\|v(t)\|_q = O(t^{-\frac{1}{2}+\frac{n}{2q}-\frac{\rho}{2}}), \qquad n \le \forall q \le \infty, \tag{3.68}$$

$$\|\nabla v(t)\|_n = O(t^{-\frac{1}{2}-\frac{\rho}{2}}) \tag{3.69}$$

as $t \to \infty$.

Proof. We start with the case $q = n$, that is,

$$\|v(t)\|_n = O(t^{-\frac{\rho}{2}}) \tag{3.70}$$

as $t \to \infty$. By using (3.40), we have

$$\|G_1(t)\|_n \le Ct^{-\frac{\rho}{2}}\big(\sup_{0<\tau<t} \tau^{\frac{1}{2}}\|\nabla v(\tau)\|_n\big)\big(\sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n\big) \le Ct^{-\frac{\rho}{2}}\|v\|_Y \sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n, \tag{3.71}$$

$$\|G_2(t)\|_n \le Ct^{-\frac{\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}} \sup_{0<\tau<t} \tau^{\frac{1}{2}}\|v(\tau)\|_\infty \le Ct^{-\frac{\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}}\|v\|_Y \tag{3.72}$$

and

$$\|G_3(t)\|_n \le Ct^{-\frac{\rho_1}{2}}\|u_s\|_{\frac{n}{1+\rho_1}}\|v\|_Y \tag{3.73}$$

for all $t > 0$. Moreover, we obtain

$$\|G_4(t)\|_n \le Ca\int_0^{\min\{1,t\}} (t-\tau)^{-\frac{1}{2}}\|v(\tau)\|_n\, d\tau \le Cat^{-\frac{1}{2}}\|v\|_Y \tag{3.74}$$

for all $t > 0$ by use of (3.42). From (3.40) we see that

$$\|H_1(t)\|_n \le CMt^{-\frac{\rho_1}{2}}\|u_s\|_{\frac{n}{1+\rho_1}} \tag{3.75}$$

and that

$$\|H_2(t)\|_n \le Ct^{-\frac{2-\rho_2}{2}}\|u_s\|_{\frac{n}{1-\rho_2}}\|\nabla u_s\|_{\frac{n}{2}} + Cat^{-\frac{1+\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}} \tag{3.76}$$

for $t > 0$. Note that $\rho_2 < 1$, see (3.16). Collecting (3.71)–(3.76) for $t > 1$ and (3.60) with $C = 4C_3$ yields

$$\begin{aligned}\sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n &\le C_4\|v\|_Y \sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n + C_5 \\ &\le 4C_3C_4(M+1)a^{\frac{n-2}{n+1}} \sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n + C_5\end{aligned}$$

with some constants $C_4 = C_4(\rho) > 0$ and $C_5 = C_5(\|v\|_Y, u_s, a, M, \rho_1, \rho_2, \rho_3) > 0$ independent of t, where C_3 comes from estimates of $H_j(t)$ $(j = 1, 2)$ in (3.64). Therefore, if we assume

$$(M+1)a^{\frac{n-2}{n+1}} < \min\left\{\varepsilon, \frac{1}{4C_3C_4}\right\} =: \varepsilon',$$

we have $\|v(t)\|_n \le Ct^{-\rho/2}$ for all $t > 0$, which implies (3.70).

We next show that

$$\|v(t)\|_\infty + \|\nabla v(t)\|_n = O(t^{-\frac{1}{2}-\frac{\rho}{2}})$$

as $t \to \infty$, which together with (3.70) implies (3.68) and (3.69). It suffices to show that

$$t^{\frac{1}{2}}\|v(t)\|_\infty + t^{\frac{1}{2}}\|\nabla v(t)\|_n \le C\left\|v\left(\frac{t}{2}\right)\right\|_n \tag{3.77}$$

for all $t \ge 2$. The following argument is similar to Enomoto and Shibata [10]. When $t \ge T > 1$, we have

$$v(t) = e^{-(t-T)A_a}v(T) - \int_T^t e^{-(t-\tau)A_a}P\big[v\cdot\nabla v + v\cdot\nabla u_s + u_s\cdot\nabla v\big]\,d\tau. \tag{3.78}$$

By the same argument as in the proof of Lemma 3.3.3 and by (3.10), (3.65) as well as (3.60) with $C = 4C_3$, the integral of (3.78) is estimated as

$$\begin{aligned}
&\int_T^t \|e^{-(t-\tau)A_a}P[\cdots]\|_\infty\,d\tau + \int_T^t \|\nabla e^{-(t-\tau)A_a}P[\cdots]\|_n\,d\tau \\
&\quad\le C_1(t-T)^{-\frac{1}{2}}\big(\sup_{T\le\tau\le t}\|v(\tau)\|_n\big)^{\frac{1}{2}}\big(\sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|v(\tau)\|_\infty\big)^{\frac{1}{2}}\big(\sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|\nabla v(\tau)\|_n\big) \\
&\qquad\qquad + C_2a^{\frac{n-2}{n+1}}(t-T)^{-\frac{1}{2}}\big\{\sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|v(\tau)\|_\infty + \sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|\nabla v(\tau)\|_n\big\} \\
&\quad\le C_1(t-T)^{-\frac{1}{2}}\|v\|_Y\sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|\nabla v(\tau)\|_n \\
&\qquad\qquad + \frac{1}{2}(t-T)^{-\frac{1}{2}}\big\{\sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|v(\tau)\|_\infty + \sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|\nabla v(\tau)\|_n\big\} \\
&\quad\le \frac{3}{4}(t-T)^{-\frac{1}{2}}\sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|\nabla v(\tau)\|_n + \frac{1}{2}(t-T)^{-\frac{1}{2}}\sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|v(\tau)\|_\infty.
\end{aligned}$$

Therefore, we have

$$\sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|\nabla v(\tau)\|_n + \sup_{T\le\tau\le t}(\tau-T)^{\frac{1}{2}}\|v(\tau)\|_\infty \le C\|v(T)\|_n$$

for all $t \ge T$. This combined with $t^{1/2} \le \sqrt{2}(t-T)^{1/2}$ for $t \ge 2T$ asserts that

$$t^{\frac{1}{2}}\|\nabla v(t)\|_n + t^{\frac{1}{2}}\|v(t)\|_\infty \le C\|v(T)\|_n$$

for all $t \ge 2T$. We then put $T = t/2$ $(t \ge 2)$ to conclude (3.77). □

Sharp decay properties (3.23)–(3.24) for the case $n = 3$ are established in the following proposition.

Proposition 3.3.7. *Let $n = 3$ and set $\varepsilon_* := \varepsilon'(\rho, 3, D)$ which is the constant in Lemma 3.3.6 with $\rho := \min\{\rho_1, \rho_3\}$ (recall that $0 < \rho_1 < 1/2$, $0 < \rho_3 < 1/4$ for $n = 3$). If $0 < (M+1)a^{1/4} < \varepsilon_*$, then the solution $v(t)$ obtained in Proposition 3.3.4 enjoys* (3.23) *and* (3.24).

Proof. The case $\rho_1 \le \rho_3$ directly follows from Lemma 3.3.6. To discuss the other case $\rho_3 < \rho_1$, we show by induction that if $0 < (M+1)a^{1/4} < \varepsilon_*$, then

$$\|v(t)\|_3 = O(t^{-\sigma_k}), \qquad \sigma_k := \min\left\{\frac{k}{2}\rho_3,\ \frac{\rho_1}{2}\right\} \tag{3.79}$$

as $t \to \infty$ for all $k \ge 1$. We already know (3.79) with $k = 1$ from Lemma 3.3.6.

Let $k \ge 2$ and suppose (3.79) with $k-1$. By taking (3.60) (near $t = 0$) and (3.77) into account, we have

$$J_{k-1}(v) := \sup_{\tau>0}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_3 + \sup_{\tau>0}\tau^{\frac{1}{2}}(1+\tau)^{\sigma_{k-1}}\big(\|v(\tau)\|_\infty + \|\nabla v(\tau)\|_3\big) < \infty.$$

We use this to see that

$$\begin{aligned}\|G_1(t)\|_3 &\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}(1+\tau)^{-2\sigma_{k-1}}\,d\tau \\ &\qquad\qquad \times \big(\sup_{\tau>0}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_3\big)\big(\sup_{\tau>0}\tau^{\frac{1}{2}}(1+\tau)^{\sigma_{k-1}}\|\nabla v(\tau)\|_3\big) \\ &\le Ct^{-2\sigma_{k-1}}J_{k-1}(v)^2,\end{aligned}$$

and that

$$\begin{aligned}\|G_2(t)\|_3 &\le C\int_0^t (t-\tau)^{-\frac{1+\rho_3}{2}}\tau^{-\frac{1}{2}}(1+\tau)^{-\sigma_{k-1}}\,d\tau\,\|\nabla u_s\|_{\frac{3}{2+\rho_3}}\sup_{\tau>0}\tau^{\frac{1}{2}}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_\infty \\ &\le Ct^{-\frac{\rho_3}{2}-\sigma_{k-1}}\|\nabla u_s\|_{\frac{3}{2+\rho_3}}J_{k-1}(v)\end{aligned}$$

for $t > 0$ due to $\sigma_{k-1} \le \rho_1/2 < 1/4$. From these and (3.73)–(3.76), we obtain (3.79) with k. We thus conclude (3.23) with $q = 3$, which together with (3.77) completes the proof. □

To derive even more rapid decay properties of the solution $v(t)$ for $n \ge 4$, we need the following lemma, which gives the L^{q_0}-decay of $v(t)$ with a specific q_0, see (3.82) below.

Lemma 3.3.8. *Let $n \ge 4$. Suppose $1 < \rho_1 \le 1 + \rho_3$ in addition to* (3.16) *(the set of those parameters is nonvoid when $n \ge 4$). Let ε be the constant in Proposition 3.3.4 and $v(t)$ the solution obtained there. Given γ satisfying*

$$\max\left\{0,\ \frac{\rho_1 + 3 - n}{2}\right\} < \gamma < \frac{1}{2} \tag{3.80}$$

(note that (3.16) *yields $\rho_1 < n-2$), there exists a constant $\varepsilon'' = \varepsilon''(\gamma, n, D) \in (0, \varepsilon]$ such that if $0 < (M+1)a^{(n-2)/(n+1)} < \varepsilon''$, then $v(t) \in L^{q_0}(D)$ for all $t > 0$ and*

$$\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0} < \infty, \tag{3.81}$$

where

$$q_0 := \frac{n}{1+\rho_1-2\gamma}\,(<n). \tag{3.82}$$

Proof. We show that there exists a constant $\varepsilon''(\gamma, n, D) \in (0, \varepsilon]$ such that if $0 < (M + 1)a^{(n-2)/(n+1)} < \varepsilon''$, then $v_m(t) \in L^{q_0}(D)$ for all $t > 0$ along with

$$K_m := \sup_{\tau>0}(1+\tau)^\gamma \|v_m(\tau)\|_{q_0} < \infty, \qquad K_m \le \frac{1}{2}K_{m-1} + C(M+1)a^{\frac{n-1}{n+1}} \tag{3.83}$$

for all $m \ge 1$, where $v_m(t)$ is the approximate solution defined by (3.62) and C is a positive constant independent of a and m. We use (3.40) to see that

$$\int_0^t \|e^{-(t-\tau)A_a}Ph_1(\tau)\|_{q_0}\,d\tau \le CM\|u_s\|_{\frac{n}{1+\rho_1}} \int_0^{\min\{1,t\}} (t-\tau)^{-\gamma}\,d\tau \le CM\|u_s\|_{\frac{n}{1+\rho_1}}(1+t)^{-\gamma} \tag{3.84}$$

for $t > 0$. Moreover, it holds that

$$\int_0^t \Big\|e^{-(t-\tau)A_a}P\Big[\psi(\tau)\big(1-\psi(\tau)\big)a\frac{\partial u_s}{\partial x_1}\Big]\Big\|_{q_0}\,d\tau \le Ca\|\nabla u_s\|_r$$

for $t \le 2$, where $r := \min\{n/(2-\rho_4), q_0\}$ and that

$$\begin{aligned}\int_0^t \Big\|e^{-(t-\tau)A_a}P\Big[\psi(\tau)\big(1-\psi(\tau)\big)a\frac{\partial u_s}{\partial x_1}\Big]\Big\|_{q_0}\,d\tau &\le Ca\|\nabla u_s\|_{\frac{n}{2+\rho_3}} \int_0^1 (t-\tau)^{-\gamma-\frac{1+\rho_3-\rho_1}{2}}\,d\tau \\ &\le Ca\|\nabla u_s\|_{\frac{n}{2+\rho_3}} t^{-\gamma}\end{aligned}$$

for $t > 2$ as well as that

$$\begin{aligned}\int_0^t \|e^{-(t-\tau)A_a}P[\psi(\tau)\big(1-\psi(\tau)\big)u_s\cdot\nabla u_s]\|_{q_0}\,d\tau &\le C\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac{n}{2}} \int_0^{\min\{1,t\}} (t-\tau)^{-1-\gamma+\frac{\rho_1-\kappa}{2}}\,d\tau \\ &\le C\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac{n}{2}}(1+t)^{-\gamma}\end{aligned}$$

for $t > 0$, where $\max\{0, \rho_1 - 2\} < \kappa < \min\{n-3, \rho_1 - 2\gamma\}$ (note that (3.16) yields $\rho_1 < n-1$). These estimates imply

$$\int_0^t \|e^{-(t-\tau)A_a}Ph_2(\tau)\|_{q_0}\,d\tau \le C(a\|\nabla u_s\|_r + a\|\nabla u_s\|_{\frac{n}{2+\rho_3}} + \|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac{n}{2}})(1+t)^{-\gamma}$$

for $t > 0$, which together with (3.84) and (3.10) leads us to $v_1(t) \in L^{q_0}(D)$ for all $t > 0$ with

$$K_1 \le C(M+1)a^{\frac{n-1}{n+1}}. \tag{3.85}$$

This proves (3.83) with $m = 1$ since $K_0 = 0$.

Let $m \ge 2$ and suppose that $v_{m-1}(t) \in L^{q_0}(D)$ for all $t > 0$ and (3.83) with $m - 1$. Then we have $G_1(v_{m-1}, v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with

$$\sup_{\tau>0}(1+\tau)^\gamma \|G_1(v_{m-1}, v_{m-1})(\tau)\|_{q_0} \le CK_{m-1} \sup_{\tau>0} \tau^{\frac{1}{2}} \|\nabla v_{m-1}(\tau)\|_n. \tag{3.86}$$

Let $t \ge 2$ and split the integral into

$$\int_0^t \|e^{-(t-\tau)A_a}P[\psi(\tau)u_s\cdot\nabla v_{m-1}]\|_{q_0}\,d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t.$$

Let $\lambda \in (0, \rho_1]$ satisfy $\lambda < n - 3 + 2\gamma - \rho_1$; in fact, we can take such λ due to (3.80). Then (3.42) with $F = v_{m-1} \otimes u_s$ implies

$$\begin{aligned}
\int_0^{\frac{t}{2}} &\le C \int_0^{\frac{t}{2}} (t-\tau)^{-1} \|u_s\|_n \|v_{m-1}(\tau)\|_{q_0} \, d\tau \le C t^{-\gamma} \|u_s\|_n K_{m-1}, \\
\int_{\frac{t}{2}}^{t-1} &\le C \int_{\frac{t}{2}}^{t-1} (t-\tau)^{-1-\frac{\lambda}{2}} \|u_s\|_{\frac{n}{1+\lambda}} \|v_{m-1}(\tau)\|_{q_0} \, d\tau \le C t^{-\gamma} \|u_s\|_{\frac{n}{1+\lambda}} K_{m-1}, \\
\int_{t-1}^{t} &\le C \int_{t-1}^{t} (t-\tau)^{-1+\frac{\rho_2}{2}} \|u_s\|_{\frac{n}{1-\rho_2}} \|v_{m-1}(\tau)\|_{q_0} \, d\tau \le C t^{-\gamma} \|u_s\|_{\frac{n}{1-\rho_2}} K_{m-1}
\end{aligned}$$

for $t \ge 2$. Moreover, we use (3.42) again to see that

$$\begin{aligned}
\int_0^t \|e^{-(t-\tau)A_a} P[\psi(\tau) u_s \cdot \nabla v_{m-1}]\|_{q_0} \, d\tau &\le C \int_0^t (t-\tau)^{-1+\frac{\rho_2}{2}} \|u_s\|_{\frac{n}{1-\rho_2}} \|v_{m-1}(\tau)\|_{q_0} \, d\tau \\
&\le C \|u_s\|_{\frac{n}{1-\rho_2}} K_{m-1}
\end{aligned}$$

for $t \le 2$. We thus conclude $G_3(v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with

$$\sup_{\tau>0} (1+\tau)^\gamma \|G_3(v_{m-1})(\tau)\|_{q_0} \le C(\|u_s\|_n + \|u_s\|_{\frac{n}{1+\lambda}} + \|u_s\|_{\frac{n}{1-\rho_2}}) K_{m-1}. \tag{3.87}$$

By the same calculation, we have $G_2(v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with

$$\sup_{\tau>0} (1+\tau)^\gamma \|G_2(v_{m-1})(\tau)\|_{q_0} \le C(\|u_s\|_n + \|u_s\|_{\frac{n}{1+\lambda}} + \|u_s\|_{\frac{n}{1-\rho_2}}) K_{m-1}. \tag{3.88}$$

We also have

$$\begin{aligned}
\int_0^t \left\| e^{-(t-\tau)A_a} P\Big[\big(1-\psi(\tau)\big) a \frac{\partial v_{m-1}}{\partial x_1} \Big] \right\|_{q_0} &\le Ca \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{1}{2}} \|v_{m-1}(\tau)\|_{q_0} \, d\tau \\
&\le Ca K_{m-1} (1+t)^{-\frac{1}{2}} \le Ca K_{m-1} (1+t)^{-\gamma}
\end{aligned}$$

for $t > 0$ by (3.42). This together with (3.85)–(3.88), (3.10) and (3.66) yields $v_m(t) \in L^{q_0}(D)$ for $t > 0$ and

$$\begin{aligned}
K_m &\le C(M+1) a^{\frac{n-1}{n+1}} + \widetilde{C}_1 \Big\{ \big(\sup_{\tau>0} \tau^{\frac{1}{2}} \|\nabla v_{m-1}(\tau)\|_n \big) + \|u_s\|_n + \|u_s\|_{\frac{n}{1+\lambda}} + \|u_s\|_{\frac{n}{1-\rho_2}} + a \Big\} K_{m-1} \\
&\le C(M+1) a^{\frac{n-1}{n+1}} + \widetilde{C}_1 (4C_3 + \widetilde{C}_2)(M+1) a^{\frac{n-2}{n+1}} K_{m-1}.
\end{aligned}$$

Suppose

$$(M+1) a^{\frac{n-2}{n+1}} < \min\left\{ \varepsilon, \, \frac{1}{2\widetilde{C}_1(4C_3 + \widetilde{C}_2)} \right\} =: \varepsilon'',$$

then we get (3.83) with m and, thereby, conclude

$$K_m \le 2C(M+1) a^{\frac{n-1}{n+1}}$$

for all $m \ge 1$. Since we know that $\|v_m(t) - v(t)\|_n \to 0$ as $m \to \infty$ for each $t > 0$, we obtain $v(t) \in L^{q_0}(D)$ for $t > 0$ with

$$\sup_{\tau>0} (1+\tau)^\gamma \|v(\tau)\|_{q_0} \le 2C(M+1) a^{\frac{n-1}{n+1}} < \infty,$$

which completes the proof. □

In view of Lemma 3.3.6 and Lemma 3.3.8, we prove sharp decay properties (3.25)–(3.26) for $n \geq 4$.

Proposition 3.3.9. *Let $n \geq 4$. Suppose $\rho_3 > 1$ and $1 < \rho_1 \leq 1+\rho_3$ in addition to* (3.16) (*the set of those parameters is nonvoid when $n \geq 4$*). *Let ε be the constant in Proposition 3.3.4. There exists a constant $\varepsilon_* = \varepsilon_*(n, D) \in (0, \varepsilon]$ such that if $0 < (M+1)a^{(n-2)/(n+1)} < \varepsilon_*$, then the solution $v(t)$ obtained in Proposition 3.3.4 enjoys* (3.25) *and* (3.26).

Proof. Fix $1/2 < \rho < 1$ and $\gamma > 0$ such that

$$\max\left\{\frac{1}{2} - \frac{\rho}{2},\, \frac{\rho_1 + 3 - n}{2}\right\} < \gamma < \frac{1}{2}. \tag{3.89}$$

Let $\varepsilon'(\rho, n, D)$ and $\varepsilon''(\gamma, n, D)$ be the constants in Lemma 3.3.6 and Lemma 3.3.8, respectively. We show by induction that if

$$(M+1)a^{\frac{n-2}{n+1}} < \min\{\varepsilon'(\rho, n, D), \varepsilon''(\gamma, n, D)\} =: \varepsilon_*(n, D),$$

then $v(t)$ satisfies

$$\|v(t)\|_n = O(t^{-\sigma_k}), \qquad \sigma_k := \min\left\{\frac{k}{2}\rho,\, \frac{\rho_1}{2}\right\} \tag{3.90}$$

as $t \to \infty$ for all $k \geq 1$. This implies (3.25) with $q = n$, which together with (3.77) completes the proof. Since $\rho < \rho_1$, (3.90) with $k = 1$ follows from Lemma 3.3.6. We note that $\sigma_1 < 1/2$ and $\sigma_k > 1/2$ for $k \geq 2$.

Let $k \geq 2$ and suppose (3.90) with $k - 1$. Then

$$L_{k-1}(v) := \sup_{\tau>0}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_n + \sup_{\tau>0}\tau^{\frac{1}{2}}(1+\tau)^{\sigma_{k-1}}\big(\|v(\tau)\|_\infty + \|\nabla v(\tau)\|_n\big) < \infty$$

holds due to (3.60) (near $t = 0$) as well as (3.77). In what follows, we always assume $t \geq 2$. From (3.81), it follows that

$$\|G_1(t)\|_n \leq \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2q_0}}\|v(\tau)\|_{q_0}\|\nabla v(\tau)\|_n\, d\tau + \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\|v(\tau)\|_n\|\nabla v(\tau)\|_n\, d\tau =: I + II \tag{3.91}$$

with

$$I \leq Ct^{-\frac{n}{2q_0}}\big(\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0}\big)L_{k-1}(v) \leq Ct^{-\frac{\rho_1}{2}}\big(\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0}\big)L_{k-1}(v), \tag{3.92}$$

where (3.82) and (3.89) are taken into account and

$$II \leq Ct^{-2\sigma_{k-1}}L_{k-1}(v)^2. \tag{3.93}$$

For $G_2(t)$, we split the integral into

$$\int_0^t \|e^{-(t-\tau)A_a}P[\psi(\tau)v\cdot\nabla u_s]\|_n\, d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t.$$

Then we find

$$\int_0^{\frac{t}{2}} \le C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1+\rho_3}{2}}\tau^{-\frac{1}{2}}(1+\tau)^{-\sigma_{k-1}}\,d\tau\,\|\nabla u_s\|_{\frac{n}{2+\rho_3}}\big(\sup_{\tau>0}\tau^{\frac{1}{2}}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_\infty\big)$$
$$\le \begin{cases} Ct^{-\frac{\rho_3}{2}-\sigma_{k-1}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}}L_{k-1}(v) \le Ct^{-\sigma_k}\|\nabla u_s\|_{\frac{n}{2+\rho_3}}L_{k-1}(v) & \text{if } k=2,\\ Ct^{-\frac{1+\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}}L_{k-1}(v) \le Ct^{-\frac{\rho_1}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}}L_{k-1}(v) & \text{if } k\ge 3\end{cases}$$

and

$$\int_{\frac{t}{2}}^{t-1} + \int_{t-1}^{t} \le Ct^{-\sigma_{k-1}-\frac{1}{2}}\big(\|\nabla u_s\|_{\frac{n}{2+\rho_3}} + \|\nabla u_s\|_{\frac{n}{2}}\big)L_{k-1}(v),$$

where we have used $\rho_3 > 1$ and $\rho_1 \le 1+\rho_3$. Estimates above imply that

$$\|G_2(t)\|_n \le Ct^{-\sigma_k}\big(\|\nabla u_s\|_{\frac{n}{2+\rho_3}} + \|\nabla u_s\|_{\frac{n}{2}}\big)L_{k-1}(v) \tag{3.94}$$

Similarly, we observe

$$\|G_3(t)\|_n \le Ct^{-\sigma_k}\big(\|u_s\|_{\frac{n}{1+\rho_1}} + \|u_s\|_n\big)L_{k-1}(v). \tag{3.95}$$

Moreover, by the same manner as in the proof of Lemma 3.3.6, we obtain

$$\|G_4(t)\|_n \le Ct^{-\frac{n}{2q_0}}\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0} \le Ct^{-\frac{\rho_1}{2}}\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0}, \tag{3.96}$$

$$\|H_1(t)\|_n \le CMt^{-\frac{\rho_1}{2}}\|u_s\|_{\frac{n}{1+\rho_1}}, \tag{3.97}$$

$$\begin{aligned}\|H_2(t)\|_n &\le Ct^{-\frac{2+\kappa}{2}}\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac{n}{2}} + Ct^{-\frac{1+\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}}\\ &\le Ct^{-\frac{\rho_1}{2}}\big(\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac{n}{2}} + \|\nabla u_s\|_{\frac{n}{2+\rho_3}}\big)\end{aligned} \tag{3.98}$$

for all $t \ge 2$, where κ is chosen such that $\max\{0, \rho_1 - 2\} < \kappa < \min\{n-3, \rho_1\}$. Collecting (3.91)–(3.98), we conclude (3.90) with k. The proof is complete. □

We next consider the uniqueness. We begin with the classical result on the uniqueness of solutions within Y_0 as in Fujita and Kato [21].

Lemma 3.3.10. *Let ψ be a function on $\mathbb{R}$ satisfying* (3.1) *and let δ be the constant in Theorem 3.1.1 with* (3.15)–(3.17). *Then there exists a constant $\tilde\varepsilon = \tilde\varepsilon(n, D) \in (0, \delta]$ such that if $0 < a^{(n-2)/(n+1)} < \tilde\varepsilon$,* (3.21) *admits at most one solution within Y_0.*

Proof. The following argument is based on [21]. Suppose that $v, \tilde v \in Y_0$ are solutions. Then we have

$$\|v-\tilde v\|_{Y,t} \le \big\{C_1\big([\nabla v]_{n,t} + [\tilde v]_{n,t}^{\frac{1}{2}}[\tilde v]_{\infty,t}^{\frac{1}{2}}\big) + C_2 a^{\frac{n-2}{n+1}}\big\}\|v-\tilde v\|_{Y,t}, \quad t>0 \tag{3.99}$$

by applying (3.10) and Lemma 3.3.3. If we assume

$$a^{\frac{n-2}{n+1}} < \min\Big\{\delta, \frac{1}{2C_2}\Big\} =: \tilde\varepsilon \tag{3.100}$$

and choose $t_0 > 0$ such that

$$C_1\big\{[\nabla v]_{n,t_0} + \big(\sup_{0<\tau<\infty}\|\tilde v(\tau)\|_n\big)^{\frac{1}{2}}[\tilde v]_{\infty,t_0}^{\frac{1}{2}}\big\} < \frac{1}{2},$$

then (3.99) yields $[v-\tilde{v}]_{Y,t_0}=0$. Hence, we conclude $v=\tilde{v}$ on $(0,t_0]$ and obtain

$$v(t)-\tilde{v}(t)=\int_{t_0}^t e^{-(t-\tau)A_a}P\Big[-(v-\tilde{v})\cdot\nabla v-\tilde{v}\cdot\nabla(v-\tilde{v})-\psi(\tau)(v-\tilde{v})\cdot\nabla u_s \\ -\psi(\tau)u_s\cdot\nabla(v-\tilde{v})+(1-\psi(\tau))a\frac{\partial}{\partial x_1}(v-\tilde{v})\Big]\,d\tau.$$

By the same argument as in the proof of Lemma 3.3.3 together with (3.10), we see that

$$\|v-\tilde{v}\|_{Y,t_0,t}\le C_*\|v-\tilde{v}\|_{Y,t_0,t} \tag{3.101}$$

for all $t>t_0$, where

$$\|v\|_{Y,t_0,t}:=\sup_{t_0\le\tau\le t}\|v(\tau)\|_n+\sup_{t_0\le\tau\le t}\|v(\tau)\|_\infty+\sup_{t_0\le\tau\le t}\|\nabla v(\tau)\|_n, \tag{3.102}$$

$$C_*=C\Big[\big(t_0^{-\frac12}\|v\|_Y+t_0^{-\frac14}\|\tilde{v}\|_Y\big)\big\{(t-t_0)^{\frac34}+(t-t_0)^{\frac14}\big\} \\ +a^{\frac{n-1}{n+1}}\big\{(t-t_0)^{\frac12}+(t-t_0)^{\frac{\rho_2}{2}}+(t-t_0)^{\frac{\rho_4}{2}}\big\}+a\big\{(t-t_0)+(t-t_0)^{\frac12}\big\}\Big] \tag{3.103}$$

and the constant C is independent of v, $\tilde{v}$, t and t_0. We choose $\eta>0$ such that

$$\xi:=C\Big[\big(t_0^{-\frac12}\|v\|_Y+t_0^{-\frac14}\|\tilde{v}\|_Y\big)\big(\eta^{\frac34}+\eta^{\frac14}\big)+a^{\frac{n-1}{n+1}}\big(\eta^{\frac12}+\eta^{\frac{\rho_2}{2}}+\eta^{\frac{\rho_4}{2}}\big)+a\big(\eta+\eta^{\frac12}\big)\Big]<1.$$

On account of (3.101), we have $\|v-\tilde{v}\|_{Y,t_0,t_0+\eta}\le\xi\|v-\tilde{v}\|_{Y,t_0,t_0+\eta}$, which leads us to $v=\tilde{v}$ on $[t_0,t_0+\eta]$. By the same calculation, we can obtain (3.101)–(3.103), in which t_0 should be replaced by $t_0+\eta$ and hence

$$\|v-\tilde{v}\|_{Y,t_0+\eta,t_0+2\eta}\le C\Big[\big\{(t_0+\eta)^{-\frac12}\|v\|_Y+(t_0+\eta)^{-\frac14}\|\tilde{v}\|_Y\big\}\big(\eta^{\frac34}+\eta^{\frac14}\big) \\ +a^{\frac{n-1}{n+1}}\big(\eta^{\frac12}+\eta^{\frac{\rho_2}{2}}+\eta^{\frac{\rho_4}{2}}\big)+a\big(\eta+\eta^{\frac12}\big)\Big]\|v-\tilde{v}\|_{Y,t_0+\eta,t_0+2\eta} \\ <\xi\|v-\tilde{v}\|_{Y,t_0+\eta,t_0+2\eta}$$

holds. This implies $v=\tilde{v}$ on $[t_0+\eta,t_0+2\eta]$. Repeating this procedure, we conclude $v=\tilde{v}$. □

Remark 3.3.11. It is clear that the equation (3.67) admits at most one solution within Y_0 under the same condition as in Lemma 3.3.10.

Let us close the section with completion of the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. Since we know $\varepsilon\le\tilde{\varepsilon}$ from (3.65) and (3.100), Proposition 3.3.4 and Lemma 3.3.10 yield the unique existence of solutions in Y_0 when $(M+1)a^{(n-2)/(n+1)}<\varepsilon$. Moreover, Proposition 3.3.7 and Proposition 3.3.9 give us sharp decay properties of the solution provided a is still smaller. We finally show the uniqueness of the solution constructed above within Y by following the argument due to Brezis [7]. It suffices to show that if $v\in Y$ is a solution, it necessarily satisfies

$$\lim_{t\to0}[v]_t=0, \tag{3.104}$$

where

$$[v]_t := \sup_{0<\tau<t} \tau^{\frac{1}{2}}(\|v(\tau)\|_\infty + \|\nabla v(\tau)\|_n).$$

We assume

$$(M+1)a^{\frac{n-2}{n+1}} < \min\left\{\delta, \frac{1}{2C_2}, \frac{1}{16C_1C_0}, \frac{1}{16C_1C_3}\right\} =: \hat{\varepsilon}(n,D)\,(\leq \varepsilon) \tag{3.105}$$

and let $v \in Y$ be a solution. Here, the constants C_i are as in Remark 3.3.5 as well as in the proof of Proposition 3.3.4. Since $v \in BC([0,\infty); L^n_\sigma(D))$ with $v(0) = 0$, there exists $s_0 > 0$ such that

$$\|v(s)\|_n + (M+1)a^{\frac{n-2}{n+1}} < \hat{\varepsilon}$$

for all $0 < s \leq s_0$. Hence by Remark 3.3.5, the integral equation (3.67) with $b = v(s)$ admits a solution $T(t)v(s) \in Y_0$ along with

$$\|T(\cdot)v(s)\|_Y \leq 4\big(C_0\|v(s)\|_n + C_3(M+1)a^{\frac{n-2}{n+1}}\big) < 4(C_0+C_3)\hat{\varepsilon} \leq \frac{1}{2C_1}. \tag{3.106}$$

On the other hand, given $s \in (0, s_0]$, we can see that $z_s(t) := v(t+s)$ for $t \geq 0$ also satisfies (3.67) with $b = v(s)$ and $z_s \in Y_0$. In view of Remark 3.3.11, we have $z_s(t) = T(t)v(s)$ for $s \in (0, s_0]$, which implies

$$t^{\frac{1}{2}}\big(\|v(t+s)\|_\infty + \|\nabla v(t+s)\|_n\big) \leq \sup_{f\in K}[T(\cdot)f]_t, \quad K = v((0,s_0]) := \{v(t) \in L^n_\sigma(D) \mid t \in (0,s_0]\}$$

for all $s \in (0, s_0]$ and $t > 0$. Passing to the limit $s \to 0$, we find

$$[v(\cdot)]_t \leq \sup_{f\in K}[T(\cdot)f]_t. \tag{3.107}$$

Furthermore, applying Lemma 3.3.3 to (3.67) with $b = f \in v((0, s_0])$ as well as Proposition 3.3.1 and (3.10), we have

$$[T(\cdot)f]_t \leq C_0[S(\cdot)f]_t + \Big(C_1 \sup_{f\in K}\|T(\cdot)f\|_Y + C_2 a^{\frac{n-2}{n+1}}\Big)[T(\cdot)f]_t + \|H_1\|_{Y,t} + \|H_2\|_{Y,t},$$

where $S(t)f := e^{-tA_a}f$, and deduce from (3.105) and (3.106) that

$$[T(\cdot)f]_t \leq \frac{C_0[S(\cdot)f]_t + \|H_1\|_{Y,t} + \|H_2\|_{Y,t}}{1 - \Big(C_1 \sup_{f\in K}\|T(\cdot)f\|_Y + C_2 a^{\frac{n-2}{n+1}}\Big)} \tag{3.108}$$

for all $f \in K$ and $t > 0$. Collecting (3.108), (3.43), (3.107) and $H_1, H_2 \in Y_0$ leads to (3.104). The proof is complete. □

Chapter 4

Attainability of a stationary solution around a rigid body rotating from rest

4.1 Introduction

We consider the large time behavior of a viscous incompressible flow around a rotating rigid body in $\mathbb{R}^3$. Assume that both a compact rigid body $\mathscr{O}$ and a viscous incompressible fluid which occupies the outside of $\mathscr{O}$ are initially at rest; then, the body starts to rotate with the angular velocity which gradually increases until it reaches a small terminal one at a certain finite time and it is fixed afterwards. We then show that the fluid motion converges to a steady solution obtained by Galdi [25] as time $t \to \infty$ (Theorem 4.2.1 in Section 4.2). This was conjectured by Hishida [31, Section 6], but it has remained open. Such a question is called the starting problem and it was originally raised by Finn [18], in which rotation was replaced by translation of the body. Finn's starting problem was first studied by Heywood [29]; since his paper, a stationary solution is said to be attainable if the fluid motion converges to it as $t \to \infty$. Later on, by using Kato's approach [37] (see also Fujita and Kato [21]) together with the L^q-L^r estimates for the Oseen equation established by Kobayashi and Shibata [39], Finn's starting problem was completely solved by Galdi, Heywood and Shibata [27].

Let us introduce the mathematical formulation. Let $\mathscr{O} \subset \mathbb{R}^3$ be a compact and connected set with non-empty interior. The motion of $\mathscr{O}$ mentioned above is described in terms of the angular velocity

$$\omega(t) = \psi(t)\omega_0, \quad \omega_0 = (0, 0, a)^\top$$

with a constant $a \in \mathbb{R}$, where ψ is a function on $\mathbb{R}$ satisfying the following conditions:

$$\psi \in C^1(\mathbb{R};\mathbb{R}), \quad |\psi(t)| \le 1 \ \text{ for } t \in \mathbb{R}, \quad \psi(t) = 0 \ \text{ for } t \le 0, \quad \psi(t) = 1 \ \text{ for } t \ge 1. \tag{4.1}$$

Here and hereafter, $(\cdot)^\top$ denotes the transpose. Then the domain occupied by the fluid can be expressed as $D(t) = \{y = O(t)x;\, x \in D\}$, where $D = \mathbb{R}^3 \setminus \mathscr{O}$ is assumed to be an exterior domain with smooth boundary ∂D and

$$O(t) = \begin{pmatrix} \cos\Psi(t) & -\sin\Psi(t) & 0 \\ \sin\Psi(t) & \cos\Psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Psi(t) = \int_0^t \psi(s) a \, ds.$$

We consider the initial boundary value problem for the Navier-Stokes equation

$$\begin{cases} \partial_t w + w\cdot\nabla_y w = \Delta_y w - \nabla_y \pi, & y\in D(t), t>0,\\ \nabla_y\cdot w = 0, & y\in D(t), t\ge 0,\\ w|_{\partial D(t)} = \psi(t)\omega_0\times y, & t\ge 0,\\ w(y,t)\to 0 & \text{as } |y|\to\infty,\\ w(y,0) = 0, & y\in D, \end{cases} \tag{4.2}$$

where $w=(w_1(y,t),w_2(y,t),w_3(y,t))^\top$ and $\pi=\pi(y,t)$ denote unknown velocity and pressure of the fluid, respectively. To reduce the problem to an equivalent one in the fixed domain D, we take the frame $x=O(t)^\top y$ attached to the body and make the change of the unknown functions: $u(x,t)=O(t)^\top w(y,t),\ p(x,t)=\pi(y,t)$. Then the problem (4.2) is reduced to

$$\begin{cases} \partial_t u + u\cdot\nabla u = \Delta u + (\psi(t)\omega_0\times x)\cdot\nabla u - \psi(t)\omega_0\times u - \nabla p, & x\in D, t>0,\\ \nabla\cdot u = 0, & x\in D, t\ge 0,\\ u|_{\partial D} = \psi(t)\omega_0\times x, & t\ge 0,\\ u\to 0 & \text{as } |x|\to\infty,\\ u(x,0) = 0, & x\in D. \end{cases} \tag{4.3}$$

The purpose of this chapter is to show that (4.3) admits a global solution which tends to a solution u_s for the stationary problem

$$\begin{cases} u_s\cdot\nabla u_s = \Delta u_s + (\omega_0\times x)\cdot\nabla u_s - \omega_0\times u_s - \nabla p_s, & x\in D,\\ \nabla\cdot u_s = 0, & x\in D,\\ u_s|_{\partial D} = \omega_0\times x, &\\ u_s\to 0 & \text{as } |x|\to\infty. \end{cases} \tag{4.4}$$

The rate of convergence in L^r with $r\in(3,\infty]$ is also studied. In [25], Galdi successfully proved that if $|\omega_0|$ is sufficiently small, problem (4.4) has a unique smooth solution (u_s,p_s) with pointwise estimates

$$|u_s(x)|\le \frac{C|\omega_0|}{|x|},\quad |\nabla u_s(x)|+|p_s(x)|\le \frac{C|\omega_0|}{|x|^2}. \tag{4.5}$$

We note that the decay rate (4.5) is scale-critical, which is also captured in terms of the Lorentz space (weak-Lebesgue space) $L^{3,\infty}$. This was in fact done by Farwig and Hishida [13] even for the external force being in a Lorentz-Sobolev space of order (-1).

Let us mention some difficulties of our problem and how to overcome them in this thesis. In [27], the L^q-L^r estimates for the Oseen semigroup play an important role. In the rotational case with constant angular velocity, Hishida and Shibata [36] also established the L^q-L^r estimates of the semigroup generated by the Stokes operator with the additional term $(\omega_0\times x)\cdot\nabla - \omega_0\times$. If we use this semigroup as in [27], we have to treat the term $(\psi(t)-1)(\omega_0\times x)\cdot\nabla v$, which is however no longer perturbation from the semigroup on account of the unbounded coefficient $\omega_0\times x$, where $v=u-\psi(t)u_s$. In this thesis, we make use of the evolution operator $\{T(t,s)\}_{t\ge s\ge 0}$

on the solenoidal space $L^q_\sigma(D)$ $(1 < q < \infty)$, which is a solution operator to the linear problem

$$\begin{cases} \partial_t u = \Delta u + (\psi(t)\omega_0 \times x)\cdot \nabla u - \psi(t)\omega_0 \times u - \nabla p, \quad x \in D, t > s, \\ \nabla \cdot u = 0, \quad x \in D, t \geq s, \\ u|_{\partial D} = 0, \quad t > s, \\ u \to 0 \quad \text{as } |x| \to \infty, \\ u(x,s) = f, \quad x \in D. \end{cases} \tag{4.6}$$

Hansel and Rhandi [28] succeeded in the proof of generation of this evolution operator with the L^q-L^r smoothing rate. They constructed the evolution operator in their own way since the corresponding semigroup is not analytic (Hishida [30], Farwig and Neustupa [15]). Recently, Hishida [33,34] developed the L^q-L^r decay estimates of the evolution operator, see Section 4.2. With those estimates, we solve the integral equation which perturbation from the stationary solution u_s obeys. However, it is difficult to perform analysis with the standard Lebesgue space on account of the scale-critical pointwise estimates (4.5). Thus, we first construct a solution for the weak formulation in the framework of Lorentz space by the strategy due to Yamazaki [48]. We next identify this solution with a local solution possessing better regularity in a neighborhood of each time. The later procedure is actually adopted by Kozono and Yamazaki [40]. Furthermore, we derive the L^∞ decay which is not observed in [27]. When the stationary solution possesses the scale-critical rate $O(1/|x|)$, Koba [38] first derived the L^∞ decay of perturbation with less sharp rate in the context of stability analysis, see also Remark 4.4.4. Although he used both the L^1-L^r estimates of the Oseen semigroup e^{-tA_a} and the L^q-L^r estimates (yielding the L^q-L^∞ estimates) of the composite operator $e^{-tA_a}P\mathrm{div}$, where P denotes the Fujita-Kato projection, it turns out that either of them is enough to accomplish the proof. In this paper, we employ merely the L^1-L^r estimates of the adjoint evolution operator $T(t,s)^*$ to simplify the argument.

This chapter is organized as follows. In Section 4.2 we give the main theorems. Section 4.3 is devoted to some preliminary results on the stationary problem and the evolution operator. In Section 4.4 we give the proof of the main theorems.

4.2 Main theorems

In this section, we give our main theorems. It is reasonable to look for a solution to (4.3) of the form

$$u(x,t) = v(x,t) + \psi(t)u_s, \quad p(x,t) = \phi(x,t) + \psi(t)p_s.$$

Then the perturbation (v,ϕ) satisfies the following initial boundary value problem

$$\begin{cases} \partial_t v = \Delta v + (\psi(t)\omega_0 \times x)\cdot \nabla v - \psi(t)\omega_0 \times v \\ \qquad\qquad + (Gv)(x,t) + H(x,t) - \nabla \phi, \quad x \in D, t > 0, \\ \nabla \cdot v = 0, \quad x \in D, t \geq 0, \\ v|_{\partial D} = 0, \quad t > 0, \\ v \to 0 \quad \text{as } |x| \to \infty, \\ v(x,0) = 0, \quad x \in D, \end{cases} \tag{4.7}$$

where

$$(Gv)(x,t) = -v \cdot \nabla v - \psi(t) v \cdot \nabla u_s - \psi(t) u_s \cdot \nabla v, \tag{4.8}$$
$$H(x,t) = \psi(t)(\psi(t)-1)\{-u_s \cdot \nabla u_s - \omega_0 \times u_s + (\omega_0 \times x)\cdot \nabla u_s\} - \psi'(t) u_s. \tag{4.9}$$

In what follows, we concentrate ourselves on the problem (4.7) instead of (4.3). In fact, if we obtain the solution v of (4.7) which converges to 0 as $t \to \infty$, the solution u of (4.3) converges to u_s as $t \to \infty$. By using the evolution operator $\{T(t,s)\}_{t \geq s \geq 0}$ on $L^q_\sigma(D)$ $(1 < q < \infty)$ associated with (4.6), problem (4.7) is converted into

$$v(t) = \int_0^t T(t,\tau) P[(Gv)(\tau) + H(\tau)]\, d\tau. \tag{4.10}$$

We are now in a position to give our attainability theorem.

Theorem 4.2.1. *Let ψ be a function on $\mathbb{R}$ satisfying (4.1) and put $\alpha := \max_{t \in \mathbb{R}} |\psi'(t)|$. For $q \in (6,\infty)$, there exists a constant $\delta(q) > 0$ such that if $(\alpha+1)|a| \leq \delta$, problem (4.10) admits a solution v which possesses the following properties:*

(i) $v \in BC_{w^*}\big((0,\infty); L^{3,\infty}_\sigma(D)\big)$, $\|v(t)\|_{3,\infty} \to 0$ as $t \to 0$, $\sup_{0<t<\infty} \|v(t)\|_{3,\infty} \leq C(\alpha+1)|a|$,

where $BC_{w^*}(I;X)$ is the set of bounded and weak-$*$ continuous functions on the interval I with values in X, the constant C is independent of a and ψ;

(ii) $v \in C\big((0,\infty); L^r_\sigma(D)\big) \cap C_{w^*}\big((0,\infty); L^\infty(D)\big)$, $\nabla v \in C_w\big((0,\infty); L^r(D)\big)$ for $r \in (3,\infty)$;

(iii) (Decay) $\|v(t)\|_r = O(t^{-\frac{1}{2}+\frac{3}{2r}})$ as $t \to \infty$ for all $r \in (3,q)$,

$\|v(t)\|_{q,\infty} = O(t^{-\frac{1}{2}+\frac{3}{2q}})$ as $t \to \infty$,

$\|v(t)\|_r = O(t^{-\frac{1}{2}+\frac{3}{2q}})$ as $t \to \infty$ for all $r \in (q,\infty]$.

Remark 4.2.2. We can obtain the L^q decay of $v(t)$ like $O(t^{-1/2+3/(2q)}\log t)$ as $t \to \infty$, but it is not clear whether $\|v(t)\|_q = O(t^{-1/2+3/(2q)})$ holds.

To prove Theorem 4.2.1, the key step is to construct a solution of the weak formulation

$$(v(t),\varphi) = \int_0^t \big(v(\tau)\otimes v(\tau) + \psi(\tau)\{v(\tau)\otimes u_s + u_s \otimes v(\tau)\}, \nabla T(t,\tau)^*\varphi\big)\, d\tau$$
$$+ \int_0^t (H(\tau), T(t,\tau)^*\varphi)\, d\tau, \quad \forall \varphi \in C^\infty_{0,\sigma}(D) \tag{4.11}$$

as in Yamazaki [48], where $T(t,\tau)^*$ denotes the adjoint of $T(t,\tau)$ and, here and in what follows, $(\cdot,\cdot)$ stands for various duality pairings. In this chapter, a function v is called a solution of (4.11) if $v \in L^\infty_{\mathrm{loc}}\big([0,\infty); L^{3,\infty}_\sigma(D)\big)$ satisfies (4.11) for a.e. t. By following Yamazaki's approach, we can easily see that the solution obtained in Theorem 4.2.1 is unique in the small, see Proposition 4.4.2. In the following theorem, we give another result on the uniqueness without assuming the smallness of solutions.

Theorem 4.2.3. *Let $q \in (3,\infty)$. Then there exists a constant $\widetilde{\delta} > 0$ independent of q and ψ such that if $|a| \leq \widetilde{\delta}$, problem (4.11) admits at most one solution within the class*

$$\big\{v \in L^\infty_{\mathrm{loc}}\big([0,\infty); L^{3,\infty}_\sigma(D)\big) \cap L^\infty_{\mathrm{loc}}\big(0,\infty; L^q_\sigma(D)\big) \;\big|\; \lim_{t\to 0} \|v(t)\|_{3,\infty} = 0\big\}.$$

Remark 4.2.4. Theorem 4.2.3 asserts that if the angular velocity is small enough and if $\widetilde{v}$ is a solution within the class above which is not necessarily small, then it coincides with the solution obtained in Theorem 4.2.1.

4.3 Preliminary results

In this section, we prepare some results on the stationary solutions and the evolution operator. For the stationary problem (4.4), Galdi [25] proved the following result.

Proposition 4.3.1 ([25]). *There exists a constant $\eta \in (0,1]$ such that if $|\omega_0| = |a| \le \eta$, the stationary problem (4.4) admits a unique solution (u_s, p_s) with the estimate*

$$\sup_{x\in D}\left\{(1+|x|)|u_s(x)|\right\} + \sup_{x\in D}\left\{(1+|x|^2)(|\nabla u_s(x)| + |p_s(x)|)\right\} \le C|a|,$$

where the constant C is independent of a.

From now on, we assume that the angular velocity $\omega_0 = (0,0,a)^\top$ always satisfies $|\omega_0| = |a| \le \eta$. Proposition 4.3.1 then yields

$$u_s \in L^{3,\infty}(D)\cap L^\infty(D), \quad \nabla u_s \in L^{\frac{3}{2},\infty}(D)\cap L^\infty(D), \quad |x|\nabla u_s \in L^{3,\infty}(D)\cap L^\infty(D)$$

and

$$H(t) \in L^{3,\infty}(D), \quad \|H(t)\|_{3,\infty} \le C(a^2 + \alpha|a|) \tag{4.12}$$

for all $t > 0$. Here, $H(t)$ is defined by (4.9) and $\alpha = \sup_{t\in\mathbb{R}} |\psi'(t)|$.

We next collect some results on the evolution operator associated with (4.6). We define the linear operator by

$$\begin{aligned}
&\mathscr{D}_q(L(t)) = \{u \in L^q_\sigma(D)\cap W^{1,q}_0(D)\cap W^{2,q}(D) \mid (\omega_0\times x)\cdot\nabla u \in L^q(D)\},\\
&L(t)u = -P[\Delta u + (\psi(t)\omega_0\times x)\cdot\nabla u - \psi(t)\omega_0\times u].
\end{aligned}$$

Then the problem (4.6) is formulated as

$$\partial_t u + L(t)u = 0, \quad t\in(s,\infty); \quad u(s) = f \tag{4.13}$$

in $L^q_\sigma(D)$. We can see that (4.1) implies

$$\psi(t)\omega_0 \in C^\theta([0,\infty);\mathbb{R}^3)\cap L^\infty(0,\infty;\mathbb{R}^3) \tag{4.14}$$

for all $\theta\in(0,1)$. In fact, we have

$$\sup_{0\le t<\infty}|\psi(t)\omega_0| = |a|, \quad \sup_{0\le s<t<\infty}\frac{|\psi(t)\omega_0 - \psi(s)\omega_0|}{(t-s)^\theta} \le |a|\max_{t\in\mathbb{R}}|\psi'(t)| \tag{4.15}$$

for all $\theta\in(0,1)$. We fix, for instance, $\theta = 1/2$. Under merely the local Hölder continuity of the angular velocity, Hansel and Rhandi [28] proved the following proposition (see also Hishida [34] concerning the assertion 1). Indeed they did not derive the assertion 4, but it directly follows from the real interpolation. For completeness, we give its proof.

Proposition 4.3.2 ([28]). *Let $1 < q < \infty$. Suppose (4.1). The operator family $\{L(t)\}_{t\ge 0}$ generates a strongly continuous evolution operator $\{T(t,s)\}_{t\ge s\ge 0}$ on $L^q_\sigma(D)$ with the following properties:*

1. *Let* $q \in (3/2, \infty)$ *and* $s \geq 0$. *For every* $f \in Z_q(D)$ *and* $t \in (s, \infty)$, *we have* $T(t,s)f \in Y_q(D)$ *and* $T(\cdot, s)f \in C^1((s,\infty); L^q_\sigma(D))$ *with*
$$\partial_t T(t,s)f + L(t)T(t,s)f = 0, \quad t \in (s,\infty)$$
in $L^q_\sigma(D)$, *where*
$$\begin{aligned} Y_q(D) &= \{u \in L^q_\sigma(D) \cap W^{1,q}_0(D) \cap W^{2,q}(D) \mid |x|\nabla u \in L^q(D)\}, \\ Z_q(D) &= \{u \in L^q_\sigma(D) \cap W^{1,q}(D) \mid |x|\nabla u \in L^q(D)\}. \end{aligned}$$

2. *For every* $f \in Y_q(D)$ *and* $t > 0$, *we have* $T(t, \cdot)f \in C^1\big([0,t]; L^q_\sigma(D)\big)$ *with*
$$\partial_s T(t,s)f = T(t,s)L(s)f \quad s \in [0,t]$$
in $L^q_\sigma(D)$.

3. *Let* $1 < q \leq r < \infty$, *and* $m, \mathcal{T} \in (0,\infty)$. *There is a constant* $C = C(q,r,m,\mathcal{T},D)$ *such that*
$$\|\nabla T(t,s)f\|_r \leq C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|f\|_q \tag{4.16}$$
holds for all $0 \leq s < t \leq \mathcal{T}$ *and* $f \in L^q_\sigma(D)$ *whenever*
$$(1 + \max_{t\in\mathbb{R}} |\psi'(t)|\,)|a| \leq m \tag{4.17}$$
is satisfied.

4. *Let* $1 < q < r < \infty, 1 \leq \rho_1, \rho_2 \leq \infty$ and $m, \mathcal{T} \in (0,\infty)$. *There is a constant* $C = C(q, r, \rho_1, \rho_2, m, \mathcal{T}, D)$ *such that*
$$\|\nabla T(t,s)f\|_{r,\rho_2} \leq C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|f\|_{q,\rho_1} \tag{4.18}$$
holds for all $0 \leq s < t \leq \mathcal{T}$ *and* $f \in L^{q,\rho_1}_\sigma(D)$ *whenever (4.17) is satisfied.*

Proof of the assertion 4. We choose r_0, r_1 such that $1 < q < r_0 < r < r_1 < \infty$. From the assertion 3 and the real interpolation, we have
$$\|\nabla T(t,s)f\|_{r_0,\rho_1} \leq C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r_0})-\frac{1}{2}}\|f\|_{q,\rho_1}, \tag{4.19}$$
$$\|\nabla T(t,s)f\|_{r_1,\rho_1} \leq C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r_1})-\frac{1}{2}}\|f\|_{q,\rho_1}. \tag{4.20}$$
By the reiteration theorem for real interpolation (see for instance [3, Theorem 3.5.3]), we obtain
$$L^{r,\rho_2}_\sigma(D) = (L^{r_0,\rho_1}_\sigma(D), L^{r_1,\rho_1}_\sigma(D))_{\beta,\rho_2}, \quad \|u\|_{r,\rho_2} \leq C\|u\|^{1-\beta}_{r_0,\rho_1}\|u\|^{\beta}_{r_1,\rho_1}, \quad \frac{1}{r} = \frac{1-\beta}{r_0} + \frac{\beta}{r_1} \tag{4.21}$$
which combined with (4.19) and (4.20) concludes (4.18). □

We know that the adjoint operator $T(t,s)^*$ is also a strongly continuous evolution operator and satisfies the backward semigroup property
$$T(\tau,s)^*T(t,\tau)^* = T(t,s)^* \quad (t \geq \tau \geq s \geq 0), \quad T(t,t)^* = I,$$
see Hishida [33, Subsection 2.3]. Under the assumption (4.14) with some $\theta \in (0,1)$, Hishida [33, 34] established the following L^q-L^r decay estimates. The assertion 3 is not found there but can be proved in the same way as above. We note that the idea of deduction of (4.32) below is due to Yamazaki [48] once we have the assertion 5.

Proposition 4.3.3 ([33, 34]). *Let $m \in (0, \infty)$ and suppose (4.1).*

1. *Let $1 < q \le r \le \infty$ $(q \neq \infty)$. Then there exists a constant $C = C(m, q, r, D)$ such that*
$$\|T(t,s)f\|_r \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_q \tag{4.22}$$
$$\|T(t,s)^*g\|_r \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}\|g\|_q \tag{4.23}$$
hold for all $t > s \ge 0$ and $f, g \in L^q_\sigma(D)$ whenever (4.17) is satisfied.

2. *Let $1 < q \le r < \infty,\ 1 \le \rho \le \infty$. Then there exists a constant $C = C(m, q, r, \rho, D)$ such that*
$$\|T(t,s)f\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_{q,\rho} \tag{4.24}$$
$$\|T(t,s)^*g\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}\|g\|_{q,\rho} \tag{4.25}$$
hold for all $t > s \ge 0$ and $f, g \in L^{q,\rho}_\sigma(D)$ whenever (4.17) is satisfied.

3. *Let $1 < q < r < \infty, 1 \le \rho_1, \rho_2 \le \infty$. Then there exists a constant $C = C(m, q, r, \rho_1, \rho_2, D)$ such that*
$$\|T(t,s)f\|_{r,\rho_2} \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_{q,\rho_1} \tag{4.26}$$
$$\|T(t,s)^*g\|_{r,\rho_2} \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}\|g\|_{q,\rho_1} \tag{4.27}$$
hold for all $t > s \ge 0$ and $f, g \in L^{q,\rho_1}_\sigma(D)$ whenever (4.17) is satisfied.

4. *Let $1 < q \le r \le 3$. Then there exists a constant $C = C(m, q, r, D)$ such that*
$$\|\nabla T(t,s)f\|_r \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|f\|_q \tag{4.28}$$
$$\|\nabla T(t,s)^*g\|_r \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|g\|_q \tag{4.29}$$
hold for all $t > s \ge 0$ and $f, g \in L^q_\sigma(D)$ whenever (4.17) is satisfied.

5. *Let $1 < q \le r \le 3,\ 1 \le \rho < \infty$. Then there exists a constant $C = C(m, q, r, \rho, D)$ such that*
$$\|\nabla T(t,s)f\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|f\|_{q,\rho} \tag{4.30}$$
$$\|\nabla T(t,s)^*g\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|g\|_{q,\rho} \tag{4.31}$$
hold for all $t > s \ge 0$ and $f, g \in L^{q,\rho}_\sigma(D)$ whenever (4.17) is satisfied.

6. *Let $1 < q \le r \le 3$ with $1/q - 1/r = 1/3$. Then there exists a constant $C = C(m, q, D)$ such that*
$$\int_0^t \|\nabla T(t,s)^*g\|_{r,1}\, ds \le C\|g\|_{q,1} \tag{4.32}$$
holds for all $t > 0$ and $g \in L^{q,1}_\sigma(D)$ whenever (4.17) is satisfied.

To prove the L^∞ decay estimate in Theorem 4.2.1, we also prepare the following L^1-L^r estimates. The following estimates for data being in $C_0^\infty(D)^3$ are enough for later use, but it is clear that, for instance, the composite operator $T(t,s)P$ extends to a bounded operator from $L^1(D)$ to $L^r_\sigma(D)$ with the same estimate.

Lemma 4.3.4. *Let $m \in (0, \infty)$ and suppose (4.1).*

1. *Let $1 < r < \infty$. Then there is a constant $C = C(m, r, D) > 0$ such that*

$$\|T(t,s)Pf\|_r \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})}\|f\|_1, \tag{4.33}$$
$$\|T(t,s)^*Pg\|_r \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})}\|g\|_1 \tag{4.34}$$

for all $t > s \ge 0$ and $f, g \in C_0^\infty(D)^3$ whenever (4.17) is satisfied.

2. *Let $1 < r < \infty$ and $1 \le \rho \le \infty$. Then there is a constant $C = C(m, r, \rho, D) > 0$ such that*

$$\|T(t,s)Pf\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})}\|f\|_1, \tag{4.35}$$
$$\|T(t,s)^*Pg\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})}\|g\|_1 \tag{4.36}$$

for all $t > s \ge 0$ and $f, g \in C_0^\infty(D)^3$ whenever (4.17) is satisfied.

3. *Let $1 < r \le 3$, $1 \le \rho < \infty$. Then there is a constant $C = C(m, r, \rho, D) > 0$ such that*

$$\|\nabla T(t,s)Pf\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})-\frac{1}{2}}\|f\|_1, \tag{4.37}$$
$$\|\nabla T(t,s)^*Pg\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})-\frac{1}{2}}\|g\|_1 \tag{4.38}$$

for all $t > s \ge 0$ and $f, g \in C_0^\infty(D)^3$ whenever (4.17) is satisfied.

Proof. The proof is simply based on duality argument (see Koba [38, Lemma 2.15]), however, we give it for completeness. Let $1 < q \le r < \infty$ and $1/r + 1/r' = 1$. By using (4.23), we see that

$$|(T(t,s)Pf, \varphi)| = |(f, T(t,s)^*\varphi)| \le \|f\|_1\|T(t,s)^*\varphi\|_\infty \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})}\|f\|_1\|\varphi\|_{r'} \tag{4.39}$$

for all $\varphi \in L^{r'}_\sigma(D)$, which implies (4.33). We next show (4.35). We fix q such that $1 < q < r$. Combining the estimate (4.26) with (4.33), we have

$$\|T(t,s)Pf\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}\left\|T\left(\frac{t+s}{2}, s\right)Pf\right\|_q \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})}\|f\|_1.$$

Finally, in view of (4.30) and (4.35), we have

$$\|\nabla T(t,s)Pf\|_{r,\rho} \le C(t-s)^{-\frac{1}{2}}\left\|T\left(\frac{t+s}{2}, s\right)Pf\right\|_{r,\rho} \le C(t-s)^{-\frac{3}{2}(1-\frac{1}{r})-\frac{1}{2}}\|f\|_1$$

which implies (4.37). The proof for the adjoint $T(t,s)^*$ is accomplished in the same way. $\square$

4.4 Proof of the main theorems

In this section we prove the main theorems (Theorem 4.2.1 and Theorem 4.2.3). We first give some key estimates and then show Theorem 4.2.3. After that, following Yamazaki [48], we construct a solution with some decay properties for (4.11) and then derive the L^∞ decay of the solution. We finally identify the solution above with a local solution possessing better regularity for the integral equation (4.10) in a neighborhood of each time $t > 0$.

Let us define the function spaces

$$X = \{v \in BC_{w^*}((0,\infty); L^{3,\infty}_\sigma(D)) \mid \lim_{t\to 0} \|v(t)\|_{3,\infty} = 0\},$$
$$X_q = \{v \in X \mid t^{\frac{1}{2}-\frac{3}{2q}} v \in BC_{w^*}((0,\infty); L^{q,\infty}_\sigma(D))\}, \quad 3 < q < \infty.$$

Both are Banach spaces endowed with norms $\|\cdot\|_X = \|\cdot\|_{X,\infty}$ and $\|\cdot\|_{X_q} = \|\cdot\|_{X_q,\infty}$, respectively, where

$$\|v\|_{X,t} = \sup_{0<\tau<t} \|v(\tau)\|_{3,\infty}, \quad \|v\|_{X_q,t} = \|v\|_{X,t} + [v]_{q,t}, \quad [v]_{q,t} = \sup_{0<\tau<t} \tau^{\frac{1}{2}-\frac{3}{2q}} \|v(\tau)\|_{q,\infty}$$

for $t \in (0,\infty]$.

Lemma 4.4.1. *1. Let $v, w \in X$ and set*

$$\langle \mathcal{I}(v,w)(t), \varphi\rangle := \int_0^t (v(\tau)\otimes w(\tau), \nabla T(t,\tau)^*\varphi)\, d\tau,$$
$$\langle \mathcal{J}(v)(t), \varphi\rangle := \int_0^t (\psi(\tau)\{v(\tau)\otimes u_s + u_s \otimes v(\tau)\}, \nabla T(t,\tau)^*\varphi)\, d\tau$$

for all $\varphi \in C^\infty_{0,\sigma}(D)$. Then $\mathcal{I}(v,w), \mathcal{J}(v) \in X$ and there exists a positive constant C such that

$$\|\mathcal{I}(v,w)\|_{X,t} \le C\|v\|_{X,t}\|w\|_{X,t}, \qquad \|\mathcal{J}(v)\|_{X,t} \le C\|u_s\|_{3,\infty}\|v\|_{X,t} \tag{4.40}$$

hold for any $v, w \in X$ and $t \in (0,\infty]$.

2. Let $q \in (3,\infty)$. If $v \in X_q$, $w \in X$, then $\mathcal{I}(v,w), \mathcal{J}(v) \in X_q$ and there exists a positive constant $C = C(q)$ such that

$$\|\mathcal{I}(v,w)\|_{X_q,t} \le C\|v\|_{X_q,t}\|w\|_{X,t}, \quad \|\mathcal{J}(v)\|_{X_q,t} \le C\|u_s\|_{3,\infty}\|v\|_{X_q,t} \tag{4.41}$$

hold for every $v \in X_q$, $w \in X$ and $t \in (0,\infty]$.

3. We set

$$\langle \mathcal{K}(t), \varphi\rangle := \int_0^t (H(\tau), T(t,\tau)^*\varphi)\, d\tau$$

for $\varphi \in C^\infty_{0,\sigma}(D)$. Let $q \in (3,\infty)$. Then $\mathcal{K} \in X_q$ and there exist positive constants C independent of q and $C' = C'(q)$ such that

$$\|\mathcal{K}\|_{X,t} \le C(a^2 + \alpha|a|), \quad \|\mathcal{K}\|_{X_q,t} \le C'(a^2 + \alpha|a|) \tag{4.42}$$

hold for every $t \in (0,\infty]$.

Proof. Estimates (4.40) and (4.41) can be proved in the same way as done by Yamazaki [48, Lemma 6.1.], see also Hishida and Shibata [36, Section 8], however, we briefly give the proof of $(4.40)_1$ and $(4.41)_1$. By (4.32), we have

$$|\langle \mathcal{I}(v,w)(t), \varphi\rangle| \le \|v\|_{X,t}\|w\|_{X,t} \int_0^t \|\nabla T(t,\tau)^*\varphi\|_{3,1} \le C\|v\|_{X,t}\|w\|_{X,t}\|\varphi\|_{\frac{3}{2},1},$$

which yields $(4.40)_1$. We choose r such that $1/3+1/q+1/r=1$ to find

$$|\langle \mathcal{I}(v,w)(t),\varphi\rangle| \le [v]_{X_q,t}\|w\|_{X,t}\int_0^t \tau^{-\frac{1}{2}+\frac{3}{2q}}\|\nabla T(t,\tau)^*\varphi\|_{r,1}\,d\tau = [v]_{q,t}\|w\|_{X,t}\left(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\right).$$

In view of (4.31), we have

$$\int_0^{\frac{t}{2}} \le C\int_0^{\frac{t}{2}} \tau^{-\frac{1}{2}+\frac{3}{2q}}(t-\tau)^{-1}\,d\tau\|\varphi\|_{q',1} \le Ct^{-\frac{1}{2}+\frac{3}{2q}}\|\varphi\|_{q',1},$$

where $1/q+1/q'=1$, whreas, (4.32) implies

$$\int_{\frac{t}{2}}^t \le \left(\frac{t}{2}\right)^{-\frac{1}{2}+\frac{3}{2q}}\int_0^t \|\nabla T(t,\tau)^*\varphi\|_{r,1}\,d\tau \le Ct^{-\frac{1}{2}+\frac{3}{2q}}\|\varphi\|_{q',1}$$

from which together with $(4.40)_1$, we obtain $(4.41)_1$. The estimate (4.40) leads us to

$$\lim_{t\to 0}\|\mathcal{I}(v,w)(t)\|_{3,\infty}=0, \quad \lim_{t\to 0}\|\mathcal{J}(v)(t)\|_{3,\infty}=0.$$

Let us consider the weak-$*$ continuity of $\mathcal{I}(v,w)$ with values in $L^{3,\infty}_\sigma(D)$ (resp. $L^{q,\infty}_\sigma(D)$) when $v\in X$ (resp. $v\in X_q$), $w\in X$. Here, we need a different argument from [48] because of the non-autonomous character as well as the non-analyticity of the corresponding semigroup. Since $C^\infty_{0,\sigma}(D)$ is dense in $L^{\kappa,1}_\sigma$ ($\kappa=3/2,\ q'$) and since we know (4.40) and (4.41), it suffices to show that

$$|\langle \mathcal{I}(v,w)(t)-\mathcal{I}(v,w)(\sigma),\varphi\rangle| \to 0 \quad \text{as} \quad \sigma\to t \tag{4.43}$$

for all $0<t<\infty$ and $\varphi\in C^\infty_{0,\sigma}(D)$. Let $0<\sigma<t$. By using the backward semigroup property, we have

$$\begin{aligned}|\langle \mathcal{I}(v,w)(t)-\mathcal{I}(v,w)(\sigma),\varphi\rangle| \le &\int_0^\sigma |(v(\tau)\otimes w(\tau),\nabla T(\sigma,\tau)^*(T(t,\sigma)^*\varphi-\varphi)|\,d\tau \\ &+\int_\sigma^t |(v(\tau)\otimes w(\tau),\nabla T(t,\tau)^*\varphi)|\,d\tau =: I_1+I_2.\end{aligned}$$

The estimate (4.32) yields

$$\begin{aligned} I_1 &\le \|v\|_{X,t}\|w\|_{X,t}\int_0^\sigma \|\nabla T(\sigma,\tau)^*(T(t,\sigma)^*\varphi-\varphi)\|_{3,1}\,d\tau \\ &\le C\|v\|_{X,t}\|w\|_{X,t}\,\|T(t,\sigma)^*\varphi-\varphi\|_{\frac{3}{2},1}\to 0 \quad \text{as } \sigma\to t.\end{aligned}$$

Furthermore, (4.31) yields

$$I_2 \le \|v\|_{X,t}\|w\|_{X,t}\int_\sigma^t \|\nabla T(t,\tau)^*\varphi\|_{3,1}\,d\tau \le C\|v\|_{X,t}\|w\|_{X,t}(t-\sigma)^{\frac{1}{2}}\|\varphi\|_{3,1}\to 0 \quad \text{as } \sigma\to t.$$

We can discuss the other case $0<t<\sigma$ similarly and thus we obtain (4.43). By the same manner, we can obtain the desired weak-$*$ continuity of $\mathcal{J}$. We thus conclude the assertion 1 and 2.

We next consider $\mathcal{K}(t)$. We use (4.25) as well as (4.12) to obtain

$$|\langle \mathcal{K}(t), \varphi\rangle| \le C(a^2+\alpha|a|)\int_0^{\min\{1,t\}} |\|T(t,\tau)^*\varphi\|_{\frac{3}{2},1}\,d\tau \le C(a^2+\alpha|a|)\min\{1,t\}\|\varphi\|_{\frac{3}{2},1}$$

for $\varphi \in C^\infty_{0,\sigma}(D)$ and $t>0$ which yields $\mathcal{K}(t) \in L^{3,\infty}_\sigma(D)$ with

$$\|\mathcal{K}\|_{X,t} \le C(a^2+\alpha|a|) \quad \text{for } t \in (0,\infty], \quad \lim_{t\to 0}\|\mathcal{K}(t)\|_{3,\infty} = 0.$$

To derive the estimate $[\mathcal{K}]_{q,t} \le C(a^2+\alpha|a|)$, we consider two cases: $0<t\le 2$ and $t\ge 2$. For $0<t\le 2$, (4.25) yields

$$|\langle \mathcal{K}(t), \varphi\rangle| \le C(a^2+\alpha|a|)\int_0^t \|T(t,\tau)^*\varphi\|_{\frac{3}{2},1}\,d\tau \le C(a^2+\alpha|a|)\|\varphi\|_{q',1}.$$

For $t\ge 2$, we have

$$|\langle \mathcal{K}(t), \varphi\rangle| \le C(a^2+\alpha|a|)\int_0^1 \|T(t,\tau)^*\varphi\|_{\frac{3}{2},1}\,d\tau \le C(a^2+\alpha|a|)\,t^{-\frac{1}{2}+\frac{3}{2q}}\|\varphi\|_{q',1}.$$

We thus obtain (4.42). It remains to show the weak-$*$ continuity. To this end, it is sufficient to show that

$$|\langle \mathcal{K}(t)-\mathcal{K}(\sigma), \varphi\rangle| \to 0 \ \text{ as } \sigma \to t \tag{4.44}$$

for all $t\in(0,\infty)$ due to (4.42). To prove (4.44), we suppose $0<\sigma<t$. We use the backward semigroup property to observe

$$\langle \mathcal{K}(t)-\mathcal{K}(\sigma), \varphi\rangle = \int_0^\sigma \big(H(\tau), T(\sigma,\tau)^*(T(t,\sigma)^*\varphi-\varphi)\big)\,d\tau + \int_\sigma^t (H(\tau), T(t,\tau)^*\varphi)\,d\tau.$$

By applying (4.25), we find that

$$\left|\int_0^\sigma \big(H(\tau), T(\sigma,\tau)^*(T(t,\sigma)^*\varphi-\varphi)\big)\,d\tau\right| \le C(a^2+\alpha|a|)\,\sigma\,\|T(t,\sigma)^*\varphi-\varphi\|_{\frac{3}{2},1} \to 0 \quad \text{as } \sigma\to t,$$
$$\left|\int_\sigma^t (H(\tau), T(t,\tau)^*\varphi)\,d\tau\right| \le C(a^2+\alpha|a|)(t-\sigma)\|\varphi\|_{\frac{3}{2},1} \to 0 \quad \text{as } \sigma \to t.$$

The other case $t<\sigma$ is discussed similarly. Hence we have (4.44). The proof is complete. □

Proof of Theorem 4.2.3. The idea of the proof is traced back to Fujita and Kato [21, Theorem 3.1.]. Let v_1 and v_2 be the solutions of (4.11). Then we have

$$\begin{aligned}(v_1(t)-v_2(t),\varphi) = \int_0^t \big(&v_1(\tau)\otimes(v_1(\tau)-v_2(\tau)) + (v_1(\tau)-v_2(\tau))\otimes v_2(\tau) \\ &+ \psi(\tau)(v_1(\tau)-v_2(\tau))\otimes u_s + \psi(\tau)u_s\otimes(v_1(\tau)-v_2(\tau)), \nabla T(t,\tau)^*\varphi\big)\,d\tau\end{aligned} \tag{4.45}$$

for $\varphi\in C^\infty_{0,\sigma}(D)$. By applying (4.40) to (4.45) and by Proposition 4.3.1, we have

$$\begin{aligned}\|v_1-v_2\|_{X,t} &\le C(\|v_1\|_{X,t}+\|v_2\|_{X,t}+\|u_s\|_{3,\infty})\|v_1-v_2\|_{X,t} \\ &\le C(\|v_1\|_{X,t}+\|v_2\|_{X,t}+|a|)\|v_1-v_2\|_{X,t}.\end{aligned}$$

Suppose

$$|a| < \frac{1}{2C} =: \widetilde{\delta}.$$

Since $\|v_j(t)\|_{3,\infty} \to 0$ as $t \to 0$ ($j = 1, 2$), one can choose $t_0 > 0$ such that $C(\|v_1\|_{X,t_0} + \|v_2\|_{X,t_0}) < 1/2$, which implies $v_1 = v_2$ on $(0, t_0]$. Hence, (4.45) is written as

$$\begin{aligned}(v_1(t) - v_2(t), \varphi) = \int_{t_0}^{t} \big(&v_1(\tau) \otimes (v_1(\tau) - v_2(\tau)) + (v_1(\tau) - v_2(\tau)) \otimes v_2(\tau) \\ &+ \psi(\tau)(v_1(\tau) - v_2(\tau)) \otimes u_s + \psi(\tau)u_s \otimes (v_1(\tau) - v_2(\tau)), \nabla T(t, \tau)^*\varphi\big)\, d\tau.\end{aligned} \tag{4.46}$$

We fix $\mathcal{T} \in (t_0, \infty)$ and set $[v]_{q,t_0,t} = \sup_{t_0 \le \tau \le t} \|v(\tau)\|_q$ for $t \in (t_0, \mathcal{T})$. It follows from (4.46) that

$$[v_1 - v_2]_{q,t_0,t} \le C_*(t - t_0)^{\frac{1}{2} - \frac{3}{2q}}[v_1 - v_2]_{q,t_0,t}\,, \quad t \in (t_0, \mathcal{T}), \tag{4.47}$$

where $C_* = C_*(t_0, \mathcal{T}) = C([v_1]_{q,t_0,\mathcal{T}} + [v_2]_{q,t_0,\mathcal{T}} + 2\|u_s\|_q\big)$. In fact, the estimate (4.29) yields

$$\begin{aligned}\int_{t_0}^{t} &\big|\big(v_1(\tau) \otimes (v_1(\tau) - v_2(\tau)), \nabla T(t, \tau)^*\varphi\big)\big|\, d\tau \\ &\le C[v_1(\tau)]_{q,t_0,\mathcal{T}}[v_1 - v_2]_{q,t_0,t} \int_{t_0}^{t} \|\nabla T(t, \tau)^*\varphi\|_{(1-\frac{2}{q})^{-1}}\, d\tau \\ &\le C[v_1(\tau)]_{q,t_0,\mathcal{T}}[v_1 - v_2]_{q,t_0,t}(t - t_0)^{\frac{1}{2} - \frac{3}{2q}}\|\varphi\|_{(1-\frac{1}{q})^{-1}}\end{aligned}$$

for all $\varphi \in C^\infty_{0,\sigma}(D)$ and $t \in (t_0, \mathcal{T})$. Since the other terms in (4.46) are treated similarly, we obtain (4.47). We take

$$\xi = \min\left\{\left(\frac{1}{2C_*}\right)^{(\frac{1}{2} - \frac{3}{2q})^{-1}}, \mathcal{T} - t_0\right\}$$

which leads us to $v_1 = v_2$ on $(0, t_0 + \xi)$. Even though we replace t_0 by $t_0 + \xi, t_0 + 2\xi, \cdots$, we can discuss similarly. Hence, $v_1 = v_2$ on $(0, \mathcal{T})$. Since $\mathcal{T}$ is arbitrary, we conclude $v_1 = v_2$. □

To prove Theorem 4.2.1, we begin to construct a solution of (4.11) by applying Lemma 4.4.1.

Proposition 4.4.2. *Let ψ be a function on $\mathbb{R}$ satisfying (4.1). We put $\alpha = \max_{t \in \mathbb{R}} |\psi'(t)|$.*

1. *There exists $\delta_1 > 0$ such that if $(\alpha + 1)|a| \le \delta_1$, problem (4.11) admits a unique solution within the class*

$$\begin{aligned}\big\{v \in BC_{w^*}\big((0, \infty); L^{3,\infty}_\sigma(D)\big) \;\big|\; &\lim_{t \to 0} \|v(t)\|_{3,\infty} = 0, \\ &\sup_{0 < \tau < \infty} \|v(\tau)\|_{3,\infty} \le C(\alpha + 1)|a|\big\},\end{aligned}$$

where $C > 0$ is independent of a and ψ.

2. Let $3 < q < \infty$. Then there exists $\delta_2(q) \in (0, \delta_1]$ such that if $(\alpha+1)|a| \le \delta_2$,

$$t^{\frac{1}{2}-\frac{3}{2q}} v \in BC_{w^*}\big((0,\infty); L^{q,\infty}_\sigma(D)\big),$$

where $v(t)$ is the solution obtained above.

Proof. We first show the assertion 1 by the contraction mapping principle. Given $v \in X$, we define

$$\langle (\Phi v)(t), \varphi \rangle = \text{the RHS of (4.11)}, \quad \varphi \in \mathrm{C}^\infty_{0,\sigma}(\mathrm{D}).$$

Lemma 4.4.1 implies that $\Phi v \in X$ with

$$\|\Phi v\|_X \le C_1 \|v\|_X^2 + C_2 |a| \|v\|_X + C_3(a^2 + \alpha|a|), \tag{4.48}$$
$$\|\Phi v - \Phi w\|_X \le (C_1\|v\|_X + C_1\|w\|_X + C_2|a|)\|v - w\|_X \tag{4.49}$$

for every $v, w \in X$. Here, C_1, C_2, C_3, C_4 are constants independent of v, w, a and ψ. Hence, if we take a satisfying

$$(\alpha+1)|a| < \min\left\{\frac{1}{2C_2}, \frac{1}{16C_1C_3}, \eta\right\} =: \delta_1,$$

where $\eta \in (0,1]$ is a constant given in Proposition 4.3.1, then we obtain a unique solution v within the class

$$\{v \in X \mid \|v\|_X \le 4C_3(\alpha+1)|a|\}$$

which completes the proof of the assertion 1.

We next show the assertion 2. By applying Lemma 4.4.1, we see that $\Phi v \in X_q$ together with (4.48)–(4.49) in which X norm was replaced by X_q norm and the constants C_i $(i = 1, 2, 3)$ are also replaced by some others $\widetilde{C}_i(q)(\ge C_i)$. If we assume

$$(\alpha+1)|a| < \min\left\{\frac{1}{2\widetilde{C}_2}, \frac{1}{16\widetilde{C}_1\widetilde{C}_3}, \eta\right\} =: \delta_2\,(\le \delta_1), \tag{4.50}$$

we can obtain a unique solution $\hat{v}$ within the class

$$\{v \in X_q \mid \|v\|_{X_q} \le 4\widetilde{C}_3(\alpha+1)|a|\}.$$

Under the condition (4.50), let v be the solution obtained in the assertion 1. Then we have (4.45) in which v_1, v_2 are replaced by v and $\hat{v}$. By applying (4.40), we see that

$$\|v - \hat{v}\|_X \le \big\{C_1(\|v\|_X + \|\hat{v}\|_X) + C_2|a|\big\}\|v - \hat{v}\|_X \le \big\{8\,\widetilde{C}_1\widetilde{C}_3(1+\alpha)|a| + \widetilde{C}_2|a|\big\}\|v - \hat{v}\|_X.$$

Furthermore, the condition (4.50) yields

$$8\,\widetilde{C}_1\widetilde{C}_3(1+\alpha)|a| + \widetilde{C}_2|a| < 1$$

which leads us to $v = \hat{v}$. The proof is complete. □

We note that Proposition 4.4.2 implies

$$t^{\frac{1}{2}-\frac{3}{2r}}v \in BC_w\big((0,\infty); L^r_\sigma(D)\big) \tag{4.51}$$

for all $r \in (3,q)$ by the interpolation inequality

$$\|f\|_r \le C\|f\|_{3,\infty}^{1-\beta}\|f\|_{q,\infty}^{\beta}, \quad \frac{1}{r} = \frac{1-\beta}{3} + \frac{\beta}{q},$$

see (4.21).

Let $q \in (6,\infty)$, then the solution obtained in Proposition 4.4.2 also fulfills the following decay properties.

Proposition 4.4.3. *Let ψ be a function on $\mathbb{R}$ satisfying (4.1) and we put $\alpha := \max_{t\in\mathbb{R}} |\psi'(t)|$. Suppose that $6 < q < \infty$. Then, under the same condition as in the latter part of Proposition 4.4.2, the solution v obtained in Proposition 4.4.2 satisfies $v(t) \in L^r(D)$ $(t > 0)$ with*

$$\|v(t)\|_r = O(t^{-\frac{1}{2}+\frac{3}{2q}}) \quad \text{as } t \to \infty \tag{4.52}$$

for $r \in (q,\infty]$.

Proof. We first show (4.52) with $r = \infty$, that is, $v(t) \in L^\infty(D)$ for $t > 0$ with

$$\|v(t)\|_\infty = O(t^{-\frac{1}{2}+\frac{3}{2q}}) \quad \text{as } t \to \infty. \tag{4.53}$$

We note by continuity that $C^\infty_{0,\sigma}(D)$ can be replaced by $PC^\infty_0(D)$ as the class of test functions in (4.11). Hence, it follows that

$$\sup_{\varphi\in C_0^\infty(D), \|\varphi\|_1\le 1} |(v(t),\varphi)| \le N_1 + N_2 + N_3 + N_4, \tag{4.54}$$

where

$$\begin{aligned}
N_1 &= \sup_{\varphi\in C_0^\infty(D), \|\varphi\|_1\le 1} \int_0^t |(v(\tau)\otimes v(\tau), \nabla T(t,\tau)^*P\varphi)|\, d\tau,\\
N_2 &= \sup_{\varphi\in C_0^\infty(D), \|\varphi\|_1\le 1} \int_0^t |(v(\tau)\otimes u_s, \nabla T(t,\tau)^*P\varphi)|\, d\tau,\\
N_3 &= \sup_{\varphi\in C_0^\infty(D), \|\varphi\|_1\le 1} \int_0^t |u_s\otimes v(\tau), \nabla T(t,\tau)^*P\varphi)|\, d\tau,\\
N_4 &= \sup_{\varphi\in C_0^\infty(D), \|\varphi\|_1\le 1} \int_0^t |(H(\tau), T(t,\tau)^*P\varphi)|\, d\tau.
\end{aligned}$$

We begin by considering N_1. In view of (4.38), we have

$$\begin{aligned}
\int_0^t |(v(\tau)\otimes v(\tau), \nabla T(t,\tau)^*P\varphi)|\, d\tau &\le C[v]_{q,\infty}^2 \int_0^t \tau^{-1+\frac{3}{q}}\, \|\nabla T(t,\tau)^*P\varphi\|_{(1-\frac{2}{q})^{-1},1}\, d\tau\\
&\le C[v]_{q,\infty}^2 \int_0^t \tau^{-1+\frac{3}{q}}(t-\tau)^{-\frac{3}{q}-\frac{1}{2}}\, d\tau\, \|\varphi\|_1\\
&\le C[v]_{q,\infty}^2 t^{-\frac{1}{2}}\|\varphi\|_1
\end{aligned}$$

for all $\varphi \in C_0^\infty(D)$ and $t > 0$. Here, the integrability is ensured because of $q \in (6, \infty)$. Hence we obtain

$$N_1 \le Ct^{-\frac{1}{2}} \quad \text{for } t > 0. \tag{4.55}$$

We next consider N_2. By applying (4.38), it follows that

$$\begin{aligned}\int_0^t |(v(\tau) \otimes u_s, \nabla T(t,\tau)^* P\varphi)|\, d\tau &\le [v]_{q,\infty} \|u_s\|_{q,\infty} \int_0^t \tau^{-\frac{1}{2}+\frac{3}{2q}} \|\nabla T(t,\tau)^* P\varphi\|_{(1-\frac{2}{q})^{-1},1}\, d\tau \\ &\le C[v]_{q,\infty} \|u_s\|_{q,\infty} t^{-\frac{3}{2q}} \|\varphi\|_1\end{aligned}$$

for $t > 0$. We thus have

$$N_2 \le Ct^{-\frac{3}{2q}} \quad \text{for } t > 0. \tag{4.56}$$

We next intend to derive the rate of decay N_2 as fast as possible. To this end, we split the integral into

$$\int_0^t |(v(\tau) \otimes u_s, \nabla T(t,\tau)^* P\varphi)|\, d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t \tag{4.57}$$

for $t > 2$. We apply (4.38) again to find

$$\begin{aligned}&\int_0^{\frac{t}{2}} \le \|u_s\|_{3,\infty} \|v\|_X \int_0^{\frac{t}{2}} \|\nabla T(t,\tau)^* P\varphi\|_{3,1}\, d\tau \le Ct^{-\frac{1}{2}} \|\varphi\|_1, \\ &\int_{\frac{t}{2}}^{t-1} \le \|u_s\|_{3,\infty} [v]_{q,\infty} \int_{\frac{t}{2}}^{t-1} \tau^{-\frac{1}{2}+\frac{3}{2q}} \|\nabla T(t,\tau)^* P\varphi\|_{(1-\frac{1}{3}-\frac{1}{q})^{-1},1}\, d\tau \le Ct^{-\frac{1}{2}+\frac{3}{2q}} \|\varphi\|_1\end{aligned}$$

and

$$\int_{t-1}^t \le \|u_s\|_{q,\infty} [v]_{q,\infty} \int_{t-1}^t \tau^{-\frac{1}{2}+\frac{3}{2q}} \|\nabla T(t,\tau)^* P\varphi\|_{(1-\frac{2}{q})^{-1},1}\, d\tau \le Ct^{-\frac{1}{2}+\frac{3}{2q}} \|\varphi\|_1$$

for all $\varphi \in C_0^\infty(D)$ and $t > 2$. Summing up the estimates above, we are led to

$$N_2 \le Ct^{-\frac{1}{2}+\frac{3}{2q}} \quad \text{for } t > 2. \tag{4.58}$$

Similarly, we have

$$N_3 \le Ct^{-\frac{3}{2q}} \quad \text{for } t > 0, \tag{4.59}$$

$$N_3 \le Ct^{-\frac{1}{2}+\frac{3}{2q}} \quad \text{for } t > 2. \tag{4.60}$$

It is easily seen from (4.1), (4.12) and (4.36) that

$$\begin{aligned}\int_0^t |(H(\tau), T(t,\tau)^* P\varphi)|\, d\tau &\le C(a^2 + \alpha|a|) \int_0^{\min\{1,t\}} \|T(t,\tau)^* P\varphi\|_{\frac{3}{2},1}\, d\tau \\ &\le C(a^2 + \alpha|a|) \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{1}{2}}\, d\tau\, \|\varphi\|_1\end{aligned}$$

for all $\varphi \in C_0^\infty(D)$ and $t > 0$, which yields

$$N_4 \le Ct^{\frac{1}{2}} \quad \text{for } t > 0, \tag{4.61}$$

$$N_4 \le Ct^{-\frac{1}{2}} \quad \text{for } t > 2. \tag{4.62}$$

Combining (4.54)–(4.62) implies $v(t) \in L^\infty(D)$ $(t > 0)$ and (4.53). In view of the interpolation relation

$$(L^{q,\infty}(D), L^\infty(D))_{1-\frac{q}{r},r} = L^r(D), \quad \|f\|_r \le C\|f\|_{q,\infty}^{\frac{q}{r}}\|f\|_\infty^{1-\frac{q}{r}}, \quad q < r < \infty,$$

we obtain (4.52) for $r \in (q, \infty]$ as well. This completes the proof. □

Remark 4.4.4. When the stationary solution possesses the scale-critical rate $O(1/|x|)$, the L^∞ decay of perturbation with less sharp rate $O(t^{-\frac{1}{2}+\varepsilon})$ was derived first by Koba [38] in the context of stability analysis, where $\varepsilon > 0$ is arbitrary. If we have a look only at the L^∞ decay rate, our rate is comparable with his result since $q \in (6, \infty)$ is arbitrary. However, we are not able to prove Proposition 4.4.3 by his method. This is because he doesn't split the integrals in N_2 and N_3, so that the rate of L^∞ decay is slower than the one of $L^{q,\infty}$ decay. From this point of view, Proposition 4.4.3 is regarded as a slight improvement of his result.

We next show that the solution v obtained in Proposition 4.4.2 actually satisfies the integral equation (4.10) by identifying v with a local solution $\widetilde{v}$ of (4.10) in a neighborhood of each time $t > 0$. To this end, we need the following lemma on the uniqueness. The proof is similar to the argument in the second half (after (4.46)) of the proof of Theorem 4.2.3 and thus we may omit it.

Lemma 4.4.5. *Let $3 < r < \infty$, $0 \le t_0 < t_1 < \infty$ and $v_0 \in L^r_\sigma(D)$. Then the problem*

$$\begin{aligned}(v(t), \varphi) = (v_0, T(t,t_0)^*\varphi) + \int_{t_0}^t \big(v(\tau)\otimes v(\tau) + \psi(\tau)\{v(\tau)\otimes u_s + u_s \otimes v(\tau)\}, \nabla T(t,\tau)^*\varphi\big)\, d\tau \\ + \int_{t_0}^t (H(\tau), T(t,\tau)^*\varphi)\, d\tau, \quad \forall \varphi \in C_{0,\sigma}^\infty(D)\end{aligned} \tag{4.63}$$

on (t_0, t_1) admits at most one solution within the class $L^\infty(t_0, t_1; L^r_\sigma(D))$. Here, H is given by (4.9).

Given $v_0 \in L^r_\sigma(D)$ with $r \in (3, \infty)$, let us construct a local solution of the integral equation

$$v(t) = T(t,t_0)v_0 + \int_{t_0}^t T(t,\tau)P[(Gv)(\tau) + H(\tau)]\, d\tau, \tag{4.64}$$

where G and H are defined by (4.8) and (4.9), respectively. For $0 \le t_0 < t_1 < \infty$ and $r \in (3, \infty)$, we define the function space

$$Y_r(t_0,t_1) = \big\{v \in C\big([t_0,t_1]; L^r_\sigma(D)\big)\;\big|\; (\cdot - t_0)^{\frac{1}{2}}\nabla v(\cdot) \in BC_w\big((t_0,t_1]; L^r(D)\big)\big\} \tag{4.65}$$

which is a Banach space equipped with norm

$$\|v\|_{Y_r(t_0,t_1)} = \sup_{t_0 \le \tau \le t_1} \|v(\tau)\|_r + \sup_{t_0 < \tau \le t_1} (\tau - t_0)^{\frac{1}{2}}\|\nabla v(\tau)\|_r \tag{4.66}$$

and set

$$U_1(v,w)(t) = \int_{t_0}^t T(t,\tau)P[v(\tau)\cdot\nabla w(\tau)]\, d\tau, \quad U_2(v)(t) = \int_{t_0}^t T(t,\tau)P[\psi(\tau)v(\tau)\cdot\nabla u_s]\, d\tau,$$

$$U_3(v)(t) = \int_{t_0}^t T(t,\tau)P[\psi(\tau)u_s\cdot\nabla v(\tau)]\, d\tau, \quad U_4(t) = \int_{t_0}^t T(t,\tau)PH(\tau)\, d\tau. \tag{4.67}$$

Lemma 4.4.6. *Let $3 < r < \infty$ and $0 \le t_0 < t_1 \le t_0 + 1$. Suppose that $v, w \in Y_r(t_0, t_1)$. Then $U_1(v, w), U_2(v), U_3(v), U_4 \in Y_r(t_0, t_1)$. Furthermore, there exists a constant $C = C(r, t_0)$ such that*

$$\|U_1(v,w)\|_{Y_r(t_0,t)} \le C(t-t_0)^{\frac{1}{2}-\frac{3}{2r}} \|v\|_{Y_r(t_0,t)} \|w\|_{Y_r(t_0,t)}, \tag{4.68}$$

$$\|U_2(v)\|_{Y_r(t_0,t)} \le C(t-t_0)^{1-\frac{3}{2r}} \|\nabla u_s\|_r \|v\|_{Y_r(t_0,t)}, \tag{4.69}$$

$$\|U_3(v)\|_{Y_r(t_0,t)} \le C(t-t_0)^{\frac{1}{2}-\frac{3}{2r}} \|u_s\|_r \|v\|_{Y_r(t_0,t)}, \tag{4.70}$$

$$\|U_4\|_{Y_r(t_0,t)} \le C(t-t_0)^{\frac{1}{2}+\frac{3}{2r}} (a^2 + \alpha|a|) \tag{4.71}$$

for all $t \in (t_0, t_1]$.

Proof. In view of (4.22), we have

$$\|U_1(t)\|_r \le C \int_{t_0}^t (t-\tau)^{-\frac{3}{2r}} \|v\|_r \|\nabla w\|_r \, d\tau \le C(t-t_0)^{-\frac{3}{2r}+\frac{1}{2}} \|v\|_{Y_r(t_0,t)} \|w\|_{Y_r(t_0,t)}. \tag{4.72}$$

Furthermore, (4.16) with $\mathcal{T} = t_0 + 1$ yields

$$\|\nabla U_1(t)\|_r \le C(t-t_0)^{-\frac{3}{2r}} \|v\|_{Y_r(t_0,t)} \|w\|_{Y_r(t_0,t)}. \tag{4.73}$$

By (4.72) and (4.73), we obtain (4.68). Similarly, we can show (4.69)–(4.71). We note that the estimate (4.71) follows from (4.18) with $\mathcal{T} = t_0 + 1$ together with (4.12).

We next show the continuity of U_1 with respect to t. Let $t_2 \in [t_0, t_1]$. If $t_2 < t$, we have

$$\begin{aligned} U_1(t) - U_1(t_2) &= \int_{t_0}^{t_2} (T(t,t_2) - 1) T(t_2, \tau) P[v(\tau) \cdot \nabla w(\tau)] \, d\tau + \int_{t_2}^t T(t,\tau) P[v(\tau) \cdot \nabla w(\tau)] \, d\tau \\ &=: U_{11}(t) + U_{12}(t). \end{aligned}$$

Lebesgue's convergence theorem yields that $\|U_{11}(t)\|_r \to 0$ as $t \to t_2$, while

$$\|U_{12}(t)\|_r \le C(t-t_2)^{\frac{1}{2}-\frac{3}{2r}} \|v\|_{Y_r(t_0,t_1)} \|w\|_{Y_r(t_0,t_1)} \to 0 \quad \text{as } t \to t_2.$$

To discuss the case $t < t_2$, we need the following device. Let $(t_0 + t_2)/2 \le \tilde{t} < t < t_2$, where $\tilde{t}$ will be determined later, then

$$\begin{aligned} U_1(t) - U_1(t_2) = &\left(\int_{t_0}^{\tilde{t}} + \int_{\tilde{t}}^t \right) T(t,\tau) P[v(\tau) \cdot \nabla w(\tau)] \, d\tau \\ &- \left(\int_{t_0}^{\tilde{t}} + \int_{\tilde{t}}^{t_2} \right) T(t_2,\tau) P[v(\tau) \cdot \nabla w(\tau)] \, d\tau. \end{aligned}$$

We observe that

$$\begin{aligned} &\int_{\tilde{t}}^t \|T(t,\tau) P[v(\tau) \cdot \nabla w(\tau)]\|_r \, d\tau + \int_{\tilde{t}}^{t_2} \|T(t_2,\tau) P[v(\tau) \cdot \nabla w(\tau)]\|_r \, d\tau \\ &\le C \|v\|_{Y_r(t_0,t_1)} \|w\|_{Y_r(t_0,t_1)} \left(\int_{\tilde{t}}^t (t-\tau)^{-\frac{3}{2r}} (\tau - t_0)^{-\frac{1}{2}} \, d\tau + \int_{\tilde{t}}^{t_2} (t_2-\tau)^{-\frac{3}{2r}} (\tau - t_0)^{-\frac{1}{2}} \, d\tau \right) \\ &\le \frac{2C}{1-\frac{3}{2r}} \|v\|_{Y_r(t_0,t_1)} \|w\|_{Y_r(t_0,t_1)} \left(\frac{t_0+t_2}{2} - t_0 \right)^{-\frac{1}{2}} (t_2 - \tilde{t})^{1-\frac{3}{2r}}. \end{aligned}$$

For any $\varepsilon > 0$, we choose $\tilde{t}$ such that

$$\frac{2C}{1-\frac{3}{2r}}\|v\|_{Y_r(t_0,t_1)}\|w\|_{Y_r(t_0,t_1)}\left(\frac{t_0+t_2}{2}-t_0\right)^{-\frac{1}{2}}(t_2-\tilde{t}\,)^{1-\frac{3}{2r}} < \varepsilon$$

which yields

$$\|U_1(t)-U_1(t_2)\|_r \leq \int_{t_0}^{\tilde{t}} \big\|\big(T(t,\tau)-T(t_2,\tau)\big)P[v(\tau)\cdot\nabla w(\tau)]\big\|_r \, d\tau + \varepsilon \quad \text{for } \tilde{t} < t < t_2$$

and therefore,

$$\limsup_{t\to t_2} \|U_1(t)-U_1(t_2)\|_r \leq \limsup_{t\to t_2} \int_{t_0}^{\tilde{t}} \big\|\big(T(t,\tau)-T(t_2,\tau)\big)P[v(\tau)\cdot\nabla w(\tau)]\big\|_r \, d\tau + \varepsilon. \quad (4.74)$$

Since $\big\|\big(T(t,\tau)-T(t_2,\tau)\big)P[v(\tau)\cdot\nabla w(\tau)]\big\|_r = \big\|\big(T(t,\tilde{t}\,)-T(t_2,\tilde{t}\,)\big)T(\tilde{t},\tau)P[v(\tau)\cdot\nabla w(\tau)]\big\|_r$ tends to 0 as $t \to t_2$ for $t_0 < \tau < \tilde{t}$, it follows from Lebesgue's convergence theorem that the integral term in (4.74) tends to 0 as $t \to t_2$. Since $\varepsilon > 0$ is arbitrary, we have

$$U_1 \in C\big([t_0,t_1];L^r_\sigma(D)\big). \quad (4.75)$$

Furthermore, we find $\nabla U_1 \in C_w\big((t_0,t_1];L^r(D)\big)$ on account of (4.73) and (4.75) together with the relation

$$(\nabla U_1(t)-\nabla U_1(t_2),\varphi) = -(U_1(t)-U_1(t_2),\nabla\cdot\varphi)$$

for all $t_2 \in (t_0,t_1]$ and $\varphi \in C_0^\infty(D)^{3\times 3}$. Since U_2, U_3 and U_4 are discussed similarly, the proof is complete. □

The following proposition provides a local solution of (4.64).

Proposition 4.4.7. *Let $3 < r < \infty$, $t_0 \geq 0$ and $v_0 \in L^r_\sigma(D)$. There exists $t_1 \in (t_0, t_0+1]$ such that (4.64) admits a unique solution $v \in Y_r(t_0,t_1)$. Moreover, the length of the existence interval can be estimated from below by*

$$t_1 - t_0 \geq \zeta(\|v_0\|_r),$$

where $\zeta(\cdot) : [0,\infty) \to (0,1)$ is a non-increasing function defined by (4.79) below.

Proof. We put

$$(\Psi v)(t) = \text{the RHS of (4.64)}.$$

By applying Lemma 4.4.6, we have

$$\|\Psi v\|_{Y_r(t_0,t)} \leq (C_1\|v\|^2_{Y_r(t_0,t)} + C_2\|v\|_{Y_r(t_0,t)} + C_3)(t-t_0)^{\frac{1}{2}-\frac{3}{2r}} + C_4\|v_0\|_r,$$
$$\|\Psi v - \Psi w\|_{Y_r(t_0,t)} \leq \{C_1(\|v\|_{Y_r(t_0,t)} + \|w\|_{Y_r(t_0,t)}) + C_2\}(t-t_0)^{\frac{1}{2}-\frac{3}{2r}}\|v-w\|_{Y_r(t_0,t)}$$

for all $t \in (t_0, t_0+1]$ and $v, w \in Y_r(t_0,t)$. We note that the constants C_i may be dependent on $\|u_s\|_r$, $\|\nabla u_s\|_r$, α and a. We choose $t_1 \in (t_0, t_0+1]$ such that

$$C_2(t_1-t_0)^{\frac{1}{2}-\frac{3}{2r}} < \frac{1}{2}, \quad (4.76)$$

$$8C_1(t_1-t_0)^{\frac{1}{2}-\frac{3}{2r}}\{C_3(t_1-t_0)^{\frac{1}{2}-\frac{3}{2r}} + C_4\|v_0\|_r\} < \frac{1}{2} \quad (4.77)$$

which imply

$$\lambda := \big\{1 - C_2(t_1 - t_0)^{\frac{1}{2}-\frac{3}{2r}}\big\}^2 - 4C_1(t_1 - t_0)^{\frac{1}{2}-\frac{3}{2r}}\big\{C_3(t_1 - t_0)^{\frac{1}{2}-\frac{3}{2r}} + C_4\|v_0\|_r\big\} > 0. \tag{4.78}$$

We set

$$\Lambda := \frac{1 - C_2(t_1 - t_0)^{\frac{1}{2}-\frac{3}{2r}} - \sqrt{\lambda}}{2C_1(t_1 - t_0)^{\frac{1}{2}-\frac{3}{2r}}} < 4(C_3 + C_4\|v_0\|_r),$$
$$Y_{r,\Lambda}(t_0, t_1) := \{v \in Y_r(t_0, t_1) \mid \|v\|_{Y_r(t_0,t_1)} \le \Lambda\}.$$

Then we find that the map $\Psi : Y_{r,\Lambda}(t_0, t_1) \to Y_{r,\Lambda}(t_0, t_1)$ is well-defined and also contractive. Hence we obtain a local solution. Indeed, the conditions (4.76) and (4.77) are accomplished by

$$t_1 - t_0 < \min\left\{1, \left(\frac{1}{2C_2}\right)^{(\frac{1}{2}-\frac{3}{2r})^{-1}}, \left(\frac{1}{16C_1(C_3 + C_4\|v_0\|_r)}\right)^{(\frac{1}{2}-\frac{3}{2r})^{-1}}\right\}.$$

Thus, it is possible to take t_1 such that

$$t_1 - t_0 \ge \frac{1}{2}\min\left\{1, \left(\frac{1}{2C_2}\right)^{(\frac{1}{2}-\frac{3}{2r})^{-1}}, \left(\frac{1}{16C_1(C_3 + C_4\|v_0\|_r)}\right)^{(\frac{1}{2}-\frac{3}{2r})^{-1}}\right\} =: \zeta(\|v_0\|_r). \tag{4.79}$$

The proof is complete. □

Lemma 4.4.8. *Let $3 < r < \infty$, $t_0 \ge 0$ and $v_0 \in L^r_\sigma(D)$. The local solution v obtained in Proposition 4.4.7 also possesses the following properties:*

$$v \in C\big((t_0, t_1]; L^\kappa_\sigma(D)\big) \cap C_{w^*}\big((t_0, t_1]; L^\infty(D)\big) \tag{4.80}$$

for every $\kappa \in (r, \infty)$ and

$$\nabla v \in C_w\big((t_0, t_1]; L^\gamma(D)\big) \tag{4.81}$$

for every $\gamma \in (r, \infty)$ satisfying

$$\frac{2}{r} - \frac{1}{\gamma} < \frac{1}{3}. \tag{4.82}$$

Proof. By using (4.22) and (4.26) and the semigroup property, we find $v(t) \in L^\infty(D)$ with

$$\|v(t)\|_\infty \le C(t - t_0)^{-\frac{3}{2r}}\big\{\|v_0\|_r + \|v\|^2_{Y_r(t_0,t_1)} + \|v\|_{Y_r(t_0,t_1)}(\|u_s\|_r + \|\nabla u_s\|_r) + (a^2 + \alpha|a|)\big\} \tag{4.83}$$

for all $t \in (t_0, t_1]$. Moreover, for each $t_2 \in (t_0, t_1]$, we know from $v \in C([t_0, t_1]; L^r_\sigma(D))$ that

$$\big(v(t), \varphi\big) - \big(v(t_2), \varphi\big) \to 0 \quad \text{as } t \to t_2 \tag{4.84}$$

for all $\varphi \in C^\infty_0(D)$, which combined with (4.83) yields $v \in C_{w^*}\big((t_0, t_1]; L^\infty(D)\big)$. Since

$$\|v(t) - v(t_2)\|_\kappa \le \|v(t) - v(t_2)\|_r^{\frac{r}{\kappa}}\|v(t) - v(t_2)\|_\infty^{1-\frac{r}{\kappa}}$$

for $\kappa \in (r,\infty)$ and $t_2 \in (t_0,t_1]$, it follows from (4.83) that

$$v \in C\big((t_0,t_1];L^\kappa_\sigma(D)\big) \quad \text{for } \kappa \in (r,\infty). \tag{4.85}$$

The estimates (4.16) and (4.18) with $\mathcal{T} = t_0+1$ imply that if we assume (4.82), we have $\nabla v(t) \in L^\gamma(D)$ with

$$\begin{aligned}\|\nabla v(t)\|_\gamma \le C(t-t_0)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{\gamma})-\frac{1}{2}}\big\{\|v_0\|_r + \|v\|^2_{Y_r(t_0,t_1)} + \|v\|_{Y_r(t_0,t_1)}(\|u_s\|_r + \|\nabla u_s\|_r)\\ + (a^2+\alpha|a|)\big\}\end{aligned} \tag{4.86}$$

for all $t \in (t_0,t_1]$. Here, we note that (4.82) is needed for estimates of ∇U_1 and ∇U_3 given in (4.67). On account of (4.85), (4.86) and

$$(\nabla v(t) - \nabla v(t_2), \varphi) = -(v(t)-v(t_2), \nabla\cdot\varphi)$$

for all $t_2 \in (t_0,t_1]$ and $\varphi \in C_0^\infty(D)^{3\times 3}$, we find the weak continuity of ∇v with values in $L^\gamma(D)$. The proof is complete. □

We close this section with completion of the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. It remains to show that the solution v obtained in Proposition 4.4.2 also satisfies (4.10) with

$$v \in C\big((0,\infty);L^\kappa_\sigma(D)\big) \cap C_{w^*}\big((0,\infty);L^\infty(D)\big), \quad \nabla v \in C_w\big((0,\infty);L^\kappa(D)\big) \tag{4.87}$$

for all $3 < \kappa < \infty$. Let $t_* \in (0,\infty)$. By applying Proposition 4.4.7 and Lemma 4.4.8 with $r=6$, we can see that for each $t_0 \in [t_*/2, t_*)$, there exists $\widetilde{v} \in Y_6(t_0,t_1)$ which satisfies (4.64) and therefore, (4.63) with $v_0 = v(t_0)$ such that

$$\widetilde{v} \in C\big((t_0,t_1];L^\kappa_\sigma(D)\big) \cap C_{w^*}\big((t_0,t_1];L^\infty(D)\big), \quad \nabla\widetilde{v} \in C_w\big((t_0,t_1];L^\kappa(D)\big)$$

for all $\kappa \in [6,\infty)$. Moreover, the length of the existence interval can be estimated by

$$t_1 - t_0 \ge \zeta(\|v(t_0)\|_6) \ge \zeta\left(C_5\left(\frac{t_*}{2}\right)^{-\frac{1}{4}}\right) =: \varepsilon,$$

where $\zeta(\cdot)$ is the non-increasing function in Proposition 4.4.7 because of

$$\|v(t)\|_6 \le C_5\left(\frac{t_*}{2}\right)^{-\frac{1}{4}}$$

for all $t \ge t_*/2$, see (4.51). We note that the solution v obtained in Proposition 4.4.2 also satisfies (4.63) with $v_0 = v(t_0)$ since $C^\infty_{0,\sigma}(D)$ can be replaced by $L^{6/5}_\sigma(D)$ as the class of test functions in (4.11). Let us take $t_0 := \max\{t_*/2, t_* - \varepsilon/2\}$ so that $t_* \in (t_0,t_1)$, in which $v = \widetilde{v}$ on account of Lemma 4.4.5. Since t_* is arbitrary, we conclude (4.87) for $\kappa \in [6,\infty)$. It is also proved by applying Proposition 4.4.7 with $r \in (3,6)$ that the solution belongs to the class (4.87) for $\kappa \in (3,6)$ as well. The proof is complete. □

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