Global existence of solutions for semilinear damped wave equations with variable coefficients (変数係数をもつ半線型消散波動方程式における解の時間大域的存在)

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1 Introduction

We consider the Cauchy problem for the semilinear damped wave equation

$$\begin{cases} \partial_t^2 u(t,x) + \partial_t u(t,x) - \nabla \cdot (b(x)\nabla u(t,x)) = |u(t,x)|^p, \ t>0, \ x \in \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x), \ \partial_t u(0,x) = \varepsilon u_1(x), \ x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where u is a real-valued unknown function of (t,x), $p>1$, $(u_0,u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, $\varepsilon > 0$ is a small parameter and $b \in B^1(\mathbb{R}^n) = \{f \in C^1(\mathbb{R}^n); f, \partial_i f \in L^\infty(\mathbb{R}^n) \ (i=1,\dots n)\}$ is a given function specified later. If $b(x) = 1$ $(x \in \mathbb{R}^n)$, equation (1) is the damped wave equation

$$\begin{cases} \partial_t^2 w + \partial_t w - \Delta w = |w|^p, \ x \in \mathbb{R}^n, \ t>0 \\ w(0,x) = w_0(x), \ \partial_t w(0,x) = w_1(x), \ x \in \mathbb{R}^n. \end{cases} \tag{2}$$

Todorova and Yordanov [16] shows that if $\rho_F(n) < p$, the equation (2) has global solutions for sufficiently small initial data by using the weighted energy method. Here, $\rho_F(n) = 1 + 2/n$ is the n-dimensional Fujita's critical exponent. In [16], the compactness was assumed for the initial data, but the existence of a global solution can be proved without the compactness of the initial data (see [4–6, 10, 11]). Also, [16] shows that if $1 < p < \rho_F(n)$, the solutions of equation (2) blow up in a finite time for any nontrivial initial data. In the critical case $p = \rho_F(n)$, Zhang [17] shows that the solutions of equation (2) blow up in a finite time for any nontrivial initial data. By this observation, Fujita's critical exponent ρ_F can be regarded as the critical exponent of equation (2), which separates the existence and nonexistence of global solutions.

The damped wave equation $\partial_t^2 u(t,x) + \partial_t u(t,x) - \Delta u(t,x) = 0$ is a mixture of the heat equaiton $\partial_t v - \Delta v = 0$ and the wave equation $\partial_t^2 w - \Delta w = 0$. So we are interested in the relationship between the solutions of these equaitons. Using Fourier analysis, Narazaki [10] shows that the high-frequency part of the solution of damped wave equation corresponds to the solution of wave equation with exponential decay, and the low-frequency part corresponds to the solution of heat equation for the suitable initial data. For the heat equaiton with power type nonlinearity

$$\begin{cases} \partial_t v(t,x) - \Delta v(t,x) = v(t,x)^p, \ t>0, \ x \in \mathbb{R}^n, \\ v(0,x) = \varphi(x), \end{cases} \tag{3}$$

it is known that if $\rho_F(n) < p$, the equation (3) has global solutions for sufficiently small initial data, and if $1 < p \le \rho_F(n)$, solutions of equation (3) blow up in a finite time for any initial data (see [2, 3, 8, 14]). From the above, the solution of damped wave equation (2) has properties similar to the that of heat equation (3). The above facts suggest that $\partial_t u$ has a stronger influence than $\partial_t^2 u$ in the damped wave equation (2). For damped wave equation with a spatially decaying dissipative term

$$\begin{cases} \partial_t^2 u(t,x) + (1+|x|)^{-\alpha}\partial_t u(t,x) - \Delta u(t,x) = |u|^p, \ t>0, \ x \in \mathbb{R}^n, \\ u(0,x) = f_0(x), \ \partial_t u(0,x) = f_1(x), \end{cases} \tag{4}$$

where $\alpha \in [0,1)$ is given constant, Ikehata, Todorova, and Yordanov [7] shows that the critical exponent of (4) is $1 + 2/(n-\alpha)$.

For the damped wave equation having variable coefficients

$$\begin{cases} \partial_t^2 u + a(x)\partial_t u - \nabla \cdot (b(x)\nabla u) = 0, \ t>0, \ x \in \mathbb{R}^n, \\ u(0,x) = f_0(x), \ \partial_t u(0,x) = f_1(x), \end{cases} \tag{5}$$

Radu, Todorova, and Yordanov [13] derive weighted energy estimate of solutions for (5). Based on [13], Lei and Yang [9] shows that if $1 + 2/n < p$, (5) with power type nonlinearity has global solutions for sufficiently small initial data by assuming $a(x) \sim a_2$, $b(x) \sim b_2$ for large x. In [9, 13], they assume that there exists a solution $A(x)$ of the differential inequality

$$\nabla \cdot (b(x)\nabla A(x)) \geq a(x).$$

They also assume that $A(x)$ satisfies some conditions, and use $A(x)$ as a weight function. Let us return to our problem (1). We assume $b \in B^1(\mathbb{R}^n)$ satisfying

$$0 < m \leq b(x) \leq M < +\infty, \ x \in \mathbb{R}^n \tag{6}$$

for $n \geq 1$ and $b(x) = c(|x|)$ for $n \geq 2$. If $1 + 2/n < p < 1 + 4/n$, we also assume

$$\frac{M-m}{m} < \frac{2(p-1-2/n)}{1+4/n-p}. \tag{7}$$

We note that this extra assumption (7) is always satisfied when b is a constant function. Our purpose is to show that if $1 + 2/n < p$, the equation (1) has global solutions for sufficiently small initial data by using the weighted energy method. To state our main result (Theorem 2.1) precisely, we introduce a weight function

$$\psi(t,x) = \psi_\lambda(t,x) = \begin{cases} \frac{1}{2(2+\lambda)(t+1)} \int_0^{|x|} \frac{r}{c(r)}\, dr, \ n \geq 2, \\ \frac{1}{2(2+\lambda)(t+1)} \int_0^{x} \frac{y}{b(y)}\, dy, \ n = 1 \end{cases} \tag{8}$$

which satisfies

$$\nabla \cdot (b(x)\nabla\psi(t,x)) = \frac{n}{2(2+\lambda)(t+1)}. \tag{9}$$

The weight function (8) is a variable coefficient version of that introduced by [12], where they consider the case $b(x) = 1$ and established the global existence for the system of semilinear damped wave equation

$$\partial_t^2 u_l + \partial_t u_l - \Delta u_l = |u_{l-1}|^{p_l}, \ (l = 1, \dots, k), \tag{10}$$

for supercritical case. Sharp estimate of the lifespan of the solution was also given from above and below for subcritical case.

We note that this thesis is a generalized version of [15] which discusses the cases when $n = 1$. This thesis has the following organization. In Section 2, we define notations and state the main theorem. In Section 3, we show some properties of the weight function (8), and prove the main theorem by using them.

2 Main theorem

We give a function space and definition of a weak solutions of equation (1). For $0 < T \leq +\infty$ we set

$$X(T) = C([0,T); H^1(\mathbb{R}^n)) \cap C^1([0,T); L^2(\mathbb{R}^n)).$$

Given $p > 1$, $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, $\varepsilon > 0$, a function $u \in X(T)$ is called a weak solution of the Cauchy problem (1) on the interval $[0,T)$ if u satisfies

$$\begin{aligned}
&\int_{[0,T)\times\mathbb{R}^n} u(t,x)\left(\partial_t^2\varphi(t,x) - \partial_t\varphi(t,x) - \nabla\cdot(b(x)\nabla\varphi(t,x)\right) dxdt \\
=&\,\varepsilon \int_{\mathbb{R}^n} \{(u_0(x)+u_1(x))\varphi(0,x) - u_0(x)\partial_t\varphi(0,x)\}\,dx \\
&+ \int_{[0,T)\times\mathbb{R}^n} |u(t,x)|^p\varphi(t,x)\,dxdt
\end{aligned}$$

for any $\varphi \in C_0^\infty([0,T)\times\mathbb{R})$.

Our main theorem reads as follows.

Theorem 2.1. *There exist constants $\lambda > 0$ and $0 < \delta < n/2$ such that the following is true: We assume that $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfies*

$$I_\lambda = I_\lambda(u_0,u_1) = \int_{\mathbb{R}^n} e^{2\psi_\lambda(0,x)}(|u_0|^2 + |\nabla u_0|^2 + |u_1|^2)\,dx < +\infty$$

and $1+2/n < p < +\infty$ $(n=1,2)$, $1+2/n < p < n/(n-2)$ $(n \geq 3)$. *If* $1+2/n < p < 1+4/n$, *we further assume* (7). *Then there exists a constant* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$, *the equation* (1) *admits a unique global solution* $u \in X(+\infty)$ *along with*

$$\int_{\mathbb{R}^n} e^{2\psi_\lambda}(|\partial_t u|^2 + |\nabla u|^2)\, dx \leq C\varepsilon^2 I_\lambda (t+1)^{-n/2-1+\delta}, \tag{11}$$

$$\int_{\mathbb{R}^n} e^{2\psi_\lambda}|u|^2\, dx \leq C\varepsilon^2 I_\lambda (t+1)^{-n/2+\delta} \tag{12}$$

for $t > 0$, *where* $C > 0$ *is a constant.*

We use the following local existence result to prove the existence of a global solution of equation (1).

Proposition 2.2. *Let* $1 < p < +\infty$ $(n=1,2)$, $1 < p < n/(n-2)$ $(n \geq 3)$ *and* $\varepsilon > 0$ *and let* $\lambda > 0$. *If the initial data* $(\varepsilon u_0, \varepsilon u_1)$ *satisfies* $I_\lambda(u_0, u_1) < +\infty$, *then there exists a maximal existence time* $T_\varepsilon \in (0, +\infty]$ *such that the equation* (1) *has a unique solution* $u \in X(T_\varepsilon)$ *satisfying*

$$\sup_{[0,T]} \left(\|e^{\psi_\lambda(t)}\partial_t u(t)\|_{L^2(\mathbb{R}^n)} + \|e^{\psi_\lambda(t)}\nabla u(t)\|_{L^2(\mathbb{R}^n)} + \|e^{\psi_\lambda(t)} u(t)\|_{L^2(\mathbb{R}^n)} \right) < +\infty$$

for any $T < T_\varepsilon$. *Moreover, if* $T_\varepsilon < +\infty$, *we have*

$$\limsup_{t \nearrow T_\varepsilon} \left(\|e^{\psi_\lambda(t)}\partial_t u(t)\|_{L^2(\mathbb{R}^n)} + \|e^{\psi_\lambda(t)}\nabla u(t)\|_{L^2(\mathbb{R}^n)} + \|e^{\psi_\lambda(t)} u(t)\|_{L^2(\mathbb{R}^n)} \right) = +\infty.$$

Proposition 2.2 can be proved by standard argument (see [6], for example, where $b(x)$ is a constant function).

3 Proof of Theorem 2.1

In this section, we prove Theorem 2.1 by using Proposition 2.2 and an a priori estimate of the solution of equation (1) (Proposition 3.3). We introduce the definitions of some constants and the properties of the weight functions. For $k \in (2,3)$, we set

$$\delta = \delta(p,k) = \begin{cases} \frac{(p-1-2/n)n}{p-1}(3-k), & (1+2/n < p < 1+4/n), \\ \frac{n}{2}(3-k), & (1+4/n \leq p), \end{cases} \tag{13}$$

$$\lambda = \frac{8\delta}{nk - 4\delta}, \tag{14}$$

and

$$\lambda_1 = \frac{n}{4} - \frac{n}{2(2+\lambda)}. \tag{15}$$

Here, we note that $0 < \delta < n/2,\ \delta = k\lambda_1$. From (8), we obtain that

$$\begin{aligned}
-\partial_t \psi(t,x) &\geq \left\{ 2 + m\left(\frac{2+\lambda}{M} - \frac{2}{m}\right)\right\} b(x)|\nabla\psi(t,x)|^2 \\
&= (2+\alpha) b(x)|\nabla\psi(t,x)|^2, \qquad (16)\\
-\partial_t \psi(t,x) &= \frac{1}{2(2+\lambda)(t+1)^2}\int_0^{|x|} \frac{r}{c(r)}\,dr \\
&\leq \frac{1}{2(2+\lambda)(t+1)^2}\frac{x^2}{2m} \\
&\leq \beta|\nabla\psi(t,x)|^2 \qquad (17)
\end{aligned}$$

for $t \geq 0,\ x \in \mathbb{R}$, where

$$\begin{aligned}
\alpha &= m\left(\frac{2+\lambda}{M} - \frac{2}{m}\right), \qquad (18)\\
\beta &= \frac{(2+\lambda)}{m}M^2.
\end{aligned}$$

We use the following lemma to prove Theorem 2.1.

Lemma 3.1. *Let $p > 1 + 2/n$. If $1 + 2/n < p < 1 + 4/n$, we also assume (7). Then, there exists a constant $k = k(p) \in (2,3)$ such that the following*

$$\frac{4\delta}{nk - 4\delta} > \frac{M-m}{m} \tag{19}$$

holds.

Proof. Case 1: $1 + 2/n < p < 1 + 4/n$. Since (13), we have

$$\frac{4\delta}{nk - 4\delta} = \frac{\gamma(3-k)}{nk - \gamma(3-k)} =: g_1(k),$$

where $\gamma = \frac{4n(p-1-2/n)}{p-1} > 0$. Since g_1 is a continuous on the interval $(2,3)$ and

$$g_1'(k) = \frac{-3\gamma}{(k-\gamma(3-k))^2} < 0, \quad \frac{M-m}{m} < \frac{2(p-1-2/n)}{1+4/n-p},$$

the following

$$\frac{M-m}{m} < g_1(k) < g_1(2) = \frac{2(p-1-2/n)}{1+4/n-p}$$

holds for a sufficiently small $k \in (2,3)$.

Case 2: $1 + 4/n \leq p$. By (13), we have

$$\frac{4\delta}{nk - 4\delta} = \frac{2(3-k)}{3(k-2)}.$$

Therefore, by taking $k \in (2,3)$ which satisfies

$$\frac{M-m}{m} < \frac{2(3-k)}{3(k-2)}, \tag{20}$$

we have (19). From the above, the proof is complete. □

Remark 3.2. *We note that the relation* (19) *implies $\alpha > 0$, where α is given by* (18).

Let us define the weighted energy by

$$\begin{aligned} W(t) =&(t+1)^{n/2+1-\delta} \int_{\mathbb{R}^n} e^{2\psi} (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2)\, dx \\ &+ (t+1)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} |u(t,x)|^2\, dx. \end{aligned}$$

Furthermore, we set $M(t) = \sup_{0 \leq s \leq t} W(s)$. In the following proposition, we show the a priori estimate of the solution of (1).

Proposition 3.3. *There exists constants $\varepsilon_0 > 0$ and $\lambda > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution of equation* (1) *satisfies*

$$M(t) \leq C\varepsilon^2 I_\lambda \tag{21}$$

for $t \geq 0$. Here $C > 0$ is independent of t and ε.

Proof. We take a constant $k \in (2,3)$ which satisfies Lemma 3.1. By multiplying equation (1) by $e^{2\psi}\partial_t u$, we have

$$\begin{aligned} e^{2\psi}\partial_t u |u|^p =& e^{2\psi}\partial_t u \partial_t^2 u + e^{2\psi}|\partial_t u|^2 - e^{2\psi}\partial_t u \nabla \cdot (b(x)\nabla u) \\ =& \frac{1}{2}\partial_t \left[e^{2\psi} \left(|\partial_t u|^2 + b(x)|\nabla u|^2 \right) \right] - \nabla \cdot \left(e^{2\psi} b(x) \partial_t u \nabla u \right) \\ &+ e^{2\psi} \left(1 - \partial_t \psi - \frac{b(x)|\nabla\psi|^2}{-\partial_t\psi} \right) |\partial_t u|^2 \\ &+ \frac{e^{2\psi}}{-\partial_t \psi} b(x) |\partial_t \psi \nabla u - \nabla\psi \partial_t u|^2. \end{aligned} \tag{22}$$

We set

$$
\begin{aligned}
T_1 =& + e^{2\psi}\left(1-\partial_t\psi-\frac{b(x)|\nabla\psi|^2}{-\partial_t\psi}\right)|\partial_t u|^2,\\
T_2 =& \frac{e^{2\psi}}{-\partial_t\psi}b(x)|\partial_t\psi\nabla u-\nabla\psi\partial_t u|^2.
\end{aligned}
$$

From (16) and $\alpha > 0$, we obtain

$$
\begin{aligned}
T_1 = e^{2\psi}\left(1-\partial_t\psi-\frac{b(x)|\nabla\psi|^2}{-\partial_t\psi}\right)|\partial_t u|^2 =& e^{2\psi}\left(1-\partial_t\psi-\frac{1}{2+\alpha}\right)|\partial_t u|^2\\
\geq& e^{2\psi}\left(\frac{1}{2}-\partial_t\psi\right)|\partial_t u|^2. \qquad (23)
\end{aligned}
$$

Moreover, the Schwarz inequality implies

$$
\begin{aligned}
&|\partial_t\psi\nabla u-\nabla\psi\partial_t u|^2\\
\geq&|\partial_t\psi\partial_x u|^2+|\nabla\psi\partial_t u|^2-2|\partial_t\psi\nabla u||\nabla\psi\partial_t u|\\
=&|\partial_t\psi\nabla u|^2+|\nabla\psi\partial_t u|^2-2\left(\frac{2}{\sqrt{5}}|\partial_t\psi\nabla u|\right)\left(\frac{\sqrt{5}}{2}|\nabla\psi\partial_t u|\right)\\
\geq&\frac{1}{5}|\partial_t\psi\nabla u|^2-\frac{1}{4}|\nabla\psi\partial_t u|^2. \qquad (24)
\end{aligned}
$$

Together with (6), (16), (24) and $\alpha > 0$ leads us to

$$
\begin{aligned}
T_2 \geq& \frac{e^{2\psi}}{5}m(-\partial_t\psi)|\nabla u|^2-\frac{e^{2\psi}}{4}\frac{1}{2+\alpha}|\partial_t u|^2\\
\geq& \frac{e^{2\psi}}{5}m(-\partial_t\psi)|\nabla u|^2-\frac{1}{8}e^{2\psi}|\partial_t u|^2. \qquad (25)
\end{aligned}
$$

On the other hand, the left hand side of (22) is estimated as

$$
e^{2\psi}\partial_t u|u|^p = \left(\frac{1}{2}e^{\psi}|\partial_t u|\right)(2e^{\psi}|u|^p) \leq \frac{1}{8}e^{2\psi}|\partial_t u|^2+2e^{2\psi}|u|^{2p}. \qquad (26)
$$

Collecting (22)–(26) yields

$$
\begin{aligned}
&\frac{1}{2}\partial_t\left[e^{2\psi}\left(|\partial_t u|^2+b(x)|\nabla u|^2\right)\right]-\nabla\cdot\left(e^{2\psi}b(x)\partial_t u\nabla u\right)\\
&+e^{2\psi}\left(\frac{1}{4}-\partial_t\psi\right)|\partial_t u|^2+\frac{m}{5}e^{2\psi}(-\partial_t\psi)|\nabla u|^2\\
\leq&2e^{2\psi}|u|^{2p}. \qquad (27)
\end{aligned}
$$

By multiplying (1) by $e^{2\psi}u$, we have

$$\begin{aligned} e^{2\psi}u|u|^p =&\partial_t \left[e^{2\psi} \left(u\partial_t u + \frac{1}{2}|u|^2 \right) \right] - \nabla \cdot \left(e^{2\psi} b(x)(u\nabla u + \nabla\psi |u|^2) \right) \\ &+ e^{2\psi} \left\{ -\partial_t \psi - 2a(x)|\nabla\psi|^2 + \nabla \cdot (b(x)\nabla\psi) \right\} |u|^2 \\ &+ e^{2\psi} b(x)|\nabla u + 2\nabla\psi u|^2 \\ &+ e^{2\psi} \{ 2(-\partial_t \psi) u\partial_t u - |\partial_t u|^2 \}. \end{aligned} \tag{28}$$

We set

$$\begin{aligned} S_1 =&e^{2\psi} \left\{ -\partial_t \psi - 2a(x)|\nabla\psi|^2 + \nabla \cdot (b(x)\nabla\psi) \right\} |u|^2, \\ S_2 =&e^{2\psi} b(x)|\nabla u + 2\nabla\psi u|^2, \\ S_3 =&e^{2\psi} \{ 2(-\partial_t \psi) u\partial_t u - |\partial_t u|^2 \}. \end{aligned}$$

On account of (9) and (16) we have

$$S_1 \geq e^{2\psi} \left\{ m\alpha |\nabla\psi|^2 + \frac{n}{2(2+\lambda)(t+1)} \right\} |u|^2. \tag{29}$$

Let

$$0 < \lambda_2 < \min \left\{ \frac{m}{5}, \frac{m\alpha}{5+\beta} \right\}$$

and put

$$\eta = \sqrt{\frac{1}{2}\left(1 - \frac{\lambda_2}{m} \right)}.$$

We note that

$$m\alpha - \lambda_2(5+\beta) > 0 \tag{30}$$

and

$$\frac{2}{5} < \eta^2 < \frac{1}{2}. \tag{31}$$

Then the Schwarz inequality together with (17), (30) and (31) implies that

$$\begin{aligned} S_2 \geq& e^{2\psi} m (|\nabla u|^2 + 4\nabla u \cdot \nabla\psi u + 4|\nabla\psi|^2 |u|^2) \\ \geq& e^{2\psi} m \left\{ (1-2\eta^2)|\nabla u|^2 + (1-2\eta^2) \left(\frac{-2}{\eta^2} \right) |\nabla\psi|^2 |u|^2 \right\} \\ >& e^{2\psi} \left(\lambda_2 |\nabla u|^2 - 5\lambda_2 |\nabla\psi|^2 |u|^2 \right), \end{aligned} \tag{32}$$

and

$$
\begin{aligned}
|S_3| \leq & e^{2\psi} \left(\lambda_2(-\partial_t \psi)|u|^2 + \frac{1}{\lambda_2}(-\partial_t \psi)|\partial_t u|^2 + |\partial_t u|^2 \right) \\
\leq & e^{2\psi} \left(\lambda_2 \beta |\nabla \psi|^2 |u|^2 + \left(\frac{-\partial_t \psi}{\lambda_2} + 1 \right) |\partial_t u|^2 \right).
\end{aligned}
\tag{33}
$$

We thus conclude

$$
\begin{aligned}
& \partial_t \left[e^{2\psi} \left(u \partial_t u + \frac{1}{2}|u|^2 \right) \right] - \nabla \cdot \left(e^{2\psi} b(x)(u \nabla u + \nabla \psi |u|^2) \right) \\
& + e^{2\psi} \left\{ \lambda_3 |\nabla \psi|^2 + \frac{n}{2(2+\lambda)(t+1)} \right\} |u|^2 \\
& + \lambda_2 e^{2\psi} |\nabla u|^2 - e^{2\psi} \left(\frac{-\partial_t \psi}{\lambda_2} + 1 \right) |\partial_t u|^2 \\
\leq & e^{2\psi} |u|^{p+1}
\end{aligned}
\tag{34}
$$

due to (28)–(30), (32) and (33), where $\lambda_3 = m\alpha - (5+\beta)\lambda_2 > 0$.

Let ν be a positive constant satisfying $1/4 - \nu > 0$ and $1 - \nu/\lambda_2 > 0$. By calculating $(27) + \nu \times (34)$, we have inequality

$$
\begin{aligned}
2e^{2\psi}|u|^{2p} + \nu e^{2\psi}|u|^{p+1} \geq & \partial_t \left[e^{2\psi} \left\{ \frac{1}{2}(|\partial_t u|^2 + b(x)|\nabla u|^2) + \nu \left(u \partial_t u + \frac{1}{2}|u|^2 \right) \right\} \right] \\
& - \nabla \cdot \left[e^{2\psi} b(x) \{ \partial_t u \nabla u + \nu (u \nabla u + \nabla \psi |u|^2) \} \right] \\
& + e^{2\psi} \left\{ \left(\frac{1}{4} - \nu \right) + \left(1 - \frac{\nu}{\lambda_2} \right) (-\partial_t \psi) \right\} |\partial_t u|^2 \\
& + e^{2\psi} \left(\nu \lambda_2 + \frac{m}{5}(-\partial_t \psi) \right) |\nabla u|^2 \\
& + \nu e^{2\psi} \left(\lambda_3 |\nabla \psi|^2 + \frac{n}{2(2+\lambda)(t+1)} \right) |u|^2.
\end{aligned}
\tag{35}
$$

By integrating (35) over $\mathbb{R}^n$, the following inequality

$$
\begin{aligned}
& 2 \int_{\mathbb{R}^n} e^{2\psi} |u|^{2p} \, dx + \nu \int_{\mathbb{R}^n} e^{2\psi} |u|^{p+1} \, dx \\
\geq & \frac{d}{dt} \left[\int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{1}{2}(|\partial_t u|^2 + b(x)|\nabla u|^2) + \nu \left(u \partial_t u + \frac{1}{2}|u|^2 \right) \right\} dx \right] \\
& + c_1 \int_{\mathbb{R}^n} e^{2\psi} \{ 1 + (-\partial_t \psi) \} (|\partial_t u|^2 + |\nabla u|^2) \, dx \\
& + \nu \int_{\mathbb{R}^n} e^{2\psi} \left(\lambda_3 |\nabla \psi|^2 |u|^2 + \frac{1}{2(2+\lambda)(t+1)} |u|^2 \right) dx
\end{aligned}
\tag{36}
$$

holds with some constant $c_1 > 0$.

Let $t_0 \geq 1$. We multiply (36) by $(t+t_0)^{n/2-\delta}$ to have

$$
\begin{aligned}
&2(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}(|u|^{2p}+|u|^{p+1})\,dx\\
\geq&\frac{d}{dt}\left[(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left\{\frac{1}{2}(|\partial_t u|^2+b(x)|\nabla u|^2)+\nu\left(u\partial_t u+\frac{1}{2}|u|^2\right)\right\}dx\right]\\
&-\left(\frac{n}{2}-\delta\right)(t+t_0)^{n/2-1-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left\{\frac{1}{2}(|\partial_t u|^2+b(x)|\nabla u|^2)+\nu\left(u\partial_t u+\frac{1}{2}|u|^2\right)\right\}dx\\
&+c_1(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\{1+(-\partial_t\psi)\}(|\partial_t u|^2+|\nabla u|^2)\,dx\\
&+\nu(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(\lambda_3|\nabla\psi|^2|u|^2+\frac{n}{2(2+\lambda)(t+1)}|u|^2\right)dx. \qquad (37)
\end{aligned}
$$

The second term of the right hand side in (37) is estimated as

$$
\begin{aligned}
&\left|-\left(\frac{n}{2}-\delta\right)(t+t_0)^{n/2-1-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(\frac{1}{2}(|\partial_t u|^2+b(x)|\nabla u|^2)+\nu\left(u\partial_t u+\frac{1}{2}|u|^2\right)\right)dx\right|\\
\leq&\left(\frac{n}{2}-\delta\right)(t+t_0)^{n/2-1-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left\{\left(\frac{1}{2}+\frac{\nu}{4\lambda_4}\right)|\partial_t u|^2+\frac{M}{2}|\nabla u|^2\right\}dx\\
&+\nu\left(\frac{n}{2}-\delta\right)(t+t_0)^{n/2-1-\delta}\int_{\mathbb{R}^n}\left(\frac{1}{2}+\lambda_4\right)|u|^2\,dx, \qquad (38)
\end{aligned}
$$

where λ_4 is a constant and satisfies

$$0<\lambda_4<\left(1-\frac{2}{k}\right)\frac{\delta}{n}. \qquad (39)$$

Since (37) and (38), we obtain inequality

$$
\begin{aligned}
&2(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}(|u|^{2p}+|u|^{p+1})\,dx\\
\geq&\frac{d}{dt}\left[(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left\{\frac{1}{2}(|\partial_t u|^2+b(x)|\nabla u|^2)+\nu\left(u\partial_t u+\frac{1}{2}|u|^2\right)\right\}dx\right]\\
&-\nu\left(\frac{n}{2}-\delta\right)(t+t_0)^{n/2-1-\delta}\int_{\mathbb{R}^n}\left(\frac{1}{2}+\lambda_4\right)|u|^2\,dx\\
&+\nu(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left\{\lambda_3|\nabla\psi|^2|u|^2+\frac{n}{2(2+\lambda)(t+1)}|u|^2\right\}dx\\
&-\left(\frac{n}{2}-\delta\right)(t+t_0)^{n/2-1-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(\left(\frac{1}{2}+\frac{\nu}{4\lambda_4}\right)|\partial_t u|^2+\frac{M}{2}|\nabla u|^2\right)dx\\
&+c_1(t+t_0)^{1/2-\delta}\int_{\mathbb{R}} e^{2\psi}(1+(-\partial_t\psi))(|\partial_t u|^2+|\partial_x u|^2)\,dx =: \Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5.
\end{aligned}
$$
(40)

By (39), the following inequality

$$\frac{n}{2(2+\lambda)} - \left(\frac{n}{2} - \delta\right)\left(\frac{1}{2} + \lambda_4\right) \geq \left(\frac{1}{2} - \frac{1}{k}\right)\delta - \frac{n\lambda_4}{2} > 0$$

holds. Thus, the second and third terms on the right hand side of (40) is calculated as

$$\begin{aligned}
\Lambda_2 + \Lambda_3 \geq & \nu(t+t_0)^{n/2-1-\delta} \int_{\mathbb{R}^n} e^{2\psi} \left(\frac{1}{2(2+\lambda)} - \left(\frac{n}{2} - \delta\right)\left(\frac{1}{2} + \lambda_4\right)\right) |u|^2 \, dx \\
& + \nu(t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} \lambda_3 |\nabla\psi|^2 |u|^2 \, dx \\
\geq & c_2(t+t_0)^{n/2-1-\delta} \int_{\mathbb{R}^n} e^{2\psi} |u|^2 \, dx \\
& + c_2(t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} |\nabla\psi|^2 |u|^2 \, dx, \qquad (41)
\end{aligned}$$

where

$$c_2 = \min\left\{\nu\left\{\frac{n}{2(2+\lambda)} - \left(\frac{n}{2} - \delta\right)\left(\frac{1}{2} + \lambda_4\right)\right\},\ \nu\lambda_3\right\}.$$

Also, the fourth and fifth terms on the right hand side of (40) is calculated as

$$\begin{aligned}
\Lambda_4 + \Lambda_5 \geq & (t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} \left(c_1 - \frac{n/2-\delta}{t_0}\left(\frac{1}{2} + \frac{\nu}{\lambda_4}\right)\right) |\partial_t u|^2 \, dx \\
& + (t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} \left(c_1 - \frac{n/2-\delta}{t_0}\frac{M}{2}\right) |\nabla u|^2 \, dx \\
& + c_1(t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} (-\partial_t\psi)(|\partial_t u|^2 + |\nabla u|^2) \, dx. \qquad (42)
\end{aligned}$$

In what follows, we take $t_0 \geq 1$ sufficiently large so that the coefficient of the integral on the right hand side in (42) are positive. From (40)–(42), we obtain

$$\begin{aligned}
& \frac{d}{dt}\left[(t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} \left\{\frac{1}{2}(|\partial_t u|^2 + b(x)|\nabla u|^2) + \nu\left(u\partial_t u + \frac{1}{2}|u|^2\right)\right\} dx\right] \\
& + c_3(t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} \{1 + (-\partial_t\psi)\}(|\partial_t u|^2 + |\nabla u|^2) \, dx \\
& + c_3(t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} \left(|\nabla\psi|^2 + \frac{1}{t+t_0}\right) |u|^2 \, dx \\
\leq & \, 2(t+t_0)^{n/2-\delta} \int_{\mathbb{R}^n} e^{2\psi} (|u|^{2p} + |u|^{p+1}) \, dx \qquad (43)
\end{aligned}$$

with some constant c_3.

By integrating (27) over $\mathbb{R}^n$ and by multiplying $(t+t_0)^{n/2+1-\delta}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[(t+t_0)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}(|\partial_t u|^2 + b(x)|\nabla u|^2)\,dx\right] \\
&-\frac{1}{2}\left(\frac{n}{2}+1-\delta\right)(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}(|\partial_t u|^2 + b(x)|\nabla u|^2)\,dx \\
&+(t+t_0)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(\left(\frac{1}{4}-\partial_t\psi\right)|\partial_t u|^2 + \frac{m}{5}(-\partial_t\psi)|\nabla u|^2\right)dx \\
\leq\, & 2(t+t_0)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^{2p}\,dx. \qquad (44)
\end{aligned}$$

Let μ be a positive constant satisfying $2c_3 - \mu(1+M)(n/2+1-\delta) > 0$. Calculating (43) $+\, \mu \times$ (44) yields

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\left[(t+t_0)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}(|\partial_t u|^2 + b(x)|\nabla u|^2)\,dx\right] \\
&+\frac{d}{dt}\left[(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(\frac{1}{2}(|\partial_t u|^2 + b(x)|\nabla u|^2) + \nu\left(u\partial_t u + \frac{1}{2}|u|^2\right)\right)dx\right] \\
&+\left(c_3 - \frac{\mu(1+M)}{2}\left(\frac{n}{2}+1-\delta\right)\right)(t+t_0)^{1/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}(1+(-\partial_t\psi))(|\partial_t u|^2 + |\nabla u|^2)\,dx \\
&+\mu(t+t_0)^{3/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(\left(\frac{1}{4}-\partial_t\psi\right)|\partial_t u|^2 + \frac{m}{5}(-\partial_t\psi)|\nabla u|^2\right)dx \\
&+c_3(t+t_0)^{1/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(|\nabla\psi|^2 + \frac{1}{t+t_0}\right)|u|^2\,dx \\
\leq\, & 2(\mu+1)(t+t_0)^{3/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^{2p}\,dx + 2(\mu+1)(t+t_0)^{1/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^{p+1}\,dx. \qquad (45)
\end{aligned}$$

It follows by integrating (45) over $[0,t]$ that

$$
\begin{aligned}
&\frac{\mu}{2}(t+t_0)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}(|\partial_t u|^2+b(x)|\nabla u|^2)\,dx\\
&+(t+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(\frac{1}{2}(|\partial_t u|^2+b(x)|\nabla u|^2)+\nu\left(u\partial_t u+\frac{1}{2}|u|^2\right)\right)dx\\
&+c_4\int_0^t (s+t_0)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}(|\partial_t u|^2+(-\partial_t\psi)(|\partial_t u|^2+|\nabla u|^2|))\,dxds\\
&+c_4\int_0^t (s+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\{1+(-\partial_t\psi)\}(|\partial_t u|^2+|\partial_x u|^2)\,dxds\\
&+c_4\int_0^t (s+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}\left(|\nabla\psi|^2+\frac{1}{t+t_0}\right)|u|^2\,dxds\\
\le& c_5\varepsilon^2\int_{\mathbb{R}^n} e^{2\psi(0,x)}(|u_0|^2+|\nabla u_0|^2+|u_1|^2)\,dx\\
&+c_5\int_0^t (s+t_0)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^{2p}\,dxds+c_5\int_0^t (s+t_0)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^{p+1}\,dxds,
\end{aligned}
\tag{46}
$$

where c_4 and c_5 are some constants dependent on δ, however, independent of t. Furthermore we observe that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} e^{2\psi}\left\{\frac{1}{2}(|\partial_t u|^2+b(x)|\nabla u|^2)+\nu\left(u\partial_t u+\frac{1}{2}|u|^2\right)\right\}dx\\
\ge&\int_{\mathbb{R}^n} e^{2\psi}\left(\frac{1}{2}|\partial_t u|^2+\frac{m}{2}|\nabla u|^2+\frac{\nu}{2}|u|^2-\frac{\nu}{4}|u|^2-\nu|\partial_t u|^2\right)dx\\
\ge& c_6\int_{\mathbb{R}^n} e^{2\psi}(|\partial_t u|^2+|\nabla u|^2+|u|^2)\,dx
\end{aligned}
\tag{47}
$$

due to the Schwarz inequality, where $c_6=\min\{1/2-\nu, m/2, \nu/4\}>0$. This observation together with

$$
(t+1)^\alpha \le (t+t_0)^\alpha \le t_0^\alpha (t+1)^\alpha, \quad \left(t\ge 0,\ \alpha=\frac{n}{2}-\delta,\ \frac{n}{2}+1-\delta\right)
$$

implies

$$
W(t)\le\ C\varepsilon^2 I_\lambda + CN(t) \tag{48}
$$

for $t\ge 0$, where

$$
N(t)=\int_0^t (s+1)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^{2p}\,dxds+\int_0^t (s+1)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^{p+1}\,dxds, \tag{49}
$$

and $C=C(\delta)$ is a positive constant independent of t.

We use the following Gagliard–Nirenberg inequality to obtain the estimate of the nonlinear term $N(t)$.

Lemma 3.4 ([1]). *Let $u \in H^1(\mathbb{R}^n)$. If $1 < p < +\infty$ $(n = 1, 2)$, $1 < p < n/(n-2)$ $(n \geq 3)$, then we have*

$$\|u\|_{L^{2p}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^n)}^{\sigma_{2p}}\|u\|_{L^2(\mathbb{R}^n)}^{1-\sigma_{2p}},$$
$$\|u\|_{L^{p+1}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^n)}^{\sigma_{p+1}}\|u\|_{L^2(\mathbb{R}^n)}^{1-\sigma_{p+1}}$$

where $\sigma_{2p} = \frac{n(p-1)}{2p}$, $\sigma_{p+1} = \frac{n(p-1)}{2(p+1)}$.

By applying Lemma 3.4 with $q = 2p$ and (8), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{2\psi}|u|^{2p}\,dx \leq& C\|\nabla(e^{\frac{\psi}{p}}u)\|_{L^2(\mathbb{R}^n)}^{2p\sigma_{2p}}\|e^{\frac{\psi}{p}}u\|_{L^2(\mathbb{R}^n)}^{2p(1-\sigma_{2p})}\\
\leq& C\left\|\frac{1}{p}(\nabla\psi)e^{\psi/p}u + e^{\psi/p}\nabla u\right\|_{L^2(\mathbb{R}^n)}^{2p\sigma_{2p}}\|e^{\psi}u\|_{L^2(\mathbb{R}^n)}^{2p(1-\sigma_{2p})}\\
\leq& C\left((t+1)^{-1}\|e^{\psi}u\|_{L^2(\mathbb{R}^n)}^2 + \|e^{\psi}\nabla u\|_{L^2(\mathbb{R}^n)}^2\right)^{p\sigma_{2p}}\|e^{\psi}u\|_{L^2(\mathbb{R}^n)}^{2p(1-\sigma_{2p})}\\
\leq& C(t+1)^{-p\sigma_{2p}(n/2+1-\delta)-p(1-\sigma_{2p})(n/2-\delta)}\\
&\times\left((t+1)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^2\,dx + (t+1)^{n/2+1-\delta}\int_{\mathbb{R}^n} e^{2\psi}(|\partial_t u|^2 + |\nabla u|^2)\,dx\right)^{p\sigma_{2p}}\\
&\times\left((t+1)^{n/2-\delta}\int_{\mathbb{R}^n} e^{2\psi}|u|^2\,dx\right)^{p(1-\sigma_{2p})}\\
\leq& C(t+1)^{-p(\sigma_{2p}+n/2-\delta)}W(t)^p. \qquad (50)
\end{aligned}$$

In the same way, we have

$$\int_{\mathbb{R}^n} e^{2\psi}|u|^{p+1}\,dx \leq C(t+1)^{-\frac{p+1}{2}(\sigma_{p+1}+n/2-\delta)}W(t)^{\frac{p+1}{2}}. \qquad (51)$$

We note that $-(p-1)n + 1 + (p-1)\delta$, $(-(p-1)n + (p-1)\delta)/2 < -1$ and collect (48), (50) and (51) to see

$$\begin{aligned}
\sup_{0\leq t_1\leq t} W(t_1) = M(t) \leq& \sup_{0\leq t_1\leq t}\Bigg[C\varepsilon^2 I_\lambda + C\int_0^{t_1}(1+s)^{-(p-1)+1+(p-1)\delta}W(s)^p\,ds\\
&+C\int_0^{t_1}(1+s)^{(-(p-1)+(p-1)\delta)/2}W(s)^{\frac{p+1}{2}}\,ds\Bigg]\\
\leq& C\varepsilon^2 I_\lambda + CM(t)^p\int_0^{+\infty}(1+s)^{-(p-1)+1+(p-1)\delta}\,ds\\
&+ CM(t)^{(p+1)/2}\int_0^{+\infty}(1+s)^{(-(p-1)+(p-1)\delta)/2}\,ds\\
\leq& C\varepsilon^2 I_\lambda + CM(t)^p + CM(t)^{(p+1)/2} \qquad (52)
\end{aligned}$$

for all $t \geq 0$. From (52) and $1 < p, (p+1)/2$, we obtain the a priori estimate

$$M(t) \leq C\varepsilon^2 I_\lambda$$

for a sufficiently small $\varepsilon > 0$. □

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