The role of forward self-similar solutions in the Cauchy problem for semi-linear heat equations with exponential nonlinearity

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Contents

1 Introduction

In this paper, we consider the Cauchy problem:

$$\begin{cases} u_t - \Delta u = e^u, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geq 1$ and u_0 is a continuous initial function. We will study the blow-up problem for (1). We say that the solution u to (1) blows up in finite time if there exists $T(u_0) < \infty$ such that $u \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap C(\mathbb{R}^N \times [0,T))$ is a unique classical solution to (1) which is bounded in $\mathbb{R}^N \times [0, T(u_0))$ and satisfies

$$\limsup_{t \nearrow T(u_0)} \sup_{x \in \mathbb{R}^N} u(x,t) = +\infty.$$

We say that u is a global solution if $u \in C^{2,1}(\mathbb{R}^N \times (0,\infty)) \cap C(\mathbb{R}^N \times [0,\infty))$ is a unique classical solution to (1) which is finite in $\mathbb{R}^N \times [0,\infty)$. It is known that the initial function u_0 has to decay to $-\infty$ as $|x| \to \infty$ for the global solution to exist. Throughout this paper, we assume that there exist $\varepsilon \in (0,2)$ and $C > 0$ such that

$$-Ce^{|x|^{2-\varepsilon}} \leq u_0(x) \leq C, \quad x \in \mathbb{R}^N. \tag{2}$$

In this paper, we are interested in the existence of solution to (1) lying on the borderline between global existence and blow-up in finite time.

We introduce some known results for a semi-linear heat equation with power type nonlinearity. We consider the Cauchy problem:

$$\begin{cases} u_t - \Delta u = u^p, & (x,t) \in \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{3}$$

where $u_t = \frac{\partial}{\partial t}u$, $\Delta u = \sum_{i=1}^{i=N} \frac{\partial^2}{\partial {x_i}^2} u$, $p > 1$ and u_0 is a non-negative and bounded continuous initial function. It is well known that the exponent $p_F := (N+2)/N$ which is called the Fujita exponent, plays an important role in the existence of global solution of (3). In fact, If $1 < p \leq p_F$ then non-trivial non-negative solutions must blow-up in finite time. On the other hand, if $p > p_F$, there exist global solutions for suitable small initial data. The existence of global solution to problem (3) strongly depends on the decay rate of initial function u_0 at $x = \infty$. In fact, Fujita [3] showed that (3) has a global solution if u_0 has the form of a small multiple of Gaussian, which decays exponentially at $x = \infty$. Weissler [15] showed that (3) has global solutions if u_0 has polynomial decay at $x = \infty$. Lee and Ni [6] showed that the borderline decay rate of u_0 is to be $|x|^{-2/(p-1)}$ at $x = \infty$. In order to study the borderline decay rate, we consider the stationary problem of (3) ,that is, positive solutions u to the equation

$$\Delta u + u^p = 0 \quad \text{in } \mathbb{R}^N, \tag{4}$$

where $N \geq 3$. When $p > N/(N-2)$, equation (4) has a singular solution of the form:

$$u^*(x) := l^*|x|^{-\frac{2}{p-1}}, \quad l^* := \left(\frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \right)^{1/(p-1)}.$$

When $p \geq (N+2)/(N-2)$, equation (4) has one parameter family of radially symmetric regular solutions $\{u_\alpha\}_\alpha$ with initial condition $u_\alpha(0) = \alpha > 0$, where every u_α satisfy $\lim_{|x|\to\infty} |x|^{\frac{2}{p-1}} u_\alpha(|x|) = L$ and their stability was studied in [4]. Define the exponent p_{JL} by

$$p_{JL} = \begin{cases} \infty, & 3 \leq N \leq 10, \\ 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & N \geq 11. \end{cases}$$

This exponent p_{JL} which is called the Joseph-Lundgren exponent plays an important role in the stability of radially symmetric stationary solutions of (3).

The equation in (3) is invariant under the similarity transform

$$u_\lambda(x,t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t), \quad \text{for all } \lambda > 0.$$

In particular, a solution u is said to be self-similar if

$$u(x,t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t), \quad \text{for all } \lambda > 0. \tag{5}$$

We call the solution u to (3) the forward self-similar solution if u is of the form:

$$u(x,t) = t^{-1/(p-1)} \varphi(x/\sqrt{t}) \tag{6}$$

where φ satisfies the elliptic equation

$$\Delta\varphi + \frac{1}{2} x \cdot \nabla\varphi + \frac{1}{p-1}\varphi + \varphi^p = 0 \text{ in } \mathbb{R}^N. \tag{7}$$

Such forward self-similar solutions are useful tools to describe the large time behavior of the solution to (3). In particular, if $\varphi = \varphi(r), r = |x|$, then φ satisfies $\varphi'(0) = 0$ and

$$\varphi'' + \left(\frac{N-1}{r} + \frac{r}{2}\right)\varphi' + \frac{1}{p-1}\varphi + \varphi^p = 0 \quad \text{for } r > 0. \tag{8}$$

Then we can use ODE theory in investigating forward self-similar solutions. We are interested in positive solutions φ to (8) satisfying $\varphi'(0) = 0$ and

$$\lim_{r\to\infty} r^{2/(p-1)} \varphi(r) = l \tag{9}$$

with some $l > 0$. For each $l > 0$, we introduce the solution set

$$S_l = \{\varphi \in C^2[0,\infty) : \varphi > 0 \text{ is a solution to (8) satisfying } \varphi'(0) = 0 \text{ and (9)}\}. \tag{10}$$

We call $\underline{\varphi}_l$ a minimal solution of S_l if $\underline{\varphi}_l \leq \varphi$ for all $\varphi \in S_l$. Naito [8] showed the existence of a minimal solution of S_l by the comparison principle.

Theorem A (Naito [8]). *Let S_l be defined by* (10). If $S_l \neq \emptyset$, then S_l has a minimal solution.

Naito [9] also showed the following results.

Theorem B (Naito [9]). *Let $p_F < p < p_{JL}$. Assume that there exists a non minimal solution φ_l of S_l. Define a self-similar solution u_l by*

$$u_l(x,t) = t^{-\frac{1}{(p-1)}}\varphi_l\left(\frac{|x|}{\sqrt{t}}\right). \tag{11}$$

(i) If $u_0(x) \geq u_l(x,t_0)$ and $u_0(x) \not\equiv u_l(x,t_0)$ for $x \in \mathbb{R}^N$ with some $t_0 > 0$, then the solution u to (3) blows up in finite time.

(ii) If $u_0(x) \leq u_l(x,t_0)$ and $u_0(x) \not\equiv u_l(x,t_0)$ for $x \in \mathbb{R}^N$ with some $t_0 > 0$, then the solution u to (3) exists globally in time.

The purpose of this paper is to prove the same conclusions of Theorem A and B to problem (1). We consider stationary solutions, that is, solutions to elliptic equation;

$$-\Delta u = e^u. \tag{12}$$

For $N \geq 3$, the function u_* defined by

$$u_*(x) := -2\log|x| + \log(2N-4),$$

is a singular solution to problem (12). Fujishima [2] showed that the decay rate $-2\log|x|$ at space infinity gives the critical decay rate for the existence of global solutions to (1). In this paper we are concerned with the case where initial function u_0 decays to $-2\log|x|$ at space infinity, that is,

$$\lim_{|x|\to\infty}(2\log|x| + u_0(x)) = L$$

with $L \in \mathbb{R}$. the equation in (1) is invariant under

$$u_\lambda(x,t) = \log\lambda^2 + u(\lambda x, \lambda^2 t) \quad for\ \lambda > 0.$$

as in mentioned in the manuscript. The function $u = u(x,t)$ is called a self-similar solution to the equation in (1) if u is of the form

$$u(x,t) = -\log t + \varphi\left(\frac{x}{\sqrt{t}}\right), \tag{13}$$

where $\varphi(y) := u(y,1)$ satisfies the elliptic equation

$$\Delta\varphi + \frac{1}{2}y\cdot\nabla\varphi + e^\varphi + 1 = 0 \quad \text{in } \mathbb{R}^N. \tag{14}$$

In particular, if $\varphi = \varphi(r), r = |y|$, then φ satisfies

$$\begin{cases} \varphi'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right)\varphi' + e^\varphi + 1 = 0, \quad r > 0, \\ \varphi'(0) = 0 \end{cases} \tag{15}$$

We are interested in solutions φ to (15) satisfying

$$\lim_{r\to\infty}(2\log r+\varphi(r))=L \tag{16}$$

with $L\in\mathbb{R}$. For any $L\in\mathbb{R}$, we introduce the solution set

$$S_L:=\big\{\varphi\in C^2([0,\infty)):\varphi \text{ is a solution to (15) satisfying (16)}\big\}. \tag{17}$$

Then we are in position to state our main theorems:

Theorem 1. *If $S_L\neq\emptyset$, then there exists a minimal solution of S_L.*

Theorem 2. *Let $3\le N\le 9$. Assume that there exists a non-minimal solution φ_L of S_L. Define a self-similar solution u_L by*

$$u_L(x,t)=-\log t+\varphi_L\left(\frac{|x|}{\sqrt{t}}\right). \tag{18}$$

(i) If $u_0(x)\ge u_L(x,t_0)$ and $u_0(x)\not\equiv u_L(x,t_0)$ for $x\in\mathbb{R}^N$ with some $t_0>0$, then the solution u to (1) blows up in finite time.

(ii) If $u_0(x)\le u_L(x,t_0)$ and $u_0(x)\not\equiv u_L(x,t_0)$ for $x\in\mathbb{R}^N$ with some $t_0>0$, then the solution u to (1) exists globally in time.

We remark that the assumption $p_{JL}=\infty$ when $3\le N\le 10$, here assumption $p_F<p<p_{JL}$ in Theorem B allows exponential nonlinearity in this case. In the case $N=10$, it is known by [2] that there is no non-minimal solution of S_L for any $L\in\mathbb{R}$. [2] also says that there exists an $L\in\mathbb{R}$ such that $S_L\neq\emptyset$ when $3\le N\le 9$.

We explain the main strategy to prove Theorem 1 and 2. We first approximate the solution to equation (1) by that of equation (3) by using the formula

$$e^u=\lim_{n\to\infty}\left(1+\frac{u}{n}\right)^n;$$

that is, we consider the following approximate equation

$$u_t^{(n)}-\Delta u^{(n)}=\left(1+\frac{u^{(n)}}{n}\right)^n \quad \text{in } \mathbb{R}^N\times(0,\infty). \tag{19}$$

Then we can use directly the knowledge for power type nonlinear equation (3) to induce desired property for exponential type nonlinear equation (1).
The paper is organized as follows: In Section 2 we present some preliminary results. In Section 3 we prove the existence of approximate self-similar solution. In Section 4 we investigate properties of solution set S_L, in particular we establish the existence of a minimal solution of S_L by using approximate solutions. In section 5, we prove Theorem 2.

2 The existence of approximate solutions.

In this section we consider the non-linear heat equation:

$$u_t^{(n)} - \Delta u^{(n)} = \left(1 + \frac{u^{(n)}}{n}\right)^n \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{20}$$

The equation in (20) is invariant under the transformation:

$$u_\lambda^{(n)}(x,t) = n(\lambda^{2/(n-1)} - 1) + \lambda^{2/(n-1)} u^{(n)}(\lambda x, \lambda^2 t) \quad \text{for all } \lambda > 0.$$

In particular, we call $u^{(n)}$ a self-similar solution when $u^{(n)} = u_\lambda^{(n)}$ for all $\lambda > 0$. Forward self-similar solutions are of the form:

$$u^{(n)}(x,t) = n(t^{-1/(n-1)} - 1) + t^{-1/(n-1)} \varphi^{(n)}(\frac{x}{\sqrt{t}}), \tag{21}$$

where $\varphi^{(n)}$ satisfies elliptic equation

$$\Delta \varphi^{(n)} + \frac{1}{2} x \cdot \nabla \varphi^{(n)} + \frac{1}{n-1}(\varphi^{(n)} + n) + \left(1 + \frac{\varphi^{(n)}}{n}\right)^n = 0 \quad \text{in } \mathbb{R}^N.$$

Note here that $\varphi^{(n)}(r)$ of (21) converges to $\varphi(r)$ of (13) as $n \to \infty$. In particular, if $\varphi^{(n)} = \varphi^{(n)}(r), r = |x|$, then $\varphi^{(n)}$ satisfies

$$\begin{cases} \varphi^{(n)\prime\prime} + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right)\varphi^{(n)\prime} + \dfrac{1}{n-1}(\varphi^{(n)} + n) + \left(1 + \dfrac{\varphi^{(n)}}{n}\right)^n = 0, \quad r > 0, \\ \varphi^{(n)\prime}(0) = 0. \end{cases} \tag{22}$$

We establish that the forward self-similar solution of semi-linear heat equations with exponential nonlinearity is approximated by that of semi-linear heat equations with power type nonlinearity.

Theorem 3. *Let φ_α be the solution to* (15) with $\varphi_\alpha(0) = \alpha \in \mathbb{R}$. Then there exists a sequence $\{\varphi_\alpha^{(n)}\}_{n \geq 1}$ of (22) such that $\varphi_\alpha^{(n)} > -n$ and

$$\lim_{n \to \infty} \sup_{0 \leq r \leq r_0} |\varphi_\alpha^{(n)}(r) - \varphi_\alpha(r)| = 0 \quad \text{for } r_0 > 0. \tag{23}$$

Proof of Theorem 3. Let $n_0 \in \mathbb{N}$ be chosen such that $n_0 + \alpha > 0$. Let $\psi_\alpha^{(n)}(r)$ be the positive solution to the following differential equation:

$$\begin{cases} \psi_\alpha^{(n)\prime\prime} + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right)\psi_\alpha^{(n)\prime} + \dfrac{1}{n-1}\psi_\alpha^{(n)} + \left(\dfrac{\psi_\alpha^{(n)}}{n}\right)^n = 0, & n \geq n_0, \\ \psi_\alpha^{(n)}(0) = \alpha + n > 0, \quad \psi_\alpha^{(n)\prime}(0) = 0, & n \geq n_0. \end{cases} \tag{24}$$

By (24), $\psi_\alpha^{(n)}$ satisfies the following integral equations:

$$\psi_\alpha^{(n)}(r) = \alpha + n - \int_0^r \frac{1}{\rho_N(s)} \int_0^s \rho_N(t) \left[\frac{1}{n-1} \psi_\alpha^{(n)}(t) + \left(\frac{\psi_\alpha^{(n)}(t)}{n} \right)^n \right] dt\, ds, \tag{25}$$

$$\psi_\alpha^{(n)\prime}(r) = -\frac{1}{\rho_N(r)} \int_0^r \rho_N(s) \left[\frac{1}{n-1} \psi_\alpha^{(n)}(s) + \left(\frac{\psi_\alpha^{(n)}(s)}{n} \right)^n \right] dt\, ds, \tag{26}$$

where $\rho_N(r) = r^{N-1} e^{\frac{r^2}{4}}$. Since $\psi_\alpha^{(n)\prime}(r) < 0$, we have

$$0 < \psi_\alpha^{(n)}(r) \le \alpha + n. \tag{27}$$

Put $\varphi_\alpha^{(n)} = \psi_\alpha^{(n)}(r) - n$. Since (25), we have

$$\varphi_\alpha^{(n)}(r) = \alpha - \int_0^r \frac{1}{\rho_N(s)} \int_0^s \rho_N(t) \left[\frac{1}{n-1} \psi_\alpha^{(n)}(t) + \left(\frac{\psi_\alpha^{(n)}(t)}{n} \right)^n \right] dt\, ds. \tag{28}$$

We remark that $(1 + a/n)^n \le e^a$ $(a > 0)$. (27) and (28) imply that

$$\begin{aligned} |\varphi_\alpha^{(n)}(r)| &\le |\alpha| + \int_0^r \frac{1}{\rho_N(s)} \int_0^s \rho_N(t) \left[\frac{1}{n-1}(|\alpha| + n) + \left(1 + \frac{|\alpha|}{n} \right)^n \right] dt\, ds, \\ &\le |\alpha| + (e^{|\alpha|} + |\alpha| + 2) \int_0^r \frac{1}{\rho_N(s)} \int_0^s \rho_N(t)\, dt\, ds, \\ &\le |\alpha| + (e^{|\alpha|} + |\alpha| + 2) \int_0^r \int_0^s dt\, ds, \\ &\le |\alpha| + \frac{1}{2}(e^{|\alpha|} + |\alpha| + 2) r_0^2, \end{aligned} \tag{29}$$

for all $r \in [0, r_0]$. Thus we obtain that $\{\varphi_\alpha^{(n)}\}_{n \ge n_0}$ is uniformly bounded on $[0, r_0]$. From (26) and (29) , we see that

$$\begin{aligned} |\varphi_\alpha^{(n)\prime}(r)| &= |\psi_\alpha^{(n)\prime}(r)| \\ &\le \frac{1}{\rho_N(r)} \int_0^r \rho_N(s) \left[\frac{1}{n-1}(|\alpha| + n) + \left(1 + \frac{|\alpha|}{n} \right)^n \right] ds, \\ &\le (e^{|\alpha|} + |\alpha| + 2) r_0, \end{aligned}$$

for all $r \in [0, r_0]$. Thus we have deduced that $\{\varphi_\alpha^{(n)}\}_{n \ge n_0}$ is equi-continuous on $[0, r_0]$. By the Ascoli-Arzela theorem, there exists a subsequence of $\{\varphi_\alpha^{(n)}\}_{n \ge n_0}$ which converges to $\tilde{\varphi}_\alpha \in C[0, r_0]$ uniformly on $[0, r_0]$. Letting $n \to \infty$ in (28) we have

$$\tilde{\varphi}_\alpha(r) = \alpha - \int_0^r \frac{1}{\rho_N(s)} \int_0^s \rho_N(t)(1 + e^{\tilde{\varphi}_\alpha(t)})\, dt\, ds.$$

Thus $\tilde{\varphi}_\alpha \in C^2$ is the solution to (15) with $\tilde{\varphi}_\alpha(0) = \alpha$ and $\tilde{\varphi}_\alpha'(0) = 0$. By the uniqueness of solution to ordinary differential equations, we conclude $\tilde{\varphi}_\alpha \equiv \varphi_\alpha$. ■

The following theorem shows that $\varphi \in S_L$ is approximated by the solution $\varphi_\alpha^{(n)}$ with the aid of Theorem 3 .

Theorem 4. *Let $\varphi_\alpha \in S_L$ with $\varphi_\alpha(0) = \alpha$. Assume that $\{\varphi_\alpha^{(n)}\}_{n\geq 1}$ is given by* Theorem 3. Then there exists $L^{(n)}(\alpha) \in \mathbb{R}$ $(n \geq 1)$ such that

$$\lim_{r\to\infty} \left[r^{\frac{2}{n-1}} (\varphi_\alpha^{(n)}(r) + n) \right] - n = L^{(n)}(\alpha), \quad \lim_{n\to\infty} L^{(n)}(\alpha) = L. \tag{30}$$

Remark 1. *Let $\psi^{(n)}$ be the solution to the equation:*

$$\begin{cases} \psi^{(n)''} + \left(\dfrac{N-1}{r} + \dfrac{r}{2} \right) \psi^{(n)'} + \dfrac{1}{n-1} \psi^{(n)} + \left(\dfrac{\psi^{(n)}}{n} \right)^n = 0 \\ \psi^{(n)'}(0) = 0. \end{cases} \tag{31}$$

For $L > 0$, we are concerned with the solution set

$$S_L^{(n)} := \{ \psi^{(n)} \in C^2[0,\infty) : \psi^{(n)} > 0 \text{ is a solution to (31) satisfying} \lim_{r\to\infty} r^{\frac{2}{n-1}} \psi^{(n)}(r) = L \ \}. \tag{32}$$

Let $\varphi_\alpha \in S_L$ with $\varphi_\alpha(0) = \alpha$. Assume that $\{\varphi_\alpha^{(n)}\}_{n\geq 1}$ is given by Theorem 3. Put $\psi_\alpha^{(n)}(r) = \varphi_\alpha^{(n)} + n$. Then $\psi_\alpha^{(n)}$ satisfies $\psi_\alpha^{(n)} > 0$, (31),

$$\lim_{r\to\infty} \left[r^{\frac{2}{n-1}} \psi_\alpha^{(n)}(r) \right] = L^{(n)}(\alpha) + n, \quad \textit{and} \quad \lim_{n\to\infty} L^{(n)}(\alpha) = L,$$

that is, $\psi_\alpha^{(n)} \in S_{L^{(n)}(\alpha)+n}^{(n)}$.

In order to prove Theorem 4, we need the following proposition.

Proposition 1. *Let $\psi^{(n)} = \psi_\alpha^{(n)} \in C^2[0,\infty)$ $(n \geq 1)$ be the solution to* (31) with $\psi_\alpha^{(n)}(0) = \alpha$. Then there exists $C = C(\alpha) > 0$ such that

$$\left(\frac{|\psi^{(n)}(r)|}{n} \right)^n \leq C(1+r)^{-2n/(n-1)} \quad \text{for } r > 0, \tag{33}$$

$$|\psi^{(n)'}(r)| \leq C(1+r)^{-2/(n-1)-1} \quad \text{for } r > 0. \tag{34}$$

We remark that Constant C do not depend on n. To prove Proposition 1, we introduce Energy function

$$E^{(n)}(r) = \frac{{\psi^{(n)'}}^2(r)}{2} + \frac{1}{2(n-1)} {\psi^{(n)}}^2(r) + \frac{1}{n^n(n+1)} {\psi^{(n)}}^{n+1}(r), \quad r > 0, n > 1. \tag{35}$$

Then, we prepare the following lemmas

Lemma 1. *Let $\psi^{(n)} = \psi_\alpha^{(n)} \in C^2[0,\infty)$ $(n \geq 1)$ be the solution to* (31) with $\psi_\alpha^{(n)}(0) = \alpha$. Assume that $E^{(n)}(r)$ is given by (35). Then $E^{(n)}(r)$ is non increasing function in r. In particular, $E^{(n)}(r) \leq E^{(n)}(0) \quad (r > 0)$.

Proof.

$$\begin{aligned}\frac{d}{dr}E^{(n)}(r) &= \left(\psi^{(n)''}(r) + \frac{1}{n-1}\psi^{(n)}(r) + \left(\frac{\psi^{(n)}(r)}{n}\right)^n\right)\psi^{(n)'}(r) \\ &= -\left(\frac{N-1}{r} + \frac{r}{2}\right)\psi^{(n)'2} \le 0,\end{aligned}$$

Thus $E^{(n)}(r)$ is non increasing in $r > 0$. In particular, $E^{(n)}(r) \le E^{(n)}(0) \quad (r > 0)$. ■

Lemma 2 ([5] Proposition 3.1)**.** *Let* $\psi^{(n)} = \psi^{(n)}_\alpha \in C^2[0,\infty)$ $(n \ge 1)$ *be the solution to* (31) with $\psi^{(n)}_\alpha(0) = \alpha$. Then there exists $C = C(\alpha, n) > 0$ such that

$$|\psi^{(n)}(r)| \le C(\alpha,n)(1+r)^{-2/(n-1)} \quad \text{for } r > 0, \tag{36}$$

$$|\psi^{(n)'}(r)| \le C(\alpha,n)(1+r)^{-2/(n-1)-1} \quad \text{for } r > 0. \tag{37}$$

where $C(\alpha, n) = \sqrt{2(n-1)E^{(n)}(0)}$.

Proof of Proposition 1. By Lemma 2, we get the esitimates

$$|\psi^{(n)}(r)| \le C(\alpha,n)(1+r)^{-2/(n-1)} \quad \text{for } r > 0, \tag{38}$$

$$|\psi^{(n)'}(r)| \le C(\alpha,n)(1+r)^{-2/(n-1)-1} \quad \text{for } r > 0. \tag{39}$$

where $C(\alpha, n) = \sqrt{2(n-1)E^{(n)}(0)}$. Since $\left(1 + \frac{a}{n}\right)^n \le e^a \quad (a > 0)$, we have

$$\begin{aligned}\frac{1}{n}C(\alpha,n) &= \frac{1}{n}\sqrt{(n-1)E^{(n)}(0)} \\ &= \sqrt{\frac{(n-1)}{n^2}\left(\frac{1}{(n-1)}(\alpha+n)^2 + \frac{2}{n^n(n+1)}(\alpha+n)^{n+1}\right)} \\ &\le \sqrt{\left(1+\frac{|\alpha|}{n}\right)^2 + \frac{2}{n}\left(1+\frac{|\alpha|}{n}\right)^{n+1}} \\ &\le \left(1+\frac{|\alpha|}{n}\right)\sqrt{1+\frac{2}{n}\left(1+\frac{|\alpha|}{n}\right)^{n-1}} \\ &\le \left(1+\frac{|\alpha|}{n}\right)\sqrt{1+\frac{2}{n}e^{|\alpha|}}.\end{aligned}$$

we obtain

$$\left(\frac{1}{n}C(\alpha,n)\right)^n \le e^{|\alpha|+e^{|\alpha|}} \tag{40}$$

By (38) and (40), we have

$$\left(\frac{|\psi^{(n)}(r)|}{n}\right)^n \le C(\alpha)(1+r)^{-\frac{2n}{n-1}} \tag{41}$$

Since (41) and $\lim_{n\to\infty}\left(\frac{(n+|\alpha|)^2}{(n-1)^2}+2\left(1+\frac{|\alpha|}{n}\right)^{n+1}\right)=1+2e^{|\alpha|}$, we get

$$\begin{aligned}
\frac{1}{n-1}C(\alpha,n) &\leq \frac{1}{n-1}\sqrt{2(n-1)E^{(n)}(0)}\\
&= \sqrt{\frac{1}{n-1}\left(\frac{1}{(n-1)}(n+|\alpha|)^2+\frac{2}{n+1}\left(\frac{(n+|\alpha|)^{n+1}}{n^n}\right)\right.}\\
&\leq \sqrt{\frac{(n+|\alpha|)^2}{(n-1)^2}+2\left(1+\frac{|\alpha|}{n}\right)^{n+1}}\\
&\leq C(\alpha) \qquad (42)
\end{aligned}$$

Since (38), (42), we have

$$\frac{|\psi^{(n)}(r)|}{n-1}\leq C(\alpha)(1+r)^{-\frac{2}{n-1}} \qquad (43)$$

By (26), (41) and (44) we have

$$\begin{aligned}
|\psi^{(n)\prime}(r)| &\leq r^{1-N}e^{-\frac{r^2}{4}}\int_0^r s^{N-1}e^{\frac{s^2}{4}}\left[\frac{1}{n-1}|\psi^{(n)}(s)|+\left(\frac{|\psi^{(n)}(s)|}{n}\right)^n\right]ds\\
&\leq C(\alpha)e^{-\frac{r^2}{4}}\int_0^r e^{\frac{s^2}{4}}\left[(1+s)^{-\frac{2}{n-1}}+(1+s)^{-\frac{2n}{n-1}}\right]ds\\
&\leq C(\alpha)e^{-\frac{r^2}{4}}\int_0^r e^{\frac{s^2}{4}}(1+s)^{-\frac{2}{n-1}}\,ds\\
&\leq C(\alpha)e^{-\frac{r^2}{4}}\left(\int_0^{\frac{r}{2}} e^{\frac{s^2}{4}}\,ds+\int_{\frac{r}{2}}^r e^{\frac{s^2}{4}}(1+s)^{-\frac{2}{n-1}}\,ds\right)\\
&\leq C(\alpha)\left[e^{-\frac{3r^2}{16}}+(1+\frac{r}{2})^{-\frac{2}{n-1}-1}e^{-\frac{r^2}{4}}\int_{\frac{r}{2}}^r(1+s)e^{\frac{s^2}{4}}\,ds\right] \qquad (44)
\end{aligned}$$

If $r<2$, Right hand side of (44) is bounded. If $r\geq 2$, Since

$$\int_{\frac{r}{2}}^r 2se^{\frac{s^2}{4}}\,ds=4e^{\frac{r^2}{4}}-4e^{\frac{r^2}{16}}\leq 4e^{\frac{r^2}{4}},$$

Right hand side of (44) is bounded. Therefore we obtain

$$|\psi^{(n)\prime}(r)|\leq C(\alpha)(1+r)^{-\frac{2}{n-1}-1}.$$

■

Lemma 3. *Let $\varphi\in C^2[0,\infty)$ be the solution to* (15) with $\varphi(0)=\alpha$. Then there exists a constant $C=C(\alpha)>0$ such that

$$|\varphi'(r)|\leq C(1+r)^{-1} \quad for\ r>0.$$

Proof of Theorem 4. The following argument can be found in the proof of [[5] proposition 3.4]. From Theorem 3, there exists a sequence $\{\varphi_\alpha^{(n)}\}_{n\geq 1}$ of (22) such that $\varphi_\alpha^{(n)} = \varphi^{(n)} > -n$ and (23). Put $\psi^{(n)} = \varphi^{(n)} + n$. Then $\psi^{(n)}$ satisfies (24). The identity

$$(r^{2/(n-1)}\psi^{(n)})' = r^{2/(n-1)-1}(r\psi^{(n)\prime} + \frac{2}{n-1}\psi^{(n)})$$

and (24) implies that

$$\begin{aligned}&\frac{d}{dr}\left[r^{2/(n-1)}\psi^{(n)}(r) + 2r^{2/(n-1)-1}\psi^{(n)\prime}(r)\right]\\&= 2(\frac{2}{n-1} - N)r^{2/(n-1)-2}\psi^{(n)\prime}(r) - 2r^{2/(n-1)-1}\left(\frac{\psi^{(n)}(r)}{n}\right)^n.\end{aligned} \tag{45}$$

Integrating (45) from 1 to r, we have

$$\begin{aligned}&r^{2/(n-1)}\psi^{(n)}(r) + 2r^{2/(n-1)-1}\psi^{(n)\prime}(r) - \psi^{(n)}(1) - 2\psi^{(n)\prime}(1)\\&= 2(\frac{2}{n-1} - N)\int_1^r t^{2/(n-1)-2}\psi^{(n)\prime}(t)\,dt - 2\int_1^r t^{2/(n-1)-1}\left(\frac{\psi^{(n)}(t)}{n}\right)^n dt.\end{aligned} \tag{46}$$

Note that we have

$$\int_1^\infty t^{2/(n-1)-2}\psi^{(n)\prime}(t)\,dt < \infty \quad \text{and} \quad \int_1^\infty t^{2/(n-1)-1}\left(\frac{\psi^{(n)}(t)}{n}\right)^n dt < \infty,$$

by Proposition 1. Letting $r \to \infty$ in (46), we get

$$\begin{aligned}&\lim_{r\to\infty}(r^{2/(n-1)}\psi^{(n)}(r)) - \psi^{(n)}(1) - 2\psi^{(n)\prime}(1)\\&= 2(\frac{2}{n-1} - N)\int_1^\infty t^{2/(n-1)-2}\psi^{(n)\prime}(t)\,dt - 2\int_1^\infty t^{2/(n-1)-1}\left(\frac{\psi^{(n)}(t)}{n}\right)^n dt.\end{aligned} \tag{47}$$

Since $\psi^{(n)} = \varphi^{(n)} + n$, we obtain

$$\begin{aligned}&L^{(n)}(\alpha) - \varphi^{(n)}(1) - 2\varphi^{(n)\prime}(1)\\&= 2(\frac{2}{n-1} - N)\int_1^\infty t^{2/(n-1)-2}\psi^{(n)\prime}(t)\,dt - 2\int_1^\infty t^{2/(n-1)-1}\left(\frac{\psi^{(n)}(t)}{n}\right)^n dt.\end{aligned} \tag{48}$$

By Proposition 1, there exists a constant $C > 0$ such that

$$\left|t^{2/(n-1)-2}\psi^{(n)\prime}(t)\right| \leq C(1+t)^{-3}, \tag{49}$$

$$\left|t^{2/(n-1)-1}\left(\frac{\psi^{(n)}(t)}{n}\right)^n\right| \leq C(1+t)^{-3} \tag{50}$$

and we have

$$\begin{aligned}
\lim_{n\to\infty} t^{2/(n-1)-2}{\psi^{(n)}}'(t) &= \lim_{n\to\infty} t^{2/(n-1)-2}\frac{d}{dt}[\varphi^{(n)}(t)+n] \\
&= \lim_{n\to\infty} t^{2/(n-1)-2}{\varphi^{(n)}}'(t) \\
&= t^{-2}\varphi'(t), \quad t\in\mathbb{R}, \\
\lim_{n\to\infty} t^{2/(n-1)-1}\left(\frac{\psi^{(n)}(t)}{n}\right)^n &= \lim_{n\to\infty} t^{2/(n-1)-1}\left(1+\frac{\varphi^{(n)}(t)}{n}\right)^n \\
&= t^{-1}e^{\varphi(t)}, \quad t\in\mathbb{R}.
\end{aligned}$$

Letting $n\to\infty$ in (48), we have

$$\begin{aligned}
&\lim_{n\to\infty} L^{(n)}(\alpha)-\varphi(1)-2\varphi'(1) \\
&= -2N\int_1^\infty t^{-2}\varphi'(t)\,dt - 2\int_1^\infty t^{-1}e^{\varphi}\,dt,
\end{aligned} \tag{51}$$

by the Lebesgue convergence theorem and (23). Thus $\lim_{n\to\infty} L^{(n)}(\alpha)$ exists. On the other hand, since

$$(2\log r+\varphi(r))' = r^{-1}(r\varphi'(r)+2),$$

we have

$$\frac{d}{dr}(2\log r+\varphi(r)+2r^{-1}\varphi'(r)) = -2Nr^{-2}\varphi'(r)-2r^{-1}e^{\varphi(r)}. \tag{52}$$

We remark that $\varphi'(r)/r\to 0$ as $r\to\infty$ by Lemma 3. Integrating (52) from 1 to ∞, we have

$$L-\varphi(1)-2\varphi'(1) = -2N\int_1^\infty t^{-2}\varphi'(t)\,dt - 2\int_1^\infty t^{-1}e^{\varphi(t)}\,dt. \tag{53}$$

From (51) and (53), we conclude that

$$\lim_{n\to\infty} L^{(n)}(\alpha) = L.$$

■

3 Properties of solution set S_L.

In this section, we will demonstrate the existence of a minimal solution of solution set S_L. To prove Theorem 1, we prepare the following lemma.

Lemma 4 ([8] Lemma 3.1)**.** *Let* $S_l^{(n)}$ *be defined by* (32). If $S_l^{(n)}\neq\emptyset$, then $S_l^{(n)}$ has a minimal solution.

Proof of Theorem 1. Let $\varphi \in S_L$ with $\varphi(0) = \alpha$. Assume that $\varphi^{(n)} = \varphi_\alpha^{(n)}$ and let $L^{(n)} = L^{(n)}(\alpha)$ be defined by Theorem 4. $\psi^{(n)} = \varphi^{(n)} + n$. Take $n \in \mathbb{N}$ so large that $L^{(n)} + n > 0$. Then we have

$$\lim_{r\to\infty} r^{2/(n-1)}\psi^{(n)}(r) > 0,$$

that is, $\psi^{(n)} \in S^{(n)}_{L^{(n)}+n}$. Hence there exists a minimal solution $\underline{\psi}^{(n)} \in S^{(n)}_{L^{(n)}+n}$ by Lemma 4. We remark that $\underline{\psi}^{(n)}$ does not depend on $\varphi(0) = \alpha$. Since $\underline{\psi}^{(n)}$ satisfies (24), we have the following integral equations:

$$\begin{aligned}
\underline{\psi}^{(n)}(r) &= \underline{\psi}^{(n)}(0) - \int_0^r \frac{1}{\rho_N(s)} \int_0^s \rho_N(t) \left[\frac{1}{n-1}\underline{\psi}^{(n)}(t) + \left(\frac{\underline{\psi}^{(n)}(t)}{n}\right)^n\right] dt\, ds, \\
\underline{\psi}^{(n)\prime}(r) &= -\frac{1}{\rho_N(r)} \int_0^r \rho_N(s) \left[\frac{1}{n-1}\underline{\psi}^{(n)}(s) + \left(\frac{\underline{\psi}^{(n)}(s)}{n}\right)^n\right] ds,
\end{aligned}$$

where $\rho_N(r) = r^{N-1}e^{r^2/4}$. Put $\underline{\varphi}^{(n)} = \underline{\psi}^{(n)} - n$. Since $\underline{\psi}^{(n)}(r) \le \psi^{(n)}(r)$ $(r > 0)$, we have $\underline{\varphi}^{(n)}(0) \le \alpha$. We now claim that $\{\underline{\varphi}^{(n)}(0)\}$ is bounded below. We integrate equation (45) with $\psi^{(n)}$ replaced by $\underline{\psi}^{(n)}$ from 1 to r. Then

$$\begin{aligned}
&r^{2/(n-1)}\underline{\psi}^{(n)}(r) + 2r^{2/(n-1)-1}\underline{\psi}^{(n)\prime}(r) - \underline{\psi}^{(n)}(1) - 2\underline{\psi}^{(n)\prime}(1) \\
&= 2(\frac{2}{n-1} - N) \int_1^r t^{2/(n-1)-2}\underline{\psi}^{(n)\prime}(t)\, dt - 2\int_1^r t^{2/(n-1)-1}\left(\frac{\underline{\psi}^{(n)}(t)}{n}\right)^n dt,
\end{aligned} \tag{54}$$

since $\lim_{r\to 0} r^{2/(n-1)}\underline{\varphi}^{(n)}(r) = 0,\ 2\lim_{r\to 0} r^{2/(n-1)-1}\underline{\varphi}^{(n)\prime}(r) = 0$. Letting $r \to \infty$ in (54), we have

$$\begin{aligned}
&L^{(n)} + n - \underline{\psi}^{(n)}(1) - 2\underline{\psi}^{(n)\prime}(1) \\
&= 2(\frac{2}{n-1} - N) \int_1^\infty t^{2/(n-1)-2}\underline{\psi}^{(n)\prime}(t)\, dt - 2\int_1^\infty t^{2/(n-1)-1}\left(\frac{\underline{\psi}^{(n)}(t)}{n}\right)^n dt.
\end{aligned} \tag{55}$$

By $\lim_{n\to\infty} L^{(n)} = L$, there exists $C > 0$ such that

$$|L^{(n)}| \le C \quad \text{for } n \in \mathbb{N}. \tag{56}$$

Since Proposition 1 and (56), we have

$$\begin{aligned}
|\underline{\psi}^{(n)}(1) - n| &\le |L^{(n)}| + |\underline{\psi}^{(n)\prime}(1)| \\
+2(\frac{2}{n-1} - N) &\int_1^\infty t^{2/(n-1)-2}|\underline{\psi}^{(n)\prime}(t)|\, dt + 2\int_1^\infty t^{2/(n-1)-1}\left(\frac{|\underline{\psi}^{(n)}(t)|}{n}\right)^n dt \\
&\le 2C + 2C\int_1^\infty t^{-3}\, dt \\
&\le C
\end{aligned}$$

Thus $\{\underline{\psi}^{(n)}(1)-n\}_{n\in\mathbb{N}}$ is bounded. Then there exists $C>0$ such that $|\underline{\psi}^{(n)}(1)-n|\le C$. Since $\psi^{(n)}(r)$ is non increasing in $r>0$, we obtain

$$-C \le \underline{\psi}^{(n)}(1)-n \le \underline{\psi}^{(n)}(0)-n$$

Therefore $\{\underline{\psi}^{(n)}(0)-n\}_{n\in\mathbb{N}}$ is bounded. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\underline{\varphi}^{(n_k)}(0)$ of $\underline{\varphi}^{(n)}(0)$. Then $\underline{\varphi}^{(n_k)}$ satisfies the following:

$$\begin{aligned}\underline{\varphi}^{(n_k)}(r) &= \underline{\varphi}^{(n_k)}(0)-\int_0^r \frac{1}{\rho_N(s)}\int_0^s \rho_N(t)\left[\frac{1}{n-1}(\underline{\varphi}^{(n_k)}(t)+n)+\left(1+\frac{\underline{\varphi}^{(n_k)}(t)}{n}\right)^n\right]dt\,ds,\\ \underline{\varphi}^{(n_k)\prime}(r) &= -\frac{1}{\rho_N(r)}\int_0^r \rho_N(s)\left[\frac{1}{n-1}(\underline{\varphi}^{(n_k)}(s)+n)+\left(1+\frac{\underline{\varphi}^{(n_k)}(s)}{n}\right)^n\right]ds,\end{aligned}$$

where $\rho_N(r)=r^{N-1}e^{\frac{r^2}{4}}$. By the same argument as that in Theorem 3, $\underline{\varphi}^{(n_k)}$ converges to some $\underline{\varphi}$ uniformly in $[0,r_0]$. In particular, $\underline{\varphi}^{(n_k)}$ converges pointwisely to $\underline{\varphi}$. We show that $\lim_{r\to\infty}(\underline{\varphi}(r)+2\log r)=L$. Since (54), we have

$$\begin{aligned}&r^{2/(n-1)}\underline{\varphi}^{(n)}(r)+nr^{2/(n-1)}-n+2r^{2/(n-1)-1}\underline{\varphi}^{(n)\prime}(r)-\underline{\varphi}^{(n)}(1)-2\underline{\varphi}^{(n)\prime}(1)\\ &=2\left(\frac{2}{n-1}-N\right)\int_1^r t^{2/(n-1)-2}\underline{\varphi}^{(n)\prime}(t)\,dt-2\int_1^r t^{2/(n-1)-1}\left(1+\frac{\underline{\varphi}^{(n)}(t)}{n}\right)^n dt\end{aligned} \tag{57}$$

Letting $n\to\infty$ in (57), we have

$$\begin{aligned}&\underline{\varphi}(r)+2\log r+2r^{-1}\underline{\varphi}'(r)-\underline{\varphi}(1)-2\underline{\varphi}'(1)\\ &\qquad=-2N\int_1^r t^{-2}\underline{\varphi}'(t)\,dt-2\int_1^r t^{-1}e^{\underline{\varphi}(t)}\,dt,\end{aligned} \tag{58}$$

for $r>0$. Letting $r\to\infty$ in (58), we obtain

$$\begin{aligned}&\lim_{r\to\infty}\big(\underline{\varphi}(r)+2\log r\big)-\underline{\varphi}(1)-2\underline{\varphi}'(1)\\ &\qquad=-2N\int_1^\infty t^{-2}\underline{\varphi}'(t)\,dt-2\int_1^\infty t^{-1}e^{\underline{\varphi}(t)}\,dt,\end{aligned} \tag{59}$$

by $\lim_{r\to\infty}r^{-1}\underline{\varphi}'(r)=0$. On the other hand, Letting $n\to\infty$ in (53) with φ replaced by $\underline{\varphi}$, we obtain

$$L-\underline{\varphi}(1)-2\underline{\varphi}'(1)=-2N\int_1^\infty t^{-2}\underline{\varphi}'(t)\,dt-2\int_1^\infty t^{-1}e^{\underline{\varphi}(t)}\,dt. \tag{60}$$

From (59) and (60), we obtain $\lim_{r\to\infty}(\underline{\varphi}(r)+2\log r)=L$. Therefore $\underline{\varphi}\in S_L$. From $\underline{\varphi}^{(n_k)}\le\varphi^{(n_k)}$, letting $n_k\to\infty$, we conclude that $\underline{\varphi}\le\varphi$. Note that $\underline{\varphi}$ does not depend on φ. Therefore $\underline{\varphi}$ is a minimal solution of S_L ,i.e., $\underline{\varphi}\le\varphi$ for all $\varphi\in S_L$. ∎

Corollary 1. *Assume that there exist at least two solutions* $\underline{\varphi}$ *and* φ *of* S_L*, where* $\underline{\varphi}$ *is a minimal solution of* S_L*. Then there exist at least two solutions* $\underline{\psi}^{(n)}$ *and* $\psi^{(n)}$ *of* $S^{(n)}_{L^{(n)}+n}$*, where* $\underline{\psi}^{(n)}$ *is a minimal solution of* $S^{(n)}_{L^{(n)}+n}$*.*

Proof. In the proof of Theorem 1, there exist $\underline{\psi}^{(n)}$ and $\psi^{(n)}$ of $S^{(n)}_{L^{(n)}+n}$ such that $\underline{\varphi}^{(n)} := \underline{\psi}^{(n)} - n$ and $\varphi^{(n)} := \psi^{(n)} - n$ converge to $\underline{\varphi}$ and φ, respectively, where $\underline{\psi}^{(n)}$ is a minimal solution of $S^{(n)}_{L^{(n)}+n}$. ■

We will show the following properties of S_L.

Proposition 2. *Let* S_L *be given by* (17). Assume that there exist at least two solutions $\underline{\varphi}_L$ and φ_L of S_L, where $\underline{\varphi}_L$ is a minimal solution of S_L.

(i) If $\varphi \in S_L$ satisfies $\varphi(r) \le \varphi_L(r)\ for\ r > 0$ then $\varphi(r) \equiv \underline{\varphi}_L(r)$ or $\varphi(r) \equiv \varphi_L(r)\ for\ r > 0$.

(ii) Assume that φ is a solution to (15) satisfying $\varphi'(0) = 0$ and $\varphi(r) \ge \varphi_L(r)\ for\ r \ge 0$. Then $\varphi(r) \equiv \varphi_L(r)\ for\ r \ge 0$.

(iii) Let $\varphi \in S_{L_0}$ with some $L_0 \in (0, L]$. Assume that $\varphi(r) \le \underline{\varphi}_L(r)\ for\ r \ge 0$. Then $\varphi \in S_{L_0}$ is a minimal solution.

(iv) There exists no positive solution $\varphi \in C^2(0,\infty)$ to (15) satisfying $\varphi(r) > \varphi_L(r)\ for\ r \in (0,\infty)$ and $\varphi(r) \to \infty\ as\ r \to 0$.

In order to prove Proposition 2, we prepare the following lemma.

Lemma 5 (Naito [9] Proposition 4.1.)**.** *Let* $S^{(n)}_{L^{(n)}+n}$ *be given by* (32). Assume that there exist at least two solutions $\underline{\psi}^{(n)}_L$ and $\psi^{(n)}_L$ of $S^{(n)}_{L^{(n)}+n}$, where $\underline{\psi}^{(n)}_L$ is a minimal solution of $S^{(n)}_{L^{(n)}+n}$.

(i) If $\psi^{(n)} \in S^{(n)}_{L^{(n)}+n}$ satisfies $\psi^{(n)}(r) \le \psi^{(n)}_L(r)\ for\ r > 0$ then $\psi^{(n)}(r) \equiv \underline{\psi}^{(n)}_L(r)$ or $\psi^{(n)}(r) \equiv \psi^{(n)}_L(r)\ for\ r > 0$.

(ii) Assume that $\psi^{(n)}$ is a solution to (31) satisfying $\psi^{(n)}(r) \ge \psi^{(n)}_L(r)\ for\ r \ge 0$. Then $\psi^{(n)}(r) \equiv \psi^{(n)}_L(r)\ for\ r \ge 0$.

(iii) Let $\psi^{(n)} \in S^{(n)}_{L_0^{(n)}+n}$ with some $L_0^{(n)} \in (0, L^{(n)}]$. Assume that $\psi^{(n)}(r) \le \underline{\psi}^{(n)}_L(r)\ for\ r \ge 0$. Then $\psi^{(n)} \in S^{(n)}_{L_0^{(n)}+n}$ is a minimal solution.

(iv) There exists no positive solution $\psi^{(n)} \in C^2(0,\infty)$ to (31) satisfying $\psi^{(n)}(r) > \psi^{(n)}_L(r)\ for\ r \in (0,\infty)$ and $\psi^{(n)}(r) \to \infty\ as\ r \to 0$.

Proof of Proposition 2. Let $\underline{\psi}^{(n)}_L$ and $\psi^{(n)}_L$ be given by Corollary 1.
(i) Since $\varphi \in S_L$, there exists $\psi^{(n)}$ and $L^{(n)}$ by Theorem 4. Since $\underline{\varphi}_L \in S_L$ is a minimal solution of S_L, we have $\underline{\varphi}_L(r) \le \varphi(r)\ for\ r \ge 0$. Assume to the contrary

that $\underline{\varphi}_L \neq \varphi$ and $\varphi \neq \varphi_L$. Then by the uniqueness of the initial value problems to (15), we get $\underline{\varphi}_L(r) < \varphi(r) < \varphi_L(r)$ for $r \geq 0$, hence there exists $N \in \mathbb{N}$ such that $\underline{\psi}_L^{(n)}(r) < \psi^{(n)}(r) < \psi_L^{(n)}(r)$ for $r \geq 0, n \geq N$, and $\psi^{(n)} \in S^{(n)}_{L^{(n)}+n}$. By Lemma 5 (i), we have $\psi^{(n)}(r) \equiv \underline{\psi}_L^{(n)}(r)$ or $\psi^{(n)}(r) \equiv \psi_L^{(n)}(r)$ for $r > 0$. This is contradiction. Therefore $\varphi(r) \equiv \underline{\varphi}_L(r)$ or $\varphi(r) \equiv \varphi_L(r)$ for $r > 0$.
(ii) The proof is given by contradiction argument. Assume to the contrary that $\varphi \neq \varphi_L$. Then, by the uniqueness of the initial value problems to equation (15), we have $\underline{\varphi}_L(r) < \varphi(r) < \varphi_L(r)$ for all $r > 0$. Then there exist $N \in \mathbb{N}$ such that $\underline{\psi}_L^{(n)}(r) < \psi_L^{(n)}(r) < \psi^{(n)}(r)$ for $r \geq 0, n \geq N$. By Lemma 5 (ii) we have $\psi^{(n)}(r) \equiv \psi_L^{(n)}(r)$ for $r \geq 0$. Letting $n \to \infty$, we obtain $\varphi(r) \equiv \varphi_L(r)$ for $r \geq 0$. This is contradiction. Therefore $\varphi(r) \equiv \varphi_L(r)$ for $r \geq 0$.
(iii) If $L_0 = L$, we see that $\varphi \in S_{L_0}$ is a minimal solution. Let $L_0 < L$. Assume to the contrary that $\varphi \in S_{L_0}$ is a non-minimal solution. Then this contradicts this Proposition 2 (ii). Therefore, $\varphi \in S_{L_0}$ is a minimal solution.
(iv) Assume to the contrary that there exists a positive solution $\varphi \in C^2(0,\infty)$ to (15) satisfying the following condition:

$$\varphi(r) > \varphi_L(r),\ r \in (0,\infty), \quad \lim_{r\to 0} \varphi(r) = \infty.$$

For $\delta > 0$, let $\psi^{(n)} \in C^2[\delta,\infty)$ be the positive solution to initial value problem:

$$\begin{cases} \psi^{(n)\prime\prime} + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right)\psi^{(n)\prime} + \dfrac{1}{n-1}\psi^{(n)} + \left(\dfrac{\psi^{(n)}}{n}\right)^n = 0 \\ \psi^{(n)}(\delta) = \varphi(\delta) + n, \quad {\psi^{(n)}}'(\delta) = \varphi'(\delta). \end{cases}$$

Since $\lim_{r\to 0} \psi^{(n)}(r) = \lim_{r\to 0} \varphi(r) + n \geq \lim_{r\to 0} \varphi(r) = \infty$, there exists a positive solution $\psi^{(n)} \in C^2(0,\infty)$ to (31) satisfying $\psi^{(n)}(r) > \psi_L^{(n)}(r)$ for $r \in (0,\infty)$ and $\psi^{(n)}(r) \to \infty$ as $r \to 0$. From Lemma 5 there exists no positive solution $\psi^{(n)} \in C^2(0,\infty)$ to (31) satisfying $\psi^{(n)}(r) > \psi_L^{(n)}(r)$ for $r \in (0,\infty)$ and $\psi^{(n)}(r) \to \infty$ as $r \to 0$. This is contradiction. Therefore, there exists no positive solution $\varphi \in C^2(0,\infty)$ to (15) satisfying $\varphi(r) > \varphi_L(r)$ for $r \in (0,\infty)$ and $\varphi(r) \to \infty$ as $r \to 0$. ■

4 Proof of Theorem 2

We begin this section by introducing the definition of weak supersolution and subsolution. We say that a function u is a continuous weak supersolution to (1) in $\mathbb{R}^N \times [0,T]$ if u is a continuous on $\mathbb{R}^N \times [0,T]$, $u(x,0) \geq u_0(x)$ $x \in \mathbb{R}^N$ and satisfies

$$\int_{\mathbb{R}^N} u(x,t)\xi(x,t)\,dx\Big|_{t=0}^{t=T'} \geq \int_0^{T'}\int_{\mathbb{R}^N} [u(x,t)(\xi_t + \Delta\xi)(x,t) + e^{u(x,t)}\xi(x,t)]\,dx\,dt, \tag{61}$$

for all $T' \in [0,T]$ and for all $\xi \in C^{2,1}(\mathbb{R}^N \times [0,T])$ with $\xi \geq 0$ such that $\operatorname{supp}\xi(\cdot,t)$ is compact in $\mathbb{R}^N$ for all $t \in [0,T]$. A continuous weak subsolution to (1) in $\mathbb{R}^N \times [0,T]$ is defined in the same way by reversing the inequalities above.
We say that a function φ is a continuous weak supersolution to (14) in $\mathbb{R}^N$ if $\varphi \in C(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} \left[\varphi\left(\Delta\eta - \frac{1}{2}y\cdot\nabla\eta - \frac{N}{2}\eta\right) + (e^{\varphi}+1)\eta\right] dy \leq 0$$

for any $\eta \in C^2(\mathbb{R}^N)$ with $\eta \geq 0$ such that $\operatorname{supp}\eta(\cdot)$ is compact in $\mathbb{R}^N$. A continuous weak subsolution to (14) in $\mathbb{R}^N$ is defined in the same way by reversing the inequalities above.

Next we introduce comparison principle for problem (1).

Lemma 6 ([2] Lemma 2.3 (i)). *Let $\overline{u}$ and $\underline{u}$ be continuous weak supersolution and subsolution to* (1) in $\mathbb{R}^N \times [0,T]$, respectively. Assume that $\overline{u}$ and $\underline{u}$ are bounded above and satisfy $\overline{u}(x,t) - \underline{u}(x,t) \geq -Ae^{B|x|^2}$ in $\mathbb{R}^N \times [0,T]$ for some constants $A, B > 0$. Then $\underline{u} \leq \overline{u}$ in $\mathbb{R}^N \times [0,T]$ and there exists a classical solution to (1) satisfying $\underline{u} \leq u \leq \overline{u}$ in $\mathbb{R}^N \times [0,T]$.

We show the following proposition.

Proposition 3. *Suppose that S_L have at least two elements $\underline{\varphi}_L$ and φ_L, where $\underline{\varphi}_L$ is a minimal solution of S_L.*

(i) *Assume that $w_0 \in C(\mathbb{R}^N)$ satisfies $w_0(x) < \varphi_L(|x|)$ for $x \in \mathbb{R}^N$. Then there exists a continuous weak supersolution $\overline{w}_0$ to* (14) such that $\overline{w}_0 = \overline{w}_0(r), r = |x|$ and satisfies $\overline{w}_0 \not\equiv \varphi_L$ and

$$w_0(x) < \overline{w}_0(|x|) \leq \varphi_L(|x|), \quad x \in \mathbb{R}^N. \tag{62}$$

(ii) Assume that $w_0 \in C(\mathbb{R}^N)$ satisfies $w_0(x) > \varphi_L(|x|)$ for $x \in \mathbb{R}^N$. Then there exists a continuous weak subsolution $\underline{w}_0$ to (14) such that $\underline{w}_0 = \underline{w}_0(r), r = |x|$ is nonincreasing in $r > 0$ and satisfies $\underline{w}_0 \not\equiv \varphi_L$ and

$$\varphi_L(|x|) \leq \underline{w}_0(|x|) < w_0(x), \quad x \in \mathbb{R}^N. \tag{63}$$

In order to prove Proposition 3, we prepare the following Lemma.

Lemma 7. *Let $\alpha_1 < \alpha_2$. Assume that $\varphi(r;\alpha_i)$ $(i=1,2)$ is the solution to* (15) satisfying $\varphi'(0) = 0$ with initial data $\varphi(0;\alpha_i) = \alpha_i$ $(i=1,2)$. Suppose that there exists $r_0 > 0$ such that

$$\varphi(r;\alpha_1) < \varphi(r;\alpha_2)\ (0 \leq r < r_0), \quad \varphi(r_0;\alpha_1) = \varphi(r_0;\alpha_2).$$

If $\alpha_3 > \alpha_2$, then $\varphi(r;\alpha_3) - \varphi(r;\alpha_2)$ has at least one zero in $(0, r_0)$.

Proof. This proof is carried out by the similar argument used in the proof of [[9] Lemma 5.1]. Assume to the contrary that $\varphi(r;\alpha_3) - \varphi(r;\alpha_2) > 0,$ for $0 \leq r < r_0$. We set $\phi_1(r) = \varphi(r;\alpha_2) - \varphi(r;\alpha_1),\ \phi_2(r) = \varphi(r;\alpha_3) - \varphi(r;\alpha_2)$. Since $\varphi(r;\alpha_i)$ $(i = 1, 2, 3)$ is the solution to (15) we have

$$(\rho_N \phi_j')' + \rho_N m_j \phi_j = 0 \quad \text{for } r > 0,\ j = 1, 2, \tag{64}$$

where $\rho_N(r) = r^{N-1} e^{r^2/4}$ and m_j satisfies:

$$e^{\varphi(r;\alpha_i)} < m_j(r) < e^{\varphi(r;\alpha_{j+1})} \quad 0 \leq r \leq r_0,\ j = 1, 2.$$

Then, we obtain $m_1(r) < m_2(r)$ for $0 \leq r < r_0$ and

$$\phi_1'(r_0) \leq 0,\ \phi_2(r_0) \geq 0. \tag{65}$$

By (64) we have

$$\begin{aligned} (\rho_N \phi_1')' \phi_2 + \rho_N m_1 \phi_1 \phi_2 &= 0 \quad (r > 0), & (66)\\ (\rho_N \phi_2')' \phi_1 + \rho_N m_2 \phi_1 \phi_2 &= 0 \quad (r > 0). & (67) \end{aligned}$$

Since (66) and (67), we have

$$(\rho_N (\phi_1' \phi_2 - \phi_1 \phi_2'))' = -\rho_N (m_1 - m_2) \phi_1 \phi_2. \tag{68}$$

We integrate (68) from 0 to r_0, we obtain

$$\rho_N (\phi_1' \phi_2 - \phi_1 \phi_2') \mid_{r=0}^{r=r_0} = -\int_0^{r_0} \rho_N (m_1 - m_2) \phi_1 \phi_2 > 0.$$

On the other hand, since (65) and $\phi_i(0)' = 0$ we have

$$\rho_N (\phi_1' \phi_2 - \phi_1 \phi_2') \mid_{r=0}^{r=r_0} = \rho_N(r_0) \phi_1'(r_0) \phi_2(r_0) \leq 0.$$

This is contradiction. Therefore $\varphi(r;\alpha_3) - \varphi(r;\alpha_2)$ has at least one zero in $(0, r_0)$. ■

Lemma 8. *Assume that S_L has at least two elements $\underline{\varphi}_L$ and φ_L. Suppose that $\alpha_* = \underline{\varphi}_L(0), \alpha^* = \varphi_L(0)$ and $\alpha_0 \in (\alpha_*, \alpha^*)$. Then there exists $r_0 > 0$ such that*

$$\underline{\varphi}_L(r) < \varphi(r;\alpha_0) < \varphi_L(r) \text{ for } 0 \leq r < r_0, \quad \varphi(r_0;\alpha_0) = \varphi_L(r_0). \tag{69}$$

In addition, we have

(i) If $\alpha \in (\alpha_0, \alpha^*)$ then there exists $r_1 \in (0, r_0)$ such that

$$\varphi(r;\alpha) < \varphi_L(r)\ (0 \leq r < r_1), \quad \varphi(r_1;\alpha) = \varphi_L(r_1); \tag{70}$$

(ii) If $\alpha > \alpha^*$ then there exists $r_2 \in (0, r_0)$ such that

$$\varphi(r;\alpha) > \varphi_L(r)\ (0 \leq r < r_2), \quad \varphi(r_2;\alpha) = \varphi_L(r_2). \tag{71}$$

Proof. Since we see that $\underline{\varphi}_L(0) < \varphi(0;\alpha_0) < \varphi_L(0)$, one of the following condition (a)-(c) holds:
(a) $\underline{\varphi}_L(r) < \varphi(r;\alpha) < \varphi_L(r)$ $r > 0$;
(b) There exists $r_0 > 0$ such that

$$\underline{\varphi}_L(r) < \varphi(r;\alpha_0) < \varphi_L(r) \ 0 \leq r < r_0 \quad \underline{\varphi}_L(r) = \varphi(r_0;\alpha_0);$$

(c) There exists $r_0 > 0$ satisfying (69).
The condition (a) does not hold by Proposition 2 (i). Assume that condition (b) holds. By Lemma 7, $\varphi_L(r) - \varphi(r;\alpha_0)$ has at least one zero in $(0, r_0)$. This is contradiction. Therefore the condition (c) holds, and we have (69).

(i) Let $\alpha \in (\alpha_0, \alpha^*)$. Assume that $\varphi(r;\alpha) < \varphi_L(r)$ for $0 \leq r < r_1$. By (69), there exists $r_1 \in (0, r_0]$ such that

$$\varphi(r;\alpha_0) < \varphi(r;\alpha) \ (0 \leq r < r_1), \quad \varphi(r_1;\alpha_0) = \varphi(r_1;\alpha).$$

By Lemma 7, $\varphi_L(r) - \varphi(r;\alpha)$ has at least one zero in $(0, r_0)$. This is contradiction. We have (70).

(ii) Let $\alpha_1 = \alpha_0, \alpha_2 = \alpha^*$ and $\alpha_3 = \alpha$. By Lemma 7, $\varphi(r;\alpha) - \varphi_L(r)$ has at least one zero in $(0, r_0)$. Therefore (71) holds.

■

Lemma 9 ([2] Lemma 2.5)**.** *(i) Let $\varphi_1 = \varphi_1(|y|)$ and $\varphi_2 = \varphi_2(|y|)$ be radially symmetric subsolutions to* (14). Assume that there exists $R > 0$ such that $\varphi_1(R) = \varphi_2(R)$ and $\varphi_1'(R) \leq \varphi_2'(R)$. Then, $\underline{\varphi}$ defined by

$$\underline{\phi}(r) := \begin{cases} \varphi_1(r), & r \in [0, R], \\ \varphi_2(r) & r \in [R, \infty) \end{cases}$$

is a continuous weak subsolution to (14).

(ii) Let $\varphi_1 = \varphi_1(|y|)$ and $\varphi_2 = \varphi_2(|y|)$ be radially symmetric supersolutions to (14). Assume that there exists $R > 0$ such that $\varphi_1(R) = \varphi_2(R)$ and $\varphi_1'(R) \geq \varphi_2'(R)$. Then $\overline{\phi}$ defined by

$$\overline{\phi}(r) := \begin{cases} \varphi_1(r), & r \in [0, R], \\ \varphi_2(r) & r \in [R, \infty) \end{cases}$$

is a continuous weak supersolution to (14).

Proof of Proposition 3. Let $\alpha_* = \varphi_L(0), \alpha^* = \varphi_L(0)$ and $\alpha_0 \in (\alpha_*, \alpha^*)$. By Lemma 8, there exists $r_0 > 0$ satisfying (69).

(i) Put $w_M(r) = \max_{|x|=r} w_0(x)$ for $r > 0$. Then we have $\varphi_L(r) > w_M(r)$ for $r > 0$. Setting $\varepsilon = \min_{0 \leq r \leq r_0} |\varphi_L(r) - w_M(r)|$. By continuous dependence of initial data, there exists $\delta > 0$ such that if $|\alpha - \alpha^*| < \delta$ then

$$|\varphi_L(r) - \varphi(r;\alpha)| < \varepsilon \quad \text{for } 0 \leq r \leq r_0. \tag{72}$$

Let $\alpha \in (\alpha^* - \delta, \alpha^*) \cap (\alpha_0, \alpha^*)$. By Lemma 8 (i), there exists $r_0 \in (0, r_1)$ such that (71). Then we have

$$w_M(r) \leq \varphi_L(r) - \varepsilon < \varphi(r;\alpha) < \varphi_L(r) \quad \text{for } 0 \leq r \leq r_1$$

and $\varphi(r_1;\alpha) = \varphi_L(r_1)$. Therefore we obtain $\varphi'(r_1;\alpha) \geq v_L'(r_1)$. Putting

$$\overline{w}_0(r) = \begin{cases} \varphi(r;\alpha), & 0 \leq r < r_1, \\ \varphi_L(r), & r \geq r_1. \end{cases}$$

Then $\overline{w}_0$ satisfies (62) and we have $\overline{w}_0$ is a continuous weak supersolution to (15) by Lemma 9 (ii).

(ii) Put $w_m(r) = \min_{|x|=r} w_0(x)$ for $r > 0$. Then we have $\varphi_L(r) < w_m(r)$ for $r > 0$. Setting $\varepsilon = \min_{0 \leq r \leq r_0} |\varphi_L(r) - w_m(r)|$. By the continuous dependence of initial data, there exists $\delta > 0$ such that if $|\alpha - \alpha^*| < \delta$ then

$$|\varphi_L(r) - \varphi(r;\alpha)| < \varepsilon \quad \text{for } 0 \leq r \leq r_0. \tag{73}$$

Put $\alpha \in (\alpha^*, \alpha^* + \delta)$. By Lemma 8 (ii), there exists $r_2 \in (0, r_0)$ satisfying (71). Then we have

$$\varphi_L(r) < \varphi(r;\alpha) < \varphi_L(r) + \varepsilon \leq w_m(r) \quad \text{for } 0 \leq r < r_2$$

and $\varphi(r_2;\alpha) = \varphi_L(r_2)$. Therefore we have $\varphi'(r_2;\alpha) \leq \varphi_L'(r_2)$. Put

$$\underline{w}_0(r) = \begin{cases} \varphi(r;\alpha), & 0 \leq r < r_2, \\ \varphi_L(r), & r \geq r_2. \end{cases}$$

Then $\underline{w}_0$ satisfies (63) and $\underline{w}_0'(r) \leq 0$ for $r \geq 0$. We obtain that $\underline{w}_0$ is a continuous weak subsolution to (15).

■

In order to prove Theorem 2, we use the self-similar variables. Let u be the solution to (1). Then we define w by the following:

$$w(y,s) := \log(1+t) + u(x,t), \quad y = \frac{x}{\sqrt{1+t}}, \quad s = \log(1+t). \tag{74}$$

Then, w satisfy

$$\begin{cases} w_s = \Delta w + \dfrac{1}{2} y \cdot \nabla w + e^w + 1 & \text{in } \mathbb{R}^N \times (0,\infty), \\ w(y,0) = w_0(y) & \text{on } \mathbb{R}^N \end{cases} \tag{75}$$

where $w_0 = u_0$.

We say that w is a continuous weak supersolution to (75) in $0 \le s \le S$ if w is a continuous on $\mathbb{R}^N \times [0, S]$, $w(y, 0) \ge w_0(y)$ $y \in \mathbb{R}^N$ and satisfies

$$\int_{\mathbb{R}^N} w(y,s)\xi(y,s)\,dy\Big|_{s=0}^{s=\sigma} \ge \int_0^\sigma \int_{\mathbb{R}^N} [w(y,s)(\xi_s + \Delta\xi)(y,s) + e^{w(y,s)}\xi(y,s)]\,dy\,ds \tag{76}$$

for all $\xi \in C^{2,1}(\mathbb{R}^N \times [0, S])$ with $\xi \ge 0$ such that $\operatorname{supp} \xi(\cdot, s)$ is compact in $\mathbb{R}^N$ for all $s \in [0, \sigma]$. A continuous weak subsolution is defined in the same way by reversing the inequalities.

Next we introduce comparison results of sub- and supersolutions..

Lemma 10 ([2] Lemma 2.3 (ii))**.** *Let $\overline{w}$ and $\underline{w}$ be weak supersolution and subsolution to* (75) in $\mathbb{R}^N \times [0, S]$, respectively. Assume that $\overline{w}$ and $\underline{w}$ are bounded above and satisfy $\overline{w}(x, s) - \underline{w}(x, s) \ge -Ae^{B|x|^2}$ in $\mathbb{R}^N \times [0, S]$ for some constants $A, B > 0$. Then $\underline{w} \le \overline{w}$ in $\mathbb{R}^N \times [0, S]$ and there exists the solution w to (75) such that $\underline{w} \le w \le \overline{w}$ in $\mathbb{R}^N \times [0, S]$.

We will prove the following proposition.

Proposition 4. *Let $3 \le N \le 9$. Assume that there exist at least two elements $\underline{\varphi}_L$ and φ_L of S_L, where $\underline{\varphi}_L$ is a minimal solution of S_L. Suppose that w_0 holds the assumption* (2).

(i) If $w_0(y) > \varphi_L(|y|)$ for $y \in \mathbb{R}^N$ then the solution w to (75) with initial data w_0 blows up in finite time.

(ii) If $w_0(y) < \varphi_L(|y|)$ for $y \in \mathbb{R}^N$ then the solution w to (75) with initial data w_0 exists globally in time.

To prove Proposition 4 we prepare the following Lemmas.

Lemma 11 ([2] Lemma 2.4.)**.** *(i) Let w_0 be continuous weak subsolution to* (14). Assume that the solution w to (75) with initial data w_0 exists globally in time. Then w is nondecreasing in s.

(ii) Let w_0 be continuous weak supersolution to (14). Assume that the solution w to (75) with initial data w_0 exists globally in time. Then w is nonincreasing in s.

Lemma 12 ([2] Lemma 2.7)**.** *Let the solution $w = w(|y|, s)$ to* (75) be a global solution and radially symmetric in y . Assume that $w(|y|, s)$ is nondecreasing function in s for each fixed $r \ge 0$ and nonincreasing function in $r = |y|$ for each fixed $s \ge 0$. Put $\varphi(r) := \lim_{s\to\infty} w(r, s)$.

(i) If φ is bounded above, then $\varphi \in C^2([0, \infty))$ is the solution to (15) satisfying $\varphi'(0) = 0$.

(ii) If φ is not bounded above. Then $\varphi \in C^2((0, \infty))$ is the solution to (15) satisfying $\lim_{r\to 0} \varphi(r) = \infty$.

Proof of Proposition 4. (i) The proof is carried out contradiction argument. Assume to contrary that w exists globally in time. By Proposition 3 (ii) there exists a continuous weak subsolution $\underline{w}_0$ such that $\underline{w}_0 = \underline{w}_0(r),\ r = |x|$, nonincreasing in r, $\underline{w}_0 \neq \varphi_L$ and (63). Let $\underline{w}$ be the solution to (75) with initial data $w_0 = \underline{w}_0$. From Lemma 10, $\underline{w} = \underline{w}(r,s),\ r = |y|$ is a radially symmetric and nonincreasing function in $r \geq 0$. We remark that $w_0 = u_0$ satisfies assumption (2). By the comparison principle, we have $\varphi_L < \underline{w} < w$ and $\underline{w}$ exists globally in time. From Lemma 11, $\underline{w}(r,s)$ is nonincreasing in s. Let $\underline{\varphi}(r) = \lim_{s\to\infty} \underline{w}(r,s)$ for $r \geq 0$. Since $\underline{w}(r,s)$ is nonincreasing in r, $\underline{\varphi}(r)$ is a nonincreasing function and satisfies

$$\varphi_L(r) < \underline{w}_0(r) \leq \underline{w}(r,s) \leq \underline{\varphi}(r), \quad r > 0,\ s \geq 0. \tag{77}$$

Assume that $\underline{\varphi}$ is bounded above. By Lemma 12 (i), we get $\underline{\varphi} \in C^2[0,\infty)$ to (15) satisfying $\underline{\varphi}'(0) = 0$. From (77), we have $\varphi_L(r) < \underline{\varphi}(r)$ for $r \geq 0$. This is a contradiction by Proposition 2 (ii). Therefore we conclude that $\underline{\varphi} \notin L^\infty[0,\infty)$. By Lemma 12 (ii), we have $\underline{\varphi} \in C^2(0,\infty)$ satisfying $\lim_{r\to 0} \underline{\varphi}(r) = \infty$. From (77), we obtain $\varphi_L(r) < \underline{\varphi}(r)$ for $0 < r < \infty$. This is a contradiction by Proposition 2 (iv). Therefore the solution w to (75) with initial data w_0 blows up in finite time.
(ii) Since φ_L is the stationary solution to (14) and w_0 satisfies assumption (2), we obtain

$$w(y,s) \leq \varphi_L(|y|), \quad y \in \mathbb{R}^N,\ s > 0$$

by the comparison principle. Therefore the solution w to (75) exists globally in time. ■

Proof of Theorem 2. (i) The proof is carried out by contradiction argument. Assume to contrary that u exists globally in time. By the comparison principle, $u(x,t) > u_L(x,t_0+t)\ x \in \mathbb{R}^N,\ t > 0$. Hence assume that $u_0(x) > u_L(x,t_0)$ for $x \in \mathbb{R}^N$. Then we have

$$\log t_0 + u_0(\sqrt{t_0}x) > \varphi_L(|x|), \quad x \in \mathbb{R}^N. \tag{78}$$

Let $w(x,t) = \log t_0 + u(\sqrt{t_0}x, t_0 t)$. Then w satisfies the following:

$$w_t = \Delta w + e^w \quad \text{in } \mathbb{R}^N \times (0,\infty), \quad w(x,0) = \log t_0 + u_0(\sqrt{t_0}x) \quad \text{in } \mathbb{R}^N. \tag{79}$$

Put

$$\hat{w}(y,s) = \log(t+1) + w(x,t), \quad y = \frac{x}{\sqrt{1+t}}, \quad s = \log(1+t).$$

Then $\hat{w}$ is a global solution to (75) satisfying $\hat{w}_0(y) = w(y,0)$ for $y \in \mathbb{R}^N$. From (78) and (79), we have $\hat{w}_0(y) > \varphi_L(|y|)$ for $y \in \mathbb{R}^N$. By proposition 4 (i), The solution $\hat{w}$ to (75) blows up in finite time. This is contradiction. Therefore u blows up in finite time.
(ii) Assume that $u_0(x) < u_L(x,t_0)$ for $y \in \mathbb{R}^N$ by the comparison principle. Then we have

$$\log t_0 + u_0(\sqrt{t_0}x) < \varphi_L(|x|) \quad \text{for } x \in \mathbb{R}^N. \tag{80}$$

Put $w(x,t) = \log t_0 + u(\sqrt{t_0}x, t_0 t)$. Then w satisfies (79). Let

$$\hat{w}(y,s) = \log(t+1) + w(x,t), \quad y = \frac{x}{\sqrt{1+t}}, \quad s = \log(1+t).$$

Then $\hat{w}$ is the solution to (75) satisfying $\hat{w}_0(y) = w(y,0)$ for $y \in \mathbb{R}^N$. From (80), we obtain $w_0(y) < \varphi_L(|y|)$ for $y \in \mathbb{R}^N$. By proposition 4 (ii), $\hat{w}$ exists globally. Therefore w exists globally. ■

Acknowledgment

The author would like to express his gratitude to professor Mitsuru Sugimoto for valuable advice and continuous encouragement. He would like to offer my special thanks to Professor Yuki Naito for his helpful comment. He has had the support and continuous encouragement. The author would like to thank Sugimoto's office members for useful discussions. The author also thank Mr. Soichiro Suzuki in Sugimoto's office members for supporting my research activities.

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