A weak limit theorem for anisotropic quantum walks on lattices

(格子上の異方的量子ウォークの弱収束定理)

Ippei Mimura

Graduate School of Mathematics
Nagoya University

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Introduction

Quantum walks and weak limit theorems

A quantum walk is classified into two types, of discrete-time and continuous-time. In this thesis we shall focus on the former one. Although several models of discrete-time quantum walks have been proposed, we will deal only with position-dependent quantum walks as in [28]. For a while, such a class of quantum walk is simply called a quantum walk. General references for quantum walks are [19, 20, 21, 28, 37]. Konno [22], Suzuki [36] are also excellent concise review articles, in particular on the weak limit theorem.

A quantum walk can be considered as a quantization of a random walk. To understand this, let us consider a random walk on the one-dimensional lattice $\mathbb{Z}$. Suppose that a random walker moves to the right and the left with probability q and p, respectively, where p, q are nonnegative real numbers with $p+q=1$. Let X_t denote a random variable which describes the position of the random walker at time $t \in \mathbb{N} \cup \{0\}$. Then the probability $\mathbb{P}(X_t = n)$ that the random walker exists at a point $n \in \mathbb{Z}$ at time t satisfies the following relation:

$$\mathbb{P}(X_{t+1} = n) = p\mathbb{P}(X_t = n+1) + q\mathbb{P}(X_t = n-1). \tag{0.1}$$

On the other hand, the probability that a quantum walker exists at a position $n \in \mathbb{Z}$ at time t is described in terms of a state. In order to consider a quantum walk on $\mathbb{Z}$, we first choose the Hilbert space $\mathcal{H} = \ell_2(\mathbb{Z}; \mathbb{C}^2)$. Then a state of a quantum walker is given by a normalized vector of $\mathcal{H}$. Let $\Psi^0 \in \mathcal{H}$ be an initial state ($\|\Psi^0\| = 1$) and $\Psi_t \in \mathcal{H}$ denote the state of the quantum walker at time t, then the time evolution of Ψ^t is defined by

$$\Psi^{t+1}(n) = P\Psi^t(n+1) + Q\Psi^t(n-1), \quad n \in \mathbb{Z}, \tag{0.2}$$

where P, Q are 2×2-matrices such that $P+Q$ is a unitary matrix. Besides, the probability that the quantum walker exists at $n \in \mathbb{Z}$ at time t is defined by

$$\mathbb{P}(X_t = n) = \|\Psi^t(n)\|^2_{\mathbb{C}^2}.$$

Thus, the time evolution of a quantum walker (0.2) is interpreted as noncommutative version of (0.1).

Quantum walks defined in this way have several distinct properties compared to random walks. One of them is the distribution of X_t. In the case of the best-known random walk with $p = q = 1/2$ starting at the origin, the probability distribution has the highest peak at the start point $n = 0$ (Fig. 0.1). Whereas, for example, in the case of the

Hadamard walk with

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},$$

$$\Psi^0(n) = \begin{cases} {}^t(1/\sqrt{2}, i/\sqrt{2}) & \text{if } n = 0 \\ {}^t(0,0) & \text{otherwise} \end{cases},$$

the probability distribution of the quantum walker is low around the origin and increases as it moves left or right (Fig. 0.1).

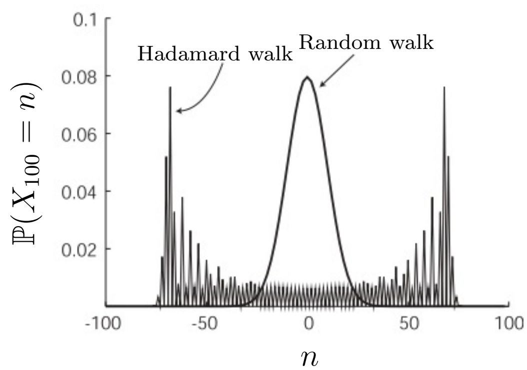


Figure 0.1: The distribution of X_{100}. Adapted from [22, p. 74].

Another distinct property of quantum walks is the order of t in the weak limit theorem. In the case of the random walk above, due to the central limit theorem, the distribution of the random variable $X_t/\sqrt{t}$ weakly converges to the Gaussian distribution. This fact is well-known as the de Moivre-Laplace theorem. On the other hand, as was discovered by Konno [17] in 2002 for the first time, the weak limit theorem for quantum walks behaves in a different way, namely, the distribution of the random variable X_t/t, not $X_t/\sqrt{t}$, weakly converges. We shall describe the theorem in detail now. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a unitary matrix. Set

$$P := \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix},$$

and consider the time evolution (0.2). Then the result is stated as follows.

Theorem. (Konno [17, 18])
Suppose that the initial state starts at the origin with probability 1 (i.e. $\|\Psi^0(0)\|^2_{\mathbb{C}^2} = 1$). If $abcd \neq 0$, then the distribution of the random variable X_t/t weakly converges to the distribution whose density function is given by

$$\frac{\sqrt{1-|a|^2}(1-\gamma x)}{\pi(1-x^2)\sqrt{|a|^2 - x^2}} \chi_{(-|a|,|a|)}(x), \tag{0.3}$$

where $\gamma = \gamma(a, b, \Psi^0)$ is a constant.

Konno proved this theorem by a combinational method. Afterwards, Grimmet, Janson, and Scudo [14] provided another proof of this theorem using the Fourier transform, and succeeded in removing the assumption of initial states. It is called the GJS method and turns out to be quite useful to determine the weak limit of the distributions.

In the case of quantum walks, since the distribution of not $X_t/\sqrt{t}$ but X_t/t is weakly convergent, the weak limit measure can be interpreted as the "asymptotic velocity" distribution of a quantum walker. Also, the function (0.3) has asymptotes $x = \pm|a|$, so the shape is quite different from the Gaussian function. For these reasons, quantum walks and their weak limit theorems have been attracting the attention of many researchers.

Let us give a more general definition of quantum walks. These quantum walks are described by time evolution operators on the state space $\mathcal{H} = \ell_2(\mathbb{Z}; \mathbb{C}^2)$. We first define two unitary operators S and C on $\mathcal{H}$ as follows. Write an element of $\mathcal{H}$ as $\Psi = {}^t(\Psi_1, \Psi_2)$. Then S is defined by the following formula:

$$(S\Psi)(n) = \begin{pmatrix} \Psi_1(n+1) \\ \Psi_2(n-1) \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{0.4}$$

which is called the *shift operator*. We define the second operator by assigning a 2×2 unitary matrix to each $n \in \mathbb{Z}$. That is, we choose a unitary-matrix-valued function $C_\bullet : \mathbb{Z} \to U(2); n \mapsto C_n$, which is called the *coin map* in this thesis. Then C is defined by

$$(C\Psi)(n) = C_n\Psi(n), \quad n \in \mathbb{Z},$$

and is called the *coin operator*. Also the *time evolution operator* is defined by $U = SC$. Given an initial state $\Psi^0 \in \mathcal{H}$, the state of the quantum walker at time $t \in \mathbb{N} \cup \{0\}$ is defined by $U^t\Psi^0$. The probability of quantum walker at $n \in \mathbb{Z}$ and at time t is defined by

$$\mathbb{P}(X_t = n) = \|U^t\Psi^0(n)\|^2_{\mathbb{C}^2}.$$

Let us consider the case where a given coin map $C_\bullet$ is constant. That is, $C_\bullet \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we divide this matrix as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} =: P + Q,$$

then

$$\Psi^t = U^t\Psi^0$$

holds by the relation (0.2). Therefore, the model of quantum walks which are described by a time evolution operator associated with a coin map is a generalization of quantum walks treated by Konno. (Actually, the model above is a generalization of quantum walks described by (0.2), see [27].) Of course, since the value of a given coin map depends on $n \in \mathbb{Z}$, the quantum walk above is called a position-dependent coined quantum walk. As mentioned at the beginning of this introduction, we will deal only with such quantum walks (on lattices) in this thesis.

It is difficult to study the weak limit theorem in general when the coin map is not constant. Therefore various models have been studied individually. For example, Konno, Łuczak, and Segawa proved the weak limit theorem for quantum walks with one-defect, where the coin map is constant on $\mathbb{Z}$ except for the origin in [23].

A quantum walk is called of two-phase if the coin map takes two different values on $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{<0}$. Two weak limit theorem are proved in [10] and [11] for two-phase quantum walks with and without one-defect as well.

Suzuki [35] proved the following theorem:

Theorem. (Suzuki [35])
Let $C_\bullet : \mathbb{Z} \to U(2)$ be a coin map and $\Psi^0 \in \mathcal{H}$ an initial state. If there exists $C_\infty \in U(2)$ and $\varepsilon > 0$ such that

$$C_n = C_\infty + O(|n|^{-1-\varepsilon}) \quad \text{as} \quad |n| \to \infty, \tag{0.5}$$

then the distribution of X_t/t weakly converges to a probability measure

$$\mu = \|\Pi_{\mathrm{pp}}(U)\Psi^0\|^2\delta_0 + \|E_V(\cdot)\Omega_+^*\Psi^0\|^2.$$

Here $\Pi_{\mathrm{pp}}(U)$ denotes the projection onto the pure point subspace of U and δ_0 the Dirac measure at the origin. In addition, V denotes the asymptotic velocity operator, $E_V(\cdot)$ the spectral measure of V and Ω_+^* the adjoint of the wave operator Ω_+.

The condition (0.5) means that C_n converges to C_∞ in an order faster than $|n|^{-1}$. Of course, it includes one-defect models as special cases. Since this condition corresponds to the short-range condition of the potential of the Schrödinger operators, the same term is used in quantum walks. Suzuki used the GJS method and the spectral scattering theory in the proof of this theorem.

In contrast, Wada [38] gave an example that there are no wave operators when the order is slower than $|n|^{-1}$ (which is said to be long-range type), and elucidated that the borderline between existence and non-existence of wave operators is -1. However, even in the long-range type case, he also showed that the modified wave operators can be constructed for a specific coin map, and succeeded in proving a weak limit theorem in [39].

Richard, Suzuki, and Tiedra de Aldecoa [29, 30] extended Suzuki's result from a different view point: Given a coin map $C_\bullet$, if

$$C_n = \begin{cases} C_\infty + O(|n|^{-1-\varepsilon}) & \text{as} \quad n \to \infty, \\ C_{-\infty} + O(|n|^{-1-\varepsilon}) & \text{as} \quad n \to -\infty, \end{cases}$$

for some $C_{\pm\infty} \in U(2)$ and $\varepsilon > 0$, then a weak limit theorem also holds. Note that all of one-defect models, two-phase models, and two-phase models with one defect satisfy this condition. Also, Richard *et al.* called such a coin map which converges to different matrices as n goes to ∞ and $-\infty$ respectively, anisotropic. Therefore, to summarize the results of Suzuki, Wada, and Richard *et al.* above, if a given coin map is isotropic short-range, isotropic long-range (a special case), and anisotropic short-range, then a weak limit theorem holds, respectively.

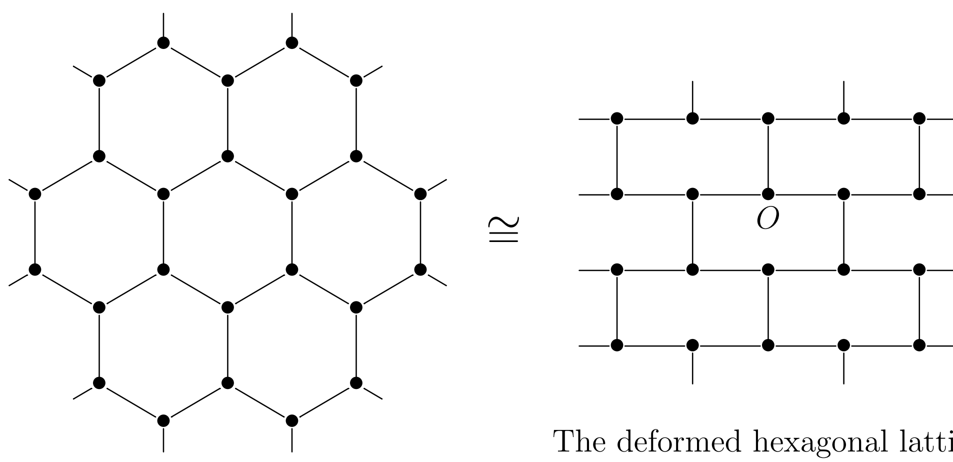
In general, a quantum walk can be defined on every connected graph. The weak limit theorem has been studied for quantum walks on several other types of graphs, for example, higher-dimensional lattices [13, 14, 26, 34, 40], crystal lattices [15], the half-line [25], jointed half lines [7], and trees [6].

Motivation and main results

One of the aims in this thesis is to provide a two-dimensional generalization of the weak limit theorem proved by Suzuki [35]. We shall focus on not only the square lattice but also the hexagonal and triangular lattices. One of the reasons why we work on those is that squares, equilateral triangles and regular hexagons are the polygons that can fill the plane.

Let us consider the coin map in Suzuki [35] and Richard *et al.* [29, 30] again. Let $\mathbb{Z} \cup \{\infty\}$ denote the one-point compactification of $\mathbb{Z}$. Then a coin map $C_\bullet : \mathbb{Z} \to U(2)$ is isotropic if and only if it can be continuously extended to $\mathbb{Z} \cup \{\infty\}$, namely, it is continuous on $\mathbb{Z} \cup \{\infty\}$. Similarly, we denote by $\mathbb{Z} \cup \{\pm\infty\}$ the usual two-point compactification of $\mathbb{Z}$, then Richard *et al.*'s anisotropicness of a coin map is equivalent to its continuity on $\mathbb{Z} \cup \{\pm\infty\}$. In this way, an anisotropic coin map can be rephrased in terms of a compactification of its domain. From this point of view, we may say that other examples of anisotropic coin maps on hexagonal, square, and triangular lattices are provided in this thesis.

Our first study is on the weak limit theorem for quantum walks associated with anisotropic short-range coin maps on the lattices.



The hexagonal lattice

The deformed hexagonal lattice with vertex set $\mathbb{Z}^2$

Figure 0.2: The hexagonal lattice graph and its deformation

Ando studied the discrete Schrödinger operator with finite support potentials on the hexagonal lattice in [2]. A discrete Schrödinger operator can be defined on any connected graph. By a graph theoretic deforming, Ando regarded the hexagonal lattice as a subgraph of the square lattice $\mathbb{Z}^2$ (Fig. 0.2), and thereby represented the Schrödinger operator as an operator on $\ell_2(\mathbb{Z}^2)$. However, it was necessary to divide the expression of the Laplacian into two cases. It is due to the fact that the hexagonal lattice has vertices with two distinct properties.

Let $\mathbb{Z}^2_e$ (resp. $\mathbb{Z}^2_o$) denote the set of points of $\mathbb{Z}^2$ such that the sum of the two coordinates is even (resp. odd). Ando avoided the complexity of the Laplacian by dividing the vertex set of the deformed hexagonal lattice into $\mathbb{Z}^2 = \mathbb{Z}^2_e \cup \mathbb{Z}^2_o$ and making the Hilbert space larger as $\ell_2(\mathbb{Z}^2) = \ell_2(\mathbb{Z}^2_e) \oplus \ell_2(\mathbb{Z}^2_o) \cong \ell_2(\mathbb{Z}^2) \oplus \ell_2(\mathbb{Z}^2)$.

We apply Ando's method above to the time evolution operators derived from hexagonal, square, and triangular lattices. Let Γ be a graph which is one of the hexagonal, square, and triangular lattices, d the degree of Γ. The vertex set of Γ can be regarded as $\mathbb{Z}^2$ by deforming the graph. Thus when we consider a quantum walk on Γ, the domain of each coin map is $\mathbb{Z}^2$ (i.e. $C_\bullet : \mathbb{Z}^2 \to U(d); (n,m) \mapsto C_{(n,m)}$) and the shift operator can be explicitly written down just like (0.4). However, as with the Laplacian above, it is necessary to divide the expression of the shift operator into two cases only if Γ is the hexagonal lattice. We now use Ando's method, then the complexity of the shift operator is removed. The method not only avoids the complexity but also makes it possible to deal with certain anisotropic coin maps in the weak limit theorem. Hence, although the complexity of the shift operator does not occur in the square and triangular lattices cases, we apply Ando's method to even those cases.

Let $\mathbb{Z}^2 \cup \{\infty_e, \infty_o\}$ denote the disjoint union of the one-point compactifications of $\mathbb{Z}^2_e$ and $\mathbb{Z}^2_o$, where ∞_e and ∞_o are points at infinity of $\mathbb{Z}^2_e$ and $\mathbb{Z}^2_o$, respectively. A coin map $C_\bullet : \mathbb{Z}^2 \to U(d)$ is said to be *anisotropic* (in this thesis) if it can be continuously extended to $\mathbb{Z}^2 \cup \{\infty_e, \infty_o\}$.

The following theorem is one of the main results in the thesis.

Theorem. (Theorem 3.4.3)
Let $C_\bullet : \mathbb{Z}^2 \to U(d)$ be a coin map and $\Psi^0 \in \mathcal{H}_\Gamma = \ell_2(\mathbb{Z}^2; \mathbb{C}^d)$ an initial state. Let X_t denote a random variable which describes the position of a quantum walker at time $t \in \mathbb{N} \cup \{0\}$. Suppose that there exist $C_e, C_o \in U(d)$ and $\varepsilon > 0$ such that

$$C_{(n,m)} = \begin{cases} C_e + O(\|(n,m)\|_1^{-2-\varepsilon}) & \text{as} \quad \mathbb{Z}^2_e \ni (n,m) \to \infty_e, \\ C_o + O(\|(n,m)\|_1^{-2-\varepsilon}) & \text{as} \quad \mathbb{Z}^2_o \ni (n,m) \to \infty_o. \end{cases} \tag{0.6}$$

In addition, we suppose that the matrix $C_e \oplus C_o \in U(2d)$ satisfies Assumption 3.2.3. Then, the distribution of X_t/t weakly converges to a probability measure

$$\mu = \|\Pi_{\mathrm{pp}}(U)\Psi^0\|^2_{\mathcal{H}_\Gamma}\delta_{(0,0)} + \|(E_{V_1^\Gamma} \otimes E_{V_2^\Gamma})(\cdot)(\Omega_+^\Gamma)^* J_\Gamma \Psi^0\|^2_{\mathcal{H}_\Gamma \oplus \mathcal{H}_\Gamma},$$

where $\delta_{(0,0)}$ is the Dirac measure at the origin, V_1^Γ, V_2^Γ and Ω_+^Γ are the asymptotic velocity operators and the wave operator on $\mathcal{H}_\Gamma \oplus \mathcal{H}_\Gamma$, respectively. Additionally, J_Γ denotes a unitary operator from $\mathcal{H}_\Gamma$ to $\mathcal{H}_\Gamma \oplus \mathcal{H}_\Gamma$. Also, for each $\alpha > 1$, the distribution of X_t/t^α weakly converges to $\delta_{(0,0)}$.

The condition (0.6) corresponds to the property that the coin map is anisotropic and short range, and Assumption 3.2.3 is an assumption about the eigenvalues and eigenvectors of a certain unitary matrix with respect to C_e and C_o.

The second study is on the essential spectrum of the time evolution operator which describes a quantum walker. Since the essential spectrum is invariant under a compact perturbation, it is an inherent numerical of an operator on an infinite-dimensional space. In [29], the authors investigated the essential spectrum of the time evolution operator associated with an anisotropic coin map on $\mathbb{Z}$. The following theorem is a lattice version of it. A coin map dealt with here is assumed to be anisotropic, but is not necessarily of short range.

Theorem. (Theorem 4.2.2)
Suppose that a given coin map $C_\bullet : \mathbb{Z}^2 \to U(d)$ is anisotropic, that is, there exist $C_e, C_o \in U(d)$ such that

$$\begin{aligned} \|C_{(n,m)} - C_e\|_{M_d(\mathbb{C})} \to 0 \quad &\text{as} \quad \mathbb{Z}_e^2 \ni (n,m) \to \infty_e, \\ \|C_{(n,m)} - C_o\|_{M_d(\mathbb{C})} \to 0 \quad &\text{as} \quad \mathbb{Z}_o^2 \ni (n,m) \to \infty_o, \end{aligned}$$

where $\|\cdot\|_{M_d(\mathbb{C})}$ denotes the C^*-norm on the matrix algebra $M_d(\mathbb{C})$. Then the essential spectrum of the time evolution operator associated with $C_\bullet$ is given by the following formula:

$$\sigma_{\mathrm{ess}}(U) = \bigcup_{(\theta,\varphi)\in[0,2\pi)^2} \sigma(\widehat{U}_\infty(\theta,\varphi)),$$

where $\widehat{U}_\infty(\theta,\varphi)$ is a $2d \times 2d$ unitary matrix derived from C_e and C_o.

Organization

This thesis is organized as follows: In Chapter 1, we recall some facts of operator theory and discrete crossed products for later chapters. In Section 1.3, we provide the two-point compactification of $\mathbb{Z}^2$.

The purpose of Chapter 2 is to determine the shift, coin, and time evolution operators which describe quantum walks on hexagonal, square, and triangular lattices. In Section 2.1, for understanding of the model, we first consider the hexagonal lattice case, which is the most important one. In Section 2.2, we define the three operators derived from a regular graph. In Section 2.3, in particular, we determine the three operators derived from hexagonal, square, and triangular lattices. In Section 2.4, by using Ando's method, we modify the unitary operators of Section 2.3. The definition of anisotropic is given in this section.

The weak limit theorem is proved in Chapter 3. The proof is based on an argument using the GJS method and spectral scattering theory inspired from [35]. In Section 3.1, we represent the characteristic function of X_t/t's distribution using the 2-variable functional calculus. In Section 3.2, we construct the asymptotic velocity operators by the GJS method. Assumption 3.2.3 is introduced in this section. In Section 3.3, we define the short-range condition and construct the wave operators. In Section 3.4, we prove a weak limit theorem for quantum walks with an anisotropic short-range coin map (Theorem 3.4.3). As a special case, we provide a weak limit theorem with respect to "quasi-uniform" coin maps (Corollary 3.4.4). An application of the weak limit theorem is the concept of localization. We also give a necessary condition that localization occurs (Corollary 3.4.5).

In Chapter 4, we investigate the essential spectrum of the time evolution operator associated with an anisotropic coin map by using the crossed product C^*-algebras (Theorem 4.2.2). This proof is affected by the one of [29, Theorem 2.2].

In Chapter 5, we consider only quantum walks on square and triangular lattices. Since there is the complexity of the shift operator only in the hexagonal lattice case, we can discuss without using Ando's method in square and triangle lattices cases. However, then, anisotropic coin maps cannot be treated. Therefore, we provide isotropic versions of Theorems 3.4.3 and 4.2.2 (Theorems 5.1.3 and 5.2.1).

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Chapter 1

Preliminaries

In this chapter, we briefly recall some notations and facts from functional analysis for later chapters.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ which is linear in the second argument. For every $\Psi \in \mathcal{H}$, let $\langle\Psi|$ and $|\Psi\rangle$ denote the bra and ket vectors, respectively. The set of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathbb{B}(\mathcal{H})$. If $\mathcal{H} = \mathbb{C}^n$, then the matrix algebra $M_n(\mathbb{C})$ can be identified with $\mathbb{B}(\mathbb{C}^n)$ as C^*-algebras. Under this identification, the following equality holds:

$$|u\rangle\langle v| = \begin{pmatrix} u_1\overline{v}_1 & \cdots & u_1\overline{v}_n \\ \vdots & & \vdots \\ u_n\overline{v}_1 & \cdots & u_n\overline{v}_n \end{pmatrix}$$

for any $u = {}^t(u_1,\cdots,u_n), v = {}^t(v_1,\cdots,v_n) \in \mathbb{C}^n$. Also we denote by E_n the identity matrix in $M_n(\mathbb{C})$.

Let X be a set and $\mathcal{H}$ a Hilbert space. The Hilbert space of all square summable $\mathcal{H}$-valued functions on X is denoted by $\ell_2(X;\mathcal{H})$ with norm

$$\|\Psi\| := \left(\sum_{x\in X} \|\Psi(x)\|_{\mathcal{H}}^2\right)^{1/2}, \quad \Psi \in \ell_2(X;\mathcal{H}).$$

When $\mathcal{H} = \mathbb{C}$, we simply denote $\ell_2(X)$ instead of $\ell_2(X;\mathbb{C})$. The following isomorphisms are often used in this thesis:

$$\ell_2(X;\mathcal{H}) \cong \ell_2(X)\otimes\mathcal{H} \cong \bigoplus_{x\in X}\mathcal{H},$$
$$\ell_2(X;\mathcal{H}\oplus\mathcal{K}) \cong \ell_2(X;\mathcal{H})\oplus\ell_2(X;\mathcal{K}).$$

Let X be a locally compact space and $\mathcal{A}$ a C^*-algebra. We denote by $C_0(X;\mathcal{A})$ the set of $\mathcal{A}$-valued continuous functions on X vanishing at infinity and equipped with the supremum norm. When X is compact, one denotes $C(X;\mathcal{A})$ instead of $C_0(X;\mathcal{A})$ and coincides with the set of all continuous functions from X to $\mathcal{A}$. As easily seen, if $\mathcal{A}$ is unital, then for all $f \in C(X;\mathcal{A})$, its spectrum is given by

$$\sigma(f) = \bigcup_{x\in X} \sigma(f(x)).$$

Let μ and $\mu_1, \mu_2, \ldots$ be probability measures on the Borel measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. A sequence $\{\mu_t\}_{t\in\mathbb{N}}$ is said to be *weakly convergent* to μ if

$$\int_{\mathbb{R}^n} f d\mu_t \to \int_{\mathbb{R}^n} f d\mu \quad \text{as} \quad t \to \infty$$

for any $f \in C_0(\mathbb{R}^n)$. Let $M(\mathbb{R}^n)$ be the measure algebra on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, the set of all regular Borel complex measures. Since it holds that $M(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)^*$ Banach spaces, the weak convergence above means the weak one in the Banach space $M(\mathbb{R}^n)$. Given a probability measure μ on $\mathbb{R}^n$, the *characteristic function* of μ is defined by

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle_n} d\mu(x), \quad \xi \in \mathbb{R}^n,$$

where $\langle \xi, x\rangle_n := \xi_1 x_1 + \cdots + \xi_n x_n$, the standard inner product on $\mathbb{R}^n$. For probability measures $\mu, \mu_1, \mu_2, \ldots$, it is well known that $\{\mu_t\}_{t\in\mathbb{N}}$ is weakly convergent to μ if and only if $\widehat{\mu}_t(\xi) \to \widehat{\mu}(\xi)$ for each $\xi \in \mathbb{R}^n$.

1.1 Operator theory

We will provide some topics from operator theory.

1.1.1 Multi-variable functional calculus

The facts of this subsection is based on [3].

Let $\mathcal{H}$ be a Hilbert space. The domain of a linear operator A on $\mathcal{H}$ is denoted by $\mathcal{D}(A)$ and the graph of A is defined by

$$\mathcal{G}(A) := \{{}^t(\Psi, A\Psi) \in \mathcal{H} \oplus \mathcal{H}(\text{a column vector}) \mid \Psi \in \mathcal{D}(A)\}.$$

When A is a self-adjoint operator, there exists a unique spectral measure $E_A(\cdot)$ on $\mathbb{R}$ such that $A = \int_{\mathbb{R}} \lambda dE_A(\lambda)$. Let $A_1, \ldots, A_n$ be n self-adjoint operators on $\mathcal{H}$ and $E_{A_1}(\cdot), \ldots, E_{A_n}(\cdot)$ be their spectral measure, respectively. Then $A_1, \ldots, A_n$ are said to be *strongly commuting* if the projections $E_{A_1}(B_1), \ldots, E_{A_n}(B_n)$ are mutually commuting for any $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$. If $A_1, \ldots, A_n$ are strongly commuting then there exists a unique spectral measure $(E_{A_1} \otimes \cdots \otimes E_{A_n})(\cdot)$ on $\mathbb{R}^n$ such that

$$(E_{A_1} \otimes \cdots \otimes E_{A_n})(B_1 \times \cdots \times B_n) = E_{A_1}(B_1) \cdots E_{A_n}(B_n)$$

for each Borel sets $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$. For each bounded Borel function f on $\mathbb{R}^n$,

$$f(A_1, \dots, A_n) := \int_{\mathbb{R}^n} f(\lambda) d(E_{A_1} \otimes \cdots \otimes E_{A_n})(\lambda)$$

defines a bounded linear operator on $\mathcal{H}$. The operation $f \mapsto f(A_1, \dots, A_n)$ is called the *n-variable functional calculus*. Note that the equality

$$\langle \Psi, f(A_1, \dots, A_n)\Phi \rangle = \int_{\mathbb{R}^n} f(\lambda) \langle \Psi, d(E_{A_1} \otimes \cdots \otimes E_{A_n})(\lambda)\Phi \rangle$$

holds for any $\Psi, \Phi \in \mathcal{H}$.

Proposition 1.1.1. *Let A and B be strongly commuting self-adjoint operators on a Hilbert space. For all $\zeta, \eta \in \mathbb{R}$, then the following formula holds:*

$$\exp i(\zeta A + \eta B) = \exp(i\zeta A) \exp(i\eta B) = \exp(i\eta B) \exp(i\zeta A),$$

where $\exp i(\zeta A + \eta B)$ means the 2-variable functional calculus.

Proof. By applying [3, Theorem 1.16 (viii)] to $f(s,t) = e^{i\zeta s}$ and $g(s,t) = e^{i\eta t}$. $\square$

1.1.2 Spectral theory for unitary operators

For a unitary operator U on a Hirbert space $\mathcal{H}$, there exists a unique spectral measure $E_U(\cdot)$ on $\mathbb{R}$ such that $U = \int_0^{2\pi} e^{i\theta} dE_U(\theta)$. Note that for each $\Psi \in \mathcal{H}$, $\langle \Psi, E_U(\cdot)\Psi \rangle = \|E_U(\cdot)\Psi\|^2$ defines a finite regular measure on $\mathbb{R}$. Let us define the following subspaces of $\mathcal{H}$:

$$\begin{aligned}
\mathcal{H}_{\mathrm{pp}}(U) &:= \{\Psi \in \mathcal{H} \mid \|E_U(\cdot)\Psi\|^2 \text{ is a pure point measure}\}, \\
\mathcal{H}_{\mathrm{ac}}(U) &:= \{\Psi \in \mathcal{H} \mid \|E_U(\cdot)\Psi\|^2 \text{ is an absolutely continuous measure}\}, \\
\mathcal{H}_{\mathrm{sc}}(U) &:= \{\Psi \in \mathcal{H} \mid \|E_U(\cdot)\Psi\|^2 \text{ is a singular continuous measure}\}.
\end{aligned}$$

They are closed subspaces, and by Lebesgue decomposition, we have

$$\mathcal{H} = \mathcal{H}_{\mathrm{pp}}(U) \oplus \mathcal{H}_{\mathrm{ac}}(U) \oplus \mathcal{H}_{\mathrm{sc}}(U).$$

Also,

$$\mathcal{H}_{\mathrm{pp}}(U) = \overline{\mathrm{Span}}\{\Psi \in \mathcal{H} \mid \Psi \text{ is an eigenvector of } U\}$$

holds, where $\overline{\mathrm{Span}}\{\cdots\}$ means the norm-closed linear span of $\{\cdots\}$. We respectively denote $\Pi_{\mathrm{pp}}(U), \Pi_{\mathrm{ac}}(U)$ and $\Pi_{\mathrm{sc}}(U)$ the orthogonal projections on $\mathcal{H}_{\mathrm{pp}}(U), \mathcal{H}_{\mathrm{ac}}(U)$ and $\mathcal{H}_{\mathrm{sc}}(U)$. Then U commutes with $\Pi_\natural(U)$, namely,

$$[U, \Pi_\natural(U)] = U\Pi_\natural(U) - \Pi_\natural(U)U = 0$$

for $\natural =$ pp, ac, sc. Note that if one sets a self-adjoint operator $A := \int_0^{2\pi} \lambda dE_U(\lambda)$, then the usual closed subspace $\mathcal{H}_\natural(A)$ coincides with $\mathcal{H}_\natural(U)$ for $\natural =$pp, ac, sc. Thus the facts above can be proved in the framework of self-adjoint operators (see e.g. [31]).

1.1.3 Essential spectrum

Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$. The *essential spectrum* of A is defined by

$$\sigma_{\mathrm{ess}}(A) := \sigma(A) \setminus \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated point of } \sigma(A) \text{ with } \dim \operatorname{Ker}(A-\lambda) < \infty\}.$$

Let $\mathbb{K}(\mathcal{H})$ denote the set of all compact operators on $\mathcal{H}$, which is called the *compact algebra*. Set $\mathcal{Q}(\mathcal{H}) := \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$, called the *Calkin algebra*, and denote π the natural projection from $\mathbb{B}(\mathcal{H})$ to $\mathcal{Q}(\mathcal{H})$. The essential spectrum is invariant under compact perturbations, that is, for any $A \in \mathbb{B}(\mathcal{H})$ and $K \in \mathbb{K}(\mathcal{H})$, one has

$$\sigma_{\mathrm{ess}}(A+K) = \sigma_{\mathrm{ess}}(A).$$

Also, thanks to Atkinson's theorem,

$$\sigma_{\mathrm{ess}}(A) = \sigma(\pi(A))$$

holds for all normal operator A. As for the proofs, see e.g. [8].

1.2 Discrete crossed product C^*-algebras

This subsection is based on [12], but a general reference that includes the subject of this subsection is, for example, [5].

Let $\mathcal{A}$ be a C^*-algebra and G a discrete group. We denote by $\operatorname{Aut}(\mathcal{A})$ the set of all $*$-automorphisms of $\mathcal{A}$. A map $\alpha : G \to \operatorname{Aut}(\mathcal{A})$ is called an *action* of G on $\mathcal{A}$ if it is a group homomorphism. We will denote by α_g instead of $\alpha(g)$ for $g \in G$. An action α is said to be *trivial* if $\alpha_g = \mathrm{id}_{\mathcal{A}}$ for all $g \in G$. A C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is said to be α*-invariant* if $\alpha_g(\mathcal{B}) \subset \mathcal{B}$ for every $g \in G$, and then one can consider a map $\alpha : G \to \operatorname{Aut}(\mathcal{B})$, which is called the *restriction* of α to $\mathcal{B}$.

Let α be an action of G on $\mathcal{A}$. One defines $\mathcal{A} \rtimes_\alpha^{\mathrm{alg}} G$ the set of all finitely supported $\mathcal{A}$-valued functions on G, and every element of this set can be expressed as follows:

$$\sum_{g\in G} a_g \delta_g,$$

where $a_g \in \mathcal{A}$ and $a_g = 0$ except for finitely many $g \in G$. This becomes a linear space. Let $X = \sum_{g\in G} a_g\delta_g, Y = \sum_{h\in G} b_h\delta_h$ be two elements in $\mathcal{A} \rtimes_\alpha^{\mathrm{alg}} G$, one then defines a multiplication and an involution as follows:

$$XY := \sum_{g,h\in G} a_g \alpha_g(b_h)\delta_{gh}, \quad X^* := \sum_{g\in G} \alpha_{g^{-1}}(a_g^*)\delta_{g^{-1}}.$$

Equipped with a multiplication and an involution, $\mathcal{A} \rtimes_\alpha^{\mathrm{alg}} G$ becomes a $*$-algebra and is called the *algebraic crossed product*. For an element X of $\mathcal{A} \rtimes_\alpha^{\mathrm{alg}} G$, we set

$$\|X\|_{\max} := \sup\{p(X) \mid p \text{ is a } C^*\text{-seminorm on } \mathcal{A} \rtimes_\alpha^{\mathrm{alg}} G\},$$

where C^*-*seminorm* p on a $*$-algebra $\mathcal{C}$ is a seminorm on $\mathcal{C}$ satisfying $p(XY) \leq p(X)p(Y)$ and $p(X^*) = p(X)$ for all $X, Y \in \mathcal{C}$. Then $\|\cdot\|_{\max}$ becomes a norm on $\mathcal{A} \rtimes_\alpha^{\mathrm{alg}} G$. The ($C^*$-*algebraic full*) *crossed product* $\mathcal{A} \rtimes_\alpha G$ is defined by the completion of $\mathcal{A} \rtimes_\alpha^{\mathrm{alg}} G$ relative to the norm $\|\cdot\|_{\max}$. We use $\mathcal{A} \rtimes_0 G$ instead of $\mathcal{A} \rtimes_\alpha G$ when α is trivial.

Theorem 1.2.1. ([12, Lemma 17.8, Proposition 17.13, and Theorem 20.7])
Let α be an action of an amenable group G on a C^*-algebra $\mathcal{A}$. Then for each $g \in G$, there exists a contractive linear map $E_g : \mathcal{A} \rtimes_\alpha G \to \mathcal{A}$ such that

$$E_g\left(\sum_{h \in G} a_h \delta_h\right) = a_g$$

for any $\sum_{h \in G} a_h \delta_h \in \mathcal{A} \rtimes_\alpha^{\mathrm{alg}} G$. Moreover, for an element X in $\mathcal{A} \rtimes_\alpha G$, $X = 0$ if and only if $E_g(X) = 0$ for all $g \in G$.

1.3 A two-point compactification of $\mathbb{Z}^2$

In this last section, we construct a two-point compactification of $\mathbb{Z}^2$, which is necessary to describe anisotropic quantum walks of this thesis.

For a locally compact space X, we denote by $\widehat{X}$ the one-point compactification of X. Let ∞_e and ∞_o be two symbols which are not elements of $\mathbb{Z}^2$, and let $\widetilde{\mathbb{Z}^2} := \mathbb{Z}^2 \cup \{\infty_e, \infty_o\}$. Also we set

$$\begin{aligned} \mathbb{Z}^2_e &:= \{(n,m) \in \mathbb{Z}^2 \mid n+m \in 2\mathbb{Z}\}, \\ \mathbb{Z}^2_o &:= \{(n,m) \in \mathbb{Z}^2 \mid n+m+1 \in 2\mathbb{Z}\}. \end{aligned}$$

Note that $\mathbb{Z}^2 = \mathbb{Z}^2_e \cup \mathbb{Z}^2_o$, and $\mathbb{Z}^2_e \cap \mathbb{Z}^2_o = \varnothing$. One defines a fundamental system of neighborhoods of $\infty_\star$ ($\star = e, o$) in $\widetilde{\mathbb{Z}^2}$ by all sets of the form

$$\{(n,m) \in \mathbb{Z}^2_\star \mid \|(n,m)\|_1 \geq N\} \cup \{\infty_\star\},$$

for $N \in \mathbb{N}$, where $\|(n,m)\|_1 := |n|+|m|$, the ℓ_1-norm on $\mathbb{Z}^2$. Then $\widetilde{\mathbb{Z}^2}$ is a compactification of $\mathbb{Z}^2$. We note that $\widehat{\mathbb{Z}^2_e}$ and $\widehat{\mathbb{Z}^2_o}$ are homeomorphic to $\mathbb{Z}^2_e \cup \{\infty_e\}$ and $\mathbb{Z}^2_o \cup \{\infty_o\}$, respectively.

Chapter 2

Quantum walks on lattices

In this chapter, we describe the shift, coin, and time evolution operators on hexagonal, square, and triangular lattices. The vertex sets of those graphs can be regarded as $\mathbb{Z}^2$, so the expression of the shift operator is explicit. We will be able to understand that in Section 2.1-2.3. However, only in the hexagonal lattice case, it is necessary to divide its expression into two cases. Section 2.4 gives a way to modify it. This idea was used to study the discrete Schrödinger operators on a hexagonal lattice in [2]. The workaround not only avoids the complexity of the shift operator but also makes it possible to treat certain anisotropic quantum walks in later chapters. Therefore, in Section 2.4, we apply the method even to the cases of square and triangular lattices that do not require the modification of shift operators.

2.1 Quantum walks on a hexagonal lattice graph

In this section, we first give straightforward definitions of the shift, coin, and time evolution operators on the hexagonal lattice, which are compatible with more general definitions in Section 2.2. For this purpose, by using Fig. 0.2 in Introduction, we rewrite the hexagonal lattice graph so that the vertex set of deformed lattice coincides with $\mathbb{Z}^2$.

Definition 2.1.1. The hexagonal lattice graph $\Gamma_H = (V(\Gamma_H), E(\Gamma_H))$ with the vertex set $V(\Gamma_H)$ and the undirected edge set $E(\Gamma_H)$ is defined as follows (Fig. 1):

1) (Vertex set) $V(\Gamma_H) := \mathbb{Z}^2$.

2) (Edge set)

$$E(\Gamma_H) := \Big\{\{(n,m),(n+1,m)\} \mid n,m \in \mathbb{Z}\Big\} \cup \Big\{\{(n,m),(n,m+1)\} \mid n+m \in 2\mathbb{Z}\Big\}.$$

In addition, we define the directed edge set.

3) (Directed edge set) Recall that $\mathbb{Z}^2_e = \{(n,m) \in \mathbb{Z}^2 \mid n+m \in 2\mathbb{Z}\}$ and $\mathbb{Z}^2_o = \{(n,m) \in \mathbb{Z}^2 \mid n+m+1 \in 2\mathbb{Z}\}$). For $v = (n,m) \in V(\Gamma_H)$, one defines the directed edges

whose initial vertex are v by

$$
\begin{aligned}
&e_1(v) := [(n,m) \to (n+1,m)],\\
&e_2(v) := [(n,m) \to (n-1,m)],\\
&e_3(v) := \begin{cases} [(n,m) \to (n,m+1)] & \text{if} \quad v \in \mathbb{Z}_e^2, \\ [(n,m) \to (n,m-1)] & \text{if} \quad v \in \mathbb{Z}_o^2. \end{cases}
\end{aligned}
\tag{2.1.1}
$$

(See Fig. 2.) The directed edge set $D(\Gamma_H)$ is defined by

$$D(\Gamma_H) := \{e_1(v) \mid v \in V(\Gamma_H)\} \cup \{e_2(v) \mid v \in V(\Gamma_H)\} \cup \{e_3(v) \mid v \in V(\Gamma_H)\}.$$

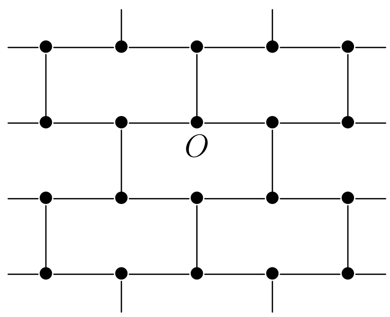

Figure 1: The hexagonal lattice graph $\Gamma_H = (V(\Gamma_H), E(\Gamma_H))$

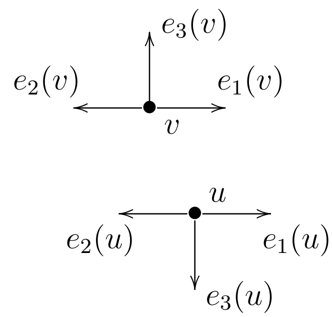

Figure 2: The directed edges ($v \in \mathbb{Z}_e^2,\ u \in \mathbb{Z}_o^2$)

Next, we shall define the state space, and the shift, coin, and time evolution operators on the state space.

Definition 2.1.2. 1) (*State space*) $\mathcal{H}_{\Gamma_H} := \ell_2(V(\Gamma_H); \mathbb{C}^3) = \ell_2(\mathbb{Z}^2; \mathbb{C}^3)$.

2) (*Shift operator*) Define $S : \mathcal{H}_{\Gamma_H} \to \mathcal{H}_{\Gamma_H}$ by

$$
(S\Psi)(n,m) := \begin{cases} \begin{pmatrix} \Psi_1(n+1,m) \\ \Psi_2(n-1,m) \\ \Psi_3(n,m+1) \end{pmatrix} & \text{if } (n,m) \in \mathbb{Z}_e^2, \\ \begin{pmatrix} \Psi_1(n+1,m) \\ \Psi_2(n-1,m) \\ \Psi_3(n,m-1) \end{pmatrix} & \text{if } (n,m) \in \mathbb{Z}_o^2 \end{cases} \quad \text{for } \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} \in \mathcal{H}_{\Gamma_H}.
\tag{2.1.2}
$$

3) (*Coin operator*) Given a map $C_\bullet : \mathbb{Z}^2 \to U(3)$, called a *coin map*, we define $C : \mathcal{H}_{\Gamma_H} \to \mathcal{H}_{\Gamma_H}$ by

$$(C\Psi)(n,m) := C_{(n,m)} \Psi(n,m) \quad \text{for } \Psi \in \mathcal{H}_{\Gamma_H},\ (n,m) \in \mathbb{Z}^2.$$

Since $\ell_2(\mathbb{Z}^2;\mathbb{C}^3) \cong \bigoplus_{\mathbb{Z}^2} \mathbb{C}^3$ and $M_3(\mathbb{C}) \cong \mathbb{B}(\mathbb{C}^3)$, the coin operator C can also be expressed as follows:

$$C = \bigoplus_{(n,m)\in\mathbb{Z}^2} C_{(n,m)}.$$

4) (*Time evolution operator*) $U := SC$.

2.2 Definition of quantum walks on regular graphs

In this section, we will define the shift, coin, and time evolution operators derived from any regular graph. It will be able to see that Definition 2.1.2 is a special case of its definition at the beginning of the next section. A quantum walk described by the time evolution operator in this section is called a position-dependent or coined model quantum walk, we referred to [1, 24, 28] for its definition.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a d-regular undirected graph with the vertex set $V(\Gamma)$ and the undirected edge set $E(\Gamma)$. Every element in $E(\Gamma)$ is represented as $\{v,u\}$ for some $v,u \in V(\Gamma)$. Note that $\{v,u\} = \{u,v\}$ in $E(\Gamma)$. We set $D(\Gamma) = \{[v\to u] \mid \{v,u\} \in E(\Gamma)\}$ as the set of all directed edges of Γ. Given a directed edge $e = [v\to u]$, the initial vertex v is denoted by $o(e)$. For each $e = [v\to u] \in D(\Gamma)$, a directed edge $[u\to v]$ is called the *inverse edge* of e, and denoted by $\bar{e}$. We also note that $\bar{\bar{e}} = e$ and $e \neq \bar{e}$ in $D(\Gamma)$.

For any vertex $v \in V(\Gamma)$, the number of elements in the set $\{e \in D(\Gamma) \mid o(e) = v\}$ is d. We assume that the d elements in this set are ordered as $e_1(v), \dots, e_d(v)$. That is, there exist d maps $e_1(\cdot), \dots, e_d(\cdot) : V(\Gamma) \to D(\Gamma)$ such that

$$\{e \in D(\Gamma) \mid o(e) = v\} = \{e_1(v), \dots, e_d(v)\} \quad \text{for all} \quad v \in V(\Gamma).$$

Then the operator $T : \ell_2(D(\Gamma)) \to \ell_2(V(\Gamma);\mathbb{C}^d)$ defined by

$$(T\Phi)(v) := \begin{pmatrix} \Phi(e_1(v)) \\ \vdots \\ \Phi(e_d(v)) \end{pmatrix} \quad (\Phi \in \ell_2(D(\Gamma)),\ v \in V(\Gamma))$$

is isomorphism. It is called the *natural isomorphism* with respect to $e_i(\cdot)$. The inverse of T is given by

$$(T^{-1}\Psi)(e_i(v)) = \Psi_i(v) \quad (\Psi = {}^t(\Psi_1, \dots, \Psi_d) \in \ell_2(V(\Gamma);\mathbb{C}^d),\ v \in V(\Gamma),\ 1 \le i \le d).$$

Let us define unitary operators which describe a quantum walk on Γ.

Definition 2.2.1. 1) (*State space*) $\mathcal{H}_\Gamma := \ell_2(V(\Gamma);\mathbb{C}^d)$.

2) (*Shift operator*) Define $\mathcal{S} : \ell_2(D(\Gamma)) \to \ell_2(D(\Gamma))$ by

$$(\mathcal{S}\Phi)(e) := \Phi(\bar{e}) \quad \text{for} \quad \Phi \in \ell_2(D(\Gamma)),\ e \in D(\Gamma).$$

3) (*Coin operator*) Given a map $C_\bullet : V(\Gamma) \to U(d)$, called a *coin map*, we define $\mathcal{C} : \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ by

$$(\mathcal{C}\Psi)(v) := C_v \Psi(v) \quad \text{for} \quad \Psi \in \mathcal{H}_\Gamma,\ v \in V(\Gamma).$$

The coin operator $\mathcal{C}$ can also be expressed as follows:

$$\mathcal{C} = \bigoplus_{v \in V(\Gamma)} C_v.$$

4) (*Time evolution operator*) $\mathcal{U} := (T\mathcal{S}T^{-1})\mathcal{C}$.

Remark 2.2.2. Let us see that the ordering of edges is not essential. For this, we define abstract quantum walks, and the unitary equivalence of them which is introduced in [32].

Let V be a countable set, let $\{\mathcal{H}_v\}_{v\in V}$ be a family of separable Hilbert spaces, and let $\mathcal{U}$ be a unitary operator on $\mathcal{H} = \bigoplus_{v\in V} \mathcal{H}_v$. Then the pair $(\mathcal{U}, \{\mathcal{H}_v\}_{v\in V})$ is called a *abstract quantum walk*. For two abstract quantum walks $(\mathcal{U}_1, \{\mathcal{H}^{(1)}_{v_1}\}_{v_1\in V_1})$ and $(\mathcal{U}_2, \{\mathcal{H}^{(2)}_{v_2}\}_{v_2\in V_2})$, they are *unitary equivalent* if there exist a unitary operator W from $\mathcal{H}_1 = \bigoplus_{v_1\in V_1} \mathcal{H}^{(1)}_{v_1}$ to $\mathcal{H}_2 = \bigoplus_{v_2\in V_2} \mathcal{H}^{(2)}_{v_2}$ and a bijective map $\phi : V_1 \to V_2$ such that $W\mathcal{U}_1W^{-1} = \mathcal{U}_2$ and $W\mathcal{H}^{(1)}_{v_1} = \mathcal{H}^{(2)}_{\phi(v_1)}$ for every $v_1 \in V_1$.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a d-regular undirected graph with the directed edge set $D(\Gamma)$ and the ordered edges $\{e_1(v), \dots, e_d(v)\}$ for each $v \in V(\Gamma)$. We fix a coin map $C_\bullet : V(\Gamma) \to U(d); v \mapsto C_v = (c^v_{ij})_{1\le i,j\le d}$ arbitrarily. Let $\{\widetilde{e}_1(v), \dots, \widetilde{e}_d(v)\}$ be another ordered edges for $v \in V(\Gamma)$, and $T, \widetilde{T} : \ell_2(D(\Gamma)) \to \mathcal{H}_\Gamma$ be the natural isomorphisms with respect to $e_i(\cdot)$ and $\widetilde{e}_i(\cdot)$, respectively.

For any $v \in V(\Gamma)$, there exists a permutation $\sigma_v \in \mathfrak{S}_d$ such that $\widetilde{e}_i(v) = e_{\sigma_v(i)}(v)$ for all $i = 1, \dots, d$, so the natural isomorphism $\widetilde{T}$ is given by

$$(\widetilde{T}\Phi)(v) = \begin{pmatrix} \Phi(\widetilde{e}_1(v)) \\ \vdots \\ \Phi(\widetilde{e}_d(v)) \end{pmatrix} = \begin{pmatrix} \Phi(e_{\sigma_v(1)}(v)) \\ \vdots \\ \Phi(e_{\sigma_v(d)}(v)) \end{pmatrix} \quad (\Phi \in \ell_2(D(\Gamma)),\ v \in V(\Gamma)).$$

We now define a unitary operator $W : \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ by

$$(W\Psi)(v) := \begin{pmatrix} \Psi_{\sigma_v(1)}(v) \\ \vdots \\ \Psi_{\sigma_v(d)}(v) \end{pmatrix} \quad (\Psi = {}^t(\Psi_1, \dots, \Psi_d) \in \mathcal{H}_\Gamma,\ v \in V(\Gamma)),$$

then the following diagram commutes.

$$\begin{tikzcd} & \ell_2(D(\Gamma)) \arrow[dl, "T"'] \arrow[dr, "\widetilde{T}"] & \\ \mathcal{H}_\Gamma \arrow[rr, "W"'] & & \mathcal{H}_\Gamma \end{tikzcd}$$

Also we define a new coin map $\widetilde{C}_\bullet : V(\Gamma) \to U(d)$ by

$$\widetilde{C}_v = (\widetilde{c}^v_{ij})_{1\le i,j\le d} := (c^v_{\sigma_v(i)\sigma_v(j)})_{1\le i,j\le d}$$

for $v \in V(\Gamma)$, and one denotes $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{U}}$ the coin and time evolution operators associated with $\widetilde{C}_\bullet$, respectively. For any $\Psi = {}^t(\Psi_1, \ldots, \Psi_d) \in \mathcal{H}_\Gamma$ and $v \in V(\Gamma)$, we have

$$(W\mathcal{C}\Psi)(v) = \begin{pmatrix} (\mathcal{C}\Psi)_{\sigma_v(1)}(v) \\ \vdots \\ (\mathcal{C}\Psi)_{\sigma_v(d)}(v) \end{pmatrix} = \begin{pmatrix} (C_v\Psi(v))_{\sigma_v(1)} \\ \vdots \\ (C_v\Psi(v))_{\sigma_v(d)} \end{pmatrix},$$

$$(\widetilde{\mathcal{C}}W\Psi)(v) = \widetilde{C}_v((W\Psi)(v)) = \widetilde{C}_v \begin{pmatrix} \Psi_{\sigma_v(1)}(v) \\ \vdots \\ \Psi_{\sigma_v(d)}(v) \end{pmatrix}.$$

The i-components $(1 \le i \le d)$ of $(W\mathcal{C}\Psi)(v)$ and $(\widetilde{\mathcal{C}}W\Psi)(v)$ are given by

$$\sum_{j=1}^{d} c^v_{\sigma_v(i)j}\Psi_j(v) \quad \text{and} \quad \sum_{j=1}^{d} \widetilde{c}^v_{ij}\Psi_{\sigma_v(j)}(v) = \sum_{j=1}^{d} c^v_{\sigma_v(i)\sigma_v(j)}\Psi_{\sigma_v(j)}(v),$$

respectively, so $W\mathcal{C}$ and $\widetilde{\mathcal{C}}W$ must be identical. Also $W(T\mathcal{S}T^{-1}) = \widetilde{T}\mathcal{S}(W^{-1}\widetilde{T})^{-1} = (\widetilde{T}\mathcal{S}\widetilde{T}^{-1})W$ holds. Therefore $W\mathcal{U} = \widetilde{\mathcal{U}}W$, which means that two abstract quantum walks $(\mathcal{U}, \{\mathbb{C}^d\}_{v\in V(\Gamma)})$ and $(\widetilde{\mathcal{U}}, \{\mathbb{C}^d\}_{v\in V(\Gamma)})$ are unitary equivalent.

2.3 Quantum walks on hexagonal, square, and triangular lattices

In this section, we provide the time evolution operator which describes quantum walks on square and triangular lattices. We first consider the hexagonal lattice graph $\Gamma_H = (V(\Gamma_H), E(\Gamma_H)) = (\mathbb{Z}^2, E(\Gamma_H))$ (see Definition 2.1.1) which is a 3-regular graph. Let T be the natural isomorphism from $\ell_2(D(\Gamma_H))$ to $\mathcal{H}_{\Gamma_H} = \ell_2(\mathbb{Z}^2; \mathbb{C}^3)$ with respect to the ordered edges (2.1.1). In order to compute $\mathcal{U} = (T\mathcal{S}T^{-1})\mathcal{C}$, we fix a coin map $C_\bullet : \mathbb{Z}^2 \to U(3)$ arbitrarily. For any $v = (n, m) \in \mathbb{Z}^2$, the inverse edge of $e_i(v)$ $(i = 1, 2, 3)$ is given by

$$\begin{aligned}
\overline{e_1(v)} &= [(n+1, m) \to (n, m)] = e_2\big((n+1, m)\big), \\
\overline{e_2(v)} &= [(n-1, m) \to (n, m)] = e_1\big((n-1, m)\big), \\
\overline{e_3(v)} &= \begin{cases} [(n, m+1) \to (n, m)] = e_3\big((n, m+1)\big) & \text{if} \quad v = (n, m) \in \mathbb{Z}^2_e, \\ [(n, m-1) \to (n, m)] = e_3\big((n, m-1)\big) & \text{if} \quad v = (n, m) \in \mathbb{Z}^2_o. \end{cases}
\end{aligned}$$

Thus for any $\Psi = {}^t(\Psi_1, \Psi_2, \Psi_3) \in \mathcal{H}_{\Gamma_H}$, if $v = (n, m) \in \mathbb{Z}^2_e$, then we have

$$(T\mathcal{S}T^{-1}\Psi)(n, m) = \begin{pmatrix} (\mathcal{S}T^{-1}\Psi)(e_1(v)) \\ (\mathcal{S}T^{-1}\Psi)(e_2(v)) \\ (\mathcal{S}T^{-1}\Psi)(e_3(v)) \end{pmatrix} = \begin{pmatrix} (T^{-1}\Psi)\big(\overline{e_1(v)}\big) \\ (T^{-1}\Psi)\big(\overline{e_2(v)}\big) \\ (T^{-1}\Psi)\big(\overline{e_3(v)}\big) \end{pmatrix}$$

$$= \begin{pmatrix} (T^{-1}\Psi)\big(e_2\big((n+1,m)\big)\big) \\ (T^{-1}\Psi)\big(e_1\big((n-1,m)\big)\big) \\ (T^{-1}\Psi)\big(e_3\big((n,m+1)\big)\big) \end{pmatrix} = \begin{pmatrix} \Psi_2(n+1,m) \\ \Psi_1(n-1,m) \\ \Psi_3(n,m+1) \end{pmatrix}.$$

Similarly, if $(n,m) \in \mathbb{Z}_o^2$, one gets

$$(T\mathcal{S}T^{-1}\Psi)(n,m) = \begin{pmatrix} \Psi_2(n+1,m) \\ \Psi_1(n-1,m) \\ \Psi_3(n,m-1) \end{pmatrix}.$$

Now we set $\sigma := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then the following equality holds as operators on $\mathcal{H}_{\Gamma_H}$:

$$T\mathcal{S}T^{-1} = S\left(\bigoplus_{\mathbb{Z}^2} \sigma\right)$$

Hence the time evolution operator $\mathcal{U}$ is expressed as follows:

$$\mathcal{U} = (T\mathcal{S}T^{-1})\mathcal{C} = SC = U,$$

where C is the coin operator associated with the coin map $\sigma C_\bullet : \mathbb{Z}^2 \to U(3); (n,m) \mapsto \sigma C_{(n,m)}$. Since $C_\bullet \leftrightarrow \sigma C_\bullet$ is one-to-one correspondence between the sets of all coin maps, one can see that the shift operator derived from Γ_H can be defined as (2.1.2).

Similarly, we can define the shift, coin, and time evolution operators derived from square and triangular lattices. For this purpose, one needs the definitions of the square and triangular lattice graphs.

Definition 2.3.1. Square lattice

The vertex and undirected edge sets of the square lattice graph Γ_S are respectively defined as follows (Fig. 3):

1) (Vertex set) $V(\Gamma_S) := \mathbb{Z}^2$.

2) (Edge set)

$$E(\Gamma_S) := \Big\{\{(n,m),(n+1,m)\} \mid n,m \in \mathbb{Z}\Big\} \cup \Big\{\{(n,m),(n,m+1)\} \mid n,m \in \mathbb{Z}\Big\}.$$

Additionally, we define the directed edge set.

3) (Directed edge set) For $v = (n,m) \in V(\Gamma_S)$, we define the directed edges whose initial vertex is v by

$$e_1(v) := [(n,m) \to (n+1,m)],$$

$$\begin{aligned} e_2(v) &:= [(n,m) \to (n-1,m)], \\ e_3(v) &:= [(n,m) \to (n,m+1)], \\ e_4(v) &:= [(n,m) \to (n,m-1)]. \end{aligned}$$

(See Fig. 4.) The directed edge set $D(\Gamma_S)$ is defined by

$$D(\Gamma_S) := \{e_i(v) \mid v \in V(\Gamma_S),\ i = 1,2,3,4\}.$$

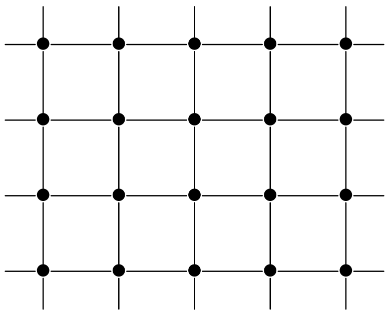

Figure 3

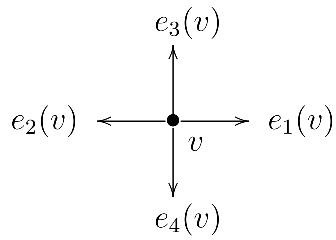

Figure 4

Triangular lattice

Similarly, the vertex, undirected edge, directed edge sets of the triangular lattice graph Γ_T are respectively defined as follows (Fig. 5):

4) (Vertex set) $V(\Gamma_T) := \mathbb{Z}^2$.

5) (Edge set)

$$E(\Gamma_T) := E(\Gamma_S) \cup \Big\{\{(n,m),(n+1,m+1)\} \mid n,m \in \mathbb{Z}\Big\}.$$

6) (Directed edge set) For $v = (n,m) \in V(\Gamma_T)$, one defines the directed edges whose initial vertex is v by

$$\begin{aligned} e_1(v) &:= [(n,m) \to (n+1,m)], \\ e_2(v) &:= [(n,m) \to (n-1,m)], \\ e_3(v) &:= [(n,m) \to (n,m+1)], \\ e_4(v) &:= [(n,m) \to (n,m-1)], \\ e_5(v) &:= [(n,m) \to (n+1,m+1)], \\ e_6(v) &:= [(n,m) \to (n-1,m-1)]. \end{aligned}$$

(See Fig. 6.) The edge set $D(\Gamma_T)$ is defined by

$$D(\Gamma_T) := \{e_i(v) \mid v \in V(\Gamma_T),\ i = 1,2,3,4,5,6\} = D(\Gamma_S) \cup \{e_i(v) \mid v \in V(\Gamma_T),\ i = 5,6\}.$$

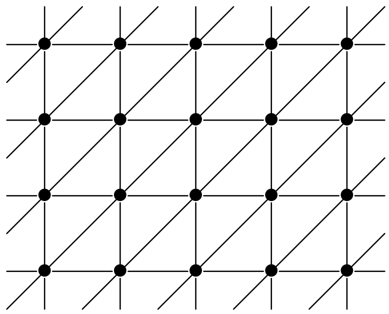

Figure 5

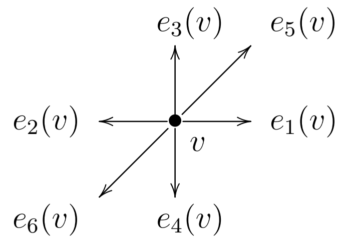

Figure 6

In the same way as the case of the hexagonal lattice, one can define the shift operator etc. The following shift operators are also treated in [16].

Definition 2.3.2. Square lattice

1) (*State space*) $\mathcal{H}_{\Gamma_S} := \ell_2(\mathbb{Z}^2; \mathbb{C}^4)$.

2) (*Shift operator*)

$$(S\Psi)(n,m) := \begin{pmatrix} \Psi_1(n+1,m) \\ \Psi_2(n-1,m) \\ \Psi_3(n,m+1) \\ \Psi_4(n,m-1) \end{pmatrix} \text{ for } \Psi = {}^t(\Psi_1, \Psi_2, \Psi_3, \Psi_4) \in \mathcal{H}_{\Gamma_S},\ (n,m) \in \mathbb{Z}^2.$$

3) (*Coin operator*) For a coin map $C_\bullet : \mathbb{Z}^2 \to U(4)$, we set $C := \bigoplus_{(n,m)\in\mathbb{Z}^2} C_{(n,m)}$.

4) (*Time evolution operator*) $U := SC$.

Triangular lattice

5) (*State space*) $\mathcal{H}_{\Gamma_T} := \ell_2(\mathbb{Z}^2; \mathbb{C}^6)$.

6) (*Shift operator*)

$$(S\Psi)(n,m) := \begin{pmatrix} \Psi_1(n+1,m) \\ \Psi_2(n-1,m) \\ \Psi_3(n,m+1) \\ \Psi_4(n,m-1) \\ \Psi_5(n+1,m+1) \\ \Psi_6(n-1,m-1) \end{pmatrix}$$

for $\Psi = {}^t(\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6) \in \mathcal{H}_{\Gamma_T}, (n,m) \in \mathbb{Z}^2$.

7) The *coin* and *time evolution operators* are defined in exactly the same manner as above.

At the end of this section, we define the term anisotropic.

Definition 2.3.3. Let Γ be one of Γ_H, Γ_S and Γ_T, and let d be its degree.

1) A coin map $C_\bullet : \mathbb{Z}^2 \to U(d)$ is said to be *anisotropic* if it can be continuously extended to $\widetilde{\mathbb{Z}^2}$ (see Section 1.3), that is, there exist $C_e, C_o \in U(d)$ such that

$$\begin{aligned} &\|C_{(n,m)} - C_e\|_{M_d(\mathbb{C})} \to 0 \ \text{ as } \ \mathbb{Z}_e^2 \ni (n,m) \to \infty_e, \\ &\|C_{(n,m)} - C_o\|_{M_d(\mathbb{C})} \to 0 \ \text{ as } \ \mathbb{Z}_o^2 \ni (n,m) \to \infty_o. \end{aligned} \tag{2.3.1}$$

It is also equivalent to $C_\bullet \in C(\widetilde{\mathbb{Z}^2}; M_d(\mathbb{C}))$. In particular, a coin map $C_\bullet : \mathbb{Z}^2 \to U(d)$ is said to be *quasi-uniform* if

$$C_{(n,m)} = \begin{cases} C_e & \text{if } (n,m) \in \mathbb{Z}_e^2, \\ C_o & \text{if } (n,m) \in \mathbb{Z}_o^2 \end{cases} \tag{2.3.2}$$

for some $C_e, C_o \in U(d)$.

2) Let $C_\bullet : \mathbb{Z}^2 \to U(d)$ be an anisotropic coin map satisfying (2.3.1). Then the unitary matrix $C_\infty = C_e \oplus C_o \in U(2d)$ is called the *limit matrix* of $C_\bullet$.

2.4 Modification of the unitary operators

Let us define two maps $\phi_e : \mathbb{Z}^2 \to \mathbb{Z}_e^2$, $\phi_o : \mathbb{Z}^2 \to \mathbb{Z}_o^2$ by

$$\phi_e(n,m) := (n-m, n+m), \ \phi_o(n,m) := (n-m, n+m+1), \quad (n,m) \in \mathbb{Z}^2.$$

Since they are bijective (with the inverses

$$\begin{aligned} \phi_e^{-1}(n,m) &= \frac{1}{2}(n+m, -n+m), \quad (n,m) \in \mathbb{Z}_e^2, \\ \phi_o^{-1}(n,m) &= \frac{1}{2}(n+m-1, -n+m-1), \quad (n,m) \in \mathbb{Z}_o^2), \end{aligned}$$

the mappings ϕ_e and ϕ_o naturally induce the following unitary operators $J_e : \ell_2(\mathbb{Z}_e^2; \mathbb{C}^d) \to \mathcal{H}_\Gamma$ and $J_o : \ell_2(\mathbb{Z}_o^2; \mathbb{C}^d) \to \mathcal{H}_\Gamma$:

$$\begin{aligned} (J_e\Phi)(n,m) &= \Phi(\phi_e(n,m)) = \Phi(n-m, n+m), \quad \Phi \in \ell_2(\mathbb{Z}_e^2; \mathbb{C}^d), \ (n,m) \in \mathbb{Z}^2, \\ (J_o\Psi)(n,m) &= \Psi(\phi_o(n,m)) = \Psi(n-m, n+m+1), \quad \Psi \in \ell_2(\mathbb{Z}_o^2; \mathbb{C}^d), \ (n,m) \in \mathbb{Z}^2, \end{aligned}$$

respectively. Also we set a Hilbert space $\mathcal{K}_\Gamma := \mathcal{H}_\Gamma \oplus \mathcal{H}_\Gamma$ (whose any element has the form ${}^t(\Phi, \Psi)$, $(\Phi, \Psi \in \mathcal{H}_\Gamma)$), and define a unitary operator $J_\Gamma : \mathcal{H}_\Gamma \to \mathcal{K}_\Gamma$ by composing the natural decomposition $\mathcal{H}_\Gamma \cong \ell_2(\mathbb{Z}_e^2; \mathbb{C}^d) \oplus \ell_2(\mathbb{Z}_o^2; \mathbb{C}^d)$ and the direct sum operator $J_e \oplus J_o : \ell_2(\mathbb{Z}_e^2; \mathbb{C}^d) \oplus \ell_2(\mathbb{Z}_o^2; \mathbb{C}^d) \to \mathcal{K}_\Gamma$. Specifically, J_Γ is given by the following formula:

$$J_\Gamma(\Phi) = {}^t\left(J_e(\Phi|_{\mathbb{Z}_e^2}), J_o(\Phi|_{\mathbb{Z}_o^2})\right), \tag{2.4.1}$$

for $\Phi \in \mathcal{H}_\Gamma$. Let us transform the shift, coin, and time evolution operators by J_Γ. In terms of the coin operator, for any $\Phi \in \mathcal{H}_\Gamma, (n,m) \in \mathbb{Z}^2$, we have

$$(J_e C J_e^{-1}\Phi)(n,m) = (CJ_e^{-1}\Phi)(\phi_e(n,m)) = C_{\phi_e(n,m)}((J_e^{-1}\Phi)(\phi_e(n,m))) = C_{\phi_e(n,m)}\Phi(n,m).$$

Similarly, $(J_o C J_o^{-1}\Phi)(n,m) = C_{\phi_o(n,m)}\Phi(n,m)$ holds. Therefore, for all ${}^t(\Phi,\Psi) \in \mathcal{K}_\Gamma$ and $(n,m) \in \mathbb{Z}^2$, one has

$$((J_\Gamma C J_\Gamma^{-1})({}^t(\Phi,\Psi)))(n,m) = \begin{pmatrix} C_{\phi_e(n,m)}\Phi(n,m) \\ C_{\phi_o(n,m)}\Psi(n,m) \end{pmatrix},$$

namely,

$$J_\Gamma C J_\Gamma^{-1} = \bigoplus_{(n,m)\in\mathbb{Z}^2} (C_{\phi_e(n,m)} \oplus C_{\phi_o(n,m)}). \tag{2.4.2}$$

In terms of the shift operator, we need to compute in each individually. First, let us consider the hexagonal lattice case. Being careful about $S(\ell_2(\mathbb{Z}_e^2;\mathbb{C}^3)) \subset \ell_2(\mathbb{Z}_o^2;\mathbb{C}^3)$, we obtain

$$\begin{aligned}(J_o S J_e^{-1}\Phi)(n,m) &= (SJ_e^{-1}\Phi)(n-m, n+m+1) \\ &= \begin{pmatrix} (J_e^{-1}\Phi)_1(n-m+1, n+m+1) \\ (J_e^{-1}\Phi)_2(n-m-1, n+m+1) \\ (J_e^{-1}\Phi)_3(n-m, n+m) \end{pmatrix} \\ &= \begin{pmatrix} \Phi_1(\phi_e^{-1}(n-m+1, n+m+1)) \\ \Phi_2(\phi_e^{-1}(n-m-1, n+m+1)) \\ \Phi_3(\phi_e^{-1}(n-m, n+m)) \end{pmatrix} \\ &= \begin{pmatrix} \Phi_1(n+1,m) \\ \Phi_2(n,m+1) \\ \Phi_3(n,m) \end{pmatrix}\end{aligned}$$

for each $\Phi = {}^t(\Phi_1,\Phi_2,\Phi_3) \in \mathcal{H}_{\Gamma_H}$ and $(n,m) \in \mathbb{Z}^2$. Similarly, for any $\Psi = {}^t(\Psi_1,\Psi_2,\Psi_3) \in \mathcal{H}_{\Gamma_H}, (n,m) \in \mathbb{Z}^2$, one gets

$$(J_e S_0 J_o^{-1}\Psi)(n,m) = \begin{pmatrix} \Psi_1(n,m-1) \\ \Psi_2(n-1,m) \\ \Psi_3(n,m) \end{pmatrix}.$$

Hence, one can deduce from the observations above that for each ${}^t(\Phi,\Psi) \in \mathcal{K}_{\Gamma_H}$ and $(n,m) \in \mathbb{Z}^2$ (be careful with the inclusions $S(\ell_2(\mathbb{Z}_e^2;\mathbb{C}^3)) \subset \ell_2(\mathbb{Z}_o^2;\mathbb{C}^3)$ and $S(\ell_2(\mathbb{Z}_o^2;\mathbb{C}^3)) \subset \ell_2(\mathbb{Z}_e^2;\mathbb{C}^3)$ again), we have

$$((J_{\Gamma_H} S J_{\Gamma_H}^{-1})({}^t(\Phi,\Psi)))(n,m) = {}^t\!\left(\begin{pmatrix} \Psi_1(n,m-1) \\ \Psi_2(n-1,m) \\ \Psi_3(n,m) \end{pmatrix}, \begin{pmatrix} \Phi_1(n+1,m) \\ \Phi_2(n,m+1) \\ \Phi_3(n,m) \end{pmatrix}\right). \tag{2.4.3}$$

Even in the case where $\Gamma = \Gamma_S$, one can get the following formula by similar to the computations above:

$$((J_{\Gamma_S} S J_{\Gamma_S}^{-1})({}^t(\Phi,\Psi)))(n,m) = {}^t\left(\begin{pmatrix} \Psi_1(n,m-1) \\ \Psi_2(n-1,m) \\ \Psi_3(n,m) \\ \Psi_4(n-1,m-1) \end{pmatrix}, \begin{pmatrix} \Phi_1(n+1,m) \\ \Phi_2(n,m+1) \\ \Phi_3(n+1,m+1) \\ \Phi_4(n,m) \end{pmatrix}\right) \tag{2.4.4}$$

for all ${}^t(\Phi,\Psi) \in \mathcal{K}_{\Gamma_S}$, $(n,m) \in \mathbb{Z}^2$.

Next, we consider the triangular lattice case. One should be careful about the following inclusion relations:

$$\begin{aligned} S(\ell_2(\mathbb{Z}_e^2;\mathbb{C}^6)) &\subset \ell_2(\mathbb{Z}_e^2; 0^4 \oplus \mathbb{C}^2) \oplus \ell_2(\mathbb{Z}_o^2; \mathbb{C}^4 \oplus 0^2), \\ S(\ell_2(\mathbb{Z}_o^2;\mathbb{C}^6)) &\subset \ell_2(\mathbb{Z}_e^2; \mathbb{C}^4 \oplus 0^2) \oplus \ell_2(\mathbb{Z}_o^2; 0^4 \oplus \mathbb{C}^2). \end{aligned}$$

Then for any ${}^t(\Phi,\Psi) \in \mathcal{K}_{\Gamma_S}$, we have

$$\begin{aligned} &((J_{\Gamma_T} S J_{\Gamma_T}^{-1})({}^t(\Phi,\Psi)))(n,m) \\ &= {}^t\left(\begin{pmatrix} (SJ_o^{-1}\Psi)_1(n-m,n+m) \\ (SJ_o^{-1}\Psi)_2(n-m,n+m) \\ (SJ_o^{-1}\Psi)_3(n-m,n+m) \\ (SJ_o^{-1}\Psi)_4(n-m,n+m) \\ (SJ_e^{-1}\Phi)_5(n-m,n+m) \\ (SJ_e^{-1}\Phi)_6(n-m,n+m) \end{pmatrix}, \begin{pmatrix} (SJ_e^{-1}\Phi)_1(n-m,n+m+1) \\ (SJ_e^{-1}\Phi)_2(n-m,n+m+1) \\ (SJ_e^{-1}\Phi)_3(n-m,n+m+1) \\ (SJ_e^{-1}\Phi)_4(n-m,n+m+1) \\ (SJ_o^{-1}\Psi)_5(n-m,n+m+1) \\ (SJ_o^{-1}\Psi)_6(n-m,n+m+1) \end{pmatrix}\right) \\ &= {}^t\left(\begin{pmatrix} (J_o^{-1}\Psi)_1(n-m+1,n+m) \\ (J_o^{-1}\Psi)_2(n-m-1,n+m) \\ (J_o^{-1}\Psi)_3(n-m,n+m+1) \\ (J_o^{-1}\Psi)_4(n-m,n+m-1) \\ (J_e^{-1}\Phi)_5(n-m+1,n+m+1) \\ (J_e^{-1}\Phi)_6(n-m-1,n+m-1) \end{pmatrix}, \begin{pmatrix} (J_e^{-1}\Phi)_1(n-m+1,n+m+1) \\ (J_e^{-1}\Phi)_2(n-m-1,n+m+1) \\ (J_e^{-1}\Phi)_3(n-m,n+m+2) \\ (J_e^{-1}\Phi)_4(n-m,n+m) \\ (J_o^{-1}\Psi)_6(n-m+1,n+m+2) \\ (J_o^{-1}\Psi)_6(n-m-1,n+m) \end{pmatrix}\right) \\ &= {}^t\left(\begin{pmatrix} \Psi_1(n,m-1) \\ \Psi_2(n-1,m) \\ \Psi_3(n,m) \\ \Psi_4(n-1,m-1) \\ \Phi_5(n+1,m) \\ \Phi_6(n-1,m) \end{pmatrix}, \begin{pmatrix} \Phi_1(n+1,m) \\ \Phi_2(n,m+1) \\ \Phi_3(n+1,m+1) \\ \Phi_4(n,m) \\ \Psi_5(n+1,m) \\ \Psi_6(n-1,m) \end{pmatrix}\right). \end{aligned} \tag{2.4.5}$$

Let us summarize the above as a definition.

Definition 2.4.1. Let Γ be one of Γ_H, Γ_S and Γ_T, let $J_\Gamma : \mathcal{H}_\Gamma \to \mathcal{K}_\Gamma$ be as (2.4.1) above. The unitary operators $J_\Gamma S J_\Gamma^{-1}$ and $J_\Gamma C J_\Gamma^{-1}$ in (2.4.2)–(2.4.5) are denoted by S_0 and C_0 respectively, and are also called the *shift operator* and the *coin operator*, respectively.

Also, the *time evolution operator* is defined by $U_0 = S_0 C_0 (= J_\Gamma U J_\Gamma^{-1})$.

$$\begin{tikzcd} \mathcal{H}_\Gamma \arrow[r, "S,\ C,\ U"] \arrow[d, "J_\Gamma"'] & \mathcal{H}_\Gamma \arrow[d, "J_\Gamma"] \\ \mathcal{K}_\Gamma \arrow[r, "S_0,\ C_0,\ U_0"'] & \mathcal{K}_\Gamma \end{tikzcd}$$

Definition 2.4.2. 1) Let $C_\bullet : \mathbb{Z}^2 \to U(d)$ be a quasi-uniform coin map satisfying (2.3.2). In this case, the coin and time evolution operators associated with $C_\bullet$ are denoted by C_∞ and U_∞ instead of C_0 and U_0, respectively. Of course,

$$C_\infty = \bigoplus_{\mathbb{Z}^2} (C_e \oplus C_o).$$

If there would be no danger of confusion, then we will also denote the unitary matrix $C_e \oplus C_o$ of size $2d$ by C_∞.

2) Let $C_\bullet : \mathbb{Z}^2 \to U(d)$ be an anisotropic coin map satisfying (2.3.1). Then one can consider a unitary operator $U_\infty = S_0 C_\infty$ which is a time evolution operator associated with the quasi-uniform coin map $\left(= \begin{cases} C_e \text{ on } \mathbb{Z}_e^2 \\ C_o \text{ on } \mathbb{Z}_o^2 \end{cases} \right)$ other than $U_0 = S_0 C_0$. The operator U_∞ is called the *auxiliary time evolution operator*. Of course, when a coin map is quasi-uniform, the time evolution operator coincides with the auxiliary time evolution one.

Chapter 3

Weak Limit Theorem

The aim of this chapter is to prove a weak limit theorem, which is a main theorem of this thesis. This theorem is proved for certain quantum walks, specifically, when the time evolution operator is associated with a short-range coin map. The short-range condition is stronger than anisotropy. The proof is based on an argument using the GJS method and spectral scattering theory inspired from [35].

Throughout this chapter, we suppose that Γ is one of hexagonal, square, and triangular lattices, unless otherwise stated, and d is its degree.

We fix a norm one vector $\Psi^0 \in \mathcal{H}_\Gamma$, which is said to be an *initial state*. The *state* of the quantum walker after time $t \in \mathbb{N}_0 (:= \mathbb{N} \cup \{0\})$ is expressed by $U^t\Psi^0$. If we denote by X_t a random variable which describes the position of a quantum walker at time t, then for $(n,m) \in \mathbb{Z}^2$ and $t \in \mathbb{N}_0$, the probability that the quantum walker exists at position (n,m) at time t is defined by

$$\mathbb{P}(X_t = (n,m)) = \|(U^t\Psi^0)(n,m)\|^2_{\mathbb{C}^d}.$$

One remarks that only the probability measure on $\mathbb{Z}^2$ is defined but not the stochastic process $\{X_t\}$. For each $\alpha > 0$, we can consider the probability distribution of X_t/t^α which is defined by

$$\mathbb{P}(X_t/t^\alpha \in B) := \mathbb{P}(X_t \in t^\alpha B) (= \mathbb{P}(X_t \in (t^\alpha B) \cap \mathbb{Z}^2)) \tag{3.0.1}$$

for $B \in \mathcal{B}(\mathbb{R}^2)$. Our aim in this chapter is to find a probability measure μ on $\mathbb{R}^2$ such that the probability distribution of X_t/t is weakly convergent to μ as $t \to \infty$.

By regarding X_t/t^α as a "random variable", the characteristic function of the probability distribution of X_t/t^α is written as follows:

$$\int_{\mathbb{R}^2} \exp\left(i\left\langle \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \begin{pmatrix}x\\ y\end{pmatrix}\right\rangle_2\right) d\mathbb{P}\left(X_t/t^\alpha = (x,y)\right) = \mathbb{E}\left[\exp\left(i\left\langle \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \frac{X_t}{t^\alpha}\right\rangle_2\right)\right] \tag{3.0.2}$$

for each ${}^t\big(\xi_1, \xi_2\big) \in \mathbb{R}^2$.

3.1 Characteristic functions

Here we will express the characteristic function of the distribution (3.0.2) using the position operators on $\mathcal{K}_\Gamma$ and their 2-variable functional calculus. We start on recalling the position operators on $\mathcal{H}_\Gamma = \ell_2(\mathbb{Z}^2; \mathbb{C}^d)$.

The domains of the position operators P_1 and P_2 on $\mathcal{H}_\Gamma$ are respectively defined by

$$\mathcal{D}(P_1) = \left\{ \Phi \in \mathcal{H}_\Gamma \;\middle|\; \sum_{(n,m)\in\mathbb{Z}^2} |n|^2 \|\Phi(n,m)\|^2_{\mathbb{C}^d} < \infty \right\},$$
$$\mathcal{D}(P_2) = \left\{ \Phi \in \mathcal{H}_\Gamma \;\middle|\; \sum_{(n,m)\in\mathbb{Z}^2} |m|^2 \|\Phi(n,m)\|^2_{\mathbb{C}^d} < \infty \right\},$$

and define P_1, P_2 by

$$P_1\Phi(n,m) := n\Phi(n,m), \quad P_2\Phi(n,m) := m\Phi(n,m),$$

respectively. The transformations of P_1, P_2 by J_Γ are as follows:

Lemma 3.1.1. *We have*

$$J_\Gamma(\mathcal{D}(P_1)) \\ = \left\{ \Phi = {}^t(\Phi_1, \Phi_2) \in \mathcal{K}_\Gamma \;\middle|\; \sum_{(n,m)\in\mathbb{Z}^2} \{|n-m|^2 \|\Phi_1(n,m)\|^2_{\mathbb{C}^d} + |n-m|^2 \|\Phi_2(n,m)\|^2_{\mathbb{C}^d}\} < \infty \right\}, \tag{3.1.1}$$

$$J_\Gamma(\mathcal{D}(P_2)) \\ = \left\{ \Phi = {}^t(\Phi_1, \Phi_2) \in \mathcal{K}_\Gamma \;\middle|\; \sum_{(n,m)\in\mathbb{Z}^2} \{|n+m|^2 \|\Phi_1(n,m)\|^2_{\mathbb{C}^d} + |n+m+1|^2 \|\Phi_2(n,m)\|^2_{\mathbb{C}^d}\} < \infty \right\},$$

and the following formulas hold:

$$(J_\Gamma P_1 J_\Gamma^{-1}\Phi)(n,m) = {}^t((n-m)\Phi_1(n,m), (n-m)\Phi_2(n,m)),$$
$$(J_\Gamma P_2 J_\Gamma^{-1}\Phi)(n,m) = {}^t((n+m)\Phi_1(n,m), (n+m+1)\Phi_2(n,m))$$

on $J_\Gamma(\mathcal{D}(P_1))$ and $J_\Gamma(\mathcal{D}(P_2))$, respectively.

Proof. Since it is similar, we will prove only the equation (3.1.1). Let $\mathcal{D}_1$ denote the right hand side of (3.1.1). For any $\Psi \in \mathcal{D}(P_1)$, using the equation

$$(J_\Gamma\Psi)(n,m) = {}^t(\Psi(\phi_e(n,m)), \Psi(\phi_o(n,m))), \quad (n,m) \in \mathbb{Z}^2,$$

one gets

$$
\begin{aligned}
&\sum_{(n,m)\in\mathbb{Z}^2} |n-m|^2\{\|\Psi(\phi_e(n,m))\|^2_{\mathbb{C}^d} + \|\Psi(\phi_o(n,m))\|^2_{\mathbb{C}^d}\} \\
&= \sum_{(n,m)\in\mathbb{Z}^2_e} |n|^2\|\Psi(n,m)\|^2_{\mathbb{C}^d} + \sum_{(n,m)\in\mathbb{Z}^2_o} |n|^2\|\Psi(n,m)\|^2_{\mathbb{C}^d} \\
&= \sum_{(n,m)\in\mathbb{Z}^2} |n|^2\|\Psi(n,m)\|^2_{\mathbb{C}^d} < \infty.
\end{aligned}
$$

Thus we have $J_\Gamma(\mathcal{D}(P_1)) \subset \mathcal{D}_1$.

On the other hand, for any $\Phi = {}^t(\Phi_1, \Phi_2) \in \mathcal{D}_1$, one has

$$
\begin{aligned}
&\sum_{(n,m)\in\mathbb{Z}^2} |n|^2\|(J_\Gamma^{-1}\Phi)(n,m)\|^2_{\mathbb{C}^d} \\
&= \sum_{(n,m)\in\mathbb{Z}^2_e} |n|^2\|(J_e^{-1}\Phi_1)(n,m)\|^2_{\mathbb{C}^d} + \sum_{(n,m)\in\mathbb{Z}^2_o} |n|^2\|(J_o^{-1}\Phi_2)(n,m)\|^2_{\mathbb{C}^d} \\
&= \sum_{(n,m)\in\mathbb{Z}^2_e} |n|^2\|\Phi_1(\phi_e^{-1}(n,m))\|^2_{\mathbb{C}^d} + \sum_{(n,m)\in\mathbb{Z}^2_o} |n|^2\|\Phi_2(\phi_o^{-1}(n,m))\|^2_{\mathbb{C}^d} \\
&= \sum_{(n,m)\in\mathbb{Z}^2} |n-m|^2\|\Phi_1((n,m))\|^2_{\mathbb{C}^d} + \sum_{(n,m)\in\mathbb{Z}^2} |n-m|^2\|\Phi_2((n,m))\|^2_{\mathbb{C}^d} < \infty.
\end{aligned}
$$

Hence $\mathcal{D}_1 \subset J_\Gamma(\mathcal{D}(P_1))$, so $J_\Gamma(\mathcal{D}(P_1))$ and $\mathcal{D}_1$ must coincide. The last assertion follows from direct computation. $\square$

The self-adjoint operators $J_\Gamma P_1 J_\Gamma^{-1}$ and $J_\Gamma P_2 J_\Gamma^{-1}$ are denoted by Q_1 and Q_2, respectively.

$$
\begin{tikzcd}
\mathcal{H}_\Gamma \arrow[r, "P_1,\ P_2"] \arrow[d, "J_\Gamma"'] & \mathcal{H}_\Gamma \arrow[d, "J_\Gamma"] \\
\mathcal{K}_\Gamma \arrow[r, "Q_1,\ Q_2"'] & \mathcal{K}_\Gamma
\end{tikzcd}
$$

For a subset L of $\mathbb{Z}^2$, we let $P(L)$ denote the projection from the Hilbert space $\mathcal{K}_\Gamma$ onto its closed subspace $\ell_2(L;\mathbb{C}^d)$. For simplicity, we write $P(n,m)$ instead of $P(\{(n,m)\})$. If we define

$$E_1(n) = P(\{n\}\times\mathbb{Z}), \quad E_2(m) = P(\mathbb{Z}\times\{m\}),$$

then the spectral decompositions of P_1 and P_2 are respectively given by

$$P_1 = \sum_{n\in\mathbb{Z}} nE_1(n) \quad \text{and} \quad P_2 = \sum_{m\in\mathbb{Z}} mE_2(m).$$

Since P_1 and P_2 are strongly commuting (because $E_1(n)E_2(m) = E_2(m)E_1(n) = P(n,m)$ for any $n, m \in \mathbb{Z}$), so are Q_1 and Q_2.

The next lemma is a purpose of this section. We note that the lemma holds for the time evolution operator associated with any coin map.

For $t \in \mathbb{N}_0$, we set $Q_j(t) := U_0^{-t} Q_j U_0^t$ $(j = 1, 2)$.

Lemma 3.1.2. *Let $\Psi^0 \in \mathcal{H}_\Gamma$ be an initial state and α be a positive number. If we write $\Phi^0 = J_\Gamma \Psi^0$, then the characteristic function of the random variable X_t/t^α is given by*

$$\mathbb{E}\left[\exp\left(i\left\langle \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \frac{X_t}{t^\alpha}\right\rangle_2\right)\right] = \left\langle \Phi^0, \exp\left\{i\left(\xi_1 \frac{Q_1(t)}{t^\alpha} + \xi_2 \frac{Q_2(t)}{t^\alpha}\right)\right\}\Phi^0\right\rangle$$

for ${}^t(\xi_1, \xi_2) \in \mathbb{R}^2$, where $\exp\left\{i\left(\xi_1 \frac{Q_1(t)}{t^\alpha} + \xi_2 \frac{Q_2(t)}{t^\alpha}\right)\right\}$ means the two-variable functional calculus.

Proof. For each $(n, m) \in \mathbb{Z}^2$, by the definition of $P(n, , m)$, the probability distribution is represented as $\mathbb{P}(X_t = (n, m)) = \|P(n, m)U^t\Psi^0\|^2_{\mathcal{K}_\Gamma}$. Hence, one has

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\left\langle \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \frac{X_t}{t^\alpha}\right\rangle_2\right)\right] \\
&= \int_{\mathbb{R}^2} \exp\left(i\left\langle \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \begin{pmatrix}x\\ y\end{pmatrix}\right\rangle_2\right) d\mathbb{P}\left(X_t/t^\alpha = (x, y)\right) \\
&= \sum_{(n,m)\in\mathbb{Z}^2} \exp\left\{i\left(\xi_1 \frac{n}{t^\alpha} + \xi_2 \frac{m}{t^\alpha}\right)\right\} \mathbb{P}(X_t = (n, m)) \\
&= \sum_{(n,m)\in\mathbb{Z}^2} \exp\left\{i\left(\xi_1 \frac{n}{t^\alpha} + \xi_2 \frac{m}{t^\alpha}\right)\right\} \|P(n, m)U^t\Psi^0\|^2_{\mathcal{K}_\Gamma} \\
&= \sum_{(n,m)\in\mathbb{Z}^2} \exp\left\{i\left(\xi_1 \frac{n}{t^\alpha} + \xi_2 \frac{m}{t^\alpha}\right)\right\} \left\langle U^t\Psi^0, E_1(n)E_2(m)U^t\Psi^0\right\rangle \\
&= \sum_{(n,m)\in\mathbb{Z}^2} \exp\left\{i\left(\xi_1 \frac{n}{t^\alpha} + \xi_2 \frac{m}{t^\alpha}\right)\right\} \left\langle U^t\Psi^0, E_1 \otimes E_2(\{(n, m)\})U^t\Psi^0\right\rangle \\
&= \left\langle U^t\Psi^0, \exp\left\{i\left(\xi_1 \frac{P_1}{t^\alpha} + \xi_2 \frac{P_2}{t^\alpha}\right)\right\} U^t\Psi^0\right\rangle \\
&= \left\langle \Psi^0, U^{-t}\exp\left\{i\left(\xi_1 \frac{P_1}{t^\alpha} + \xi_2 \frac{P_2}{t^\alpha}\right)\right\} U^t\Psi^0\right\rangle \\
&= \left\langle J_\Gamma\Psi^0, J_\Gamma U^{-t}\exp\left\{i\left(\xi_1 \frac{P_1}{t^\alpha} + \xi_2 \frac{P_2}{t^\alpha}\right)\right\} U^t J_\Gamma^{-1} J_\Gamma \Psi^0\right\rangle \\
&= \left\langle \Phi^0, U_0^{-t} J_\Gamma \exp\left\{i\left(\xi_1 \frac{P_1}{t^\alpha} + \xi_2 \frac{P_2}{t^\alpha}\right)\right\} J_\Gamma^{-1} U_0^t \Phi^0\right\rangle \\
&= \left\langle \Phi^0, U_0^{-t}\exp\left\{i\left(\xi_1 \frac{Q_1}{t^\alpha} + \xi_2 \frac{Q_2}{t^\alpha}\right)\right\} U_0^t \Phi^0\right\rangle \\
&= \left\langle \Phi^0, \exp\left\{i\left(\xi_1 \frac{Q_1(t)}{t^\alpha} + \xi_2 \frac{Q_2(t)}{t^\alpha}\right)\right\} \Phi^0\right\rangle.
\end{aligned}$$

This finishes the proof. $\square$

3.2 Asymptotic velocity operators

In this section, we will construct the asymptotic velocity operators by using the GJS method. First, we shall see a relation between the position and momentum operators. Let $L_2((0,2\pi)^2; d\theta d\varphi/4\pi^2)$ be the Hilbert space of all square integrable functions $f : (0,2\pi)^2 \to \mathbb{C}$ with norm

$$\|f\| = \left(\int_{(0,2\pi)^2} |f(\theta,\varphi)|^2 \frac{d\theta d\varphi}{4\pi^2} \right)^{1/2}.$$

For simplicity, we write $L_2((0,2\pi)^2) := L_2((0,2\pi)^2; d\theta d\varphi/4\pi^2)$. One denotes the functions $f_{n,m}(\theta,\varphi) := e^{in\theta}e^{im\varphi} \in L_2((0,2\pi)^2)$, then $\{f_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$ is a complete orthonormal basis of $L_2((0,2\pi)^2)$. Let $\mathscr{F} : \ell_2(\mathbb{Z}^2) \to L_2((0,2\pi)^2)$ be the Fourier transform which is the unitary operator defined as the unique extension of $\mathscr{F}(\delta_{n,m}) = f_{n,m}$, where $\{\delta_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$ is the canonical orthonormal basis for $\ell_2(\mathbb{Z}^2)$. The tensor product of $\mathscr{F}$ and the identity operator on $\mathbb{C}^{2d}$ is also denoted by $\mathscr{F}$, namely

$$\mathscr{F} : \ell_2(\mathbb{Z}^2) \otimes \mathbb{C}^{2d} (\cong \mathcal{K}_\Gamma) \to L_2((0,2\pi)^2) \otimes \mathbb{C}^{2d}; \delta_{n,m} \otimes \begin{pmatrix} x_1 \\ \vdots \\ x_{2d} \end{pmatrix} \mapsto f_{n,m} \otimes \begin{pmatrix} x_1 \\ \vdots \\ x_{2d} \end{pmatrix}.$$

In the same way as the case of "ℓ_2", we will use the following isomorphisms:

$$L_2((0,2\pi)^2) \otimes \mathbb{C}^{2d} \cong L_2((0,2\pi)^2; \mathbb{C}^{2d}) \cong L_2((0,2\pi)^2; \mathbb{C}^d \oplus \mathbb{C}^d) \cong \int_{(0,2\pi)^2}^{\oplus} \mathbb{C}^{2d} \frac{d\theta d\varphi}{4\pi^2}, \text{ etc.}$$

Now, we define the following dense subspace of $\mathcal{K}_\Gamma$:

$$\mathcal{K}_0 := \bigcup_{k=0}^{\infty} \{\Phi \in \mathcal{K}_\Gamma | \Phi(n,m) = 0, \quad \|(n,m)\|_1 \geq k\}.$$

Since the image of $\mathcal{K}_0$ under the Fourier transform is contained in $C^1((0,2\pi)^2; \mathbb{C}^{2d})$ which is the set of all C^1-class functions from $(0,2\pi)^2$ to $\mathbb{C}^{2d}$, the momentum operators $D_1, D_2 : \mathscr{F}(\mathcal{H}_0) \to L_2((0,2\pi)^2; \mathbb{C}^d \oplus \mathbb{C}^d)$

$$D_1 := \frac{1}{i}\frac{\partial}{\partial\theta} - \frac{1}{i}\frac{\partial}{\partial\varphi}, \quad D_2 := \frac{1}{i}\frac{\partial}{\partial\theta} + \frac{1}{i}\frac{\partial}{\partial\varphi} + P.$$

are well defined, where P is the projection from $L_2((0,2\pi)^2; \mathbb{C}^d \oplus \mathbb{C}^d)$ onto $L_2((0,2\pi)^2; 0^d \oplus \mathbb{C}^d)$.

Proposition 3.2.1. *The momentum operators D_1 and D_2 are closable, and those closures $\overline{D}_1, \overline{D}_2$ are as follows:*

$$\overline{D}_1 = \mathscr{F} Q_1 \mathscr{F}^{-1}, \ \overline{D}_2 = \mathscr{F} Q_2 \mathscr{F}^{-1}.$$

Proof. We will prove only the latter. First, we show that the subspace $\mathcal{K}_0$ is a core for Q_2. For $\Phi = {}^t(\Phi_1, \Phi_2) \in \mathcal{D}(Q_2)$, we put $\Phi^k(n,m) := \begin{cases} \Phi(n,m) & \text{if } \|(n,m)\|_1 < k \\ 0 & \text{if } \|(n,m)\|_1 \geq k \end{cases}$. Then $\Phi^k \in \mathcal{K}_0$, $\Phi^k \to \Phi$ and

$$
\begin{aligned}
&\|Q_2\Phi - Q_2\Phi^k\|^2 \\
&= \sum_{\|(n,m)\|_1 \geq k} \{|n+m|^2 \|\Phi_1(n,m)\|^2_{\mathbb{C}^d} + |n+m+1|^2 \|\Phi_2(n,m)\|^2_{\mathbb{C}^d}\} \longrightarrow 0 \ (as\ k \to \infty).
\end{aligned}
$$

Thus $\mathcal{G}(Q_2) = \overline{\mathcal{G}(Q_2|_{\mathcal{K}_0})}$, and so $\mathcal{K}_0$ is a core for Q_2. That is, the operator $Q_2|_{\mathcal{K}_0}$ is a closable operator and its closure is Q_2. Also, since every element of $\mathscr{F}(\mathcal{K}_0)$ is represented as a linear combination of the form

$$
f_{n,m} \otimes {}^t\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) = e^{in\theta} e^{im\varphi} \otimes {}^t\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) \in \mathscr{F}(\mathcal{K}_0),
$$

and

$$
\begin{aligned}
&e^{in\theta} e^{im\varphi} \otimes {}^t\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) \xmapsto{\mathscr{F}^{-1}} \delta_{n,m} \otimes {}^t\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) \\
&= {}^t\left(\delta_{n,m} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \delta_{n,m} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) \xmapsto{Q_2} {}^t\left((n+m)\delta_{n,m} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, (n+m+1)\delta_{n,m} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) \\
&= n\, {}^t\left(\delta_{n,m} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \delta_{n,m} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) + m\, {}^t\left(\delta_{n,m} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \delta_{n,m} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) + {}^t\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \delta_{n,m} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) \\
&\xmapsto{\mathscr{F}} n\, {}^t\left(f_{n,m} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, f_{n,m} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) + m\, {}^t\left(f_{n,m} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, f_{n,m} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) + {}^t\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, f_{n,m} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) \\
&= D_2\left(e^{in\theta} e^{im\varphi} \otimes {}^t\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right)\right),
\end{aligned}
$$

the equality $\mathscr{F} Q_2|_{\mathcal{K}_0} \mathscr{F}^{-1} = D_2$ holds on $\mathscr{F}(\mathcal{K}_0)$. These observations lead to the statement. $\square$

Next, we see that the Fourier transform of the time evolution operator associated with a quasi-uniform coin map is a multiplication operator.

Lemma 3.2.2. *Let $C_\bullet$ be a quasi-uniform coin map satisfying (2.3.2). Then the Fourier transform $\widehat{U}_\infty := \mathscr{F} U_\infty \mathscr{F}^{-1}$ of U_∞ is a decomposable operator whose decomposition is as follows:*

$$\widehat{U}_\infty = \int_{(0,2\pi)^2}^{\oplus} \widehat{U}_\infty(\theta,\varphi) \frac{d\theta d\varphi}{4\pi^2},$$

where $\widehat{U}_\infty(\theta,\varphi) :=$

$$\begin{cases}
\begin{pmatrix} O_3 & \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{pmatrix} C_\infty & \text{if } \Gamma = \Gamma_H, \\
\begin{pmatrix} O_4 & \begin{pmatrix} e^{i\varphi} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i(\theta+\varphi)} \end{pmatrix} \\ \begin{pmatrix} e^{-i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\varphi} & 0 & 0 \\ 0 & 0 & e^{-i(\theta+\varphi)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & O_4 \end{pmatrix} C_\infty & \text{if } \Gamma = \Gamma_S, \\
\begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\theta} \end{pmatrix} & \begin{pmatrix} e^{i\varphi} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i(\theta+\varphi)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} e^{-i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-i\varphi} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i(\theta+\varphi)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\theta} \end{pmatrix} \end{pmatrix} C_\infty & \text{if } \Gamma = \Gamma_T.
\end{cases} \tag{3.2.1}$$

Proof. It can be easily verified that $\mathscr{F} C_\infty \mathscr{F}^{-1} = \int_{(0,2\pi)^2}^{\oplus} C_\infty \frac{d\theta d\varphi}{4\pi^2}$.

Because it is similar, we shall consider only the case where $\Gamma = \Gamma_H$. Then one has

$$
\begin{aligned}
& f_{n,m} \otimes {}^t\!\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) \xmapsto{\mathscr{F}^{-1}} \delta_{n,m} \otimes {}^t\!\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) \\
&= {}^t\!\left(\delta_{n,m}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \delta_{n,m}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) \xmapsto{S_0} {}^t\!\left(\begin{pmatrix} y_1\delta_{n,m+1} \\ y_2\delta_{n+1,m} \\ y_3\delta_{n,m} \end{pmatrix}, \begin{pmatrix} x_1\delta_{n-1,m} \\ x_2\delta_{n,m-1} \\ x_3\delta_{n,m} \end{pmatrix}\right) \\
& \xmapsto{\mathscr{F}} {}^t\!\left(\begin{pmatrix} y_1 f_{n,m+1} \\ y_2 f_{n+1,m} \\ y_3 f_{n,m} \end{pmatrix}, \begin{pmatrix} x_1 f_{n-1,m} \\ x_2 f_{n,m-1} \\ x_3 f_{n,m} \end{pmatrix}\right) = {}^t\!\left(f_{n,m}\begin{pmatrix} y_1 f_{0,1} \\ y_2 f_{1,0} \\ y_3 \end{pmatrix}, f_{n,m}\begin{pmatrix} x_1 f_{-1,0} \\ x_2 f_{0,-1} \\ x_3 \end{pmatrix}\right) \\
&= {}^t\!\left(f_{n,m}\begin{pmatrix} f_{0,1} & 0 & 0 \\ 0 & f_{1,0} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, f_{n,m}\begin{pmatrix} f_{-1,0} & 0 & 0 \\ 0 & f_{0,-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) \\
&= \begin{pmatrix} O_3 & \begin{pmatrix} f_{0,1} & 0 & 0 \\ 0 & f_{1,0} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} f_{-1,0} & 0 & 0 \\ 0 & f_{0,-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{pmatrix} \times \left(f_{n,m} \otimes {}^t\!\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right)\right).
\end{aligned}
$$

Since for every $(\theta, \varphi) \in (0, 2\pi)^2$

$$
\begin{aligned}
& \begin{pmatrix} O_3 & \begin{pmatrix} f_{0,1}(\theta,\varphi) & 0 & 0 \\ 0 & f_{1,0}(\theta,\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} f_{-1,0}(\theta,\varphi) & 0 & 0 \\ 0 & f_{0,-1}(\theta,\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{pmatrix} \\
&= \begin{pmatrix} O_3 & \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{pmatrix},
\end{aligned}
$$

we obtain that

$$
\mathscr{F} S_0 \mathscr{F}^{-1} = \int_{(0,2\pi)^2}^{\oplus} \begin{pmatrix} O_3 & \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{pmatrix} \frac{d\theta d\varphi}{4\pi^2}
$$

by the denseness of $\mathscr{F}(\mathcal{K}_0)$. Thus $\widehat{U}_\infty = (\mathscr{F}S_0\mathscr{F}^{-1})(\mathscr{F}C_\infty\mathscr{F}^{-1}) = \int_{(0,2\pi)^2}^{\oplus} \widehat{U}_\infty(\theta,\varphi)\frac{d\theta d\varphi}{4\pi^2}$ holds. $\square$

Since the matrix $\widehat{U}_\infty(\theta,\varphi)$ is a unitary matrix for each $(\theta,\varphi) \in (0,2\pi)^2$, there exists an orthonormal basis $\{u_j(\theta,\varphi)\}_{j=1,\ldots,2d}$ of $\mathbb{C}^{2d}$ such that $\widehat{U}_\infty(\theta,\varphi)$ is represented as

$$\widehat{U}_\infty(\theta,\varphi) = \sum_{j=1}^{2d} \lambda_j(\theta,\varphi)|u_j(\theta,\varphi)\rangle\langle u_j(\theta,\varphi)|, \tag{3.2.2}$$

where $\lambda_j(\theta,\varphi)$ is an eigenvalue of $\widehat{U}_\infty(\theta,\varphi)$. If $\lambda_j(\theta,\varphi)$ is partial differentiable at $(\theta,\varphi) \in (0,2\pi)^2$, then we set

$$\lambda_j^\theta(\theta,\varphi) := \frac{1}{i}\frac{\partial\lambda_j(\theta,\varphi)}{\partial\theta}\frac{1}{\lambda_j(\theta,\varphi)} \quad \text{and} \quad \lambda_j^\varphi(\theta,\varphi) := \frac{1}{i}\frac{\partial\lambda_j(\theta,\varphi)}{\partial\varphi}\frac{1}{\lambda_j(\theta,\varphi)}.$$

We introduce the following assumption on $\lambda_j(\theta,\varphi)$ and $u_j(\theta,\varphi)$.

Assumption 3.2.3. For each $j = 1,\ldots,2d$,

1) the function $\lambda_j : (0,2\pi)^2 \to \mathbb{T}$ is C^2-class and its all partial derivatives up to order 2 are bounded on $(0,2\pi)^2$,

2) the function $u_j : (0,2\pi)^2 \to \mathbb{C}^{2d}$ is C^1-class and $\frac{\partial u_j}{\partial\theta}$, $\frac{\partial u_j}{\partial\varphi}$ are bounded on $(0,2\pi)^2$.

In addition,

3) there exists a permutation $\tau \in \mathfrak{S}_{2d}$ such that

$$\lim_{\varphi\to+0}\lambda_j^\sharp(\theta,\varphi) = \lim_{\varphi\to2\pi-0}\lambda_{\tau(j)}^\sharp(\theta,\varphi),\ \lim_{\theta\to+0}\lambda_j^\sharp(\theta,\varphi) = \lim_{\theta\to2\pi-0}\lambda_{\tau(j)}^\sharp(\theta,\varphi),$$
$$\lim_{\varphi\to+0}u_j(\theta,\varphi) = \lim_{\varphi\to2\pi-0}u_{\tau(j)}(\theta,\varphi),\ \lim_{\theta\to+0}u_j(\theta,\varphi) = \lim_{\theta\to2\pi-0}u_{\tau(j)}(\theta,\varphi)$$

for all $j = 1,\ldots,2d,\ 0 < \theta,\varphi < 2\pi$ and each symbol $\sharp = \theta,\varphi$.

Definition 3.2.4. Let us consider the situation in Lemma 3.2.2. Additionally, we suppose that the spectral decomposition of $\widehat{U}_\infty(\theta,\varphi)$ is given by (3.2.2). Then, we say that the unitary matrix $C_\infty \in U(2d)$ satisfies Assumption 3.2.3 if $\lambda_j(\theta,\varphi)$ and $u_j(\theta,\varphi)$ satisfy it.

Remark 3.2.5. By the differentiability of λ_j, for any $(\theta_0,\varphi_0) \in (0,2\pi)^2$, there exists an $\mathbb{R}$-valued differentiable function r_j^0 such that $\lambda_j(\theta,\varphi) = e^{ir_j^0(\theta,\varphi)}$ on a neighborhood of (θ_0,φ_0). Then, for each symbol $\sharp = \theta,\varphi$, we have $\lambda_j^\sharp(\theta_0,\varphi_0) = \frac{\partial r_j^0}{\partial\sharp}(\theta_0,\varphi_0)$, so $\lambda_j^\sharp$ is an $\mathbb{R}$-valued function.

Remark 3.2.6. Consider the case where $\Gamma = \Gamma_H$. If we write $A(\theta, \varphi) = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then $\widehat{U}_\infty(\theta, \varphi) = \begin{pmatrix} O_3 & A(\varphi, \theta)C_o \\ A(-\theta, -\varphi)C_e & O_3 \end{pmatrix}$. By [33, Theorem 3], one gets

$$\det \begin{pmatrix} \lambda E_3 & -A(\varphi, \theta)C_o \\ -A(-\theta, -\varphi)C_e & \lambda E_3 \end{pmatrix} = \det \left(\lambda^2 E_3 - A(\varphi, \theta)C_o A(-\theta, -\varphi)C_e\right)$$

and thus the eigenvalues of $\widehat{U}_\infty(\theta, \varphi)$ can always be obtained algebraically.

Example 3.2.7. Suppose that $\Gamma = \Gamma_H$. Let us consider a trivial case: $C_\bullet \equiv E_3$ on $\mathbb{Z}^2$. Then the matrix $\widehat{U}_\infty(\theta, \varphi)$ becomes

$$\begin{pmatrix} O_3 & \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{pmatrix}$$

and its eigenvalues and eigenvectors can be given as follows:

$$\begin{aligned}
&\lambda_1(\theta, \varphi) = e^{\frac{i}{2}(-\theta+\varphi)},\ \lambda_2(\theta, \varphi) = -e^{\frac{i}{2}(-\theta+\varphi)},\ \lambda_3(\theta, \varphi) = e^{\frac{i}{2}(\theta-\varphi)},\\
&\lambda_4(\theta, \varphi) = -e^{\frac{i}{2}(\theta-\varphi)},\ \lambda_5(\theta, \varphi) = 1,\ \lambda_6(\theta, \varphi) = -1,\\
&u_1(\theta, \varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2}(\theta+\varphi)} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ u_2(\theta, \varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{\frac{i}{2}(\theta+\varphi)} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ u_3(\theta, \varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{\frac{i}{2}(\theta+\varphi)} \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\\
&u_4(\theta, \varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -e^{\frac{i}{2}(\theta+\varphi)} \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ u_5(\theta, \varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ u_6(\theta, \varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

Although $u_j(\theta, \varphi)$ is not "2π-periodic" for $j = 1, 2, 3, 4$, if we put

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 5 & 6 \end{pmatrix} \in \mathfrak{S}_6,$$

then Assumption 3.2.3 is fulfilled. Hence $E_6 = E_3 \oplus E_3$ satisfies Assumption 3.2.3.

Example 3.2.8. Suppose that $\Gamma = \Gamma_H$ again. As a non-trivial case, one considers the case where $C_\bullet = \begin{cases} C_e \text{ on } \mathbb{Z}_e^2 \\ C_o \text{ on } \mathbb{Z}_o^2 \end{cases}$, here we set

$$C_e := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_o := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the unitary matrix $\widehat{U}_\infty(\theta, \varphi)$ is given by

$$\widehat{U}_\infty(\theta,\varphi) = \begin{pmatrix} O_3 & \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & 0 & e^{i\theta} \\ 0 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & e^{-i\theta} & 0 \\ e^{-i\varphi} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{pmatrix},$$

and one can choose $\lambda_j(\theta,\varphi)$ and $u_j(\theta,\varphi)$ as follows:

$$\lambda_j(\theta,\varphi) \equiv e^{i\frac{(j-1)\pi}{3}}, \quad u_j(\theta,\varphi) = \frac{1}{\sqrt{6}} \begin{pmatrix} e^{i(\varphi+(j-1)\pi)} \\ e^{i\left(\theta+\frac{5(j-1)\pi}{3}\right)} \\ e^{i\frac{(j-1)\pi}{3}} \\ e^{i\frac{4(j-1)\pi}{3}} \\ e^{i\frac{2(j-1)\pi}{3}} \\ 1 \end{pmatrix}, \quad j = 1, \ldots, 6.$$

It follows that $C_e \oplus C_o$ satisfies Assumption 3.2.3 as τ is the identity permutation.

Since $\lambda_j^\theta(\theta,\varphi)$ and $\lambda_j^\varphi(\theta,\varphi)$ are $\mathbb{R}$-valued bounded functions on $(0,2\pi)^2$ by Assumption 3.2.3, we can define the following two operators:

Definition 3.2.9. If the unitary matrix C_∞ satisfies Assumption 3.2.3, one can define two bounded self-adjoint operators V_1^Γ, V_2^Γ on $\mathcal{K}_\Gamma$ by

$$V_1^\Gamma := \mathscr{F}^{-1}\left(\int_{(0,2\pi)^2}^{\oplus} \sum_{j=1}^{2d} \left(\lambda_j^\theta(\theta,\varphi) - \lambda_j^\varphi(\theta,\varphi)\right) |u_j(\theta,\varphi)\rangle\langle u_j(\theta,\varphi)| \frac{d\theta d\varphi}{4\pi^2}\right)\mathscr{F},$$

$$V_2^\Gamma := \mathscr{F}^{-1}\left(\int_{(0,2\pi)^2}^{\oplus} \sum_{j=1}^{2d} \left(\lambda_j^\theta(\theta,\varphi) + \lambda_j^\varphi(\theta,\varphi)\right) |u_j(\theta,\varphi)\rangle\langle u_j(\theta,\varphi)| \frac{d\theta d\varphi}{4\pi^2}\right)\mathscr{F},$$

which are called the *asymptotic velocity operators*. If there would be no danger of confusions, then we may write simply V_j.

Since $\{u_j(\theta, \varphi)\}_{j=1,\ldots,2d}$ is an orthonormal basis of $\mathbb{C}^{2d}$ for each $(\theta, \varphi) \in (0, 2\pi)^2$, the asymptotic velocity operators V_1, V_2 are commuting.

We denote $\mathcal{D}$ as the set of vectors $\Phi \in \mathcal{K}_\Gamma$ satisfying the following two properties:

- $\widehat{\Phi} := \mathscr{F}\Phi : (0, 2\pi)^2 \to \mathbb{C}^{2d}$ is a C^1-class bounded function and its partial derivatives are also bounded,
- $\lim_{\varphi \to +0} \widehat{\Phi}(\theta, \varphi) = \lim_{\varphi \to 2\pi - 0} \widehat{\Phi}(\theta, \varphi)$ holds for any $\theta \in (0, 2\pi)$, and $\lim_{\theta \to +0} \widehat{\Phi}(\theta, \varphi) = \lim_{\theta \to 2\pi - 0} \widehat{\Phi}(\theta, \varphi)$ holds for all $\varphi \in (0, 2\pi)$.

Note that $\mathcal{D}$ is a dense subspace of $\mathcal{K}_\Gamma$ because $\mathcal{K}_0 \subset \mathcal{D}$. To study properties of the asymptotic velocity operators, we prepare two lemmas for $\mathcal{D}$.

Lemma 3.2.10. *Let $\mathcal{D}$ be as above. Then the following properties hold:*

1) $\mathscr{F}(\mathcal{D}) \subset \mathcal{D}(\overline{D}_j)$ $(j = 1, 2)$, and

$$\overline{D}_1 = \frac{1}{i}\frac{\partial}{\partial \theta} - \frac{1}{i}\frac{\partial}{\partial \varphi}, \quad \overline{D}_2 = \frac{1}{i}\frac{\partial}{\partial \theta} + \frac{1}{i}\frac{\partial}{\partial \varphi} + P,$$

on $\mathscr{F}(\mathcal{D})$.

2) $\mathcal{D}$ is U_∞-invariant (The matrix C_∞ does not necessary satisfy Assumption 3.2.3).

Proof. Let $\{e_j\}_{1 \le j \le 2d}$ be the canonical basis of $\mathbb{C}^{2d}$ and let $f^j_{n,m} := f_{n,m} \otimes e_j$. Then $\{f^j_{n,m}\}_{1 \le j \le 2d, (n,m) \in \mathbb{Z}^2}$ is a complete orthonormal basis of $L_2((0, 2\pi)^2; \mathbb{C}^{2d})$. To prove the assertion *1)*, take an element Φ of $\mathcal{D}$ arbitrarily. Since $\widehat{\Phi}$ and its partial derivatives of order 1 are elements of $L_2((0, 2\pi)^2; \mathbb{C}^{2d})$, they can be expressed as

$$\widehat{\Phi} = \lim_{k \to \infty} \sum_{1 \le j \le 2d, (n,m) \in B_k} \langle f^j_{n,m}, \widehat{\Phi} \rangle f^j_{n,m},$$
$$\frac{\partial \widehat{\Phi}}{\partial \theta} = \lim_{k \to \infty} \sum_{1 \le j \le 2d, (n,m) \in B_k} \left\langle f^j_{n,m}, \frac{\partial \widehat{\Phi}}{\partial \theta} \right\rangle f^j_{n,m},$$
$$\frac{\partial \widehat{\Phi}}{\partial \varphi} = \lim_{k \to \infty} \sum_{1 \le j \le 2d, (n,m) \in B_k} \left\langle f^j_{n,m}, \frac{\partial \widehat{\Phi}}{\partial \varphi} \right\rangle f^j_{n,m},$$

where B_k is the closed ball in $\mathbb{Z}^2$ of center the origin and radius k with respect to ℓ_1-norm. By the definition of $\mathcal{D}$ and integration by parts, one gets

$$\left\langle f^j_{n,m}, \frac{\partial \widehat{\Phi}}{\partial \theta} \right\rangle = in \left\langle f^j_{n,m}, \widehat{\Phi} \right\rangle \quad \text{and} \quad \left\langle f^j_{n,m}, \frac{\partial \widehat{\Phi}}{\partial \varphi} \right\rangle = im \left\langle f^j_{n,m}, \widehat{\Phi} \right\rangle.$$

(Thus the Fourier series of $\widehat{\Phi}$ is termwise differentiable.) Therefore, for example,

$$D_2(\widehat{\Phi}_k) \to \left(\frac{1}{i}\frac{\partial}{\partial\theta} + \frac{1}{i}\frac{\partial}{\partial\varphi} + P\right)\widehat{\Phi} \text{ (as } k \to \infty) \text{ in } L_2((0,2\pi)^2;\mathbb{C}^{2d}),$$

where $\widehat{\Phi}_k := \sum_{1\le j\le 2d,(n,m)\in B_k}\langle f^j_{n,m},\widehat{\Phi}\rangle f^j_{n,m} \in \mathscr{F}(\mathcal{K}_0) \subset \mathcal{D}(\overline{D_2})$. By closedness of $\overline{D}_2$, $\widehat{\Phi} \in \mathcal{D}(\overline{D_2})$ and $\overline{D}_2\widehat{\Phi} = \left(\frac{1}{i}\frac{\partial}{\partial\theta} + \frac{1}{i}\frac{\partial}{\partial\varphi} + P\right)\widehat{\Phi}$, as stated in *1)*.

For the inclusion *2)*, take any $\Phi \in \mathcal{D}$. Since it is similar, we will prove only the case where $\Gamma = \Gamma_H$. Lemma 3.2.2 gives the equation $\widehat{U_\infty\Phi}(\theta,\varphi) = \widehat{U}_\infty(\theta,\varphi)\widehat{\Phi}(\theta,\varphi)$ for $(\theta,\varphi) \in (0,2\pi)^2$, this means that $\widehat{U_\infty\Phi}$ is C^1-class bounded function on $(0,2\pi)^2$ and satisfies the second condition of the definition of $\mathcal{D}$. Moreover, by the following estimate

$$\begin{aligned}
\left\|\frac{\partial\widehat{U_\infty\Phi}}{\partial\theta}(\theta,\varphi)\right\|_{\mathbb{C}^6} &\le \left\|\frac{\partial\widehat{U}_\infty(\theta,\varphi)}{\partial\theta}\widehat{\Phi}(\theta,\varphi)\right\|_{\mathbb{C}^6} + \left\|\widehat{U}_\infty(\theta,\varphi)\frac{\partial\widehat{\Phi}}{\partial\theta}(\theta,\varphi)\right\|_{\mathbb{C}^6} \\
&= \left\|\frac{\partial\widehat{U}_\infty(\theta,\varphi)}{\partial\theta}\widehat{\Phi}(\theta,\varphi)\right\|_{\mathbb{C}^6} + \left\|\frac{\partial\widehat{\Phi}}{\partial\theta}(\theta,\varphi)\right\|_{\mathbb{C}^6} \\
&= \left\|\begin{pmatrix} O_3 & \begin{pmatrix} 0 & 0 & 0 \\ 0 & ie^{i\theta} & 0 \\ 0 & 0 & 0\end{pmatrix} \\ \begin{pmatrix} -ie^{-i\theta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} & O_3 \end{pmatrix} C_\infty\widehat{\Phi}(\theta,\varphi)\right\|_{\mathbb{C}^6} + \left\|\frac{\partial\widehat{\Phi}}{\partial\theta}(\theta,\varphi)\right\|_{\mathbb{C}^6} \\
&\le 2\left\|\widehat{\Phi}(\theta,\varphi)\right\|_{\mathbb{C}^6} + \left\|\frac{\partial\widehat{\Phi}}{\partial\theta}(\theta,\varphi)\right\|_{\mathbb{C}^6},
\end{aligned}$$

one gets

$$\sup_{(\theta,\varphi)\in(0,2\pi)^2}\left\|\frac{\partial\widehat{U_\infty\Phi}}{\partial\theta}(\theta,\varphi)\right\|_{\mathbb{C}^6} < \infty.$$

Similarly, the function $\frac{\partial\widehat{U_\infty\Phi}}{\partial\varphi}$ is bounded, and hence $U_\infty\Phi \in \mathcal{D}$. □

Lemma 3.2.11. *Suppose that C_∞ satisfies Assumption* 3.2.3*. Then for every $z \in \mathbb{C}\setminus\mathbb{R}$, we have*

$$(V_1 - z)^{-1}\mathcal{K}_0 \subset \mathcal{D} \quad \text{and} \quad (V_2 - z)^{-1}\mathcal{K}_0 \subset \mathcal{D}.$$

Proof. We will check only the second inclusion of the statement. Since the equality

$$\mathrm{id}_{\mathbb{C}^{2d}} = \sum_{j=1}^{2d} |u_j(\theta,\varphi)\,\rangle\langle\, u_j(\theta,\varphi)|$$

holds, it follows that

$$(V_2 - z)^{-1}$$

$$= \left[\mathscr{F}^{-1} \left(\int_{(0,2\pi)^2}^{\oplus} \sum_{j=1}^{2d} \left\{ \lambda_j^{\theta}(\theta,\varphi) + \lambda_j^{\varphi}(\theta,\varphi) - z \right\} |u_j(\theta,\varphi)\rangle \langle u_j(\theta,\varphi)| \frac{d\theta d\varphi}{4\pi^2} \right) \mathscr{F} \right]^{-1}$$

$$= \mathscr{F}^{-1} \left(\int_{(0,2\pi)^2}^{\oplus} \sum_{j=1}^{2d} \left\{ \lambda_j^{\theta}(\theta,\varphi) + \lambda_j^{\varphi}(\theta,\varphi) - z \right\}^{-1} |u_j(\theta,\varphi)\rangle \langle u_j(\theta,\varphi)| \frac{d\theta d\varphi}{4\pi^2} \right) \mathscr{F}.$$

Therefore one obtains that

$$\begin{aligned}
&(\mathscr{F}\,(V_2 - z)^{-1}\,\Phi)(\theta,\varphi) \\
&= \sum_{j=1}^{2d} \left\{ \lambda_j^{\theta}(\theta,\varphi) + \lambda_j^{\varphi}(\theta,\varphi) - z \right\}^{-1} \left\langle u_j(\theta,\varphi), \widehat{\Phi}(\theta,\varphi) \right\rangle |u_j(\theta,\varphi)\rangle
\end{aligned}$$

for every $\Phi \in \mathcal{K}_0$ and $(\theta,\varphi) \in (0,2\pi)^2$. From this and Assumption 3.2.3, it follows $(V_2 - z)^{-1}\,\Phi \in \mathcal{D}$. □

In particular, when the time evolution operator is associated with a quasi-uniform coin map, we write $Q_j^{\infty}(t) := U_{\infty}^{-t} Q_j U_{\infty}^{t}$ $(j = 1,2)$ for $t \in \mathbb{N}_0$. Since $Q_j^{\infty}(t)/t$ can be interpreted as a "velocity", the following lemma tells us why the operator V_j is called the asymptotic velocity operator.

Lemma 3.2.12. *Let U_{∞} be a time evolution operator associated with a quasi-uniform coin map satisfying* (2.3.2)*. Suppose that the unitary matrix $C_{\infty} = C_e \oplus C_o$ satisfies Assumption* 3.2.3*. Then for each $\xi_1, \xi_2 \in \mathbb{R}$, the unitary operators $\exp\left(i\xi_1 \frac{Q_1^{\infty}(t)}{t}\right)$ and $\exp\left(i\xi_2 \frac{Q_2^{\infty}(t)}{t}\right)$ are strongly convergent to $\exp(i\xi_1 V_1)$ and $\exp(i\xi_2 V_2)$ as t goes to ∞, respectively. Also, for any $\alpha > 1$, both $\exp\left(i\xi_1 \frac{Q_1^{\infty}(t)}{t^{\alpha}}\right)$ and $\exp\left(i\xi_2 \frac{Q_2^{\infty}(t)}{t^{\alpha}}\right)$ are strongly convergent to the identity operator I.*

Proof. We will show only the case where $j = 2$, since the case of $j = 1$ is also proven by the similar manner. We note that the convergence of the statement is equivalent to

$$\left(\frac{Q_j^{\infty}(t)}{t} - z \right)^{-1} \xrightarrow{t \to \infty} (V_j - z)^{-1} \quad \text{(SOT)}$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$ (see e.g., [9, Proposition 10.1.8]). To prove it, take any $z \in \mathbb{C} \setminus \mathbb{R}$, $\Phi \in \mathcal{K}_{\Gamma}$ and $\varepsilon > 0$. Then there exists a vector $\Phi_{\varepsilon} \in \mathcal{K}_0$ such that $\|\Phi - \Phi_{\varepsilon}\| < \varepsilon$. Since by Lemma 3.2.10 one has $\mathcal{D} \subset \mathscr{F}^{-1}(\mathcal{D}(\overline{D}_2)) = \mathcal{D}(Q_2)$, it follows that $U_{\infty}^{t}(\mathcal{D}) \subset \mathcal{D} \subset \mathcal{D}(Q_2)$. Thus $\mathcal{D} \subset U_{\infty}^{-t}(\mathcal{D}(Q_2)) = \mathcal{D}(Q_2^{\infty}(t))$. Also, by Lemma 3.2.11, $(V_2 - z)^{-1}\mathcal{K}_0 \subset \mathcal{D}$. Therefore, we have

$$\begin{aligned}
&\left\| \left(\frac{Q_2^{\infty}(t)}{t} - z \right)^{-1} \Phi - (V_2 - z)^{-1}\Phi \right\| \\
&\leq \left\| \left(\frac{Q_2^{\infty}(t)}{t} - z \right)^{-1} (\Phi - \Phi_{\varepsilon}) \right\| + \left\| \left(\frac{Q_2^{\infty}(t)}{t} - z \right)^{-1} \Phi_{\varepsilon} - (V_2 - z)^{-1}\Phi_{\varepsilon} \right\| + \left\| (V_2 - z)^{-1}(\Phi_{\varepsilon} - \Phi) \right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\varepsilon}{|\Im(z)|} + \left\| \left(\frac{Q_2^\infty(t)}{t} - z \right)^{-1} \Phi_\varepsilon - (V_2 - z)^{-1}\Phi_\varepsilon \right\| + \frac{\varepsilon}{|\Im(z)|} \\
&= \frac{2\varepsilon}{|\Im(z)|} + \left\| \left(\frac{Q_2^\infty(t)}{t} - z \right)^{-1} \left(\frac{Q_2^\infty(t)}{t} - V_2 \right) (V_2 - z)^{-1}\Phi_\varepsilon \right\| \\
&\leq \frac{2\varepsilon}{|\Im(z)|} + \frac{\varepsilon}{|\Im(z)|} \left\| \left(\frac{Q_2^\infty(t)}{t} - V_2 \right) (V_2 - z)^{-1} \Phi_\varepsilon \right\|,
\end{aligned}
$$

where the equality is implied by the second resolvent equation, and $\Im(z)$ is the imaginary part of z. Thus, since $(V_2 - z)^{-1}\mathcal{K}_0 \subset \mathcal{D}$ again, it suffices to show that

$$
\left\| \left(\frac{Q_2^\infty(t)}{t} - V_2 \right) \Phi \right\| \to 0,
$$

for all $\Phi \in \mathcal{D}$.

For this proof, take $\Phi \in \mathcal{D}$ and $(\theta, \varphi) \in (0, 2\pi)^2$ arbitrarily. Note that

$$
\mathscr{F} Q_2^\infty(t) = \mathscr{F} U_\infty^{-t} Q_2 U_\infty^t = \widehat{U}_\infty^{-t} \mathscr{F} Q_2 U_\infty^t = \widehat{U}_\infty^{-t} \mathscr{F} Q_2 \mathscr{F}^{-1} \mathscr{F} U_\infty^t = \widehat{U}_\infty^{-t} \overline{D}_2 \widehat{U}_\infty^t \mathscr{F}
$$

hold on $\mathcal{D}(Q_2^\infty(t))$ by Lemma 3.2.1. Also $\mathcal{D} \subset \mathcal{D}(Q_2^\infty(t))$ holds. Hence, by Lemma 3.2.10, one has

$$
\begin{aligned}
&(\mathscr{F} Q_2^\infty(t)\Phi)(\theta, \varphi) = \left(\widehat{U}_\infty^{-t} \left(\frac{1}{i}\frac{\partial}{\partial\theta} + \frac{1}{i}\frac{\partial}{\partial\varphi} + P \right) \widehat{U}_\infty^t \widehat{\Phi} \right)(\theta, \varphi) \\
&= \widehat{U}_\infty(\theta, \varphi)^{-t} \left(\frac{1}{i}\frac{\partial}{\partial\theta} + \frac{1}{i}\frac{\partial}{\partial\varphi} + P \right) U_\infty(\theta, \varphi)^t \widehat{\Phi}(\theta, \varphi) \\
&= \left(\sum_{j=1}^{2d} \lambda_j(\theta, \varphi)^{-t} \, | u_j(\theta, \varphi) \rangle\langle u_j(\theta, \varphi) | \right) \\
&\quad \times \left(\frac{1}{i}\frac{\partial}{\partial\theta} + \frac{1}{i}\frac{\partial}{\partial\varphi} + P \right) \left(\sum_{j=1}^{2d} \lambda_j(\theta, \varphi)^t \left\langle u_j(\theta, \varphi), \widehat{\Phi}(\theta, \varphi) \right\rangle | u_j(\theta, \varphi) \rangle \right) \\
&= \left(\sum_{j=1}^{2d} \lambda_j(\theta, \varphi)^{-t} \, | u_j(\theta, \varphi) \rangle\langle u_j(\theta, \varphi) | \right) \\
&\quad \times \left\{ \sum_{j=1}^{2d} \frac{t}{i} \lambda_j(\theta, \varphi)^{t-1} \left(\frac{\partial \lambda_j}{\partial\theta}(\theta, \varphi) + \frac{\partial \lambda_j}{\partial\varphi}(\theta, \varphi) \right) \left\langle u_j(\theta, \varphi), \widehat{\Phi}(\theta, \varphi) \right\rangle | u_j(\theta, \varphi) \rangle + o(t) \right\} \\
&= \left(\sum_{j=1}^{2d} t \left(\lambda_j^\theta(\theta, \varphi) + \lambda_j^\varphi(\theta, \varphi) \right) \left\langle u_j(\theta, \varphi), \widehat{\Phi}(\theta, \varphi) \right\rangle | u_j(\theta, \varphi) \rangle + o(t) \right) \\
&= t(\mathscr{F} V_2 \Phi)(\theta, \varphi) + o(t).
\end{aligned}
$$

Here we remark that by Assumption 3.2.3, the Landau symbol $o(t) \in \mathbb{C}^{2d}$ above divided by t is uniformly convergent to 0 as $t \to \infty$ with respect to $(\theta, \varphi) \in (0, 2\pi)^2$. From this, it follows that

$$\left\| \left(\frac{Q_2^{\infty}(t)}{t} - V_2 \right) \Phi \right\| = \left\| \mathscr{F} \left(\frac{Q_2^{\infty}(t)}{t} - V_2 \right) \Phi \right\| \longrightarrow 0 \quad as \quad t \to \infty.$$

By similar arguments, one can prove that for each $\alpha > 1$, we have

$$\left(\frac{Q_2^{\infty}(t)}{t^{\alpha}} - z \right)^{-1} \overset{t\to\infty}{\longrightarrow} (0 - z)^{-1} \quad \text{(SOT)},$$

where 0 is the zero operator. Hence, $\exp\left(i\xi_2 \frac{Q_2^{\infty}(t)}{t^{\alpha}} \right)$ is strongly convergent to $\exp(i\xi_2 0) = I$. This finishes the proof. □

By Lemma 3.2.12 and Proposition 1.1.1, we obtain the following result.

Corollary 3.2.13. *Under the same assumption as in Lemma* 3.2.12*, we have for any* ${}^t(\xi_1 \ \xi_2) \in \mathbb{R}^2$,

$$\exp\left\{ i \left(\xi_1 \frac{Q_1^{\infty}(t)}{t^{\alpha}} + \xi_2 \frac{Q_2^{\infty}(t)}{t^{\alpha}} \right) \right\} \overset{t\to\infty}{\longrightarrow} \begin{cases} \exp i\left(\xi_1 V_1 + \xi_2 V_2\right) & \text{if} \quad \alpha = 1, \\ I & \text{if} \quad \alpha > 1, \end{cases}$$

in the sense of the strongly operator topology.

3.3 Scattering theory

In this section, we consider the time evolution operator U_0 which is associated to an anisotropic coin map. According to Lemma 3.1.2, the characteristic function of the random variable X_t/t is expressed by using the unitary operator $\exp\left\{ i \left(\xi_1 \frac{Q_1(t)}{t} + \xi_2 \frac{Q_2(t)}{t} \right) \right\}$. This operator has distinct properties on $\mathcal{K}_{\mathrm{pp}}(U_0)$, $\mathcal{K}_{\mathrm{ac}}(U_0)$, and $\mathcal{K}_{\mathrm{sc}}(U_0)$, respectively. In particular, we investigate the asymptotic behavior of $\exp\left\{ i \left(\xi_1 \frac{Q_1(t)}{t} + \xi_2 \frac{Q_2(t)}{t} \right) \right\}$ on $\mathcal{K}_{\mathrm{ac}}(U_0)$ in this section. Scattering theory is useful for this, as in the case of the study of Schrödinger operators.

In order to construct the wave operators with respect to U_0 and U_∞ which is the auxiliary time evolution operator, we introduce the following condition.

Definition 3.3.1. A coin map $C_\bullet : \mathbb{Z}^2 \to U(d)$ is said to be of *short range* if there exist $C_e, C_o \in U(d)$, $\kappa > 0$, and $\varepsilon > 0$ such that

$$\begin{aligned} \|C_{(n,m)} - C_e\|_{M_d(\mathbb{C})} &\le \kappa(1 + \|(n,m)\|_1)^{-2-\varepsilon} \text{ for all } (n,m) \in \mathbb{Z}_e^2, \\ \|C_{(n,m)} - C_o\|_{M_d(\mathbb{C})} &\le \kappa(1 + \|(n,m)\|_1)^{-2-\varepsilon} \text{ for all } (n,m) \in \mathbb{Z}_o^2. \end{aligned} \tag{3.3.1}$$

Of course, the short-range condition is stronger than the anisotropic one.

Remark 3.3.2. We suppose that $C_\bullet$ is a short range coin map satisfying (3.3.1) and put $C_\infty = C_e \oplus C_o$. Since

$$\frac{1}{2}\|(n,m)\|_1 \le \|\phi_\star(n,m)\|_1 \tag{3.3.2}$$

for $\star = e, o, (n,m) \in \mathbb{Z}^2$, for all $(n,m) \in \mathbb{Z}^2$, we have

$$\begin{aligned}\|C_{\phi_e(n,m)} \oplus C_{\phi_o(n,m)} - C_\infty\|_{M_d(\mathbb{C})} &= \max\{\|C_{\phi_e(n,m)} - C_e\|_{M_d(\mathbb{C})}, \|C_{\phi_o(n,m)} - C_o\|_{M_d(\mathbb{C})}\} \\ &\le \kappa'(1+\|(n,m)\|_1)^{-2-\varepsilon},\end{aligned}$$

for some positive number κ'.

Lemma 3.3.3. *If U_0 is a time evolution operator associated with a short-range coin map, then the difference $U_0 - U_\infty$ is trace class.*

Proof. Assume that a coin map $C_\bullet$ satisfies the condition (3.3.1) and set $C_\infty := C_e \oplus C_o$. Let $T(n,m) := C_{\phi_e(n,m)} \oplus C_{\phi_o(n,m)} - C_\infty \in M_{2d}(\mathbb{C})$ for any $(n,m) \in \mathbb{Z}^2$, then we have

$$\begin{aligned}(U_0 - U_\infty)^*(U_0 - U_\infty) &= (C_0 - C_\infty)^*(C_0 - C_\infty) \\ &= \left(\bigoplus_{(n,m)\in\mathbb{Z}^2} T(n,m)\right)^* \left(\bigoplus_{(n,m)\in\mathbb{Z}^2} T(n,m)\right) \\ &= \bigoplus_{(n,m)\in\mathbb{Z}^2} T(n,m)^* T(n,m).\end{aligned}$$

Also, since $\{T(n,m)^*T(n,m)\}^{\frac{1}{2}}$ is a positive Hermitian matrix, there exist non-negative real numbers $r_1(n,m), ..., r_{2d}(n,m)$ and an orthonormal basis $\{w_1(n,m), ..., w_{2d}(n,m)\}$ of $\mathbb{C}^{2d}$ such that

$$\{T(n,m)^*T(n,m)\}^{\frac{1}{2}} = \sum_{j=1}^{2d} r_j(n,m)\, |w_j(n,m)\,\rangle\langle\, w_j(n,m)|\,.$$

We define a function $e^j_{n,m} : \mathbb{Z}^2 \to \mathbb{C}^{2d}$ by $e^j_{n,m}(k,l) := \begin{cases} w_j(n,m) & \text{if } (k,l) = (n,m) \\ 0 & \text{otherwise} \end{cases}$, then $\{e^j_{n,m}\}_{1\le j\le 2d, (n,m)\in\mathbb{Z}^2}$ is an orthonormal basis of $\mathcal{K}_\Gamma = \ell_2(\mathbb{Z}^2; \mathbb{C}^{2d})$ and

$$\bigoplus_{(n,m)\in\mathbb{Z}^2} \{T(n,m)^*T(n,m)\}^{\frac{1}{2}} = \sum_{(n,m)\in\mathbb{Z}^2} \sum_{j=1}^{2d} r_j(n,m)\, |e^j_{n,m}\,\rangle\langle\, e^j_{n,m}|$$

holds. Because $C_\bullet$ is of short range, we obtain that

$$\mathrm{Tr}|U_0 - U_\infty| = \mathrm{Tr}\left(\{(U_0 - U_\infty)^*(U_0 - U_\infty)\}^{\frac{1}{2}}\right)$$

$$
\begin{aligned}
&= \mathrm{Tr}\left(\left\{\bigoplus_{(n,m)\in\mathbb{Z}^2} T(n,m)^*T(n,m)\right\}^{\frac{1}{2}}\right)\\
&= \mathrm{Tr}\left(\bigoplus_{(n,m)\in\mathbb{Z}^2} \{T(n,m)^*T(n,m)\}^{\frac{1}{2}}\right)\\
&= \mathrm{Tr}\left(\sum_{(n,m)\in\mathbb{Z}^2}\sum_{j=1}^{2d} r_j(n,m)\,|e^j_{n,m}\rangle\langle e^j_{n,m}|\right)\\
&= \sum_{(n,m)\in\mathbb{Z}^2}\sum_{j=1}^{2d} r_j(n,m)\\
&\le 2d \sum_{(n,m)\in\mathbb{Z}^2} \max_{1\le j\le 2d} r_j(n,m)\\
&= 2d \sum_{(n,m)\in\mathbb{Z}^2} \left\|\{T(n,m)^*T(n,m)\}^{\frac{1}{2}}\right\|\\
&= 2d \sum_{(n,m)\in\mathbb{Z}^2} \|T(n,m)\|\\
&= 2d \sum_{(n,m)\in\mathbb{Z}^2} \left\|C_{\phi_e(n,m)}\oplus C_{\phi_o(n,m)} - C_\infty\right\|\\
&\le 2d\kappa' \sum_{(n,m)\in\mathbb{Z}^2} (1+\|(n,m)\|_1)^{-2-\varepsilon}\\
&= 2d\kappa' \sum_{(n,m)\in\mathbb{Z}^2} (1+|n|+|m|)^{-2-\varepsilon} < \infty,
\end{aligned}
$$

which proves the lemma. □

From the next remark we can conclude that the order -2 is the borderline between whether $U_0 - U_\infty$ is trace class or not.

Remark 3.3.4. Let $C_\bullet$ be a coin map with $C_{(n,m)} = e^{i(1+\|(n,m)\|_1)^{-2}}E_d$. The matrix similar to it is treated in [38]. In this case, we have

$$
\frac{1}{2}(1+\|(n,m)\|_1)^{-2} \le \|C_{(n,m)} - E_d\|_{M_d(\mathbb{C})} \le (1+\|(n,m)\|_1)^{-2},
$$

for any $(n,m)\in\mathbb{Z}^2$, so $C_\bullet$ does not satisfy (3.3.1). If we use the notations in the proof of Lemma 3.3.3, then

$$
\{T(n,m)^*T(n,m)\}^{\frac{1}{2}} = \sqrt{2-2\cos(1+\|\phi_e(n,m)\|_1)^{-2}}E_d \oplus \sqrt{2-2\cos(1+\|\phi_o(n,m)\|_1)^{-2}}E_d
$$

hold. Thus, by using $\mathbb{Z}^2 = \mathbb{Z}_e^2 \sqcup \mathbb{Z}_o^2$ and the inequality $1 - \cos x \geq \frac{x^2}{\pi}$ $(0 \leq x \leq \frac{\pi}{2})$, one has

$$\begin{aligned}
\mathrm{Tr}|U_0 - U_\infty| &= \sum_{(n,m)\in\mathbb{Z}^2} \sum_{j=1}^{2d} r_j(n,m) \\
&= d \sum_{(n,m)\in\mathbb{Z}^2} \left\{ \sqrt{2 - 2\cos(1 + \|\phi_e(n,m)\|_1)^{-2}} + \sqrt{2 - 2\cos(1 + \|\phi_o(n,m)\|_1)^{-2}} \right\} \\
&= d \sum_{(n,m)\in\mathbb{Z}^2} \sqrt{2 - 2\cos(1 + \|(n,m)\|_1)^{-2}} \\
&\geq \sqrt{2} d \sum_{(n,m)\in\mathbb{Z}^2} \sqrt{\frac{1}{\pi}(1 + \|(n,m)\|_1)^{-4}} \\
&= \frac{\sqrt{2}d}{\sqrt{\pi}} \sum_{(n,m)\in\mathbb{Z}^2} \frac{1}{(1 + |n| + |m|)^2} = \infty,
\end{aligned}$$

which means that $U_0 - U_\infty$ is not trace class.

Lemma 3.3.5. *Let U_0 be a time evolution operator associated with a short-range coin map. Then the following wave operators*

$$\Omega_\pm := \Omega_\pm^\Gamma := s\text{-}\lim_{t\to\pm\infty} U_0^{-t} U_\infty^t \Pi_{ac}(U_\infty)$$

exist and are complete (*hence, in particular,* $\Omega_+^* = s\text{-}\lim_{t\to\infty} U_\infty^{-t} U_0^t \Pi_{ac}(U_0)$).

Proof. It follows from Lemma 3.3.3 and [35, Proposition 3.1]. □

Lemma 3.3.6. *Let U_∞ be a time evolution operator associated with a quasi-uniform coin map* (2.3.2)*. If the matrix $C_\infty = C_e \oplus C_o$ satisfies Assumption* 3.2.3*, then U_∞ and* $\exp(i\xi V_j)$ *are commuting for each $\xi \in \mathbb{R}$ and $j = 1, 2$.*

Proof. By Lemma 3.2.12, we have for each $\xi \in \mathbb{R}$ and $j = 1, 2$

$$\begin{aligned}
[U_\infty, \exp(i\xi V_j)] &= s\text{-}\lim_{t\to\infty} \left[U_\infty, \exp\left(i\xi \frac{Q_j^\infty(t)}{t} \right) \right] \\
&= s\text{-}\lim_{t\to\infty} \left\{ U_\infty \exp\left(i\xi \frac{Q_j^\infty(t)}{t} \right) - U_\infty U_\infty^{-1} \exp\left(i\xi \frac{Q_j^\infty(t)}{t} \right) U_\infty \right\} \\
&= s\text{-}\lim_{t\to\infty} U_\infty \left\{ \exp\left(i\xi \frac{Q_j^\infty(t)}{t} \right) - \left(\exp\left(i\xi \frac{Q_j^\infty(t+1)}{t+1} \right) \right)^{\frac{t+1}{t}} \right\} \\
&= U_\infty \left(\exp(i\xi V_j) - \exp(i\xi V_j) \right) \\
&= 0.
\end{aligned}$$

This completes the proof. □

The next corollary follows from Lemma 3.3.6 and Proposition 1.1.1.

Corollary 3.3.7. *Suppose that the same assumption as in Lemma* 3.3.6 *holds. Then* $[\Pi_{\mathrm{ac}}(U_\infty), \exp(i\xi V_j)] = 0$ *for each* $\xi \in \mathbb{R}, j = 1, 2$, *and* $[\Pi_{\mathrm{ac}}(U_\infty), \exp i(\xi_1 V_1 + \xi_2 V_2)] = 0$ *also holds for all* ${}^t(\xi_1, \xi_2) \in \mathbb{R}^2$.

Since the wave and asymptotic velocity operators exist when U_0 is a time evolution operator associated with a short-range coin map whose limit matrix satisfies Assumption 3.2.3, one can set the following operator

$$V_j^+ := \Omega_+ V_j \Omega_+^* \quad (j = 1, 2).$$

Remark 3.3.8. Corollary 3.3.7 also implies that $[\Pi_{\mathrm{ac}}(U_\infty), V_j] = 0$ for each $j = 1, 2$. This means that $\mathcal{K}_{\mathrm{ac}}(U_\infty)$ is V_j-invariant for $j = 1, 2$, and hence one has for any $\xi \in \mathbb{R}, j = 1, 2$

$$\exp(i\xi V_j^+)\Pi_{\mathrm{ac}}(U_0) = \Omega_+ \exp(i\xi V_j)\Omega_+^* \Pi_{\mathrm{ac}}(U_0). \tag{3.3.3}$$

Lemma 3.3.9. *If* U_0 *is a time evolution operator associated with a short-range coin map whose limit matrix satisfies Assumption* 3.2.3, *then the operators* V_1^+, V_2^+ *are commuting and the following identity holds:*

$$s\text{-}\lim_{t\to\infty} \exp\left\{ i \left(\xi_1 \frac{Q_1(t)}{t^\alpha} + \xi_2 \frac{Q_2(t)}{t^\alpha} \right) \right\} \Pi_{\mathrm{ac}}(U_0) = \begin{cases} \exp i(\xi_1 V_1^+ + \xi_2 V_2^+)\Pi_{\mathrm{ac}}(U_0) & \text{if} \quad \alpha = 1, \\ \Pi_{\mathrm{ac}}(U_0) & \text{if} \quad \alpha > 1. \end{cases}$$

for each ${}^t(\xi_1, \xi_2) \in \mathbb{R}^2$.

Proof. We first consider the case where $\alpha = 1$. By Remark 3.3.8 and the commutativity of the asymptotic velocity operators, we obtain that

$$V_1^+ V_2^+ = \Omega_+ V_1 \Omega_+^* \Omega_+ V_2 \Omega_+^* = \Omega_+ V_1 \Pi_{\mathrm{ac}}(U_\infty) V_2 \Omega_+^* = \Omega_+ V_2 \Pi_{\mathrm{ac}}(U_\infty) V_1 \Omega_+^* = V_2^+ V_1^+,$$

and so half of the assertion is proved.

To prove the identity in the statement, take any $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2, t \in \mathbb{N}$ and we let $\Omega_t := U_0^{-t} U_\infty^t$. By using (3.3.3) and Proposition 1.1.1, we have

$$\begin{aligned}
&\exp i(\xi_1 V_1^+ + \xi_2 V_2^+)\Pi_{\mathrm{ac}}(U_0) \\
&= \exp\left(i\xi_1 V_1^+\right) \exp\left(i\xi_2 V_2^+\right) \Pi_{\mathrm{ac}}(U_0) \\
&= \Omega_+ \exp\left(i\xi_1 V_1\right) \Omega_+^* \Omega_+ \exp\left(i\xi_2 V_2\right) \Omega_+^* \Pi_{\mathrm{ac}}(U_0) \\
&= \Omega_+ \exp i\left(\xi_1 V_1 + \xi_2 V_2\right) \Omega_+^* \Pi_{\mathrm{ac}}(U_0).
\end{aligned}$$

It follows from this and Corollary 3.3.7 that

$$\exp\left\{ i \left(\xi_1 \frac{Q_1(t)}{t} + \xi_2 \frac{Q_2(t)}{t} \right) \right\} \Pi_{\mathrm{ac}}(U_0) - \exp i(\xi_1 V_1^+ + \xi_2 V_2^+)\Pi_{\mathrm{ac}}(U_0)$$

$$
\begin{aligned}
&= \Omega_t \exp\left\{ i \left(\xi_1 \frac{Q_1^\infty(t)}{t} + \xi_2 \frac{Q_2^\infty(t)}{t} \right) \right\} \Omega_t^* \Pi_{\mathrm{ac}}(U_0) \\
&\quad - \Omega_t \exp\left\{ i \left(\xi_1 \frac{Q_1^\infty(t)}{t} + \xi_2 \frac{Q_2^\infty(t)}{t} \right) \right\} \Omega_+^* \Pi_{\mathrm{ac}}(U_0) \\
&\quad + \Omega_t \exp\left\{ i \left(\xi_1 \frac{Q_1^\infty(t)}{t} + \xi_2 \frac{Q_2^\infty(t)}{t} \right) \right\} \Omega_+^* \Pi_{\mathrm{ac}}(U_0) - \Omega_t \exp i \left(\xi_1 V_1 + \xi_2 V_2 \right) \Omega_+^* \Pi_{\mathrm{ac}}(U_0) \\
&\quad + \Omega_t \exp i \left(\xi_1 V_1 + \xi_2 V_2 \right) \Omega_+^* \Pi_{\mathrm{ac}}(U_0) - \Omega_+ \exp i \left(\xi_1 V_1 + \xi_2 V_2 \right) \Omega_+^* \Pi_{\mathrm{ac}}(U_0) \\
&= \Omega_t \exp\left\{ i \left(\xi_1 \frac{Q_1^\infty(t)}{t} + \xi_2 \frac{Q_2^\infty(t)}{t} \right) \right\} (\Omega_t^* - \Omega_+^*) \Pi_{\mathrm{ac}}(U_0) \\
&\quad + \Omega_t \left[\exp\left\{ i \left(\xi_1 \frac{Q_1^\infty(t)}{t} + \xi_2 \frac{Q_2^\infty(t)}{t} \right) \right\} - \exp i \left(\xi_1 V_1 + \xi_2 V_2 \right) \right] \Omega_+^* \Pi_{\mathrm{ac}}(U_0) \\
&\quad + (\Omega_t - \Omega_+) \exp i \left(\xi_1 V_1 + \xi_2 V_2 \right) \Omega_+^* \Pi_{\mathrm{ac}}(U_0) \\
&=: I_1(t) + I_2(t) + I_3(t).
\end{aligned}
$$

Because Ω_t and $\exp\left\{ i \left(\xi_1 \frac{Q_1^\infty(t)}{t} + \xi_2 \frac{Q_2^\infty(t)}{t} \right) \right\}$ are uniformly bounded, one has from Corollary 3.2.13 that $I_1(t)$ and $I_2(t)$ strongly converge to 0. Also, by Corollary 3.3.7 again and the definition of Ω_+,

$$
\begin{aligned}
I_3(t) &= (\Omega_t - \Omega_+) \exp i \left(\xi_1 V_1 + \xi_2 V_2 \right) \Pi_{\mathrm{ac}}(U_\infty) \Omega_+^* \Pi_{\mathrm{ac}}(U_0) \\
&= (\Omega_t - \Omega_+) \Pi_{\mathrm{ac}}(U_\infty) \exp i \left(\xi_1 V_1 + \xi_2 V_2 \right) \Omega_+^* \Pi_{\mathrm{ac}}(U_0) \xrightarrow{\mathrm{SOT}} 0.
\end{aligned}
$$

Finally we consider the case $\alpha > 1$. Then one has

$$
\begin{aligned}
&\exp\left\{ i \left(\xi_1 \frac{Q_1(t)}{t^\alpha} + \xi_2 \frac{Q_2(t)}{t^\alpha} \right) \right\} \Pi_{\mathrm{ac}}(U_0) \\
&= \left[\Omega_t \exp\left\{ i \left(\xi_1 \frac{Q_1^\infty(t)}{t^\alpha} + \xi_2 \frac{Q_2^\infty(t)}{t^\alpha} \right) \right\} \Omega_t^* \Pi_{\mathrm{ac}}(U_0) - \Omega_t \Omega_+^* \right] + \Omega_t \Pi_{\mathrm{ac}}(U_\infty) \Omega_+^* \\
&\xrightarrow{\mathrm{SOT}} 0 + \Omega_+ \Omega_+^* = \Pi_{\mathrm{ac}}(U_0).
\end{aligned}
$$

Therefore, we have the desired result. □

3.4 Weak limit theorem

In order to prove a weak limit theorem, we also prepare two lemmas. As one can see below, the unitary operator $\exp\left\{ i \left(\xi_1 \frac{Q_1(t)}{t} + \xi_2 \frac{Q_2(t)}{t} \right) \right\}$ is asymptotically identity on the pure point subspace $\mathcal{K}_{\mathrm{pp}}(U_0)$, and the singular continuous space $\mathcal{K}_{\mathrm{sc}}(U_0)$ is the zero space.

Lemma 3.4.1. *Let W be a unitary operator and T_1, T_2 be strongly commuting self-adjoint operators on a Hilbert space, and let $\alpha \geq 1$. Let $T_j(t) := W^{-t} T_j W^t$ for $j = 1, 2$, then one has*

$$
s\text{-}\lim_{t \to \infty} \exp\left\{ i \left(\xi_1 \frac{T_1(t)}{t^\alpha} + \xi_2 \frac{T_2(t)}{t^\alpha} \right) \right\} \Pi_{\mathrm{pp}}(W) = \Pi_{\mathrm{pp}}(W)
$$

for each $\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$ *in* $\mathbb{R}^2$.

Proof. It follows from [35, Theorem 4.2] (even in $\alpha > 1$ case, it can be proved in the same way as in the case where $\alpha = 1$) and Proposition 1.1.1. $\square$

Lemma 3.4.2. *If* U_0 *is a time evolution operator associated with a short-range coin map, then* U_0 *has no singular continuous spectrum.*

Proof. Let $C_\bullet$ be a short-range coin map satisfying (3.3.1). Since

$$\begin{aligned}&\int_1^\infty \sup_{\|(n,m)\|_1\geq r}\|C_e^{-1}C_{\phi_e(n,m)}\oplus C_o^{-1}C_{\phi_o(n,m)}-E_{2d}\|dr\\&=\int_1^\infty \sup_{\|(n,m)\|_1\geq r}\|C_{\phi_e(n,m)}\oplus C_{\phi_o(n,m)}-C_\infty\|dr\\&\leq\int_1^\infty \kappa'(1+r)^{-2-\varepsilon}dr<\infty\quad\text{(Remark 3.3.2)},\end{aligned}$$

the assertion can be proved by applying [4, Theorem 3.4] to the unitary operator $U_0 = U_\infty\left(\bigoplus_{(n,m)\in\mathbb{Z}^2}(C_e^{-1}C_{\phi_e(n,m)}\oplus C_o^{-1}C_{\phi_o(n,m)})\right)$. $\square$

Theorem 3.4.3. *Let* $C_\bullet : \mathbb{Z}^2 \to U(d)$ *be a coin map and* $\Psi^0 \in \mathcal{H}_\Gamma$ *an initial state. Suppose that* $C_\bullet$ *is of short range and that its limit matrix satisfies Assumption* 3.2.3. *Then, the distribution of* X_t/t *weakly converges to a probability measure*

$$\begin{aligned}\mu =&\|\Pi_{\mathrm{pp}}(U_0)J_\Gamma\Psi^0\|_{\mathcal{K}_\Gamma}^2\delta_{(0,0)}+\|(E_{V_1^\Gamma}\otimes E_{V_2^\Gamma})(\cdot)(\Omega_+^\Gamma)^*J_\Gamma\Psi^0\|_{\mathcal{K}_\Gamma}^2\\=&\|\Pi_{\mathrm{pp}}(U)\Psi^0\|_{\mathcal{H}_\Gamma}^2\delta_{(0,0)}+\|(E_{V_1^\Gamma}\otimes E_{V_2^\Gamma})(\cdot)(\Omega_+^\Gamma)^*J_\Gamma\Psi^0\|_{\mathcal{K}_\Gamma}^2.\end{aligned}$$

where $\delta_{(0,0)}$ *is the Dirac measure at* $(0,0)\in\mathbb{R}^2$. *Also, for each* $\alpha > 1$, *the distribution of* X_t/t^α *weakly converges to* $\delta_{(0,0)}$.

Proof. In this proof, we simply write $\Phi^0 := J_\Gamma\Psi^0$. We remark that

$$\begin{aligned}&\|\Pi_{\mathrm{pp}}(U_0)\Phi^0\|^2\delta_{(0,0)}+\|(E_{V_1^+}\otimes E_{V_2^+})(\cdot)\Pi_{\mathrm{ac}}(U_0)\Phi^0\|^2\\&\qquad=\|\Pi_{\mathrm{pp}}(U_0)\Phi^0\|^2\delta_{(0,0)}+\|\Omega_+((E_{V_1}\otimes E_{V_2})(\cdot))\Omega_+^*\Phi^0\|^2\\&\qquad=\|\Pi_{\mathrm{pp}}(U_0)\Phi^0\|^2\delta_{(0,0)}+\|(E_{V_1}\otimes E_{V_2})(\cdot)\Omega_+^*\Phi^0\|^2\end{aligned}$$

and

$$\exp i\left(\xi_1V_1^++\xi_2V_2^+\right)\Pi_{\mathrm{ac}}(U_0)=\Pi_{\mathrm{ac}}(U_0)\exp i\left(\xi_1V_1^++\xi_2V_2^+\right)\Pi_{\mathrm{ac}}(U_0).$$

From these equations and the lemmas above, we have that

$$\lim_{t\to\infty}\mathbb{E}\left[\exp i\left\langle\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix},\frac{X_t}{t}\right\rangle_2\right]$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \left\langle \Phi^0, \exp\left\{ i\left(\xi_1 \frac{Q_1(t)}{t} + \xi_2 \frac{Q_2(t)}{t}\right)\right\} \Phi^0 \right\rangle \quad \text{(Lemma 3.1.2)} \\
&= \left\langle \Phi^0, \Pi_{\mathrm{pp}}(U_0)\Phi^0 \right\rangle + \left\langle \Phi^0, \exp i\left(\xi_1 V_1^+ + \xi_2 V_2^+\right) \Pi_{\mathrm{ac}}(U_0)\Phi^0 \right\rangle \quad \text{(Lemma 3.3.9} - \text{3.4.2)} \\
&= \|\Pi_{\mathrm{pp}}(U_0)\Phi^0\|^2 + \left\langle \Pi_{\mathrm{ac}}(U_0)\Phi^0, \exp i\left(\xi_1 V_1^+ + \xi_2 V_2^+\right) \Pi_{\mathrm{ac}}(U_0)\Phi^0 \right\rangle \\
&= \|\Pi_{\mathrm{pp}}(U_0)\Phi^0\|^2 + \int_{\mathbb{R}^2} \exp i\left(\xi_1 x + \xi_2 y\right) d\left\langle \Pi_{\mathrm{ac}}(U_0)\Phi^0, (E_{V_1^+} \otimes E_{V_2^+})(x,y)\Pi_{\mathrm{ac}}(U_0)\Phi^0 \right\rangle \\
&= \int_{\mathbb{R}^2} \exp\left(i \left\langle \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_2 \right) d\left(\|\Pi_{\mathrm{pp}}(U_0)\Phi^0\|^2 \delta_{(0,0)}(x,y) + \|(E_{V_1^+} \otimes E_{V_2^+})(x,y)\Pi_{\mathrm{ac}}(U_0)\Phi^0\|^2 \right) \\
&= \int_{\mathbb{R}^2} \exp\left(i \left\langle \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_2 \right) d\mu.
\end{aligned}
$$

Similarly, for each $\alpha > 1$, one obtains that

$$
\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\exp i \left\langle \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \frac{X_t}{t^\alpha} \right\rangle_2 \right] \\
&= \lim_{t\to\infty} \left\langle \Phi^0, \exp\left\{ i\left(\xi_1 \frac{Q_1(t)}{t^\alpha} + \xi_2 \frac{Q_2(t)}{t^\alpha}\right)\right\} \Phi^0 \right\rangle \quad \text{(Lemma 3.1.2)} \\
&= \left\langle \Phi^0, \Pi_{\mathrm{pp}}(U_0)\Phi^0 \right\rangle + \left\langle \Phi^0, \Pi_{\mathrm{ac}}(U_0)\Phi^0 \right\rangle \quad \text{(Lemma 3.3.9} - \text{3.4.2)} \\
&= 1 \\
&= \int_{\mathbb{R}^2} \exp\left(i \left\langle \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_2 \right) d\delta_{(0,0)}.
\end{aligned}
$$

This completes the proof. □

When $C_\bullet$ is quasi-uniform, the wave operator Ω_+ coincides with the projection $\Pi_{\mathrm{ac}}(U_\infty)$. From this it follows:

Corollary 3.4.4. *Under the same assumption as in Theorem* 3.4.3*, if a coin map $C_\bullet$ is quasi-uniform* (*thus automatically it is of short range and $U_0 = U_\infty$*)*, then the distribution of X_t/t weakly converges to a probability measure*

$$\mu = \|\Pi_{\mathrm{pp}}(U_\infty) J_\Gamma \Psi^0\|^2_{\mathcal{K}_\Gamma} \delta_{(0,0)} + \|(E_{V_1^\Gamma} \otimes E_{V_2^\Gamma})(\cdot)\Pi_{\mathrm{ac}}(U_\infty) J_\Gamma \Psi^0\|^2_{\mathcal{K}_\Gamma}.$$

An application of the weak limit theorem is the concept of localization. Let $\Psi^0 \in \mathcal{H}_\Gamma$ be an initial state. Then we say that *localization occurs* if

$$\limsup_{t\to\infty} \mathbb{P}(X_t = (n,m)) = \limsup_{t\to\infty} \|(U^t\Psi^0)(n,m)\|^2_{\mathbb{C}^d} > 0$$

for some $(n,m) \in \mathbb{Z}^2$. "Localization occurs" means that a quantum walker remains for a long time at a certain point. It is an inherent property of quantum walks. Localization occurs if and only if the initial state Ψ^0 overlaps with $\mathcal{H}_{\mathrm{pp}}(U)$, namely, $\Pi_{\mathrm{pp}}(U)\Psi^0 \neq 0$ ([32, Proposition 2.4]). Thus, Theorem 3.4.3 gives a following necessary condition for "localization occurs":

Corollary 3.4.5. *Suppose that the same assumption as in Theorem* 3.4.3 *holds. Then "localization occurs" implies that* $\mu(\{(0,0)\}) > 0$.

Example 3.4.6. (Example 3.2.7 continued) For $j = 1, ..., 6$, we set

$$K_j^{\mp}(\theta,\varphi) := \left(\lambda_j^{\theta}(\theta,\varphi) \mp \lambda_j^{\varphi}(\theta,\varphi)\right) |u_j(\theta,\varphi)\rangle\langle u_j(\theta,\varphi)|\,.$$

Since $\lambda_j^{\theta}(\theta,\varphi) + \lambda_j^{\varphi}(\theta,\varphi) = 0$, one has $K_j^{+}(\theta,\varphi) = 0$ for $j = 1, ..., 6$. It follows that the asymptotic velocity operator V_2 is the zero operator, and hence its spectral measure is given by $E_{V_2}(\{0\}) = I$. Also we can compute that

$$K_1^{-}(\theta,\varphi) = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & e^{\frac{i}{2}(\theta+\varphi)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ e^{-\frac{i}{2}(\theta+\varphi)} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_2^{-}(\theta,\varphi) = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -e^{\frac{i}{2}(\theta+\varphi)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -e^{-\frac{i}{2}(\theta+\varphi)} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_3^{-}(\theta,\varphi) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & e^{\frac{i}{2}(\theta+\varphi)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\frac{i}{2}(\theta+\varphi)} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_4^{-}(\theta,\varphi) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -e^{\frac{i}{2}(\theta+\varphi)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-\frac{i}{2}(\theta+\varphi)} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_5^{-}(\theta,\varphi) = K_6^{-}(\theta,\varphi) = O_6.$$

Thus, one has

$$V_1 = \mathcal{F}^{-1}\left(\int_{(0,2\pi)^2}^{\oplus} \sum_{j=1}^{6} K_j^{-}(\theta,\varphi)\frac{d\theta d\varphi}{4\pi^2}\right)\mathcal{F}$$

$$= \bigoplus_{\mathbb{Z}^2} \begin{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & O_3 \\ O_3 & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \text{ (a multiplication operator)},$$

and its spectral measure is given by

$$E_{V_1}(\{-1\}) = \bigoplus_{\mathbb{Z}^2} \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & O_3 \\ O_3 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}, \; E_{V_1}(\{1\}) = \bigoplus_{\mathbb{Z}^2} \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & O_3 \\ O_3 & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$E_{V_1}(\{0\}) = \bigoplus_{\mathbb{Z}^2} \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \\ O_3 & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}.$$

It follows from the form of the matrix $\widehat{U}_\infty(\theta, \varphi)$ that the pure point subspace of $\widehat{U}_\infty$ coincides with

$$L_2((0, 2\pi)^2; 0 \oplus 0 \oplus \mathbb{C} \oplus 0 \oplus 0 \oplus \mathbb{C}).$$

By taking the orthogonal complement, the absolutely continuous subspace of $\widehat{U}_\infty$ is equal to

$$L_2((0, 2\pi)^2; \mathbb{C} \oplus \mathbb{C} \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C} \oplus 0).$$

So one gets,

$$\begin{aligned} \mathcal{K}_{\mathrm{pp}}(U_\infty) &= \ell_2(\mathbb{Z}^2; 0 \oplus 0 \oplus \mathbb{C} \oplus 0 \oplus 0 \oplus \mathbb{C}), \\ \mathcal{K}_{\mathrm{ac}}(U_\infty) &= \ell_2(\mathbb{Z}^2; \mathbb{C} \oplus \mathbb{C} \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C} \oplus 0). \end{aligned}$$

Therefore, if we write $\Phi^0 := {}^t(\Phi^0_1, \Phi^0_2, \Phi^0_3, \Phi^0_4, \Phi^0_5, \Phi^0_6) := J_{\Gamma_H} \Psi^0 \in \mathcal{K}_{\Gamma_H}$, then the weak limit measure μ is expressed by

$$\begin{aligned} \mu &= \|\Pi_{\mathrm{pp}}(U_\infty)\Phi^0\|^2 \delta_{(0,0)} + \|(E_{V_1} \otimes E_{V_2})(\cdot)\Pi_{\mathrm{ac}}(U_\infty)\Phi^0\|^2 \\ &= \|E_{V_1}(\{0\})\Phi^0\|^2 \delta_{(0,0)} + \|E_{V_1}(\{-1\})\Phi^0\|^2 \delta_{(-1,0)} + \|E_{V_1}(\{1\})\Phi^0\|^2 \delta_{(1,0)} \\ &= \|\Phi^0_{(3,6)}\|^2_{\ell_2(\mathbb{Z}^2;\mathbb{C}^2)} \delta_{(0,0)} + \|\Phi^0_{(1,4)}\|^2_{\ell_2(\mathbb{Z}^2;\mathbb{C}^2)} \delta_{(-1,0)} + \|\Phi^0_{(2,5)}\|^2_{\ell_2(\mathbb{Z}^2;\mathbb{C}^2)} \delta_{(1,0)}, \end{aligned}$$

where $\Phi^0_{(j,k)} := {}^t(\Phi^0_j, \Phi^0_k) \in \ell_2(\mathbb{Z}^2; \mathbb{C}^2)$.

Finally, let us consider a simple initial state. Let $\Psi^0 \in \mathcal{H}_{\Gamma_H}$ be the initial state as follows:

$$\Psi^0(n,m) = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} & \text{if } (n,m) = (0,0), \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Since $\Phi^0 = J_{\Gamma_H}\Psi^0$ is given by

$$\Phi^0(n,m) = \begin{cases} \frac{1}{\sqrt{2}} {}^t\!\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) & \text{if } (n,m) = (0,0), \\ {}^t\!\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) & \text{otherwise,} \end{cases}$$

we have $\mu = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(-1,0)}$, so $\mu(\{(0,0)\}) = \mu(\{(-1,0)\}) = \frac{1}{2}$. Now, the weak limit measure μ means the probability measure of the asymptotic velocity of quantum walkers. Thus, in this case, the probability that the asymptotic velocity of the quantum walker is $(0,0)$ and $(-1,0)$ are both $\frac{1}{2}$. In fact, by (2.1.2), the following equality hods:

$$(U^t\Psi^0)(n,m) = (S^t\Psi^0)(n,m) = \begin{cases} \begin{cases} \frac{1}{\sqrt{2}} {}^t(1\ 0\ 0) & \text{if } (n,m) = (-t,0) \\ \frac{1}{\sqrt{2}} {}^t(0\ 0\ 1) & \text{if } (n,m) = (0,0) \\ {}^t(0\ 0\ 0) & \text{otherwise} \end{cases} & \text{if } t : \text{even}, \\ \begin{cases} \frac{1}{\sqrt{2}} {}^t(1\ 0\ 0) & \text{if } (n,m) = (-t,0) \\ \frac{1}{\sqrt{2}} {}^t(0\ 0\ 1) & \text{if } (n,m) = (0,1) \\ {}^t(0\ 0\ 0) & \text{otherwise} \end{cases} & \text{if } t : \text{odd}. \end{cases}$$

Also $\limsup_{t\to\infty} \mathbb{P}(X_t = (0,0)) = \limsup_{t\to\infty} \mathbb{P}(X_t = (0,1)) = \frac{1}{2} > 0$ holds, so localization occurs at $(0,0)$ and $(0,1)$.

Chapter 4

Essential spectrum of time evolution operators

In this section, we deal with the time evolution operator $U = SC$ associated with an anisotropic coin map. The purpose of this section is to express the essential spectrum of U. The argument here is similar to a method employed in the proof of [29, Theorem 2.2] which is used the discrete crossed product C^*-algebras. After constructing a suitable C^*-subalgebra of the Calkin algebra in the first section, we determine the essential spectrum of U by using the fact that the element $\pi(U_\infty)$ of the Calkin algebra is contained in the subalgebra.

4.1 A subalgebra of the Calkin algebra

Throughout this section, we fix $r, s \in \mathbb{N}$ arbitrarily. In order to construct a crossed product C^*-algebra, we shall define an action of the (amenable) group $\mathbb{Z}^r$ on $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$. Define a map $\alpha : \mathbb{Z}^r \to \mathrm{Aut}(C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})))$ by

$$(\alpha_n f)(x) := f(x+n)$$

for $n \in \mathbb{Z}^r, f \in C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})), x \in \widehat{\mathbb{Z}^r}$, where $\infty + n := \infty$. Then α is an action of $\mathbb{Z}^r$ on $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$. The matrix algebra $M_s(\mathbb{C})$ can be naturally regarded as a C^*-subalgebra of $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$. Besides, $M_s(\mathbb{C})$ is α-invariant, and the restriction of α to $M_s(\mathbb{C})$ becomes a trivial action. By the universality of crossed product (see [12, Proposition 11.14]), one can construct a $*$-homomorphism ∂_∞ from $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r$ to $M_s(\mathbb{C}) \rtimes_0 \mathbb{Z}^r$ such that

$$\partial_\infty(f\delta_n) = f(\infty)\delta_n$$

for $f \in C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$ and $n \in \mathbb{Z}^r$. Since $C_0(\mathbb{Z}^r; M_s(\mathbb{C}))$ is a closed ideal of $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$ and α-invariant, $C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r$ is also a closed ideal of $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r$(see [12, Proposition 21.12]).

Proposition 4.1.1. *The $*$-homomorphism $\partial_\infty : C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r \to M_s(\mathbb{C}) \rtimes_0 \mathbb{Z}^r$ is surjective and its kernel coincides with $C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r$.*

Proof. Let us consider the following short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\mathbb{Z}^r; M_s(\mathbb{C})) & \hookrightarrow & C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) & \longrightarrow & M_s(\mathbb{C}) & \longrightarrow & 0 \\ & & & & \Psi & & \Psi & & \\ & & & & f & \longmapsto & f(\infty) & & \end{array}$$

Then its corresponding sequence

$$0 \longrightarrow C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r \hookrightarrow C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r \xrightarrow{\partial_\infty} M_s(\mathbb{C}) \rtimes_0 \mathbb{Z}^r \longrightarrow 0$$

is also exact (see [12, Theorem 22.9]). ◻

Let we write $\mathcal{K}_{r,s} := \ell_2(\mathbb{Z}^r; \mathbb{C}^s)$. We shall construct a faithful $*$-representation of $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r$ on $\mathcal{K}_{r,s}$. For each $f \in C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$, since $f(n) \in M_s(\mathbb{C}) \cong \mathbb{B}(\mathbb{C}^s)$ for $n \in \mathbb{Z}^r$, the direct sum $\bigoplus_{n\in\mathbb{Z}^r} f(n)$ is a bounded operator on $\bigoplus_{n\in\mathbb{Z}^r} \mathbb{C}^s \cong \mathcal{K}_{r,s}$ with norm

$$\left\| \bigoplus_{n\in\mathbb{Z}^r} f(n) \right\|_{\mathbb{B}(\mathcal{K}_{r,s})} = \sup_{n\in\mathbb{Z}^r} \|f(n)\|_{\mathbb{B}(\mathbb{C}^s)} = \|f\|_{C(\widehat{\mathbb{Z}^r}, M_s(\mathbb{C}))}.$$

Hence $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$ can be regarded as a C^*-subalgebra of $\mathbb{B}(\mathcal{K}_{r,s})$. The operator $\bigoplus_{n\in\mathbb{Z}^r} f(n)$ is written by f if there is no danger of confusion. We next define r unitary operators $T_1, \dots, T_r$ on $\mathcal{K}_{r,s}$. For each $j = 1, \dots, r$, $T_j : \mathcal{K}_{r,s} \to \mathcal{K}_{r,s}$ is defined by

$$(T_j\Phi)(n) := \Phi(n_1, \dots, n_{j-1}, \overset{\overset{j}{\downarrow}}{n_j+1}, n_{j+1}, \dots, n_r)$$

for $\Phi \in \mathcal{K}_{r,s}$, $n = (n_1, \dots, n_r) \in \mathbb{Z}^r$. Then $T_1, \dots, T_r$ are commuting and their adjoint operators are given by

$$(T_j^*\Phi)(n) = (T_j^{-1}\Phi)(n) = \Phi(n_1, \dots, n_{j-1}, \overset{\overset{j}{\downarrow}}{n_j-1}, n_{j+1}, \dots, n_r)$$

for all $j = 1, \dots, r$. For convenience, we write $T^n := T_1^{n_1} T_2^{n_2} \cdots T_r^{n_r} \in \mathbb{B}(\mathcal{K}_{r,s})$ for any $n = (n_1, n_2, \dots, n_r) \in \mathbb{Z}^r$, and we define a linear map $\rho : C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha^{\mathrm{alg}} \mathbb{Z}^r \to \mathbb{B}(\mathcal{K}_{r,s})$ by $\rho(f\delta_n) = fT^n$ for $f \in C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$, $n \in \mathbb{Z}^r$.

Lemma 4.1.2. *The linear map $\rho : C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha^{\mathrm{alg}} \mathbb{Z}^r \to \mathbb{B}(\mathcal{K}_{r,s})$ is a $*$-homomorphism.*

Proof. First of all, we note that the following formula holds in $\mathbb{B}(\mathcal{K}_{r,s})$:

$$\alpha_n(f) = T^n f (T^n)^*$$

for all $f \in C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$ and $n \in \mathbb{Z}^r$. Indeed, for every $\Phi \in \mathcal{K}_{r,s}, x \in \mathbb{Z}^r$, we have

$$\begin{aligned}\mathbb{C}^s \ni &(\alpha_n(f)T^n\Phi)(x) = f(x+n)(T^n\Phi(x)) = f(x+n)(\Phi(x+n)) \\ &= (f\Phi)(x+n) = (T^n f\Phi)(x).\end{aligned}$$

By using the equality above and the commutativity of $T_1, \dots, T_r$, one obtains that

$$\begin{aligned}&\rho((f\delta_n)^*) = \rho(\alpha_{-n}(f^*)\delta_{-n}) = \alpha_{-n}(f^*)T^{-n} = T^{-n}f^* = (fT^n)^* = \rho(f\delta_n)^*,\\ &\rho((f\delta_n)(g\delta_m)) = \rho(f\alpha_n(g)\delta_{n+m}) = f\alpha_n(g)T^{n+m} = fT^n gT^m = \rho(f\delta_n)\rho(g\delta_m).\end{aligned}$$

Thus the desired result follows. $\square$

With this Lemma 4.1.2 and the universality of crossed product, the morphism ρ can be extended to a $*$-homomorphism from $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r$ to $\mathbb{B}(\mathcal{K}_{r,s})$, and we use the same symbol for this extension.

Lemma 4.1.3. *The $*$-homomorphism $\rho : C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r \to \mathbb{B}(\mathcal{K}_{r,s})$ is injective and its image of $C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r$ is the compact algebra $\mathbb{K}(\mathcal{K}_{r,s})$.*

Proof. In order to see the injectivity of ρ, we take $X \in \mathrm{Ker}\rho$ arbitrarily, and a sequence $\{X_N\}_{N=1}^\infty$ of $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha^{\mathrm{alg}} \mathbb{Z}^r$ such that $X_N \to X$ in $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r$. For each $N \in \mathbb{N}$, the element X_N can be expressed as $X_N = \sum_{n\in\mathbb{Z}^r} f_n^N \delta_n$, where $f_n^N = 0$ except for finitely many $n \in \mathbb{Z}^r$. Note that

$$\lim_{N\to\infty} \left\| \sum_{n\in\mathbb{Z}^r} f_n^N T^n \right\| = \lim_{N\to\infty} \|\rho(X_N)\| = \|\rho(X)\| = 0.$$

For $n \in \mathbb{Z}^r$ and $k = 1, \dots, s$, we define a norm one element δ_n^k in $\mathcal{K}_{r,s}$ as follows: $\delta_n^k(m) := \begin{cases} e_k & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$, where $\{e_k\}_{k=1}^s$ is the natural basis of $\mathbb{C}^s$. Then $\{\delta_n^k\}_{n\in\mathbb{Z}^r, 1\le k\le s}$ is a complete orthonormal basis for $\mathcal{K}_{r,s}$. Let $m \in \mathbb{Z}^r$ be fixed arbitrarily. For any $x \in \mathbb{Z}^r$ and $l = 1, \dots, s$, one has

$$\begin{aligned}\left\| \sum_{n\in\mathbb{Z}^r} f_n^N T^n \right\|^2_{\mathbb{B}(\mathcal{K}_{r,s})} &\geq \left\| \left(\sum_{n\in\mathbb{Z}^r} f_n^N T^n \right) (\delta_{x+m}^l) \right\|^2_{\mathcal{K}_{r,s}} \\ &\geq \left\| \left(\sum_{n\in\mathbb{Z}^r} f_n^N T^n \delta_{x+m}^l \right) (x) \right\|^2_{\mathbb{C}^s} \\ &= \left\| \sum_{n\in\mathbb{Z}^r} f_n^N(x)(\delta_{x+m}^l(x+n)) \right\|^2_{\mathbb{C}^s} \\ &= \left\| f_m^N(x)(\delta_{x+m}^l(x+m)) \right\|^2_{\mathbb{C}^s} \\ &= \left\| f_m^N(x)(e_l) \right\|^2_{\mathbb{C}^s}\end{aligned}$$

$$
\begin{aligned}
&= \left\|\text{the } l\text{-th column of } f_m^N(x)\right\|_{\mathbb{C}^s}^2 \\
&\geq \max_{1\leq k\leq s} \left|\left(f_m^N(x)\right)_{kl}\right|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left\|f_m^N\right\|_{C(\widehat{\mathbb{Z}^r};M_s(\mathbb{C}))} = \sup_{x\in\mathbb{Z}^r} \left\|f_m^N(x)\right\|_{M_s(\mathbb{C})} \\
&\leq \sup_{x\in\mathbb{Z}^r} \sum_{1\leq k,l\leq s} \left(\left|\left(f_m^N(x)\right)_{kl}\right|^2\right)^{\frac{1}{2}} \\
&\leq \sup_{x\in\mathbb{Z}^r} \sum_{1\leq l\leq s} s \left(\max_{1\leq k\leq s} \left|\left(f_m^N(x)\right)_{kl}\right|^2\right)^{\frac{1}{2}} \\
&\leq s \sup_{x\in\mathbb{Z}^r} \sum_{1\leq l\leq s} \left\|\sum_{n\in\mathbb{Z}^r} f_n^N T^n\right\|_{\mathbb{B}(\mathcal{K}_{r,s})} \\
&= s^2 \left\|\sum_{n\in\mathbb{Z}^r} f_n^N T^n\right\|_{\mathbb{B}(\mathcal{K}_{r,s})} \longrightarrow 0 \quad (\text{as } N\to\infty).
\end{aligned}
$$

Let us now consider the contractive linear map $E_m : C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r \to C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C}))$ $(m \in \mathbb{Z}^r)$ in Theorem 1.2.1. For every $m \in \mathbb{Z}^r$, we have

$$
E_m(X) = \lim_{N\to\infty} E_m(X_N) = \lim_{N\to\infty} E_m\left(\sum_{n\in\mathbb{Z}^r} f_n^N \delta_n\right) = \lim_{N\to\infty} f_m^N = 0.
$$

Thus $X = 0$, from this it follows that ρ is injective.

Next, let us show that $\rho(C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r) = \mathbb{K}(\mathcal{K}_{r,s})$. Every element of the image of $C_0(\mathbb{Z}^r; M_s(\mathbb{C}))$ under the embedding $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \hookrightarrow \mathbb{B}(\mathcal{K}_{r,s})$ above is approximated by finite-rank operators. Hence $\rho(C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r) \subset \mathbb{K}(\mathcal{K}_{r,s})$. Conversely, for any $n, m \in \mathbb{Z}^r$ and $k, l = 1, \ldots, s$, one defines $f \in C_0(\mathbb{Z}^r; M_s(\mathbb{C}))$ by

$$
f(x) := \begin{cases} E_{k,l} & \text{if } x = n \\ O_s & \text{otherwise,} \end{cases}
$$

where $E_{k,l} \in M_n(\mathbb{C})$ is the matrix having 1 at (k,l)-component, and 0 everywhere else. Then the following equality holds:

$$
|\delta_n^k\rangle\langle\delta_m^l| = fT^{m-n} = \rho(f\delta_{m-n}).
$$

Since the compact algebra $\mathbb{K}(\mathcal{K}_{r,s})$ is the closed linear span of $|\delta_n^k\rangle\langle\delta_m^l|$'s, it follows that $\mathbb{K}(\mathcal{K}_{r,s}) \subset \rho(C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r)$. □

Lemma 4.1.1 and Lemma 4.1.3 can be summarized as follows:

Lemma 4.1.4. *We have*

$$\mathcal{Q}(\mathcal{K}_{r,s}) \supset \frac{\rho(C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r)}{\rho(C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r)} \overset{\overline{\rho}}{\twoheadleftarrow\!\hookleftarrow} \frac{C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r}{C_0(\mathbb{Z}^r; M_s(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^r} \overset{\overline{\partial}_\infty}{\hookrightarrow\!\!\!\rightarrow} M_s(\mathbb{C}) \rtimes_0 \mathbb{Z}^r \ , \tag{4.1.1}$$

where the maps $\overline{\rho}$ *and* $\overline{\partial}_\infty$ *are naturally induced* $*$*-isometric isomorphisms from* ρ *and* ∂_∞*, respectively.*

4.2 Essential spectrum of U

Throughout this section, Γ is one of hexagonal, square, and triangular lattices, unless otherwise stated. In this section, we shall deal with anisotropic coin maps. Let $C_\bullet : \mathbb{Z}^2 \to U(d)$ be an anisotropic coin map, that is, there exist $C_e, C_o \in U(d)$ such that

$$C_{(n,m)} \to C_e \ \text{ as } \ \mathbb{Z}^2_e \ni (n,m) \to \infty_e \quad \text{and} \quad C_{(n,m)} \to C_o \ \text{ as } \ \mathbb{Z}^2_o \ni (n,m) \to \infty_o.$$

For convenience, we define a map $C'_\bullet : \mathbb{Z}^2 \to U(2d)$ by $C'_{(n,m)} := C_{\phi_e(n,m)} \oplus C_{\phi_o(n,m)}$. Note that $U_0 = S_0 \left(\bigoplus_{(n,m)\in\mathbb{Z}^2} C'_{(n,m)} \right)$. By using the inequality (3.3.2), one gets

$$\|C'_{(n,m)} - C_\infty\|_{M_{2d}(\mathbb{C})} \to 0 \ \text{ as } \ \mathbb{Z}^2 \ni (n,m) \to \infty,$$

where $C_\infty = C_e \oplus C_o$ is the limit matrix of $C_\bullet$. Hence $C'_\bullet : \mathbb{Z}^2 \to U(2d)$ is a member of $C(\widehat{\mathbb{Z}^2}; M_{2d}(\mathbb{C}))$. We will use the notations in the previous section as $r = 2, s = 2d$. Note that $\mathcal{K}_\Gamma = \mathcal{H}_{2,2d}$.

The following lemma is obtained by direct computations:

Lemma 4.2.1. *The shift operator* (2.4.3)$-$(2.4.5) *is expressed as follows:*

$$S_0 = \begin{cases} T_1 \begin{pmatrix} O_3 & O_3 \\ E_{1,1} & O_3 \end{pmatrix} + T_1^* \begin{pmatrix} O_3 & E_{2,2} \\ O_3 & O_3 \end{pmatrix} + T_2 \begin{pmatrix} O_3 & O_3 \\ E_{2,2} & O_3 \end{pmatrix} + T_2^* \begin{pmatrix} O_3 & E_{1,1} \\ O_3 & O_3 \end{pmatrix} & \\ \quad + \begin{pmatrix} O_3 & E_{3,3} \\ E_{3,3} & O_3 \end{pmatrix} & \text{if } \Gamma = \Gamma_H, \\ T_1 \begin{pmatrix} O_4 & O_4 \\ E_{1,1} & O_4 \end{pmatrix} + T_1^* \begin{pmatrix} O_4 & E_{2,2} \\ O_4 & O_4 \end{pmatrix} + T_2 \begin{pmatrix} O_4 & O_4 \\ E_{2,2} & O_4 \end{pmatrix} + T_2^* \begin{pmatrix} O_4 & E_{1,1} \\ O_4 & O_4 \end{pmatrix} & \\ \quad + T_1 T_2 \begin{pmatrix} O_4 & O_4 \\ E_{3,3} & O_4 \end{pmatrix} + T_1^* T_2^* \begin{pmatrix} O_4 & E_{4,4} \\ O_4 & O_4 \end{pmatrix} + \begin{pmatrix} O_4 & E_{3,3} \\ E_{4,4} & O_4 \end{pmatrix} & \text{if } \Gamma = \Gamma_S, \\ T_1 \begin{pmatrix} E_{5,5} & O_6 \\ E_{1,1} & E_{5,5} \end{pmatrix} + T_1^* \begin{pmatrix} E_{6,6} & E_{2,2} \\ O_6 & E_{6,6} \end{pmatrix} + T_2 \begin{pmatrix} O_6 & O_6 \\ E_{2,2} & O_6 \end{pmatrix} + T_2^* \begin{pmatrix} O_6 & E_{1,1} \\ O_6 & O_6 \end{pmatrix} & \\ \quad + T_1 T_2 \begin{pmatrix} O_6 & O_6 \\ E_{3,3} & O_6 \end{pmatrix} + T_1^* T_2^* \begin{pmatrix} O_6 & E_{4,4} \\ O_6 & O_6 \end{pmatrix} + \begin{pmatrix} O_6 & E_{3,3} \\ E_{4,4} & O_6 \end{pmatrix} & \text{if } \Gamma = \Gamma_T. \end{cases}$$

In particular, one has $S_0 \in \rho(C(\widehat{\mathbb{Z}^2}; M_{2d}(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^2)$ *(Actually,* $S_0 \in \rho(M_{2d}(\mathbb{C}) \rtimes_0 \mathbb{Z}^2)$*).*

Theorem 4.2.2. *Let* $C_\bullet : \mathbb{Z}^2 \to U(d)$ *be an anisotropic coin map with a limit matrix* $C_\infty = C_e \oplus C_o \in U(2d)$*. Then, the essential spectrum of the time evolution operator* $U \in \mathbb{B}(\mathcal{H}_\Gamma)$ *associated with* $C_\bullet$ *is given by*

$$\sigma_{\mathrm{ess}}(U) = \sigma_{\mathrm{ess}}(U_0) = \sigma(U_\infty) = \bigcup_{(\theta,\varphi)\in[0,2\pi)^2} \sigma(\widehat{U}_\infty(\theta,\varphi)),$$

where $U_\infty = S_0 C_\infty$ *is the auxiliary time evolution operator, and* $\widehat{U}_\infty(\theta,\varphi)$ *is the unitary matrix* (3.2.1) *in Lemma* 3.2.2*. In particular,* $\sigma_{\mathrm{ess}}(U_\infty) = \sigma(U_\infty)$ *also holds.*

Proof. In the same way as $C(\widehat{\mathbb{Z}^r}; M_s(\mathbb{C})) \hookrightarrow \mathbb{B}(\mathcal{H}_{r,s})$, we can see that $C([0,2\pi]^2; M_{2d}(\mathbb{C}))$ is a C^*-subalgebra of $\mathbb{B}(L_2((0,2\pi)^2; \mathbb{C}^{2d}))$ by$f \mapsto \int_{(0,2\pi)^2}^{\oplus} f(\theta,\varphi) \frac{d\theta d\varphi}{4\pi^2}$. By this embedding and Lemma 3.2.2, $\widehat{U}_\infty(\theta,\varphi)$ maps to $\widehat{U}_\infty$. Since any injective $*$-homomorphism between C^*-algebras preserves the spectrum, one gets

$$\sigma(U_\infty) = \sigma(\widehat{U}_\infty) = \bigcup_{(\theta,\varphi)\in[0,2\pi]^2} \sigma(\widehat{U}_\infty(\theta,\varphi)) = \bigcup_{(\theta,\varphi)\in[0,2\pi)^2} \sigma(\widehat{U}_\infty(\theta,\varphi)).$$

Since the map $C'_\bullet - C_\infty : \mathbb{Z}^2 \to M_{2d}(\mathbb{C})$ is in $C_0(\mathbb{Z}^2; M_{2d}(\mathbb{C}))$, it follows form Lemma 4.1.3 that $U_0 - U_\infty = S_0(C_0 - C_\infty) \in \mathbb{K}(\mathcal{K}_\Gamma)$. Hence, one obtains that $\sigma_{\mathrm{ess}}(U) = \sigma_{\mathrm{ess}}(U_0) = \sigma_{\mathrm{ess}}(U_\infty) = \sigma(\pi(U_\infty))$, here π is the natural projection from $\mathbb{B}(\mathcal{K}_\Gamma)$ to the Calkin algebra $\mathcal{Q}(\mathcal{K}_\Gamma)$. So let us compute $\sigma(\pi(U_\infty))$. Since it is similar, we will consider only the case of $\Gamma = \Gamma_H$. By the embedding (4.1.1) and Lemma 4.2.1, $\pi(U_\infty) \in \mathcal{Q}(\mathcal{K}_{\Gamma_H})$ is equal to

$$\begin{aligned} &\begin{pmatrix} O_3 & O_3 \\ E_{1,1} & O_3 \end{pmatrix} C_\infty \delta_{(1,0)} + \begin{pmatrix} O_3 & E_{2,2} \\ O_3 & O_3 \end{pmatrix} C_\infty \delta_{(-1,0)} + \begin{pmatrix} O_3 & O_3 \\ E_{2,2} & O_3 \end{pmatrix} C_\infty \delta_{(0,1)} \\ &\qquad + \begin{pmatrix} O_3 & E_{1,1} \\ O_3 & O_3 \end{pmatrix} C_\infty \delta_{(0,-1)} + \begin{pmatrix} O_3 & E_{3,3} \\ E_{3,3} & O_3 \end{pmatrix} C_\infty \delta_{(0,0)} \in M_6(\mathbb{C}) \rtimes_0 \mathbb{Z}^2. \end{aligned} \tag{4.2.1}$$

It is well known that there is a unique $*$-isometric isomorphism between $M_{2d}(\mathbb{C}) \rtimes_0 \mathbb{Z}^2$ and $C(\mathbb{T}^2; M_{2d}(\mathbb{C}))$ such that $A\delta_{n,m} \leftrightarrow A z_1^n z_2^m$, where $A \in M_{2d}(\mathbb{C})$ and $\mathbb{T}^2 \to \mathbb{T}; (z_1, z_2) \mapsto z_1^n z_2^m$. Under this $*$-isometric isomorphism, (4.2.1) is equal to

$$\begin{aligned} &\begin{pmatrix} O_3 & O_3 \\ E_{1,1} & O_3 \end{pmatrix} C_\infty z_1 + \begin{pmatrix} O_3 & E_{2,2} \\ O_3 & O_3 \end{pmatrix} C_\infty z_1^{-1} + \begin{pmatrix} O_3 & O_3 \\ E_{2,2} & O_3 \end{pmatrix} C_\infty z_2 \\ &\qquad + \begin{pmatrix} O_3 & E_{1,1} \\ O_3 & O_3 \end{pmatrix} C_\infty z_2^{-1} + \begin{pmatrix} O_3 & E_{3,3} \\ E_{3,3} & O_3 \end{pmatrix} C_\infty \\ &= \begin{pmatrix} O_3 & \begin{pmatrix} z_2^{-1} & 0 & 0 \\ 0 & z_1^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{pmatrix} C_\infty \in C(\mathbb{T}^2; M_6(\mathbb{C})), \end{aligned}$$

whose spectrum is as follows:

$$\bigcup_{(z_1,z_2)\in\mathbb{T}^2} \sigma\left(\left(\begin{matrix} O_3 & \begin{pmatrix} z_2^{-1} & 0 & 0 \\ 0 & z_1^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & O_3 \end{matrix}\right) C_\infty\right) = \bigcup_{(\theta,\varphi)\in[0,2\pi)} \sigma(\widehat{U}_\infty((\theta,\varphi))).$$

From the above, it follows that $\sigma(\pi(U_\infty)) = \bigcup_{(\theta,\varphi)\in[0,2\pi)} \sigma(\widehat{U}_\infty((\theta,\varphi)))$, hence the statement is proved. $\square$

Chapter 5

The isotropic case

In this last chapter, we shall consider only the case where $\Gamma = \Gamma_S, \Gamma_T$. In this case, one can discuss without using $J_\Gamma, \mathcal{K}_\Gamma$ and the modification of unitary operators S_0, C_0, and U_0 because there is no the complexity of the representation of the shift operator S. Therefore, in this chapter, when we use terminologies, "shift, coin, and time evolution operators" in the sense of Definition 2.3.2. Also we will deal with only "isotropic" coin maps. Then we can obtain a weak limit theorem and an expression of the essential spectrum of U in a similar way to the proof for the anisotropic case.

Suppose that Γ is either Γ_S or Γ_T, and d is its degree.

Definition 5.0.1. 1) A coin map $C_\bullet : \mathbb{Z}^2 \to U(d)$ is said to be *isotropic* if it can be continuously extended to $\widehat{\mathbb{Z}^2}$, that is, there exists $C_\infty \in U(d)$ such that

$$\|C_{(n,m)} - C_\infty\|_{M_d(\mathbb{C})} \to 0 \quad \text{as} \quad \mathbb{Z}^2 \ni (n,m) \to \infty.$$

In this case, the unitary matrix $C_\infty \in U(d)$ is called the *limit matrix* of $C_\bullet$.

As a special case, if

$$C_{(n,m)} = C_\infty \quad \text{for all} \quad (n,m) \in \mathbb{Z}^2 \tag{5.0.1}$$

for some $C_\infty \in U(d)$, then a coin map $C_\bullet$ is said to be *uniform*.

2) In the same way as the previous chapters, the coin and time evolution operators on $\mathcal{H}_\Gamma$ associated with a uniform coin map are respectively denoted by C_∞ and U_∞ instead of C and U.

3) Suppose that $C_\bullet : \mathbb{Z}^2 \to U(d)$ is an isotropic coin map with a limit matrix $C_\infty \in U(d)$. Then we can consider the *auxiliary time evolution operator* $U_\infty = SC_\infty$ which is a time evolution operator associated with the uniform coin map ($\equiv C_\infty$ on $\mathbb{Z}^2$) other than $U = SC$.

It follows from the definition of $\widetilde{\mathbb{Z}^2}$ that the following inclusion relations:

$$\begin{array}{ccc}
\{\text{uniform coin maps}\} & \subset & \{\text{quasi-uniform coin maps}\} \\
\cap & & \cap \\
\{\text{isotropic short-range coin maps}\} & \subset & \{\text{anisotropic short-range coin maps}\} \\
\cap & & \cap \\
\{\text{isotropic coin maps}\} & \subset & \{\text{anisotropic coin maps}\},
\end{array}$$

where isotropic short-range coin maps will be defined in Definition 5.1.2.

5.1 Weak limit theorem

The facts of Chapter 3 can be proved in the above model as well, so one can get a weak limit theorem of the isotropic version. However, since there are a few changes, we shall list them below.

The momentum operators $D_1, D_2 : \mathscr{F}(\mathcal{H}_0) \to L_2((0,2\pi)^2, d\theta d\varphi/4; \mathbb{C}^d)$ are defined by

$$D_1 := \frac{1}{i}\frac{\partial}{\partial\theta}, \quad D_2 := \frac{1}{i}\frac{\partial}{\partial\varphi},$$

where $\mathcal{H}_0$ is the set of all functions of $\mathcal{H}_\Gamma$ with finite support. When a coin map $C_\bullet$ is uniform satisfying (5.0.1), the Fourier transform $\widehat{U}_\infty = \mathscr{F} U_\infty \mathscr{F}^{-1}$ of U_∞ is decomposable as $\widehat{U}_\infty = \int_{(0,2\pi)^2}^{\oplus} \widehat{U}_\infty(\theta,\varphi)\frac{d\theta d\varphi}{4\pi^2}$, and the unitary matrix $\widehat{U}_\infty(\theta,\varphi) \in U(d)$ is given by

$$\widehat{U}_\infty(\theta,\varphi) = \begin{cases} \begin{pmatrix} e^{-i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{-i\varphi} & 0 \\ 0 & 0 & 0 & e^{i\varphi} \end{pmatrix} C_\infty & \text{if } \Gamma = \Gamma_S, \\ \begin{pmatrix} e^{-i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\varphi} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i(\theta+\varphi)} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i(\theta+\varphi)} \end{pmatrix} C_\infty & \text{if } \Gamma = \Gamma_T. \end{cases} \tag{5.1.1}$$

Definition 5.1.1. If the unitary matrix $C_\infty \in U(d)$ satisfies Assumption 3.2.3 (change $2d$ to d), one defines two bounded self-adjoint operators V_1^Γ, V_2^Γ on $\mathcal{H}_\Gamma$ by

$$V_1^\Gamma = \mathscr{F}^{-1}\left(\int_{(0,2\pi)^2}^{\oplus} \sum_{j=1}^{d} \lambda_j^\theta(\theta,\varphi)\, |u_j(\theta,\varphi)\rangle\langle u_j(\theta,\varphi)|\, \frac{d\theta d\varphi}{4\pi^2}\right)\mathscr{F},$$

$$V_2^\Gamma = \mathscr{F}^{-1}\left(\int_{(0,2\pi)^2}^{\oplus} \sum_{j=1}^{d} \lambda_j^\varphi(\theta,\varphi)\, |u_j(\theta,\varphi)\rangle\langle u_j(\theta,\varphi)|\, \frac{d\theta d\varphi}{4\pi^2}\right)\mathscr{F}.$$

Definition 5.1.2. An isotropic coin map $C_\bullet : \mathbb{Z}^2 \to U(d)$ with a limit matrix $C_\infty \in U(d)$ is said to be of *short range* if there exist $\kappa > 0$ and $\varepsilon > 0$ such that

$$\|C_{(n,m)} - C_\infty\|_{M_d(\mathbb{C})} \le \kappa(1 + \|(n,m)\|_1)^{-2-\varepsilon} \text{ for all } (n,m) \in \mathbb{Z}^2.$$

Theorem 5.1.3. *Let $C_\bullet : \mathbb{Z}^2 \to U(d)$ be an isotropic coin map and $\Psi^0 \in \mathcal{H}_\Gamma$ an initial state. Suppose that $C_\bullet$ is of short range and its limit matrix satisfies Assumption* 3.2.3. *Then, the distribution of X_t/t is weakly convergent to a probability measure*

$$\mu = \|\Pi_{\mathrm{pp}}(U)\Psi^0\|^2_{\mathcal{H}_\Gamma}\delta_{(0,0)} + \|(E_{V_1^\Gamma} \otimes E_{V_2^\Gamma})(\cdot)(\Omega_+^\Gamma)^*\Psi^0\|^2_{\mathcal{H}_\Gamma}.$$

Furthermore, if localization occurs, then we have $\mu(\{(0,0)\}) > 0$.

Corollary 5.1.4. *Under the same assumption as in Theorem* 5.1.3*, if $C_\bullet$ is uniform (thus automatically it is of short range and $U = U_\infty$), then the distribution of X_t/t weakly converges to a probability measure*

$$\mu = \|\Pi_{\mathrm{pp}}(U_\infty)\Psi^0\|^2_{\mathcal{H}_\Gamma}\delta_{(0,0)} + \|(E_{V_1^\Gamma} \otimes E_{V_2^\Gamma})(\cdot)\Pi_{\mathrm{ac}}(U_\infty)\Psi^0\|^2_{\mathcal{H}_\Gamma}.$$

Watabe, Kobayashi, Katori, and Konno [40] proved a weak limit theorem for generalized Grover walks on the square lattice $\mathbb{Z}^2$. The model of this quantum walk is realized by taking

$$C_\bullet \equiv G(p) := \begin{pmatrix} -p & q & \sqrt{pq} & \sqrt{pq} \\ q & -p & \sqrt{pq} & \sqrt{pq} \\ \sqrt{pq} & \sqrt{pq} & -q & p \\ \sqrt{pq} & \sqrt{pq} & p & -q \end{pmatrix} \quad (0 < p < 1,\ q = 1 - p),$$

as a uniform coin map on the graph $\Gamma = \Gamma_S$, and assuming that the initial state starts at the origin with probability 1 (i.e. $\|\Psi^0(0,0)\|_{\mathbb{C}^4} = 1$). The matrix $G(p)$ is called the Grover matrix when $p = \frac{1}{2}$. In the case where $C_\bullet \equiv G(p)$, the unitary matrix $\widehat{U}_\infty(\theta,\varphi)$ coincides with

$$\frac{1}{2}\begin{pmatrix} -pe^{-i\theta} & qe^{-i\theta} & \sqrt{pq}e^{-i\theta} & \sqrt{pq}e^{-i\theta} \\ qe^{i\theta} & -pe^{i\theta} & \sqrt{pq}e^{i\theta} & \sqrt{pq}e^{i\theta} \\ \sqrt{pq}e^{-i\varphi} & \sqrt{pq}e^{-i\varphi} & -qe^{-i\varphi} & pe^{-i\varphi} \\ \sqrt{pq}e^{i\varphi} & \sqrt{pq}e^{i\varphi} & pe^{i\varphi} & -qe^{i\varphi} \end{pmatrix},$$

and its eigenvalue $\lambda_j(\theta,\varphi)$ is given by

$$\lambda_1(\theta,\varphi) \equiv 1, \quad \lambda_2(\theta,\varphi) \equiv -1, \quad \lambda_3(\theta,\varphi) = e^{i\omega(\theta,\varphi)}, \quad \lambda_4(\theta,\varphi) = e^{-i\omega(\theta,\varphi)},$$

where $\omega(\theta,\varphi) = \arccos\{-(p\cos\theta + q\cos\varphi)\}$. But, for example, the second order derivative $\partial^2\lambda_3/\partial\theta^2$ of $\lambda_3(\theta,\varphi)$ is not bounded on $(0,2\pi)^2$, so the matrix $G(p)$ does not satisfy Assumption 3.2.3. Thus Corollary 5.1.4 does not contain the weak limit theorem which is provided by Watabe *et al.* [40].

5.2 Essential spectrum of U

The following theorem can also be obtained by the similar way of the proof of Theorem 4.2.2.

Theorem 5.2.1. *Let $C_\bullet : \mathbb{Z}^2 \to U(d)$ be an isotropic coin map with a limit matrix $C_\infty \in U(d)$. Then, the essential spectrum of the time evolution operator $U \in \mathbb{B}(\mathcal{H}_\Gamma)$ associated with $C_\bullet$ is given by*

$$\sigma_{\mathrm{ess}}(U) = \sigma(U_\infty) = \bigcup_{(\theta,\varphi)\in[0,2\pi)^2} \sigma(\widehat{U}_\infty(\theta,\varphi)),$$

where $U_\infty = SC_\infty$ is the auxiliary time evolution operator, and $\widehat{U}_\infty(\theta,\varphi)$ is the unitary matrix (5.1.1)*. In particular, $\sigma_{\mathrm{ess}}(U_\infty) = \sigma(U_\infty)$.*

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