Palindromes and v-Palindromes
(回文数と v-回文数)

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Abstract

We introduce the concept of v-palindromes and prove that the v-palindromicity of the terms of the sequence of repeated concatenations of the digits of an arbitrary natural number is periodic. Then, we examine this periodic phenomenon more closely, introducing concepts such as the indicator function of a number and the type of a v-palindrome. Our subsequent main purposes are to

(i) provide a general procedure to express the indicator function of a number as a certain linear combination,

(ii) prove an invariance property about the type of a v-palindrome,

(iii) prove the existence of v-palindromes in infinitely many bases.

However we also

(iv) provide a survey of past results on the usual palindromes and other palindromic objects,

(v) provide a treatment of periodic functions because of its relevance to (i) above,

(vi) consider repeated concatenations in residue classes.

In the conclusion, we collect some conjectures and problems and describe how the content of this dissertation might be generalizable.

Contents

Chapter 1

Introduction

Besides the Introduction and Conclusion, this dissertation consists of four parts as follows.

(I) A survey of past results on palindromes and other "palindromic objects" (Chapters 2 and 3),

(II) A treatment of periodic functions (Chapter 5),

(III) Presentation of results on v-palindromes (Chapters 4 and 6 to 8),

(IV) An algorithm to find repeated concatenations in residue classes (Chapter 9).

In (I), we state past results but only prove a few of them. As will be explained in Section 1.3.1, the v-palindromes can be regarded as an analogy to the usual palindromes. This is why we include this survey.

In (II), we start with the definition of periodic functions and then prove four formulae for their fundamental periods. We include this treatment because of its relavance to Chapter 6. The content of this part is from [46].

(III) is based on [48, 46, 47, 50].

(IV) is based on [49].

The logical dependency of these parts is that (I), (II), and (IV) are independent whereas (III) depends on (II).

The rest of this Introduction is mostly an outline of most of the rest of this dissertation. In Section 1.1, we list some notation to be used in this dissertation. In Section 1.2, we describe the origin of the concept of v-palindromes. In Section 1.3, we define v-palindromes rigorously.

In Section 1.4, we state [48, Theorem 1] as Theorem 1.2. This theorem says that the v-palindromicity of the terms of the sequence of repeated concatenations of the decimal digits of an arbitrary natural number is periodic. We also call this theorem the *periodic phenomenon* in this dissertation. When we have a periodic phenomenon, there will be a smallest period, also called the fundamental period. We provide a method to derive the fundamental period of Theorem 1.2. In fact, we provide a procedure to express what we call the *indicator function* for a number (see Section 1.5) as a certain linear combination from which the fundamental period can be easily derived as a least common multiple. We call this procedure the *general procedure* in this dissertation and it is described in Section 6.6. We also define a concept of the *type* of a v-palindrome (see Section 1.6). Although this concept is originally defined as a relative concept, we prove that in fact it is absolute. We call this the *invariance property* in this dissertation and it is stated rigorously as Theorem 7.1.

Section 1.7 provides a specific family of v-palindromes based on the smallest v-palindrome 18. Just like usual palindromicity, the v-palindromicity of a number depends on the base used to represent it. In Section 1.8, we provide a table of the smallest v-palindrome in base b, which we call (v, b)-palindrome, for $2 \leq b \leq 19$.

In Section 1.9, we describe our motivation for (IV), i.e., Chapter 9. In Section 1.10, we explain the relation of the paper by Vaidyanathan [51] from signal processing to this dissertation. In Section 1.11, we describe some connection of the concept of v-palindromes to works of others.

Our derivation of the fundamental period of Theorem 1.2 belongs naturally to the more general derivation of the fundamental period of an arbitrary periodic function. It is surprising that there are extensive investigations of functions $\mathbb{Z} \to \mathbb{C}$ in signal processing (cf. [39, 51, 52, 53]), where such functions are called *discrete signals* or *discrete-time signals*.

1.1 Notation

We list some notation to be used in this dissertation as follows.

- For an integer a, the set of integers $\geq a$ is denoted by $\mathbb{Z}_{\geq a}$.
- For an integer $a \geq 1$, the set of a-th roots of unity in $\mathbb{C}$ is denoted by $\mathcal{R}(a)$.
- For integers $a \neq 0$ and b, the notation $a \mid b$ means that a divides b.
- For a prime p and integer $n \neq 0$, the exponent of p in the prime factorization of n is denoted by $\mathrm{ord}_p(n)$.
- For a prime p and integers $\alpha \geq 0$ and $n \neq 0$, the notation $p^\alpha \parallel n$ means that $\mathrm{ord}_p(n) = \alpha$.
- For integers $a_1, a_2 \dots, a_n$ not all 0, their greatest common divisor is denoted by $(a_1, a_2, \dots, a_n)$.
- For a statement P, the *Iverson symbol* $[P]$ is defined by setting $[P] = 1$ if P is true and $[P] = 0$ if P is false.
- The *sign function* is the function $\mathrm{sgn}\colon \mathbb{R} \setminus \{0\} \to \{-1, 1\}$ defined by setting $\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$.

1.2 Origin of the concept

As I recall, it was some time in the first half of the year 2007 when I was 15 years old. My mother and younger brother were in a video rental shop near our home in Taipei and my father and I were waiting outside the shop, standing beside our parked car. I was a bored and glanced at the license plate of our car, which was 0198-QB. For no clear reason, I took the number 198 and did the following. I factorized $198 = 2 \cdot 3^2 \cdot 11$, reversed the digits of 198, and factorized $891 = 3^4 \cdot 11$. Then I summed the numbers appearing in each factorization: $2 + 3 + 2 + 11 = 18$ and $3 + 4 + 11 = 18$ respectively. So surprisingly they are equal! We illustrate this pictorially as follows.

$$\begin{array}{cccccc} 198 & = & 2 \cdot 3^2 \cdot 11 & \longmapsto & 2 + (3 + 2) + 11 \\ \uparrow & & & & \| \\ \text{reverses} & & & & 18 \\ \downarrow & & & & \| \\ 891 & = & 3^4 \cdot 11 & \longmapsto & (3 + 4) + 11 \end{array} \tag{1.1}$$

Strictly speaking, the 2 and 11's in the factorizations above have exponents being 1. However, because they are usually not written, we do not sum them. We provide another example of such a number as follows.

$$\begin{array}{cccccc} 56056 & = & 2^3 \cdot 7^2 \cdot 11 \cdot 13 & \longmapsto & (2 + 3) + (7 + 2) + 11 + 13 \\ \uparrow & & & & \| \\ \text{reverses} & & & & 38 \\ \downarrow & & & & \| \\ 65065 & = & 5 \cdot 7 \cdot 11 \cdot 13^2 & \longmapsto & 5 + 7 + 11 + (13 + 2) \end{array} \tag{1.2}$$

After returning home from the video rental shop, I spent some time to try to show that there is an infinitude of such numbers but could not show it. I did not develop this concept much further for the next 11 years. Then in October 2018, I published a very short note [45] in the Sūgaku Seminar magazine. In this note I merely defined such numbers and showed their infinitude, though I recall already knowing how to show their infinitude as early as the summer of 2015.

1.3 v-palindromes

In Subsection 1.3.1, we introduce the concept of v-palindromes rigorously by first defining the more general concept of (f, b)-palindromes and then singling out the case of v-palindromes. Then in Subsection 1.3.2, we mention the infinitude of v-palindromes.

1.3.1 (f, b)-palindromes

Recall the base b representation of a number as follows.

Definition 1.1. For integers $b \geq 2$, $L \geq 1$, and $0 \leq a_0, a_1, \dots, a_{L-1} < b$, put

$$(a_{L-1} \cdots a_1 a_0)_b = \sum_{i=0}^{L-1} a_i b^i. \tag{1.3}$$

Theorem 1.1 ([30, Theorem 4.7])**.** *Let $b \geq 2$ be an integer. Then for every integer $n \geq 1$, there exist unique integers $L \geq 1$ and $0 \leq a_0, a_1, \dots, a_{L-1} < b$ with $a_{L-1} \neq 0$ such that*

$$n = (a_{L-1} \cdots a_1 a_0)_b. \tag{1.4}$$

Definition 1.2. With notation as in the above theorem, we say that $(a_{L-1} \cdots a_1 a_0)_b$ is the *base b representation* of n, that $a_{L-1}, \dots, a_1, a_0$ are the *base b digits* of n, and that n has *L base b digits*. We also say *decimal* in place of *base* 10.

We next define reverses as follows.

Definition 1.3. Let $n \geq 1$ be an integer with base b representation $(a_{L-1} \cdots a_1 a_0)_b$. Then the *b-reverse* of n is $r_b(n) = (a_0 a_1 \cdots a_{L-1})_b$. The 10-reverse of n will simply be called the *reverse* of n and $r_{10}(n)$ simply denoted by $r(n)$.

We next define b-palindromes as follows.

Definition 1.4. Let $n \geq 1$ be an integer with base b representation $(a_{L-1} \cdots a_1 a_0)_b$. Then n is a *b-palindrome* if $n = r_b(n)$. We also consider 0 to be a b-palindrome for every $b \geq 2$. A 10-palindrome will simply be called a *palindrome*. The set of b-palindromes is denoted by $\mathcal{P}_b$.

We can now define (f, b)-palindromes as follows.

Definition 1.5. Let $f \colon \mathbb{N} \to \mathbb{C}$ be a function and $b \geq 2$ an integer. An integer $n \geq 1$ is an *(f, b)-palindrome* if

(i) $b \nmid n$,

(ii) $n \neq r_b(n)$, and

(iii) $f(n) = f(r_b(n))$.

In particular, an $(f, 10)$-palindrome will simply be called an *f-palindrome*.

We illustrate the concept that n is an (f, b)-palindrome pictorially as follows.

$$\begin{array}{ccc} n & \longmapsto & f(n) \\ \uparrow & & \| \\ b\text{-reverses} & & \text{same number} \\ \downarrow & & \| \\ r_b(n) & \longmapsto & f(r_b(n)) \end{array}$$

It is clear that if n is an (f, b)-palindrome then so is $r_b(n)$. We explain our naming of (f, b)-palindrome. An integer $n \geq 1$ is a *b-palindrome* if and only if $n = r_b(n)$. Condition (iii) has $f(n) = f(r_b(n))$ instead, and hence *(f, b)-palindrome*. Condition (i) is included so that n and $r_b(n)$ have the same number of digits. Condition (ii) is included to exclude the uninteresting case $n = r_b(n)$, from which obviously $f(n) = f(r_b(n))$. We next define the function v as follows.

Definition 1.6. The additive function $v\colon \mathbb{N} \to \mathbb{Z}$ is defined by setting $v(p) = p$ for each prime p and $v(p^\alpha) = p + \alpha$ for each prime power p^α with $\alpha \geq 2$.

Remark 1.1. The "v" is for "value". The quantity $v(n)$ is thought of as the "value" of n. The quantities $v(n)$ have been created as sequence A338038 in the OEIS.

The concept of a *v-palindrome* now follows from Definitions 1.5 and 1.6, but because it is the most important definition in this dissertation, we state it on its own as follows.

Definition 1.7. An integer $n \geq 1$ is a *v-palindrome* if $10 \nmid n$, $n \neq r(n)$, and $v(n) = v(r(n))$. The set of v-palindromes is denoted by $\mathbb{V}$.

Remark 1.2. The sequence of v-palindromes has been created as A338039 in OEIS and we show the first few terms as follows.

$$\begin{gathered}18, 81, 198, 576, 675, 819, 891, 918, 1131, 1304, 1311, 1818, 1998, 2262,\\ 2622, 3393, 3933, 4031, 4154, 4514, 4636, 6364, 8181, 8749, 8991, 9478,\\ 12441, 14269, 14344, 14421, 15167, 15602, 16237, 18018, 18449, 18977,\\ 19998, 20651, 23843, 24882, 26677, 26892, 27225, \ldots.\end{gathered}$$

It is easily seen that Definition 1.7 suitably formalizes the concept of the previous section and so 198 and 56056 are v-palindromes. In particular, not summing exponents which are 1 leads to the unnatural two-cases definition of v. If we do not impose $10 \nmid n$, then 560 would be a v-palindrome. However we impose $10 \nmid n$ so that n and $r(n)$ have the same number of digits and so consider 560 as not a v-palindrome. We can discard $n \neq r(n)$ and regard the positive palindromes as *trivial* v-palindromes. However we do not adopt this alternative viewpoint in this dissertation.

In this dissertation we will only be dealing with (v, b)-palindromes, but I have already started collaborating with Professor Prapanpong Pongsriiam on more general (f, b)-palindromes.

1.3.2 The infinitude of v-palindromes

The first natural question to ask about any kind of number after defining them is whether there are infinitely many of them. As proved in [45], we have the sequence of v-palindromes

$$18, 198, 1998, 19998, \ldots, \tag{1.5}$$

where we simply continue to increase the number of 9's in the middle. Also mentioned in [45] is the sequence of v-palindromes

$$18, 1818, 181818, \ldots, \tag{1.6}$$

where we simply continue to concatenate another 18. This sequence was the original inspiration for the periodic phenomenon (Theorem 1.2). We also have the sequences of v-palindromes

$$198, 198198, 198198198, \ldots, \tag{1.7}$$

$$576, 576576, 576576576, \ldots. \tag{1.8}$$

In fact (1.5), (1.6), and (1.7) are subfamilies of the more general family of v-palindromes of Theorem 1.8.

1.4 The periodic phenomenon

In Subsection 1.4.1, we state the periodic phenomenon and make some comments. Then in Subsection 1.4.2, we provide examples.

1.4.1 The statement

We give the following definition.

Definition 1.8. Let $n \geq 1$ be an integer with base b representation $(a_{L-1}a_{L-2}\cdots a_0)_b$. For integers $k \geq 1$, put

$$\begin{aligned} n(k)_b &= (\underbrace{a_{L-1}\cdots a_1a_0\, a_{L-1}\cdots a_1a_0 \;\cdots\cdots\; a_{L-1}\cdots a_1a_0}_{k \text{ copies of } a_{L-1}\cdots a_1a_0})_b \\ &= n(1+b^L+\cdots+b^{(k-1)L}) = n\cdot\frac{1-b^{kL}}{1-b^L}. \end{aligned} \tag{1.9}$$

We say that $n(k)_b$ is the (b,k)-*repeated concatenation* of n. We denote $n(k)_{10}$ simply by $n(k)$ and call the $(10,k)$-repeated concatenation of n simply the k-*repeated concatenation* of n. As a loose term, we call such numbers defined here *repeated concatenations*.

For example, $18(3) = 181818$ and $56056(2) = 5605656056$. We can now state the periodic phenomenon, which is one of the main results of this dissertation, as follows.

Theorem 1.2 ([48, Theorem 1]). *Let $n \geq 1$ be an integer with $10 \nmid n$ and $n \neq r(n)$. There exists an integer $\omega \geq 1$ such that for all integers $k \geq 1$,*

$$n(k) \in \mathbb{V} \quad \textit{if and only if} \quad n(k+\omega) \in \mathbb{V}. \tag{1.10}$$

Let us make some comments on Theorem 1.2. We have the sequence

$$n(1), n(2), n(3), \ldots. \tag{1.11}$$

Now replace each term above by 1 if it is a v-palindrome and 0 otherwise. So for instance it might become

$$1, 0, 0, 1, 1, 1, 0, 1, \ldots. \tag{1.12}$$

The theorem says that this sequence of 0's and 1's is periodic. We illustrate with $n = 48$ and $n = 117$ in the next subsection. We also give the following definition.

Definition 1.9. Let n be as in Theorem 1.2. An integer $\omega \geq 1$ satisfying the condition of Theorem 1.2 is called a *period* of n. The smallest period of n is called the *fundamental period* of n and denoted by $\omega_0(n)$. If there exists a $k \geq 1$ such that $n(k)$ is a v-palindrome, the least such integer is denoted by $c(n)$; otherwise we write $c(n) = \infty$. The integer (or ∞) $c(n)$ is called the *order* of n.

Remark 1.3. The sequence of n such that $c(n) < \infty$ has been created as sequence A338371 in OEIS.

Just like in the above definition, in this dissertation, we will often use the phrase "n be as in Theorem 1.2" to mean that $n \geq 1$ is an integer with $10 \nmid n$ and $n \neq r(n)$.

1.4.2 Examples

We illustrate Theorem 1.2 with $n = 48$ and $n = 117$, taken from two rows of Table 6.5. A full justification of the following comments follows from Chapter 6.

For $n = 48$, the sequence (1.11) becomes

$$48, 4848, 484848, \ldots. \tag{1.13}$$

Replacing each term above by 1 if it is a v-palindrome and 0 otherwise, it becomes

$$0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots. \tag{1.14}$$

So it might look like $48(k) \in \mathbb{V}$ if and only if $3 \mid k$ and so the smallest period is 3. However in fact $48(21) \notin \mathbb{V}$ and the smallest period is 21. We have $\omega_0(48) = 21$ and $c(48) = 3$.

For $n = 117$, the sequence (1.11) becomes

$$117, 117117, 117117117, \ldots. \tag{1.15}$$

Here we do have the simple rule that $117(k) \in \mathbb{V}$ if and only if $2054 \mid k$. We have $\omega_0(117) = c(117) = 2054$.

1.5 The indicator functions I^n

We define certain functions I^n as follows.

Definition 1.10. Let n be as in Theorem 1.2. The *indicator function* for n is the periodic function $I^n\colon \mathbb{Z} \to \{0,1\}$ such that

$$I^n(k) = \begin{cases} 1 & \text{if } n(k) \in \mathbb{V}, \\ 0 & \text{if } n(k) \notin \mathbb{V}, \end{cases} \quad \text{for} \quad k \geq 1. \tag{1.16}$$

The superscript in I^n simply specifies n and does not denote function composition. The function I^n is defined at the positive integers by (1.16) and then uniquely periodically extended to all integers in view of Theorem 1.2. Then, the following follows from Theorem 5.1.

Theorem 1.3. *The set of all periods of n is $\omega_0(n)\mathbb{N}$.*

We next define certain functions I_a as follows.

Definition 1.11. For integers $a \geq 1$, we denote the indicator function of $a\mathbb{Z} \subseteq \mathbb{Z}$ by I_a. That is, the function $I_a\colon \mathbb{Z} \to \{0,1\}$ is defined by letting

$$I_a(x) = \begin{cases} 1 & \text{if } a \mid x, \\ 0 & \text{if } a \nmid x, \end{cases} \quad \text{for} \quad x \in \mathbb{Z}. \tag{1.17}$$

The *general procedure* of Section 6.6, which is one of the main contributions of this dissertation, expresses I^n in the form

$$\sum_{j=1}^{q} \lambda_j I_{c_j}, \tag{1.18}$$

where $q \geq 0$, $1 \leq c_1 < \cdots < c_q$, and $\lambda_1, \ldots, \lambda_q \neq 0$ are integers. According to Theorem 5.8 such an expression is unique. Then according to Theorem 6.14 we will have

$$\omega_0(n) = \operatorname{lcm}\{c_1, \ldots, c_q\}, \quad c(n) = \inf\{c_1, \ldots, c_q\}. \tag{1.19}$$

The infimum is considered in the extended real number system, so that $\inf \varnothing = \infty$. This matches the notation of Definition 1.9.

1.6 The invariance property

The concept of the *type* of a v-palindrome is defined rigorously in Definition 6.6. In Subsection 1.6.1, we give a rough description of this concept and the *invariance property* (Theorem 7.1), which is one of the main results of this dissertation, about it. Then in Subsection 1.6.2, we illustrate the invariance property with the v-palindrome 13(15).

1.6.1 Rough description

Let n be as in Theorem 1.2. We have the sequence

$$n(1), n(2), n(3), \ldots, n(k), \ldots. \tag{1.20}$$

The v-palindromes in (1.20) are categorized into a finite number of "types" in a way depending on the "starting number" n. The set of possible "types" is denoted by $\mathcal{U}^*(n)$. This notation $\mathcal{U}^*(n)$ actually has the precise meaning as the set of nondegenerate characteristic solutions for n (Definition 6.6). For a v-palindrome $n(k)$ we denote its *type* with respect to n by $\mathbf{Type}(n(k), n)$, which is an element of $\mathcal{U}^*(n)$.

Now let m be a v-palindrome and write $m = n_0(k_0)$, where $n_0, k_0 \geq 1$ are integers and n_0 is minimal. Then

$$\{(n,k) \in \mathbb{N}^2 \colon m = n(k)\} = \{(n_0(d), k_0/d) : d \mid k_0\}. \tag{1.21}$$

Consequently, all the possible types of m are

$$\mathbf{Type}(m, n_0(d)) \in \mathcal{U}^*(n_0(d)), \quad \text{for} \quad d \mid k_0. \tag{1.22}$$

In fact it will hold that

$$\mathcal{U}^*(n_0(1)), \mathcal{U}^*(n_0(2)), \mathcal{U}^*(n_0(3)), \ldots \subseteq \mathcal{U}(n_0), \tag{1.23}$$

where $\mathcal{U}(n_0)$ is the set of characteristic solutions for n_0 (Definition 6.3). Therefore we have

$$\mathbf{Type}(m, n_0(d)) \in \mathcal{U}(n_0), \quad \text{for} \quad d \mid k_0. \tag{1.24}$$

The *invariance property* (Theorem 7.1) says that these types are all the same element of $\mathcal{U}(n_0)$. We illustrate this property with the v-palindrome 13(15) in the next subsection.

1.6.2 The example of 13(15)

The number 13(15) is a v-palindrome and all of its possible types are

$$\mathbf{Type}(13(15), 13), \quad \mathbf{Type}(13(15), 13(3)), \quad \mathbf{Type}(13(15), 13(5)), \quad \mathbf{Type}(13(15), 13(15)). \tag{1.25}$$

It will hold that

$$\mathcal{U}(13) = \{(1)_{p\in\{13,31\}}, (2)_{p\in\{13,31\}}\}. \tag{1.26}$$

Here $(1)_{p\in\{13,31\}}$ is the family indexed by $\{13, 31\}$ with all values being 1; similarly for $(2)_{p\in\{13,31\}}$. The following theorems can be derived using content of Chapter 6.

Theorem 1.4. *Let $k \geq 1$ be an integer. Then the repeated concatenation* 13(k) *is a v-palindrome if and only if*

(i) $6045 \mid k$*, in which case* $\mathbf{Type}(13(k), 13) = (1)_{p\in\{13,31\}}$*, or*

(ii) $15 \mid k$ *but* $13 \nmid k$ *and* $31 \nmid k$*, in which case* $\mathbf{Type}(13(k), 13) = (2)_{p\in\{13,31\}}$.

Theorem 1.5. *Let $k \geq 1$ be an integer. Then the repeated concatenation* 13(3)(k) *is a v-palindrome if and only if*

(i) $2015 \mid k$*, in which case* $\mathbf{Type}(13(3)(k), 13(3)) = (1)_{p\in\{13,31\}}$*, or*

(ii) $5 \mid k$ *but* $13 \nmid k$ *and* $31 \nmid k$*, in which case* $\mathbf{Type}(13(3)(k), 13(3)) = (2)_{p\in\{13,31\}}$.

Theorem 1.6. *Let $k \geq 1$ be an integer. Then the repeated concatenation* 13(5)(k) *is a v-palindrome if and only if*

(i) $1209 \mid k$*, in which case* $\mathbf{Type}(13(5)(k), 13(5)) = (1)_{p\in\{13,31\}}$*, or*

(ii) $3 \mid k$ *but* $13 \nmid k$ *and* $31 \nmid k$*, in which case* $\mathbf{Type}(13(5)(k), 13(5)) = (2)_{p\in\{13,31\}}$.

Theorem 1.7. *Let $k \geq 1$ be an integer. Then the repeated concatenation* 13(15)(k) *is a v-palindrome if and only if*

(i) $403 \mid k$*, in which case* $\mathbf{Type}(13(15)(k), 13(15)) = (1)_{p\in\{13,31\}}$*, or*

(ii) $13 \nmid k$ *and* $31 \nmid k$*, in which case* $\mathbf{Type}(13(15)(k), 13(15)) = (2)_{p\in\{13,31\}}$.

It is then easily checked that all of the types (1.25) are $(2)_{p\in\{13,31\}}$.

1.7 A family of v-palindromes

We have the following family of v-palindromes based on 18.

Theorem 1.8 ([49, Theorem 3])**.** *If $\rho \geq 1$ is a palindrome all of whose digits are* 0 *or* 1*, then* 18ρ *is a v-palindrome.*

Proof. When read from left to right, the decimal representation of ρ must be formed by a_1 digits of 1's, followed by a_2 digits of 0's, followed by a_3 digits of 1's, and so on until lastly, a_{2r-1} digits of 1's, where $r, a_1, a_2, \ldots, a_{2r-1} \geq 1$ are integers such that $a_i = a_{2r-i}$ for $1 \leq i \leq 2r-1$. Writing ρ out,

$$\rho = \underbrace{1\cdots1}_{a_1}\overbrace{0\cdots0}^{a_2}\underbrace{1\cdots1}_{a_3}\cdots\cdots\underbrace{1\cdots1}_{a_3}\overbrace{0\cdots0}^{a_2}\underbrace{1\cdots1}_{a_1}.$$

Thus

$$18\rho = 1\underbrace{9\cdots9}_{a_1-1}8\overbrace{0\cdots0}^{a_2-1}1\underbrace{9\cdots9}_{a_3-1}8\cdots\cdots1\underbrace{9\cdots9}_{a_3-1}8\overbrace{0\cdots0}^{a_2-1}1\underbrace{9\cdots9}_{a_1-1}8,$$

$$81\rho = 8\underbrace{9\cdots9}_{a_1-1}1\overbrace{0\cdots0}^{a_2-1}8\underbrace{9\cdots9}_{a_3-1}1\cdots\cdots8\underbrace{9\cdots9}_{a_3-1}1\overbrace{0\cdots0}^{a_2-1}8\underbrace{9\cdots9}_{a_1-1}1,$$

and we see that $r(18\rho) = 81\rho \neq 18\rho$. Clearly $10 \nmid 18\rho$. Now suppose that $3^g \parallel \rho$ and write $\rho = 3^g b$. Then

$$v(18\rho) = v(2 \cdot 3^2 \cdot 3^g b) = v(2 \cdot 3^{2+g} b) = v(2 \cdot 3^{2+g}) + v(b),$$
$$v(81\rho) = v(3^4 \cdot 3^g b) = v(3^{4+g} b) = v(3^{4+g}) + v(b).$$

Since $v(2 \cdot 3^{2+g}) = v(3^{4+g}) = 7 + g$, we see that $v(18\rho) = v(81\rho)$. Therefore 18ρ is a v-palindrome. □

This theorem sort of relates the usual palindromes to v-palindromes. If we restrict ρ to

- be a repunit, then we recover (1.5),
- have alternating digits of 1 and 0, then we recover (1.6),
- have alternating strings of 11 and 0, then we recover (1.7).

1.8 (v, b)-palindromes

Instead of just the v-palindromes, we can also consider more general (v, b)-palindromes, giving the following notation.

Notation 1.12. For integers $b \geq 2$, the set of (v, b)-palindromes is denoted by $\mathbb{V}_b$.

The first natural consideration would be about their existence. We have the following theorem which says that as long as one exists, then infinitely many exist.

Theorem 1.9 ([50, Theorem 5])**.** *Let $b \geq 2$ be an integer. If there exists a (v, b)-palindrome, then there exist infinitely many (v, b)-palindromes.*

On the other hand, the existence of a (v, b)-palindrome has been established for infinitely many bases b as implied by the following.

Theorem 1.10 ([50, Corollary 12])**.** *If $b \equiv 120 \pmod{330}$ is a positive integer, then there exists a (v, b)-palindrome.*

Theorems 1.9 and 1.10 are two of the main results of this dissertation. In fact, a (v, b)-palindrome is found for each b in Theorem 1.10 in Theorem 8.9. The smallest (v, b)-palindrome in the bases $b \leq 19$ are tabulated as follows.

Table 1.1: The smallest (v, b)-palindrome in the bases $b \leq 19$.

b	$\min(\mathbb{V}_b)$ written in base 10	$\min(\mathbb{V}_b)$ written in base b
2	175	1, 0, 1, 0, 1, 1, 1, 1
3	1280	1, 2, 0, 2, 1, 0, 2
4	6	1, 2
5	288	2, 1, 2, 3
6	10	1, 4
7	731	2, 0, 6, 3
8	14	1, 6
9	93	1, 1, 3
10	18	1, 8
11	135	1, 1, 3
12	22	1, 10
13	63	4, 11
14	26	1, 12
15	291	1, 4, 6
16	109	6, 13
17	581	2, 0, 3
18	34	1, 16
19	144	7, 11

It is still unsettled whether a (v, b)-palindrome exists in every base b.

1.9 Repeated concatenations in residue classes

We describe the motivation for (IV), i.e., Chapter 9. The following theorem was originally in the language of arithmetic sequences, but we prefer to restate it in terms of residue classes.

Theorem 1.11 ([23, Theorem B]). *Let a and $m \geq 1$ be integers. Then the residue class $a + m\mathbb{Z}$ contains a positive palindrome if and only if $a + m\mathbb{Z} \not\subseteq 10\mathbb{Z}$, in which case it will contain infinitely many palindromes.*

Analogously, we can consider the following problem about v-palindromes.

Problem 1.12. Characterize those residue classes containing a v-palindrome.

Recall the sequence of v-palindromes

$$18, 1818, 181818, \dots. \tag{1.27}$$

Hence to look for a v-palindrome in a residue class, we can first look for a number of the form $18(k)$. This inspires an isolated problem, removed from the topic of v-palindromes, as follows.

Problem 1.13. Characterize those residue classes containing a number of the form $18(k)$.

The following theorems are partial answers to the above problem.

Theorem 1.14. *For any integer $\alpha \geq 0$, the residue class $1 + 19^\alpha\mathbb{Z}$ contains infinitely many numbers of the form $18(k)$.*

Theorem 1.15. *Let a and $\alpha \geq 2$ be integers. Then the residue class $a + 7^\alpha\mathbb{Z}$ contains a number of the form $18(k)$ if and only if $a \equiv 0, 4, 5 \pmod 7$, in which case it will contain infinitely many such numbers.*

We can generalize Problem 1.13 into the following, however perhaps with a different logical feeling.

Problem 1.16. Let $n \geq 1$, $b \geq 2$, a, and $m \geq 1$ be integers. How to determine whether in $a + m\mathbb{Z}$ there is a number of the form $n(k)_b$? Or even better, how to find the set of integers $k \geq 1$ such that $n(k)_b \in a + m\mathbb{Z}$?

Algorithm 9.2 is given as an answer to the latter question in the above problem.

1.10 The paper by Vaidyanathan [51]

We explain the relation of the paper by Vaidyanathan [51] from signal processing to this dissertation first in words in Subsection 1.10.1, and then illustrate it pictorially in Subsection 1.10.2.

1.10.1 Formulae for the fundamental period of a periodic function

An arbitrary function $f\colon \mathbb{Z} \to \mathbb{C}$ periodic modulo ω has the form

$$f(x) = \sum_{k=1}^{\omega} h_k \zeta_\omega^{-xk}, \quad \text{for} \quad x \in \mathbb{Z}, \tag{1.28}$$

where the h_k's are complex numbers and $\zeta_\omega = \exp(2\pi i/\omega)$. Let the set of integers k with $1 \le k \le \omega$ such that $h_k \ne 0$ be $\{k_1, \dots, k_l\}$. Then [51, Theorem 9] says that the fundamental period of f is

$$\frac{\omega}{(k_1, \dots, k_l, \omega)}. \tag{1.29}$$

On the other hand, I proved that the fundamental period of f is

$$\operatorname{lcm}\{\delta(\zeta_\omega^{-k_1}), \dots, \delta(\zeta_\omega^{-k_l})\} \tag{1.30}$$

(δ is defined in Notation 5.2). My formula above is Theorem 5.5 in this dissertation. Hence (1.29) and (1.30) must be the same quantity because they are both the fundamental period of f; a direct proof is given in Section 5.2.

Another theorem from [51] is the following, where the Ramanujan spaces S_ω are defined in Definition 5.3 and the 0 denotes the zero function $\mathbb{Z} \to \{0\}$.

Theorem 1.17 ([51, Theorem12]). *Let $\omega_1, \dots, \omega_m \ge 1$ be distinct integers and let $0 \ne f_j \in S_{\omega_j}$ for each $1 \le j \le m$. Then the fundamental period of the periodic function $f_1 + \dots + f_m$ is* $\operatorname{lcm}\{\omega_1, \dots, \omega_m\}$.

The paper [51] contains proofs of both its Theorems 9 and 12, but its proof of its Theorem 12 is not a simple application of its Theorem 9. In contrast, I proved Theorem 1.17 with a simple application of Theorem 5.5. Subsequently, I used Theorem 1.17 to prove Theorem 5.9, which I used to derive Corollary 6.14.

1.10.2 The logical picture

We illustrate the logic described in the last subsection pictorially as follows.

$$\begin{array}{ccccccc} \text{Theorem 5.5} & \longrightarrow & \text{Theorem 1.17} & \longrightarrow & \text{Theorem 5.9} & \longrightarrow & \text{Corollary 6.14} \\ \Updownarrow & & \| & & & & \\ \text{[51, Theorem 9]} & \not\longrightarrow & \text{[51, Theorem 12]} & & & & \end{array}$$

The first row are statements in my dissertation whereas the second row that in [51], moreover the notation have the following meanings.

- $A \longrightarrow B$ means that A is used to prove B,
- $A \not\longrightarrow B$ means that it is not true that A is used to prove B,
- $\Leftrightarrow$ denotes logical equivalence,
- $=$ denotes that they are the same theorem.

Motivated to compute the quantity $\omega_0(n)$, I proved Theorem 5.5 independently, which turned out to be logically equivalent to [51, Theorem 9]. It is surprising that motivation in signal processing and my motivation to compute $\omega_0(n)$ led to similar mathematical considerations.

1.11 Connection to other work

In this section we describe some connection of the concept of v-palindromes to works of others. In Subsection 1.11.1 we describe functions similar to v. In Subsection 1.11.2 we indicate that the reverse $r(n)$ has also been considered by others. In Subsection 1.11.3 we describe a connection to the paper by Spiegelhofer [41].

1.11.1 Functions similar to v

Functions similar to v have been studied by others. Let the prime factorization of an integer $n \geq 1$ be written as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, \tag{1.31}$$

where $m \geq 0$ and $\alpha_1, \ldots, \alpha_m \geq 1$ are integers and $p_1, \ldots, p_m$ distinct primes. Then

$$v(n) = \sum_{1 \leq i \leq m, \alpha_i = 1} p_i + \sum_{1 \leq i \leq m, \alpha_i \geq 2} (p_i + \alpha_i). \tag{1.32}$$

The entry A008474 is the function

$$v_1(n) = \sum_{i=1}^{m} (p_i + \alpha_i). \tag{1.33}$$

The function

$$A(n) = \sum_{i=1}^{m} p_i \alpha_i \tag{1.34}$$

is studied in [3]. Also, the entry A000026 is the function

$$a(n) = \prod_{i=1}^{m} p_i \alpha_i. \tag{1.35}$$

1.11.2 The reverse $r(n)$

In [26], numbers n such that n divides $r(n)$ are mentioned. In particular, all of the numbers in

$$2178, 21978, 219978, 2199978, \ldots, \tag{1.36}$$

i.e., the sequence of numbers $219\ldots978$, with any number of 9's in between, satisfy $4n = r(n)$. The resemblance of the sequences (1.5) and (1.36) is interesting. While the relation $n \mid r(n)$ is studied in [26], the relation $v(n) = v(r(n))$ is used in the definition of v-palindromes. In [19], non-palindromic prime numbers p such that $r(p)$ is also prime are called *emirps*.

An integer $n \geq 1$ such that $n = r(n)$, i.e., a positive palindrome (Definition 1.4), obviously satisfy $n \mid r(n)$ and also $v(n) = v(r(n))$. Therefore when considering such conditions, only the case $n \neq r(n)$ is interesting, as we have imposed in the definition of v-palindromes (Definition 1.7).

1.11.3 A connection to the paper by Spiegelhofer [41]

The equation

$$f(n) = f(r_b(n)) \tag{1.37}$$

is used in Definition 1.5. If (1.37) holds for all integers $n \geq 1$, then an (f, b)-palindrome would be synonymous with an integer $n \geq 1$ which is not a b-palindrome nor a multiple of b. Indeed, the following result from [41] gives a sufficient condition for (1.37) to hold for all integers $n \geq 1$.

Theorem 1.18 ([41, Theorem 2])**.** *Let $b \geq 2$ be an integer and let $f\colon \mathbb{N} \to \mathbb{C}$ be a function such that*

(i) *There exist 2×2 matrices of complex numbers*

$$A(i) = \begin{pmatrix} a_1(i) & a_2(i) \\ a_3(i) & a_4(i) \end{pmatrix} \quad \textit{for } 0 \leq i < b \tag{1.38}$$

and $\alpha, \beta \in \mathbb{C}$ such that for all integers $n \geq 1$, if $n = (a_{L-1}a_{L-2}\cdots a_0)_b$ is the base b representation of n, then

$$f(n) = \begin{pmatrix} 1 & 0 \end{pmatrix} A(a_0)A(a_1)\cdots A(a_{L-1}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{1.39}$$

and

(ii) *There exist $b_1, b_2, b_3, b_4 \in \mathbb{C}$ with $b_1, b_2 \neq 0$ and $b_1b_4 - b_2b_3 = 1$, such that*

$$b_1b_2(a_1(i)\beta - a_3(i)\alpha - a_4(i)\beta) + a_2(i)(\beta + 2b_2b_3\beta - b_3b_4\alpha) = 0 \quad \text{for } 0 \leq i < b. \tag{1.40}$$

Then $f(n) = f(r_b(n))$ for all integers $n \geq 1$.

In [41], Theorem 2 is used to derive a few corollaries. We mention two particularly interesting consequences. The function $b\colon \mathbb{N} \to \mathbb{C}$ is defined by letting $b(1) = 1$, $b(2) = \sqrt{2}$, and imposing that

$$b(3n) = b(n), \tag{1.41}$$

$$b(3n+1) = \sqrt{2}b(n) + b(n+1), \tag{1.42}$$

$$b(3n+2) = b(n) + \sqrt{2}b(n+1), \tag{1.43}$$

for integers $n \geq 1$. Then $b(n) = b(r_3(n))$ for all integers $n \geq 1$ [41, Theorem 1]. The function $s\colon \mathbb{N} \to \mathbb{C}$ is defined by letting $s(1) = s(2) = 1$ and imposing that

$$s(2n) = s(n), \tag{1.44}$$

$$s(2n+1) = s(n) + s(n+1), \tag{1.45}$$

for integers $n \geq 1$. Then $s(n) = s(r_2(n))$ for all integers $n \geq 1$.

Chapter 2

Palindromes

This chapter is a survey of past results on palindromes and more general b-palindromes, which are defined in Definition 1.4. The b-palindromes have been researched with lots of theorems proved, mostly by advanced analytical methods.

In Section 2.1 we first list some notation to be used in this chapter. For each of the remaining sections, we state a result or multiple related results and also comment on them. The full proof is included only for Theorem 2.17.

2.1 Notation

We list some notation to be used in this chapter as follows.

- The set of primes is denoted by $\mathbb{P}$.
- For an integer $k \geq 1$, the set of positive k-th powers of natural numbers is denoted by $\mathbb{N}^k$.
- For an integer $n \geq 1$, the number of distinct prime divisors of n is denoted by $\omega(n)$.
- For a set A of integers bounded below, we put

$$A(x) = \#\{n \in \mathbb{Z} \colon n \leq x \text{ and } n \in A\}, \quad \text{for} \quad x \in \mathbb{R}. \tag{2.1}$$

2.2 An iteration

Iteration using the function defined below is described in [44, 18, 31].

Definition 2.1. Let $b \geq 2$ be an integer. Define the function $\iota_b \colon \mathbb{N} \to \mathbb{N}$ by $\iota_b(n) = n + r_b(n)$.

Starting from an integer $n \geq 1$, we can apply ι_b repeatedly, yielding a sequence

$$n, \iota_b(n), \iota_b^2(n), \ldots. \tag{2.2}$$

For $(n, b) = (195, 10)$, the sequence (2.2) becomes

$$195, 786, 1473, 5214, 9339, 18678, 106359, \ldots, \tag{2.3}$$

where the fifth term is the first palindrome. In contrast, for $(n, b) = (22, 2)$, the sequence (2.2) becomes

$$22, 35, 84, 105, 180, 225, 360, \ldots, \tag{2.4}$$

where it has been shown that no term is a 2-palindrome [2]. In view of this difference, the following definition is given.

Definition 2.2. Let $b \geq 2$ and $n \geq 1$ be integers. Then n is a *Lychrel number in base b* if the sequence (2.2) has no term which is a b-palindrome.

Hence 195 is not a Lychrel number in decimal and 22 is a Lychrel number in binary. In decimal, the smallest positive integer which has not yet been iterated to reach a palindrome is 196. In fact in February 2015, the number 196 has been iterated using a computer to a billion digits without reaching a palindrome [17]. Therefore the following is conjectured.

Conjecture 2.1. *The number* 196 *is a Lychrel number in decimal.*

Other numbers which are suspected to be Lychrel numbers in decimal are recorded in A023108. The smallest positive integer which is suspected to be a Lychrel number in each base $2 \leq b \leq 47$ are recorded in A060382. A Lychrel number exists in every base which is a power of 2 [2]. Also, Lychrel numbers in bases $4, 11, 17, 20, 26$ have been found no later than January 1996 [1].

To gather the information presented in this section, the Wikipedia article [54] is used.

2.3 Number of palindromes up to a number x

Since $\mathcal{P}_b$ denotes the set of b-palindromes, $\mathcal{P}_b(x)$ denotes the number of b-palindromes no greater than a real number x. The prime number theorem states an approximation to the number of primes $\pi(x)$ no greater than x. In contrast, there is actually an exact formula for $\mathcal{P}_b(x)$ [36]. The authors were surprised that no such formula existed in the literature and so wanted to provide one. We give the following definition before we state the formula.

Definition 2.3. Let $b \geq 2$ and $n \geq 1$ be integers. The smallest b-palindrome no less than n is denoted by $\lceil n \rceil_b$. Moreover, if the base b representation of n is $(a_{L-1} \cdots a_1 a_0)_b$, then we put

$$z_b(n) = (a_{L-1} a_{L-2} \cdots a_{L-1-\lfloor (L-1)/2 \rfloor} 00 \cdots 0)_b. \tag{2.5}$$

So $z_b(n)$ is the number formed by changing all the digits of n on the right half, except the digit in the very middle when L is odd, to 0.

Then the formula is as follows, where $[\cdot]$ is the Iverson symbol defined in Section 1.1.

Theorem 2.2 ([36, Theorem 2.2])**.** *Let $b \geq 2$ and $n \geq 1$ be integers and let $(a_{L-1} \cdots a_1 a_0)_b$ be the base b representation of n. Then*

$$\mathcal{P}_b(n) = b^{\lceil (L-1)/2 \rceil} + \sum_{i=0}^{\lfloor (L-1)/2 \rfloor} a_{L-1-i} b^{\lfloor (L-1)/2 \rfloor - i} + [n \geq \lceil z_b(n) \rceil_b] - 1. \tag{2.6}$$

The proof of Theorem 2.2 is elementary, done by dividing the positive integers $\leq n$ into a number of cases, counting the number of b-palindromes in each case, and then summing. There are also formulae counting only the even (or only the odd) b-palindromes [36, Theorems 2.3 and 2.4].

2.4 Palindromic terms in sequences

When we have a sequence $(a_n)_{n \geq 0}$ of integers, we may consider the terms which are b-palindromes. In Subsection 2.4.1 we first consider arithmetic sequences. In Subsection 2.4.2 we consider Lucas sequences. Finally in Subsection 2.4.3 we consider linear recurrence sequences, which includes the Lucas sequences.

Essentially equivalently, when we have a function $f \colon \mathbb{N} \to \mathbb{C}$, we may consider the integers $n \geq 1$ such that $f(n)$ is a b-palindrome. In this context the sum of proper divisors function $s(n)$, the Euler phi function $\varphi(n)$, the sum of all divisors function $\sigma(n)$, and the Carmichael λ-function $\lambda(n)$ have been studied [33].

2.4.1 Arithmetic sequences

We paraphrased [23, Theorem B] into the language of residue classes and stated it as Theorem 1.11, but the original statement is more like the following.

Theorem 2.3 ([23, Theorem B])**.** *Let $(a + nd)_{n \geq 0}$ be an arithmetic sequence where $a, d \geq 1$ are integers. Then $(a + nd)_{n \geq 0}$ contains infinitely many palindromic terms if and only if it is not the case that $a \equiv d \equiv 0 \pmod{10}$.*

Even if an arithmetic sequence contains infinitely many palindromic terms, it cannot be that all terms are palindromic because of Theorem 2.9. Theorem 2.3 was proved constructively, by actually finding infinitely many palindromic terms. In fact Theorem 2.3 generalizes straightforwardly as follows.

Theorem 2.4 ([23, Theorem C]). *Let $b \geq 2$ be an integer and let $(a+nd)_{n\geq 0}$ be an arithmetic sequence where $a, d \geq 1$ are integers. Then $(a+nd)_{n\geq 0}$ contains infinitely many b-palindromic terms if and only if it is not the case that $a \equiv d \equiv 0 \pmod{b}$.*

For the analogous situation for prime numbers, we have the following famous theorem of Dirichlet.

Theorem 2.5 ([5, Section 7.1]). *Let $(a+nd)_{n\geq 0}$ be an arithmetic sequence where $a, d \geq 1$ are integers. Then $(a+nd)_{n\geq 0}$ contains infinitely many prime terms if and only if $(a, d) = 1$.*

Asking whether there is a term in an arithmetic sequence $(a+nd)_{n\geq 0}$, where $a, d \geq 1$ are integers, belonging to a set $S \subseteq \mathbb{Z}$ is similar to asking whether $S \cap (a + d\mathbb{Z}) \neq \emptyset$. This was why we paraphrased [23, Theorem B] into Theorem 1.11. Phrased in the language of residue classes, investigations have been done for S being the set of Fibonacci or Lucas numbers [11, 10].

2.4.2 Lucas sequences

We define Lucas sequences following [29]. Let $r, s \in \mathbb{Z}$ with $s, r^2 + 4s \neq 0$ and let $w_0, w_1 \in \mathbb{Z}$. Define the sequence $(w_n)_{n\geq 0}$ by imposing that $w_{n+2} = rw_{n+1} + sw_n$ holds for $n \geq 0$. Let α, β be the complex zeros of $x^2 - rx - s$. Then there exist algebraic numbers c, d such that $w_n = c\alpha^n + d\beta^n$ for $n \geq 0$. Assume further that $cd\alpha\beta \neq 0$ and that α/β is not a root of unity. Then such a sequence $(w_n)_{n\geq 0}$ is called a *binary recurrent sequence*.

Definition 2.4. A *Lucas sequence of the first kind* is a binary recurrent sequence $(w_n)_{n\geq 0}$ with $w_0 = 0$ and $w_1 = 1$. A *Lucas sequence of the second kind* is a binary recurrent sequence $(w_n)_{n\geq 0}$ with $w_0 = 2$ and $w_1 = r$ (r of the previous paragraph).

For instance, the Fibonacci sequence $(F_n)_{n\geq 0}$ defined by imposing that $F_0 = 0$ and $F_1 = 1$ and that $F_{n+2} = F_{n+1} + F_n$ holds for $n \geq 0$ is a Lucas sequence of the first kind.

Theorem 2.6 ([29, p. 210]). *Let $b \geq 2$ be an integer, let $w = (w_n)_{n\geq 0}$ be a Lucas sequence of the first or second kind such that $s = \pm 1$ (s of the first paragraph of this subsection), and let $P = \{n \in \mathbb{N} \colon |w_n| \in \mathcal{P}_b\}$. Then*

$$P(x) \ll_{w,b} \frac{x}{(\log x)^{\frac{1}{2\omega(b)}}}, \quad x \to \infty \tag{2.7}$$

(by notation introduced in Section 2.1, $P(x)$ denotes the number of elements of P no greater than x).

Two analytical tools used in the proof of Theorem 2.6 were a lower bound for linear forms in two logarithms due to Baker and Jensen's inequality for convex functions.

2.4.3 Linear recurrence sequences

With the motivation to generalize Theorem 2.6, Cilleruelo et. al. proved the following theorem, which applies to a wider class of sequences $w = (w_n)_{n\geq 0}$ and provides a sharper estimate. The concept of multiplicative independence is involved and so we first define it.

Definition 2.5 ([15, p. 435]). Two real numbers $u, v > 0$ are *multiplicatively independent* if assuming that $u^x = v^y$ and $x, y \in \mathbb{Z}$, necessarily $x = y = 0$.

Theorem 2.7 ([15, Theorem 1.1]). *Let $b \geq 2$ be an integer, let $w = (w_n)_{n\geq 0}$ be the linear recurrence sequence of integers of minimal recurrence relation*

$$w_{n+k} = c_1 w_{n+k-1} + c_2 w_{n+k-2} + \cdots + c_k w_n, \quad \text{for} \quad n \geq 1, \tag{2.8}$$

where $C(x) = x^k - c_1 x^{k-1} - \cdots - c_k \in \mathbb{Z}[x]$ has a unique dominant zero $\alpha_1 > 0$ multiplicatively independent with b, and let $P = \{n \in \mathbb{N} \colon |w_n| \in \mathcal{P}_b\}$. Then there exists $c = c(w, b) > 0$ such that

$$P(x) \ll_{w,b} x^{1-c}, \quad x \to \infty. \tag{2.9}$$

Two tools used in the proof of Theorem 2.7 are the closed-form formula of a linear recurrence sequence [15, Theorem 1.1] and an inequality due to Baker [15, Theorem 2.4].

2.5 Arithmetic progressions of palindromes

We first give the definition of an arithmetic progression in $\mathbb{Z}$.

Definition 2.6 ([42]). Let a and $r, N \geq 1$ be integers. Then

$$a + r \cdot [0, N) = \{a, a + r, a + 2r, \ldots, a + (N-1)r\}. \tag{2.10}$$

Such sets are called *arithmetic progressions* in $\mathbb{Z}$. The number N is the *length* of the arithmetic progression.

We can consider arithmetic progressions in any set of integers. For primes, we have the following famous theorem of Green-Tao.

Theorem 2.8 ([20, Theorem 1.1]). *For any integer $N \geq 1$, the primes contain infinitely many arithmetic progressions of length N.*

Similarly, we can consider arithmetic progressions in the b-palindromes. Domotorp asked Tao this and Tao gave an answer showing that any arithmetic progression in the 10-palindromes must have length $< 10^8$ [43]. Motivated by this, Pongsriiam found the longest possible length of an arithmetic progression of 10-palindromes as follows.

Theorem 2.9 ([35]). *The longest possible length of an arithmetic progression of* 10*-palindromes is* 10*.*

The proof of Theorem 2.9 was elementary but long with many cases and calculations. Pongsriiam believes that with the same proof method, Theorem 2.9 holds even if we replace 10 by other bases. That is, that the longest possible length of an arithmetic progression of b-palindromes is b.

Arithmetic progressions in other sets of integers have also been investigated, e.g., polygonal numbers [9], the least positive reduced residue system modulo a natural number [34, 12], and Lucas numbers [21].

2.6 Palindromic primes

We have the following definition.

Definition 2.7. Let $b \geq 2$ be an integer. A *b-palindromic prime* is a b-palindrome which is also a prime.

Remark 2.1. The sequence of 10-palindromic primes is A002385 in the OEIS.

For each integer $b \geq 2$, we may ask the following question.

Question 2.10. *Are there infinitely many b-palindromic primes?*

The answer of the above question is not known for any b. However, there are approximate results.

Theorem 2.11 ([7, Theorem 5.1]). *Let $b \geq 2$ be an integer. We have*

$$\#(\mathcal{P}_b(x) \cap \mathbb{P}) \ll_b \#\mathcal{P}_b(x) \frac{\log\log\log x}{\log\log x}, \quad x \to \infty. \tag{2.11}$$

From the above theorem it follows that almost all b-palindromes are composite. An analytical tool used in the proof of Theorem 2.11 was Brun's combinatorial sieve.

One might have the intuition that it is 'easier' for a 10-palindrome to be prime than a random natural number. There is evidence for this [18]. Let $L \geq 1$ be an odd integer. Denote by $\mathcal{P}_{10,L}$ the set of 10-palindromes with L digits and by $\mathbb{N}_{10,L}$ the set of natural numbers with L decimal digits. Define the quotients

$$Q_{\mathcal{P}}(L) = \frac{\#(\mathcal{P}_{10,L} \cap \mathbb{P})}{\#\mathcal{P}_{10,L}}, \quad Q_{\mathbb{N}}(L) = \frac{\#(\mathbb{N}_{10,L} \cap \mathbb{P})}{\#\mathbb{N}_{10,L}}, \quad Q(L) = \frac{Q_{\mathcal{P}}(L)}{Q_{\mathbb{N}}(L)}. \tag{2.12}$$

The following table is computed by PARI/GP.

Table 2.1: First few values of $Q_{\mathcal{P}}(L)$, $Q_{\mathbb{N}}(L)$, and $Q(L)$.

L	1	3	5	7	9
$Q_{\mathcal{P}}(L)$	0.444	0.166	0.103	0.074	0.057
$Q_{\mathbb{N}}(L)$	0.444	0.159	0.093	0.065	0.050
$Q(L)$	1.000	1.049	1.112	1.140	1.147

The above table suggests the following conjecture.

Conjecture 2.12 ([18])**.** *$Q(L)$ is strictly increasing over odd integers $L \geq 1$.*

2.7 Palindromic squares and higher powers

In this section we describe results about b-palindromes which are simultaneously a perfect power. In Subsection 2.7.1 we consider b-palindromic squares and in Subsection 2.7.2, higher powers.

2.7.1 Palindromic squares

We first give the following definition.

Definition 2.8. Let $b \geq 2$ be an integer. A nonnegative integer is a *b-palindromic square* if it is simultaneously a b-palindrome and also the square of an integer.

In the papers [22, 27, 28, 14] containing results on b-palindromic squares, only positive integers were considered, but because we defined also 0 to be a b-palindrome, we include 0 as a b-palindromic square too. There have been two notions of a b-palindromic square being *trivial* used in the papers [22, 27, 28] which we define as follows.

Definition 2.9 ([28, p. 262])**.** Let n be a b-palindromic square. Then

(i) n is *first-trivial* if $\sqrt{n}$ is a b-palindrome; otherwise it is *first-nontrivial*,

(ii) n is *second-trivial* if we have that: If $n = (a_L a_{L-1} \cdots a_0)_b$ and $\sqrt{n} = (c_K c_{K-1} \cdots c_0)_b$, then

$$a_L x^L + a_{L-1} x^{L-1} + \cdots + a_0 = (c_K x^K + c_{K-1} x^{K-1} + \cdots + c_0)^2; \tag{2.13}$$

otherwise it is *second-nontrivial*.

If a b-palindromic square is second-trivial, then it must also be first-trivial. In [22], examples of 10-palindromic squares which are not first-trivial were found by a computer, and the question of the infinitude of such numbers is left as a further problem. In [27], an answer in the positive not only in base 10 but also in all bases $b \geq 3$ with $b \neq 5$ was given. Finally in [28], the case $b = 5$ was settled with a positive answer. So altogether we have the following.

Theorem 2.13. *For any integer $b \geq 3$, there are infinitely many b-palindromic squares which are not first-trivial.*

We now describe a result on second-trivial b-palindromic squares from [28].

Definition 2.10 ([28, Definition 2.2 (ii)])**.** The polynomial $f(x) \in \mathbb{Z}[x]$ *produces second-nontrivial b-palindromic squares* if it is nonconstant and for every integer $k > \log_b H(f(x))$, the integer $f(b^k)$ is a second-nontrivial b-palindromic square.

Remark 2.2. Here $H(f(x))$ is the *height* of $f(x)$ defined as the maximum of the absolute values of the coefficients of $f(x)$.

The following theorem completely characterizes the polynomials in $\mathbb{Z}[x]$ which produce second-nontrivial b-palindromic squares simultaneously for all $b \geq 10$.

Theorem 2.14 ([28, Theorem 5.2])**.** *A polynomial $f(x) \in \mathbb{Z}[x]$ produces second-nontrivial b-palindromic squares for all $b \geq 10$ if and only if $f(x)$ is the square of a polynomial of one of the following forms:*

(i) $g_{r,s}(x) = x^{2r} + x^{2r-s} + x^{r+s} - x^r + x^{r-s} + x^s + 1$, *where $r, s \in \mathbb{Z}$ such that $r > 2s > 0$,*

(ii) $g_{r,s}(x) + x^{2r-2s} + x^{2s}$, *where $r, s \in \mathbb{Z}$ such that $r > 2s > 0$ and $r \neq 3s$,*

(iii) $g_{r,s}(x) + x^{3r/2} + x^{r/2}$, *where $r, s \in \mathbb{Z}$ such that $r > 2s > 0$ and $2 \mid r$,*

(iv) $g_{r,s}(x) + x^{r+s/2} + x^{r-s/2}$, *where $r, s \in \mathbb{Z}$ such that $r > 2s > 0$ and $2 \mid s$,*

(v) $g_{r,s}(x) + x^{3r/2-s/2} + x^{r/2+s/2}$, *where $r, s \in \mathbb{Z}$ such that $r > 2s > 0$ and $2 \mid (r + s)$ and $r \neq 3s$.*

For instance, for $r = 3$ and $s = 1$,

$$g_{3,1}(x) = x^6 + x^5 + x^4 - x^3 + x^2 + x + 1, \tag{2.14}$$

thus

$$g_{3,1}(x)^2 = (x^6 + x^5 + x^4 - x^3 + x^2 + x + 1)^2. \tag{2.15}$$

According to the above theorem, the polynomial $g_{3,1}(x)^2$ produces second-nontrivial b-palindromic squares for all $b \geq 10$. In particular, this holds for $b = 10$. Since $\log_{10} H(g_{3,1}(x)^2) = 1.945\cdots$, we have that for every integer $k \geq 2$, the number $g_{3,1}(10^k)^2$ is a second-nontrivial 10-palindromic square. For instance,

$$g_{3,1}(10^2)^2 = 1020300010207020100030201, \tag{2.16}$$

$$g_{3,1}(10^3)^2 = 1002003000001002007002001000003002001. \tag{2.17}$$

So it seems that the 10-palindromes $g_{3,1}(10^k)^2$ for $k \geq 2$ are basically the same one, the others obtained by inserting more and more 0's. Two lemmas which were crucial in leading to a proof of Theorem 2.14 are the following.

Lemma 2.15 ([28, Lemma 4.2])**.** *Let $f(x) \in \mathbb{Z}[x]$. If the values $f(x)$ are squares for sufficiently large $x \in \mathbb{Z}$, then there is a $g(x) \in \mathbb{Z}[x]$ such that $f(x) = g(x)^2$.*

Lemma 2.16 ([28, Lemma 4.4])**.** *Let $f(x) \in \mathbb{Z}[x]$. If $f(x)^2$ is reciprocal and with all coefficients nonnegative, then $f(x)$ is also reciprocal.*

Remark 2.3. Here a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ with $a_n \neq 0$ is *reciprocal* if $a_{n-i} = a_i$ for every $0 \leq i \leq n$; the zero polynomial 0 is also considered reciprocal.

There is a generalization of Lemma 2.15 in [40]. For each $3 \leq b \leq 9$, a polynomial which produces second-nontrivial b-palindromic squares is constructed [28, Theorems 3.2 and 3.3].

2.7.2 Higher palindromic powers

More general than squares, of course we can consider higher powers. They are investigated in [14], where the distribution of k-th powers in b-palindromes is described. Let $L \geq 0$ be an integer. Let $\mathcal{P}_{b,L}$ denote the set of b-palindromes with L digits. Further, for integers $k \geq 2$, let $\mathcal{P}_{b,L}^k = \mathcal{P}_{b,L} \cap \mathbb{N}^k$, i.e., the set of b-palindromes with L digits simultaneously a k-th power. Then we have the following result where we also include the proof.

Theorem 2.17 ([14])**.** *Let $b, k \geq 2$ be integers. Then*

$$\#\mathcal{P}_{b,L}^k \ll (\#\mathcal{P}_{b,L})^{1/k}, \quad L \to \infty. \tag{2.18}$$

Proof. Consider L to be a large integer and put $M = \lfloor (L-1)/(2k) \rfloor$. For each integer a with $0 \leq a < b^M$, let

$$P_{b,L,a}^k = \{n \in \mathcal{P}_{b,L}^k : n^{1/k} \equiv a \pmod{b^M}\}. \tag{2.19}$$

Then

$$\#P_{b,L}^k = \sum_{0 \leq a < b^M} \#P_{b,L,a}^k. \tag{2.20}$$

We now consider any particular integer a with $0 \leq a < b^M$ and find an upper bound for $\#\mathcal{P}_{b,L,a}^k$. Let $n \in \mathcal{P}_{b,L,a}^k$. Then $n \equiv a^k \pmod{b^M}$. Clearly $M < L$. Let r be the number formed by reversing the rightmost M digits, with leading zeros if necessary, of a^k. Since n is palindromic, the leftmost M digits of n is r. Consequently,

$$rb^{L-M} < n < (r+1)b^{L-M}, \tag{2.21}$$

which implies that

$$r^{1/k} b^{\frac{L-M}{k}} < n^{1/k} < (r+1)^{1/k} b^{\frac{L-M}{k}}. \tag{2.22}$$

Since $n^{1/k} \equiv a \pmod{b^M}$, the value of $\#\mathcal{P}_{b,L,a}^k$ is no greater than the number of integers in the interval

$$(r^{1/k} b^{\frac{L-M}{k}}, (r+1)^{1/k} b^{\frac{L-M}{k}}) \tag{2.23}$$

congruent to a modulo b^M. Hence

$$\#\mathcal{P}_{b,L,a}^k \le \frac{(r+1)^{1/k} b^{\frac{L-M}{k}} - r^{1/k} b^{\frac{L-M}{k}}}{b^M} + 1. \tag{2.24}$$

Now let a_0 be a value of $0 \le a < b^M$ which maximizes $\#\mathcal{P}_{b,L,a}^k$, with the r above denoted r_0. Then

$$\begin{aligned}\#P_{b,L}^k &= \sum_{0\le a<b^M} \#P_{b,L,a}^k \le b^M \cdot \#\mathcal{P}_{b,L,a_0}^k \le (r_0+1)^{1/k} b^{\frac{L-M}{k}} - r_0^{1/k} b^{\frac{L-M}{k}} + b^M \\ &= ((r_0+1)^{1/k} - r_0^{1/k}) b^{(L-M)/k} + b^M. \end{aligned}\tag{2.25}$$

Now

$$(r_0+1)^{1/k} - r_0^{1/k} = \frac{1}{(r_0+1)^{(k-1)/k} + (r_0+1)^{(k-2)/k} r_0^{1/k} + \cdots + r_0^{(k-1)/k}} \le \frac{1}{k r_0^{(k-1)/k}} \le \frac{1}{k (b^{M-1})^{(k-1)/k}}. \tag{2.26}$$

Substituting this into (2.25) we get

$$\begin{aligned}\#P_{b,L}^k &\le \frac{1}{k} \cdot b^{\frac{L-1}{k} - M + 1} + b^M = \frac{1}{k} \cdot b^{\frac{L-1}{k} - \lfloor \frac{L-1}{2k} \rfloor + 1} + b^{\lfloor \frac{L-1}{2k} \rfloor} \le \frac{1}{k} \cdot b^{\frac{L-1}{k} - (\frac{L-1}{2k} - 1) + 1} + b^{\frac{L-1}{2k}} \\ &= \frac{1}{k} \cdot b^{\frac{L-1}{2k} + 2} + b^{\frac{L-1}{2k}} = \left(\frac{b^2}{k} + 1\right) b^{(L-1)/(2k)}. \end{aligned}\tag{2.27}$$

Now

$$\#\mathcal{P}_{b,L} = (b-1) b^{\lfloor \frac{L-1}{2} \rfloor} \ge (b-1) b^{\frac{L-3}{2}}, \quad \text{and so} \quad \#\mathcal{P}_{b,L}^{1/k} \ge (b-1)^{1/k} b^{\frac{L-3}{2k}}. \tag{2.28}$$

Since

$$\left(\frac{b^2}{k} + 1\right) b^{(L-1)/(2k)} \ll (b-1)^{1/k} b^{\frac{L-3}{2k}}, \quad L \to \infty, \tag{2.29}$$

in view of (2.27) and the second inequality in (2.28), the proof is complete. □

There is also a lower bound for $\#\mathcal{P}_{b,L}^k$ given in [14] as follows.

Theorem 2.18 ([14])**.** *Let $k \ge 2$ be a fixed integer. Then there exist real numbers $c, b_0 > 0$ such that for integers $b \ge b_0$ and $L \ge 0$,*

$$\#\mathcal{P}_{b,L}^k \gg L^{c b^{1/\lfloor k/2 \rfloor}}. \tag{2.30}$$

The proof of the above theorem involves giving a lower bound for the number of subsets (of a set) satisfying a specially-defined property.

2.8 Sums of palindromes

We first give the following definition.

Definition 2.11. Let $S \subseteq \mathbb{N}$. For integers $k \ge 1$, we define

$$kS = \{a_1 + \cdots + a_k \mid a_1, \dots, a_k \in S\}. \tag{2.31}$$

The set S is an *additive basis* if there exists an integer $k \ge 1$ such that

$$\bigcup_{i=1}^k iS = \mathbb{N}, \tag{2.32}$$

in which case the smallest possible k is the *degree* of S.

For the set $\mathcal{P}_{10} \setminus \{0\}$ of positive 10-palindromes, we have the following result of Banks.

Theorem 2.19 ([6, Theorem 1])**.** *The set $\mathcal{P}_{10} \setminus \{0\}$ is an additive basis of degree ≤ 49.*

The proof of Theorem 2.19 is elementary but involves some inductive steps. With the motivation to improve the 49 in Theorem 2.19, Cilleruelo et al. showed the following.

Theorem 2.20 ([13, Theorem 1.2]). *Let $b \geq 5$ be an integer. Then $\mathcal{P}_b \setminus \{0\}$ is an additive basis of degree* 3.

The proof of Theorem 2.20 is by providing an algorithm which, when given an integer $n \geq 1$ as an input, outputs $p_1, p_2, p_3 \in \mathcal{P}_b$ such that $n = p_1 + p_2 + p_3$. The following theorem illustrates that the 3 in Theorem 2.20 is optimal.

Theorem 2.21 ([13, Theorem 1.4]). *Let $b \geq 3$ be an integer. There exists a constant $c < 1$ such that*

$$\#\{n \leq x \colon n = p_1 + p_2 \text{ for some } p_1, p_2 \in \mathcal{P}_b\} \leq cx \tag{2.33}$$

for sufficiently large x.

The proof of Theorem 2.21 is mostly by observing that any number of the form $((b-1)(b-1)\cdots 0(b-1))_b$ with at least 4 digits and the digits other than the 4 shown randomly chosen, is not the sum of two b-palindromes. With the motivation to complete the investigation by treating the cases $b = 2, 3, 4$ left out in Theorem 2.20, Rajasekaran et al. [38] showed using automata theory that $\mathcal{P}_2 \setminus \{0\}$, $\mathcal{P}_3 \setminus \{0\}$, and $\mathcal{P}_4 \setminus \{0\}$ are additive bases with degrees $4, 3, 3$ respectively.

2.9 The reciprocal sum of the palindromes

That the reciprocal sum of the primes, i.e.,

$$\sum_{p \in \mathbb{P}} \frac{1}{p} \tag{2.34}$$

diverges is well-known. On the contrary, the reciprocal sum of the positive b-palindromes

$$s_b = \sum_{n \in \mathcal{P}_b \setminus \{0\}} \frac{1}{n} \tag{2.35}$$

converges for every $b \geq 2$. Phunphayap and Ponsgriiam [32] investigated these s_b.

Theorem 2.22 ([32, Theorem 3]). *The sequence $(s_b)_{b \geq 2}$ is strictly increasing.*

In the proof of Theorem 2.22, the cases $b < 16$ and $b \geq 16$ were treated separately. In the latter, the main tool used was the following.

Lemma 2.23 ([32, p. 5]). *Let $a < b$ be integers and $f \colon [a, b] \to \mathbb{R}$ be monotone. Then*

$$\min\{f(a), f(b)\} \leq \sum_{n=a}^{b} f(n) - \int_a^b f(t)\, dt \leq \max\{f(a), f(b)\}. \tag{2.36}$$

There is also the following asymptotic formula for s_b, where γ is Euler's constant.

Theorem 2.24 ([32, Theorem 5]). *For $b \geq 2$,*

$$s_b = \log b \cdot \frac{b^3 + 3b^2 + 3b + 2}{b^3 + b^2} + \left(\gamma + \frac{\gamma}{b+1} - \frac{1}{2b} - \frac{1}{12b(b+1)}\right) + O\left(\frac{\log b}{b^3}\right), \tag{2.37}$$

where the implicit constant can be taken to be 6.

The main tool used in the proof of Theorem 2.24 was

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + \frac{1}{2x} - \frac{1}{12x^2} + \frac{\theta(x)}{60x^4}, \tag{2.38}$$

where $\theta(x) \in [0, 1]$, which follows from the Euler-Maclaurin summation formula. The following is also proved.

Corollary 2.25 ([32, Corollary 6]). *We have the following limits.*

$$\lim_{b \to \infty} s_b = \infty, \quad \lim_{b \to \infty} (s_b - s_{b-1}) = 0. \tag{2.39}$$

Corollary 2.25 follows from Theorem 2.24 by using $\log(b-1) = \log b + O(1/b)$.

The sums (2.34) and (2.35) are instances of the more general theme of the reciprocal sum of an infinite set of natural numbers, on which an exposition can be found in [8].

Chapter 3

Other Palindromic Objects

This chapter mentions past results on other palindromic objects by which we mean objects with a left-right symmetry akin to the palindromes.

For each section, we state a result or multiple related results and also comment on them. Full proofs are included only for Theorems 3.2 and 3.3.

3.1 Staircase polynomials

Reciprocal polynomials in $\mathbb{Z}[x]$ are defined in Remark 2.3. We give the definition for polynomials over a field as follows.

Definition 3.1. Let F be a field and let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in F[x], \tag{3.1}$$

where $n \geq -1$ and $a_n \neq 0$ if $n \geq 0$. Then $f(x)$ is *palindromic* (or *reciprocal*) if $a_i = a_{n-i}$ for $0 \leq i \leq n$.

Hence any constant polynomial is trivially palindromic. We now define a special kind of palindromic polynomials as follows.

Definition 3.2 ([37, Definition 3.1])**.** Let $n \geq 0$ and $1 \leq h \leq \lceil (n+1)/2 \rceil$ be integers. We denote

$$S_{n,h}(x) = x^n + 2x^{n-1} + \cdots + \underbrace{hx^{n+1-h} + \cdots + hx^{h-1}}_{n+3-2h \text{ terms}} + \cdots + 2x + 1 \tag{3.2}$$

and call such polynomials *staircase polynomials*.

If $\Phi_d(x)$ denotes the d-th cyclotomic polynomial for integers $d \geq 1$, then we have the following theorem.

Theorem 3.1 ([37, Theorem 3.2])**.** *Let $n \geq 0$ and $1 \leq h \leq \lceil (n+1)/2 \rceil$ be integers. Then*

$$S_{n,h}(x) = \prod_{\substack{d \mid h \\ d \neq 1}} \Phi_d(x) \prod_{\substack{e \mid n+2-h \\ e \neq 1}} \Phi_e(x). \tag{3.3}$$

The fact that staircase polynomials are the product of two polynomials all of whose coefficients are 1 [37, Lemma 3.2] and the factorization of polynomials of the form $x^m - 1$ into cyclotomic polynomials [37, Theorem 2.3] are used in the proof of Theorem 3.1. Factoring polynomials of high degree are in general difficult, but Theorem 3.1 provides the factorization for the staircase polynomials.

3.2 Palindromic compositions

We first give the following definition.

Definition 3.3. Let $n \geq 0$ be an integer. A *composition* of n is an ordered tuple $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ of length $k \geq 0$ of positive integers such that $n = \sigma_1 + \sigma_2 + \cdots + \sigma_k$. The entries of σ are called *parts*. The composition σ is

(i) *palindromic* if $\sigma_i = \sigma_{k+1-i}$ for $1 \le i \le k$,

(ii) a *partition* if $\sigma_1 \ge \cdots \ge \sigma_k$.

We have the following closed-form formula.

Theorem 3.2 ([24, Theorem 1.2]). *Let $1 \le a_0 < a_1 < \cdots$ be integers and let $A = \{a_k \mid k \ge 0\}$. Put*

$$F(x) = x^{a_0} + x^{a_1} + \cdots \in \mathbb{Q}[[x]]. \tag{3.4}$$

For integers $n \ge 0$, let P_n denote the number of palindromic compositions of n all of whose parts belong to A. Then

$$P(x) := \sum_{n=0}^{\infty} P_n x^n = \frac{1 + F(x)}{1 - F(x^2)}. \tag{3.5}$$

Proof. Extend the definition of P_n to allow $P_n = 0$ for $n < 0$. Let $n \ge 0$ be an integer. We first count those palindromic compositions of n with at least two parts. For integers $k \ge 0$, the number of palindromic compositions of n with at least two parts and with the first (therefore the last) part being a_k is P_{n-2a_k}. Only for $n \in A$ will there be a composition of length 1 and only for $n = 0$ will there be a composition of length 0. Hence

$$P(x) = \sum_{n=0}^{\infty} P_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} P_{n-2a_k} \right) x^n + 1 + x^{a_0} + x^{a_1} + \cdots = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{n-2a_k} x^{n-2a_k} x^{2a_k} + 1 + F(x) \tag{3.6}$$

$$= \sum_{k=0}^{\infty} x^{2a_k} \sum_{n=0}^{\infty} P_{n-2a_k} x^{n-2a_k} + 1 + F(x) = \sum_{k=0}^{\infty} x^{2a_k} P(x) + 1 + F(x) = P(x) \sum_{k=0}^{\infty} x^{2a_k} + 1 + F(x) \tag{3.7}$$

$$= P(x)F(x^2) + 1 + F(x). \tag{3.8}$$

From this (3.5) follows. □

There is also the following weaker concept.

Definition 3.4. Let $m \ge 1$ be an integer. A composition $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ is *palindromic modulo m* if $\sigma_i \equiv \sigma_{k+1-i} \pmod{m}$ for $1 \le i \le k$. For integers $n \ge 0$, let the number of compositions of n palindromic modulo m be denoted by $\mathrm{pc}(n, m)$.

Andrews and Simay [4] investigated compositions palindromic modulo 2 and later Just [25] generalized their results to modulo m. In [25], Just first proved the following closed-form formula.

Theorem 3.3 ([25, Theorem 1]). *Let $m \ge 1$ be an integer. Then*

$$\sum_{n=1}^{\infty} \mathrm{pc}(n, m) x^n = \frac{x + 2x^2 - x^{m+1}}{1 - 2x^2 - x^m}. \tag{3.9}$$

Proof. Let the left-hand side of (3.9) be denoted by $P(x)$. Write $\mathrm{pc}(n, m) = \mathrm{pc}(n)$, omitting the m as understood, and naturally set $\mathrm{pc}(n) = 0$ for $n < 0$. Let $A = \{(a, b) \in \mathbb{N}^2 \colon a \equiv b \pmod{m}\}$. Then when $n \ge 1$,

$$\mathrm{pc}(n) = 1 + \sum_{(a,b)\in A} \mathrm{pc}(n - a - b). \tag{3.10}$$

Consequently,

$$P(x) = \sum_{n=1}^{\infty} \left(1 + \sum_{(a,b)\in A} \mathrm{pc}(n - a - b) \right) x^n = \sum_{n=1}^{\infty} x^n + \sum_{n=1}^{\infty} \sum_{(a,b)\in A} \mathrm{pc}(n - a - b) x^n. \tag{3.11}$$

Working on the second summand on the rightmost side of (3.11), we have

$$\sum_{n=1}^{\infty} \sum_{(a,b)\in A} \mathrm{pc}(n - a - b) x^n = \sum_{(a,b)\in A} \sum_{n=1}^{\infty} \mathrm{pc}(n - a - b) x^n = \sum_{(a,b)\in A} \sum_{n=1}^{\infty} \mathrm{pc}(n - a - b) x^{n-a-b} x^{a+b}$$

$$= \sum_{(a,b)\in A} x^{a+b} \sum_{n=1}^{\infty} \mathrm{pc}(n - a - b) x^{n-a-b} = \sum_{(a,b)\in A} x^{a+b}(1 + P(x)) = (1 + P(x)) \sum_{(a,b)\in A} x^{a+b}. \tag{3.12}$$

Working on the sum on the rightmost side of (3.12), we have

$$\sum_{(a,b)\in A} x^{a+b} = \sum_{\substack{(a,b)\in A\\ a<b}} x^{a+b} + \sum_{a=1}^{\infty} x^{2a} + \sum_{\substack{(a,b)\in A\\ a>b}} x^{a+b} = \sum_{a=1}^{\infty} x^{2a} + 2\sum_{\substack{(a,b)\in A\\ a<b}} x^{a+b}. \tag{3.13}$$

Working on the sum in the second summand on the rightmost side of (3.13), we have

$$\sum_{\substack{(a,b)\in A\\ a<b}} x^{a+b} = \sum_{a=1}^{\infty}\sum_{k=1}^{\infty} x^{a+a+km} = \sum_{a=1}^{\infty}\sum_{k=1}^{\infty} x^{2a+km} = \sum_{a=1}^{\infty} x^{2a} \sum_{k=1}^{\infty} x^{km} = \frac{x^2}{1-x^2}\frac{x^m}{1-x^m}. \tag{3.14}$$

Consequently, substituting (3.14) into (3.13), we have

$$\sum_{(a,b)\in A} x^{a+b} = \sum_{a=1}^{\infty} x^{2a} + 2\frac{x^2}{1-x^2}\frac{x^m}{1-x^m} = \frac{x^2}{1-x^2} + 2\frac{x^2}{1-x^2}\frac{x^m}{1-x^m}. \tag{3.15}$$

Substituting (3.12) into (3.11), we have

$$P(x) = \sum_{n=1}^{\infty} x^n + (1+P(x))\sum_{(a,b)\in A} x^{a+b} = \frac{x}{1-x} + (1+P(x))\sum_{(a,b)\in A} x^{a+b}, \tag{3.16}$$

which, using (3.15), implies that

$$P(x) = \frac{\frac{x}{1-x} + \sum_{(a,b)\in A} x^{a+b}}{1-\sum_{(a,b)\in A} x^{a+b}} = \frac{\frac{x}{1-x} + \frac{x^2}{1-x^2} + 2\frac{x^2}{1-x^2}\frac{x^m}{1-x^m}}{1-\left(\frac{x^2}{1-x^2} + 2\frac{x^2}{1-x^2}\frac{x^m}{1-x^m}\right)} = \frac{x+2x^2-x^{m+1}}{1-2x^2-x^m}. \tag{3.17}$$

□

Theorem 3.2 restricts the parts of a palindromic composition to belong to a set, whereas Theorem 3.3 relaxes palindromicity to merely modulo m. We can ask the following question.

Question 3.4. *Let $A \subseteq \mathbb{N}$ be infinite and let $m \geq 1$ be an integer. For integers $n \geq 0$, denote by $\mathrm{pc}(n,m,A)$ the number of compositions of n palindromic modulo m with all parts belonging to A. Is there a closed-form formula for the generating function*

$$\sum_{n=0}^{\infty} \mathrm{pc}(n,m,A)x^n? \tag{3.18}$$

3.3 In continued fraction expansions

In this section we describe a left-right symmetry in the terms of certain continued fraction expansions following the presentation in [16] as follows.

Let $\alpha > 1$ be an irrational number. If we let $q_0 = \lfloor \alpha \rfloor$, then

$$\alpha = q_0 + \frac{1}{\alpha_1} \tag{3.19}$$

for some irrational $\alpha_1 > 1$. If we let $q_1 = \lfloor \alpha_1 \rfloor$, then

$$\alpha_1 = q_1 + \frac{1}{\alpha_2} \tag{3.20}$$

for some irrational $\alpha_2 > 1$. Having constructed the irrational $\alpha_n > 1$, we let $q_n = \lfloor \alpha_n \rfloor$ so that

$$\alpha_n = q_n + \frac{1}{\alpha_{n+1}} \tag{3.21}$$

for some irrational $\alpha_{n+1} > 1$. In this way we get a sequence of positive integers $(q_0, q_1, q_2, \ldots)$, and it can be shown that

$$\lim_{n\to\infty}\left(q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{\cdots q_n}}}\right) = \alpha. \tag{3.22}$$

So this defines a map ϕ from the set I of irrational numbers greater than 1 to the set $\mathbb{N}^\infty$ of all sequences of positive integers. It can be shown that ϕ is a bijection. The sequence $\phi(\alpha)$ is called the *continued fraction expansion* of α. We now give the following definition.

Definition 3.5. A *quadratic irrational* is an irrational number α which is algebraic of degree 2 over $\mathbb{Q}$. A sequence $(q_n)_{n\geq 0} \in \mathbb{N}^\infty$ is *periodic* if there exists an integer $\omega \geq 1$ such that $q_n = q_{n+\omega}$ for sufficiently large integers n.

Remark 3.1. Notice that here, unlike in Definition 5.1, for a periodic sequence $(q_n)_{n\geq 0} \in \mathbb{N}^\infty$ it is only required that $q_n = q_{n+\omega}$ for sufficiently large integers n.

Then we have the following theorem proved by Lagrange in 1770.

Theorem 3.5 ([16, p. 92])**.** *Let $\alpha > 1$ be a quadratic irrational. Then $\phi(\alpha)$ is periodic.*

In the following special case of Theorem 3.5, a left-right symmetry arises in the terms of $\phi(\alpha)$.

Theorem 3.6 ([16, p. 92])**.** *Let $N \geq 1$ be an integer, not a square. Then the continued fraction expansion of $\sqrt{N}$ is of the form*

$$q_0, \overline{\underbrace{q_1, q_2, \cdots, q_2, q_1}_{\text{palindromic part}}, 2q_0}, \tag{3.23}$$

where the q_i's are positive integers and the palindromic part might be empty.

The proof of Theorem 3.6 is elementary, consisting of many inegenious algebraic manipulations. A table of continued fraction expansions (3.23) is given in [16, p. 97]. The expansion (3.23) can be used to find all positive integral solutions (x, y) to *Pell's equation* $x^2 - Ny^2 = 1$.

Chapter 4

Proof of the Periodic Phenomenon

This chapter is devoted to the proof of the periodic phenomenon, i.e., Theorem 1.2, which we state again as follows.

Theorem 1.2. ([48, Theorem 1]). *Let $n \geq 1$ be an integer with $10 \nmid n$ and $n \neq r(n)$. There exists an integer $\omega \geq 1$ such that for all integers $k \geq 1$,*

$$n(k) \in \mathbb{V} \quad \text{if and only if} \quad n(k+\omega) \in \mathbb{V}. \tag{4.1}$$

In Section 4.1, we introduce certain numbers we denote by $\rho_{k,L}$ and $h_{p^\alpha,L}$. In Section 4.2, we introduce certain functions we denote by $\varphi_{p,\delta}$. In Section 4.3, we prove the case $n = 819$ of Theorem 1.2. In Section 4.4, we prove Theorem 1.2 fully, which is a straightforward generalization of the proof of the case $n = 819$. The proof of Theorem 1.2 constructs a particular ω, i.e., period of n, which we put into a theorem in Section 4.5.

4.1 The numbers $\rho_{k,L}$ and $h_{p^\alpha,L}$

We define certain numbers denoted $\rho_{k,L}$ as follows and then consider their divisibility properties in Subsection 4.1.1.

Definition 4.1. For $k, L \geq 1$, put

$$\rho_{k,L} = \overbrace{1\underbrace{0\ldots0}_{L-1}1\underbrace{0\ldots0}_{L-1}1\ldots1\underbrace{0\ldots0}_{L-1}1}^{k},$$

meaning that 1 appears k times and that between each consecutive pair of them 0 appears $L-1$ times.

Then we clearly have the following.

Lemma 4.1. *Let $n \geq 1$ be an integer with L decimal digits and let $k \geq 1$. Then the k-repeated concatenation of n, i.e., $n(k)$, is just $n\rho_{k,L}$.*

4.1.1 Divisibility properties

We consider the divisibility of the numbers $\rho_{k,L}$ by prime powers p^α.

Lemma 4.2. *Let p^α be a prime power with $p \neq 2, 5$, let $L \geq 1$, let $\beta = \mathrm{ord}_p(10^L - 1)$, and let h be the order of 10^L regarded as an element of $(\mathbb{Z}/p^{\alpha+\beta}\mathbb{Z})^\times$. Then $h > 1$ and for $k \geq 1$, we have $p^\alpha \mid \rho_{k,L}$ if and only if $h \mid k$.*

Proof. We first show that $h > 1$. That $h = 1$ means that $10^L \equiv 1 \pmod{p^{\alpha+\beta}}$, or equivalently, $p^{\alpha+\beta} \mid 10^L - 1$, or equivalently, $p^{\alpha+\mathrm{ord}_p(10^L-1)} \mid 10^L - 1$. This cannot be, whence $h > 1$. We have

$$(10^L - 1)\rho_{k,L} = (10^L - 1)\sum_{i=0}^{k-1} 10^{Li} = 10^{Lk} - 1.$$

As $\beta = \operatorname{ord}_p(10^L - 1)$, we have $p^\alpha \mid \rho_{k,L}$ if and only if $10^{Lk} - 1 \equiv 0 \pmod{p^{\alpha+\beta}}$, or equivalently, $10^{Lk} \equiv 1 \pmod{p^{\alpha+\beta}}$, or equivalently, $h \mid k$. The last equivalence is due to the structure of cyclic groups. □

Remark 4.1. If $p = 2, 5$, then 10^L cannot be regarded as an element of $(\mathbb{Z}/p^{\alpha+\beta}\mathbb{Z})^\times$. But obviously for every $k \geq 1$ we have $p^\alpha \nmid \rho_{k,L}$.

We shall use the numbers h in the above lemma often and so give them a notation as follows.

Notation 4.2. For p^α and L as in the above lemma, the h is denoted by $h_{p^\alpha,L}$.

We give the following table of some values of $h_{p^\alpha,L}$, computed using Mathematica.

Table 4.1: Values of $h_{p^\alpha,3}$ for various prime powers p^α.

p^α	7	7^2	13	13^2	17	17^2
$h_{p^\alpha,3}$	2	14	2	26	16	272

Regarding divisibility in general, not just for $\rho_{k,L}$, we recall the following lemmas.

Lemma 4.3. *Let $n \geq 1$ be an integer, let p be a prime, and let $g = \operatorname{ord}_p(n)$. Then*

(i) $g = 0$ *if and only if* $p \nmid n$,

(ii) $g = 1$ *if and only if* $p \mid n$ *and* $p^2 \nmid n$,

(iii) $g \leq 1$ *if and only if* $p^2 \nmid n$,

(iv) $g \geq 1$ *if and only if* $p \mid n$, *and*

(v) $g \geq 2$ *if and only if* $p^2 \mid n$.

Lemma 4.4. *Let $a, b \geq 1$ be integers with $a \mid b$. Then for integers $c \geq 1$, we have $a \mid c$ if and only if $a \mid (c + b)$.*

4.2 The functions $\varphi_{p,\delta}$

We define certain functions denoted $\varphi_{p,\delta}$ in Subsection 4.2.1 and then state a lemma about them in Subsection 4.2.2.

4.2.1 Definition

For a fixed prime p, the sequence of powers of p is

$$1, p, p^2, \ldots, p^\alpha, \ldots.$$

Applying the function v (Definition 1.6) to them yields

$$0, p, p + 2, \ldots, p + \alpha, \ldots.$$

Now we take differences of consecutive terms to get

$$p, 2, 1, \ldots, 1, \ldots, \tag{4.2}$$

with all 1's from the third term onwards. We give notation for the terms of this sequence.

Definition 4.3. For a prime p and integer $\alpha \geq 0$, put

$$\varphi_{p,1}(\alpha) = v(p^{\alpha+1}) - v(p^\alpha).$$

In this notation then, the sequence (4.2) is $(\varphi_{p,1}(\alpha))_{\alpha=0}^{\infty}$. More generally we define the following.

Definition 4.4. For a prime p and integers $\alpha \geq 0$ and $\delta \geq 1$, put

$$\varphi_{p,\delta}(\alpha) = v(p^{\alpha+\delta}) - v(p^{\alpha}).$$

In this notation, for instance, the sequence $(\varphi_{p,3}(\alpha))_{\alpha=0}^{\infty}$ is

$$p+3, 4, 3, \ldots, 3, \ldots,$$

with all 3's from the third term onwards. More generally, for $\delta \geq 2$, the sequence $(\varphi_{p,\delta}(\alpha))_{\alpha=0}^{\infty}$ is just

$$p+\delta, \delta+1, \delta, \ldots, \delta, \ldots. \tag{4.3}$$

Thus for every pair (p, δ) of a prime p and an integer $\delta \geq 1$, we have a function $\varphi_{p,\delta}\colon \mathbb{Z}_{\geq 0} \to \mathbb{N}$. Rephrasing (4.2) and (4.3), the values of $\varphi_{p,\delta}$ may be summarized as

$$\varphi_{2,1}(\alpha) = \begin{cases} 2 & (\alpha = 0, 1) \\ 1 & (\alpha \geq 2), \end{cases} \tag{4.4}$$

and if $p \neq 2$,

$$\varphi_{p,1}(\alpha) = \begin{cases} p & (\alpha = 0) \\ 2 & (\alpha = 1) \\ 1 & (\alpha \geq 2), \end{cases} \tag{4.5}$$

and if $\delta \geq 2$,

$$\varphi_{p,\delta}(\alpha) = \begin{cases} p+\delta & (\alpha = 0) \\ \delta+1 & (\alpha = 1) \\ \delta & (\alpha \geq 2). \end{cases} \tag{4.6}$$

We give a notation for the ranges of $\varphi_{p,\delta}$.

Definition 4.5. For a prime p and integer $\delta \geq 1$, put $R_{p,\delta} = \varphi_{p,\delta}(\mathbb{Z}_{\geq 0})$.

4.2.2 A lemma

In view of (4.4), (4.5), and (4.6), it is clear that $|R_{2,1}| = 2$ and $|R_{p,\delta}| = 3$ otherwise. Also, any nonempty inverse image of $\varphi_{p,\delta}$ is one of

$$\{0\}, \quad \{1\}, \quad \{0, 1\}, \quad \mathbb{Z}_{\geq 2}.$$

We have the following lemma, which is straightforward to prove.

Lemma 4.5. *Let p be a prime, $\delta \geq 1$, $u \in R_{p,\delta}$, and $\mu \geq 0$. Then we have the following.*

(i) *In case $\varphi_{p,\delta}^{-1}(u) = \{0\}$, for $g \geq 0$,*

$$\begin{aligned} \varphi_{p,\delta}(\mu + g) = u \quad & \text{if and only if} \quad \mu + g = 0 \\ & \text{if and only if} \quad \begin{cases} g = 0 & (\mu = 0) \\ \text{impossible} & (\mu \geq 1). \end{cases} \end{aligned} \tag{4.7}$$

(ii) *In case $\varphi_{p,\delta}^{-1}(u) = \{1\}$, for $g \geq 0$,*

$$\begin{aligned} \varphi_{p,\delta}(\mu + g) = u \quad & \text{if and only if} \quad \mu + g = 1 \\ & \text{if and only if} \quad \begin{cases} g = 1 - \mu & (\mu = 0, 1) \\ \text{impossible} & (\mu \geq 2). \end{cases} \end{aligned} \tag{4.8}$$

(iii) *In case $\varphi_{p,\delta}^{-1}(u) = \{0, 1\}$, for $g \geq 0$,*

$$\begin{aligned} \varphi_{p,\delta}(\mu + g) = u \quad & \text{if and only if} \quad \mu + g \in \{0, 1\} \\ & \text{if and only if} \quad \begin{cases} g \leq 1 & (\mu = 0) \\ g = 0 & (\mu = 1) \\ \text{impossible} & (\mu \geq 2). \end{cases} \end{aligned} \tag{4.9}$$

(iv) *In case* $\varphi_{p,\delta}^{-1}(u) = \mathbb{Z}_{\geq 2}$, *for* $g \geq 0$,

$$\begin{aligned}\varphi_{p,\delta}(\mu+g) = u \quad &\text{if and only if} \quad \mu + g \geq 2 \\ &\text{if and only if} \quad \begin{cases} g \geq 2-\mu & (\mu = 0,1) \\ \text{always true} & (\mu \geq 2). \end{cases}\end{aligned} \tag{4.10}$$

Here "impossible" *means that no* $g \geq 0$ *can be found to fulfill* $\varphi_{p,\delta}(\mu+g) = u$, *and* "always true" *means that all* $g \geq 0$ *fulfills* $\varphi_{p,\delta}(\mu+g) = u$.

4.3 The case $n = 819$ of Theorem 1.2

In this section we prove the case $n = 819$ of Theorem 1.2 as follows.

We have the prime factorizations

$$\begin{aligned} 819 &= 3^2 \cdot 7 \cdot 13, \\ 918 &= 2 \cdot 3^3 \cdot 17. \end{aligned}$$

For integers $k \geq 1$, let the prime factorization of $\rho_{k,3}$ be

$$\rho_{k,3} = 3^{g_1} \cdot 7^{g_2} \cdot 13^{g_3} \cdot 17^{g_4} \cdot b,$$

where $(b, 3 \cdot 7 \cdot 13 \cdot 17) = 1$. The numbers g_1, g_2, g_3, g_4, b obviously depend on k, but we have suppressed the notation for simplicity. By Lemma 4.1,

$$\begin{aligned} 819(k) = 819\rho_{k,3} &= 3^{2+g_1} \cdot 7^{1+g_2} \cdot 13^{1+g_3} \cdot 17^{g_4} \cdot b, \\ r(819(k)) = 918(k) = 918\rho_{k,3} &= 2 \cdot 3^{3+g_1} \cdot 7^{g_2} \cdot 13^{g_3} \cdot 17^{1+g_4} \cdot b. \end{aligned}$$

Applying the additive function v to these equations,

$$\begin{aligned} v(819(k)) &= v(3^{2+g_1}) + v(7^{1+g_2}) + v(13^{1+g_3}) + v(17^{g_4}) + v(b), \\ v(r(819(k))) &= v(2) + v(3^{3+g_1}) + v(7^{g_2}) + v(13^{g_3}) + v(17^{1+g_4}) + v(b). \end{aligned}$$

Hence $819(k)$ is a v-palindrome if and only if the above two quantities are equal, that is, after rearranging,

$$\begin{aligned} &(v(7^{1+g_2}) - v(7^{g_2})) + (v(13^{1+g_3}) - v(13^{g_3})) \\ &\qquad = 2 + (v(3^{3+g_1}) - v(3^{2+g_1})) + (v(17^{1+g_4}) - v(17^{g_4})). \end{aligned}$$

In terms of the functions $\varphi_{p,\delta}$ of Section 4.2, this becomes

$$\varphi_{7,1}(g_2) + \varphi_{13,1}(g_3) = 2 + \varphi_{3,1}(2+g_1) + \varphi_{17,1}(g_4). \tag{4.11}$$

Since $2 + g_1 \geq 2$, by (4.5), we have $\varphi_{3,1}(2+g_1) = 1$. Therefore (4.11) becomes

$$\varphi_{7,1}(g_2) + \varphi_{13,1}(g_3) = 3 + \varphi_{17,1}(g_4). \tag{4.12}$$

Now consider the equation

$$u_2 + u_3 = 3 + u_4. \tag{4.13}$$

We want to solve it for $u_2 \in R_{7,1}$, $u_3 \in R_{13,1}$, and $u_4 \in R_{17,1}$. In view of (4.5),

$$R_{7,1} = \{7,2,1\}, \quad R_{13,1} = \{13,2,1\}, \quad R_{17,1} = \{17,2,1\}.$$

By trying all possibilities we see that the only solutions are $(u_2, u_3, u_4) = (7, 13, 17)$ and $(2, 2, 1)$. Whence (4.12) is satisfied if and only if

$$(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17) \quad \text{or} \quad (2, 2, 1).$$

We first consider when $(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17)$. By Lemmas 4.5 (or more easily just by looking at (4.5)), 4.3, 4.2, and Table 4.1,

$$\begin{aligned} \varphi_{7,1}(g_2) = 7 \quad &\text{if and only if} \quad g_2 = 0 \quad \text{if and only if} \quad 7 \nmid \rho_{k,3} \\ &\text{if and only if} \quad h_{7,3} \nmid k \quad \text{if and only if} \quad 2 \nmid k, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\varphi_{13,1}(g_3) = 13 \quad &\text{if and only if} \quad g_3 = 0 \quad \text{if and only if} \quad 13 \nmid \rho_{k,3} \\ &\text{if and only if} \quad h_{13,3} \nmid k \quad \text{if and only if} \quad 2 \nmid k,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\varphi_{17,1}(g_4) = 17 \quad &\text{if and only if} \quad g_4 = 0 \quad \text{if and only if} \quad 17 \nmid \rho_{k,3} \\ &\text{if and only if} \quad h_{17,3} \nmid k \quad \text{if and only if} \quad 16 \nmid k.\end{aligned} \tag{4.16}$$

Hence $(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17)$ simply when k is odd. We next consider when

$$(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (2, 2, 1). \tag{4.17}$$

Similarly we have

$$\begin{aligned}\varphi_{7,1}(g_2) = 2 \quad &\text{if and only if} \quad g_2 = 1 \quad \text{if and only if} \quad \begin{cases} 7 \mid \rho_{k,3}, \\ 7^2 \nmid \rho_{k,3} \end{cases} \\ &\text{if and only if} \quad \begin{cases} h_{7,3} \mid k, \\ h_{7^2,3} \nmid k \end{cases} \quad \text{if and only if} \quad \begin{cases} 2 \mid k, \\ 14 \nmid k, \end{cases}\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\varphi_{13,1}(g_3) = 2 \quad &\text{if and only if} \quad g_3 = 1 \quad \text{if and only if} \quad \begin{cases} 13 \mid \rho_{k,3}, \\ 13^2 \nmid \rho_{k,3} \end{cases} \\ &\text{if and only if} \quad \begin{cases} h_{13,3} \mid k, \\ h_{13^2,3} \nmid k \end{cases} \quad \text{if and only if} \quad \begin{cases} 2 \mid k, \\ 26 \nmid k, \end{cases}\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\varphi_{17,1}(g_4) = 1 \quad &\text{if and only if} \quad g_4 \geq 2 \quad \text{if and only if} \quad 17^2 \mid \rho_{k,3} \\ &\text{if and only if} \quad h_{17^2,3} \mid k \quad \text{if and only if} \quad 272 \mid k,\end{aligned} \tag{4.20}$$

where two divisibility relations to the right of a left brace means that both must hold. Hence

$$(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (2, 2, 1) \tag{4.21}$$

precisely when $272 \mid k$ and $(k, 7 \cdot 13) = 1$. Hence we have established the following characterization.

Theorem 4.6. *For $k \geq 1$, the number $819(k)$ is a v-palindrome if and only if k is odd or if $272 \mid k$ and $(k, 7 \cdot 13) = 1$.*

From the above theorem, we immediately see that $c(819) = 1$ (Definition 1.9). We see that $819(k)$ is a v-palindrome if and only if all 3 conditions (4.14), (4.15), and (4.16) hold, or if all 3 conditions (4.18), (4.19), and (4.20) hold. Now these conditions have the same truth values when k increases by $\operatorname{lcm}(16, 14, 26, 272) = 24752$. Hence $\omega = 24752$ is a period of 819. Using the general procedure of Section 6.6, it can be shown that actually it is the smallest period, that is, $\omega_0(819) = 24752$.

4.4 Proof of Theorem 1.2

In this section we prove Theorem 1.2 fully, and this is essentially writing the proof of the case $n = 819$ in the previous section in the general setting. We divide the proof into three subsections as follows.

4.4.1 First half of the proof

Let the prime factorizations of n and $r(n)$ be

$$\begin{aligned} n &= p_1^{e_1} p_2^{e_2} \dots p_m^{e_m}, \\ r(n) &= p_1^{f_1} p_2^{f_2} \dots p_m^{f_m}, \end{aligned}$$

where we have done the factorization over the set of primes which divide one of n or $r(n)$, setting $e_i = 0$ or $f_i = 0$ if necessary. Since $n \neq r(n)$, we have $e_i \neq f_i$ for some i. Let the i's such that $e_i \neq f_i$ be arranged in order as

$$i_1 < i_2 < \ldots < i_t. \tag{4.22}$$

Let the number of decimal digits n has be denoted by L. For integers $k \geq 1$, let the prime factorization of $\rho_{k,L}$ be

$$\rho_{k,L} = p_1^{g_1} p_2^{g_2} \ldots p_m^{g_m} b, \tag{4.23}$$

where $(b, p_1 p_2 \ldots p_m) = 1$. The $g_1, g_2, \ldots, g_m, b$ obviously depend on k, but we suppress them from our notation for simplicity. Then by Lemma 4.1,

$$n(k) = n\rho_{k,L} = p_1^{e_1+g_1} p_2^{e_2+g_2} \ldots p_m^{e_m+g_m} b,$$
$$r(n(k)) = r(n)(k) = r(n)\rho_{k,L} = p_1^{f_1+g_1} p_2^{f_2+g_2} \ldots p_m^{f_m+g_m} b.$$

Taking their v, we have

$$v(n(k)) = \sum_{i=1}^{m} v(p_i^{e_i+g_i}) + v(b),$$
$$v(r(n(k))) = \sum_{i=1}^{m} v(p_i^{f_i+g_i}) + v(b).$$

Hence $n(k)$ is a v-palindrome, that is, $v(n(k)) = v(r(n(k)))$, if and only if

$$\sum_{i=1}^{m} (v(p_i^{e_i+g_i}) - v(p_i^{f_i+g_i})) = 0. \tag{4.24}$$

If $e_i = f_i$, of course the term $v(p_i^{e_i+g_i}) - v(p_i^{f_i+g_i}) = 0$, so by (4.22), the equation (4.24) is equivalent to

$$\sum_{j=1}^{t} (v(p_{i_j}^{e_{i_j}+g_{i_j}}) - v(p_{i_j}^{f_{i_j}+g_{i_j}})) = 0. \tag{4.25}$$

This is a cumbersome notation, and we will just write p_{i_j} as p_j, e_{i_j} as e_j, f_{i_j} as f_j, and g_{i_j} as g_j, because we will not refer to the other prime factors or exponents from here on. Consequently, (4.25) becomes

$$\sum_{j=1}^{t} (v(p_j^{e_j+g_j}) - v(p_j^{f_j+g_j})) = 0. \tag{4.26}$$

We also write

$$\delta_j = e_j - f_j,$$
$$\mu_j = \min(e_j, f_j),$$
$$\alpha_j = \mu_j + g_j,$$

for $1 \leq j \leq t$. Then it is clear that the left-hand side of (4.26) can be rewritten, using the functions $\varphi_{p,\delta}$ of Section 4.2, as

$$\begin{aligned}&\sum_{j=1}^{t} (v(p_j^{e_j+g_j}) - v(p_j^{f_j+g_j})) \\ &\qquad = \sum_{j=1}^{t} \operatorname{sgn}(\delta_j)(v(p_j^{\alpha_j+|\delta_j|}) - v(p_j^{\alpha_j})) = \sum_{j=1}^{t} \operatorname{sgn}(\delta_j)\varphi_{p_j,|\delta_j|}(\alpha_j),\end{aligned} \tag{4.27}$$

Now consider the equation

$$\sum_{j=1}^{t} \operatorname{sgn}(\delta_j) u_j = 0. \tag{4.28}$$

Supposedly we can solve it for

$$(u_1, u_2, \dots, u_t) \in R_{p_1,|\delta_1|} \times R_{p_2,|\delta_2|} \times \cdots \times R_{p_t,|\delta_t|}.$$

Let the set of all solutions be

$$U = \{u = (u_1, \dots, u_t)\}.$$

Then we see that

$$\sum_{j=1}^{t} \operatorname{sgn}(\delta_j)\varphi_{p_j,|\delta_j|}(\alpha_j) = 0$$

holds if and only if for some $u \in U$,

$$\varphi_{p_j,|\delta_j|}(\alpha_j) = u_j$$

for all $1 \le j \le t$. Summarizing what we have done up to now, we have shown the following.

Lemma 4.7. *For $k \ge 1$, the number $n(k)$ is a v-palindrome if and only if for some $u \in U$, we have $\varphi_{p_j,|\delta_j|}(\alpha_j) = u_j$ for all $1 \le j \le t$.*

4.4.2 Any parcitular $\varphi_{p_j,|\delta_j|}(\alpha_j) = u_j$

Now let us just consider any particular condition $\varphi_{p_j,|\delta_j|}(\alpha_j) = \varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j$. If in the statement of Lemma 4.5, we substitute the p, δ, u, and μ by p_j, $|\delta_j|$, u_j, and μ_j, respectively, then we have

$$\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j \quad \text{if and only if} \quad \begin{cases} g_j = 0 & \text{(if (i, 0), or (ii, 1), or (iii, 1))} \\ g_j = 1 & \text{(if (ii, 0))} \\ g_j \le 1 & \text{(if (iii, 0))} \\ g_j \ge 1 & \text{(if (iv, 1))} \\ g_j \ge 2 & \text{(if (iv, 0))} \\ \text{always true} & \text{(if (iv, } \ge 2)) \\ \text{impossible} & \text{(otherwise)}, \end{cases} \tag{4.29}$$

where on the right, a notation like (N, μ), where N is a Roman numeral and $\mu = 0, 1$, denotes the case (N) in Lemma 4.5 and in addition the case where $\mu_j = \mu$; (iv, ≥ 2) denotes the case (iv) in Lemma 4.5 and in addition the case where $\mu_j \ge 2$. As the last two cases ("always true" and "impossible") never change as k varies, we exclude them from our consideration. By Lemma 4.3, we can continue the equivalences in (4.29) respectively (here we do not write out the cases as in (4.29)), recalling that $g_j = \operatorname{ord}_{p_j}(\rho_{k,L})$,

$$\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j \quad \text{if and only if} \quad \begin{cases} p_j \nmid \rho_{k,L} \\ p_j \mid \rho_{k,L} \text{ and } p_j^2 \nmid \rho_{k,L} \\ p_j^2 \nmid \rho_{k,L} \\ p_j \mid \rho_{k,L} \\ p_j^2 \mid \rho_{k,L}. \end{cases} \tag{4.30}$$

In case $p_j \ne 2, 5$, we can apply Lemma 4.2 to (4.30) to obtain, respectively,

$$\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j \quad \text{if and only if} \quad \begin{cases} h_{p_j,L} \nmid k \\ h_{p_j,L} \mid k \text{ and } h_{p_j^2,k} \nmid k \\ h_{p_j^2,L} \nmid k \\ h_{p_j,L} \mid k \\ h_{p_j^2,L} \mid k. \end{cases} \tag{4.31}$$

However, in case $p_j = 2, 5$, by Remark 4.1, the conditions (4.30) become

$$\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j \quad \text{if and only if} \quad \begin{cases} \text{always true} \\ \text{impossible} \\ \text{always true} \\ \text{impossible} \\ \text{impossible.} \end{cases} \tag{4.32}$$

4.4.3 Finishing the proof

In view of Lemma 4.4, we see that the truth value of any particular condition $\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j$ does not change if we increase k by

$$\omega = \mathrm{lcm}\{h_{p_j,L}, h_{p_j^2,L} \mid 1 \le j \le t \text{ with } p_j \ne 2, 5\}. \tag{4.33}$$

In view of Lemma 4.7, whether $n(k)$ is a v-palindrome depends only on the truth values of the individual $(\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j)$'s. Hence this ω serves as a possible ω as required by Theorem 1.2.

4.5 The constructed period

The proof of Theorem 1.2 in the previous section constructed a particular period (4.33) of n, which we put into a theorem as follows. We first give the following definition.

Definition 4.6. Let n be as in Theorem 1.2. A *crucial prime* of n is a prime p for which $\mathrm{ord}_p(n) \ne \mathrm{ord}_p(r(n))$. The set of all crucial primes of n is denoted by $K(n)$.

Then we have the following.

Theorem 4.8. *Let n be as in Theorem 1.2 and let L be the number of decimal digits of n. Then*

$$\omega_f(n) = \mathrm{lcm}\{h_{p^2,L} \mid p \in K(n) \setminus \{2, 5\}\} \tag{4.34}$$

is a period of n.

Proof. The quantity (4.33) can be rewritten as

$$\mathrm{lcm}\{h_{p,L}, h_{p^2,L} \mid p \in K(n) \setminus \{2, 5\}\}. \tag{4.35}$$

Hence it suffices to prove that $h_{p,L} \mid h_{p^2,L}$ for every $p \in K(n) \setminus \{2, 5\}$. Therefore let $p \in K(n) \setminus \{2, 5\}$ be arbitrary. Since

$$(10^L)^{h_{p^2,L}} \equiv 1 \pmod{p^{2+\mathrm{ord}_p(10^L-1)}}, \tag{4.36}$$

plainly

$$(10^L)^{h_{p^2,L}} \equiv 1 \pmod{p^{1+\mathrm{ord}_p(10^L-1)}}. \tag{4.37}$$

Now $h_{p,L}$ is the order of 10^L regarded as an element of $(\mathbb{Z}/p^{1+\mathrm{ord}_p(10^L-1)}\mathbb{Z})^\times$, thus $h_{p,L} \mid h_{p^2,L}$ follows from the structure of cyclic groups. ◻

Remark 4.2. The "f" in the notation $\omega_f(n)$ comes from "found", because $\omega_f(n)$ is a period of n we found.

Chapter 5

Periodic Functions

This chapter is a treatment of the general topic of periodic functions. Our derivation of the fundamental period of n (Definition 1.9) in Chapter 6 belongs naturally to the more general derivation of the fundamental period of an arbitrary periodic function.

In Section 5.1, we start from the definition of periodic functions and prove a formula for the fundamental period of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$ (Theorem 5.5). The theorem [51, Theorem 9] gives another formula for the fundamental period of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$ and we direcly show that the formulae in Theorem 5.5 and [51, Theorem 9] give the same quantity in Section 5.2. In Section 5.3, we introduce Ramanujan spaces and prove Theorem 1.17 with a simple application of Theorem 5.5.

Recall that the functions I_a have been introduced in Definition 1.11. In Section 5.4, we indicate some properties of such functions and give a formula for the fundamental period of an arbitrary linear combination of such functions with integer coefficients (Theorem 5.9). We also in the same section introduce certain sets we denote by $S(A, B)$ and express their indicator functions in terms of functions of the form I_a (Lemma 5.11).

5.1 Periodic functions

In this section, we start from the definition of periodic functions in Subsection 5.1.1 and prove a formula for the fundamental period of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$ (Theorem 5.5) in Subsection 5.1.3.

In Subsection 5.1.1 we also characterize periods (Theorem 5.1). In Subsection 5.1.2, we show that the general form of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$ is a finite linear combination (5.6) with complex cofficients of functions of the form ζ^x where ζ is a root of unity.

5.1.1 Definition and characterization of periods

We first give the following definition.

Definition 5.1. A function $f\colon \mathbb{Z} \to \mathbb{C}$ (respectively $f\colon \mathbb{N} \to \mathbb{C}$) is *periodic* if there is an integer $\omega \geq 1$ such that for all $x \in \mathbb{Z}$ (respectively $x \in \mathbb{N}$),

$$f(x + \omega) = f(x). \tag{5.1}$$

Such an ω is called a *period* of f, and we also say that f is *periodic modulo* ω. When f is periodic, the smallest period of f is called its *fundamental period*.

We have the following characterization of periods.

Theorem 5.1. *Let $f\colon \mathbb{Z} \to \mathbb{C}$ be a periodic function. Then we have the following.*

(i) *The restriction of f to $\mathbb{N}$, i.e., $f|_{\mathbb{N}}$, is periodic. Moreover, an integer $\omega \geq 1$ is a period of f if and only if it is a period of $f|_{\mathbb{N}}$.*

(ii) *If the fundamental period of f is ω_0, then the set of periods of f is $\omega_0\mathbb{N}$.*

Proof. (i) Plainly any period of f is also a period of its restriction $f|_{\mathbb{N}}$. We only need to prove that any period of $f|_{\mathbb{N}}$ is conversely a period of f. So let ω be a period of $f|_{\mathbb{N}}$. Choose a period μ of f, then

μ is also a period of $f|_{\mathbb{N}}$. For any $x \in \mathbb{Z}$, there exists a positive integer $q > 0$ such that $x + q\mu > 0$. Also, $x + \omega + q\mu > 0$. Since ω is a period of $f|_{\mathbb{N}}$, $f(x + q\mu) = f(x + q\mu + \omega)$. Now, since μ is a period of f,

$$f(x) = f(x + q\mu) = f(x + q\mu + \omega) = f(x + \omega). \tag{5.2}$$

Since the above holds for all $x \in \mathbb{Z}$, ω is a period of f.

(ii) Let ω be a period of f. Use the division algorithm to write $\omega = q\omega_0 + r$, where $q, r \in \mathbb{Z}$ are such that $0 \le r < \omega_0$ and $q > 0$. Assume that $r > 0$, then $r = \omega - q\omega_0$. For any $x \in \mathbb{Z}$,

$$f(x) = f(x + \omega) = f(x + \omega - q\omega_0) = f(x + r), \tag{5.3}$$

because both ω and ω_0 are periods of f. Hence r is a period of f smaller than ω_0, this is a contradiction. Hence $r = 0$ and so $\omega_0 \mid \omega$. The converse, that any positive integral multiple of ω_0 is a period of f, is plain.

□

5.1.2 The general form of a periodic function

In this subsection, we show that the general form of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$ is a finite linear combination (5.6) with complex cofficients of functions of the form ζ^x where ζ is a root of unity.

Let the function $e\colon \mathbb{R} \to \mathbb{C}$ be defined by

$$e(t) = e^{2\pi i t}. \tag{5.4}$$

Then the set of roots of unity in $\mathbb{C}$ is

$$\mathcal{R} = \{e(\alpha) \mid \alpha \in \mathbb{Q}\}. \tag{5.5}$$

We give the following notation.

Notation 5.2. For a $\zeta = e(\alpha) \in \mathcal{R}$, where $0 \le \alpha < 1$ is rational, write $\alpha = a/b$ in lowest terms, i.e. $a, b \in \mathbb{Z}$, $b > 0$, and $(a, b) = 1$, then denote $\nu(\zeta) = a$ and $\delta(\zeta) = b$. Thus ζ is a primitive $\delta(\zeta)$-th root of unity. For each integer $m \ge 1$, we denote by ζ_m the primitive m-th root of unity $e(1/m)$.

Consider functions $g\colon \mathcal{R} \to \mathbb{C}$ with $g(\zeta) = 0$ outside a finite set. Let the set of such functions be denoted $\mathcal{G}$. For a $g \in \mathcal{G}$, define a function $f\colon \mathbb{Z} \to \mathbb{C}$ by

$$f(x) = \sum_{\zeta \in \mathcal{R}} g(\zeta)\zeta^x. \tag{5.6}$$

The sum is actually finite because $g(\zeta) = 0$ for all but finitely many ζ's. We denote this f by $\Phi(g)$.

Theorem 5.2. *Let $g \in \mathcal{G}$. Then $\Phi(g)$ is periodic modulo*

$$\operatorname{lcm}\{\delta(\zeta) \mid \zeta \in \mathcal{R}, g(\zeta) \neq 0\}. \tag{5.7}$$

Proof. Let the least common multiple (5.7) be denoted by ω and $\Phi(g) = f$. For each $\zeta \in \mathcal{R}$ with $g(\zeta) \neq 0$, ζ is a $\delta(\zeta)$-th root of unity, therefore $\zeta^{x+\delta(\zeta)} = \zeta^x$ for every $x \in \mathbb{Z}$. As ω is a multiple of $\delta(\zeta)$, we have $\zeta^{x+\omega} = \zeta^x$ for every $x \in \mathbb{Z}$. Consequently, for every $x \in \mathbb{Z}$,

$$f(x + \omega) = \sum_{\zeta \in \mathcal{R}, g(\zeta) \neq 0} g(\zeta)\zeta^{x+\omega} = \sum_{\zeta \in R, g(\zeta) \neq 0} g(\zeta)\zeta^x = f(x). \tag{5.8}$$

□

Let the set of periodic functions $f\colon \mathbb{Z} \to \mathbb{C}$ be denoted by $\mathcal{F}$. The above established a mapping $\Phi\colon \mathcal{G} \to \mathcal{F}$. We shall prove that it is bijective. Before that, we first state the first half of [5, Theorem 8.4 on p. 160] as a theorem in this dissertation, which will be used.

Theorem 5.3 ([5, Theorem 8.4 on p. 160])**.** *Let $f\colon \mathbb{Z} \to \mathbb{C}$ be periodic modulo ω. Then there exist unique coefficients $h_r \in \mathbb{C}$ for $0 \le r < \omega$ such that for all $x \in \mathbb{Z}$,*

$$f(x) = \sum_{r=0}^{\omega-1} h_r \zeta_\omega^{xr}. \tag{5.9}$$

Theorem 5.4. *The mapping $\Phi\colon \mathcal{G} \to \mathcal{F}$ is bijective.*

Proof. Let $f \in \mathcal{F}$ be periodic modulo ω. By Theorem 5.3, there exist unique coefficients $h_r \in \mathbb{C}$ for $0 \le r < \omega$ such that for all $x \in \mathbb{Z}$,

$$f(x) = \sum_{r=0}^{\omega-1} h_r \zeta_\omega^{xr}. \tag{5.10}$$

If we define the function $g\colon \mathcal{R} \to \mathbb{C}$ by setting $g(\zeta_\omega^r) = h_r$ for $0 \le r < \omega$ and $g(\zeta) = 0$ for all other $\zeta \in \mathcal{R}$, it is easy to see that $g \in \mathcal{G}$ and that $\Phi(g) = f$. Whence Φ is surjective.

We now prove injectivity. Assume that $\Phi(g_1) = \Phi(g_2) = f \in \mathcal{F}$, then for all $x \in \mathbb{Z}$,

$$f(x) = \sum_{\zeta\in\mathcal{R}} g_1(\zeta)\zeta^x = \sum_{\zeta\in\mathcal{R}} g_2(\zeta)\zeta^x. \tag{5.11}$$

Let $S = \{\zeta \in \mathcal{R} : (g_1(\zeta), g_2(\zeta)) \neq (0,0)\}$. Then S is finite and the above sums can be written as

$$\sum_{\zeta\in S} g_1(\zeta)\zeta^x = \sum_{\zeta\in S} g_2(\zeta)\zeta^x. \tag{5.12}$$

Consequently, for all $x \in \mathbb{Z}$,

$$\sum_{\zeta\in S} (g_1(\zeta) - g_2(\zeta))\zeta^x = 0. \tag{5.13}$$

If $S = \emptyset$, plainly $g_1 = g_2 = 0$ is identically zero. Thus assume otherwise and list the elements of S as $\{\xi_1, \dots, \xi_m\}$. Put $x_j = g_1(\xi_j) - g_2(\xi_j)$ for $1 \le j \le m$. Then (5.13) becomes

$$\sum_{j=1}^{m} x_j \xi_j^x = 0. \tag{5.14}$$

Since this holds for all $x \in \mathbb{Z}$, in particular it holds for all $0 \le x < m$, and we have a homogeneous system of linear equations. Since the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_m \\ \cdots & \cdots & \cdots & \cdots \\ \xi_1^{m-1} & \xi_2^{m-1} & \cdots & \xi_m^{m-1} \end{vmatrix} \neq 0 \tag{5.15}$$

as the ξ_j's are distinct, $x_j = 0$ for all $1 \le j \le m$, i.e. $g_1(\xi_j) = g_2(\xi_j)$ for all $1 \le j \le m$. In other words, $g_1(\zeta) = g_2(\zeta)$ for all $\zeta \in S$. Since $g_1(\zeta) = g_2(\zeta) = 0$ for all $\zeta \in \mathcal{R} \setminus S$, we have shown that $g_1 = g_2$. Whence Φ is injective. □

Hence we have established the general form of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$ as (5.6).

5.1.3 A formula for the fundamental period

We prove that the fundamental period of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$ is indeed given by (5.7).

Theorem 5.5. *Let $g \in \mathcal{G}$. Then the fundamental period of $\Phi(g)$ is indeed given by* (5.7).

Proof. Let $\Phi(g) = f$. Theorem 5.2 already showed that

$$\omega = \operatorname{lcm}\{\delta(\zeta) \mid \zeta \in \mathcal{R}, g(\zeta) \neq 0\} \tag{5.16}$$

is a period of f. We need to show that it is the smallest one. Assume that the smallest period is actually ω_0, where $0 < \omega_0 < \omega$. By Theorem 5.1 (ii), we have $\omega_0 \mid \omega$. By Theorem 5.3, there exist unique coefficients h_r for $0 \le r < \omega_0$ such that for all $x \in \mathbb{Z}$,

$$f(x) = \sum_{r=0}^{\omega_0-1} h_r \zeta_{\omega_0}^{xr}. \tag{5.17}$$

Hence we see that $g(\zeta_{\omega_0}^r) = h_r$ for $0 \le r < \omega_0$ and $g(\zeta) = 0$ for all other $\zeta \in \mathcal{R}$.

Now $\omega_0 < \omega$ and so in view of (5.16) there exists some $\xi \in \mathcal{R}$ with $g(\xi) \neq 0$ such that $\delta(\xi) \nmid \omega_0$. We have

$$\xi^{\omega_0} = \left(e\left(\frac{\nu(\xi)}{\delta(\xi)}\right)\right)^{\omega_0} = e\left(\frac{\nu(\xi)\omega_0}{\delta(\xi)}\right). \tag{5.18}$$

Now the argument on the right above is not an integer. For if it is, then $\delta(\xi) \mid \nu(\xi)\omega_0$. Since $(\delta(\xi), \nu(\xi)) = 1$, we have $\delta(\xi) \mid \omega_0$, which is a contradiction. Therefore $\nu(\xi)\omega_0/\delta(\xi)$ is not an integer, and so $\xi^{\omega_0} \neq 1$. That is, ξ is not an ω_0-th root of unity. But g vanishes at all $\zeta \in \mathcal{R}$ which is not an ω_0-th root of unity. This is a contradiction. Hence ω is indeed the fundamental period of f. □

5.2 The formula in [51, Theorem 9]

The theorem [51, Theorem 9] gives another formula for the fundamental period of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$ and we direcly show that the formulae in Theorem 5.5 and [51, Theorem 9] give the same quantity as follows.

Let $f\colon \mathbb{Z} \to \mathbb{C}$ be a periodic function of period ω. Then by Theorem 5.3, there exist unique coefficients h_r for $0 \le r < \omega$ such that for all $x \in \mathbb{Z}$,

$$f(x) = \sum_{r=0}^{\omega-1} h_r \zeta_\omega^{xr}. \tag{5.19}$$

We can write $f(x)$ as

$$f(x) = \sum_{k=1}^{\omega} h_{\omega-k} \zeta_\omega^{-xk}, \tag{5.20}$$

or if we rename $h_{\omega-k}$ as h_k,

$$f(x) = \sum_{k=1}^{\omega} h_k \zeta_\omega^{-xk}. \tag{5.21}$$

Let the set of integers k such that $1 \le k \le \omega$ and $h_k \neq 0$ be $\{k_1, \dots, k_l\}$. Then according to [51, Theorem 9], the fundamental period of f is

$$L = \frac{\omega}{(k_1, \dots, k_l, \omega)}. \tag{5.22}$$

On the other hand, according to Theorem 5.5, the fundamental period of f is

$$R = \operatorname{lcm}(\delta(\zeta_\omega^{-k_1}), \dots, \delta(\zeta_\omega^{-k_l})). \tag{5.23}$$

We show that $L = R$, i.e.,

$$\frac{\omega}{(k_1, \dots, k_l, \omega)} = \operatorname{lcm}(\delta(\zeta_\omega^{-k_1}), \dots, \delta(\zeta_\omega^{-k_l})). \tag{5.24}$$

When there are no $1 \le k \le \omega$ such that $h_k \neq 0$, i.e. when $l = 0$, this is plain. Thus assume that $l > 0$. Notice that for $1 \le j \le l$,

$$\delta(\zeta_\omega^{-k_j}) = \delta\left(e\left(\frac{-k_j}{\omega}\right)\right) = \frac{\omega}{(k_j, \omega)}, \tag{5.25}$$

which is easily seen to divide L. Hence $R \mid L$.

Notice that for $1 \le j \le l$, because of (5.25),

$$\frac{\omega}{(k_j, \omega)} \mid R, \quad \text{and therefore} \quad \omega \mid R(k_j, \omega). \tag{5.26}$$

Since

$$(k_1, \dots, k_l, \omega) = ((k_1, \omega), \dots, (k_l, \omega)), \tag{5.27}$$

we have a linear combination

$$(k_1, \dots, k_l, \omega) = \sum_{j=1}^{l} y_j (k_j, \omega), \tag{5.28}$$

where the y_j's are integers. We prove that $L \mid R$, or equivalently,

$$\omega \mid R(k_1, \dots, k_l, \omega). \tag{5.29}$$

Since (5.26) holds for all $1 \le j \le l$, using also (5.28),

$$\omega \mid \sum_{j=1}^{l} y_j R(k_j, \omega) = R \sum_{j=1}^{l} y_j(k_j, \omega) = R(k_1, \dots, k_l, \omega). \tag{5.30}$$

Since we have proved that $R \mid L$ and $L \mid R$, it holds that $L = R$.

Hence both Theorem 5.5 and [51, Theorem 9] give a formula for the fundamental period of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$. These formulae might look different on the surface, but indeed give the quantity.

5.3 Ramanujan spaces

We introduce Ramanujan spaces and prove Theorem 1.17 with a simple application of Theorem 5.5.

The set $\mathbb{C}^{\mathbb{Z}}$ of functions $f : \mathbb{Z} \to \mathbb{C}$ is obviously a vector space over $\mathbb{C}$ with both vector addition and scalar multiplication defined pointwise. The set, denoted by $\mathcal{F}$ in Section 5.1, of all periodic functions $\mathbb{Z} \to \mathbb{C}$, is a subspace of $\mathbb{C}^{\mathbb{Z}}$. We define the so-called Ramanujan spaces as follows, where $\mathcal{R}^*(\omega)$ denotes the set of primitive ω-th roots of unity in $\mathbb{C}$.

Definition 5.3. Let $\omega \ge 1$ be an integer. The set of all functions

$$f(x) = \sum_{\zeta \in \mathcal{R}^*(\omega)} g(\zeta)\zeta^x, \quad \text{for } x \in \mathbb{Z}, \tag{5.31}$$

where the $g(\zeta)$'s are complex numbers, is a subspace of $\mathcal{F}$ called a *Ramanujan space* and denoted by S_ω.

We restate Theorem 1.17 as follows, where 0 denotes the zero function $\mathbb{Z} \to \{0\}$.

Theorem 1.17. ([51, Theorem12]). *Let $\omega_1, \dots, \omega_m \ge 1$ be distinct integers and let $0 \ne f_j \in S_{\omega_j}$ for each $1 \le j \le m$. Then the fundamental period of the periodic function $f_1 + \cdots + f_m$ is* $\operatorname{lcm}\{\omega_1, \dots, \omega_m\}$.

Proof. For each $1 \le j \le m$, we have

$$f_j(x) = \sum_{\zeta \in \mathcal{R}^*(\omega_j)} g_j(\zeta)\zeta^x, \tag{5.32}$$

where $g_j = \Phi^{-1}(f_j)$. Since the $\mathcal{R}^*(\omega_j)$ are pairwise disjoint, $\Phi^{-1}(f) = g$, where

$$g(\zeta) = \begin{cases} g_j(\zeta) & \text{if } \zeta \in \mathcal{R}^*(\omega_j) \text{ for some } 1 \le j \le m, \\ 0 & \text{otherwise.} \end{cases} \tag{5.33}$$

By Theorem 5.5, the fundamental period of f is

$$L = \operatorname{lcm}\{\delta(\zeta) \mid \zeta \in \mathcal{R}, g(\zeta) \ne 0\}, \tag{5.34}$$

whereas the fundamental period asserted by the theorem is

$$T = \operatorname{lcm}\{\omega_1, \dots, \omega_m\}. \tag{5.35}$$

So we have to prove that $L = T$.

Let ζ be a root of unity such that $g(\zeta) \ne 0$. Then $\zeta \in \mathcal{R}^*(\omega_j)$ for some $1 \le j \le m$. Since $\delta(\zeta) = \omega_j$ and $\omega_j \mid T$, we have $\delta(\zeta) \mid T$. Since $\delta(\zeta) \mid T$ for any root of unity ζ such that $g(\zeta) \ne 0$, we have $L \mid T$. On the other hand, let $1 \le j \le m$. Since $f_j \ne 0$, we have $g(\zeta) = g_j(\zeta) \ne 0$ for some $\zeta \in \mathcal{R}^*(\omega_j)$. Since $\delta(\zeta) = \omega_j$ and $\delta(\zeta) \mid L$, we have $\omega_j \mid L$. Since $\omega_j \mid L$ for any $1 \le j \le m$, we have $T \mid L$. Since both $L \mid T$ and $T \mid L$, we have $L = T$. □

Since any periodic function $\mathbb{Z} \to \mathbb{C}$ can be expressed in the form $f_1 + \cdots + f_m$ as in Theorem 1.17, the theorem gives another formula for the fundamental period of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$.

5.4 The functions I_a

Recall that the functions I_a have been introduced in Definition 1.11. In this section, we indicate some properties of such functions in Subsection 5.4.1 and give a formula for the fundamental period of an arbitrary linear combination of such functions with integer coefficients (Theorem 5.9) in Subsection 5.4.2.

In Subsection 5.4.3 we introduce certain sets we denote by $S(A, B)$ and express their indicator functions in terms of functions of the form I_a (Lemma 5.11).

5.4.1 Properties

Recall the concept of indicator functions as follows.

Definition 5.4. If $A \subseteq \Omega$, then the *indicator function* of A in Ω is the function $I_A : \Omega \to \{0, 1\}$ defined by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \Omega \setminus A. \end{cases} \tag{5.36}$$

As said in Definition 1.11, I_a is the indicator function of $a\mathbb{Z}$ in $\mathbb{Z}$. Clearly the function $I_a : \mathbb{Z} \to \{0, 1\}$ is periodic with fundamental period a. The function I_a in the form of (5.6) is given by [5, Theorem 8.1 on p. 158] as follows, where $\mathcal{R}(a)$ denotes the set of a-th roots of unity in $\mathbb{C}$.

Lemma 5.6 ([5, Theorem 8.1 on p. 158]). *For $a \geq 1$, we have that for all $x \in \mathbb{Z}$,*

$$I_a(x) = \frac{1}{a} \sum_{\zeta \in \mathcal{R}(a)} \zeta^x. \tag{5.37}$$

We also have the following multiplication property, which will be used to write the indicator function I^n (Definition 8.2) as a linear combination (5.39) of functions of the form I_a with integer coefficients in Subsection 6.4.2.

Lemma 5.7. *For any integers $a, b \geq 1$, we have $I_a I_b = I_{\mathrm{lcm}(a,b)}$.*

Proof. We need to prove that for all $x \in \mathbb{Z}$,

$$I_a(x) I_b(x) = I_{\mathrm{lcm}(a,b)}(x). \tag{5.38}$$

If $\mathrm{lcm}(a, b) \mid x$, then both $a \mid x$ and $b \mid x$, thus both sides of the above equation evaluates to 1. If $\mathrm{lcm}(a, b) \nmid x$, then either $a \nmid x$ or $b \nmid x$, thus in the above equation, one of the factors on the left-hand side is 0, and the right-hand side is also 0. This completes the proof. □

5.4.2 Linear combinations

We prove a formula for the fundamental period of an arbitrary linear combination (5.39) of functions of the form I_a with integer coefficients (Theorem 5.9).

Firstly, we have the following uniqueness property. It can be easily proved by induction.

Theorem 5.8. *Suppose that a function $f : \mathbb{Z} \to \mathbb{C}$ is expressed in the form*

$$f = \sum_{j=1}^{q} \lambda_j I_{c_j}, \tag{5.39}$$

where $q \geq 0$ and $1 \leq c_1 < \cdots < c_q$ and $\lambda_1, \ldots, \lambda_q \neq 0$ are integers. Then the expression is unique, in the sense that if $q' \geq 0$ and $1 \leq c'_1 < \cdots < c'_{q'}$ and $\lambda'_1, \ldots, \lambda'_{q'} \neq 0$ are integers such that

$$f = \sum_{j=1}^{q'} \lambda'_j I_{c'_j}, \tag{5.40}$$

then necessarily $q = q'$ and for all $1 \leq j \leq q$, we have $c_j = c'_j$ and $\lambda_j = \lambda'_j$.

The fundamental period of a function of the form (5.39) is as follows.

Theorem 5.9. *The fundamental period of a function of the form* (5.39) *is* $\operatorname{lcm}\{c_1, \ldots, c_q\}$.

Proof. Define the set

$$D = \{d \in \mathbb{N} : d \mid c_j \text{ for some } 1 \le j \le q\}. \tag{5.41}$$

That is, D is the union of the divisors of $c_1, \ldots, c_q$. In view of Lemma 5.6, the function f can be written as

$$f(x) = \sum_{d \in D} \sum_{\zeta \in \mathcal{R}^*(d)} \left(\sum_{1 \le j \le q, d \mid c_j} \frac{\lambda_j}{c_j} \right) \zeta^x. \tag{5.42}$$

Let us denote, for $d \in D$,

$$f_d(x) = \sum_{\zeta \in \mathcal{R}^*(d)} \left(\sum_{1 \le j \le q, d \mid c_j} \frac{\lambda_j}{c_j} \right) \zeta^x, \tag{5.43}$$

so that $f_d \in S_d$. Then $f = \sum_{d \in D} f_d$. In view of Theorem 1.17, the fundamental period of f is

$$\operatorname{lcm}\{d \in D : f_d \ne 0\} = \operatorname{lcm}\left\{ d \in D : \sum_{1 \le j \le q, d \mid c_j} \frac{\lambda_j}{c_j} \ne 0 \right\}. \tag{5.44}$$

We have to show that

$$\operatorname{lcm}\left\{ d \in D : \sum_{1 \le j \le q, d \mid c_j} \frac{\lambda_j}{c_j} \ne 0 \right\} = \operatorname{lcm}\{c_1, \ldots, c_q\}. \tag{5.45}$$

That the right-hand side above (denote it by R) is a multiple of the left-hand side (denote it by L) is plain. For each $d \in D$, we can write

$$\sum_{1 \le j \le q, d \mid c_j} \frac{\lambda_j}{c_j} = \frac{\lambda_1}{c_1}[d \mid c_1] + \cdots + \frac{\lambda_q}{c_q}[d \mid c_q], \tag{5.46}$$

where $[\cdot]$ is the Iverson bracket listed in Section 1.1. Now suppose that p^α is any prime power with $p^\alpha \mid R$ but $p^{\alpha+1} \nmid R$. Let j_0 be the largest integer with $1 \le j_0 \le q$ and $p^\alpha \mid c_{j_0}$. Then

$$\sum_{1 \le j \le q, c_{j_0} \mid c_j} \frac{\lambda_j}{c_j} = \frac{\lambda_1}{c_1}[c_{j_0} \mid c_1] + \cdots + \frac{\lambda_{j_0}}{c_{j_0}}[c_{j_0} \mid c_{j_0}] + \cdots + \frac{\lambda_q}{c_q}[c_{j_0} \mid c_q] = \frac{\lambda_{j_0}}{c_{j_0}} \ne 0. \tag{5.47}$$

This holds because, for $1 \le j < j_0$, as $c_j < c_{j_0}$, plainly $[c_{j_0} \mid c_j] = 0$; and for $j_0 < j \le q$, if $c_{j_0} \mid c_j$, then $p^\alpha \mid c_j$, which contradicts our choice of j_0, thus $[c_{j_0} \mid c_j] = 0$. Therefore as L is a multiple of c_{j_0}, it is also a multiple of p^α. Consequently, as L is a multiple of every prime power divisor of R, we have $R \mid L$. Since both $L \mid R$ and $R \mid L$, the equality (5.45) holds. □

About functions of the form (5.39), we also have the following which is obvious.

Theorem 5.10. *Let f be a function of the form* (5.39). *If $q > 0$, then the smallest positive integer k such that $f(k) \ne 0$ is c_1. If $q = 0$, then $f(k) = 0$ for all integers $k \ge 1$.*

5.4.3 The sets $S(A, B)$

We define certain sets $S(A, B)$ and express their indicator functions in terms of functions of the form I_a (Lemma 5.11).

Definition 5.5. For finite sets $A, B \subseteq \mathbb{N}$, put

$$S(A, B) = \{x \in \mathbb{Z} : (\text{for all } a \in A, a \mid x) \text{ and } (\text{for all } b \in B, b \nmid x)\}. \tag{5.48}$$

That is, $S(A, B)$ is the set of all integers divisible by every element of A, but indivisible by every element of B.

We have the following expression of the indicator function of $S(A, B)$ in $\mathbb{Z}$.

Lemma 5.11. *Let $A, B \subseteq \mathbb{N}$ be finite sets. Then for all $x \in \mathbb{Z}$,*

$$I_{S(A,B)}(x) = I_{\mathrm{lcm}(A)}(x) \prod_{b \in B} (1 - I_b(x)). \tag{5.49}$$

Hence $I_{S(A,B)}$ is periodic modulo $\mathrm{lcm}(A \cup B)$.

Proof. If $x \in S(A, B)$, then $a \mid x$ for all $a \in A$, and so $\mathrm{lcm}(A) \mid x$. Thus $I_{\mathrm{lcm}(A)}(x) = 1$. Moreover, for every $b \in B$, we have $b \nmid x$, and so $I_b(x) = 0$. Hence we see that the right-hand side of (5.49) is 1.

On the other hand, assume that x is an integer with $x \notin S(A, B)$. Then either x is not divisible by some particular $a \in A$, or is divisible by some particular $b \in B$. In the first case, x cannot be a multiple of $\mathrm{lcm}(A)$, thus $I_{\mathrm{lcm}(A)}(x) = 0$ and we see that the right-hand side of (5.49) is 0. In the second case, $I_b(x) = 1$ for some $b \in B$, hence one of the factors in the product on the right-hand side of (5.49) becomes 0, and we see again that the right-hand side of (5.49) evaluates to 0. This proves (5.49).

To prove the periodicity, we see that if we add to x the quantity $\mathrm{lcm}(A \cup B)$, the values of $I_{\mathrm{lcm}(A)}$ and all the I_b ($b \in B$) do not change, and hence $I_{S(A,B)}$ is periodic modulo $\mathrm{lcm}(A \cup B)$. □

Chapter 6

v-Palindromicity in Repeated Concatenations

For an n as in Theorem 1.2, we have the sequence of repeated concatenations of n

$$n(1), n(2), n(3), \ldots, n(k), \ldots. \tag{6.1}$$

In this chapter we examine the v-palindromicity of the terms above. The values $\omega_0(n)$ and $c(n)$ defined in Definition 1.9 represent a preliminary rough description of the pattern and we tabulate some of their values in Section 6.1.

The crux of this chapter is a procedure to express the indicator function I^n (Definition 1.10) as a linear combination (6.25) of functions of the form I_a with integer coefficients. We call it the *general procedure* and it is described in Section 6.6.

In Section 6.2, we define the D symbol which will be convenient when actually performing the general procedure, as illustrated in Section 6.7. In Section 6.3, we investigate the necessary and sufficient condition such that $n(k)$ is a v-palindrome. We also in the same section define the *type* of a v-palindrome (Definition 6.6). In Section 6.4, we construct the indicator function I^n based on the discussion in Section 6.3. In Section 6.5, we show how both $\omega_0(n)$ and $c(n)$ can be easily derived from an expression of I^n in the form (6.25). Finally in Section 6.8, we provide a table of indicator functions I^n expressed in the form (6.25).

6.1 Table of $\omega_0(n)$ and $c(n)$

Recall that a particular period $\omega_f(n)$ of n was given in Theorem 4.8. Theorem 1.3 says that we must have $\omega_0(n) \mid \omega_f(n)$. The following is a table of $\omega_f(n)$, $\omega_0(n)$, and $c(n)$, for $n \leq 56$ with $n < r(n)$, computed using Mathematica. We impose that $n < r(n)$ because the values for n and $r(n)$ are exactly the same, i.e.,

$$\omega_f(n) = \omega_f(r(n)), \quad \omega_0(n) = \omega_0(r(n)), \quad c(n) = c(r(n)). \tag{6.2}$$

Table 6.1: Values of $\omega_f(n)$, $\omega_0(n)$, and $c(n)$, for $n \leq 56$ with $n < r(n)$.

n	12	13	14	15	16	17	18	19	23
$\omega_f(n)$	21	6045	4305	136	1830	337960	9	15561	253
$\omega_0(n)$	1	6045	1	1	1	337960	1	15561	1
$c(n)$	∞	15	∞	∞	∞	280	1	819	∞
n	24	25	26	27	28	29	34	35	36
$\omega_f(n)$	21	39	6045	9	4305	102718	122808	14469	21
$\omega_0(n)$	1	1	6045	1	1	1	1	1	1
$c(n)$	∞	∞	15	∞	∞	∞	∞	∞	∞
n	37	38	39	45	46	47	48	49	56
$\omega_f(n)$	32412	581913	6045	9	253	119991	21	22701	273
$\omega_0(n)$	32412	1	6045	1	1	1	21	22701	273
$c(n)$	12	∞	15	∞	∞	∞	3	3243	3

From the above table it seems that the following might hold.

Conjecture 6.1. *Let n be as in Theorem 1.2. Then either $\omega_0(n) = 1$ or $\omega_0(n) = \omega_f(n)$.*

However, this turned out to be false and we provide a counterexample in Section 6.7.

6.2 The symbol $D(p, \delta, u, \mu)$

We define certain symbols $D(p, \delta, u, \mu)$ symbol which will be convenient when actually performing the general procedure, as illustrated in Section 6.7.

Recall that we defined the functions $\varphi_{p,\delta}$ in Definition 4.4 and denoted their ranges as $R_{p,\delta}$. The cases [i] through [vii] in the following lemma correspond to the conditions on the right in (4.29).

Lemma 6.2. *For an ordered quadruple (p, δ, u, μ), where p is a prime, δ a natural number, $u \in R_{p,\delta}$, and $\mu \geq 0$ an integer, exactly one of the following is the case. Moreover each case is possible.*

[i] $\varphi_{p,\delta}^{-1}(u) = \{0\}$ *and* $\mu = 0$, *or* $\varphi_{p,\delta}^{-1}(u) = \{1\}$ *and* $\mu = 1$, *or* $\varphi_{p,\delta}^{-1}(u) = \{0, 1\}$ *and* $\mu = 1$,

[ii] $\varphi_{p,\delta}^{-1}(u) = \{1\}$ *and* $\mu = 0$,

[iii] $\varphi_{p,\delta}^{-1}(u) = \{0, 1\}$ *and* $\mu = 0$,

[iv] $\varphi_{p,\delta}^{-1}(u) = \mathbb{Z}_{\geq 2}$ *and* $\mu = 1$,

[v] $\varphi_{p,\delta}^{-1}(u) = \mathbb{Z}_{\geq 2}$ *and* $\mu = 0$,

[vi] $\varphi_{p,\delta}^{-1}(u) = \mathbb{Z}_{\geq 2}$ *and* $\mu \geq 2$,

[vii] *otherwise.*

Moreover, each case is possible.

Proof. In view of (4.4), (4.5), and (4.6), clearly $\varphi_{p,\delta}^{-1}(u)$ is one of $\{0\}$, $\{1\}$, $\{0,1\}$, and $\mathbb{Z}_{\geq 2}$. Then one sees that the first six cases are mutually exclusive. That each case is possible is plain. □

Based on the above lemma we define the following notation.

Notation 6.1. For each quadruple (p, δ, u, μ) as in Lemma 6.2, denote by $D(p, \delta, u, \mu)$ the case number (in lower case Roman numerals in brackets as in the lemma). That is, $D(p, \delta, u, \mu) = $ [ii] if and only if $\varphi_{p,\delta}^{-1}(u) = \{1\}$ and $\mu = 0$, and $D(p, \delta, u, \mu) = $ [iii] if and only if $\varphi_{p,\delta}^{-1}(u) = \{0,1\}$ and $\mu = 0$, etc.

6.3 The conditions when $n(k)$ is a v-palindrome

Throughout this section we fix an n as in Theorem 1.2 with L decimal digits.

In Subsection 6.3.1, we define certain numbers associated to n. In Subsection 6.3.2, we introduce the *characteristic equation* and the symbols $T_{p,\mathbf{u}}$, $A_{\mathbf{u}}$, and $B_{\mathbf{u}}$. In Subsection 6.3.3, we state the necessary and sufficient condition on k such that $n(k)$ is a v-palindrome (Theorem 6.3) and consider the set of all integers $k \geq 1$ such that $n(k)$ is a v-palindrome. In Subsection 6.3.4, we define the *type* of a v-palindrome. Finally in Subsection 6.3.5, we summarize the content of this section.

Proofs are not really given because they all follow from the proof of Theorem 1.2 in Section 4.4.

6.3.1 Associated numbers

We define certain numbers associated to n. Suppose that n and $r(n)$ have the prime factorizations

$$n = \prod_p p^{a_p}, \quad r(n) = \prod_p p^{b_p}, \tag{6.3}$$

where the products are over the primes, the $a_p, b_p \geq 0$ are integers, and $a_p = b_p = 0$ for all but finitely many primes p. For integers $k \geq 1$, we have the $\rho_{k,L}$ of Definition 4.1, but because L is fixed in this section, we denote it simply by ρ_k.

For each crucial prime p of n (Definition 4.6), put

$$\delta_p = a_p - b_p \neq 0, \tag{6.4}$$
$$\mu_p = \min(a_p, b_p) \geq 0, \tag{6.5}$$
$$g_p = \operatorname{ord}_p(\rho_k), \tag{6.6}$$
$$\alpha_p = \mu_p + g_p. \tag{6.7}$$

The δ_p and μ_p depend only on n, thus can be considered as fixed. The g_p clearly depends on not only n but also on k. However, we omit k from the notation for simplicity, keeping in mind that g_p depends also on the variable k. Consequently, α_p also depends on k.

6.3.2 The characteristic equation

Recall that the set of crucial primes of n is denoted by $K(n)$ in Definition 4.6. But because n is fixed in this section, we denote it simply by K. We give the following definition.

Definition 6.2. The equation

$$\sum_{p \in K} \operatorname{sgn}(\delta_p) u_p = 0 \tag{6.8}$$

will be called the *characteristic equation* for n, where the u_p are variables.

We want to solve (6.8) for the u_p but with certain restrictions as follows.

Definition 6.3. A solution $(u_p)_{p \in K}$ to (6.8) with $u_p \in R_{p,|\delta_p|}$ for all $p \in K$ will be called a *characteristic solution* for n. The set of characteristic solutions will be denoted by U (or $\mathcal{U}(n)$ to specify n).

We have the numbers $h_{q,L}$ defined in Notation 4.2 for every prime power q relatively prime to 10. Since L is fixed in this section we denote it simply by h_q. We then define the following notation.

Notation 6.4. Let $\mathbf{u} = (u_p)_{p\in K}$ be a characteristic solution for n. We denote, for $p \in K \setminus \{2, 5\}$,

$$T_{p,\mathbf{u}} = (A_{p,\mathbf{u}}, B_{p,\mathbf{u}}) = \begin{cases} (\varnothing, \{h_p\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{i}], \\ (\{h_p\}, \{h_{p^2}\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{ii}], \\ (\varnothing, \{h_{p^2}\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{iii}], \\ (\{h_p\}, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{iv}], \\ (\{h_{p^2}\}, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{v}]. \end{cases} \tag{6.9}$$

For $p \in \{2, 5\}$, denote

$$T_{p,\mathbf{u}} = (A_{p,\mathbf{u}}, B_{p,\mathbf{u}}) = \begin{cases} (\varnothing, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{i}], \\ (\varnothing, \{1\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{ii}], \\ (\varnothing, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{iii}], \\ (\varnothing, \{1\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{iv}], \\ (\varnothing, \{1\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{v}]. \end{cases} \tag{6.10}$$

Also, we denote, for any $p \in K$,

$$T_{p,\mathbf{u}} = (A_{p,\mathbf{u}}, B_{p,\mathbf{u}}) = \begin{cases} (\varnothing, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{vi}], \\ (\varnothing, \{1\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{vii}]. \end{cases} \tag{6.11}$$

Remark 6.1. Therefore $T_{p,\mathbf{u}}$ is an ordered pair of sets of at most one positive integer. The "T" comes from "table". A table of p versus $\mathbf{u}$, with entries $T_{p,\mathbf{u}}$, will be convenient when actually performing the general procedure, as illustrated in Section 6.7 with Table 6.4.

We give further notation as follows.

Notation 6.5. For each characteristic solution $\mathbf{u}$ for n, put

$$A_{\mathbf{u}} = \bigcup_{p\in K} A_{p,\mathbf{u}}, \quad B_{\mathbf{u}} = \bigcup_{p\in K} B_{p,\mathbf{u}}, \tag{6.12}$$

and $S_{\mathbf{u}} = S(A_{\mathbf{u}}, B_{\mathbf{u}})$.

6.3.3 The necessary and sufficient condition

We state the necessary and sufficient condition on k such that $n(k)$ is a v-palindrome as follows.

Theorem 6.3. *For $k \geq 1$, the number $n(k)$ is a v-palindrome if and only if for some characteristic solution $\mathbf{u} = (u_p)_{p\in K}$ for n,*

$$\varphi_{p,|\delta_p|}(\alpha_p) = u_p, \quad \text{for all } p \in K. \tag{6.13}$$

Moreover, given a characteristic solution $\mathbf{u} = (u_p)_{p\in K}$ for n, (6.13) *holds if and only if*

$$k \in S_{\mathbf{u}}. \tag{6.14}$$

The first sentence in the above theorem is Lemma 4.7, and the second sentence follows from arguments in Subsection 4.4.2.

The condition (6.13) might seem to be independent of k, but the α_p actually depend on k. To write (6.13) out so that the dependence on k is more visible, we can recover (6.13) into

$$\varphi_{p,|\delta_p|}(\mu_p + \text{ord}_p(\rho_k)) = u_p, \quad \text{for all } p \in K. \tag{6.15}$$

Since the condition (6.13) (or equivalently (6.15)) cannot hold, for the same k, for two distinct characteristic solutions, the conditions (6.13) are mutually exclusive over $\mathbf{u}$. Consequently, the conditions (6.14) are also mutually exclusive over $\mathbf{u}$. Therefore the sets $S_{\mathbf{u}} \cap \mathbb{N}$ are pairwise disjoint, we write this as a corollary.

Corollary 6.4. *The sets $S_{\mathbf{u}} \cap \mathbb{N}$ are pairwise disjoint over $\mathbf{u} \in U$.*

In fact we have the following, which says that not only are the intersections $S_{\mathbf{u}} \cap \mathbb{N}$ of the sets $S_{\mathbf{u}}$ with $\mathbb{N}$ pairwise disjoint, but the sets $S_{\mathbf{u}}$ themselves are already pairwise disjoint as subsets of $\mathbb{Z}$.

Theorem 6.5. *The sets $S_{\mathbf{u}}$ are pairwise disjoint over $\mathbf{u} \in U$.*

Proof. Suppose on the contrary that for some distinct $\mathbf{u}, \mathbf{v} \in U$ that there exists an integer

$$x \in S_{\mathbf{u}} \cap S_{\mathbf{v}} = S(A_{\mathbf{u}}, B_{\mathbf{u}}) \cap S(A_{\mathbf{v}}, B_{\mathbf{v}}). \tag{6.16}$$

If we let

$$\omega = \operatorname{lcm}(A_{\mathbf{u}} \cup B_{\mathbf{u}} \cup A_{\mathbf{v}} \cup B_{\mathbf{v}}), \tag{6.17}$$

then we see that $x + \omega \in S_{\mathbf{u}} \cap S_{\mathbf{v}}$ too. Therefore adding ω as many times as necessary to x, we obtain a natural number in $S_{\mathbf{u}} \cap S_{\mathbf{v}}$, this contradicts Corollary 6.4. □

The set of integers $k \geq 1$ such that $n(k)$ is a v-palindrome can now be written as follows.

Corollary 6.6. *The set of integers $k \geq 1$ such that $n(k)$ is a v-palindrome is*

$$\bigsqcup_{\mathbf{u} \in U} (S_{\mathbf{u}} \cap \mathbb{N}) = \left(\bigsqcup_{\mathbf{u} \in U} S_{\mathbf{u}} \right) \cap \mathbb{N}. \tag{6.18}$$

Proof. This follows directly from Theorem 6.3 and Corollaries 6.4 and 6.5. □

6.3.4 The type of a v-palindrome

An integer $k \geq 1$ such that $n(k)$ is a v-palindrome can be categorized as to which $S_{\mathbf{u}}$ it belongs to. However, it could happen that $S_{\mathbf{u}} = \varnothing$, therefore we give the following definition.

Definition 6.6. Let $\mathbf{u}$ be a characteristic solution for n. If $S_{\mathbf{u}}$ is empty, then we call $\mathbf{u}$ *degenerate*; otherwise it is *nondegenerate*. The set of nondegenerate characteristic solutions will be denoted by U^* (or $U^*(n)$ to specify n). For a $\mathbf{u} \in U^*$, an $n(k)$ which is a v-palindrome will be said to be of *type* $\mathbf{u}$ (with respect to n) if $k \in S_{\mathbf{u}}$. We also denote

$$S = \bigsqcup_{\mathbf{u} \in U^*} S_{\mathbf{u}}. \tag{6.19}$$

Remark 6.2. We have included "with respect to n" in our definition of type above because the same v-palindrome m might be $m = n_1(k_1) = n_2(k_2)$ for $n_1 \neq n_2$, and therefore the type of m can be considered with respect to n_1 and also with respect to n_2. However, we shall prove in Chapter 7 that the types are the same. We call this the *invariance property* (Theorem 7.1). Assuming this property for now, we shall omit saying "with respect to n" hereafter in this chapter.

Notice that if $\mathbf{u}$ is nondegenerate, then there exists a v-palindrome $n(k)$ of type $\mathbf{u}$, because $S_{\mathbf{u}}$ contains positive integers. We have thus categorized the v-palindromes $n(k)$ into $|\mathcal{U}^*|$ types. For the characteristic equation (6.8) for n, it could happen that there are no characteristic solutions at all, or that there are characteristic solutions but unfortunately all are degenerate, or that there are nondegenerate solutions. In the former two cases $n(k)$ is not a v-palindrome for any $k \geq 1$, i.e. $c(n) = \infty$. In the third case only does there exist an integer $k \geq 1$ for which $n(k)$ is a v-palindrome.

6.3.5 Summary

Our original motivation was to examine the v-palindromicity of the repeated concatenations (6.1) of n. In order to do this, we first solve for the characteristic solutions for n. Then, each nondegenerate characteristic solution $\mathbf{u}$ gives rise to a nonempty infinite subset $S_{\mathbf{u}} \cap \mathbb{N}$ of integers $k \geq 1$ for which $n(k)$ is a v-palindrome. The sets $S_{\mathbf{u}} \cap \mathbb{N}$ are pairwise disjoint over the nondegenerate solutions $\mathbf{u}$ and their union gives the set of all integers $k \geq 1$ for which $n(k)$ is a v-palindrome. This section is of a more theoretical and abstract nature which forms the basis of the general procedure of Section 6.6.

6.4 Construction of the indicator function I^n

Again, throughout this section we fix an n as in Theorem 1.2. In Subsection 6.4.1, we construct the indicator function I^n (Definition 8.2) based on the discussion in the previous section. In Subsection 6.4.2, we express I^n as a linear combination (6.25) of functions of the form I_a (Definition 1.11) with integer coefficients.

6.4.1 Construction

Following directly from Lemma 5.11, we have the following.

Theorem 6.7. *Let* $\mathbf{u}$ *be a nondegenerate characteristic solution for n. Then for all* $x \in \mathbb{Z}$*,*

$$I_{S_{\mathbf{u}}}(x) = I_{\mathrm{lcm}(A_{\mathbf{u}})}(x) \prod_{b \in B_{\mathbf{u}}} (1 - I_b(x)). \tag{6.20}$$

Hence $I_{S_{\mathbf{u}}}$ *is periodic modulo* $\mathrm{lcm}(A_{\mathbf{u}} \cup B_{\mathbf{u}})$*. Moreover, for integers* $k \geq 1$*, the number* $n(k)$ *is v-palindromic of type* $\mathbf{u}$ *if and only if* $I_{S_{\mathbf{u}}}(k) = 1$*.*

Since we have the disjoint union (6.19), we have the following.

Theorem 6.8. *We have that for all* $x \in \mathbb{Z}$*,*

$$I_S(x) = \sum_{\mathbf{u} \in U^*} I_{S_{\mathbf{u}}}(x). \tag{6.21}$$

Hence I_S *is periodic modulo*

$$\mathrm{lcm}\left(\bigcup_{\mathbf{u} \in U^*} (A_{\mathbf{u}} \cup B_{\mathbf{u}}) \right). \tag{6.22}$$

Moreover, for integers $k \geq 1$*, the number* $n(k)$ *is v-palindromic if and only if* $I_S(k) = 1$*. Hence* I_S *is the indicator function* I^n *for* n*.*

Proof. If $x \in S$, then $x \in S_{\mathbf{u}}$ for exactly one $\mathbf{u} \in U^*$, so the right-hand side of (6.21) evaluates to 1. If x is an integer with $x \notin S$, then $I_{S_{\mathbf{u}}}(x) = 0$ for all $\mathbf{u} \in U^*$, so the right-hand side of (6.21) evaluates to 0. Let the quantity in (6.22) be denoted by ω. For each $\mathbf{u} \in U^*$, the function $I_{S_{\mathbf{u}}} = I_{S(A_{\mathbf{u}}, B_{\mathbf{u}})}$ is periodic modulo $\mathrm{lcm}(A_{\mathbf{u}} \cup B_{\mathbf{u}})$ by Theorem 6.7. Since ω is a multiple of $\mathrm{lcm}(A_{\mathbf{u}} \cup B_{\mathbf{u}})$ for every $\mathbf{u} \in U^*$, we see that I_S is periodic modulo ω.

□

6.4.2 The indicator function I^n as a linear combination

We express I^n as a linear combination (6.25) of functions of the form I_a (Definition 1.11) with integer coefficients. We first have the following.

Theorem 6.9. *Let* $\mathbf{u}$ *be a nondegenerate characteristic solution for n. Then*

$$I_{S_{\mathbf{u}}} = \sum_{B \subseteq B_{\mathbf{u}}} (-1)^{|B|} I_{\mathrm{lcm}(A_{\mathbf{u}} \cup B)}. \tag{6.23}$$

Proof. This follows by expanding the equation (6.20) in Theorem 6.7 and then simplifying using Lemma 5.7. □

Consequently we have the following.

Theorem 6.10. *We have the expression*

$$I^n = \sum_{\mathbf{u} \in U^*} \sum_{B \subseteq B_{\mathbf{u}}} (-1)^{|B|} I_{\mathrm{lcm}(A_{\mathbf{u}} \cup B)} \tag{6.24}$$

for the indicator function for n.

Proof. This follows from Theorems 6.8 and 6.9. □

Consequently we have the following.

Theorem 6.11. *There exist integers* $q \geq 0$ *and* $1 \leq c_1 < c_2 < \cdots < c_q$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_q \neq 0$ *such that*

$$I^n = \sum_{j=1}^{q} \lambda_j I_{c_j}. \tag{6.25}$$

Proof. We simply collect like terms in equation (6.24). □

According to Theorem 5.8, we have in particular that the indicator function I^n for n can be expressed in the form (6.25) uniquely. Examples of indicator functions I^n in this form are given in Table 6.5.

6.5 Derivation of $\omega_0(n)$ and $c(n)$

Again, throughout this section we fix an n as in Theorem 1.2.

In Subsection 6.5.1, we state that a period of n is simply a period of its indicator function I^n (Theorem 6.12). In Subsection 6.5.2, we express I^n in the form (5.6) of a periodic function $\mathbb{Z} \to \mathbb{C}$. In Subsection 6.5.3, we show how both $\omega_0(n)$ and $c(n)$ can be easily derived from an expression of I^n in the form (6.28).

6.5.1 The periods of n

Recall the concept of a *period* of n from Definition 1.9. On the other hand, we also have the concept of a *period* of a periodic function $\mathbb{Z} \to \mathbb{C}$ from Definition 5.1. The two concepts are related as follows.

Theorem 6.12. *Let n be as in Theorem 1.2. Then for integers $\omega \geq 1$, the following are equivalent.*

(i) *ω is a period of I^n*

(ii) *ω is a period of $I^n|_{\mathbb{N}}$*

(iii) *ω is a period of n.*

Hence $\omega_0(n)$ is the fundamental period of I^n.

Proof. This follows from Theorems 5.1 and 6.8. □

Because of the above theorem, to find $\omega_0(n)$, we just have to find the fundamental period of I^n.

6.5.2 The indicator function I^n in the form (5.6)

We express I^n in the form (5.6) of a periodic function $\mathbb{Z} \to \mathbb{C}$. Then in principle we can use Theorem 5.5 to compute the fundamental period of I^n, which will then be $\omega_0(n)$. However, this will be conceivably tedious, and so the following theorem is only of theoretical interest. Recall that $\mathcal{R}(a)$ denotes the set of a-th roots of unity in $\mathbb{C}$.

Theorem 6.13. *The indicator function I^n for n is*

$$I^n(x) = \sum_{\mathbf{u}\in U^*} \sum_{B\subseteq B_{\mathbf{u}}} \frac{(-1)^{|B|}}{\operatorname{lcm}(A_{\mathbf{u}} \cup B)} \sum_{\zeta\in\mathcal{R}(\operatorname{lcm}(A_{\mathbf{u}}\cup B))} \zeta^x \tag{6.26}$$

$$= \sum_{\zeta\in\mathcal{R}(\omega)} \left(\sum_{\mathbf{u}\in U^*, B\subset B_{\mathbf{u}}, \zeta\in\mathcal{R}(\operatorname{lcm}(A_{\mathbf{u}}\cup B))} \frac{(-1)^{|B|}}{\operatorname{lcm}(A_{\mathbf{u}} \cup B)} \right) \zeta^x, \tag{6.27}$$

where ω is the quantity (6.22).

Proof. The first equality follows by using Lemma 5.6 into the equation (6.24) in Theorem 6.10. The second equality is simply an iterated version of the first, summing over ζ first. □

6.5.3 Using the linear combination

By the following, both $\omega_0(n)$ and $c(n)$ can be easily derived.

Theorem 6.14. *Suppose that the indicator function for n is expressed as*

$$I^n = \sum_{j=1}^{q} \lambda_j I_{c_j}, \tag{6.28}$$

where $q \geq 0$ and $1 \leq c_1 < \ldots < c_q$ and $\lambda_1, \ldots, \lambda_q \neq 0$ are integers. Then

$$\omega_0(n) = \operatorname{lcm}\{c_1, \ldots, c_q\}, \quad c(n) = \inf\{c_1, \ldots, c_q\}. \tag{6.29}$$

Proof. This follows directly from Theorems 5.9 and 5.10. □

In this way, once we have expressed the indicator function I^n in the form (6.28), it will be easy to derive both $\omega_0(n)$ and $c(n)$. In the next section, we describe the general procedure to express I^n in the form (6.28).

6.6 The general procedure

Again, throughout this section we fix an n as in Theorem 1.2. We describe a procedure to express the indicator function I^n (Definition 1.10) as a linear combination (6.35) of functions of the form I_a with integer coefficients in the following. We call it the *general procedure*. This procedure is an "extraction" from the previous discussions and so works due to the reasoning there.

Step 1. Factorize both n and $r(n)$,

$$n = p_1^{a_1} \cdots p_m^{a_m}, \tag{6.30}$$

$$r(n) = p_1^{b_1} \cdots p_m^{b_m}, \tag{6.31}$$

where $p_1 < \cdots < p_m$ are primes, and $a_i, b_i \geq 0$ are integers, not both 0.

Step 2. Look for those primes p_i for which $a_i \neq b_i$, i.e. the crucial primes. Since we are only going to focus on these primes, we denote them again by $p_1 < \cdots < p_m$, and the exponents are a_i, b_i. Define the numbers $\delta_i = a_i - b_i$ and $\mu_i = \min(a_i, b_i)$ for $1 \leq i \leq m$.

Step 3. The characteristic equation for n is

$$\operatorname{sgn}(\delta_1)u_1 + \operatorname{sgn}(\delta_2)u_2 + \cdots + \operatorname{sgn}(\delta_m)u_m = 0. \tag{6.32}$$

We want to solve it for $u_i \in R_{p_i,|\delta_i|}$, i.e. to find the characteristic solutions. If there are no solutions, then conclude that $c(n) = \infty$ and $\omega_0(n) = 1$. Otherwise, let the solutions be $\mathbf{u}_1, \ldots, \mathbf{u}_t$, in any order.

Step 4. For each characteristic solution $\mathbf{u}$, we have the sets $A_\mathbf{u}$ and $B_\mathbf{u}$ of Definition 6.5. The solution $\mathbf{u}$ is nondegenerate if and only if $S(A_\mathbf{u}, B_\mathbf{u}) \neq \varnothing$. Now $S(A_\mathbf{u}, B_\mathbf{u}) \neq \varnothing$ if and only if $b \nmid \operatorname{lcm}(A_\mathbf{u})$ for all $b \in B_\mathbf{u}$. Use this to rule out those characteristic solutions $\mathbf{u}$ which are degenerate. If no characteristic solutions remain, conclude that $c(n) = \infty$ and $\omega_0(n) = 1$. Otherwise, let the nondegenerate characteristic solutions be $\mathbf{u}_1^*, \ldots, \mathbf{u}_s^*$, in any order.

Step 5. The indicator function I^n for n is then given by Theorem 6.8 as

$$I^n = \sum_{i=1}^{s} I_{S_{\mathbf{u}_i^*}}. \tag{6.33}$$

By Theorem 6.7 this can be written as

$$I^n = \sum_{i=1}^{s} I_{\operatorname{lcm}(A_{\mathbf{u}_i^*})} \prod_{b \in B_{\mathbf{u}_i^*}} (1 - I_b). \tag{6.34}$$

Multiplying everything out on the right-hand side above with the help of Lemma 5.7 and collecting like terms, I^n can be expressed in the form (6.25), i.e.

$$I^n = \sum_{j=1}^{q} \lambda_j I_{c_j}, \tag{6.35}$$

where $q \geq 1$ and $1 \leq c_1 < \cdots < c_q$ and $\lambda_1, \ldots, \lambda_q \neq 0$ are integers (how this is actually done is illustrated in the example of $n = 126$ in Section 6.7). Finally, conclude that

$$c(n) = c_1, \quad \omega_0(n) = \operatorname{lcm}\{c_1, \ldots, c_q\}. \tag{6.36}$$

Remark 6.3. We have described the general procedure in five steps as above. Whether $c(n) = \infty$ can be ascertained at certain points during the procedure. Namely, in Step 3, if there are no characteristic solutions at all, we immediately conclude that $c(n) = \infty$ and the procedure ends; and in Step 5, if all the characteristic solutions are degenerate, then we immediately conclude that $c(n) = \infty$ and the procedure ends. Otherwise, $c(n)$ and $\omega_0(n)$ and the indicator function I^n in the form (6.35) are found in Step 5.

6.7 Counterexample to Conjecture 6.1

In Subsection 6.7.1, we perform the general procedure of the previous section to $n = 126$, which is the smallest counterexample to Conjecture 6.1 found by PARI/GP. In Subsection 6.7.2 we consider another function $\omega_b(n)$ which will always be a period of n, but Conjecture 6.1 with $\omega_f(n)$ replaced by $\omega_b(n)$ is still false.

6.7.1 The general procedure performed to $n = 126$

We perform the general procedure to $n = 126$ as follows.

Step 1. We factorize

$$126 = 2 \cdot 3^2 \cdot 7, \tag{6.37}$$

$$621 = 3^3 \cdot 23. \tag{6.38}$$

Step 2. The crucial primes are $2, 3, 7, 23$. We arrange the numbers p_i, a_i, b_i, δ_i, and μ_i into a table.

Table 6.2: The p_i, a_i, b_i,δ_i, and μ_i for $n = 126$.

i	p_i	a_i	b_i	δ_i	μ_i
1	2	1	0	1	0
2	3	2	3	−1	2
3	7	1	0	1	0
4	23	0	1	−1	0

Step 3. The characteristic equation is

$$u_1 - u_2 + u_3 - u_4 = 0, \tag{6.39}$$

where we want to solve for $u_1 \in \{1, 2\}$, $u_2 \in \{1, 2, 3\}$, $u_3 \in \{1, 2, 7\}$, and $u_4 \in \{1, 2, 23\}$. The characteristic solutions are

$$\mathbf{u}_1 = (1, 1, 1, 1), \quad \mathbf{u}_2 = (1, 1, 2, 2), \quad \mathbf{u}_3 = (1, 2, 2, 1), \quad \mathbf{u}_4 = (2, 1, 1, 2) \tag{6.40}$$

$$\mathbf{u}_5 = (2, 2, 1, 1), \quad \mathbf{u}_6 = (2, 2, 2, 2), \quad \mathbf{u}_7 = (2, 3, 2, 1). \tag{6.41}$$

For each characteristic solution $\mathbf{u}_l$ $(1 \le l \le 7)$, also write $\mathbf{u}_l = (u_{l1}, u_{l2}, u_{l3}, u_{l4})$.

Step 4. We make two tables of the crucial primes p_i $(1 \le i \le 4)$ versus the characteristic solutions $\mathbf{u}_l$ $(1 \le l \le 7)$ as follows.

The first is where in the $(p_i, \mathbf{u}_l)$-entry we have the $D(p_i, |\delta_i|, u_{li}, \mu_i)$ of Definition 6.1. The second is where in the $(p_i, \mathbf{u}_l)$-entry we have the $T_{p_i,\mathbf{u}_l}$ defined in Notation 6.4 and also at the bottom, the sets $A_\mathbf{u}$, $B_\mathbf{u}$, and $S_\mathbf{u}$ defined in Definition 6.5. The first table helps us construct the second table because the definition of $T_{p_i,\mathbf{u}_l}$ depends on $D(p_i, |\delta_i|, u_{li}, \mu_i)$.

Table 6.3: Table of $D(p_i, |\delta_i|, u_{li}, \mu_i)$.

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{u}_5$	$\mathbf{u}_6$	$\mathbf{u}_7$
2	[v]	[v]	[v]	[iii]	[iii]	[iii]	[iii]
3	[vi]	[vi]	[vii]	[vi]	[vii]	[vii]	[vii]
7	[v]	[ii]	[ii]	[v]	[v]	[ii]	[ii]
23	[v]	[ii]	[v]	[ii]	[v]	[ii]	[v]

Table 6.4: Table of $T_{p_i,\mathbf{u}_l}$ and $A_\mathbf{u}$, $B_\mathbf{u}$, and $S_\mathbf{u}$.

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{u}_5$	$\mathbf{u}_6$	$\mathbf{u}_7$
2	$(\varnothing,\{1\})$	$(\varnothing,\{1\})$	$(\varnothing,\{1\})$	$(\varnothing,\varnothing)$	$(\varnothing,\varnothing)$	$(\varnothing,\varnothing)$	$(\varnothing,\varnothing)$
3	$(\varnothing,\varnothing)$	$(\varnothing,\varnothing)$	$(\varnothing,\{1\})$	$(\varnothing,\varnothing)$	$(\varnothing,\{1\})$	$(\varnothing,\{1\})$	$(\varnothing,\{1\})$
7	$(\{14\},\varnothing)$	$(\{2\},\{14\})$	$(\{2\},\{14\})$	$(\{14\},\varnothing)$	$(\{14\},\varnothing)$	$(\{2\},\{14\})$	$(\{2\},\{14\})$
23	$(\{506\},\varnothing)$	$(\{22\},\{506\})$	$(\{506\},\varnothing)$	$(\{22\},\{506\})$	$(\{506\},\varnothing)$	$(\{22\},\{506\})$	$(\{506\},\varnothing)$
$A_\mathbf{u}$	$\{14,506\}$	$\{2,22\}$	$\{2,506\}$	$\{14,22\}$	$\{14,506\}$	$\{2,22\}$	$\{2,506\}$
$B_\mathbf{u}$	$\{1\}$	$\{1,14,506\}$	$\{1,14\}$	$\{506\}$	$\{1\}$	$\{1,14,506\}$	$\{1,14\}$
$S_\mathbf{u}$	$\varnothing$	$\varnothing$	$\varnothing$	$S(\{14,22\},\{506\})$	$\varnothing$	$\varnothing$	$\varnothing$

We see immediately from the above table that the only nondegenerate solution is $\mathbf{u}_4$.

Step 5. The indicator function for 126 is then

$$I^{126} = I_{14}I_{22}(1 - I_{506}) = I_{154} - I_{3542}. \tag{6.42}$$

We conclude that $c(126) = 154$ and $\omega_0(126) = \mathrm{lcm}(154, 3542) = 3542$. Since $\omega_f(126) = 31878$ (calculation omitted), we see that $n = 126$ is a counterexample to Conjecture 6.1.

6.7.2 A refinement

So it was not difficult to find a counterexample to Conjecture 6.1, because there is a counterexample as small as 126. According to Theorem 6.8, the quantity

$$\omega_b(n) = \mathrm{lcm}\left(\bigcup_{\mathbf{u}\in U^*}(A_\mathbf{u} \cup B_\mathbf{u})\right) \tag{6.43}$$

is always a period of n. It is easily seen that we always have $\omega_b(n) \mid \omega_f(n)$. Therefore in a sense $\omega_b(n)$ is a refinement of $\omega_f(n)$ because the former provides a smaller period. Thus we can speculate whether $\omega_0(n)$ is always 1 or $\omega_b(n)$. But this too is false, as the smallest counterexample found by PARI/GP is $n = 5957$.

6.8 Table of indicator functions I^n

We provide a table of indicator functions I^n expressed in the form (6.25) computed by performing the general procedure of Section 6.6 using PARI/GP. We also include $c(n)$ and $\omega_0(n)$.

Table 6.5: The indicator function I^n and $c(n)$ and $\omega_0(n)$ for some numbers n.

n	I^n	$c(n)$	$\omega_0(n)$
13	$I_{15} - I_{195} - I_{465} + 2I_{6045}$	15	6045
17	$I_{280} - I_{4760} - I_{19880} + 2I_{337960}$	280	337960
18	I_1	1	1
19	$I_{819} - I_{15561}$	819	15561
26	$I_{15} - I_{195} - I_{465} + 2I_{6045}$	15	6045
37	$I_{12} - I_{444} - I_{876} + 2I_{32412}$	12	32412
39	$I_{15} - I_{195} - I_{465} + 2I_{6045}$	15	6045
48	$I_3 - I_{21}$	3	21
49	$I_{3243} - I_{22701}$	3243	22701
56	$I_3 - I_{21} - I_{39} + 2I_{273}$	3	273
79	$I_{624} - I_{49296} - I_{60528} + 2I_{4781712}$	624	4781712
103	$I_{10234} - I_{1054102}$	10234	1054102
107	$I_{37100} - I_{3969700} - I_{26007100} + 2I_{2782759700}$	37100	2782759700
109	$I_{1686672} - I_{183847248}$	1686672	183847248
113	$I_{17360} - I_{1961680} - I_{5398960} + 2I_{610082480}$	17360	610082480
117	I_{2054}	2054	2054
119	$I_{123760} - I_{112745360}$	123760	112745360
122	$I_{80} - I_{1040} - I_{1360} - I_{4880} + I_{17680} + 2I_{63440} + 2I_{82960} - 3I_{1078480}$	80	1078480

We see that the indicator functions for 13, 26, and 39 are identical. There is another curiosity in the above table: for all these indicator functions, the largest subscript is a multiple of all smaller subscripts. This is not always true, and the smallest counterexample found by PARI/GP is $n = 21726$, with

$$I^{21726} = I_{816} - I_{5712} - I_{8976} - I_{10608} + I_{16401} - I_{32802} + I_{62832} + I_{74256} + I_{116688} - I_{816816} - I_{1098867} + I_{2197734}.$$

Here $816 \nmid 2197734$.

Chapter 7

Proof of the Invariance Property

This chapter is devoted to the proof of the invariance property, roughly described in Subsection 1.6.1. We state it formally as follows.

Theorem 7.1. *Let m be a v-palindrome and write $m = n_0(k_0)$, where $n_0, k_0 \geq 1$ are integers and n_0 is minimal. Then all of the types*

$$\textbf{Type}(m, n_0(d)), \quad \textit{for} \quad d \mid k_0, \tag{7.1}$$

are the same element of $\mathcal{U}(n_0)$.

In Section 7.1, we prove a formula about the numbers $h_{p^\alpha,L}$ defined in Notation 4.2. In Section 7.2, we describe a slightly altered form of the general procedure of Section 6.6 suitable for the proof of Theorem 7.1. We call it the *altered general procedure*. In Section 7.3, we perform the altered general procedure to all the repeated concatenations $n(k)$ of an integer n as in Theorem 1.2 simultaneously. Finally in Section 7.4, we prove Theorem 7.1.

7.1 A formula for $h_{p^\alpha,Lk}$

We prove the following formula about the numbers $\rho_{k,L}$ defined in Definition 4.1 and the numbers $h_{p^\alpha,L}$ defined in Notation 4.2.

Lemma 7.2. *Let p^α be a prime power with $p \notin \{2, 5\}$ and $k, L \geq 1$ integers. We have*

$$h_{p^\alpha,Lk} = \frac{h_{p^{\alpha+\mathrm{ord}_p(\rho_{k,L})},L}}{(k, h_{p^{\alpha+\mathrm{ord}_p(\rho_{k,L})},L})}. \tag{7.2}$$

Proof. The number $h_{p^\alpha,Lk}$ is the smallest positive integer such that

$$(10^{Lk})^{h_{p^\alpha,Lk}} \equiv 1 \pmod{p^{\alpha+\mathrm{ord}_p(10^{Lk}-1)}}. \tag{7.3}$$

We have

$$10^{Lk} - 1 = (10^L - 1)(10^{L(k-1)} + 10^{L(k-2)} + \cdots + 1) = (10^L - 1)\rho_{k,L}, \tag{7.4}$$

thus

$$\mathrm{ord}_p(10^{Lk} - 1) = \mathrm{ord}_p(10^L - 1) + \mathrm{ord}_p(\rho_{k,L}). \tag{7.5}$$

Hence we can rewrite (7.3) as

$$(10^{Lk})^{h_{p^\alpha,Lk}} \equiv 1 \pmod{p^{(\alpha+\mathrm{ord}_p(\rho_{k,L}))+\mathrm{ord}_p(10^L-1)}}. \tag{7.6}$$

Now the number $h_{p^{\alpha+\mathrm{ord}_p(\rho_{k,L})},L}$ is the smallest positive integer such that

$$(10^L)^{h_{p^{\alpha+\mathrm{ord}_p(\rho_{k,L})},L}} \equiv 1 \pmod{p^{(\alpha+\mathrm{ord}_p(\rho_{k,L}))+\mathrm{ord}_p(10^L-1)}}. \tag{7.7}$$

Hence by the property of cyclic groups, $h_{p^\alpha,Lk}$ is the smallest positive integer for which $h_{p^{\alpha+\mathrm{ord}_p(\rho_{k,L})},L} \mid kh_{p^\alpha,Lk}$, and this is clearly that given by (7.2). □

7.2 Altered general procedure

We describe a slightly altered form of the general procedure of Section 6.6 suitable for the proof of Theorem 7.1. We call it the *altered general procedure*. In essense, the altered version is the original version with Step 5 left out and with construction of tables akin to Tables 6.3 and 6.4. Just like the original version, the altered version is also to be performed to an integer n as in Theorem 1.2. We now fix such an n and describe the altered general procedure as follows.

Step 1 Factorize both n and $r(n)$,

$$n = p_1^{a_1} \cdots p_m^{a_m}, \tag{7.8}$$

$$r(n) = p_1^{b_1} \cdots p_m^{b_m}, \tag{7.9}$$

where $p < \cdots < p_m$ are primes, and $a_i, b_i \geq 0$ are integers, not both 0.

Step 2 Let K be the set of crucial primes of n. Since $n \neq r(n)$, the set K is a nonempty finite set. We are only going to focus on these primes, and so we denote them again by $p_1 < \cdots < p_m$, and the exponents are a_i, b_i. Define the numbers $\delta_i = a_i - b_i$ and $\mu_i = \min(a_i, b_i)$ for $1 \leq i \leq m$.

Step 3 Let $\mathcal{U}$ be the set of characteristic solutions for n. If $\mathcal{U} = \emptyset$, then conclude that $c(n) = \infty$ and $\omega_0(n) = 1$. Otherwise, suppose that $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_t\}$, where we write $\mathbf{u}_l = (u_{li})_{i=1}^m$ for $1 \leq l \leq t$.

Step 4 We make two tables of the crucial primes p_i ($1 \leq i \leq m$) versus the characteristic solutions $\mathbf{u}_l$ ($1 \leq l \leq t$) as follows.

The first is where in the $(p_i, \mathbf{u}_l)$-entry we have the $D(p_i, |\delta_i|, u_{li}, \mu_i)$ defined in Definition 6.1. It is called the *first table* for n. We illustrate the generic first table as follows.

Table 7.1: The first table.

	$\mathbf{u}_1$	$\cdots$	$\mathbf{u}_l$	$\cdots$	$\mathbf{u}_t$
p_1	$D(p_1, \|\delta_1\|, u_{11}, \mu_1)$	$\cdots$	$D(p_1, \|\delta_1\|, u_{l1}, \mu_1)$	$\cdots$	$D(p_1, \|\delta_1\|, u_{t1}, \mu_1)$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
p_i	$D(p_i, \|\delta_i\|, u_{1i}, \mu_i)$	$\cdots$	$D(p_i, \|\delta_i\|, u_{li}, \mu_i)$	$\cdots$	$D(p_i, \|\delta_i\|, u_{ti}, \mu_i)$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
p_m	$D(p_m, \|\delta_m\|, u_{1m}, \mu_m)$	$\cdots$	$D(p_m, \|\delta_m\|, u_{lm}, \mu_m)$	$\cdots$	$D(p_m, \|\delta_m\|, u_{tm}, \mu_m)$

The second is where in the $(p_i, \mathbf{u}_l)$-entry we have the $T_{p_i,\mathbf{u}_l} = (A_{p_i,\mathbf{u}_l}, B_{p_i,\mathbf{u}_l})$ defined in Notation 6.4. This table of entries $T_{p_i,\mathbf{u}_l}$ is called the *second table* for n and we illustrate the generic one as follows.

Table 7.2: The second table.

	$\mathbf{u}_1$	$\cdots$	$\mathbf{u}_l$	$\cdots$	$\mathbf{u}_t$
p_1	$(A_{p_1,\mathbf{u}_1}, B_{p_1,\mathbf{u}_1})$	$\cdots$	$(A_{p_1,\mathbf{u}_l}, B_{p_1,\mathbf{u}_l})$	$\cdots$	$(A_{p_1,\mathbf{u}_t}, B_{p_1,\mathbf{u}_t})$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
p_i	$(A_{p_i,\mathbf{u}_1}, B_{p_i,\mathbf{u}_1})$	$\cdots$	$(A_{p_i,\mathbf{u}_l}, B_{p_i,\mathbf{u}_l})$	$\cdots$	$(A_{p_i,\mathbf{u}_t}, B_{p_i,\mathbf{u}_t})$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
p_m	$(A_{p_m,\mathbf{u}_1}, B_{p_m,\mathbf{u}_1})$	$\cdots$	$(A_{p_m,\mathbf{u}_l}, B_{p_m,\mathbf{u}_l})$	$\cdots$	$(A_{p_m,\mathbf{u}_t}, B_{p_m,\mathbf{u}_t})$

The first table helps us to construct the second table because the determination of $T_{p_i,\mathbf{u}_l}$ depends on $D(p_i, |\delta_i|, u_{li}, \mu_i)$.

We have the sets $A_{\mathbf{u}_l}, B_{\mathbf{u}_l}, S_{\mathbf{u}_l}$ of Definition 6.5. Namely, for each $\mathbf{u}_l$, put

$$A_{\mathbf{u}_l} = \bigcup_{i=1}^{m} A_{p_i,\mathbf{u}_l}, \quad B_{\mathbf{u}_l} = \bigcup_{i=1}^{m} B_{p_i,\mathbf{u}_l,k}, \tag{7.10}$$

$$S_{\mathbf{u}_l} = S(A_{\mathbf{u}_l}, B_{\mathbf{u}_l}). \tag{7.11}$$

Visually speaking, $A_{\mathbf{u}_l}$ and $B_{\mathbf{u}_l}$ are respectively the union of the left and right coordinates of the entries in the $\mathbf{u}_l$-column. We then put

$$S = \bigsqcup_{l=1}^{t} S_{\mathbf{u}_l}. \tag{7.12}$$

The set of nondegenerate solutions for n is denoted by $\mathcal{U}^*$.

7.3 Altered general procedure performed to all the $n(k)$

Again, throughout this section we fix an n as in Theorem 1.2 with L decimal digits. We perform the altered general procedure of Section 7.2 to all the repeated concatenations $n(k)$ simultaneously as follows.

Step 1 Factorize both n and $r(n)$,

$$n = p_1^{a_1} \cdots p_m^{a_m}, \tag{7.13}$$
$$r(n) = p_1^{b_1} \cdots p_m^{b_m}, \tag{7.14}$$

where $p < \cdots < p_m$ are primes, and $a_i, b_i \geq 0$ are integers, not both 0. For integers $k \geq 1$, we abbreviate $\rho_{k,L}$ as ρ_k. Since $n(k) = n\rho_k$ and $r(n(k)) = r(n)(k) = r(n)\rho_k$,

$$n(k) = p_1^{a_1} \cdots p_m^{a_m} \rho_k, \tag{7.15}$$
$$r(n(k)) = p_1^{b_1} \cdots p_m^{b_m} \rho_k. \tag{7.16}$$

Hence we see that for $1 \leq i \leq m$,

$$\mathrm{ord}_{p_i}(n(k)) = a_i + \mathrm{ord}_{p_i}(\rho_k), \tag{7.17}$$
$$\mathrm{ord}_{p_i}(r(n(k))) = b_i + \mathrm{ord}_{p_i}(\rho_k), \tag{7.18}$$

and that for any other prime p,

$$\mathrm{ord}_p(n(k)) = \mathrm{ord}_p(r(n(k))) = \mathrm{ord}_p(\rho_k). \tag{7.19}$$

Step 2 In view of the equalities (7.17), (7.18), and (7.19) in Step 1, the crucial primes of $n(k)$ are the same as that of n. That is, $K(n(k)) = K(n)$, and we denote it simply by K. We are only going to focus on these primes, and so we denote them again by $p_1 < \cdots < p_m$, and we redefine

$$a_i = \mathrm{ord}_{p_i}(n), \quad b_i = \mathrm{ord}_{p_i}(r(n)), \tag{7.20}$$

for $1 \leq i \leq m$. Define the numbers $\delta_i = a_i - b_i$ and $\mu_i = \min(a_i, b_i)$ for $1 \leq i \leq m$. Since we are applying the procedure to all the $n(k)$, we also define, for $1 \leq i \leq m$ and all $k \geq 1$,

$$a_{ik} = \mathrm{ord}_{p_i}(n(k)) = a_i + \mathrm{ord}_{p_i}(\rho_k), \tag{7.21}$$
$$b_{ik} = \mathrm{ord}_{p_i}(r(n(k))) = b_i + \mathrm{ord}_{p_i}(\rho_k), \tag{7.22}$$
$$\delta_{ik} = a_{ik} - b_{ik} = a_i - b_i = \delta_i, \tag{7.23}$$
$$\mu_{ik} = \min(a_{ik}, b_{ik}) = \min(a_i, b_i) + \mathrm{ord}_{p_i}(\rho_k) = \mu_i + \mathrm{ord}_{p_i}(\rho_k). \tag{7.24}$$

Hence we see clearly how the $a_{ik}, b_{ik}, \delta_{ik}, \mu_{ik}$ changes as k increases. More precisely, we see that the δ_{ik} do not change and that the changes in the a_{ik}, b_{ik}, μ_{ik} depend only on $\mathrm{ord}_{p_i}(\rho_k)$. We denote $x_{ik} = \mathrm{ord}_{p_i}(\rho_k)$ for $1 \leq i \leq m$ and all $k \geq 1$.

Step 3 In view of (7.23), the characteristic equation for all the $n(k)$ are the same and it is

$$\mathrm{sgn}(\delta_1)u_1 + \mathrm{sgn}(\delta_2)u_2 + \cdots + \mathrm{sgn}(\delta_m)u_m = 0. \tag{7.25}$$

Moreover, the characteristic solutions for all the $n(k)$ are the same. That is,

$$\mathcal{U}(n) = \mathcal{U}(n(2)) = \mathcal{U}(n(3)) = \cdots = \mathcal{U}. \tag{7.26}$$

If $\mathcal{U} = \emptyset$, conclude that for all integers $k \geq 1$, the number $n(k)$ is not a v-palindrome. Otherwise, suppose that $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_t\}$, where we write $\mathbf{u}_l = (u_{li})_{i=1}^m$ for $1 \leq l \leq t$.

Step 4 We illustrate the first and second tables for $n(k)$ as follows.

Table 7.3: The first table for $n(k)$.

	$\mathbf{u}_1$	$\cdots$	$\mathbf{u}_l$	$\cdots$	$\mathbf{u}_t$
p_1	$D(p_1, \|\delta_1\|, u_{11}, \mu_1 + x_{1k})$	$\cdots$	$D(p_1, \|\delta_1\|, u_{l1}, \mu_1 + x_{1k})$	$\cdots$	$D(p_1, \|\delta_1\|, u_{t1}, \mu_1 + x_{1k})$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
p_i	$D(p_i, \|\delta_i\|, u_{1i}, \mu_i + x_{ik})$	$\cdots$	$D(p_i, \|\delta_i\|, u_{li}, \mu_i + x_{ik})$	$\cdots$	$D(p_i, \|\delta_i\|, u_{ti}, \mu_i + x_{ik})$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
p_m	$D(p_m, \|\delta_m\|, u_{1m}, \mu_m + x_{mk})$	$\cdots$	$D(p_m, \|\delta_m\|, u_{lm}, \mu_m + x_{mk})$	$\cdots$	$D(p_m, \|\delta_m\|, u_{tm}, \mu_m + x_{mk})$

For the second table, a third subscript of k is added to the $A_{p_i,\mathbf{u}_l}$ and $B_{p_i,\mathbf{u}_l}$ to indicate the dependence on k.

Table 7.4: The second table for $n(k)$.

	$\mathbf{u}_1$	$\cdots$	$\mathbf{u}_l$	$\cdots$	$\mathbf{u}_t$
p_1	$(A_{p_1,\mathbf{u}_1,k}, B_{p_1,\mathbf{u}_1,k})$	$\cdots$	$(A_{p_1,\mathbf{u}_l,k}, B_{p_1,\mathbf{u}_l,k})$	$\cdots$	$(A_{p_1,\mathbf{u}_t,k}, B_{p_1,\mathbf{u}_t,k})$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
p_i	$(A_{p_i,\mathbf{u}_1,k}, B_{p_i,\mathbf{u}_1,k})$	$\cdots$	$(A_{p_i,\mathbf{u}_l,k}, B_{p_i,\mathbf{u}_l,k})$	$\cdots$	$(A_{p_i,\mathbf{u}_t,k}, B_{p_i,\mathbf{u}_t,k})$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
p_m	$(A_{p_m,\mathbf{u}_1,k}, B_{p_m,\mathbf{u}_1,k})$	$\cdots$	$(A_{p_m,\mathbf{u}_l,k}, B_{p_m,\mathbf{u}_l,k})$	$\cdots$	$(A_{p_m,\mathbf{u}_t,k}, B_{p_m,\mathbf{u}_t,k})$

For each $\mathbf{u}_l$ and all integers $k \geq 1$, put

$$A_{\mathbf{u}_l,k} = \bigcup_{i=1}^{m} A_{p_i,\mathbf{u}_l,k}, \quad B_{\mathbf{u}_l,k} = \bigcup_{i=1}^{m} B_{p_i,\mathbf{u}_l,k}, \tag{7.27}$$

$$S_{\mathbf{u}_l,k} = S(A_{\mathbf{u}_l,k}, B_{\mathbf{u}_l,k}). \tag{7.28}$$

We then put

$$S_k = \bigsqcup_{l=1}^{t} S_{\mathbf{u}_l,k}. \tag{7.29}$$

The set of nondegenerate solutions for $n(k)$ is denoted by $\mathcal{U}^*(n(k))$. Although the crucial primes and characteristic solutions are the same for all the $n(k)$, the nondegenerate solutions are not in general.

7.4 Proof of the invariance property

Again, throughout this section we fix an n as in Theorem 1.2 with L decimal digits.

Assume that we have performed the alternated general procedure to n and to all the $n(k)$ as described in Sections 7.2 and 7.3. Moreover, all those notation are inherited. To prove the invariance property, i.e., Theorem 7.1, it suffices to prove that if $n(kj)$ is a v-palindrome, where $k, j \geq 1$ are integers, then the type of $n(kj)$ with respect to n is the same as that with respect to $n(k)$. That is,

$$\mathbf{Type}(n(kj), n) = \mathbf{Type}(n(kj), n(k)), \tag{7.30}$$

which we proceed to prove. Assume that $\mathbf{Type}(n(kj), n) = \mathbf{u}_l$. This is equivalent to saying that $kj \in S_{\mathbf{u}_l}$. We need to show that $\mathbf{Type}(n(kj), n(k)) = \mathbf{u}_l$ too, i.e., $j \in S_{\mathbf{u}_l,k}$. We illustrate the $\mathbf{u}_l$-column in the second tables for n and $n(k)$ as follows.

Table 7.5: The $\mathbf{u}_l$-column in the second table for n.

	$\mathbf{u}_l$
p_1	$(A_{p_1,\mathbf{u}_l}, B_{p_1,\mathbf{u}_l})$
$\vdots$	$\vdots$
p_i	$(A_{p_i,\mathbf{u}_l}, B_{p_i,\mathbf{u}_l})$
$\vdots$	$\vdots$
p_m	$(A_{p_m,\mathbf{u}_l}, B_{p_m,\mathbf{u}_l})$

Visually speaking, that $kj \in S_{\mathbf{u}_l}$ means exactly that kj is divisible by every number which appears in the left coordinate of an entry in the above column, and indivisible by every number which appears in the right coordinate of an entry in the above column.

Table 7.6: The $\mathbf{u}_l$-column in the second table for $n(k)$.

	$\mathbf{u}_l$
p_1	$(A_{p_1,\mathbf{u}_l,k}, B_{p_1,\mathbf{u}_l,k})$
$\vdots$	$\vdots$
p_i	$(A_{p_i,\mathbf{u}_l,k}, B_{p_i,\mathbf{u}_l,k})$
$\vdots$	$\vdots$
p_m	$(A_{p_m,\mathbf{u}_l,k}, B_{p_m,\mathbf{u}_l,k})$

Similarly, that $j \in S_{\mathbf{u}_l,k}$ means exactly that j is divisible by every number which appears in the left coordinate of an entry in the above column, and indivisible by every number which appears in the right coordinate of an entry in the above column. Hence it suffices to prove that $j \in S(A_{p_i,\mathbf{u}_l,k}, B_{p_i,\mathbf{u}_l,k})$ for all $1 \le i \le m$.

We focus on an arbitrary $(A_{p_i,\mathbf{u}_l,k}, B_{p_i,\mathbf{u}_l,k})$ and denote $p = p_i$, $\mathbf{u} = \mathbf{u}_l$, $\delta = \delta_i$, $u = u_{li}$, $\mu_i = \mu$, and $x_k = x_{ik}$, and $(A, B) = (A_{p_i,\mathbf{u}_l}, A_{p_i,\mathbf{u}_l})$ and $(A_k, B_k) = (A_{p_i,\mathbf{u}_l,k}, B_{p_i,\mathbf{u}_l,k})$. Hence we need to prove that $j \in S(A_k, B_k)$. We divide our consideration into various cases, according to the prime p and $D = D(p, |\delta|, u, \mu)$, in reference to how the second table for n is determined via (6.9), (6.10), and (6.11). Just as the determination of (A, B) depends on D, the determination of (A_k, B_k) depends on $D_k = D(p, |\delta|, u, \mu + x_k)$. By Lemma 7.2, if $p \notin \{2, 5\}$, then

$$h_{p,Lk} = \frac{h_{p^{1+\mathrm{ord}_p(\rho_{k,L})},L}}{(k, h_{p^{1+\mathrm{ord}_p(\rho_{k,L})},L})} = \frac{h_{p^{1+x_k},L}}{(k, h_{p^{1+x_k},L})}, \tag{7.31}$$

$$h_{p^2,Lk} = \frac{h_{p^{2+\mathrm{ord}_p(\rho_{k,L})},L}}{(k, h_{p^{2+\mathrm{ord}_p(\rho_{k,L})},L})} = \frac{h_{p^{2+x_k},L}}{(k, h_{p^{2+x_k},L})}. \tag{7.32}$$

We will be using the above equalities in the following case analysis.

($p \notin \{2, 5\}$ and $D =$ [i]) We have $(A, B) = (\emptyset, \{h_{p,L}\})$. Thus $h_{p,L} \nmid kj$, and so $h_{p,L} \nmid k$. Consequently, by Lemma 4.2, $p \nmid \rho_{k,L}$, and so $x_k = 0$. Therefore $D_k =$ [i] too, and so $(A_k, B_k) = (\emptyset, \{h_{p,Lk}\})$. Since $x_k = 0$, we have $h_{p,Lk} = h_{p,L}/(k, h_{p,L})$. Assume on the contrary that $h_{p,Lk} \mid j$, then

$$h_{p,L} \mid (k, h_{p,L})j \mid kj, \tag{7.33}$$

a contradiction to $h_{p,L} \nmid kj$. Whence $h_{p,Lk} \nmid j$, i.e., $j \in S(A_k, B_k)$.

($p \notin \{2, 5\}$ and $D =$ [ii]) We have $(A, B) = (\{h_{p,L}\}, \{h_{p^2,L}\})$. Thus $h_{p,L} \mid kj$ and $h_{p^2,L} \nmid kj$, and so $h_{p^2,L} \nmid k$. Consequently, by Lemma 4.2, $p^2 \nmid \rho_{k,L}$, and so $x_k \le 1$.

In case $x_k = 0$, we have $D_k =$ [ii] too, and so $(A_k, B_k) = (\{h_{p,Lk}\}, \{h_{p^2,Lk}\})$. We have $h_{p,Lk} = h_{p,L}/(k, h_{p,L})$ and $h_{p^2,Lk} = h_{p^2,L}/(k, h_{p^2,L})$. Since $h_{p,L} \mid kj$, we have $h_{p,Lk} \mid j$. Assume on the contrary that $h_{p^2,Lk} \mid j$, then

$$h_{p^2,L} \mid (k, h_{p^2,L})j \mid kj, \tag{7.34}$$

a contradiction to $h_{p^2,L} \nmid kj$. Whence $h_{p^2,Lk} \nmid j$, and so $j \in S(A_k, B_k)$.

In case $x_k = 1$, we have $D_k = $ [i], and so $(A_k, B_k) = (\varnothing, \{h_{p,Lk}\})$. We have $h_{p,Lk} = h_{p^2,L}/(k, h_{p^2,L})$. Assume on the contrary that $h_{p,Lk} \mid j$, then

$$h_{p^2,L} \mid (k, h_{p^2,L})j \mid kj, \tag{7.35}$$

a contradiction to $h_{p^2,L} \nmid kj$. Whence $h_{p,Lk} \nmid j$, i.e., $j \in S(A_k, B_k)$.

($p \notin \{2,5\}$ and $D = $ [iii]) We have $(A, B) = (\varnothing, \{h_{p^2,L}\})$. Thus $h_{p^2,L} \nmid kj$, and so $h_{p^2,L} \nmid k$. Consequently, by Lemma 4.2, $p^2 \nmid \rho_{k,L}$, and so $x_k \leq 1$.

In case $x_k = 0$, we have $D_k = $ [iii] too, and so $(A_k, B_k) = (\varnothing, \{h_{p^2,Lk}\})$. We have $h_{p^2,Lk} = h_{p^2,L}/(k, h_{p^2,L})$. Assume on the contrary that $h_{p^2,Lk} \mid j$, then

$$h_{p^2,L} \mid (k, h_{p^2,L})j \mid kj, \tag{7.36}$$

a contradiction to $h_{p^2,L} \nmid kj$. Whence $h_{p^2,Lk} \nmid j$, i.e., $j \in S(A_k, B_k)$.

In case $x_k = 1$, we have $D_k = $ [i], and so $(A_k, B_k) = (\varnothing, \{h_{p,Lk}\})$. We have $h_{p,Lk} = h_{p^2,L}/(k, h_{p^2,L})$. Assume on the contrary that $h_{p,Lk} \mid j$, then

$$h_{p^2,L} \mid (k, h_{p^2,L})j \mid kj, \tag{7.37}$$

a contradiction to $h_{p^2,L} \nmid kj$. Whence $h_{p,Lk} \nmid j$, i.e., $j \in S(A_k, B_k)$.

($p \notin \{2,5\}$ and $D = $ [iv]) We have $(A, B) = (\{h_{p,L}\}, \varnothing)$. Thus $h_{p,L} \mid kj$.

In case $x_k = 0$, we have $D_k = $ [iv] too, and so $(A_k, B_k) = (\{h_{p,Lk}\}, \varnothing)$. We have $h_{p,Lk} = h_{p,L}/(k, h_{p,L})$. Since $h_{p,L} \mid kj$, it is evident that $h_{p,Lk} \mid j$, i.e., $j \in S(A_k, B_k)$.

In case $x_k \geq 1$, we have $D_k = $ [vi], and so $(A_k, B_k) = (\varnothing, \varnothing)$. Whence $j \in S(A_k, B_k)$ holds trivially.

($p \notin \{2,5\}$ and $D = $ [v]) We have $(A, B) = (\{h_{p^2,L}\}, \varnothing)$. Thus $h_{p^2,L} \mid kj$.

In case $x_k = 0$, we have $D_k = $ [v] too, and so $(A_k, B_k) = (\{h_{p^2,Lk}\}, \varnothing)$. We have $h_{p^2,Lk} = h_{p^2,L}/(k, h_{p^2,L})$. Since $h_{p^2,L} \mid kj$, it is evident that $h_{p^2,Lk} \mid j$, i.e., $j \in S(A_k, B_k)$.

In case $x_k = 1$, we have $D_k = $ [iv], and so $(A_k, B_k) = (\{h_{p,Lk}\}, \varnothing)$. We have $h_{p,Lk} = h_{p^2,L}/(k, h_{p^2,L})$. Since $h_{p^2,L} \mid kj$, it is evident that $h_{p,Lk} \mid j$, i.e., $j \in S(A_k, B_k)$.

In case $x_k \geq 2$, we have $D_k = $ [vi], and so $(A_k, B_k) = (\varnothing, \varnothing)$. Whence $j \in S(A_k, B_k)$ holds trivially.

($p \in \{2,5\}$ and $D \in \{$[i], [iii]$\}$) We have $(A, B) = (\varnothing, \varnothing)$. Since $p \in \{2,5\}$, we have $x_k = \mathrm{ord}_p(\rho_{k,L}) = 0$, and so $D_k = D$ still. Thus $(A_k, B_k) = (\varnothing, \varnothing)$. Whence $j \in S(A_k, B_k)$ holds trivially.

($D = $ [vi]) We have $(A, B) = (\varnothing, \varnothing)$. We see that irregardless of x_k, we always still have $D_k = $ [vi], and so $(A_k, B_k) = (\varnothing, \varnothing)$. Whence $j \in S(A_k, B_k)$ holds trivially.

($p \in \{2,5\}$ and $D \in \{$[ii], [iv], [v]$\}$, or $D = $ [vii]) We have $(A, B) = (\varnothing, \{1\})$. Thus $1 \nmid kj$. But this is impossible, so actually this case cannot happen.

This completes the proof of the invariance property.

Chapter 8

v-Palindromes in Other Bases

Recall that (v, b)-palindromes are briefly considered in Section 1.8. In fact most of the previously described considerations with v-palindromes also hold for more general (v, b)-palindromes with almost the same proofs.

In Section 8.1, we state the periodic phenomenon for (v, b)-palindromes (Theorem 8.1). In Section 8.2, we briefly consider the corresponding concept of the indicator function I^n for (v, b)-palindromes. In Section 8.3, we prove that if a (v, b)-palindrome exists, then infinitely many exist (Theorem 1.9). Finally in Section 8.4, we prove the existence of (v, b)-palindromes in infinitely many bases b.

8.1 The periodic phenomenon in base b

We state the periodic phenomenon for (v, b)-palindromes as follows, recalling that $\mathbb{V}_b$ denotes the set of (v, b)-palindromes.

Theorem 8.1. *Let $b \geq 2$ and $n \geq 1$ be integers such that $b \nmid n$ and $n \neq r_b(n)$. Then there exists an integer $\omega \geq 1$ such that for all integers $k \geq 1$,*

$$n(k)_b \in \mathbb{V}_b \quad \textit{if and only if} \quad n(k + \omega)_b \in \mathbb{V}_b. \tag{8.1}$$

Then just like Definition 1.9, we make the following definitions.

Definition 8.1. Let b and n be as in Theorem 8.1. An integer $\omega \geq 1$ satisfying the condition of Theorem 8.1 is called a *b-period* of n. The smallest b-period of n is called the *b-fundamental period* of n and denoted by $\omega_0(n)_b$. If there exists a $k \geq 1$ such that $n(k)_b$ is a (v, b)-palindrome, the least such integer is denoted by $c(n)_b$; otherwise we write $c(n)_b = \infty$. The integer (or ∞) $c(n)_b$ is called the *b-order* of n.

Just like in decimal, there is the problem of deriving $\omega_0(n)_b$ and $c(n)_b$.

8.2 The indicator functions I_b^n

We briefly consider the corresponding concept of the indicator function I^n for (v, b)-palindromes.

Definition 8.2. Let b and n be as in Theorem 8.1. The *b-indicator function* for n is the periodic function $I_b^n \colon \mathbb{Z} \to \{0, 1\}$ such that

$$I_b^n(k) = \begin{cases} 1 & \text{if } n(k)_b \in \mathbb{V}_b, \\ 0 & \text{if } n(k)_b \notin \mathbb{V}_b, \end{cases} \quad \text{for} \quad k \geq 1. \tag{8.2}$$

Then just like Theorem 6.11, we have the following.

Theorem 8.2. *Let b and n be as in Theorem 8.1. Then there exist integers $q \geq 0$ and $1 \leq c_1 < c_2 < \cdots < c_q$ and $\lambda_1, \lambda_2, \ldots, \lambda_q \neq 0$ such that*

$$I_b^n = \sum_{j=1}^{q} \lambda_j I_{c_j}. \tag{8.3}$$

The general procedure of Section 6.6 can be easily adapted to express the b-indicator function I_b^n as a linear combination (8.3) of functions of the form I_a with integer coefficients. Also, just like Theorem 6.14, we have the following.

Theorem 8.3. *Let b and n be as in Theorem 8.1. Suppose that*

$$I_b^n = \sum_{j=1}^{q} \lambda_j I_{c_j}, \tag{8.4}$$

where $q \geq 0$ and $1 \leq c_1 < \ldots < c_q$ and $\lambda_1, \ldots, \lambda_q \neq 0$ are integers. Then

$$\omega_0(n)_b = \operatorname{lcm}\{c_1, \ldots, c_q\}, \quad c(n)_b = \inf\{c_1, \ldots, c_q\}. \tag{8.5}$$

8.3 One implies infinitely many

We prove that if a (v, b)-palindrome exists, then infinitely many exist as follows.

Theorem 1.9 ([50, Theorem 5]). *Let $b \geq 2$ be an integer. If there exists a (v, b)-palindrome, then there exist infinitely many (v, b)-palindromes.*

Proof. Suppose that n is a (v, b)-palindrome. We have the b-indicator function I_b^n for n. That n is a (v, b)-palindrome means that $I_b^n(1) = 1$. Since I_b^n is periodic, say with ω as a period, we see that

$$1 = I_b^n(1) = I_b^n(1 + \omega) = I_b^n(1 + 2\omega) = \cdots. \tag{8.6}$$

Consequently,

$$n(1)_b, n(1 + \omega)_b, n(1 + 2\omega)_b, \ldots \tag{8.7}$$

are all (v, b)-palindromes. □

8.4 Existence of (v, b)-palindromes in infinitely many bases

In this section, we prove the existence of (v, b)-palindromes (and therefore infinitely many (v, b)-palindromes in view of Theorem 1.9) in infinitely many bases b as follows, divided into three subsections. The proof is based on the humble fact that $v(5) = v(6)$.

8.4.1 First part of the proof

Imagine that we have a base $b \geq 2$ for which we would like to show that a (v, b)-palindrome exists. The first simple try would be to look in the two-digit numbers. That is, numbers $(ac)_b = ab + c$, where $1 \leq a < c < b$ are integers. By definition, $(ac)_b$ is a (v, b)-palindrome if and only if $v((ac)_b) = v((ca)_b)$, or equivalently,

$$v(ab + c) = v(cb + a). \tag{8.8}$$

Since $v(n)$ is an additive function, for every integer $t \geq 1$ with $(t, 30) = 1$, we have $v(5t) = v(6t)$. Therefore (8.8) would hold if for some integer $t \geq 1$ with $(t, 30) = 1$,

$$\begin{cases} ab + c = 5t, \\ cb + a = 6t. \end{cases} \tag{8.9}$$

Therefore we have shown the following.

Lemma 8.4. *Let $b \geq 2$ be an integer. If there exists an ordered triple (a, c, t) of positve integers such that $a < c < b$ and $(t, 30) = 1$ and (8.9) holds, then the two-digit number $(ac)_b$ is a (v, b)-palindrome. Hence in particular there exists a (v, b)-palindrome.*

8.4.2 Permissible triples

Based on Lemma 8.4, we make the following definition.

Definition 8.3. We call a triple (a, c, t) in the premise of Lemma 8.4 a *b-permissible triple*.

Our strategy is to try to find b-permissible triples. The system (8.9) can be written in matrix from as

$$\begin{pmatrix} b & 1 \\ 1 & b \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = t\begin{pmatrix} 5 \\ 6 \end{pmatrix}. \tag{8.10}$$

Solving we have

$$\begin{pmatrix} a \\ c \end{pmatrix} = t\begin{pmatrix} b & 1 \\ 1 & b \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{t}{b^2-1}\begin{pmatrix} b & -1 \\ -1 & b \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} \tag{8.11}$$

$$= \frac{t}{b^2-1}\begin{pmatrix} 5b-6 \\ -5+6b \end{pmatrix} = \begin{pmatrix} \frac{t(5b-6)}{b^2-1} \\ \frac{t(-5+6b)}{b^2-1} \end{pmatrix}. \tag{8.12}$$

We write them separately as

$$a = \frac{t(5b-6)}{b^2-1}, \quad c = \frac{t(-5+6b)}{b^2-1}, \tag{8.13}$$

from which we also see that $0 < a < c$. Hence we have proved the following.

Lemma 8.5. *Let $b \geq 2$ be an integer. For every integer $t \geq 1$, there exist unique $a, c \in \mathbb{Q}$ such that* (8.9) *holds, and they are given by* (8.13)*. Moreover, $0 < a < c$.*

Hence the only possible b-permissible triples are

$$\left(\frac{t(5b-6)}{b^2-1}, \frac{t(-5+6b)}{b^2-1}, t\right), \tag{8.14}$$

for an integer $t \geq 1$ with $(t, 30) = 1$. The only missing conditions to fulfill are

$$\frac{t(5b-6)}{b^2-1}, \frac{t(-5+6b)}{b^2-1} \in \mathbb{Z}, \tag{8.15}$$

$$\frac{t(-5+6b)}{b^2-1} < b. \tag{8.16}$$

We write

$$\frac{t(5b-6)}{b^2-1} = \frac{t(5b-6)/(5b-6, b^2-1)}{(b^2-1)/(5b-6, b^2-1)}, \tag{8.17}$$

$$\frac{t(-5+6b)}{b^2-1} = \frac{t(-5+6b)/(-5+6b, b^2-1)}{(b^2-1)/(-5+6b, b^2-1)}. \tag{8.18}$$

Hence we see that (8.15) holds if and only if t is a multiple of

$$f(b) = \left[\frac{b^2-1}{(5b-6, b^2-1)}, \frac{b^2-1}{(-5+6b, b^2-1)}\right]; \tag{8.19}$$

here we also defined the function $f(b)$ for integers $b \geq 2$. Hence we have shown the following.

Lemma 8.6. *Let $b \geq 2$ be an integer. Then the b-permissible triples are precisely the triples*

$$\left(\frac{t(5b-6)}{b^2-1}, \frac{t(-5+6b)}{b^2-1}, t\right), \tag{8.20}$$

where

$$t \in S(b) = \left\{t \in \mathbb{N} \colon (t, 30) = 1,\ f(b) \mid t,\ t < \frac{b(b^2-1)}{-5+6b}\right\}; \tag{8.21}$$

where we also defined the set-valued function $S(b)$ for integers $b \geq 2$.

8.4.3 Existence of permissible triples

Lemma 8.6 does not promise that a b-permissible triple exists, i.e., $S(b) \neq \emptyset$. However, we have the following sufficient condition.

Lemma 8.7. *Let $b \geq 2$ be an integer. If*

$$(f(b), 30) = 1, \quad f(b) < \frac{b(b^2-1)}{-5+6b}, \tag{8.22}$$

then $f(b) \in S(b)$, and consequently there is a b-permissible triple.

Since $f(b) \mid b^2 - 1$, if $(b^2-1, 30) = 1$ then $(f(b), 30) = 1$. Hence the above lemma can be weakened to the following.

Lemma 8.8. *Let $b \geq 2$ be an integer. If*

$$(b^2-1, 30) = 1, \quad f(b) < \frac{b(b^2-1)}{-5+6b}, \tag{8.23}$$

then $f(b) \in S(b)$, and consequently there is a b-permissible triple.

We now consider the condition $(b^2-1, 30) = 1$. It is easily shown that this is equivalent to that both $b \equiv 0 \pmod 6$ and $b \equiv 0, 2, 3 \pmod 5$. In particular, $b \equiv 0 \pmod{30}$ is a sufficient condition. Suppose that $k \geq 1$ is an integer, then

$$f(30k) = \left[\frac{(30k)^2-1}{(5(30k)-6, (30k)^2-1)}, \frac{(30k)^2-1}{(-5+6(30k), (30k)^2-1)}\right] \tag{8.24}$$

$$= \left[\frac{(30k)^2-1}{(6k-2, 11)}, \frac{(30k)^2-1}{(5k+2, 11)}\right], \tag{8.25}$$

where for the second equality we used a property of the greatest common divisor function to simplify. Because of the right inequality in (8.23), we want $f(30k)$ to be small. Thus it might be good if we have $(6k-2, 11) = (5k+2, 11) = 11$, which is easily shown to be equivalent to that $k \equiv 4 \pmod{11}$. Whence assume that $k \equiv 4 \pmod{11}$, then

$$f(30k) = \frac{(30k)^2-1}{11}. \tag{8.26}$$

On the other hand, the right-hand side of the right inequality in (8.23) becomes

$$\frac{(30k)((30k)^2-1)}{-5+6(30k)}. \tag{8.27}$$

That $f(30k)$ is strictly less than the above quantity is equivalent to

$$-5 + 6(30k) < 11(30k), \tag{8.28}$$

which clearly always holds. The following theorem easily follows from the previous discussion.

Theorem 8.9. *Let $k \equiv 4 \pmod{11}$ be a positive integer. Then*

$$\left(\frac{-6+150k}{11}, \frac{-5+180k}{11}, \frac{-1+900k^2}{11}\right) \tag{8.29}$$

is a $30k$-permissible triple. In particular, the two-digit number

$$\left(\frac{-6+150k}{11}, \frac{-5+180k}{11}\right)_{30k} \tag{8.30}$$

is a $(v, 30k)$-palindrome.

Hence we have proved the existence of (v, b)-palindromes in infinitely many bases, summarized as follows.

Theorem 1.10. ([50, Corollary 12]). *If $b \equiv 120 \pmod{330}$ is a positive integer, then there exists a (v, b)-palindrome.*

In particular, there is a positive density of bases $b \geq 2$ for which a (v, b)-palindrome exists.

Chapter 9

Repeated Concatenations in Residue Classes

The motivation for this chapter is described in Section 1.9. We restate the problem stated at the end of that section as follows.

Problem 1.16. Let $n \geq 1$, $b \geq 2$, a, and $m \geq 1$ be integers. How to determine whether in $a + m\mathbb{Z}$ there is a number of the form $n(k)_b$? Or even better, how to find the set of integers $k \geq 1$ such that $n(k)_b \in a + m\mathbb{Z}$?

Algorithm 9.2 is given as an answer to the latter question in the above problem. In Section 9.1, we rephrase the latter question in Problem 1.16 into congruence notation and list some notation and conventions to be used in this chapter. In Section 9.2, we first give Algorithm 9.1 for the case when m is a prime power. In Section 9.3, we give Algorithm 9.2. Finally in Section 9.4, we give a concrete example using Algorithm 9.2.

9.1 Preliminaries

In Subsection 9.1.1, we rephrase the latter question in Problem 1.16 into congruence notation. In Subsection 9.1.2, we list some notation and conventions to be used in this chapter.

9.1.1 In congruence notation

Let $n \geq 1$, $b \geq 2$, a, and $m \geq 1$ be integers. Then for integers $k \geq 1$, the condition $n(k)_b \in a + m\mathbb{Z}$ is equivalent to

$$n(k)_b \equiv a \pmod{m}. \tag{9.1}$$

Algorithm 9.2 takes (n, b, a, m) as input and outputs the set

$$\{k \in \mathbb{N} : n(k)_b \equiv a \pmod{m}\}. \tag{9.2}$$

This set will be denoted by K.

9.1.2 Notation and conventions

We shall use the following notation.

- In a congruence relation modulo m, a notation x^{-1} denotes an inverse of x modulo m.
- If g is a primitive root modulo m and $\gcd(x, m) = 1$, then $\operatorname{ind}_{g,m} x$ denotes the index of x to the base g modulo m.

We also make the following conventions for our algorithms.

- Once an output is reached, the algorithm terminates.
- An output written as a condition on k means that we output the set of all integers $k \geq 1$ satisfying that condition.

9.2 When $m = p^\alpha$ is a prime power

In this section, in the notation of Subsection 9.1.1, we consider the case when $m = p^\alpha$ is a prime power.

In Subsection 9.2.1, we start to try to solve (9.1). This preliminary consideration leads to two cases, considered separately in Subsections 9.2.2 and 9.2.3. Finally in Subsection 9.2.4, we summarize the previous discussion into Algorithm 9.1.

9.2.1 Preliminary consideration

By (1.9), the congruence (9.1) is equivalent to

$$n \cdot \frac{1 - b^{Lk}}{1 - b^L} \equiv a \pmod{p^\alpha}. \tag{9.3}$$

Put $d = \gcd(n, p^\alpha)$. If $d \nmid a$, then there is no solution for k, i.e., $K = \varnothing$. Thus assume that $d \mid a$. Then (9.3) is equivalent to

$$\frac{n}{d} \cdot \frac{1 - b^{Lk}}{1 - b^L} \equiv \frac{a}{d} \pmod{\frac{p^\alpha}{d}},$$

which is equivalent to

$$\frac{1 - b^{Lk}}{1 - b^L} \equiv \frac{a}{d} \cdot \left(\frac{n}{d}\right)^{-1} \pmod{\frac{p^\alpha}{d}}. \tag{9.4}$$

Put $p^\alpha/d = p^{\alpha_1}$ and let $a_1 \equiv a/d \cdot (n/d)^{-1} \pmod{p^{\alpha_1}}$. Then (9.4) is equivalent to

$$\frac{1 - b^{Lk}}{1 - b^L} \equiv a_1 \pmod{p^{\alpha_1}}. \tag{9.5}$$

Suppose that $p^\beta \parallel 1 - b^L$, then (9.5) is equivalent to

$$\frac{1 - b^{Lk}}{p^\beta} \equiv a_1 \cdot \frac{1 - b^L}{p^\beta} \pmod{p^{\alpha_1}},$$

or equivalently,

$$1 - b^{Lk} \equiv a_1(1 - b^L) \pmod{p^{\alpha_1+\beta}},$$

or equivalently,

$$b^{Lk} \equiv 1 - a_1(1 - b^L) \pmod{p^{\alpha_1+\beta}}. \tag{9.6}$$

Put $p^{\alpha_1+\beta} = p^{\alpha_2}$ and let $a_2 \equiv 1 - a_1(1 - b^L) \pmod{p^{\alpha_2}}$. Then (9.6) is equivalent to

$$b^{Lk} \equiv a_2 \pmod{p^{\alpha_2}}. \tag{9.7}$$

If $\alpha_2 = 0$, then $K = \mathbb{N}$. Thus assume that $\alpha_2 \geq 1$. There will be two cases, according to as whether there is not or is a primitive root modulo p^{α_2}, and we consider them in Subsections 9.2.2 and 9.2.3, respectively. Recall that there is no primitive root modulo p^{α_2} if and only if $p = 2$ and $\alpha_2 \geq 3$.

9.2.2 In case $p = 2$ and $\alpha_2 \geq 3$

In case $p = 2$ and $\alpha_2 \geq 3$, the congruence (9.7) is equivalent to

$$b^{Lk} \equiv a_2 \pmod{2^{\alpha_2}}. \tag{9.8}$$

If $b \not\equiv a_2 \pmod 2$, then $K = \varnothing$. Thus assume that $b \equiv a_2 \pmod 2$. We consider the cases $b \equiv a_2 \equiv 0 \pmod 2$ and $b \equiv a_2 \equiv 1 \pmod 2$ in the next two paragraphs, respectively.

In case $b \equiv a_2 \equiv 0 \pmod 2$, write $b = 2^\delta b_1$, where $2^\delta \parallel b$. If $a_2 \equiv 0 \pmod{2^{\alpha_2}}$, then $K = \{k \in \mathbb{N} : k \geq \alpha_2/(\delta L)\}$. Thus assume that $a_2 \not\equiv 0 \pmod{2^{\alpha_2}}$. Write $a_2 = 2^\varepsilon a_3$, where $2^\varepsilon \parallel a_2$. Then (9.8) is equivalent to

$$2^{\delta L k} b_1^{Lk} \equiv 2^\varepsilon a_3 \pmod{2^{\alpha_2}}. \tag{9.9}$$

Since $a_2 \not\equiv 0 \pmod{2^{\alpha_2}}$, we have $\varepsilon < \alpha_2$, therefore (9.9) implies that

$$2^{\delta L k} b_1^{Lk} \equiv 0 \pmod{2^\varepsilon}.$$

Hence, we need to have $\delta Lk \geq \varepsilon$. Now assume that $\delta Lk \geq \varepsilon$. Then (9.9) holds if and only if

$$2^{\delta Lk-\varepsilon} b_1^{Lk} \equiv a_3 \pmod{2^{\alpha_2-\varepsilon}}. \tag{9.10}$$

If $\delta Lk > \varepsilon$, then the above congruence cannot hold because the two sides are of opposite parity. Hence, we need to have $k = \varepsilon/(\delta L)$. If $\varepsilon/(\delta L)$ is not an integer, then $K = \emptyset$. Thus assume that $\varepsilon/(\delta L)$ is an integer. Letting $k = \varepsilon/(\delta L)$, the congruence (9.10) becomes

$$b_1^{\varepsilon/\delta} \equiv a_3 \pmod{2^{\alpha_2-\varepsilon}}. \tag{9.11}$$

If (9.11) holds, then $K = \{\varepsilon/(\delta L)\}$, otherwise $K = \emptyset$.

In case $b \equiv a_2 \equiv 1 \pmod 2$, by the structure of $(\mathbb{Z}/2^{\alpha_2}\mathbb{Z})^\times$, there exist unique integers $0 \leq \mu_1, \mu_2 < 2$ and $0 \leq \nu_1, \nu_2 < 2^{\alpha_2-2}$ such that $b \equiv (-1)^{\mu_1}5^{\nu_1} \pmod{2^{\alpha_2}}$ and $a_2 \equiv (-1)^{\mu_2}5^{\nu_2} \pmod{2^{\alpha_2}}$. Hence, (9.8) is equivalent to

$$(-1)^{\mu_1 Lk}5^{\nu_1 Lk} \equiv (-1)^{\mu_2}5^{\nu_2} \pmod{2^{\alpha_2}},$$

which holds if and only if both of the congruences

$$\mu_1 Lk \equiv \mu_2 \pmod 2, \tag{9.12}$$

$$\nu_1 Lk \equiv \nu_2 \pmod{2^{\alpha_2-2}} \tag{9.13}$$

hold. We solve this system of congruences for k. If $\mu_1 L$ is even and μ_2 odd, then (9.12) cannot hold, thus $K = \emptyset$. Thus assume that $K \neq \emptyset$. We divide into two cases as follows.

(i) **If $\mu_1 L$ is odd**: (9.12) is equivalent to $k \equiv \mu_2 \pmod 2$. We solve (9.13) in the usual way. Put $f = \gcd(\nu_1 L, 2^{\alpha_2-2})$. If $f \nmid \nu_2$, then (9.13) cannot hold, thus $K = \emptyset$. Thus assume that $f \mid \nu_2$. Then (9.13) is equivalent to

$$k \equiv \frac{\nu_2}{f}\left(\frac{\nu_1 L}{f}\right)^{-1} \pmod{\frac{2^{\alpha_2-2}}{f}}. \tag{9.14}$$

If $2^{\alpha_2-2}/f = 1$, then (9.14) always hold, and so

$$K = \{k \in \mathbb{N} : k \equiv \mu_2 \pmod 2\}. \tag{9.15}$$

Thus assume that $2^{\alpha_2-2}/f > 1$. Then (9.14) implies that

$$k \equiv \frac{\nu_2}{f}\left(\frac{\nu_1 L}{f}\right)^{-1} \equiv \frac{\nu_2}{f} \pmod 2.$$

If

$$\mu_2 \equiv \frac{\nu_2}{f} \pmod 2, \tag{9.16}$$

then

$$K = \left\{k \in \mathbb{N} : k \equiv \frac{\nu_2}{f}\left(\frac{\nu_1 L}{f}\right)^{-1} \pmod{\frac{2^{\alpha_2-2}}{f}}\right\}.$$

If (9.16) does not hold, then $K = \emptyset$.

(ii) **If $\mu_1 L$ and μ_2 are both even**: (9.12) always hold, so we are left with solving just (9.13), which we do as in the second to sixth sentences in case (i).

9.2.3 In case p is odd or $\alpha_2 < 3$

We now consider the case when p is odd or $\alpha_2 < 3$. The congruence (9.7) implies that $b^{Lk} \equiv a_2 \pmod p$. Consequently, if $[p \mid b] \neq [p \mid a_2]$, then $K = \emptyset$. Thus assume that $[p \mid b] = [p \mid a_2]$. In case $[p \mid b] = [p \mid a_2] = 1$, we solve (9.7) in the same way as in the case when $p = 2$, $\alpha_2 \geq 3$, and $b \equiv a_2 \equiv 0 \pmod 2$, described in the second paragraph of Subsection 9.2.2. Thus assume that $[p \mid b] = [p \mid a_2] = 0$.

Let g be a primitive root modulo p^{α_2}. Then (9.7) is equivalent to

$$Lk \operatorname{ind}_{g,p^{\alpha_2}} b \equiv \operatorname{ind}_{g,p^{\alpha_2}} a_2 \pmod{p^{\alpha_2-1}(p-1)}. \tag{9.17}$$

So we just have to solve (9.17), which we do in the usual way. Put

$$f = \gcd(L \operatorname{ind}_{g,p^{\alpha_2}} b, p^{\alpha_2-1}(p-1)).$$

If $f \nmid \operatorname{ind}_{g,p^{\alpha_2}} a_2$, then (9.17) cannot hold, thus $K = \varnothing$. Thus assume that $f \mid \operatorname{ind}_{g,p^{\alpha_2}} a_2$. Then (9.17) is equivalent to

$$k \equiv \frac{\operatorname{ind}_{g,p^{\alpha_2}} a_2}{f}\left(\frac{L \operatorname{ind}_{g,p^{\alpha_2}} b}{f}\right)^{-1} \pmod{\frac{p^{\alpha_2-1}(p-1)}{f}},$$

and so

$$K = \left\{k \in \mathbb{N} \colon k \equiv \frac{\operatorname{ind}_{g,p^{\alpha_2}} a_2}{f}\left(\frac{L \operatorname{ind}_{g,p^{\alpha_2}} b}{f}\right)^{-1} \pmod{\frac{p^{\alpha_2-1}(p-1)}{f}}\right\}.$$

9.2.4 Algorithm when $m = p^\alpha$ is a prime power

Up to this point in Section 9.2, we have shown how to determine all the integers $k \geq 1$ satisfying (9.1), when m is a prime power. We now summarize the process into the following algorithm.

Algorithm 9.1. *Given integers $n \geq 1$, $b \geq 2$, $a \in \mathbb{Z}$, and a prime power $m = p^\alpha$, this algorithm computes the set K of integers $k \geq 1$ satisfying* (9.1).

(I) *Put $d = \gcd(n, p^\alpha)$. If $d \nmid a$, output $K = \varnothing$.*

(II) *Let the number of base b digits of n be denoted by L. Put $p^\alpha/d = p^{\alpha_1}$ and suppose that $p^\beta \parallel 1 - b^L$. Put $\alpha_2 = \alpha_1 + \beta$. If $\alpha_2 = 0$, output $K = \mathbb{N}$. Let $a_1, a_2 \in \mathbb{Z}$ be such that*

$$a_1 \equiv \frac{a}{d} \cdot \left(\frac{n}{d}\right)^{-1} \pmod{p^{\alpha_1}},$$
$$a_2 \equiv 1 - a_1(1 - b^L) \pmod{p^{\alpha_2}}.$$

If p is odd or $\alpha_2 < 3$, go to step (XII).

(III) *If $b \not\equiv a_2 \pmod 2$, output $K = \varnothing$. If $b \equiv a_2 \equiv 1 \pmod 2$, go to step* (VII).

(IV) *Suppose that $p^\delta \parallel b$. If $a_2 \equiv 0 \pmod{p^{\alpha_2}}$, output $k \geq \alpha_2/(\delta L)$.*

(V) *Suppose that $p^\varepsilon \parallel a_2$. If $\delta L \nmid \varepsilon$, output $K = \varnothing$.*

(VI) *If $b^{\varepsilon/\delta} \equiv a_2 \pmod{p^{\alpha_2}}$, output $k = \varepsilon/(\delta L)$. Output $K = \varnothing$.*

(VII) *Let $0 \leq \mu_1, \mu_2 < 2$ and $0 \leq \nu_1, \nu_2 < 2^{\alpha_2-2}$ be integers such that*

$$b \equiv (-1)^{\mu_1} 5^{\nu_1} \pmod{2^{\alpha_2}},$$
$$a_2 \equiv (-1)^{\mu_2} 5^{\nu_2} \pmod{2^{\alpha_2}}.$$

If $2 \mid \mu_1 L$ and $2 \nmid \mu_2$, output $K = \varnothing$.

(VIII) *Put $f = \gcd(\nu_1 L, 2^{\alpha_2-2})$. If $f \nmid \nu_2$, output $K = \varnothing$. If $2 \nmid \mu_1 L$, go to step* (X).

(IX) *Output*

$$k \equiv \frac{\nu_2}{f}\left(\frac{\nu_1 L}{f}\right)^{-1} \pmod{\frac{2^{\alpha_2-2}}{f}}.$$

(X) *If $f = 2^{\alpha_2-2}$, output $k \equiv \mu_2 \pmod 2$.*

(XI) *If $\mu_2 \not\equiv \frac{\nu_2}{f} \pmod 2$, output $K = \varnothing$. Go to step* (IX).

(XII) *If $[p \mid b] \neq [p \mid a_2]$, output $K = \varnothing$. If $[p \mid b] = [p \mid a_2] = 1$, go to step* (IV).

(XIII) *Let g be a primitive root modulo p^{α_2} and put $f = \gcd(L \operatorname{ind}_{g,p^{\alpha_2}} b, p^{\alpha_2-1}(p-1))$. If $f \nmid \operatorname{ind}_{g,p^{\alpha_2}} a_2$, output $K = \varnothing$.*

(XIV) *Output*

$$k \equiv \frac{\operatorname{ind}_{g,p^{\alpha_2}} a_2}{f}\left(\frac{L \operatorname{ind}_{g,p^{\alpha_2}} b}{f}\right)^{-1} \pmod{\frac{p^{\alpha_2-1}(p-1)}{f}}.$$

9.3 For general modulus m

We now solve the congruence (9.1) for k, for a general modulus m. When $m = 1$, clearly $K = \mathbb{N}$. Thus assume that $m > 1$. Let the prime factorization of m be $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Then the congruence (9.1) is the conjunction of

$$n(k)_b \equiv a \pmod{p_j^{\alpha_j}}, \tag{9.18}$$

for $1 \leq j \leq r$. For each $1 \leq j \leq r$, we can solve the above congruence for k by the process of Section 9.2, i.e., Algorithm 9.1, obtaining a solution set K_j. Consequently, $K = K_1 \cap \cdots \cap K_r$. In actually finding K, we can use the Chinese remainder theorem. We summarize this into the following algorithm.

Algorithm 9.2. *Given integers $n \geq 1$, $b \geq 2$, $a \in \mathbb{Z}$, and $m \geq 1$, this algorithm computes the set K of integers $k \geq 1$ satisfying* (9.1).

(I) *If $m = 1$, output $K = \mathbb{N}$.*

(II) *Let the prime factorization of m be $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. For each $1 \leq j \leq r$, compute the set K_j of integers $k \geq 1$ satisfying* (9.18) *by using Algorithm 9.1. Output $K = K_1 \cap \cdots \cap K_r$.*

9.4 A concrete example

In this section, we give a concrete example using Algorithm 9.2. Consider the congruence

$$18(k)_3 \equiv 2 \pmod{208}. \tag{9.19}$$

We find the set K of integers $k \geq 1$ satisfying the above congruence by using Algorithm 9.2 with $n = 18$, $b = 3$, $a = 2$, and $m = 208$. Since $m > 1$, we go to step (II).

We have the prime factorization $208 = 2^4 \cdot 13$. In Subsections 9.4.1 and 9.4.2, by using Algorithm 9.1, we find the sets K_1 and K_2 of integers $k \geq 1$ satisfying the congruences

$$18(k)_3 \equiv 2 \pmod{2^4} \quad \text{and} \quad 18(k)_3 \equiv 2 \pmod{13},$$

respectively. Then, in Subsection 9.4.3, we consider $K = K_1 \cap K_2$.

9.4.1 Computation of K_1

We use Algorithm 9.1 with $n = 18$, $b = 3$, $a = 2$, and $m = 2^4$ as follows.

(I) Put $d = \gcd(18, 2^4) = 2$. Since $d = 2 \mid 2 = a$, we go to step (II).

(II) Since $18 = 200_3$, we have $L = 3$. Since $2^4/2 = 2^3$, we have $\alpha_1 = 3$. Since $1 - b^L = 1 - 3^3 = -26$, $\beta = 1$. Put $\alpha_2 = \alpha_1 + \beta = 3 + 1 = 4 \neq 0$. Since

$$\frac{2}{2} \cdot \left(\frac{18}{2}\right)^{-1} = 9^{-1} \equiv 1^{-1} \equiv 1 \pmod{2^3},$$
$$1 - 1 \cdot (-26) = 1 + 26 \equiv -5 \pmod{2^4},$$

we can choose $a_1 = 1$ and $a_2 = -5$. Since $p = 2$ and $\alpha_2 = 4 \geq 3$, we go to step (III).

(III) Since $3 \equiv -5 \equiv 1 \pmod 2$, we go to step (VII).

(VII) Since

$$b = 3 \equiv (-1)^1 \cdot 5^3 \pmod{2^4},$$
$$a_2 = -5 \equiv (-1)^1 \cdot 5^1 \pmod{2^4},$$

$\mu_1 = \mu_2 = 1$, $\nu_1 = 3$, and $\nu_2 = 1$. Since $2 \nmid 3 = \mu_1 L$, we go to step (VIII).

(VIII) Put $f = \gcd(\nu_1 L, 2^{\alpha_2 - 2}) = \gcd(9, 2^2) = 1$. Then $f = 1 \mid \nu_2$. Since $2 \nmid 3 = \mu_1 L$, we go to step (X).

(X) Since $f = 1 < 2^2 = 2^{\alpha_2 - 2}$, we go to step (XI).

(XI) Since $\mu_2 = \nu_2/f$, we go to step (IX).

(IX) Since
$$\frac{1}{1}\cdot\left(\frac{9}{1}\right)^{-1} \equiv 1 \pmod{2^2},$$
we obtain that $k \equiv 1 \pmod 4$.

Therefore we have computed that
$$K_1 = \{k \in \mathbb{N} \colon k \equiv 1 \pmod 4\}.$$

9.4.2 Computation of K_2

We use Algorithm 9.1 with $n = 18$, $b = 3$, $a = 2$, and $m = 13$ as follows.

(I) Put $d = \gcd(18, 13) = 1$. Since $d = 1 \mid 2 = a$, we go to step (II).

(II) Since $18 = 200_3$, we have $L = 3$. Since $13/1 = 13^1$, we have $\alpha_1 = 1$. Since $1 - b^L = -26$, we have $\beta = 1$. Put $\alpha_2 = \alpha_1 + \beta = 1 + 1 = 2 \neq 0$. Since
$$\frac{2}{1}\cdot\left(\frac{18}{1}\right)^{-1} \equiv 2\cdot 5^{-1} \equiv 3 \pmod{13},$$
$$1 - 3(-26) = 1 + 3\cdot 26 \equiv 79 \pmod{13^2},$$
we can choose $a_1 = 3$ and $a_2 = 79$. Since $p = 13$, we go to step (XII).

(XII) Since $[13 \mid 3] = [13 \mid 79] = 0$, we go to step (XIII).

(XIII) A primitive root modulo 13^2 is $g = 2$. We have $\operatorname{ind}_{2,13^2} 3 = 124$ and $\operatorname{ind}_{2,13^2} 79 = 24$. Put $f = \gcd(3\cdot 124, 13\cdot 12) = 12$. Since $f = 12 \mid 24 = \operatorname{ind}_{2,13^2} 79$, we go to step (XIV).

(XIV) Since
$$\frac{24}{12}\cdot\left(\frac{3\cdot 124}{12}\right)^{-1} = 2\cdot 31^{-1} \equiv 2\cdot 5^{-1} \equiv 3 \pmod{13},$$
we obtain that $k \equiv 3 \pmod{13}$.

Therefore we have computed that
$$K_2 = \{k \in \mathbb{N} \colon k \equiv 3 \pmod{13}\}.$$

9.4.3 Computation of K

In Subsections 9.4.1 and 9.4.2, we computed respectively that $K_1 = \{k \in \mathbb{N} \colon k \equiv 1 \pmod 4\}$ and $K_2 = \{k \in \mathbb{N} \colon k \equiv 3 \pmod{13}\}$. By the Chinese remainder theorem, $K = K_1 \cap K_2 = \{k \in \mathbb{N} \colon k \equiv 29 \pmod{52}\}$. Therefore we showed that, for integers $k \geq 1$, the congruence (9.19) holds if and only if $k \equiv 29 \pmod{52}$.

Chapter 10

Conclusion

We have introduced the new concept of (f, b)-palindromes in Definition 1.5. Then we focused on the special case of v-palindromes, proving the *periodic phenomenon* (Theorem 1.2). In investigating this phenomenon more closely, we developed the *general procedure* of Section 6.6. We also defined the concept of the *type* of a v-palindrome and proved the *invariance property* (Theorem 7.1) about this concept. In Chapter 8, we considered more general (v, b)-palindromes, proving their existence in infinitely many bases b (Theorem 1.10).

An isolated problem on repeated concatenations in residue classes was considered in Chapter 9. Also, because the derivation of the fundamental period $\omega_0(n)$ of an integer n as in Theorem 1.2 belongs naturally to the more general theme of the derivation of the fundamental period of an arbitrary periodic function $\mathbb{Z} \to \mathbb{C}$, we included a short treatment, Chapter 5, on periodic functions.

In Section 10.1, we collect some conjectures and problems on (v, b)-palindromes. In Section 10.2, we describe variations of the congruence (10.3) which might be interesting to solve. Finally in Section 10.3, we describe how the research presented in this dissertation might be generalizable from v-palindromes to more general (f, b)-palindromes.

10.1 Conjectures and problems

We collect some conjectures and problems on (v, b)-palindromes in Subsections 10.1.1 and 10.1.2, repectively.

10.1.1 Conjectures

In the short note [45], three conjectures on v-palindromes have been proposed by commentators after extensive computer experiment. We state two of them as follows.

Conjecture 10.1. *There does not exist a prime v-palindrome.*

Conjecture 10.2. *There are infinitely many v-palindromes n such that both n and $r(n)$ are squarefree.*

The above are only about v-palindromes, but the corresponding statement for (v, b)-palindromes can also be considered.

In Table 6.1, we had $c(n) = \infty$ for 17 out of the 27 values of n. In fact it can be shown that all the numbers in (1.36) have $c(n) = \infty$, so in particular there are infinitely many such numbers. Although it might be slightly bold, we make the following conjecture.

Conjecture 10.3. *Let $S = \{n \in \mathbb{N} : 10 \nmid n, n < r(n)\}$ and let $T = \{n \in S : c(n) = \infty\}$. Then the asymptotic density of T in S is* 1*.*

10.1.2 Problems

While Theorem 2.2 gives an exact formula for the number of b-palindromes no greater than an integer $n \geq 1$, i.e., $\mathcal{P}_b(n)$, the same can be considered for (v, b)-palindromes, namely the following.

Problem 10.4. Let $b \geq 2$ be an integer. Is there an exact formula for the number of (v, b)-palindromes no greater than an integer $n \geq 1$, i.e., $\mathbb{V}_b(n)$? If not, how can it be approximated?

From 199 till 575 are 377 consecutive positive integers each not a v-palindrome. Just as consecutive composite numbers can be aribitrarily long, we propose the following problem.

Problem 10.5. Let $b \geq 2$ be an integer. Can consecutive positive integers each not a (v, b)-palindrome be arbitrarily long?

While Theorem 1.10 says that a (v, b)-palindrome exists in every base $b \equiv 120 \pmod{330}$, it is still unsettled whether a (v, b)-palindrome exists in every base b. Therefore we propose the following problem.

Problem 10.6. For which bases $b \geq 2$ does there exist a (v, b)-palindrome?

For the above problem, we provide a possible approach as follows. The proof of Theorem 1.10 in Section 8.4 was based on the equality $v(5) = v(6)$. It is conceivable that the same method basing on other common values of v will find other bases b for which a (v, b)-palindrome exists. For instance, we have

$$v(5) = v(6) = v(8) = v(9), \tag{10.1}$$
$$v(7) = v(10) = v(12) = v(18). \tag{10.2}$$

Perhaps exploiting this method will lead to resolving the existence of a (v, b)-palindrome in all bases b.

10.2 Variations

Throughout this section we fix integers $n \geq 1$, $b \geq 2$, a, and $m \geq 1$ and let the number of base b digits n have be denoted by L. We describe variations of the congruence

$$n(k)_b \equiv a \pmod{m} \tag{10.3}$$

solved for integers $k \geq 1$ in Chapter 9 which might also be interesting to solve. Recall that consideration of the above congruence was inspired by the sequence

$$18, 1818, 181818, \ldots \tag{10.4}$$

of v-palindromes. By restricting in Theorem 1.8, the palindrome ρ to have only the first and last digits being 1 and at least one 0 in between, we obtain the sequence

$$1818, 18018, 180018, \ldots \tag{10.5}$$

of v-palindromes. Just as (10.4) inspired consideration of the congruence (10.3), the sequence (10.5) inspires consideration of another congruence which we describe using the following notation.

Notation 10.1. For integers $k \geq 0$, denote by $n[k]_b$ the positive integer whose base b digits are those of n, followed by k digits of 0, and then another n again.

Then we can try to solve the congruence

$$n[k]_b \equiv a \pmod{m}$$

for integers $k \geq 0$. Conceivably in a similar way, consideration of other congruences can be inspired, by restricting in Theorem 1.8, the palindrome ρ to a special form. Perhaps with the most generality, we can try to solve the congruence

$$n\rho \equiv a \pmod{m}$$

for ρ being a b-palindrome all of whose digits are 0 and 1 and such that between any pair of consecutive digits of 1 there are at least $L-1$ digits of 0. This restriction on the number of digits of 0 between any pair of consecutive digits of 1 is imposed so that in the multiplication $n\rho$, "the copies of n do not overlap". In contrast, this restriction is not imposed in Theorem 1.8, and we see that in the multiplication 18ρ, "the copies of 18 overlap to create digits of 9".

10.3 Generalization

In this dissertation, we started with the definition of v-palindromes and developed a sort of "theory" consisting of the periodic phenomenon, the general procedure, and the invariance property. However, we have always been only considering v-palindromes throughout. As proposed by Professor Yasuo Ohno, it would be an interesting theme of research to explore how this "theory" or parts of it could hold for other (f, b)-palindromes. We give the following notation.

Notation 10.2. For a function $f\colon \mathbb{N} \to \mathbb{C}$ and integer $b \geq 2$, the set of (f, b)-palindromes is denoted by $\mathrm{P}(f, b)$.

Then, for any particular pair (f, b), the corresponding statement of the periodic phenomenon (Theorem 1.2) for (f, b)-palindromes is as follows.

Statement(f, b). Let $n \geq 1$ be an integer with $b \nmid n$ and $n \neq r_b(n)$. There exists an integer $\omega \geq 1$ such that for all integers $k \geq 1$,

$$n(k)_b \in \mathrm{P}(f, b) \quad \text{if and only if} \quad n(k+\omega)_b \in \mathrm{P}(f, b). \tag{10.6}$$

Notice that this statement is a function of the pair (f, b). Consequently, what Theorem 1.2 is saying is that Statement(v, 10) is true. As a first step in the exploration, we propose the following problem.

Problem 10.7. Try to characterize those pairs (f, b) for which Statement(f, b) is true.

In fact, recently I became the mentor of a small group of students who are reading my papers [48, 46, 49, 50, 47] and from there finding topics to research. One of them has already had much consideration about the above problem. It is my hope that (f, b)-palindromes can eventually become, like the usual palindromes, a new kind of widely researched entity.

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