On a generalization of the Hörmander condition in the Calderón–Zygmund theory of singular integral operators [1]

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[1] Calderón–Zygmund の特異積分論における Hörmander 条件の一般化について

Contents

Abstract

It is well-known that if a singular integral operator is bounded on L^2 and its kernel satisfies the so-called Hörmander condition, then the operator is also bounded from L^1 to $L^{1,\infty}$ and on L^p for any $1 < p < 2$, due to L. Hörmander (1960). In this thesis, we discuss the L^p boundedness under two weaker variants of the Hörmander condition. The first one is an L^q mean version introduced by L. Grafakos and C. B. Stockdale (2019). We show that the same boundedness as Hörmander (1960) holds under the L^1 mean version, which improves the result of Grafakos and Stockdale (2019) significantly. Moreover, the L^q mean version is actually equivalent to the original one. The other variant is a BMO version introduced by the author (2022). In this case, the L^p boundedness $(1 < p < 2)$ still holds, though the $L^1 \to L^{1,\infty}$ boundedness is no longer true in general. Also, we give a sufficient condition and an equivalent characterization of the boundness of maximal singular integral operators under the BMO Hörmander condition, which are analogous to the results under the classical Hörmander condition by L. Grafakos (2003) and G. Hu, Da. Yang and Do. Yang (2007), respectively.

1 Introduction

The study of singular integral operators is one of the central topics in harmonic analysis. Roughly speaking, it is an integral operator with a kernel which has a singularity along the diagonal, such as the Hilbert transform H (for one-dimension) and the Riesz transform R_j (for $d\,(\geq 2)$-dimension, $j = 1, 2, \ldots, d$):

$$Hf(x) \coloneqq \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy, \tag{1.1}$$

$$R_j f(x) \coloneqq \lim_{\varepsilon \to +0} c_d \int_{|x-y|>\varepsilon} \frac{(x_j - y_j) f(y)}{|x-y|^{d+1}}\, dy, \quad c_d \coloneqq \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}.$$

The Hilbert transform was introduced to study harmonic functions in the early 1900s. Let $f \in \mathcal{S}(\mathbb{R})$ be real-valued and u be its harmonic extension in the upper half-plane $\mathbb{H} \coloneqq \{\, (x,y) \in \mathbb{R}^2 : y > 0 \,\}$, that is, u be a solution of the Laplace equation

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in \mathbb{H} \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Then, the harmonic extension u can be written by the convolution of f and the Poisson kernel P_y;

$$P_y(x) \coloneqq \frac{1}{\pi} \frac{y}{x^2 + y^2},$$

$$u(x,y) = P_y * f(x) = \int_{z \in \mathbb{R}} P_y(x - z) f(z)\, dz.$$

One can see that $\{P_y\}_{y>0}$ is an approximation to the identity as $y \to +0$, that is,

$$\lim_{y \to +0} P_y(x) = \delta(x)$$

in the sense of tempered distributions (where δ is the Dirac delta), it follows that

$$\lim_{y \to +0} u(x,y) = f(x).$$

Since u is a real-valued harmonic function on $\mathbb{H}$, there exists $v \colon \mathbb{H} \to \mathbb{R}$ such that $u + iv$ is holomorphic on $\mathbb{H}$ (which is unique up to a constant). The function v is called the harmonic conjugate of u. The harmonic conjugate v is given by the convolution of f and the conjugate Poisson kernel Q_y;

$$Q_y(x) \coloneqq \frac{1}{\pi} \frac{x}{x^2 + y^2},$$

$$v(x,y) = Q_y * f(x) = \int_{z \in \mathbb{R}} Q_y(x - z) f(z)\, dz,$$

and the Hilbert transform of $f \in \mathcal{S}(\mathbb{R})$ can be defined by

$$Hf(x) \coloneqq \lim_{y \to +0} Q_y * f(x). \tag{1.2}$$

We can show that

$$\lim_{y\to+0} Q_y(x) = \frac{1}{\pi}\,\mathrm{p.v.}\,\frac{1}{x}$$

in the sense of tempered distributions, where p.v. $\frac{1}{x}$ is defined by

$$\langle \mathrm{p.v.}\,\frac{1}{x}, \varphi\rangle := \lim_{\varepsilon\to+0}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x}\,dx,$$

hence (1.2) coincides with (1.1).

Let us consider the $L^p(\mathbb{R})$ boundedness of the Hilbert transform H. When $p=2$, we can establish the $L^2(\mathbb{R})$ boundedness of H by using the Plancherel theorem, since we have $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi)$ (where $\widehat{f}$ denotes the Fourier transform of f). On the other hand, if $p=1$ or $p=\infty$, then H is not bounded on $L^p(\mathbb{R})$. This can be proved by computing $H\chi_I$, where χ_I is the indicator function of $I := [0,1]$. Since $\chi_I \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and

$$H\chi_I(x) = \frac{1}{\pi}\log\left|\frac{x}{x-1}\right| \notin L^1(\mathbb{R})\cup L^\infty(\mathbb{R}),$$

H is not bounded on $L^1(\mathbb{R})$ nor $L^\infty(\mathbb{R})$. Therefore, the problem is in the case $p\neq 1,2,\infty$. [Riesz, 1928] proved that H is actually bounded on $L^p(\mathbb{R})$ for any $1<p<\infty$. As a consequence, we obtain the L^p summability of the Fourier series:

$$f(x) = \lim_{N\to\infty}\sum_{j=-N}^{N}\widehat{f}(j)e^{2\pi ijx}$$

holds for any $f\in L^p([0,1])$ in the sense of $L^p([0,1])$, $1<p<\infty$. This is not true in the case $p=1,\infty$. Also, for the Hilbert transform, the explicit value of the operator norm is known:

$$\|H\|_{L^p\to L^p} = \begin{cases}\tan(\pi/2p), & 1<p\le 2,\\ \cot(\pi/2p), & 2\le p<\infty.\end{cases} \tag{1.3}$$

(1.3) is originally observed by [Gokhberg and Krupnik, 1968] for the special case $p=2^j$ ($j\in\mathbb{Z}_{\ge 0}$) and [Pichorides, 1972] proved for all $1<p<\infty$.

As we saw above, the Hilbert transform of $L^1(\mathbb{R})$ function does not have to be in $L^1(\mathbb{R})$. A natural question is:

$$\text{Find a proper space } V \supset L^1(\mathbb{R}) \text{ such that } H \text{ is bounded from } L^1(\mathbb{R}) \text{ to } V. \tag{1.4}$$

$$\text{Find a proper space } W \subset L^1(\mathbb{R}) \text{ such that } H \text{ is bounded from } W \text{ to } L^1(\mathbb{R}). \tag{1.5}$$

At first we consider (1.4). Let $f\in\mathcal{S}(\mathbb{R})$ be non-negative and satisfy $\mathrm{supp}\, f\subset I = [0,1]$.

Then, for any $x > 2$, we have

$$\begin{aligned} Hf(x) &= \lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy \\ &= \frac{1}{\pi} \int_{y\in I} \frac{f(y)}{x-y}\, dy \\ &\geq \frac{1}{\pi} \int_{y\in I} \frac{f(y)}{x}\, dy \\ &= \frac{1}{\pi x} \|f\|_{L^1}. \end{aligned}$$

This observation leads us to take $V = L^{1,\infty}(\mathbb{R})$; the weak L^1 space, since $1/x \in L^{1,\infty}(\mathbb{R}) \setminus L^1(\mathbb{R})$. Next we consider (1.5). Assume that $f \in \mathcal{S}(\mathbb{R})$ satisfies $Hf \in L^1(\mathbb{R})$. Then, since the Fourier transform maps $L^1(\mathbb{R})$ to $C^0(\mathbb{R})$, $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi)$ must be continuous, that is,

$$\widehat{f}(0) = \int_{x\in\mathbb{R}} f(x)\, dx = 0.$$

This observation leads us to take $W = H^1(\mathbb{R})$; the Hardy space, since $H^1(\mathbb{R}) \subset \{\, f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f = 0 \,\}$. The $L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})$ and $H^1(\mathbb{R}) \to L^1(\mathbb{R})$ boundednesses of the Hilbert transform are essentially proved by [Kolmogorov, 1925] and [Riesz, 1923], respectively (they worked on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ instead of the line $\mathbb{R}$).

Now we discuss the boundedness of general singular integral operators. The following results are the earliest and most basic ones established by Calderón and Zygmund in the 1950s.

Theorem 1.A ([Calderón and Zygmund, 1952, Lemma 2, Theorem 1])**.** *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies the gradient condition*

$$|\nabla_y K(x,y)| \leq \frac{C}{|x-y|^{d+1}} \tag{1.6}$$

for some constant $C > 0$. Then T is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for any $1 < p < 2$ with bounds

$$\begin{aligned} \|T\|_{L^1\to L^{1,\infty}} &\lesssim_d \|T\|_{L^2\to L^2} + C, \\ \|T\|_{L^p\to L^p} &\lesssim_d (p-1)^{-1}(\|T\|_{L^2\to L^2} + C). \end{aligned}$$

Theorem 1.B ([Calderon and Zygmund, 1956, Theorem 1])**.** *Let $\Omega \in L^1(\mathbb{S}^{d-1})$ satisfy $\int_{\mathbb{S}^{d-1}} \Omega = 0$ and T be a singular integral operator defined by*

$$Tf(x) = \lim_{\varepsilon\to+0} \int_{|x-y|>\varepsilon} \frac{\Omega((x-y)/|x-y|)}{|x-y|^d} f(y)\, dy,$$

where $\mathbb{S}^{d-1} := \{\, x \in \mathbb{R}^d : |x| = 1 \,\}$. Then we have following.

- *If Ω is odd, then T is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$ with a bound*

$$\|T\|_{L^p \to L^p} \lesssim_d \max\{(p-1)^{-1}, p\} \|\Omega\|_{L^1(\mathbb{S}^{d-1})}.$$

- *If Ω is even and $\Omega \in L^q(\mathbb{S}^{d-1})$ for some $q > 1$, then T is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$ with a bound*

$$\|T\|_{L^p \to L^p} \lesssim_d \max\{(p-1)^{-2}, p^2\} \|\Omega\|_{L^q(\mathbb{S}^{d-1})}.$$

Results such as Theorem 1.A are called 'smooth kernel theory' because they require some smoothness conditions. On the other hand, results such as Theorem 1.B are called 'rough kernel theory' since they require some integrability conditions instead of smoothness. Our aim of this paper is to extend smooth kernel theory.

After [Calderón and Zygmund, 1952], [Hörmander, 1960] proved that the conclusion of Theorem 1.A still holds when the gradient condition (1.6) is replaced by weaker assumption

$$[K]_{H_\infty} := \sup_{Q \subset \mathbb{R}^d} \sup_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} |K(x,y) - K(x, c(Q))| \, dx < \infty, \tag{1.7}$$

where the supremum $\sup_{Q \subset \mathbb{R}^d}$ is taken over all cubes $Q \subset \mathbb{R}^d$, $c(Q)$ is the center of Q, $2Q$ denotes the cube with the same center as Q and whose side-length is twice as long. Nowadays (1.7) is called the Hörmander condition. To prove the $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ boundedness, [Hörmander, 1960] used two important lemmas: the Marcinkiewicz interpolation ([Marcinkiewicz, 1939]) and the Calderón–Zygmund decomposition ([Hörmander, 1960, Lemma 2.2]). The Marcinkiwicz interpolation states that if a (sub)linear operator is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ and on $L^2(\mathbb{R}^d)$, then it is also bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < 2$. Since we have assumed the $L^2(\mathbb{R}^d)$ boundedness, it is enough to show the $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ boundedness. Here we use the Calderón–Zygmund decomposition, that is, roughly speaking, any $f \in L^1(\mathbb{R}^d)$ can be written as the sum of a 'good part' g and 'bad parts' $\{b_j\}_j$: $f = g + b = g + \sum_j b_j$.

Furthermore, [Fefferman and Stein, 1972] proved the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness under the same assumption as that of [Hörmander, 1960]. The essence of their proof is to establish the duality $(H^1(\mathbb{R}^d))^* = \mathrm{BMO}(\mathbb{R}^d)$ and reducing the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness to the $L^\infty(\mathbb{R}^d) \to \mathrm{BMO}(\mathbb{R}^d)$ boundedness (which was already proved by [Spanne, 1966]). After that, [Coifman and Weiss, 1977] gave another proof using the atomic decomposition of $f \in H^1(\mathbb{R}^d)$.

Theorem 1.C ([Hörmander, 1960, Theorem 2.1, Theorem 2.2], [Fefferman and Stein, 1972, Corollary 1], [Coifman and Weiss, 1977, (1.24)])**.** *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies the Hörmander condition*

$$[K]_{H_\infty} = \sup_{Q \subset \mathbb{R}^d} \sup_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} |K(x,y) - K(x, c(Q))| \, dx < \infty.$$

Then T is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$, from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, and on $L^p(\mathbb{R}^d)$ for any $1 < p < 2$ with bounds

$$\|T\|_{L^1 \to L^{1,\infty}} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_\infty},$$
$$\|T\|_{H^1 \to L^1} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_\infty},$$
$$\|T\|_{L^p \to L^p} \lesssim_d (p-1)^{-1}(\|T\|_{L^2 \to L^2} + [K]_{H_\infty}).$$

Recently, [Grafakos and Stockdale, 2019] introduced an L^q mean variant of the Hörmander condition in order to establish a "limited-range" version of Theorem 1.C.

Theorem 1.D ([Grafakos and Stockdale, 2019])**.** *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies the L^q mean Hörmander condition*

$$[K]_{H_q} := \sup_{Q \subset \mathbb{R}^d} \left(\frac{1}{|Q|} \int_{y \in Q} \left(\int_{x \in \mathbb{R}^d \setminus 2Q} |K(x,y) - K(x, c(Q))| \, dx \right)^q dy \right)^{1/q} < \infty \tag{1.9}$$

for some $2 < q < \infty$. Then T is bounded from $L^{q'}(\mathbb{R}^d)$ to $L^{q',\infty}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for any $q' < p < 2$ with bounds

$$\|T\|_{L^{q'} \to L^{q',\infty}} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_q},$$
$$\|T\|_{L^p \to L^p} \lesssim_d (p-q')^{-1/q'}(\|T\|_{L^2 \to L^2} + [K]_{H_q}),$$

where q' denotes the Hölder conjugate of q, that is, $q' = \frac{q}{q-1}$.

By the Hölder inequality, if $1 < q_1 < q_2 < \infty$, then $[K]_{H_1} \le [K]_{H_{q_1}} \le [K]_{H_{q_2}} \le [K]_{H_\infty}$. Hence, in comparison with Theorem 1.C, Theorem 1.D requires a weaker smoothness condition and implies the $L^p(\mathbb{R}^d)$ boundedness for p in 'limited-range'; $q' < p < 2$. However, two important problems remain open.

Problem 1. Existence of K such that $[K]_{H_q} < \infty$ and $[K]_{H_\infty} = \infty$.

Problem 2. Existence of T such that satisfying the assumption of Theorem 1.D and not bounded on $L^{q'}(\mathbb{R}^d)$ (or, at least, not bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$).

[Suzuki, 2021] established the following Theorem I and disproved Problem 2.

Theorem I ([Suzuki, 2021, Theorem 1, Theorem 3])**.** *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies the L^1 mean Hörmander condition*

$$[K]_{H_1} = \sup_{Q \subset \mathbb{R}^d} \frac{1}{|Q|} \int_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} |K(x,y) - K(x, c(Q))| \, dx \, dy < \infty. \tag{1.10}$$

Then T is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$, from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, and on $L^p(\mathbb{R}^d)$ for any $1 < p < 2$ with bounds

$$\|T\|_{L^1 \to L^{1,\infty}} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_1},$$
$$\|T\|_{H^1 \to L^1} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_1},$$
$$\|T\|_{L^p \to L^p} \lesssim_d (p-1)^{-1}(\|T\|_{L^2 \to L^2} + [K]_{H_1}).$$

The proof of Theorem I in [Suzuki, 2021] uses an idea inspired by the proof of [Fefferman, 1970, THEOREM 2'], that is, construct certain functions $\{\tilde{b}_j\}_j$ approximating bad parts $\{b_j\}_j$ and decompose $f \in L^1(\mathbb{R}^d)$ as $f = g + \sum_j (b_j - \tilde{b}_j) + \sum_j \tilde{b}_j$ (see Section 7 for details). An analogous method gives an sufficient condition of the $L^2(\mathbb{R}^d)$ boundedness of convolution type singular integral operators satisfying the L^1 mean Hörmander condition (Theorem 7.1). Very recently, [Wang, 2022] extended Theorem I to multilinear singular integral operators.

On the other hand, in fact, Theorem I is a consequence of the following Theorem II proved in [Suzuki, 2022].

Theorem II ([Suzuki, 2022, Theorem 1])**.** *The inequality*

$$[K]_{H_1} \le [K]_{H_\infty} \le 2^{d+3}[K]_{H_1}$$

holds for any $K \in L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta)$*, where* Δ *denotes the diagonal set* $\{\,(x,x) : x \in \mathbb{R}^d\,\}$*.*

Theorem II disproves Problem 1, that is, the L^q mean variant (1.9) coincides with the classical one (1.7). Therefore, it is natural to ask an actual generalization of the Hörmander condition. We introduce a new variant of the Hörmander condition:

$$[K]_{H_*} \coloneqq \sup_{Q \subset \mathbb{R}^d} \frac{1}{|Q|} \int_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} \left| K(x,y) - \frac{1}{|Q|} \int_{z \in Q} K(x,z)\, dz \right| dx\, dy < \infty, \tag{1.11}$$

which is a natural generalization of the L^1 mean Hörmander condition (1.10) in terms of BMO (Bounded Mean Oscillation). Recall that a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is in $\mathrm{BMO}(\mathbb{R}^d)$ if

$$\|f\|_{\mathrm{BMO}} \coloneqq \sup_{Q \subset \mathbb{R}^d} \frac{1}{|Q|} \int_{y \in Q} \left| f(y) - \frac{1}{|Q|} \int_{z \in Q} f(z)\, dz \right| dy < \infty.$$

We call (1.11) a BMO Hörmander condition. Note that we can easily see $[K]_{H_*} \le 2[K]_{H_\infty}$ (see Section 2.2), which is analogous to $\|f\|_{\mathrm{BMO}} \le 2\|f\|_{L^\infty}$. We will show the following:

Theorem III. *Let* T *be a singular integral operator with a kernel* K*. Suppose that* T *is bounded on* $L^2(\mathbb{R}^d)$ *and* K *satisfies the BMO Hörmander condition (1.11). Then* T *is bounded from* $H^1(\mathbb{R}^d)$ *to* $L^1(\mathbb{R}^d)$*, from* $L^p(\mathbb{R}^d)$ *to* $L^{p,\infty}(\mathbb{R}^d)$ *and on* $L^p(\mathbb{R}^d)$ *for any* $1 < p < 2$ *with bounds*

$$\|T\|_{L^p \to L^{p,\infty}} \lesssim_d (p-1)^{-1}(\|T\|_{L^2 \to L^2} + [K]_{H_*}),$$
$$\|T\|_{H^1 \to L^1} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_*},$$
$$\|T\|_{L^p \to L^p} \lesssim_d (p-1)^{-1}(\|T\|_{L^2 \to L^2} + [K]_{H_*}).$$

On the other hand, T *is not bounded from* $L^1(\mathbb{R}^d)$ *to* $L^{1,\infty}(\mathbb{R}^d)$ *in general. In particular, the* BMO *Hörmander condition is strictly weaker than the classical one.*

Moreover, Theorem III still holds in the non-doubling setting with an appropriate modification. Let μ be a Radon measure on $\mathbb{R}^d$ which satisfies the polynomial growth condition: there exists a constant $C_\mu > 0$ and $0 < n \leq d$ such that

$$\mu(Q(c, \ell)) \leq C_\mu \ell^n$$

for any cubes $Q(c, \ell)$. These measures are usually called 'non-doubling measures'. In this case, the following generalization of Theorem 1.C is known.

Theorem 1.E ([Nazarov et al., 1998, Theorem 6.1], [Tolsa, 2001a, Theorem 4.2])**.** *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d, \mu)$ and K satisfies the Hörmander condition with respect to μ:*

$$[K]_{H_\infty} \coloneqq \sup_{Q \subset \mathbb{R}^d} \sup_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} |K(x, y) - K(x, c(Q))| \, d\mu(x) < \infty,$$

and $|K(x, y)| \leq A|x - y|^{-n}$ for some positive constant $A > 0$. Then T is bounded from $L^1(\mathbb{R}^d, \mu)$ to $L^{1,\infty}(\mathbb{R}^d, \mu)$, from $H^1_{\text{atb}}(\mathbb{R}^d, \mu)$ to $L^1(\mathbb{R}^d, \mu)$ and on $L^p(\mathbb{R}^d, \mu)$ for any $1 < p < 2$ with bounds

$$\|T\|_{L^1(\mu) \to L^{1,\infty}(\mu)} \lesssim_\mu \|T\|_{L^2(\mu) \to L^2(\mu)} + [K]_{H_*} + A,$$
$$\|T\|_{H^1_{\text{atb}}(\mu) \to L^1(\mu)} \lesssim_\mu \|T\|_{L^2 \to L^2} + [K]_{H_*} + A,$$
$$\|T\|_{L^p(\mu) \to L^p(\mu)} \lesssim_\mu (p-1)^{-1} (\|T\|_{L^2 \to L^2} + [K]_{H_*} + A),$$

where $H^1_{\text{atb}}(\mathbb{R}^d, \mu)$ is the atomic block Hardy space introduced by [Tolsa, 2001a].

The $L^1(\mathbb{R}^d, \mu) \to L^{1,\infty}(\mathbb{R}^d, \mu)$ and $H^1_{\text{atb}}(\mathbb{R}^d, \mu) \to L^1(\mathbb{R}^d, \mu)$ boundednesses were proved by [Nazarov et al., 1998] and [Tolsa, 2001a], respectively. Also [Tolsa, 2001b] gave another proof of the $L^1(\mathbb{R}^d, \mu) \to L^{1,\infty}(\mathbb{R}^d, \mu)$ boundedness. We will give a natural generalization of Theorem 1.E in the sense of Theorem III (see Theorem 3.4 in Section 3.2 for details). To establish the theorem, we modify the BMO Hörmander condition into an RBMO version, where RBMO is the Regularized Bounded Mean Oscillation space, which is also introduced by [Tolsa, 2001a].

Next, we discuss maximal singular integral operators. For a singular integral operator T, its truncated operator T_ε ($\varepsilon > 0$) and maximal operator T_* are defined by

$$T_\varepsilon f(x) \coloneqq T(f \chi_{\mathbb{R}^d \setminus Q(x,\varepsilon)})(x) = \int_{y \in \mathbb{R}^d \setminus Q(x,\varepsilon)} K(x, y) f(y) \, dy,$$
$$T_* f(x) \coloneqq \sup_{\varepsilon > 0} |T_\varepsilon f(x)| = \sup_{\varepsilon > 0} \left| \int_{y \in \mathbb{R}^d \setminus Q(x,\varepsilon)} K(x, y) f(y) \, dy \right|.$$

As [Cotlar, 1955, Proposition 1] pointed out, the boundedness of T_* can be used to obtain almost everywhere convergence of $\lim_{\varepsilon \to +0} T_\varepsilon f(x)$. For example, the $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ boundedness of the maximal Hilbert transform H_* implies that

$$Hf(x) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{|x-y| > \varepsilon} \frac{f(y)}{x - y} \, dy$$

holds for almost every $x \in \mathbb{R}$ and any $f \in L^2(\mathbb{R})$. Note that this is not trivial for $f \in L^2(\mathbb{R}) \setminus \mathcal{S}(\mathbb{R})$.

On the $L^p(\mathbb{R}^d)$ boundedness of maximal singular operators, the following Cotlar inequality is well-known.

Theorem 1.F ([Cotlar, 1955, Theorem V])**.** *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies*

$$\max\{|\nabla_x K(x,y)|, |\nabla_y K(x,y)|\} \le \frac{C}{|x-y|^{d+1}} \tag{1.12}$$

for some constant $C > 0$. Then the pointwise inequality

$$T_* f(x) \lesssim_d (\|T\|_{L^2 \to L^2} + C)(MTf(x) + Mf(x)) \tag{1.13}$$

holds, where M denotes the Hardy–Littlewood maximal operator.

Note that operators satisfying the assumption of Theorem 1.F are bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$, which follows from Theorem 1.A (use the duality argument for $2 < p < \infty$). Therefore, combining with the $L^p(\mathbb{R}^d)$ boundedness of M, (1.13) implies the $L^p(\mathbb{R}^d)$ boundedness of T_* for any $1 < p < \infty$. In fact, we can obtain the $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ boundedness of T_* under the assumption of Theorem 1.F (see [Duoandikoetxea, 2000, Theorem 5.14, Lemma 5.15] for details).

Now we replace the gradient condition (1.12) by the Hörmander condition. In this case, the pointwise inequality (1.13) is no longer true in general. On the other hand, we can still obtain the boundedness of T_*.

Theorem 1.G ([Rivière, 1971, Theorem (5.1)], [Grafakos, 2003, THEOREM 1], [Hu et al., 2007, Theorem 1.3])**.** *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies*

$$[K]_{H_\infty} = \sup_{Q \subset \mathbb{R}^d} \sup_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} |K(x,y) - K(x, c(Q))| \, dx < \infty,$$

$$[{}^\top K]_{H_\infty} = \sup_{Q \subset \mathbb{R}^d} \sup_{x \in Q} \int_{y \in \mathbb{R}^d \setminus 2Q} |K(x,y) - K(c(Q), y)| \, dy < \infty,$$

$$A := \sup_{Q \subset \mathbb{R}^d} \int_{y \in 2Q \setminus Q} |K(c(Q), y)| \, dy < \infty.$$

Then T_* is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$, from $L^\infty_c(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$ with bounds

$$\|T_*\|_{L^1 \to L^{1,\infty}} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_\infty} + [{}^\top K]_{H_\infty} + A,$$

$$\|T_*\|_{L^p \to L^p} \lesssim_d \max\{(p-1)^{-1}, p\}(\|T\|_{L^2 \to L^2} + [K]_{H_\infty} + [{}^\top K]_{H_\infty} + A),$$

$$\|T_*\|_{L^\infty_c \to \mathrm{BMO}} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_\infty} + [{}^\top K]_{H_\infty} + A.$$

The $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ boundednesses of Theorem 1.G were firstly proved for convolution type operators by [Rivière, 1971], and [Grafakos, 2003] extended to non-convolution type. The $L^\infty_{\mathrm{c}}(\mathbb{R}^d) \to \mathrm{BMO}(\mathbb{R}^d)$ boundedness is due to [Hu et al., 2007]. Also, [Hu et al., 2007] gave equivalent characterizations of the boundedness of T_* under the Hörmander condition.

Theorem 1.H ([Hu et al., 2007, Theorem 1.1])**.** *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that K satisfies*

$$[K]_{H_\infty} = \sup_{Q \subset \mathbb{R}^d} \sup_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} |K(x,y) - K(x, c(Q))| \, dx < \infty,$$

$$[{}^\top K]_{H_\infty} = \sup_{Q \subset \mathbb{R}^d} \sup_{x \in Q} \int_{y \in \mathbb{R}^d \setminus 2Q} |K(x,y) - K(c(Q), y)| \, dy < \infty,$$

$$A = \sup_{Q \subset \mathbb{R}^d} \int_{y \in 2Q \setminus Q} |K(c(Q), y)| \, dy < \infty.$$

Then the following are equivalent:

(1.14) *T_* is bounded from $L^2(\mathbb{R}^d)$ to $L^{2,\infty}(\mathbb{R}^d)$.*

(1.15) *T_* is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$.*

(1.16) *T_* is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$.*

(1.17) *There exist $0 < p < \infty$ and $B > 0$ such that $\|T_* f\|_{L^{p,\infty}(Q)} \le B|Q|^{1/p}\|f\|_{L^\infty}$ holds for any cubes $Q \subset \mathbb{R}^d$ and $f \in L^\infty(Q)$.*

(1.18) *T_* is bounded from $L^\infty_{\mathrm{c}}(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$.*

In fact, [Hu et al., 2007] proved their results on so-called homogeneous metric measure spaces, not only on the Euclidean space with the Lebesgue measure. Furthermore, [Liu et al., 2012] and [Liu et al., 2014] established an analogy of [Hu et al., 2007] and [Grafakos, 2003] on non-homogeneous metric measure spaces, respectively.

It is obvious that one of the key of Theorem 1.G is the $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ boundedness. On the other hand, unlike Theorem 1.C, we cannot hope the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness of T_*, since the truncation ruins the cancellation of $f \in H^1(\mathbb{R})$. In fact, the maximal Hilbert transform of $f \in H^1(\mathbb{R})$; $H_* f$, does not have to be in $L^1(\mathbb{R})$. This can be shown by an analogous argument to that of (1.4).

Now a question arises: what happens if we assume the BMO Hörmander condition instead of the classical one in Theorem 1.G? The main problem here is that we do not have either of the $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ boundedness or the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness. We will show that analogies of Theorem 1.G and 1.H hold despite the lack of endpoint estimates.

Theorem IV. *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies*

$$[K]_{H_*} = \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y) - \frac{1}{|Q|}\int_{z\in Q} K(x,z)\,dz\right| dx\,dy < \infty,$$
$$[{}^{\top}K]_{H_\infty} = \sup_{Q\subset\mathbb{R}^d}\sup_{x\in Q}\int_{y\in\mathbb{R}^d\setminus 2Q} |K(x,y) - K(c(Q),y)|\,dy < \infty,$$
$$A = \sup_{Q\subset\mathbb{R}^d}\int_{y\in 2Q\setminus Q} |K(c(Q),y)|\,dy < \infty.$$

Then T_* is bounded from $L^\infty_c(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$ with bounds

$$\|T_*\|_{L^p\to L^p} \lesssim_d \max\{(p-1)^{-2}, p\}(\|T\|_{L^2\to L^2} + [K]_{H_*} + [{}^{\top}K]_{H_\infty} + A),$$
$$\|T_*\|_{L^\infty_c\to\mathrm{BMO}} \lesssim_d \|T\|_{L^2\to L^2} + [K]_{H_*} + [{}^{\top}K]_{H_\infty} + A.$$

Remark 1. Note that we have $\|T_*\|_{L^p\to L^p} \lesssim_{d,T} \max\{(p-1)^{-2}, p\}$ instead of $\|T_*\|_{L^p\to L^p} \lesssim_{d,T} \max\{(p-1)^{-1}, p\}$. This difference affects, for example, the conclusion obtained using the Yano extrapolation theorem (see [Yano, 1951]). Also, we assume $[{}^{\top}K]_{H_\infty} < \infty$ instead of $[{}^{\top}K]_{H_*} < \infty$ in order to establish a certain pointwise estimate (see Lemma 4.2). As of this writing, it remains open that these issues can be removed or not.

Theorem V. *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that K satisfies*

$$[K]_{H_*} = \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y) - \frac{1}{|Q|}\int_{z\in Q} K(x,z)\,dz\right| dx\,dy < \infty,$$
$$[{}^{\top}K]_{H_*} = \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{x\in Q}\int_{y\in\mathbb{R}^d\setminus 2Q}\left|K(x,y) - \frac{1}{|Q|}\int_{z\in Q} K(z,y)\,dz\right| dy\,dx < \infty,$$
$$A = \sup_{Q\subset\mathbb{R}^d}\int_{y\in 2Q\setminus Q} |K(c(Q),y)|\,dy < \infty.$$

Then the following are equivalent:

(1.14) *T_* is bounded from $L^2(\mathbb{R}^d)$ to $L^{2,\infty}(\mathbb{R}^d)$.*

(1.15) *T_* is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$.*

(1.17) *There exist $0 < p < \infty$ and $B > 0$ such that $\|T_*f\|_{L^{p,\infty}(Q)} \le B|Q|^{1/p}\|f\|_{L^\infty}$ holds for any cubes $Q \subset \mathbb{R}^d$ and $f \in L^\infty(Q)$.*

(1.18) *T_* is bounded from $L^\infty_c(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$.*

Finally, we remark that there exist numerous other variants of the Hörmander condition. Here are some of them:

[Fefferman, 1970] Let $0 \leq \theta < 1$ and $F_\theta \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be

$$F_\theta(r) := \begin{cases} 2r^{1-\theta}, & 0 < r < 1, \\ 2r, & r \geq 1. \end{cases}$$

[Fefferman, 1970, THEOREM 2'] states that if

$$\widehat{K}(\xi) \lesssim (1+|\xi|^2)^{-\theta d/4}, \tag{1.19}$$

$$[K]_{H(\theta)} := \sup_{r>0} \int_{|x|\geq F_\theta(r)} |K(x-y)-K(x)|\,dx < \infty, \tag{1.20}$$

then $T\colon f \mapsto K * f$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$. Also [Fefferman and Stein, 1972] proved the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness of T under the same assumption. Note that setting $\theta = 0$ gives Theorem 1.C (at least for convolution type operators). In the case $0 < \theta < 1$, (1.19) is stronger than the $L^2(\mathbb{R}^d)$ boundedness of T and (1.20) is weaker than the classical Hörmander condition (1.7), hence these operators are called weakly-strongly singular integral operators. A typical example is

$$K(x) := \frac{\exp\left(i|x|^{-\theta/(1-\theta)}\right)}{x}.$$

After that, [Miyachi, 1978] extended the result as follows: let $m\colon \mathbb{R}^d \to \mathbb{R}_{>0}$ be a bounded function satisfying a certain property and $F_m \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be

$$F_m(r) := \left(\int_{\mathbb{R}^d} (m(\xi))^2 (1+r|\xi|)^{-2d}\,d\xi\right)^{-1/d}.$$

[Miyachi, 1978, Theorem 3] states that if

$$\widehat{K}(\xi) \lesssim m(\xi),$$

$$[K]_{H(m)} := \sup_{r>0} \int_{|x|\geq F_m(r)} |K(x-y)-K(x)|\,dx < \infty,$$

then $T\colon f \mapsto K * f$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$. In particular, setting $m(\xi) = (1+|\xi|^2)^{-\theta d/4}$ $(0 \leq \theta < 1)$ gives [Fefferman, 1970, THEOREM 2'].

[Watson, 1990] Let $1 \leq q < \infty$. [Watson, 1990, THEOREM 2] states that if

$$\widehat{K}(\xi) \lesssim 1,$$

$$[K]_{H^q} := \sup_{r>0} \sup_{|y|\leq r} \sum_{j=1}^{\infty} (2^j r)^{d/q'} \left(\int_{2^j R<|x|\leq 2^{j+1}R} |K(x-y)-K(x)|^q\,dx\right)^{1/q} < \infty,$$

then $T\colon f \mapsto K * f$ is bounded from $L^1(\mathbb{R}^d, w\,dx)$ to $L^{1,\infty}(\mathbb{R}^d, w\,dx)$ and on $L^p(\mathbb{R}^d, w\,dx)$ for any $1 < p < \infty$, where w denotes a weight satisfying a certain condition (depending on p and q). By the Hölder inequality, if $1 < q_1 < q_2 < \infty$, then $[K]_{H_\infty} = [K]_{H^1} \leq [K]_{H^{q_1}} \leq [K]_{H^{q_2}}$. Unlike the L^q mean Hörmander condition of [Grafakos and Stockdale, 2019], it is known that the reverse inequality does not hold. Also see [Martell et al., 2005] for related results.

[Grubb and Moore, 1997] Let $N \in \mathbb{Z}_{\geq 1}$, $\{A_j\}_{j=1}^N \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\{\varphi_j\}_{j=1}^N \subset L^\infty(\mathbb{R}^d)$ and $\Phi\colon (\mathbb{R}^d)^N \to \mathbb{R}$ be

$$\Phi(y_1, y_2, \cdots, y_N) := \left(\det\left(\varphi_i(y_j)\right)_{1 \leq i,j \leq N}\right)^2.$$

Assume that Φ satisfies the reverse L^∞ inequality, that is, there exists a constant $C > 0$ such that

$$0 < \|\Phi\|_{L^\infty(Q)} \leq \frac{C}{|Q|} \int_Q \Phi$$

for any cubes $Q \subset (\mathbb{R}^d)^N$ centered at origin. [Grubb and Moore, 1997, THEOREM] states that if

$$\widehat{K}(\xi) \lesssim 1,$$

$$\sup_{r>0} \sup_{|y| \leq r} \int_{|x| \geq 2r} \left| K(x-y) - \sum_{j=1}^N A_j(x)\varphi_j(y) \right| dx < \infty,$$

then $T\colon f \mapsto K * f$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$, but not from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ in general (recall that our BMO Hörmander condition implies the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness but not $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$, in contrast). Note that setting $N = 1$, $A(x) = K(x)$, $\varphi(x) = 1$ gives Theorem 1.C. A typical example is

$$K(x) := \frac{\sin x}{x}$$

with $N = 2$ and

$$A_1(x) = \frac{\sin x}{x}, \quad \varphi_1(y) := \cos y,$$
$$A_2(x) = \frac{\cos x}{x}, \quad \varphi_2(y) := -\sin y.$$

This example shows that the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness does not hold in general under the assumption of [Grubb and Moore, 1997, THEOREM]. Also see [Trujillo-González, 2003], [Zhang and Zhang, 2013], [Zhou, 2015] for related results.

See [Duong and McIntosh, 1999], [Gallo et al., 2019], [Lorente et al., 2005], [Lorente et al., 2008] for further variants.

This doctoral thesis is organized as follows.

In Section 2.1, we prove Theorem II. Note that our argument works for any doubling measures (see Remark 2). On the other hand, it remains open that the equivalence holds with non-doubling measures or not.

In Section 2.2, we prove two basic properties of BMO Hörmander condition, Proposition 2.2 and 2.3, which are analogous to that of the BMO norm. Proposition 2.3 plays a crucial role in Section 4.

In Section 3.1, we prove Theorem III. Our argument is based on [Coifman and Weiss, 1977] (for $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness; Proposition 3.2) and [Grafakos and Stockdale, 2019] (for $L^p(\mathbb{R}^d) \to L^{p,\infty}(\mathbb{R}^d)$ boundedness; Proposition 3.3). Note that Proposition 3.3 follows from Proposition 3.2 and the interpolation, but we give a direct proof of Proposition 3.3 in order to show the usage of Proposition 2.3. The case of non-doubling measures is discussed in Section 3.2.

In Section 4, we prove Theorem IV and a part of Theorem V. Since we cannot have the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness of T_*, we have to establish the $L^p(\mathbb{R}^d) \to L^{p,\infty}(\mathbb{R}^d)$ boundedness directly by using Proposition 2.3.

In Section 5, we prove the so-called sharp maximal inequality for median maximal operators (Theorem 5.7), which is important in Section 6. Theorem 5.7 is a little refined version of [Hu et al., 2007, Theorem 2.2] (see Remark 5). This improvement simplifies our argument in Section 6.

In Section 6, we prove the rest part of Theorem V.

In Section 7, we give a direct proof of Theorem I. Also, we give a sufficient codition for the $L^2(\mathbb{R}^d)$ boundedness of convolution type singular integral operators under the L^1 mean Hörmander condition, which is inspired by [Benedek et al., 1962, Theorem 3]. Note that results in this section themselves are nothing new, since the L^1 mean Hörmander condition is equivalent to classical one.

Section 2, 3 are based on [Suzuki, 2022] and Section 7 is on [Suzuki, 2021].

Preliminaries

We will work on the d-dimensional Euclidean space $\mathbb{R}^d$ with the Lebesgue measure (except Section 3.2). For $x = (x_j)_{1 \le j \le d} \in \mathbb{R}^d$, $|x|$ is its Euclidean norm, that is,

$$|x| := \left(\sum_{1 \le j \le d} x_j^2 \right)^{1/2}.$$

For $E \subset \mathbb{R}^d$, $|E|$ is the Lebesgue measure of E (if E is measurable). The indicator function of E is denoted as χ_E;

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus E. \end{cases}$$

Throughout this paper, "a cube" always means a cube with sides parallel to the coordinate axes. Also we write

$$|||x||| := \max_{1 \le j \le d} |x_j|.$$

The cube with center $c \in \mathbb{R}^d$ and side-length $2\ell > 0$ is denoted by $Q(c,\ell)$. Conversely, for a cube $Q \subset \mathbb{R}^d$, $c(Q)$ and $2\ell(Q)$ denote the center of Q and the side-length of Q, respectively. For $Q = Q(c,\ell)$ and $\alpha > 0$, we define $\alpha Q \coloneqq Q(c, \alpha\ell)$.

For $j \in \mathbb{Z}$,

$$\mathcal{Q}_j \coloneqq \left\{ \prod_{k=1}^{d} [2^{-j}a_k, 2^{-j}(a_k+1)) : (a_k) \in \mathbb{Z}^d \right\}$$

is called dyadic cubes of the j-th generation. We write

$$\mathcal{Q} \coloneqq \bigcup_j \mathcal{Q}_j.$$

For each $j \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, there exists an unique dyadic cube $Q_j(x) \in \mathcal{Q}_j$ such that $Q_j(x) \ni x$. Also, for each $j \in \mathbb{Z}$ and $Q \in \mathcal{Q}_j$, there exists an unique dyadic cube $\widetilde{Q} \in \mathcal{Q}_{j-1}$ such that $Q \subset \widetilde{Q}$. The dyadic cube $\widetilde{Q} \in \mathcal{Q}_{j-1}$ is called the parent of $Q \in \mathcal{Q}_j$.

For non-negative numbers A and B, $A \lesssim_X B$ means that there exists a positive constant $C_X > 0$ depending only on X such that $A \leq C_X B$. For example, $|2Q| \lesssim_d |Q|$ for any cubes Q, since $|2Q| = 2^d|Q|$.

Let $E \subset \mathbb{R}^d$ be measurable. The function spaces appearing in this paper are defined as follows.

$L^p(E)$: the space of measurable functions $f \colon E \to \mathbb{C}$ such that

$$\|f\|_{L^p(E)} \coloneqq \begin{cases} \left(\int_{x\in E} |f(x)|^p \, dx\right)^{1/p} & \text{if } 0 < p < \infty, \\ \operatorname{ess\,sup}_{x\in E} |f(x)| & \text{if } p = \infty, \end{cases}$$

is finite.

$L^\infty_c(\mathbb{R}^d)$: the space of $f \in L^\infty(\mathbb{R}^d)$ which is compactly supported.

$L^\infty_{c,0}(\mathbb{R}^d)$: $L^\infty_c(\mathbb{R}^d)$ is the space of $f \in L^\infty_c(\mathbb{R}^d)$ whose integral is zero.

$L^{p,\infty}(E)$: the space of measurable functions $f \colon E \to \mathbb{C}$ such that

$$\|f\|_{L^{p,\infty}(E)} \coloneqq \begin{cases} \sup_{\lambda>0} \left(\lambda^p |\{ x \in E : |f(x)| > \lambda \}|\right)^{1/p} & \text{if } 0 < p < \infty, \\ \|f\|_{L^\infty(E)} & \text{if } p = \infty., \end{cases}$$

is finite.

$H^1(\mathbb{R}^d)$: a measurable function $a \colon \mathbb{R}^d \to \mathbb{C}$ is called an atom if there exists a cube Q such that

$$\operatorname{supp} a \subset Q, \quad \|a\|_{L^\infty} \leq |Q|^{-1}, \quad \int_Q a = 0.$$

$H^1(\mathbb{R}^d)$ is the space of $f \in L^1(\mathbb{R}^d)$ such that

$$f = \sum_j \lambda_j a_j \text{ in } L^1(\mathbb{R}^d), \quad \sum_j |\lambda_j| < \infty, \quad \lambda_j \in \mathbb{C}$$

and its norm is defined by

$$\|f\|_{H^1} = \inf \{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \text{ in } L^1(\mathbb{R}^d) \}.$$

Note that there are several characterizations of $H^1(\mathbb{R}^d)$. See [Grafakos, 2014b, Chapter 2]. On the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness, the following fact is well-known:

Proposition 1.A. *Let T be a bounded linear operator on $L^2(\mathbb{R}^d)$. If there exists $C > 0$ such that*

$$\|Ta\|_{L^1} \le C$$

for any atoms a, then T is bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ with $\|T\|_{H^1 \to L^1} \le C$.

Here we need to be careful. Let $H^1_{\text{fin}}(\mathbb{R}^d)$ be the space of finite linear combinations of atoms (which is a dense subspace of $H^1(\mathbb{R}^d)$). Then there exists a linear operator $T \colon H^1_{\text{fin}}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that

- There exists $C > 0$ such that $\|Ta\|_{L^1} \le C$ for any atoms a.
- The operator T cannot be extended to a bounded operator $\tilde{T} \colon H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$.

In this sense, the $L^2(\mathbb{R}^d)$ boundedness in Proposition 1.A is essential. See [Meyer et al., 1985, (5.6)], [Bownik, 2005], [Meda et al., 2008] for details.

BMO($\mathbb{R}^d$): the space of $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$\|f\|_{\text{BMO}} \coloneqq \sup_{Q \subset \mathbb{R}^d} \frac{1}{|Q|} \int_{y \in Q} \left| f(y) - \frac{1}{|Q|} \int_Q f(z)\, dz \right| dy$$

is finite, where supremum $\sup_{Q \subset \mathbb{R}^d}$ is taken over all cubes $Q \subset \mathbb{R}^d$. Also, the sharp maximal operator $M^\#$ is defined by

$$M^\# f(x) \coloneqq \sup_{Q \ni x} \frac{1}{|Q|} \int_{y \in Q} \left| f(y) - \frac{1}{|Q|} \int_Q f(z)\, dz \right| dy,$$

where supremum $\sup_{Q \ni x}$ is taken over all cubes $Q \subset \mathbb{R}^d$ such that $Q \ni x$. It is obvious that $\|M^\# f\|_{L^\infty} = \|f\|_{\text{BMO}}$.

The Hardy–Littlewood maximal operator M and its dyadic variant M^{dyadic} are defined by

$$Mf(x) \coloneqq \sup_{Q \ni x} \frac{1}{|Q|} \int_{y \in Q} |f(y)|\, dy,$$
$$M^{\text{dyadic}} f(x) \coloneqq \sup_{j \in \mathbb{Z}} \frac{1}{|Q_j(x)|} \int_{y \in Q_j(x)} |f(y)|\, dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. It is well-known that M and M^{dyadic} are bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for any $1 < p \leq \infty$. In particular, we have

$$\|M\|_{L^p \to L^{p,\infty}} \leq 2 \cdot 3^{d/p},$$
$$\|M^{\mathrm{dyadic}}\|_{L^1 \to L^{1,\infty}} \leq 1.$$

See [Grafakos, 2014a, Exercise 2.1.4.(a), Exercise 2.1.12.(a)] for example.

Let T be a linear operator from $L^\infty_{\mathrm{c}}(\mathbb{R}^d)$ to the space of all measurable functions. We say T is a singular integral operator with a kernel $K \in L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta)$ if

$$Tf(x) = \int_{y \in \mathbb{R}^d} K(x,y) f(y)\, dy$$

for any $f \in L^\infty_c(\mathbb{R}^d)$ and $x \in \mathbb{R}^d \setminus \operatorname{supp} f$.

2 Variants of the Hörmander condition: classical, L^q mean, BMO

2.1 Equivalence of the classical Hörmander condition and the L^1 mean variant

In this section, we prove Theorem II. We begin with the following Lemma 2.1.

Lemma 2.1. *The inequality*

$$[K]_{H_\infty} \leq 2 \sup_{Q \subset \mathbb{R}^d} \sup_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 3Q} |K(x,y) - K(x,c(Q))| \, dx$$

holds for any $K \in L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta)$.

Proof of Lemma 2.1. Fix a cube $Q_0 = Q(c, 2\ell) \subset \mathbb{R}^d$, $y \in Q_0$ and write $z := (y+c)/2$. Since $Q(z, 3\ell) \subset 2Q_0$ and $c, y \in Q(z, \ell)$, we have

$$\begin{aligned}
&\int_{x \in \mathbb{R}^d \setminus 2Q_0} |K(x,y) - K(x,c)| \, dx \\
&\leq \int_{x \in \mathbb{R}^d \setminus 2Q_0} |K(x,y) - K(x,z)| \, dx + \int_{x \in \mathbb{R}^d \setminus 2Q_0} |K(x,z) - K(x,c)| \, dx \\
&\leq \int_{x \in \mathbb{R}^d \setminus Q(z,3\ell)} |K(x,y) - K(x,z)| dx + \int_{x \in \mathbb{R}^d \setminus Q(z,3\ell)} |K(x,c) - K(x,z)| \, dx \\
&\leq 2 \sup_{Q \subset \mathbb{R}^d} \sup_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 3Q} |K(x,y) - K(x,c(Q))| \, dx.
\end{aligned}$$

□

Proof of Theorem II. Let $\ell > 0$ and $c, y \in \mathbb{R}^d$ satisfy $|||y - c||| \leq \ell$. We write

$$\begin{aligned}
I(c,y) &:= \int_{x \in \mathbb{R}^d \setminus Q(c,2\ell)} |K(x,y) - K(x,c)| \, dx, \\
J(c,y) &:= \int_{x \in \mathbb{R}^d \setminus Q(c,3\ell)} |K(x,y) - K(x,c)| \, dx, \\
A &:= Q(c,\ell) \cap Q(y,\ell).
\end{aligned}$$

Since $Q(y, 2\ell) \cup Q(c, 2\ell) \subset Q(c, 3\ell)$, we obtain

$$\begin{aligned}
J(c,y) &= \int_{x \in \mathbb{R}^d \setminus Q(c,3\ell)} |K(x,y) - K(x,c)| \, dx \\
&\leq \int_{x \in \mathbb{R}^d \setminus Q(c,3\ell)} |K(x,y) - K(x,z)| \, dx + \int_{x \in \mathbb{R}^d \setminus Q(c,3\ell)} |K(x,z) - K(x,c)| \, dx \\
&\leq \int_{x \in \mathbb{R}^d \setminus Q(y,2\ell)} |K(x,z) - K(x,y)| \, dx + \int_{x \in \mathbb{R}^d \setminus Q(c,2\ell)} |K(x,z) - K(x,c)| \, dx \\
&= I(y,z) + I(c,z)
\end{aligned}$$

for any $z \in A$. Therefore, letting $J = J(c,y)$, we have

$$A \subset \{ z \in A : I(c,z) \geq J/2 \} \cup \{ z \in A : I(y,z) \geq J/2 \},$$

which implies at least one of the following:

$$|A|/2 \le |\{\, z \in A : I(c,z) \ge J/2 \,\}|, \tag{2.2}$$
$$|A|/2 \le |\{\, z \in A : I(y,z) \ge J/2 \,\}|. \tag{2.3}$$

By the symmetry between c and y, we assume that (2.2) holds without loss of generality. Now we have

$$|A|J/4 \le \int_{z\in A\,:\,I(c,z)\ge J/2} J/2\,dz \le \int_{z\in A} I(c,z)\,dz \le \int_{z\in Q(c,\ell)} I(c,z)\,dz \le |Q(c,\ell)|[K]_{H_1}.$$

Since

$$|Q(c,\ell)| = 2^d|Q((c+y)/2,\ell/2)| \le 2^d|A|, \tag{2.4}$$

we get $J \le 2^{d+2}[K]_{H_1}$. By Lemma 2.1, we conclude that the inequality

$$[K]_{H_\infty} \le 2^{d+3}[K]_{H_1}$$

holds. □

Remark 2. Let μ be a Radon measure on $\mathbb{R}^d$ which satisfies the doubling property: there exists a constant $C_\mu > 0$ such that

$$\mu(Q(c,3\ell)) \le C_\mu \mu(Q(c,\ell))$$

for any cubes $Q(c,\ell)$, and consider $[K]_{H_\infty}$ and $[K]_{H_1}$ with respect to μ:

$$[K]_{H_\infty} := \sup_{Q\subset\mathbb{R}^d}\sup_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}|K(x,y)-K(x,c(Q))|\,d\mu(x),$$
$$[K]_{H_1} := \sup_{Q\subset\mathbb{R}^d}\frac{1}{\mu(Q)}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}|K(x,y)-K(x,c(Q))|\,d\mu(x)\,d\mu(y).$$

In this case, we can show that the inequality $[K]_{H_\infty} \le 8C_\mu[K]_{H_1}$ holds by the same argument. To see this, note that μ satisfies

$$\mu(Q(c,\ell)) \le \mu(Q((c+y)/2, 3\ell/2)) \le C_\mu \mu(B((c+y)/2,\ell/2)) \le C_\mu \mu(A),$$

which can be a replacement of (2.4).

2.2 Basic properties of the BMO Hörmander condition

In this section, We will give two elemental properties of the BMO Hörmander condition

$$[K]_{H_*} = \sup_{Q\subset\mathbb{R}^d}\frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y)-\frac{1}{|Q|}\int_{z\in Q}K(x,z)\,dz\right|dx\,dy < \infty, \tag{1.11}$$

which are analogous to the following well-known facts of the BMO norm.

Proposition 2.A. *If there exists a collection of numbers* $\{m_Q\}_Q$ *such that*

$$\sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q} |f(y)-m_Q|\,dy < \infty,$$

then we have

$$\|f\|_{\mathrm{BMO}} \le 2 \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q} |f(y)-m_Q|\,dy.$$

Proposition 2.B. *Let* $E\subset\mathbb{R}^d$ *be a cube or* $E=\mathbb{R}^d$*. We write*

$$\|f\|_{\mathrm{BMO}_q(E)} \coloneqq \sup_{Q\subset E}\left(\frac{1}{|Q|}\int_{y\in Q}\left|f(y)-\frac{1}{|Q|}\int_{z\in Q} f(z)\,dz\right|^q dy\right)^{1/q}$$

for $1\le q<\infty$*. Then there exists a constant* $C_d>0$ *depending only on dimension* d *such that*

$$\|f\|_{\mathrm{BMO}_q(E)} \le qC_d\|f\|_{\mathrm{BMO}(E)}.$$

Proposition 2.A is quite elementary. On the other hand, Proposition 2.B is a much deeper result proved by [John and Nirenberg, 1961]. We begin with an analogy to Proposition 2.A.

Proposition 2.2. *If there exists a collection of functions* $\{m_Q\}_Q$ *such that*

$$\sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q} |K(x,y)-m_Q(x)|\,dx < \infty,$$

then we have

$$[K]_{H_*} \le 2 \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q} |K(x,y)-m_Q(x)|\,dx\,dy.$$

Proof of Proposition 2.2.

$$\begin{aligned}
&\frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y)-\frac{1}{|Q|}\int_{z\in Q} K(x,z)\,dz\right|dx\,dy\\
&\le \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}|K(x,y)-m_Q(x)|\,dx\,dy\\
&\quad+\frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}\left|m_Q(x)-\frac{1}{|Q|}\int_{z\in Q} K(x,z)\,dz\right|dx\,dy\\
&\le \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}|K(x,y)-m_Q(x)|\,dx\,dy\\
&\quad+\frac{1}{|Q|}\int_{y\in Q}\left(\frac{1}{|Q|}\int_{z\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}|K(x,z)-m_Q(x)|\,dx\,dz\right)dy\\
&= \frac{2}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}|K(x,y)-m_Q(x)|\,dx\,dy.
\end{aligned}$$

□

We write

$$[K]_{H_{**}} := \inf_{\{m_Q\}_Q} \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q} |K(x,y)-m_Q(x)|\,dx\,dy,$$

where the infimum $\inf_{\{m_Q\}_Q}$ is taken over all collections of functions $\{m_Q\}_Q$. Then Proposition 2.2 gives us $[K]_{H_*} \le 2[K]_{H_{**}}$. Also note that $[K]_{H_{**}} \le [K]_{H_1}$ and $[K]_{H_{**}} \le [K]_{H_*}$: consider $m_Q(x) = K(x, c(Q))$ and $m_Q(x) = \frac{1}{|Q|}\int_{z\in Q} K(x,z)\,dz$, respectively. Therefore, we have

$$[K]_{H_{**}} \le [K]_{H_*} \le 2[K]_{H_{**}} \le 2[K]_{H_1} \le 2[K]_{H_\infty} \le 2^{d+4}[K]_{H_1}.$$

In particular, the classical Hörmander condition implies the BMO version. We will show later that the converse is not true (see Proposition 3.1).

Next we show an analogy to Proposition 2.B.

Proposition 2.3. *Let define*

$$[K]_{H_{*,q}} := \sup_{Q\subset\mathbb{R}^d}\left(\frac{1}{|Q|}\int_{y\in Q}\left(\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y)-\frac{1}{|Q|}\int_{z\in Q}K(x,z)\,dz\right|dx\right)^q dy\right)^{1/q}$$

for $1<q<\infty$. *Then there exists a constant* $C_d>0$ *depending only on dimension* d *such that*

$$[K]_{H_{*,q}} \le qC_d[K]_{H_*}.$$

Proof of Proposition 2.3. Since it is trivial when $[K]_{H_*}=\infty$, we assume $[K]_{H_*}<\infty$. Then, for a cube $P\in\mathbb{R}^d$,

$$\boldsymbol{K}_P(y) := \left(K(\,\cdot\,,y)-\frac{1}{|P|}\int_{z\in P}K(\,\cdot\,,z)\,dz\right)\chi_{\mathbb{R}^d\setminus 2P}(\,\cdot\,)$$

defines a vector-valued function $\boldsymbol{K}_P\colon P\to L^1(\mathbb{R}^d)$. We rewrite $[K]_{H_{*,q}}$ using $\boldsymbol{K}_P$:

$$\begin{aligned}
[K]_{H_{*,q}} &= \sup_{Q\subset\mathbb{R}^d}\left(\frac{1}{|Q|}\int_{y\in Q}\left(\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y)-\frac{1}{|Q|}\int_{z\in Q}K(x,z)\,dz\right|dx\right)^q dy\right)^{1/q}\\
&= \sup_{Q\subset\mathbb{R}^d}\sup_{P\supset Q}\left(\frac{1}{|Q|}\int_{y\in Q}\left(\int_{x\in\mathbb{R}^d\setminus 2P}\left|K(x,y)-\frac{1}{|Q|}\int_{z\in Q}K(x,z)\,dz\right|dx\right)^q dy\right)^{1/q}\\
&= \sup_{P\subset\mathbb{R}^d}\sup_{Q\subset P}\left(\frac{1}{|Q|}\int_{y\in Q}\left\|\boldsymbol{K}_P(y)-\frac{1}{|Q|}\int_{z\in Q}\boldsymbol{K}_P(z)\,dz\right\|_{L^1}^q dy\right)^{1/q}\\
&= \sup_{P\subset\mathbb{R}^d}\|\boldsymbol{K}_P\|_{\mathrm{BMO}_q(P,L^1)}.
\end{aligned}$$

Now we use Proposition 2.B (extended to vector-valued), which implies

$$[K]_{H_{*,q}} = \sup_{P\subset\mathbb{R}^d}\|\boldsymbol{K}_P\|_{\mathrm{BMO}_q(P,L^1)} \le \sup_{P\subset\mathbb{R}^d} qC_d\|\boldsymbol{K}_P\|_{\mathrm{BMO}(P,L^1)} = qC_d[K]_{H_*}. \qquad \square$$

3 Boundednesses of singular integral operators

3.1 Case I: the Lebesgue measure

In this section, we prove Theorem III. At first, we show that T does not have to be bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$.

Proposition 3.1. *Let* $\varphi \in (L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)) \setminus \{0\}$, $\psi \in (L^2(\mathbb{R}^d) \cap \mathrm{BMO}(\mathbb{R}^d)) \setminus L^\infty(\mathbb{R}^d)$ *and* $K(x, y) := \varphi(x)\psi(y)$. *Then*

$$T\colon f \mapsto \int_{\mathbb{R}^d} K(\cdot, y) f(y)\, dy \text{ is bounded on } L^2(\mathbb{R}^d), \tag{3.1}$$

$$[K]_{H_*} < \infty, \tag{3.2}$$

$$T \text{ is not bounded from } L^1(\mathbb{R}^d) \text{ to } L^{1,\infty}(\mathbb{R}^d). \tag{3.3}$$

Proof of Proposition 3.1.

(3.1) It is obvious that $\|T\|_{L^2 \to L^2} \le \|\varphi\|_{L^2} \|\psi\|_{L^2}$.

(3.2) We have

$$\begin{aligned} &\frac{1}{|Q|} \int_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} \left| K(x, y) - \frac{1}{|Q|} \int_{z \in Q} K(x, z)\, dz \right| dx\, dy \\ &= \left(\int_{x \in \mathbb{R}^d \setminus 2Q} |\varphi(x)|\, dx \right) \cdot \left(\frac{1}{|Q|} \int_{y \in Q} \left| \psi(y) - \frac{1}{|Q|} \int_{z \in Q} \psi(z)\, dz \right| dy \right) \\ &\le \|\varphi\|_{L^1} \|\psi\|_{\mathrm{BMO}} \end{aligned}$$

for any cubes $Q \subset \mathbb{R}^d$, thus $[K]_{H_*} \le \|\varphi\|_{L^1} \|\psi\|_{\mathrm{BMO}}$.

(3.3) For each $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, Tf is given by

$$Tf(x) = \int_{y \in \mathbb{R}^d} \varphi(x)\psi(y) f(y)\, dy = \varphi(x) \int_{y \in \mathbb{R}^d} \psi(y) f(y)\, dy,$$

hence

$$\|Tf\|_{L^{1,\infty}} = \|\varphi\|_{L^{1,\infty}} \left| \int_{y \in \mathbb{R}^d} \psi(y) f(y)\, dy \right|.$$

Since $\psi \notin L^\infty(\mathbb{R}^d)$, there exists a sequence of measurable sets $\{E_j\}_{j \in \mathbb{N}}$ such that

$$0 < |E_j| < \infty, \quad E_j \subset \{\, y \in \mathbb{R}^d : |\psi(y)| > j \,\}.$$

Define $f_j \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$f_j := \frac{\chi_{E_j}}{|E_j|} \cdot \frac{\overline{\psi}}{|\psi|},$$

then f_j satisfies $\|f_j\|_{L^1} = 1$ and

$$\left| \int_{y \in \mathbb{R}^d} \psi(y) f_j(y)\, dy \right| = \frac{1}{|E_j|} \int_{y \in E_j} |\psi(y)|\, dy \ge j$$

for each j, thus T is not bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. □

For the L^p boundedness, we give two different proofs. The first one is via the interpolation result between $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ boundednesses.

Theorem 3.A ([Bui and Langesen, 2013, Corollary 3])**.** *Let T be a linear operator which is bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ and on $L^2(\mathbb{R}^d)$. Then T is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < 2$ with a bound*

$$\|T\|_{L^p \to L^p} \lesssim_d (p-1)^{-1} \|T\|_{H^1 \to L^1}^{1-\theta} \|T\|_{L^2 \to L^2}^{\theta},$$

where $1/p = 1 - \theta/2$, $0 < \theta < 1$.

Remark 3. The L^p boundedness under the assumption of Theorem 3.A is originally observed by [Fefferman and Stein, 1972, (5.1)], but with a much worse bound. The best estimate known before [Bui and Langesen, 2013] is

$$\|T\|_{L^p \to L^p} \lesssim_d 2^{(p-1)^{-1}} \|T\|_{H^1 \to L^1}^{1-\theta} \|T\|_{L^2 \to L^2}^{\theta},$$

see [Grafakos, 2014b, Theorem 3.4.7]. Note that $O((p-1)^{-1})$ is sharp (consider the Hilbert transform).

Proposition 3.2. *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies $[K]_{H_{**}} < \infty$. Then T is bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ with a bound*

$$\|T\|_{H^1 \to L^1} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_{**}}.$$

Proof of Proposition 3.2. The proof is almost the same as the proof under the classical Hörmander condition by [Coifman and Weiss, 1977]. Let $a \in H^1(\mathbb{R}^d)$ be an atom, that is, there exists a cube such that

$$\operatorname{supp} a \subset Q, \quad \|a\|_{L^\infty} \leq |Q|^{-1}, \quad \int_Q a = 0.$$

Since T is bounded on $L^2(\mathbb{R}^d)$, it is enough to show that

$$\|Ta\|_{L^1} \lesssim_d \|T\|_{L^2 \to L^2} + [K]_{H_{**}}.$$

We decompose $\|Ta\|_{L^1}$ as

$$\|Ta\|_{L^1} = \|Ta\|_{L^1(2Q)} + \|Ta\|_{L^1(\mathbb{R}^d \setminus 2Q)}$$

and prove

$$\|Ta\|_{L^1(2Q)} \leq 2^{d/2} \|T\|_{L^2 \to L^2}, \tag{3.4}$$

$$\|Ta\|_{L^1(\mathbb{R}^d \setminus 2Q)} \leq [K]_{H_{**}}. \tag{3.5}$$

(3.4) By the Hölder inequality and the $L^2(\mathbb{R}^d)$ boundedness of T, we have

$$\begin{aligned}\|Ta\|_{L^1(2Q)} &\le |2Q|^{1/2}\|Ta\|_{L^2}\\ &\le 2^{d/2}|Q|^{1/2}\|T\|_{L^2\to L^2}\|a\|_{L^2(Q)}\\ &\le 2^{d/2}|Q|^{1/2}\|T\|_{L^2\to L^2}|Q|^{1/2}\|a\|_{L^\infty}\\ &\le 2^{d/2}\|T\|_{L^2\to L^2}.\end{aligned}$$

(3.5) Since

$$\begin{aligned}\|Ta\|_{L^1(\mathbb{R}^d\setminus 2Q)} &= \int_{x\in\mathbb{R}^d\setminus 2Q}\left|\int_{y\in Q} K(x,y)a(y)\,dy\right|dx\\ &= \int_{x\in\mathbb{R}^d\setminus 2Q}\left|\int_{y\in Q} K(x,y)a(y)\,dy - m_Q(x)\int_{y\in Q} a(y)\,dy\right|dx\\ &\le \|a\|_{L^\infty}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}|K(x,y)-m_Q(x)|\,dx\,dy\\ &\le \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}|K(x,y)-m_Q(x)|\,dx\,dy\end{aligned}$$

for any collections $\{m_Q\}_Q$, we have $\|Ta\|_{L^1(\mathbb{R}^d\setminus 2Q)} \le [K]_{H_{**}}$. □

Now we are going to give the second proof. We will show that T is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for any $1<p<2$ and use the Marcinkiewicz interpolation theorem ([Marcinkiewicz, 1939]).

Proposition 3.3. *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies $[K]_{H_*}<\infty$. Then T is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for any $1<p<2$ with a bound*

$$\|T\|_{L^p\to L^{p,\infty}} \lesssim_d (p-1)^{-1}(\|T\|_{L^2\to L^2}+[K]_{H_*}).$$

We use Proposition 2.3 and the following L^p variant of the Calderón–Zygmund decomposition.

Proposition 3.A. *Let $1\le p<\infty$, $f\in L^p(\mathbb{R}^d)$ and $\lambda>0$. Then there exists a pairwise disjoint family of dyadic cubes $\{Q_j\}_j$ satisfying*

$$\begin{gathered}\Omega := \bigcup_j Q_j = \{\, x\in\mathbb{R}^d : M^{\mathrm{dyadic}}(|f|^p)(x) > \lambda^p \,\},\\ \lambda^p|\Omega| \le \|f\|_{L^p}^p, \qquad (3.6)\\ |f(x)|\le\lambda \ \ a.e.\ x\in\mathbb{R}^d\setminus\Omega,\\ \lambda^p < \frac{1}{|Q_j|}\int_{Q_j}|f|^p \le 2^d\lambda^p.\end{gathered}$$

Moreover, functions g, b_j and b defined by

$$g := f\chi_{\mathbb{R}^d\setminus\Omega} + \sum_j \frac{\chi_{Q_j}}{|Q|}\int_Q f,$$
$$b_j := \left(f - \frac{1}{|Q|}\int_Q f\right)\chi_{Q_j},$$
$$b := \sum_j b_j,$$

satisfy

$$\|g\|_{L^p} \le \|f\|_{L^p}, \quad \|g\|_{L^\infty} \le 2^{d/p}\lambda, \tag{3.7}$$

$$\operatorname{supp} b_j \subset Q_j, \quad \int_{Q_j} b_j = 0, \quad |Q_j|^{-1/p}\|b_j\|_{L^p} \le 2^{d/p+1}\lambda, \quad \sum_j \|b_j\|_{L^p}^p = \|b\|_{L^p}^p \le 2^p\|f\|_{L^p}^p. \tag{3.8}$$

Proof of Proposition 3.3. Fix $1 < p < 2$, $f \in L^\infty_c(\mathbb{R}^d)$, $\lambda, \alpha > 0$ and form the L^p-Calderón–Zygmund decomposition of f at height $\alpha^{-1}\lambda$ (we choose α later). Since

$$|\{\, x \in \mathbb{R}^d : |Tf(x)| > 2\lambda \,\}| \le |\{\, x \in \mathbb{R}^d : |Tg(x)| > \lambda \,\}| + |\{\, x \in \mathbb{R}^d : |Tb(x)| > \lambda \,\}|,$$

it suffices to estimate the following:

$$\lambda^p|\{\, x \in \mathbb{R}^d : |Tg(x)| > \lambda \,\}|, \tag{3.9}$$
$$\lambda^p|\{\, x \in \mathbb{R}^d : |Tb(x)| > \lambda \,\}|. \tag{3.10}$$

(3.9) Since $1 < p < 2$, (3.7) implies $\|g\|_{L^2}^2 \le 2^{(2-p)d/p}(\alpha^{-1}\lambda)^{2-p}\|f\|_{L^p}^p$. Therefore, using the $L^2(\mathbb{R}^d)$ boundedness of T, it follows that

$$\begin{aligned}\lambda^p|\{\, x \in \mathbb{R}^d : |Tg(x)| > \lambda \,\}| &= \lambda^{-(2-p)}\lambda^2|\{\, x \in \mathbb{R}^d : |Tg(x)| > \lambda \,\}| \\ &\le \lambda^{-(2-p)}\|Tg\|_{L^2}^2 \\ &\le \lambda^{-(2-p)}\|T\|_{L^2\to L^2}^2\|g\|_{L^2}^2 \\ &= \alpha^{-(2-p)}2^{(2-p)d/p}\|T\|_{L^2\to L^2}^2\|f\|_{L^p}^p.\end{aligned}$$

(3.10) We write $2\Omega := \bigcup_j 2Q_j$. Since

$$\{\, x \in \mathbb{R}^d : |Tb(x)| > \lambda \,\} \subset 2\Omega \cup \{\, x \in \mathbb{R}^d \setminus 2\Omega : |Tb(x)| > \lambda \,\},$$

it follows that

$$\begin{aligned}&\lambda^p|\{\, x \in \mathbb{R}^d : |Tb(x)| > \lambda \,\}| \\ &\le \lambda^p|2\Omega| + \lambda^p|\{\, x \in \mathbb{R}^d \setminus 2\Omega : |Tb(x)| > \lambda \,\}| \\ &\underset{(3.6)}{\le} \alpha^p 2^d\|f\|_{L^p}^p + \lambda^{p-1}\|Tb\|_{L^1(\mathbb{R}^d\setminus 2\Omega)}.\end{aligned}$$

We estimate the second term by $[K]_{H_{*,p'}}$. For each j, we have

$$
\begin{aligned}
&\|Tb_j\|_{L^1(\mathbb{R}^d\setminus 2Q_j)}\\
&=\int_{x\in\mathbb{R}^d\setminus 2Q_j}\left|\int_{y\in Q_j}K(x,y)b_j(y)\,dy\right|dx\\
&=\int_{x\in\mathbb{R}^d\setminus 2Q_j}\left|\int_{y\in Q_j}K(x,y)b_j(y)\,dy-\int_{y\in Q_j}\left(\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right)b_j(y)\,dy\right|dx\\
&\le\int_{y\in Q_j}\int_{x\in\mathbb{R}^d\setminus 2Q_j}\left|K(x,y)-\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right|dx\,|b_j(y)|\,dy\\
&\le\left(\int_{y\in Q_j}\left(\int_{x\in\mathbb{R}^d\setminus 2Q_j}\left|K(x,y)-\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right|dx\right)^{p'}dy\right)^{1/p'}\|b_j\|_{L^p}\\
&\le[K]_{H_{*,p'}}|Q_j|^{1/p'}\|b_j\|_{L^p}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\|Tb\|_{L^1(\mathbb{R}^d\setminus 2\Omega)}&\le\sum_j\|Tb_j\|_{L^1(\mathbb{R}^d\setminus 2Q_j)}\\
&\le[K]_{H_{*,p'}}\sum_j|Q_j|^{1/p'}\|b_j\|_{L^p}\\
&\le[K]_{H_{*,p'}}\left(\sum_j|Q_j|\right)^{1/p'}\left(\sum_j\|b_j\|_{L^p}^p\right)^{1/p}\\
&\underset{(3.6),(3.8)}{\le}[K]_{H_{*,p'}}\left(\frac{\|f\|_{L^p}^p}{(\alpha^{-1}\lambda)^p}\right)^{1/p'}\left(2^p\|f\|_{L^p}^p\right)^{1/p}\\
&=2[K]_{H_{*,p'}}\alpha^{p-1}\lambda^{-(p-1)}\|f\|_{L^p}^p.
\end{aligned}
$$

Summing up the estimates of (3.9) and (3.10) above, we have

$$
\|Tf\|_{L^{p,\infty}}^p\le 2^p\alpha^{p-1}(2^{(2-p)d/p}\|T\|_{L^2\to L^2}^2\alpha^{-1}+2[K]_{H_{*,p'}}+2^d\alpha)\|f\|_{L^p}^p.
$$

Now we choose

$$
\alpha=2^{-(p-1)d/p}\|T\|_{L^2\to L^2}
$$

and conclude that

$$
\begin{aligned}
\|T\|_{L^p\to L^{p,\infty}}^p&\le 2^{p+1}(2^{-(p-1)d/p}\|T\|_{L^2\to L^2})^{p-1}(2^{d/p}\|T\|_{L^2\to L^2}+[K]_{H_{*,p'}})\\
&\le 2^{p+1}(2^{d/p}\|T\|_{L^2\to L^2}+p'[K]_{H_*})^p.
\end{aligned}
$$

□

3.2 Case II: non-doubling measures

In this section, we consider a Radon measure μ on $\mathbb{R}^d$ which satisfies the polynomial growth condition: there exists a constant $C_\mu>0$ and $0<n\le d$ such that

$$
\mu(Q(c,\ell))\le C_\mu\ell^n
$$

for any cubes $Q(c,\ell)$. In this case, unlike in the case of the Lebesgue measure, the Hardy space $H^1(\mathbb{R}^d,\mu)$ (see [Mateu et al., 2000]) is not suitable for the Calderón–Zygmund theory since [Verdera, 2000] pointed out that the Cauchy integral is not bounded from $H^1(\mathbb{R}^d,\mu)$ to $L^1(\mathbb{R}^d,\mu)$ in general. After that, [Tolsa, 2001a] developed the atomic block Hardy space $H^1_{\mathrm{atb}}(\mathbb{R}^d,\mu)$ and established the $H^1_{\mathrm{atb}}(\mathbb{R}^d,\mu) \to L^1(\mathbb{R}^d,\mu)$ boundedness of singular integral operators (Theorem 1.E). We will show that our BMO Hörmander condition still works in this setting with a little modification.

Recall that $H^1_{\mathrm{atb}}(\mathbb{R}^d,\mu)$ and $\mathrm{RBMO}(\mathrm{\mathbb{R}^d},\mu)$ is defined as follows.

The coefficient. Let (Q_0,Q) be a pair of cubes such that $Q_0 \subset Q$. The coefficient $\delta(Q_0,Q)$ is defined by

$$\delta(Q_0,Q) := \int_{y\in 2Q\setminus Q_0} \frac{1}{|y-c(Q_0)|^n}\,d\mu(y).$$

The atomic block. A function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mu)$ is called an atomic block if there exist a cube Q, a pair of cubes $\{Q_j\}_{j=1}^2$, functions $\{a_j\}_{j=1}^2$ and numbers $\{\lambda_j\}_{j=1}^2$ such that

$$\operatorname{supp} b \subset Q, \quad \operatorname{supp} a_j \subset Q_j, \quad Q_j \subset Q,$$
$$\int_Q b\,d\mu = 0,$$
$$\|a_j\|_{L^\infty(\mu)} \le ((1+\delta(Q_j,Q))\mu(2Q_j))^{-1},$$
$$b = \lambda_1 a_1 + \lambda_2 a_2$$

and write

$$|b|_{H^1_{\mathrm{atb}}(\mu)} := |\lambda_1| + |\lambda_2|.$$

The atomic block Hardy space. The atomic block Hardy space $H^1_{\mathrm{atb}}(\mathbb{R}^d,\mu)$ is defined by

$$H^1_{\mathrm{atb}}(\mathbb{R}^d,\mu) := \left\{ \sum_{j=1}^\infty b_j : b_j \text{ are atomic blocks such that } \sum_{j=1}^\infty |b_j|_{H^1_{\mathrm{atb}}(\mu)} < \infty \right\}$$

and its norm is

$$\|f\|_{H^1_{\mathrm{atb}}(\mu)} := \inf \left\{ \sum_{j=1}^\infty |b_j|_{H^1_{\mathrm{atb}}(\mu)} : b_j \text{ are atomic blocks such that } f = \sum_{j=1}^\infty b_j \right\}.$$

The regularized bounded mean oscillation space. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mu)$ is in $\mathrm{RBMO}(\mathrm{\mathbb{R}^d},\mu)$ if there exists a collection of numbers $\{m_Q\}_Q$ such that

$$\sup_{Q\subset\mathbb{R}^d} \frac{1}{\mu(2Q)} \int_{y\in Q} |f(y)-m_Q|\,d\mu(y) + \sup_{Q_0\subset Q\subset\mathbb{R}^d} \frac{1}{1+\delta(Q_0,Q)} |m_{Q_0}-m_Q| \tag{3.13}$$

is finite, and its norm is defined by $\|f\|_{\mathrm{RBMO}(\mu)} := \inf_{\{m_Q\}_Q} (3.13)$, where supremum $\sup_{Q\subset\mathbb{R}^d}$, $\sup_{Q_0\subset Q\subset\mathbb{R}^d}$ and infimum $\inf_{\{m_Q\}_Q}$ are taken over all cubes Q with

$\mu(Q) > 0$, all pairs of cubes (Q_0, Q) such that $Q_0 \subset Q$, and all collections $\{m_Q\}_Q$, respectively.

It is known that these spaces $H^1_{\rm atb}(\mathbb{R}^d, \mu)$ and $\mathrm{RBMO}(\mathrm{R^d}, \mu)$ satisfy properties analogous to those of usual $H^1(\mathbb{R}^d)$ and $\mathrm{BMO}(\mathbb{R}^d)$ with the Lebesgue measure, such as the John–Nirenberg inequality, the $H^1_{\rm atb}(\mathbb{R}^d, \mu)$-$\mathrm{RBMO}(\mathrm{R^d}, \mu)$ duality, interpolation inequalities, $T1$ and Tb theorems (see [Bui and Duong, 2011], [Hu et al., 2013], [Hytönen, 2010], [Nazarov et al., 2003], [Tolsa, 2001a], [Tolsa, 2014]). Now we introduce an RBMO Hörmander condition: there exists a collection of functions $\{m_Q\}_Q$ such that

$$\begin{aligned} &\sup_{Q\subset\mathbb{R}^d} \frac{1}{\mu(2Q)} \int_{y\in Q} \int_{x\in\mathbb{R}^d\setminus 2Q} |K(x,y) - m_Q(x)|\, d\mu(x)\, d\mu(y) \\ &\quad + \sup_{Q_0\subset Q\subset\mathbb{R}^d} \frac{1}{1+\delta(Q_0,Q)} \int_{x\in\mathbb{R}^d\setminus 2Q} |m_{Q_0}(x) - m_Q(x)|\, d\mu(x) \end{aligned} \tag{3.14}$$

is finite. We write $[K]_{H_{**}} := \inf_{\{m_Q\}_Q} (3.14)$. Note that we can easily see $[K]_{H_{**}} \le 2[K]_{H_\infty}$. We are going to prove the non-doubling version of Proposition 3.2.

Theorem 3.4. *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d, \mu)$ and K satisfies the* RBMO *Hörmander condition $[K]_{H_{**}} < \infty$ and $|K(x,y)| \le A|x-y|^{-n}$ for some constant $A > 0$. Then T is bounded from $H^1_{\rm atb}(\mathbb{R}^d, \mu)$ to $L^1(\mathbb{R}^d, \mu)$ with a bound*

$$\|T\|_{H^1_{\rm atb}(\mu)\to L^1(\mu)} \lesssim_\mu \|T\|_{L^2(\mu)\to L^2(\mu)} + [K]_{H_{**}} + A.$$

Proof of Theorem 3.4. Let $b = \sum_{j=1}^2 \lambda_j a_j \in H^1_{\rm atb}(\mathbb{R}^d, \mu)$ be an atomic block. Since T is bounded on $L^2(\mathbb{R}^d, \mu)$, it is enough to show that

$$\|Tb\|_{L^1(\mu)} \lesssim_\mu (\|T\|_{L^2(\mu)\to L^2(\mu)} + [K]_{H_{**}} + A)|b|_{H^1_{\rm atb}(\mu)}.$$

We decompose $\|Tb\|_{L^1(\mu)}$ as

$$\|Tb\|_{L^1(\mu)} \le \sum_{j=1}^2 |\lambda_j| (\|Ta_j\|_{L^1(2Q_j,\mu)} + \|Ta_j\|_{L^1(2Q\setminus 2Q_j,\mu)}) + \|Tb\|_{L^1(\mathbb{R}^d\setminus 2Q,\mu)}$$

and prove

$$\|Ta_j\|_{L^1(2Q_j,\mu)} \le \|T\|_{L^2(\mu)\to L^2(\mu)}, \tag{3.15}$$

$$\|Ta_j\|_{L^1(2Q\setminus 2Q_j,\mu)} \le 2^n A, \tag{3.16}$$

$$\|Tb\|_{L^1(\mathbb{R}^d\setminus 2Q,\mu)} \le [K]_{H_{**}} |b|_{H^1_{\rm atb}(\mu)}. \tag{3.17}$$

(3.15) By the Hölder inequality and the $L^2(\mathbb{R}^d, \mu)$ boundedness of T, we have

$$\begin{aligned} \|Ta_j\|_{L^1(2Q_j,\mu)} &\le \mu(2Q)^{1/2} \|Ta_j\|_{L^2(\mu)} \\ &\le \mu(2Q_j)^{1/2} \|T\|_{L^2(\mu)\to L^2(\mu)} \|a_j\|_{L^2(Q,\mu)} \\ &\le \mu(2Q_j)^{1/2} \|T\|_{L^2(\mu)\to L^2(\mu)} \mu(Q_j)^{1/2} \|a_j\|_{L^\infty(\mu)} \\ &\le \|T\|_{L^2(\mu)\to L^2(\mu)}. \end{aligned}$$

(3.16) For each $x \in 2Q \setminus 2Q_j$, we have a pointwise estimate

$$\begin{aligned}
|Ta_j(x)| &= \left| \int_{y \in Q_j} K(x,y) a_j(y)\, d\mu(y) \right| \\
&\le \int_{y \in Q_j} |K(x,y) a_j(y)|\, d\mu(y) \\
&\le \|a_j\|_{L^\infty(\mu)} \int_{y \in Q_j} \frac{A}{|x-y|^n}\, d\mu(y) \\
&\le \frac{1}{(1+\delta(Q_j,Q))\mu(2Q_j)} \int_{y \in Q_j} \frac{2^n A}{|x-c(Q_j)|^n}\, d\mu(y) \\
&\le \frac{2^n A}{1+\delta(Q_j,Q)} \frac{1}{|x-c(Q_j)|^n}.
\end{aligned}$$

Therefore, we obtain

$$\|Ta_j\|_{L^1(2Q \setminus 2Q_j, \mu)} \le \frac{2^n A}{1+\delta(Q_j,Q)} \delta(Q_j,Q) \le 2^n A.$$

(3.17) Since

$$\begin{aligned}
&\|Tb\|_{L^1(\mathbb{R}^d \setminus 2Q, \mu)} \\
&= \int_{x \in \mathbb{R}^d \setminus 2Q} \left| \int_{y \in Q} K(x,y) b(y)\, d\mu(y) \right| d\mu(x) \\
&= \int_{x \in \mathbb{R}^d \setminus 2Q} \left| \int_{y \in Q} K(x,y) b(y)\, d\mu(y) - m_Q(x) \int_{y \in Q} b(y)\, d\mu(y) \right| d\mu(x) \\
&\le \int_{y \in Q} \int_{x \in \mathbb{R}^d \setminus 2Q} |K(x,y) - m_Q(x)|\, d\mu(x)\, |b(y)|\, d\mu(y) \\
&\le \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^\infty(\mu)} \int_{y \in Q_j} \int_{x \in \mathbb{R}^d \setminus 2Q} |K(x,y) - m_Q(x)|\, d\mu(x)\, d\mu(y) \\
&\le \sum_{j=1}^{2} |\lambda_j| \Bigg(\frac{1}{(1+\delta(Q_j,Q))\mu(2Q_j)} \int_{y \in Q_j} \int_{x \in \mathbb{R}^d \setminus 2Q_j} |K(x,y) - m_{Q_j}(x)|\, d\mu(x)\, d\mu(y) \\
&\qquad + \frac{1}{(1+\delta(Q_j,Q))\mu(2Q_j)} \int_{y \in Q_j} \int_{x \in \mathbb{R}^d \setminus 2Q} |m_{Q_j}(x) - m_Q(x)|\, d\mu(x)\, d\mu(y) \Bigg) \\
&\le \sum_{j=1}^{2} |\lambda_j| \Bigg(\frac{1}{\mu(2Q_j)} \int_{y \in Q_j} \int_{x \in \mathbb{R}^d \setminus 2Q_j} |K(x,y) - m_{Q_j}(x)|\, d\mu(x)\, d\mu(y) \\
&\qquad + \frac{1}{1+\delta(Q_j,Q)} \int_{x \in \mathbb{R}^d \setminus 2Q} |m_{Q_j}(x) - m_Q(x)|\, d\mu(x) \Bigg)
\end{aligned}$$

for any collections $\{m_Q\}_Q$, we have $\|Tb\|_{L^1(\mathbb{R}^d \setminus 2Q)} \le [K]_{H_{**}} |b|_{H^1_{\rm atb}(\mu)}$. □

4 Boundednesses of maximal singular integral operators Part I: sufficient condition

In this section, we prove Theorem IV. At first we prove the $L^p(\mathbb{R}^d) \to L^{p,\infty}(\mathbb{R}^d)$ boundedness.

Theorem 4.1. *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies*

$$[K]_{H_*} = \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|} \int_{y\in Q} \int_{x\in\mathbb{R}^d\setminus 2Q} \left| K(x,y) - \frac{1}{|Q|} \int_{z\in Q} K(x,z)\,dz \right| dx\,dy < \infty,$$
$$[{}^{\top}K]_{H_\infty} = \sup_{Q\subset\mathbb{R}^d} \sup_{x\in Q} \int_{y\in\mathbb{R}^d\setminus 2Q} |K(x,y) - K(c(Q),y)\,|\,dy < \infty,$$
$$A = \sup_{Q\subset\mathbb{R}^d} \int_{y\in 2Q\setminus Q} |K(c(Q),y)|\,dy < \infty.$$

Then the maximal operator T_* is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for any $1 < p < \infty$ with a bound

$$\|T_*\|_{L^p\to L^{p,\infty}} \lesssim_d \max\{(p-1)^{-1}, p\}(\|T\|_{L^2\to L^2} + [K]_{H_*} + [{}^{\top}K]_{H_\infty} + A).$$

We begin with two pointwise estimates.

Lemma 4.2. *Let T be a singular integral operator with a kernel K. Suppose that T is bounded on $L^2(\mathbb{R}^d)$ and K satisfies*

$$[{}^{\top}K]_{H_\infty} = \sup_{Q\subset\mathbb{R}^d} \int_{y\in\mathbb{R}^d\setminus 2Q} |K(x,y) - K(c(Q),y)\,|\,dy < \infty.$$

Then, for any $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^d)$, the inequality

$$T_*f(x) \le (2^{d/2}\|T\|_{L^2\to L^2} + [{}^{\top}K]_{H_\infty})\|f\|_{L^\infty} + MTf(x)$$

holds.

Proof of Lemma 4.2. Fix $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and a cube $Q = Q(x_0, \ell)$. Then we have

$$\begin{aligned} T(f\chi_{\mathbb{R}^d\setminus 2Q})f(x_0) &= (T(f\chi_{\mathbb{R}^d\setminus 2Q})f(x_0) - T(f\chi_{\mathbb{R}^d\setminus 2Q})f(x)) + T(f\chi_{\mathbb{R}^d\setminus 2Q})f(x) \\ &= \int_{y\in\mathbb{R}^d\setminus 2Q} (K(x_0,y) - K(x,y))f(y)\,dy + Tf(x) - T(f\chi_{2Q})(x) \end{aligned}$$

for any $x \in Q$, which implies

$$|T(f\chi_{\mathbb{R}^d\setminus 2Q})f(x_0)| \le [{}^{\top}K]_{H_\infty}\|f\|_{L^\infty} + |Tf(x)| + |T(f\chi_{2Q})(x)|.$$

Since

$$\frac{1}{|Q|} \int_{x\in Q} |Tf(x)|\,dx \le MTf(x_0)$$

and

$$\begin{aligned}\frac{1}{|Q|}\int_{x\in Q}|T(f\chi_{2Q})(x)|\,dx &\le \left(\frac{1}{|Q|}\int_{x\in Q}|T(f\chi_{2Q})(x)|^2\,dx\right)^{1/2}\\ &\le \|T\|_{L^2\to L^2}\left(\frac{1}{|Q|}\int_{x\in 2Q}|f(x)|^2\,dx\right)^{1/2}\\ &\le 2^{d/2}\|T\|_{L^2\to L^2}M(|f|^2)(x_0)^{1/2}\\ &\le 2^{d/2}\|T\|_{L^2\to L^2}\|f\|_{L^\infty},\end{aligned}$$

we conclude that

$$T_*f(x_0) \le ([{}^\top K]_{H_\infty} + 2^{d/2}\|T\|_{L^2\to L^2})\|f\|_{L^\infty} + MTf(x_0)$$

holds. □

Lemma 4.3. *Let T be a singular integral operator with a kernel K, $f \in L^\infty_{c,0}(\mathbb{R}^d)$ and P be a cube such that* $\operatorname{supp} f \subset P$*. Then, for any $x \in \mathbb{R}^d \setminus 5P$ and cubes Q centered at x, the inequality*

$$\begin{aligned}|T(f\chi_{\mathbb{R}^d\setminus 2Q})(x)| \le &\int_{y\in P}\left|K(x,y) - \frac{1}{|P|}\int_{z\in P}K(x,z)\,dz\right||f(y)|\,dy\\ &+ \left(\frac{1}{|P|}\int_P|f|\right)\int_{y\in P}\left|K(x,y) - \frac{1}{|P|}\int_{z\in P}K(x,z)\,dz\right||f(y)|\,dy\\ &+ \left(\frac{1}{|P|}\int_P|f|\right)\int_{y\in P\cap(3Q\setminus Q)}|K(x,y)|\,dy.\end{aligned}$$

holds.

Proof of Lemma 4.3. Fix $x \in \mathbb{R}^d \setminus 5P$ and a cube Q centered at x. To prove the inequality, we will consider three cases:

(4.1) $P \subset 2Q$,

(4.2) $P \cap \partial(2Q) \neq \emptyset$,

(4.3) $P \subset \mathbb{R}^d \setminus 2Q$,

where $\partial(2Q)$ denotes the boundary of $2Q$.

Case (4.1) and (4.3) are easy. In the case (4.1), $(\mathbb{R}^d \setminus 2Q) \cap P = \emptyset$ implies that

$$T(f\chi_{\mathbb{R}^d\setminus 2Q})(x) = 0.$$

In the case (4.3), since $(\mathbb{R}^d \setminus 2Q) \cap P = P$, we have

$$\begin{aligned}T(f\chi_{\mathbb{R}^d\setminus 2Q})(x) &= Tf(x)\\ &= \int_{y\in P}K(x,y)f(y)\,dy\\ &= \int_{y\in P}\left(K(x,y) - \frac{1}{|P|}\int_{z\in P}K(x,z)\,dz\right)f(y)\,dy.\end{aligned}$$

Therefore, in both cases of (4.1) and (4.3), the desired inequality holds.

Now consider the case (4.2). Let define $m \in \mathbb{C}$ and $\tilde{f} \in L^\infty(\mathbb{R}^d)$ by

$$m := \frac{1}{|P|} \int_{y \in P} f(y) \chi_{\mathbb{R}^d \setminus 2Q}(y)\, dy,$$
$$\tilde{f}(y) := f(y)\chi_{\mathbb{R}^d \setminus 2Q}(y) - m\chi_P(y).$$

Then we have $\operatorname{supp} \tilde{f} \subset P$ and $\int_P f = 0$, thus

$$\begin{aligned} T(f\chi_{\mathbb{R}^d\setminus 2Q})(x) &= T(\tilde{f})(x) + mT\chi_P(x) \\ &= \int_{y\in P} \left(K(x,y) - \frac{1}{|P|}\int_{z\in P} K(x,z)\, dz \right) \tilde{f}(y)\, dy + m \int_{y \in P} K(x,y)\, dy. \end{aligned}$$

Since

$$|\tilde{f}(y)| \le |f(y)| + |m| \le |f(y)| + \frac{1}{|P|}\int_{y\in P} |f|,$$

we obtain

$$\begin{aligned} |T(f\chi_{\mathbb{R}^d\setminus 2Q})(x)| \le &\int_{y\in P} \left| K(x,y) - \frac{1}{|P|}\int_{z\in P} K(x,z)\, dz \right| |f(y)|\, dy \\ &+ \left(\frac{1}{|P|}\int_P |f| \right) \int_{y\in P} \left| K(x,y) - \frac{1}{|P|}\int_{z\in P} K(x,z)\, dz \right| dy \\ &+ \left(\frac{1}{|P|}\int_P |f| \right) \int_{y\in P} |K(x,y)|\, dy. \end{aligned}$$

We need to show $P \subset 3Q \setminus Q$ to complete the proof. Let $a \in P \cap \partial(2Q)$ (it exists by the assumption (4.2)). Then we have

$$|||a - c(P)||| \underset{a\in P}{\le} \ell(P), \quad |||a - x||| \underset{a\in\partial(2Q)}{=} 2\ell(Q), \quad |||x - c(P)||| \underset{x\in\mathbb{R}^d\setminus 5P}{\ge} 5\ell(P),$$

which implies that

$$2l(Q) = |||a - x||| \ge |||x - c(P)||| - |||a - c(P)||| \ge 5\ell(P) - \ell(P) = 4\ell(P).$$

Therefore, for any $y \in P$, we obtain

$$\begin{aligned} |||y - x||| &\le |||y - c(P)||| + |||c(P) - a||| + |||a - x||| \le \ell(P) + \ell(P) + \ell(Q) \le 3\ell(Q), \\ |||y - x||| &\ge |||x - a||| - |||a - c(P)||| - |||c(P) - y||| \ge 2\ell(Q) - \ell(P) - \ell(P) \ge l(Q), \end{aligned}$$

which means $P \subset 3Q \setminus Q$. □

Proof of Theorem 4.1. Fix $1 < p < \infty$, $f \in L^\infty_c(\mathbb{R}^d)$, $\alpha, \lambda > 0$ and form the L^p-Calderón–Zygmund decomposition of f at height $\alpha^{-1}\lambda$ (we choose α later). We write

$$B_T := 2^{d/2}\|T\|_{L^2\to L^2} + [K]_{H_{*,p'}} + [{}^\top K]_{H_\infty} + A$$

for simplicity and assume $\alpha > 2^{d/p+1}B_T$. It suffices to estimate the following:

$$\lambda^p |\{ x \in \mathbb{R}^d : |T_* g(x)| > \lambda \}|, \tag{4.4}$$
$$\lambda^p |\{ x \in \mathbb{R}^d : |T_* b(x)| > \lambda \}|. \tag{4.5}$$

(4.4) Since we have

$$\|T\|_{L^p\to L^p} \lesssim_d B_T$$

for any $1 < p < 2$ thanks to Theorem III and it is known that the Hardy–Littlewood maximal operator M satisfies $\|M\|_{L^p\to L^{p,\infty}} \le 2\cdot 3^{d/p}$, we get

$$\|MT\|_{L^p\to L^{p,\infty}} \le C_d B_T$$

for some dimensional constant $C_d > 0$. Therefore, using Lemma 4.2, we obtain

$$\begin{aligned}
&\lambda^p|\{\, x\in\mathbb{R}^d : T_*g(x) > \lambda\,\}| \\
&\underset{\text{Lemma 4.2}}{\le} \lambda^p|\{\, x\in\mathbb{R}^d : B_T\|g\|_{L^\infty} + MTg(x) > \lambda\,\}| \\
&\underset{(3.7)}{\le} \lambda^p|\{\, x\in\mathbb{R}^d : 2^{d/p+1}B_T\alpha^{-1}\lambda + MTg(x) > \lambda\,\}| \\
&= \lambda^p|\{\, x\in\mathbb{R}^d : MTg(x) > (1-2^{d/p+1}B_T\alpha^{-1})\lambda\,\}| \\
&\le (1-2^{d/p+1}B_T\alpha^{-1})^{-p}\|MT\|^p_{L^p\to L^{p,\infty}}\|g\|^p_{L^p} \\
&\underset{(3.7)}{\le} \left(\frac{C_dB_T}{1-2^{d/p+1}B_T\alpha^{-1}}\right)^p\|f\|^p_{L^p}.
\end{aligned}$$

(4.5) We write $5\Omega \coloneqq \bigcup_j 5Q_j$ and begin with pointwise estimate of T_*b on $\mathbb{R}^d\setminus 5\Omega$. Fix $x\in\mathbb{R}^d\setminus 5\Omega$ and a cube Q centered at x. Using Lemma 4.3, we have

$$\begin{aligned}
|T(b\chi_{\mathbb{R}^d\setminus 2Q})(x)| &\le \sum_j |T(b_j\chi_{\mathbb{R}^d\setminus 2Q})(x)| \\
&\underset{\text{Lemma 4.3}}{\le} \sum_j \int_{y\in Q_j}\left|K(x,y)-\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right||b_j(y)|\,dy \\
&\quad + \sum_j\left(\frac{1}{|Q_j|}\int_{Q_j}|b_j|\right)\int_{y\in Q_j}\left|K(x,y)-\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right|dy \\
&\quad + \sum_j\left(\frac{1}{|Q_j|}\int_{Q_j}|b_j|\right)\int_{y\in Q_j\cap(3Q\setminus Q)}|K(x,y)|\,dy \\
&\underset{(3.8)}{\le} \sum_j \int_{y\in Q_j}\left|K(x,y)-\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right||b_j(y)|\,dy \\
&\quad + 2^{d/p+1}\alpha^{-1}\lambda\sum_j\int_{y\in Q_j}\left|K(x,y)-\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right|dy \\
&\quad + 2^{d/p+1}A\alpha^{-1}\lambda.
\end{aligned}$$

Now letting

$$\begin{aligned}
F_1(x) &\coloneqq \sum_j \int_{y\in Q_j}\left|K(x,y)-\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right||b_j(y)|\,dy, \\
F_2(x) &\coloneqq \sum_j \int_{y\in Q_j}\left|K(x,y)-\frac{1}{|Q_j|}\int_{z\in Q_j}K(x,z)\,dz\right|dy.
\end{aligned}$$

Since

$$T_*b(x) \le F_1(x) + 2^{d/p+1}\alpha^{-1}\lambda F_2(x) + 2^{d/p+1}B_T\alpha^{-1}\lambda$$

holds for $x \in \mathbb{R}^d \setminus 5\Omega$, it implies

$$\begin{aligned}
&\lambda^p|\{\, x \in \mathbb{R}^d \setminus 5\Omega : T_*b(x) > \lambda \,\}| \\
&\le \lambda^p|\{\, x \in \mathbb{R}^d \setminus 5\Omega : F_1(x) + 2^{d/p+1}\alpha^{-1}\lambda F_2(x) + 2^{d/p+1}B_T\alpha^{-1}\lambda > \lambda \,\}| \\
&= \lambda^p|\{\, x \in \mathbb{R}^d \setminus 5\Omega : F_1(x) + 2^{d/p+1}\alpha^{-1}\lambda F_2(x) > (1 - 2^{d/p+1}B_T\alpha^{-1})\lambda \,\}| \\
&\le \frac{\lambda^{p-1}}{1 - 2^{d/p+1}B_T\alpha^{-1}} \int_{x\in\mathbb{R}^d\setminus 5\Omega} F_1(x)\,dx + \frac{2^{d/p+1}\alpha^{-1}\lambda^p}{1 - 2^{d/p+1}B_T\alpha^{-1}} \int_{x\in\mathbb{R}^d\setminus 5\Omega} F_2(x)\,dx.
\end{aligned}$$

We compute integrals of F_1 and F_2 as follows:

$$\begin{aligned}
&\int_{x\in\mathbb{R}^d\setminus 5\Omega} F_1(x)\,dx \\
&= \int_{x\in\mathbb{R}^d\setminus 5\Omega} \sum_j \int_{y\in Q_j} \left| K(x,y) - \frac{1}{|Q_j|}\int_{z\in Q_j} K(x,z)\,dz \right| |b_j(y)|\,dy\,dx \\
&\le \sum_j \int_{y\in Q_j} \left(\int_{x\in\mathbb{R}^d\setminus 5Q_j} \left| K(x,y) - \frac{1}{|Q_j|}\int_{z\in Q_j} K(x,z)\,dz \right| dx \right) |b_j(y)|\,dy \\
&\le \sum_j \left(\int_{y\in Q_j} \left(\int_{x\in\mathbb{R}^d\setminus 2Q_j} \left| K(x,y) - \frac{1}{|Q_j|}\int_{z\in Q_j} K(x,z)\,dz \right| dx \right)^{p'} dy \right)^{1/p'} \|b_j\|_{L^p} \\
&\sum_j |Q_j|^{1/p'} [K]_{H_{*,p'}} \|b_j\|_{L^p} \\
&\le [K]_{H_{*,p'}} \left(\sum_j |Q_j| \right)^{1/p'} \left(\sum_j \|b_j\|_{L^p}^p \right)^{1/p} \\
&\underset{(3.6),(3.8)}{\le} [K]_{H_{*,p'}} \left(\frac{\|f\|_{L^p}^p}{(\alpha^{-1}\lambda)^p} \right)^{1/p'} \left(2^p \|f\|_{L^p}^p\right)^{1/p} \\
&\le 2B_T\alpha^{p-1}\lambda^{-(p-1)}\|f\|_{L^p}^p.
\end{aligned}$$

$$\begin{aligned}
&\int_{x\in\mathbb{R}^d\setminus 5\Omega} F_2(x)\,dx \\
&= \int_{x\in\mathbb{R}^d\setminus 5\Omega} \sum_j \int_{y\in Q_j} \left| K(x,y) - \frac{1}{|Q_j|}\int_{z\in Q_j} K(x,z)\,dz \right| dy\,dx \\
&\le \sum_j \int_{y\in Q_j} \int_{x\in\mathbb{R}^d\setminus 5Q_j} \left| K(x,y) - \frac{1}{|Q_j|}\int_{z\in Q_j} K(x,z)\,dz \right| dx\,dy \\
&\le \sum_j |Q_j| [K]_{H_*} \\
&= |\Omega| [K]_{H_*} \\
&\underset{(3.6)}{\le} (\alpha^{-1}\lambda)^{-p} B_T \|f\|_{L^p}^p.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}\lambda^p|\{x\in\mathbb{R}^d : |T_*b(x)|>\lambda\}| &\le \lambda^p|5\Omega| + \lambda^p|\{x\in\mathbb{R}^d\setminus 5\Omega : |T_*b(x)|>\lambda\}| \\ &\le \left(5^d\alpha^p + \frac{\alpha^{p-1}2^{d/p+2}B_T}{1-2^{d/p+1}B_T\alpha^{-1}}\right)\|f\|_{L^p}^p\end{aligned}$$

Summing up the estimates of (4.4) and (4.5) above, we have

$$\|T_*f\|_{L^{p,\infty}}^p \le 2^p\left(\left(\frac{C_dB_T}{1-2^{d/p+1}B_T\alpha^{-1}}\right)^p + \frac{\alpha^{p-1}2^{d/p+2}B_T}{1-2^{d/p+1}B_T\alpha^{-1}} + 5^d\alpha^p\right)\|f\|_{L^p}^p.$$

Now we choose $\alpha = 2^{d/p+2}B_T$, which implies

$$\begin{aligned}\|T\|_{L^p\to L^{p,\infty}} &\le 2\Big((2C_dB_T)^p + 2(2^{d/p+2}B_T)^p + 5^d(2^{d/p+2}B_T)^p\Big)^{1/p} \\ &\lesssim_d B_T.\end{aligned}$$

□

Next we prove the $L^\infty_c(\mathbb{R}^d)\to \mathrm{BMO}(\mathbb{R}^d)$ boundedness.

Theorem 4.4. *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that T_* is bounded from $L^2(\mathbb{R}^d)$ to $L^{2,\infty}(\mathbb{R}^d)$ and K satisfies*

$$[{}^\top K]_{H_*} = \sup_{Q\subset\mathbb{R}^d}\frac{1}{|Q|}\int_{x\in Q}\int_{y\in\mathbb{R}^d\setminus 2Q}\left|K(x,y) - \frac{1}{|Q|}\int_{z\in Q}K(z,y)\,dz\right|dy\,dx < \infty,$$

$$A = \sup_{Q\subset\mathbb{R}^d}\int_{y\in 2Q\setminus Q}|K(c(Q),y)|\,dy < \infty.$$

Then T_* is bounded from $L^\infty_c(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$ with a bound

$$\|T_*\|_{L^\infty_c\to \mathrm{BMO}} \lesssim_d \|T_*\|_{L^2\to L^{2,\infty}} + [{}^\top K]_{H_*} + A.$$

Proof of Theorem 4.4. Let $f\in L^\infty_c(\mathbb{R}^d)$ and $P\subset\mathbb{R}^d$ be a cube. We decompose f as $f = f\chi_{5P} + f\chi_{\mathbb{R}^d\setminus 5P} =: f_1 + f_2$. Using $|T_*f(x) - T_*f_2(x)| \le |T_*f_1(x)|$, we have

$$\begin{aligned}&\frac{1}{|P|}\int_{x\in P}\left|T_*f(x) - \sup_{\varepsilon>0}\left|\frac{1}{|P|}\int_{z\in P}T_\varepsilon f_2(z)\,dz\right|\right|dx \\ &\le \frac{1}{|P|}\int_{x\in P}|T_*f_1(x)|\,dx &(4.6)\\ &\quad+\frac{1}{|P|}\int_{x\in P}\left|T_*f_2(x) - \sup_{\varepsilon>0}\left|\frac{1}{|P|}\int_{z\in P}T_\varepsilon f_2(z)\,dz\right|\right|dx, &(4.7)\end{aligned}$$

Therefore, it is enough to estimate (4.6) and (4.7).

(4.6) Since T_* is bounded from $L^2(\mathbb{R}^d)$ to $L^{2,\infty}(\mathbb{R}^d)$, using the Kolmogorov inequality (see [Grafakos, 2014a, Exercise 1.1.11.(a)]), we have

$$\begin{aligned}\frac{1}{|P|}\int_{x\in P}|T_*f_1(x)|\,dx &\le 2\cdot 5^d|5P|^{-1/2}\|T_*f_1\|_{L^{2,\infty}} \\ &\le 2\cdot 5^d|5P|^{-1/2}\|T_*\|_{L^2\to L^{2,\infty}}\|f_1\|_{L^2} \\ &\le 2\cdot 5^d\|T_*\|_{L^2\to L^{2,\infty}}\|f\|_{L^\infty}.\end{aligned}$$

(4.7) Since T is linear, using

$$\begin{aligned}\chi_{\mathbb{R}^d\setminus Q(z,\varepsilon)} &= \chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)} - (\chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)} - \chi_{\mathbb{R}^d\setminus Q(z,\varepsilon)}) \\ &= \chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)} + (\chi_{Q(x,\varepsilon)} - \chi_{Q(z,\varepsilon)}) \\ &= \chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)} + \chi_{Q(x,\varepsilon)\setminus Q(z,\varepsilon)} - \chi_{Q(z,\varepsilon)\setminus Q(x,\varepsilon)},\end{aligned}$$

we have

$$\begin{aligned}&\left|T_* f_2(x) - \sup_{\varepsilon>0}\left|\frac{1}{|P|}\int_{z\in P} T_\varepsilon f_2(z)\,dz\right|\right| \\ &\le \sup_{\varepsilon>0}\left|T(f_2\chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)})(x) - \frac{1}{|P|}\int_{z\in P} T(f_2\chi_{\mathbb{R}^d\setminus Q(z,\varepsilon)})(z)\,dz\right| \\ &\le \sup_{\varepsilon>0}\left|T(f_2\chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)})(x) - \frac{1}{|P|}\int_{z\in P} T(f_2\chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)})(z)\,dz\right| \\ &\quad + \sup_{\varepsilon>0}\frac{1}{|P|}\int_{z\in P}|T(f_2\chi_{Q(x,\varepsilon)\setminus Q(z,\varepsilon)})(z)|\,dz + \sup_{\varepsilon>0}\frac{1}{|P|}\int_{z\in P}|T(f_2\chi_{Q(z,\varepsilon)\setminus Q(x,\varepsilon)}(z)|\,dz.\end{aligned}$$

Now we use the integral representation of T and obtain

$$\begin{aligned}&\left|T(f_2\chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)})f(x) - \frac{1}{|P|}\int_{z\in P} T(f_2\chi_{\mathbb{R}^d\setminus Q(x,\varepsilon)})(z)\,dz\right| \\ &= \left|\int_{y\in\mathbb{R}^d\setminus Q(x,\varepsilon)}\left(K(x,y) - \frac{1}{|P|}\int_{z\in P} K(z,y)\,dz\right) f_2(y)\,dy\right| \\ &\le \|f\|_{L^\infty}\int_{y\in\mathbb{R}^d\setminus 5P}\left|K(x,y) - \frac{1}{|P|}\int_{z\in P}K(z,y)\,dz\right|dy,\end{aligned}$$

$$|T(f_2\chi_{Q(x,\varepsilon)\setminus Q(z,\varepsilon)})(z)| \le \|f\|_{L^\infty}\int_{y\in(Q(x,\varepsilon)\setminus 5P)\setminus Q(z,\varepsilon)}|K(z,y)|\,dy,$$

$$|T(f_2\chi_{Q(z,\varepsilon)\setminus Q(x,\varepsilon)}(z)| \le \|f\|_{L^\infty}\int_{y\in(Q(z,\varepsilon)\setminus 5P)\setminus Q(x,\varepsilon)}|K(z,y)|\,dy.$$

Since it is obvious that

$$\frac{1}{|P|}\int_{x\in P}\left(\|f\|_{L^\infty}\int_{y\in\mathbb{R}^d\setminus 5P}\left|K(x,y) - \frac{1}{|P|}\int_{z\in P}K(z,y)\,dz\right|dy\right)dx \le [{}^\top K]_{H_*}\|f\|_{L^\infty},$$

we consider the second and third one. We show that

$$(Q(x,\varepsilon)\setminus 5P)\setminus Q(z,\varepsilon) \subset Q(z,3\varepsilon/2)\setminus Q(z,\varepsilon), \tag{4.8}$$

$$(Q(z,\varepsilon)\setminus 5P)\setminus Q(x,\varepsilon) \subset Q(z,\varepsilon)\setminus Q(z,\varepsilon/2) \tag{4.9}$$

hold for any $x, z\in P$. Note that (4.8) and (4.9) imply

$$\int_{y\in(Q(x,\varepsilon)\setminus 5P)\setminus Q(z,\varepsilon)}|K(z,y)|\,dy + \int_{y\in(Q(z,\varepsilon)\setminus 5P)\setminus Q(x,\varepsilon)}|K(z,y)|\,dy \le 2A.$$

(4.8) Since it is trivial when $Q(x,\varepsilon)\setminus 5P=\emptyset$, we assume $Q(x,\varepsilon)\setminus 5P\neq\emptyset$. Then there exists $a\in\mathbb{R}^d$ such that

$$|||a-x|||\le\varepsilon,\quad |||a-c(P)|||\ge 5\ell(P).$$

Since $x,z\in P$;

$$|||x-c(P)|||,|||z-c(P)|||\le l(P),$$

we obtain

$$4\ell(P)\le |||a-c(P)|||-|||x-c(P)|||\le |||a-x|||\le\varepsilon.$$

Therefore, for any $y\in Q(x,\varepsilon)$, we get

$$|||y-z|||\le |||y-x|||+|||x-c(P)|||+|||c(P)-z|||\le\varepsilon+\ell(P)+\ell(P)\le 3\varepsilon/2,$$

which means $Q(x,\varepsilon)\subset Q(z,3\varepsilon/2)$ and thus (4.8) holds.

(4.9) By the same argument, it is enough to consider the case $Q(z,\varepsilon)\setminus 5P\neq\emptyset$ and which implies $4\ell(P)\le\varepsilon$. Then, for any $y\in Q(z,\varepsilon/2)$, we get

$$|||y-x|||\le |||y-z|||+|||z-c(P)|||+|||c(P)-x|||\le\varepsilon/2+\ell(P)+\ell(P)\le\varepsilon,$$

which means $Q(z,\varepsilon/2)\subset Q(x,\varepsilon)$ and thus (4.9) holds.

Summing up estimates above, we have

$$\frac{1}{|P|}\int_{x\in P}\left|T_*f(x)-\sup_{\varepsilon>0}\left|\frac{1}{|P|}\int_{z\in P}T_\varepsilon f_2(z)\,dz\right|\right|dx\le (2\cdot 5^d\|T_*\|_{L^2\to L^{2,\infty}}+[{}^\top K]_{H_*}+2A)\|f\|_{L^\infty}.$$

Using Proposition 2.A gives us the desired result. □

Before finishing this section, we prove a part of Theorem V.

Theorem 4.5. *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that T_* is bounded from $L^2(\mathbb{R}^d)$ to $L^{2,\infty}(\mathbb{R}^d)$ and K satisfies*

$$[K]_{H_*}=\sup_{Q\subset\mathbb{R}^d}\frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y)-\frac{1}{|Q|}\int_{z\in Q}K(x,z)\,dz\right|dx\,dy<\infty,$$
$$A=\sup_{Q\subset\mathbb{R}^d}\int_{y\in 2Q\setminus Q}|K(c(Q),y)|\,dy<\infty.$$

Then T_* is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for any $1<p\le 2$ with a bound

$$\|T\|_{L^p\to L^{p,\infty}}\lesssim_d (p-1)^{-1}(\|T_*\|_{L^2\to L^{2,\infty}}+[K]_{H_*}+A).$$

Proof of Theorem 4.5. Fix $1<p<2$, $f\in L^\infty_{\mathrm{c}}(\mathbb{R}^d)$, $\alpha,\lambda>0$ and form the L^p-Calderón–Zygmund decomposition of f at height $\alpha^{-1}\lambda$ (we choose α later). We write

$$B_T\coloneqq\|T_*\|_{L^2\to L^{2,\infty}}+[K]_{H_{*,p'}}+A$$

for simplicity and assume $\alpha > 2^{d/p+1}B_T$. By the proof of Theorem 4.1, we already know that

$$\lambda^p|\{ x \in \mathbb{R}^d : |T_* b(x)| > \lambda \}| \le \left(\frac{\alpha^{p-1}2^{d/p+2}B_T}{1-2^{d/p+1}B_T\alpha^{-1}} + 5^d\alpha^p \right) \|f\|_{L^p}^p.$$

Also, by the same argument as in the proof of Theorem 3.3, we obtain

$$\lambda^p|\{ x \in \mathbb{R}^d : |T_* g(x)| > \lambda \}| \le \alpha^{-(2-p)}2^{(2-p)d/p}B_T^2\|f\|_{L^p}^p.$$

Summing up the estimates above, we have

$$\|T_*f\|_{L^{p,\infty}}^p \le 2^p \left(\alpha^{-(2-p)}2^{(2-p)d/p}B_T^2 + \frac{\alpha^{p-1}2^{d/p+2}B_T}{1-2^{d/p+1}B_T\alpha^{-1}} + 5^d\alpha^p \right) \|f\|_{L^p}^p.$$

Now we choose $\alpha = 2^{d/p+2}B_T$ and conclude that

$$\begin{aligned} \|T\|_{L^p \to L^{p,\infty}} &\le 2\Big(2^{-2(2-p)}B_T^p + 2(2^{d/p+2}B_T)^p + 5^d(2^{d/p+2}B_T)^p\Big)^{1/p} \\ &\lesssim_d B_T. \end{aligned}$$

□

Theorem 4.6. *Let T_* be a sublinear operator and consider following statements:*

(1.14) *T_* is bounded from $L^2(\mathbb{R}^d)$ to $L^{2,\infty}(\mathbb{R}^d)$.*

(1.17) *There exist $0 < p < \infty$ and $B > 0$ such that $\|T_*f\|_{L^{p,\infty}(Q)} \le B|Q|^{1/p}\|f\|_{L^\infty}$ holds for any cubes $Q \subset \mathbb{R}^d$ and $f \in L^\infty(Q)$.*

(4.10) *There exist $0 < p < \infty$ and $B > 0$ such that $\|T_*f\|_{L^p(Q)} \le B|Q|^{1/p}\|f\|_{L^\infty}$ holds for any cubes $Q \subset \mathbb{R}^d$ and $f \in L^\infty(Q)$.*

Then we have (1.14) $\Rightarrow$ (1.17) $\Leftrightarrow$ (4.10).

Proof of Theorem 4.6. (1.14) $\Rightarrow$ (1.17) and (4.10) $\Rightarrow$ (1.17) are immediate: consider $p = 2$ and use $\|T_*f\|_{L^{p,\infty}(Q)} \le \|T_*f\|_{L^p(Q)}$, respectively. (1.17) $\Rightarrow$ (4.10) follows from the Kolmogorov inequality (which we used in the Proof of Theorem 4.4):

$$\|T_*f\|_{L^q(Q)}^q \le \frac{p}{p-q}|Q|^{1-q/p}\|T_*f\|_{L^{p,\infty}(Q)}^q \le \frac{p}{p-q}B^q|Q|\|f\|_{L^\infty}^q$$

holds for any $0 < q < p$. □

Theorem 4.7. *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that there exist $0 < p < \infty$ and $B > 0$ such that $\|T_*f\|_{L^p(Q)} \le B|Q|^{1/p}\|f\|_{L^\infty}$ holds for any cubes $Q \subset \mathbb{R}^d$ and $f \in L^\infty(Q)$, and K satisfies*

$$[{}^\top K]_{H_*} = \sup_{Q \subset \mathbb{R}^d} \frac{1}{|Q|} \int_{x \in Q} \int_{y \in \mathbb{R}^d \setminus 2Q} \left| K(x,y) - \frac{1}{|Q|}\int_{z \in Q} K(z,y)\, dz \right| dy\, dx < \infty,$$

$$A = \sup_{Q \subset \mathbb{R}^d} \int_{y \in 2Q \setminus Q} |K(c(Q),y)|\, dy < \infty.$$

Then T_* is bounded from $L^\infty_c(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$ with a bound

$$\|T_*\|_{L^\infty_c \to \mathrm{BMO}} \lesssim_{d,p} B + [{}^\top K]_{H_*} + A.$$

Proof of Theorem 4.7. At first we consider the case $1 \le p < \infty$. In this case, the proof is almost identical to that of Theorem 4.4. The only difference is we use $\|T_* f\|_{L^p(5P)} \le B|5P|^{1/p}\|f\|_{L^\infty}$ for (4.6). Now we consider the case $0 < p < 1$. In this case, we have

$$\begin{aligned}
&\frac{1}{|P|}\int_{x\in P}\left|T_* f(x) - \sup_{\varepsilon>0}\left|\frac{1}{|P|}\int_{z\in P} T_\varepsilon f_2(z)\,dz\right|\right|^p dx \\
&\le \frac{1}{|P|}\int_{x\in P}|T_* f_1(x)|^p\,dx + \frac{1}{|P|}\int_{x\in P}\left|T_* f_2(x) - \sup_{\varepsilon>0}\left|\frac{1}{|P|}\int_{z\in P} T_\varepsilon f_2(z)\,dz\right|\right|^p dx \\
&\le \frac{5^d}{|5P|}\int_{x\in 5P}|T_* f_1(x)|^p\,dx + \left(\frac{1}{|P|}\int_{x\in P}\left|T_* f_2(x) - \sup_{\varepsilon>0}\left|\frac{1}{|P|}\int_{z\in P} T_\varepsilon f_2(z)\,dz\right|\right| dx\right)^p \\
&\le 5^d B^p \|f\|_{L^\infty}^p + ([{}^\top K]_{H_*} + 2A)^p \|f\|_{L^\infty}^p \\
&\lesssim_p (5^{d/p}B + [{}^\top K]_{H_*} + 2A)^p \|f\|_{L^\infty}^p.
\end{aligned}$$

Now we use the John–Strömberg inequality (see [Strömberg, 1979]), which implies

$$\begin{aligned}
\|T_* f\|_{\mathrm{BMO}} &\lesssim_p \sup_{P\subset\mathbb{R}^d}\left(\frac{1}{|P|}\int_{x\in P}\left|T_* f(x) - \sup_{\varepsilon>0}\left|\frac{1}{|P|}\int_{z\in P} T_\varepsilon f_2(z)\,dz\right|\right|^p dx\right)^{1/p} \\
&\lesssim_p (5^{d/p}B + [{}^\top K]_{H_*} + 2A)\|f\|_{L^\infty}.
\end{aligned}$$

□

5 The median and related maximal operators

In this section, we discuss maximal operators $M_{0,s}^{\text{dyadic}}$ and $M_{0,s}^{\#}$ defined by the median. Our aim is the sharp maximal inequality $\|M_{0,s}^{\text{dyadic}} f\|_{L^{p,\infty}} \lesssim_{p,d,s} \|M_{0,s}^{\#} f\|_{L^{p,\infty}}$, which was proved by [Hu et al., 2007] (they did on the so-called homogeneous spaces, which include the Euclidean space with the Lebesgue measure). Our proof is mainly based on [Hu et al., 2007, Section 2] and [Duoandikoetxea, 2000, Section 6.3], but it is slightly refined by using ideas from [Bui and Langesen, 2013] and gives a better result than [Hu et al., 2007].

At first, we define maximal operators $M_{0,s}^{\text{dyadic}}$ and $M_{0,s}^{\#}$.

Definition 5.1. Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be measurable, $Q \subset \mathbb{R}^d$ be a cube and $0 \leq s \leq 1$. We write

$$m_{0,s,Q} f := \inf\{\, \lambda \geq 0 : |\{\, y \in Q : |f(y)| > \lambda \,\}| \leq s|Q| \,\} \tag{5.1}$$

and define maximal operators $M_{0,s}^{\text{dyadic}}$ and $M_{0,s}^{\#}$ by

$$M_{0,s}^{\text{dyadic}} f(x) := \sup_{j\in\mathbb{Z}} m_{0,s,Q_j(x)} f = \sup_{j\in\mathbb{Z}} \inf\{\, \lambda \geq 0 : |\{\, y \in Q_j(x) : |f(y)| > \lambda \,\}| \leq s|Q_j(x)| \,\},$$

$$M_{0,s}^{\#} f(x) := \sup_{Q\ni x} \inf_{c\in C} m_{0,s,Q}(f-c) = \sup_{Q\ni x} \inf_{c\in\mathbb{C}} \inf\{\, \lambda \geq 0 : |\{\, y \in Q : |f(y)-c| > \lambda \,\}| \leq s|Q| \,\}.$$

Remark 4. Some authors (including [Hu et al., 2007]) use a little different definition from (5.1), that is,

$$\widetilde{m_{0,s,Q}} f := \inf\{\, \lambda \geq 0 : |\{\, y \in Q : |f(y)| > \lambda \,\}| < s|Q| \,\}.$$

Note that

$$m_{0,s,Q} f \leq \widetilde{m_{0,s,Q}} f$$

and there exist s, Q, f such that

$$0 = m_{0,s,Q} f < \widetilde{m_{0,s,Q}} f.$$

Some basic properties of the median $m_{0,s,Q} f$ are easily follows from the definition.

Proposition 5.2. *The median satisfies the following:*

$$0 \leq s_2 \leq s_1 \leq 1 \Rightarrow 0 = m_{0,1,Q} f \leq m_{0,s_1,Q} f \leq m_{0,s_2,Q} f \leq m_{0,0,Q} f = \|f\|_{L^\infty(Q)},$$

$$|F(x)| \leq |f_1(x)| + |f_2(x)| \Rightarrow m_{0,s_1+s_2,Q} F \leq m_{0,s_1,Q} f_1 + m_{0,s_2,Q} f_2, \tag{5.2}$$

$$0 < s \leq 1 \Rightarrow m_{0,s,Q} f \leq s^{-1} \frac{1}{|Q|} \int_Q |f|, \tag{5.3}$$

$$0 \leq \lambda < m_{0,s,Q} f \Leftrightarrow |\{\, y \in Q : |f(y)| > \lambda \,\}| > s|Q|. \tag{5.4}$$

Proposition 5.2 is easy and well-known. See [Grafakos, 2014a, Proposition 1.4.5], for example. Notice that (5.3) implies that $M^{\text{dyadic}}_{0,s} f \le s^{-1} M^{\text{dyadic}} f$ and $M^{\#}_{0,s} f \le s^{-1} M^{\#} f$ for $0 < s \le 1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Now we are going to discuss maximal operators $M^{\text{dyadic}}_{0,s}$ and $M^{\#}_{0,s} f$. Proposition 5.2 give us the following.

Proposition 5.3. *Let $F, f_1, f_2 \colon \mathbb{R}^d \to \mathbb{C}$ be measurable and $0 \le s_1, s_2 \le 1$ satisfy $0 \le s_1 + s_2 \le 1$.*

(5.5) *If $|F(x)| \le |f_1(x)| + |f_2(x)|$, then*

$$M^{\text{dyadic}}_{0,s_1+s_2} F(x) \le M^{\text{dyadic}}_{0,s_1} f_1(x) + M^{\text{dyadic}}_{0,s_2} f_2(x).$$

(5.6) *If $F(x) = f_1(x) + f_2(x)$, then*

$$M^{\#}_{0,s_1+s_2} F(x) \le M^{\#}_{0,s_1} f_1(x) + M^{\#}_{0,s_2} f_2(x).$$

(5.7) *If $|F(x) - f_1(x)| \le |f_2(x)|$, then*

$$M^{\#}_{0,s_1+s_2} F(x) \le M^{\#}_{0,s_1} f_1(x) + M^{\text{dyadic}}_{0,s_2} f_2(x).$$

Proof of Proposition 5.3. (5.5) is a direct consequence of (5.2). To see (5.6), note that the pointwise inequality

$$|F(x) - (c_1 + c_2)| \le |f_1(x) - c_1| + |f_2(x) - c_2|$$

holds for any $c_1, c_2 \in \mathbb{C}$, which implies the desired result. (5.7) follows from

$$\begin{aligned} |F(x) - c| &\le |f_1(x) - c| + |F(x) - f_1(x)| \\ &\le |f_1(x) - c| + |f_2(x)|. \end{aligned}$$

□

Note that maximal singular integral operators satisfy $|T_* F(x) - T_* f_1(x)| \le |T_* f_2(x)|$ if $F = f_1 + f_2$, since

$$\begin{aligned} |T_* F(x) - T_* f_1(x)| &= |\sup_{\varepsilon>0} T_\varepsilon F(x) - \sup_{\varepsilon>0} T_\varepsilon f_1(x)| \\ &\le \sup_{\varepsilon>0} |T_\varepsilon F(x) - T_\varepsilon f_1(x)| \\ &= \sup_{\varepsilon>0} |T_\varepsilon f_2(x)| \\ &= |T_* f_2(x)|. \end{aligned}$$

Since $M^{\text{dyadic}}_{0,s} f \le s^{-1} M^{\text{dyadic}} f$, it is obvious that $M^{\text{dyadic}}_{0,s} f$ is also bounded from $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ and on $L^p(\mathbb{R}^d)$ for $1 < p \le \infty$. On the other hand, in fact, the following stronger estimates hold if $s < 1$.

Lemma 5.4. *Let $0 < s < 1$. Then the inequalities*

$$|\{ x \in \mathbb{R}^d : |f(x)| > \lambda \}| \le |\{ x \in \mathbb{R}^d : M_{0,s}^{\text{dyadic}} f(x) > \lambda \}|, \tag{5.8}$$
$$|\{ x \in \mathbb{R}^d : M_{0,s}^{\text{dyadic}} f(x) > \lambda \}| \le s^{-1} |\{ x \in \mathbb{R}^d : |f(x)| > \lambda \}| \tag{5.9}$$

hold.

Proof of Lemma 5.4. Let

$$E_\lambda := \{ x \in \mathbb{R}^d : |f(x)| > \lambda \}.$$

At first we show that

$$\{ x \in \mathbb{R}^d : M_{0,s}^{\text{dyadic}} f(x) > \lambda \} = \{ x \in \mathbb{R}^d : M^{\text{dyadic}}(\chi_{E_\lambda})(x) > s \}. \tag{5.10}$$

It follows from

$$\begin{aligned}
&M_{0,s}^{\text{dyadic}} f(x) > \lambda \\
&\Leftrightarrow \text{There exists } j \in \mathbb{Z} \text{ such that } m_{0,s,Q_j(x)} f > \lambda \\
&\underset{(5.4)}{\Leftrightarrow} \text{There exists } j \in \mathbb{Z} \text{ such that } |\{ y \in Q_j(x) : |f(y)| > \lambda \}| > s|Q_j(x)| \\
&\Leftrightarrow \text{There exists } j \in \mathbb{Z} \text{ such that } \frac{1}{|Q_j(x)|} \int_{y \in Q_j(x)} \chi_{E_\lambda}(y)\, dy > s \\
&\Leftrightarrow M^{\text{dyadic}}(\chi_{E_\lambda})(x) > s.
\end{aligned}$$

To see (5.8), notice that E_λ can be written as

$$E_\lambda = \{ x \in \mathbb{R}^d : \chi_{E_\lambda}(x) > s \},$$

since $0 < s < 1$. Therefore, since $\chi_{E_\lambda}(x) \le M^{\text{dyadic}}(\chi_{E_\lambda})(x)$ for almost every $x \in \mathbb{R}^d$, we obtain

$$\begin{aligned}
|E_\lambda| &= |\{ x \in \mathbb{R}^d : \chi_{E_\lambda}(x) > s \}| \\
&\le |\{ x \in \mathbb{R}^d : M^{\text{dyadic}}(\chi_{E_\lambda})(x) > s \}| \\
&\underset{(5.10)}{=} |\{ x \in \mathbb{R}^d : M_{0,s}^{\text{dyadic}} f(x) > \lambda \}|.
\end{aligned}$$

(5.9) is immediate from $\|M^{\text{dyadic}}\|_{L^1 \to L^{1,\infty}} = 1$:

$$\begin{aligned}
|\{ x \in \mathbb{R}^d : M_{0,s}^{\text{dyadic}} f(x) > \lambda \}| &\underset{(5.10)}{=} |\{ x \in \mathbb{R}^d : M^{\text{dyadic}}(\chi_{E_\lambda})(x) > s \}| \\
&\le s^{-1} |E_\lambda|.
\end{aligned}$$

□

The following Lemma 5.5 is analogous to Proposition 2.A.

Lemma 5.5. *Let $0 < s_2 \le s_1 < 1$, $s_1 + s_2 < 1$ and $f \colon \mathbb{R}^d \to \mathbb{R}_{\ge 0}$, then the inequality*

$$m_{0,s_2,Q}(f - m_{0,s_1,Q} f) \le 2 \inf_{c \in \mathbb{C}} m_{0,s_2,Q}(f - c)$$

holds and it implies that

$$\frac{1}{2} \sup_{Q \ni x} m_{0,s_2,Q}(f - m_{0,s_1,Q} f) \le M_{0,s_2}^{\#} f(x) \le \sup_{Q \ni x} m_{0,s_2,Q}(f - m_{0,s_1,Q} f).$$

Proof of Lemma 5.5. Since f is non-negative, we have

$$|f(x)-c| \geq |f(x)-|c||$$

for any $c \in \mathbb{C}$ and thus

$$\inf_{c\in\mathbb{C}} m_{0,s_2,Q}(f-c) \geq \inf_{c\in\mathbb{C}} m_{0,s_2,Q}(f-|c|) = \inf_{c\geq 0} m_{0,s_2,Q}(f-c).$$

Let fix $c \geq 0$ and show that

$$m_{0,s_2,Q}(f - m_{0,s_1,Q}f) \leq 2m_{0,s_2,Q}(f-c).$$

The triangle inequality

$$|f(x)-m_{0,s_1,Q}f| \leq |f(x)-c| + |m_{0,s_1,Q}f - c|$$

implies

$$m_{0,s_2,Q}(f-m_{0,s_1,Q}f) \leq m_{0,s_2,Q}(f-c) + |m_{0,s_1,Q}f-c|,$$

hence it suffices to prove that

$$|m_{0,s_1,Q}f - c| \leq m_{0,s_2,Q}(f-c).$$

It follows from

$$m_{0,s_1,Q}f \underset{s_1-s_2\geq 0}{\leq} m_{0,s_2,Q}(f-c) + m_{0,s_1-s_2,Q}c = m_{0,s_2,Q}(f-c)+c$$

and

$$c \underset{s_1+s_2<1}{=} m_{0,s_1+s_2,Q}c \leq m_{0,s_2,Q}(f-c) + m_{0,s_1,Q}f.$$

Note that $m_{0,s,Q}c = c$ for $0 \leq s < 1$. □

Lemma 5.6. *Let $0 < s_2 \leq s_1 < 1$, $s_1 + s_2 < 1$, $\alpha > 0$. Then the inequality*

$$\begin{aligned}&|\{\, x \in \mathbb{R}^d : M^{\text{dyadic}}_{0,s_1} f(x) > (\alpha+1)\lambda, M^{\#}_{0,s_2}(|f|)(x) \leq \alpha\lambda/2 \,\}| \\ &\leq 2^d s_1^{-1} s_2 |\{\, x \in \mathbb{R}^d : M^{\text{dyadic}}_{0,s_1} f(x) > \lambda \,\}|\end{aligned}$$

holds.

Proof of Lemma 5.6. Since it is trivial when

$$|\{\, x \in \mathbb{R}^d : M^{\text{dyadic}}_{0,s_1} f(x) > \lambda \,\}| = \infty,$$

we assume

$$|\{\, x \in \mathbb{R}^d : M^{\text{dyadic}}_{0,s_1} f(x) > \lambda \,\}| < \infty.$$

Then there exists a family of pairwise disjoint dyadic cubes $\{Q_j\}_j$ such that

$$\{\, x \in \mathbb{R}^d : M^{\text{dyadic}}_{0,s_1} f(x) > \lambda \,\} = \bigcup_j Q_j,$$

$$m_{0,s_1,Q_j} f > \lambda,$$

$$\forall P \in \mathcal{Q}, (Q_j \subsetneq P \Rightarrow m_{0,s_1,P} f \leq \lambda) \tag{5.11}$$

and it reduces to show that

$$|\{\, x \in Q_j : M^{\mathrm{dyadic}}_{0,s_1} f(x) > (\alpha+1)\lambda, M^{\#}_{0,s_2}(|f|)(x) \le \alpha\lambda/2 \,\}| \le s_1^{-1} s_2 |\widetilde{Q_j}| \tag{5.12}$$

holds for each j, where $\widetilde{Q_j}$ denotes the parent of Q_j. In order to establish (5.12), we prove the following statements:

$$\begin{aligned} &\{\, x \in Q_j : M^{\mathrm{dyadic}}_{0,s_1} f(x) > (\alpha+1)\lambda \,\} \\ &\subset \{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1}((|f| - m_{0,s_1,\widetilde{Q_j}} f)\chi_{Q_j})(x) > \alpha\lambda \,\}, \end{aligned} \tag{5.13}$$

$$\begin{aligned} &\{\, x \in Q_j : M^{\#}_{0,s_2}(|f|)(x) \le \alpha\lambda/2 \,\} \neq \emptyset \\ &\Rightarrow |\{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1}((|f| - m_{0,s_1,\widetilde{Q_j}} f)\chi_{Q_j})(x) > \alpha\lambda \,\}| \le s_1^{-1} s_2 |\widetilde{Q_j}|. \end{aligned} \tag{5.14}$$

(5.13) Fix $x \in Q_j$ such that $M^{\mathrm{dyadic}}_{0,s_1} f(x) > (\alpha+1)\lambda$. Using (5.11), we have

$$\begin{aligned} (\alpha+1)\lambda &< M^{\mathrm{dyadic}}_{0,s_1} f(x) \\ &= \sup_{i\in\mathbb{Z}} m_{0,s_1,Q_i(x)} f \\ &= \max\{\sup_{i\ge j} m_{0,s_1,Q_i(x)} f, \sup_{i<j} m_{0,s_1,Q_i(x)} f\} \\ &\underset{(5.11)}{=} \max\{M^{\mathrm{dyadic}}_{0,s_1}(f\chi_{Q_j})(x), \lambda\}. \end{aligned}$$

Since $\alpha > 0$, we obtain $M^{\mathrm{dyadic}}_{0,s_1}(f\chi_{Q_j})(x) > (\alpha+1)\lambda$. Also (5.11) implies $m_{0,s_1,\widetilde{Q_j}} f \le \lambda$. Therefore, we conclude that

$$M^{\mathrm{dyadic}}_{0,s_1}((|f| - m_{0,s_1,\widetilde{Q_j}} f)\chi_{Q_j})(x) \ge M^{\mathrm{dyadic}}_{0,s_1}(f\chi_{Q_j})(x) - m_{0,s_1,\widetilde{Q_j}} f > \alpha\lambda$$

holds.

(5.14) By the assumption, there exists $x_0 \in Q_j$ such that $M^{\#}_{0,s_2}(|f|)(x_0) \le \alpha\lambda/2$. Then we have

$$m_{0,s_2,\widetilde{Q_j}}(|f| - m_{0,s_1,\widetilde{Q_j}} f) \le 2M^{\#}_{0,s_2}(|f|)(x_0) \le \alpha\lambda$$

by Lemma 5.5, and thus

$$\begin{aligned} &|\{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1}((|f| - m_{0,s_1,\widetilde{Q_j}} f)\chi_{Q_j})(x) > \alpha\lambda \,\}| \\ &\underset{(5.9)}{\le} s_1^{-1} |\{\, x \in Q_j : ||f(x)| - m_{0,s_1,\widetilde{Q_j}} f)| > \alpha\lambda \,\}| \\ &\le s_1^{-1}\alpha |\{\, x \in \widetilde{Q_j} : ||f(x)| - m_{0,s_1,\widetilde{Q_j}} f| > m_{0,s_2,\widetilde{Q_j}}(|f| - m_{0,s_1,\widetilde{Q_j}} f) \,\}| \\ &\underset{(5.4)}{\le} s_1^{-1} s_2 |\widetilde{Q_j}|. \end{aligned}$$

Finally, we show that (5.13) and (5.14) imply the desired inequality (5.12). Since it is trivial when the left-hand side of (5.12) equals to zero, we assume that it is not zero.

Then we have $\{\, x \in Q_j : M^{\#}_{0,s_2}(|f|)(x) \le \alpha\lambda/2 \,\} \neq \emptyset$; the assumption of (5.14), thus

$$\begin{aligned} &|\{\, x \in Q_j : M^{\mathrm{dyadic}}_{0,s_1} f(x) > (\alpha+1)\lambda, M^{\#}_{0,s_2}(|f|)(x) \le \alpha\lambda/2 \,\}| \\ \underset{(5.13)}{\le}\ & |\{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1}((|f| - m_{0,s_1,\widetilde{Q_j}} f)\chi_{Q_j})(x) > \alpha\lambda \,\}| \\ \underset{(5.14)}{\le}\ & s_1^{-1} s_2 |\widetilde{Q_j}|. \end{aligned}$$

□

Theorem 5.7. *Let $s_1, s_2, p_0, p > 0$ and $f\colon \mathbb{R}^d \to \mathbb{C}$ satisfy $0 < 2^d s_2 < s_1 < 1$, $s_1 + s_2 < 1$, $0 < p_0 \le p < \infty$ and*

$$\sup_{0<\lambda<R} \lambda^{p_0} |\{\, x \in \mathbb{R}^d : |f(x)| > \lambda \,\}| < \infty$$

for any $R > 0$. Then there exists a constant $C_{d,s_1,s_2,p}$, depending only on d, s_1, s_2, p, such that the inequality

$$\|M^{\mathrm{dyadic}}_{0,s_1} f\|_{L^{p,\infty}} \le C_{d,s_1,s_2,p} \|M^{\#}_{0,s_2}(|f|)\|_{L^{p,\infty}}$$

holds and satisfies $C_{d,s_1,s_2,p} \lesssim_{d,s_1,s_2} p$ where $p \ge 1$.

Proof of Theorem 5.7. Let $\alpha, R > 0$ and write

$$\begin{aligned} \beta &:= \alpha + 1, \\ I_R &:= \sup_{0<\lambda<R} \lambda^p |\{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1} f(x) > \lambda \,\}|, \\ J_R &:= \sup_{0<\lambda<R} \lambda^p |\{\, x \in \mathbb{R}^d : M^{\#}_{0,s_2}(|f|)(x) > \lambda \,\}|. \end{aligned}$$

Using Lemma 5.6, we have

$$\begin{aligned} &(\beta\lambda)^p |\{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1} f(x) > \beta\lambda \,\}| \\ &\le (\beta\lambda)^p |\{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1} f(x) > \beta\lambda, M^{\#}_{0,s_2}(|f|)(x) \le \alpha\lambda/2 \,\}| \\ &\quad + (\beta\lambda)^p |\{\, x \in \mathbb{R}^d : M^{\#}_{0,s_2}(|f|)(x) > \alpha\lambda/2 \,\}| \\ &\le 2^d (\beta\lambda)^p s_1^{-1} s_2 |\{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1} f(x) > \lambda \,\}| \\ &\quad + (2\alpha^{-1}\beta)^p (\alpha\lambda/2)^p |\{\, x \in \mathbb{R}^d : M^{\#}_{0,s_2}(|f|)(x) > \alpha\lambda/2 \,\}| \end{aligned}$$

for each $\lambda > 0$. Taking supremum in $0 < \lambda < 2R$ implies that

$$I_{2\beta R} \le 2^d \beta^p s_1^{-1} s_2 I_{2R} + (2\alpha^{-1}\beta)^p J_{\alpha R} \le 2^d \beta^p s_1^{-1} s_2 I_{2\beta R} + (2\alpha^{-1}\beta)^p J_{\alpha R}$$

holds. Since

$$\begin{aligned} I_R &= \sup_{0<\lambda<R} \lambda^p |\{\, x \in \mathbb{R}^d : M^{\mathrm{dyadic}}_{0,s_1} f(x) > \lambda \,\}| \\ &\le \sup_{0<\lambda<R} \lambda^{p-p_0} \lambda^{p_0} s_1^{-1} |\{\, x \in \mathbb{R}^d : |f(x)| > \lambda \,\}| \\ &\le R^{p-p_0} s_1^{-1} \sup_{0<\lambda<R} \lambda^{p_0} |\{\, x \in \mathbb{R}^d : |f(x)| > \lambda \,\}| < \infty, \end{aligned}$$

we obtain

$$(1 - 2^d \beta^p s_1^{-1} s_2) I_{2\beta R} \le (2\alpha^{-1}\beta)^p J_{\alpha R}.$$

Now we take $\alpha > 0$ small enough, then we have $1 - 2^d \beta^p s_1^{-1} s_2 > 0$ since $2^d s_2 < s_1$. Therefore, letting $R \to \infty$, we obtain

$$\|M_{0,s_1}^{\mathrm{dyadic}} f\|_{L^{p,\infty}}^p \le C_{d,s_1,s_2,\alpha}^p \|M_{0,s_2}^{\#}(|f|)\|_{L^{p,\infty}}^p,$$
$$C_{d,s_1,s_2,\alpha}^p := \frac{(2\alpha^{-1}(\alpha+1))^p}{1 - 2^d s_1^{-1} s_2 (\alpha+1)^p}.$$

In particular, if we choose $\alpha > 0$ satisfying

$$1 - 2^d s_1^{-1} s_2 (\alpha+1)^p = (1 - 2^d s_1^{-1} s_2)/2,$$

that is

$$\alpha = \left(\frac{2^d s_2 + s_1}{2^{d+1} s_2} \right)^{1/p} - 1,$$

then we have

$$\begin{aligned} C_{d,s_1,s_2,\alpha}^p &= (2\alpha^{-1})^p \frac{(\alpha+1)^p}{1 - 2^d s_1^{-1} s_2 (\alpha+1)^p} \\ &= (2\alpha^{-1})^p \frac{2^{-d} s_1 s_2^{-1} (1 + 2^d s_1^{-1} s_2)}{1 - 2^d s_1^{-1} s_2}, \end{aligned}$$

and thus

$$\begin{aligned} C_{d,s_1,s_2,\alpha} &\le 2 \left(\frac{2^{-d} s_1 s_2^{-1} (1 + 2^d s_1^{-1} s_2)}{1 - 2^d s_1^{-1} s_2} \right)^{1/p} \alpha^{-1} \\ &= 2 \left(\frac{2^{-d} s_1 s_2^{-1} (1 + 2^d s_1^{-1} s_2)}{1 - 2^d s_1^{-1} s_2} \right)^{1/p} \left(\left(\frac{2^d s_2 + s_1}{2^{d+1} s_2} \right)^{1/p} - 1 \right)^{-1}, \end{aligned}$$

which satisfies $C_{d,s_1,s_2,\alpha} \lesssim_{d,s_1,s_2} p$ where $p \ge 1$. □

Remark 5. Compare our Theorem 5.7 with [Hu et al., 2007, Theorem 2.1]:

Theorem 5.A ([Hu et al., 2007, Theorem 2.1])**.** *Let $s, p_0, p > 0$ and $f \colon \mathbb{R}^d \to \mathbb{C}$ satisfy $0 < 3^p C_d s < 1$, $s < 1/2$, $0 < p_0 \le p < \infty$ and*

$$\|f\|_{L^{p_0,\infty}} = \sup_{0<\lambda<\infty} \lambda^{p_0} |\{ x \in \mathbb{R}^d : |f(x)| > \lambda \}| < \infty,$$

where $C_d > 0$ denotes some constant depending only on dimension d. Then the inequality

$$\|f\|_{L^{p,\infty}}^p \le \frac{C_d}{1 - 3^p C_d s} \|M_{0,s}^{\#}(|f|)\|_{L^{p,\infty}}^p$$

holds.

In Theorem 5.7, the assumption for s_1 and s_2; $0 < 2^d s_2 < s_1 < 1$, $s_1 + s_2 < 1$ is independent of p. On the other hand, in Theorem 5.A, it requires $0 < 3^p C_d s < 1$. Also, the assumption

$$\|f\|_{L^{p_0,\infty}} = \sup_{0<\lambda<\infty} \lambda^{p_0} |\{\, x \in \mathbb{R}^d : |f(x)| > \lambda \,\}| < \infty$$

in Theorem 5.A is relaxed to

$$\sup_{0<\lambda<R} \lambda^{p_0} |\{\, x \in \mathbb{R}^d : |f(x)| > \lambda \,\}| < \infty \text{ for any } R > 0$$

in Theorem 5.7.

6 Boundednesses of maximal singular integral operators Part II: equivalent characterization

In this section, we prove the following Theorem 6.1.

Theorem 6.1. *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that T_* is bounded from $L^\infty_c(\mathbb{R}^d) \to \mathrm{BMO}(\mathbb{R}^d)$ and K satisfies*

$$[K]_{H_*} = \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y)-\frac{1}{|Q|}\int_{z\in Q}K(x,z)\,dz\right|dx\,dy<\infty,$$

$$A=\sup_{Q\subset\mathbb{R}^d}\int_{y\in 2Q\setminus Q}|K(c(Q),y)|\,dy<\infty.$$

Then T_* is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for any $2\le p<\infty$ with a bound

$$\|T_*\|_{L^p\to L^{p,\infty}} \lesssim_d p(\|T\|_{L^\infty_c\to\mathrm{BMO}}+[K]_{H_*}+A).$$

Notice that Theorem 6.1 and results in Section 4 give Theorem V, since

(1.14) T_* is bounded from $L^2(\mathbb{R}^d)$ to $L^{2,\infty}(\mathbb{R}^d)$.

$\underset{\text{Theorem 4.5}}{\Longrightarrow}$ (6.1) T_* is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for any $1<p\le 2$.

$\underset{\text{Theorem 4.6}}{\Longrightarrow}$ (1.17) There exist $0<p<\infty$ and $B>0$ such that $\|T_*f\|_{L^{p,\infty}(Q)}\le B|Q|^{1/p}\|f\|_{L^\infty}$ holds for any cubes $Q\subset\mathbb{R}^d$ and $f\in L^\infty(Q)$.

$\underset{\text{Theorem 4.6}}{\Longrightarrow}$ (4.10) There exist $0<p<\infty$ and $B>0$ such that $\|T_*f\|_{L^{p}(Q)}\le B|Q|^{1/p}\|f\|_{L^\infty}$ holds for any cubes $Q\subset\mathbb{R}^d$ and $f\in L^\infty(Q)$.

$\underset{\text{Theorem 4.7}}{\Longrightarrow}$ (1.18) T_* is bounded from $L^\infty_c(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$.

$\underset{\text{Theorem 6.1}}{\Longrightarrow}$ (6.2) T_* is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for any $2\le p<\infty$.

$\Longrightarrow$ (1.14) T_* is bounded from $L^2(\mathbb{R}^d)$ to $L^{2,\infty}(\mathbb{R}^d)$.

and

(6.1) $\wedge$ (6.2) T_* is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for any $1<p<\infty$.

$\underset{\text{interpolation}}{\Longrightarrow}$ (1.15) T_* is bounded on $L^p(\mathbb{R}^d)$ for any $1<p<\infty$.

We begin with the following Lemma 6.2.

Lemma 6.2. *Let T be a singular integral operator with a kernel K and T_* be its maximal operator. Suppose that K satisfies*

$$[K]_{H_*} = \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_{y\in Q}\int_{x\in\mathbb{R}^d\setminus 2Q}\left|K(x,y)-\frac{1}{|Q|}\int_{z\in Q}K(x,z)\,dz\right|dx\,dy<\infty,$$

$$A=\sup_{Q\subset\mathbb{R}^d}\int_{y\in 2Q\setminus Q}|K(c(Q),y)|\,dy<\infty.$$

Then

$$\sup_{0<\lambda<R} \lambda^2 |\{ x \in \mathbb{R}^d : T_* f(x) > \lambda \}| < \infty$$

holds for any $f \in L^\infty_{\mathrm{c},0}(\mathbb{R}^d)$ *and* $R > 0$.

Proof of Lemma 6.2. Let $f \in L^\infty_{\mathrm{c},0}(\mathbb{R}^d)$, Q be a cube such that $\operatorname{supp} f \subset Q$ and $0 < \lambda < R$. We define $\alpha \geq 1$ by

$$\alpha^d = \frac{A\|f\|_{L^\infty}}{2\lambda} + 1,$$

which implies

$$\operatorname{supp} f \subset \alpha Q,$$
$$\frac{1}{|\alpha Q|} \int_{y \in \alpha Q} |f| \leq \frac{|Q|}{|\alpha Q|} \|f\|_{L^\infty} = \alpha^{-d} \|f\|_{L^\infty} \leq \lambda/2A.$$

Using Lemma 4.3, we have

$$T_* f(x) \leq 2\|f\|_{L^\infty} \int_{y \in \alpha Q} \left| K(x,y) - \frac{1}{|\alpha Q|} \int_{z \in \alpha Q} K(x,z)\, dz \right| dy + \lambda/2$$

for $x \in \mathbb{R}^d \setminus 5\alpha Q$ and therefore

$$\begin{aligned}
&\lambda^2 |\{ x \in \mathbb{R}^d \setminus 5\alpha Q : T_* f(x) > \lambda \}| \\
&\leq \lambda^2 \left| \left\{ x \in \mathbb{R}^d \setminus 5\alpha Q : 2\|f\|_{L^\infty} \int_{y \in \alpha Q} \left| K(x,y) - \frac{1}{|\alpha Q|} \int_{z \in \alpha Q} K(x,z)\, dz \right| dy > \lambda/2 \right\} \right| \\
&\leq 4\lambda \|f\|_{L^\infty} \int_{y \in \alpha Q} \int_{x \in \mathbb{R}^d \setminus 5\alpha Q} \left| K(x,y) - \frac{1}{|\alpha Q|} \int_{z \in \alpha Q} K(x,z)\, dz \right| dy \\
&\leq 4\lambda \|f\|_{L^\infty} |\alpha Q| [K]_{H_*} \\
&= 4\lambda \|f\|_{L^\infty} (A\|f\|_{L^\infty}/(2\lambda) + 1) |Q| [K]_{H_*} \\
&\leq 2(A\|f\|_{L^\infty} + 2\lambda) |Q| [K]_{H_*} \\
&\leq 2(A\|f\|_{L^\infty} + 2R) |Q| [K]_{H_*}.
\end{aligned}$$

Also it is easy to see that

$$\begin{aligned}
\lambda^2 |\{ x \in 5\alpha Q : T_* f(x) > \lambda \}| &\leq \lambda^2 |5\alpha Q| \\
&= \lambda^2 (A\|f\|_{L^\infty}/(2\lambda) + 1) |5Q| \\
&\leq R(A\|f\|_{L^\infty}/2 + R) |5Q|.
\end{aligned}$$

□

Proof of Theorem 6.1. Note that we have $L^\infty_{\mathrm{c},0}(\mathbb{R}^d) \underset{\text{dense}}{\subset} L^p(\mathbb{R}^d)$ for $2 \leq p < \infty$, hence it is enough to consider $f \in L^\infty_{\mathrm{c},0}(\mathbb{R}^d)$. Using Theorem 5.7 and Lemma 6.2, we have

$$\|T_* f\|_{L^{p,\infty}} \leq \|M^{\mathrm{dyadic}}_{0,s_1} T_* f\|_{L^{p,\infty}} \lesssim_d p \|M^{\#}_{0,2s_2} T_* f\|_{L^{p,\infty}}$$

for any $f \in L^\infty_{\mathrm{c},0}(\mathbb{R}^d)$, where $s_1 = 1/2$, $s_2 = 2^{-d-3}$. We write

$$B_T := s_2^{-1} \|T_*\|_{L^\infty_{\mathrm{c}} \to \mathrm{BMO}} + [K]_{H_{*,p'}} + A$$

for simplicity and assume $\alpha > 2^{d/p+1} B_T$. Fix $f \in L^\infty_{c,0}(\mathbb{R}^d)$, $\lambda, \alpha > 0$ and form the L^p-Calderón–Zygmund decomposition of f at height $\alpha^{-1}\lambda$ (we choose α later). Then we have

$$\begin{aligned}
&|\{ x \in \mathbb{R}^d : M^{\#}_{0,2s_2} T_* f(x) > 2\lambda \}| \\
\underset{(5.7)}{\leq}\ & |\{ x \in \mathbb{R}^d : M^{\#}_{0,s_2} T_* g(x) + M^{\text{dyadic}}_{0,s_2} T_* b(x) > 2\lambda \}| \\
\leq\ & |\{ x \in \mathbb{R}^d : M^{\#}_{0,s_2} T_* g(x) > \lambda \}| + |\{ x \in \mathbb{R}^d : M^{\text{dyadic}}_{0,s_2} T_* b(x) > \lambda \}| \\
\underset{(5.3),(5.9)}{\leq}\ & |\{ x \in \mathbb{R}^d : M^{\#} T_* g(x) > s_2\lambda \}| + s_2^{-1} |\{ x \in \mathbb{R}^d : T_* b(x) > \lambda \}|.
\end{aligned}$$

Now we use the $L^\infty_c(\mathbb{R}^d) \to \text{BMO}(\mathbb{R}^d)$ boundedness of T_*, which implies

$$\|M^{\#} T_* g\|_{L^\infty} = \|T_* g\|_{\text{BMO}} \leq s_2 B_T \|g\|_{L^\infty} \underset{(3.7)}{\leq} 2^{d/p} B_T \alpha^{-1} s_2 \lambda < s_2 \lambda$$

and thus

$$|\{ x \in \mathbb{R}^d : M^{\#} T_* g(x) > s_2\lambda \}| = 0.$$

On the other hand, by the proof of Theorem 4.1, we know that

$$\lambda^p |\{ x \in \mathbb{R}^d : T_* b(x) > \lambda \}| \leq \left(5^d \alpha^p + \frac{\alpha^{p-1} 2^{d/p+2} B_T}{1 - 2^{d/p+1} B_T \alpha^{-1}} \right) \|f\|^p_{L^p}.$$

Therefore, we have

$$\|T_* f\|_{L^{p,\infty}} \lesssim_d p \left(5^d \alpha^p + \frac{\alpha^{p-1} 2^{d/p+2} B_T}{1 - 2^{d/p+1} B_T \alpha^{-1}} \right)^{1/p} \|f\|_{L^p}.$$

The desired inequality follows from taking $\alpha = 2^{d/p+2} B_T$. □

7 Appendix: Direct proofs of results under the L^1 mean Hörmander condition

In this section, we give a direct proof of Theorem I and a sufficient condition of the $L^2(\mathbb{R}^d)$ boundedness for convolution type singular integral operators under the L^1 mean Hörmander condition (Theorem 7.1), though they immediately follow from Theorem II and known results under the classical Hörmander condition. Our argument is inspired by the proof of [Fefferman, 1970, Theorem 2']. We use a 'bounded overlap' variant of the L^1-Calderón–Zygmund decomposition.

Proposition 7.A ([Fefferman, 1970, DECOMPOSITION LEMMA])**.** *Let $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. Then there exists a pairwise disjoint family of dyadic cubes $\{Q_j\}_j$ satisfying*

$$\Omega := \bigcup_j Q_j = \{ x \in \mathbb{R}^d : Mf(x) > \lambda \},$$

$$\lambda|2\Omega| \le C_d \|f\|_{L^1} \text{ where } 2\Omega := \bigcup_j 2Q_j, \tag{7.1}$$

$$|f(x)| \le \lambda \text{ a.e. } x \in \mathbb{R}^d \setminus \Omega,$$

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f| \le C_d \lambda,$$

$$\sum_j \chi_{2Q_j} \le C_d, \tag{7.2}$$

where C_d denotes some positive constant depending only on dimension d. Moreover, functions g, b_j and b;

$$g := f\chi_{\mathbb{R}^d \setminus \Omega}, \quad b_j := f\chi_{Q_j}, \quad b := \sum_j b_j,$$

satisfy

$$\|g\|_{L^1} \le \|f\|_{L^1}, \quad \|g\|_{L^\infty} \le \lambda, \quad \|g\|_{L^2}^2 \le \lambda \|f\|_{L^1}, \tag{7.3}$$

$$\operatorname{supp} b_j \subset Q_j, \quad |Q_j|^{-1} \|b_j\|_{L^1} \le C_d \lambda, \quad \sum_j \|b_1\|_{L^1} = \|b\|_{L^1} \le \|f\|_{L^1}. \tag{7.4}$$

Remark 6. Our definition of functions g, b_j, b is different from that of [Fefferman, 1970, DECOMPOSITION LEMMA];

$$g := f\chi_{\mathbb{R}^d \setminus \Omega} + \sum_j \frac{\chi_{Q_j}}{|Q|} \int_Q f, \quad b_j := \left(f - \frac{1}{|Q|} \int_Q f \right) \chi_{Q_j}, \quad b := \sum_j b_j,$$

since our argument does not need $\int b_j = 0$.

Remark 7. Note that Proposition 7.A holds for any doubling measures (though the constant C_d may change) and our proof of Theorem I below also works in this setting.

Proof of Theorem I. Fix $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^d)$, $\lambda, \alpha > 0$ and form the L^1-Calderón–Zygmund decomposition of f at height $\alpha^{-1}\lambda$ (we choose α later). Moreover, We write $\ell_j \coloneqq \ell(Q_j)$ and

$$\tilde{b}_j(x) \coloneqq \int_{y \in Q(x,\ell_j)} \frac{b_j(y)}{|Q(y,\ell_j)|}\, dy,$$
$$\tilde{b}(x) \coloneqq \sum_j b_j(x).$$

It suffices to estimate the following:

$$\lambda |\{\, x \in \mathbb{R}^d : |Tg(x)| > \lambda \,\}|, \tag{7.5}$$
$$\lambda |\{\, x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda \,\}|, \tag{7.6}$$
$$\lambda |\{\, x \in \mathbb{R}^d : |T\tilde{b}(x)| > \lambda \,\}|. \tag{7.7}$$

(7.5) Since T is bounded on $L^2(\mathbb{R}^d)$, it follows that

$$\begin{aligned}\lambda |\{\, x \in \mathbb{R}^d : |Tg(x)| > \lambda \,\}| &\le \lambda^{-1} \|Tg\|_{L^2}^2 \\ &\le \lambda^{-1} \|T\|_{L^2 \to L^2}^2 \|g\|_{L^2}^2 \\ &\underset{(7.3)}{\le} \alpha^{-1} \|T\|_{L^2 \to L^2}^2 \|f\|_{L^1}.\end{aligned}$$

(7.6) Since

$$\{\, x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda \,\} \subset 2\Omega \cup \{\, x \in \mathbb{R}^d \setminus 2\Omega : |T(\tilde{b} - b)(x)| > \lambda \,\},$$

it follows that

$$\begin{aligned}&\lambda^p |\{\, x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda \,\}| \\ &\le \lambda |2\Omega| + \lambda |\{\, x \in \mathbb{R}^d \setminus 2\Omega : |T(\tilde{b} - b)(x)| > \lambda \,\}| \\ &\underset{(7.1)}{\le} C_d \alpha \|f\|_{L^1} + \|T(\tilde{b} - b)\|_{L^1(\mathbb{R}^d \setminus 2\Omega)}.\end{aligned}$$

We estimate the second term by the L^1 mean Hörmander condition. Note that $\tilde{b}_j$ satisfies $\operatorname{supp} \tilde{b}_j \subset 2Q_j$, which implies that

$$\begin{aligned}T\tilde{b}_j(x) &= \int_{z \in \mathbb{R}^d} K(x,z) \left(\int_{y \in Q(z,\ell_j)} \frac{b_j(y)}{|Q(y,\ell_j)|}\, dy \right) dz \\ &= \int_{y \in Q_j} \left(\frac{1}{|Q(y,\ell_j)|} \int_{z \in Q(y,\ell_j)} K(x,z)\, dz \right) b_j(y)\, dy\end{aligned}$$

for $x \in \mathbb{R}^d \setminus 2Q_j$. Also notice that $Q(y, 2\ell_j) \subset 2Q_j$ for $y \in Q_j$. Thus, we have

$$\begin{aligned}
&\|T(\tilde{b}_j - b_j)\|_{L^1(\mathbb{R}^d\setminus 2Q_j)} \\
&= \int_{x\in\mathbb{R}^d\setminus 2Q_j} \left| \int_{y\in Q_j} \left(\frac{1}{|Q(y,\ell_j)|} \int_{z\in Q(y,\ell_j)} K(x,z)\,dz - K(x,y) \right) b_j(y)\,dy \right| dx \\
&\le \int_{y\in Q_j} \frac{1}{|Q(y,\ell_j)|} \int_{z\in Q(y,\ell_j)} \int_{x\in\mathbb{R}^d\setminus Q(y,2\ell_j)} |K(x,z) - K(x,y)|\,dz\,dx\,|b_j(y)|\,dy \\
&\le [K]_{H_1} \|b_j\|_{L^1}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\|Tb\|_{L^1(\mathbb{R}^d\setminus 2\Omega)} &\le \sum_j \|Tb_j\|_{L^1(\mathbb{R}^d\setminus 2Q_j)} \\
&\le [K]_{H_1} \sum_j \|b_j\|_{L^1} \\
&\underset{(7.4)}{\le} [K]_{H_1} \|f\|_{L^1}.
\end{aligned}$$

(7.7) At first, we estimate $\|\tilde{b}\|_{L^1}$ and $\|\tilde{b}\|_{L^\infty}$. For each j, we obtain

$$\begin{aligned}
\|\tilde{b}_j\|_{L^1} &= \int_{x\in\mathbb{R}^d} \left| \int_{y\in Q(x,\ell_j)} \frac{b_j(y)}{|Q(y,\ell_j)|}\,dy \right| dx \\
&\le \int_{y\in\mathbb{R}^d} \int_{x\in Q(y,\ell_j)} \frac{|b_j(y)|}{|Q(y,\ell_j)|}\,dx\,dy \\
&= \|b_j\|_{L^1}, \\
\|\tilde{b}_j\|_{L^\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \left| \int_{y\in Q(x,\ell_j)} \frac{b_j(y)}{|Q(y,\ell_j)|}\,dy \right| \\
&\le \int_{y\in Q_j} \frac{|b_j(y)|}{|Q(y,\ell_j)|}\,dy \\
&= |Q_j|^{-1} \|b_j\|_{L^1} \\
&\underset{(7.4)}{\le} C_d \alpha^{-1} \lambda.
\end{aligned}$$

Note that here we used $|Q(y,\ell_j)| = |Q_j|$. In the case of doubling measures, we use following estimate:

$$\mu(Q_j) \le \mu(Q(y, 2\ell_j)) \lesssim_\mu \mu(Q(y,\ell_j)) \text{ for any } y \in Q_j.$$

Anyway, now we have

$$\|\tilde{b}\|_{L^1} \le \sum_j \|\tilde{b}_j\|_{L^1} \le \sum_j \|b_j\|_{L^1} \underset{(7.4)}{\le} \|f\|_{L^1},$$

$$\|\tilde{b}\|_{L^\infty} \le \left\| \sum_j |\tilde{b}_j| \right\|_{L^\infty} \le \left\| \sum_j \|\tilde{b}_j\|_{L^\infty} \chi_{2Q_j} \right\|_{L^\infty} \underset{(7.2)}{\le} C_d^2 \alpha^{-1} \lambda,$$

and therefore

$$\|\tilde{b}\|_{L^2}^2 \le \|\tilde{b}\|_{L^1}\|\tilde{b}\|_{L^\infty} \le C_d^2\alpha^{-1}\lambda\|f\|_{L^1}.$$

We use the $L^2(\mathbb{R}^d)$ boundedness of T again and obtain

$$\begin{aligned}\lambda|\{\, x \in \mathbb{R}^d : |T\tilde{b}(x)| > \lambda \,\}| &\le \lambda^{-1}\|T\tilde{b}\|_{L^2}^2 \\ &\le \lambda^{-1}\|T\|_{L^2\to L^2}^2\|\tilde{b}\|_{L^2}^2 \\ &\le \alpha^{-1}C_d^2\|T\|_{L^2\to L^2}^2\|f\|_{L^1}.\end{aligned}$$

Summing up the estimates of (7.5), (7.6) and (7.7) above, we have

$$\|Tf\|_{L^{1,\infty}} \le 3((1+C_d^2)\|T\|_{L^2\to L^2}^2\alpha^{-1} + [K]_{H_1} + C_d\alpha)\|f\|_{L^1}.$$

Now we choose

$$\alpha = C_d^{-1/2}(1+C_d^2)^{1/2}\|T\|_{L^2\to L^2}$$

and conclude that

$$\|T\|_{L^1\to L^{1,\infty}} \le 3(C_d^{1/2}(1+C_d^2)^{1/2}\|T\|_{L^2\to L^2} + [K]_{H_1}). \qquad \square$$

Finally, we prove the following sufficient condition of the $L^2(\mathbb{R}^d)$ boundedness for convolution type singular integral operators.

Theorem 7.1 ([Suzuki, 2021, Theorem 2])**.** *Let $K \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ satisfy*

$$A := \sup_{0<a<b<\infty}\left|\int_{a<|x|<b} K(x)\,dx\right| < \infty, \tag{7.8}$$

$$B := \sup_{a>0}\frac{1}{a}\int_{|x|<a}|x||K(x)|\,dx < \infty, \tag{7.9}$$

$$[K]_{H_1} := \sup_{r>0}\frac{1}{V_d r^d}\int_{|y|\le r}\int_{|x|\ge 2r}|K(x-y)-K(x)|\,dx\,dy < \infty, \tag{7.10}$$

where V_d denotes the volume of the d dimensional unit ball, and define $K_{\varepsilon,R} := K\chi_{\{\varepsilon<|x|<R\}}$ for $0<\varepsilon<R<\infty$. Then $K_{\varepsilon,R}$ satisfies

$$\sup_{0<\varepsilon<R<\infty}\sup_{\xi\in\mathbb{R}^d}|\widehat{K_{\varepsilon,R}}(\xi)| \lesssim_d A + B + [K]_{H_1}.$$

Theorem 7.1 is basically [Benedek et al., 1962, Theorem 3], but the Hörmander condition is replaced by the L^1 mean variant (though they are equivalent, it was not known at the time of [Suzuki, 2021]). Anyway, we give a direct proof of Theorem 7.1 without the equivalence. We use the following Lemma 7.2:

Lemma 7.2. *If $K \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ satisfies (7.9) and (7.10), then*

$$\sup_{0<\varepsilon<R<\infty}[K_{\varepsilon,R}]_{H_1} \le [K]_{H_1} + 7B. \tag{7.11}$$

Proof of Lemma 7.2. It is obvious that

$$\begin{aligned}
&\frac{1}{V_d r^d}\int_{|y|\le r}\int_{|x|\ge 2r}|K_{\varepsilon,R}(x-y)-K_{\varepsilon,R}(x)|\,dx\,dy\\
&=\frac{1}{V_d r^d}\int_{|y|\le r}\int_{|x|\ge 2r,\varepsilon<|x-y|<R,\varepsilon<|x|<R}|K(x-y)-K(x)|\,dx\,dy\\
&\quad+\frac{1}{V_d r^d}\int_{|y|\le r}\int_{|x|\ge 2r,\varepsilon<|x-y|<R,|x|\le\varepsilon}|K(x-y)|\,dx\,dy\\
&\quad+\frac{1}{V_d r^d}\int_{|y|\le r}\int_{|x|\ge 2r,\varepsilon<|x-y|<R,|x|\ge R}|K(x-y)|\,dx\,dy\\
&\quad+\frac{1}{V_d r^d}\int_{|y|\le r}\int_{|x|\ge 2r,|x-y|\le\varepsilon,\varepsilon<|x|<R}|K(x)|\,dx\,dy\\
&\quad+\frac{1}{V_d r^d}\int_{|y|\le r}\int_{|x|\ge 2r,|x-y|\ge R,\varepsilon<|x|<R}|K(x)|\,dx\,dy
\end{aligned}$$

and the first term is bounded by $[K]_{H_1}$. To estimate other terms, note that (7.9) implies

$$\sup_{a>0}\int_{a<|x|<ca}|K(x)|\,dx\le\sup_{a>0}\int_{a<|x|<ca}\frac{|x|}{a}|K(x)|\,dx\le cB$$

for any $c>1$. Since we have

$$\begin{aligned}
\varepsilon<|x-y|<R,|x|\le\varepsilon \quad&\Rightarrow\quad |x-y|\le|x|+|y|\le 3|x|/2\le 3\varepsilon/2,\\
\varepsilon<|x-y|<R,|x|\ge R \quad&\Rightarrow\quad |x-y|\ge|x|-|y|\ge|x|/2\ge R/2,\\
|x-y|\le\varepsilon,\varepsilon<|x|<R \quad&\Rightarrow\quad |x|\le 2(|x-y|+|y|)-|x|\le 2|x-y|<2\varepsilon,\\
|x-y|\ge R,\varepsilon<|x|<R \quad&\Rightarrow\quad |x|\ge 2(|x-y|-|y|)/3+|x|/3\ge 2|x-y|/3\ge 2R/3
\end{aligned}$$

under the condition $2|y|\le 2r\le|x|$, the second and fifth terms are bounded by $3B/2$, the third and fourth terms are bounded by $2B$. □

Proof of Theorem 7.1. Fix $0<\varepsilon<R<\infty$ and $\xi\in\mathbb{R}^d$. Since it is obvious that

$$|\widehat{K_{\varepsilon,R}}(0)|=\left|\int_{\varepsilon<|x|<R}K(x)\,dx\right|\underset{(7.8)}{\le}A,$$

we assume $\xi\ne 0$ and write $s:=|\xi|^{-1}$. If we decompose $\widehat{K_{\varepsilon,R}}(\xi)$ as

$$\begin{aligned}
&\widehat{K_{\varepsilon,R}}(\xi)\\
&=\int_{x\in\mathbb{R}^d}K_{\varepsilon,R}(x)e^{-2\pi i x\cdot\xi}\,dx\\
&=\int_{|x|<2s}K_{\varepsilon,R}(x)(e^{-2\pi i x\cdot\xi}-1)\,dx+\int_{|x|<2s}K_{\varepsilon,R}(x)\,dx+\int_{2s\le|x|}K_{\varepsilon,R}(x)e^{-2\pi i x\cdot\xi}\,dx\\
&=:I_1+I_2+I_3,
\end{aligned}$$

then we easily get

$$|I_1| \le \int_{|x|<2s} |K_{\varepsilon,R}(x)||e^{-2\pi i x\cdot\xi} - 1|\,dx \le 4\pi \frac{1}{2s} \int_{|x|<2s} |x||K_{\varepsilon,R}(x)|\,dx \underset{(7.9)}{\le} 4\pi B,$$

$$|I_2| = \left| \int_{\varepsilon<|x|<2s} K_{\varepsilon,R}(x)\,dx \right| \underset{(7.8)}{\le} A.$$

To estimate I_3, fix a radial function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\operatorname{supp}\varphi \subset B(0,1), \quad \int \varphi = 1, \quad \varphi \ge 0, \quad |\widehat{\varphi}(1)| < 1,$$

where $B(0,1)$ means the unit ball and $\widehat{\varphi}(1)$ denotes the value of $\widehat{\varphi}$ on the unit sphere, and define $\varphi_s(x) := s^{-d}\varphi(s^{-1}x)$. Moreover, rewrite I_3 as

$$I_3 = \int_{|x|\ge 2s} \int_{|y|\le s} K_{\varepsilon,R}(x)\varphi_s(y)\,dy\, e^{-2\pi i x\cdot\xi}\,dx \tag{7.12}$$

and introduce

$$I_4 := \int_{|x|\ge 2s} \int_{|y|\le s} K_{\varepsilon,R}(x-y)\varphi_s(y)\,dy\, e^{-2\pi i x\cdot\xi}\,dx, \tag{7.13}$$

$$\begin{aligned} I_5 &:= \int_{|x|<2s} \int_{|y|\le s} K_{\varepsilon,R}(x-y)\varphi_s(y)\,dy\, e^{-2\pi i x\cdot\xi}\,dx && (7.14)\\ &= \int_{|x|<2s} \int_{|x-y|\le s} K_{\varepsilon,R}(y)\varphi_s(x-y)\,dy\, e^{-2\pi i x\cdot\xi}\,dx \\ &= \int_{|x|<2s} \int_{|y|\le 3s} K_{\varepsilon,R}(y)\varphi_s(x-y)\,dy\, e^{-2\pi i x\cdot\xi}\,dx, && (7.15)\\ I_6 &:= \int_{|x|<2s} \int_{|y|\le 3s} K_{\varepsilon,R}(y)\varphi_s(x)\,dy\, e^{-2\pi i x\cdot\xi}\,dx && (7.16)\\ &= \widehat{\varphi_s}(\xi) \int_{|y|\le 3s} K_{\varepsilon,R}(y)\,dy. && (7.17) \end{aligned}$$

We decompose I_3 into $(I_3 - I_4) + (I_4 + I_5) - (I_5 - I_6) - I_6$. By (7.12), (7.13) and Lemma 7.2, we get

$$\begin{aligned} &|I_4 - I_3| \\ &\underset{(7.12),(7.13)}{=} \left| \int_{|x|\ge 2s} \int_{|y|\le s} (K_{\varepsilon,R}(x-y) - K_{\varepsilon,R}(x))\varphi_s(y)\, e^{-2\pi i x\cdot\xi}\,dy\,dx \right| \\ &\le \int_{|y|\le s} \int_{|x|\ge 2s} |K_{\varepsilon,R}(x-y) - K_{\varepsilon,R}(x)|\,\varphi_s(y)\,dx\,dy \\ &\le V_d\|\varphi\|_\infty \frac{1}{V_d s^d} \int_{|y|\le s} \int_{|x|<2s} |K_{\varepsilon,R}(x-y) - K_{\varepsilon,R}(x)|\,dx\,dy \\ &\le V_d\|\varphi\|_\infty [K_{\varepsilon,R}]_{H_1} \\ &\underset{(7.11)}{\le} V_d\|\varphi\|_\infty ([K]_{H_1} + 7B). \end{aligned}$$

For $I_5 - I_6$, use (7.15), (7.16) and the mean value theorem to obtain

$$\begin{aligned}
&|I_5 - I_6| \\
&\underset{(7.15),(7.16)}{=} \left| \int_{|x|<2s} \int_{|y|\le 3s} K_{\varepsilon,R}(y)(\varphi_s(x-y) - \varphi_s(x)) e^{-2\pi i x\cdot\xi}\, dy\, dx \right| \\
&\le \int_{|x|<2s} \int_{|y|\le 3s} |K_{\varepsilon,R}(y)||\varphi_s(x-y) - \varphi_s(x)|\, dy\, dx \\
&\le \int_{|x|<2s} \int_{|y|\le 3s} |K_{\varepsilon,R}(y)| s^{-d-1}|y|\|\nabla\varphi\|_{L^\infty}\, dy\, dx \\
&= 3s^{-d}\|\nabla\varphi\|_{L^\infty} \int_{|x|<2s} \left(\frac{1}{3s} \int_{|y|\le 3s} |y||K_{\varepsilon,R}(y)|\, dy \right) dx \\
&\underset{(7.9)}{\le} 3s^{-d}\|\nabla\varphi\|_{L^\infty} \int_{|x|\le 2s} B\, dx \\
&= 3\cdot 2^d V_d \|\nabla\varphi\|_{L^\infty} B.
\end{aligned}$$

For $I_4 + I_5$ and I_6, remark that $\widehat{\varphi_s}(\xi) = \widehat{\varphi}(s\xi) = \widehat{\varphi}(1)$ because φ is radial and $s = |\xi|^{-1}$. Then it follows immediately that

$$\begin{aligned}
I_4 + I_5 &\underset{(7.13),(7.14)}{=} \widehat{K_{\varepsilon,R} * \varphi_s}(\xi) = \widehat{\varphi}(1)\widehat{K_{\varepsilon,R}}(\xi), \\
|I_6| &\underset{(7.17)}{=} \left| \widehat{\varphi_s}(\xi) \int_{|x|\le 3s} K_{\varepsilon,R}(y)\, dy \right| \underset{(7.8)}{\le} |\widehat{\varphi}(1)|A.
\end{aligned}$$

Now we have

$$\begin{aligned}
&|\widehat{K_{\varepsilon,R}}(\xi)| \\
&\le |I_1| + |I_2| + |I_3 - I_4| + |I_4 + I_5| + |I_5 - I_6| + |I_6| \\
&\le 4\pi B + A + V_d\|\varphi\|_{L^\infty}([K]_{H_1} + 7B) + |\widehat{\varphi}(1)||\widehat{K_{\varepsilon,R}}(\xi)| + 3\cdot 2^d V_d\|\nabla\varphi\|_{L^\infty} B + |\widehat{\varphi}(1)|A
\end{aligned}$$

for any $\xi \in \mathbb{R}^d$ (it is still valid in the case $\xi = 0$). Finally, remember $|\widehat{\varphi}(1)| < 1$ to conclude that

$$|\widehat{K_{\varepsilon,R}}(\xi)| \le \frac{(1 + |\widehat{\varphi}(1)|)A + (4\pi + V_d(7\|\varphi\|_{L^\infty} + 3\cdot 2^d\|\nabla\varphi\|_{L^\infty}))B + V_d\|\varphi\|_{L^\infty}[K]_{H_1}}{1 - |\widehat{\varphi}(1)|}. \square$$

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