ELECTROMAGNETIC WAVE SCATTERING BY IONOSPHERIC IRREGULARITIES

Masashi HAYAKAWA, Jinsuke OHTSU
and Akira IWAI

Abstract

This paper deals with the scattering of electromagnetic waves obliquely incident on an inhomogeneous plasma column which has a parabolic radial density distribution and its application to ionospheric irregulariries. Explicit expressions of scattered wave fields and fields inside the irregularity are given. Formulas of scattering patterns are given analytically as functions of frequency, incident angle, and irregularity width. As special cases, scattering by a magnetic field-aligned irregularity at normal incidence, and scattering by irregularities with regular spatial distribution, are also discussed.

1. Introduction

Recent development of rocket and artificial satellite techniques made it possible directly to study the ionospheric irregularities. On the other hand, VHF scattering method of irregularity investigation is successfully established as an indirect method, and some scattering models have been proposed by some workers. Booker and Gordon (1950) have derived a scattering formula of radio waves by an atmosphere containing isotropic irregularities described by an exponential autocorrelation function. Following this Booker (1956) proposed a scattering model of anisotropic irregularities. Recently Sato and Maeda (1968) have introduced a new model in which a sufficient number of homogeneous plasma columns of the same kind and shape are embedded along the magnetic field line in an isotropic medium. They showed that the scattered field could be expressed as the product of the so-called element factor and the array factor well known in the antenna theory.

The problem of electromagnetic wave scattering by a circular cylinder at oblique incidence was originally given by Wait (1955) and recently by Lind and Greenberg (1966).

However, the treatments cited above all concern the scattering by homogeneous madia. In this respect, scattering by an inhomogeneous medium was once discussed by

Keitel (1955), who solved the scattering of a plane wave at normal incidence on a cylindrical meteor trail which has a Gaussian radial distribution of ion density. Yeh and Kaprielian (1963) have treated the scattering from a cylinder coated with an inhomogeneous dielectric sheath at normal incidence.

But no exact solutions of electromagnetic wave scattering by an inhomogeneous medium at oblique incidence could have so far been found, presumably because the resulting wave equations become non-linear and involve the coupling between TM and TE modes. We (1968 a, b) have treated the scattering by an irregularity in a plasma whose radial density distribution is of parabolic form at oblique incidence and applied it to the ionospheric irregularities.

2. Electromagnetic fields in an inhomogeneous plasma

The electric and magnetic fields of electromagnetic waves propagating in an inhomogeneous but lossless plasma are governed respectively by the following equations,

$$\nabla^2\mathbf{E}+k_0{}^2\varepsilon(r)\mathbf{E}=-\nabla\left(\frac{\nabla\varepsilon(r)}{\varepsilon(r)}\cdot\mathbf{E}\right) \tag{1}$$

$$\nabla^2\mathbf{H}+k_0{}^2\varepsilon(r)\mathbf{H}=-\frac{\nabla\varepsilon(r)}{\varepsilon(r)}\times(\nabla\times\mathbf{H}) \tag{2}$$

where the time dependence is assumed to be $\exp(i\omega t)$. k_0 is a free space propagation constant and ω is the angular frequency of incident wave. $\varepsilon(r)$ is the relative dielectric constant which is a function of position due to the spatial inhomogeneity.

Now we consider the case when the wavelength of the incident wave is smaller than the characteristic length of the variation of dielectric constant. This condition is considered to be valid for ionospheric irregularities at VHF and HF range. Then the right hand sides of eqs. (1) and (2) are neglected and both electric and magnetic fields satisfy the same equation.

$$(\nabla^2+k_0{}^2\varepsilon(r))\mathbf{E}\ \text{or}\ \mathbf{H}=0 \tag{3}$$

This wave equation is a basis for electromagnetic waves propagating in an inhomogeneous medium, i.e., inside and outside the irregularity.

2.1 Density distribution of an ionospheric irregularity

Smith (1966) has shown by rocket observations that around the altitude of 100 km there exists a region of irregularities, and that the amplitude of variation in electron density is about 20% and the dimension of the irregularities in height is some hundred meters. Also they are well known to be elongated along the magnetic field line taking a cylindrical structure.

In this paper we consider a scattering model in which a cylindrically inhomogeneous irregularity is embedded in a homogeneous isotropic plasma. The configuration of the problem and cylindrical coordinate system (r, ϕ, z) we used are shown in Fig. 1. The cylindrical irregularity, whose central axis is coincident with z-axis, occupies the space $r \leq a$. The space $r \geq a$ corresponds to the ambient plasma. a is the radius of the irregularity. Now the radial electron density distribution is assumed to be constant in the ambient plasma and to be of parabolic form inside the irregularity as follows,

$$N(r) = \begin{cases} N_a & ; \ r \geq a \\ N_a\{1 + m(1 - r^2/a^2)\} & ; \ r \leq a \end{cases} \tag{4}$$

where N_a is the ambient plasma density. m is the modulation factor, i.e., the rate of enhancement or depression of the density at the center of the column relative to that in the ambient plasma. So the irregularity with an enhanced (a depressed) density corresponds to the case $m > 0$ $(m < 0)$.

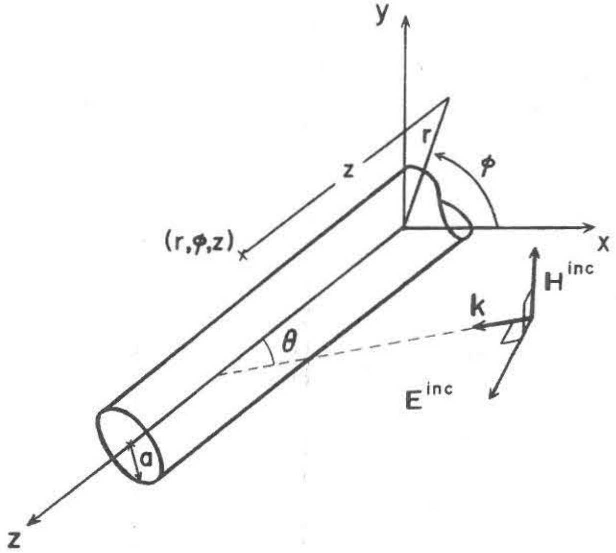

Fig. 1 An ionospheric irregularity and cylindrical coordinate.

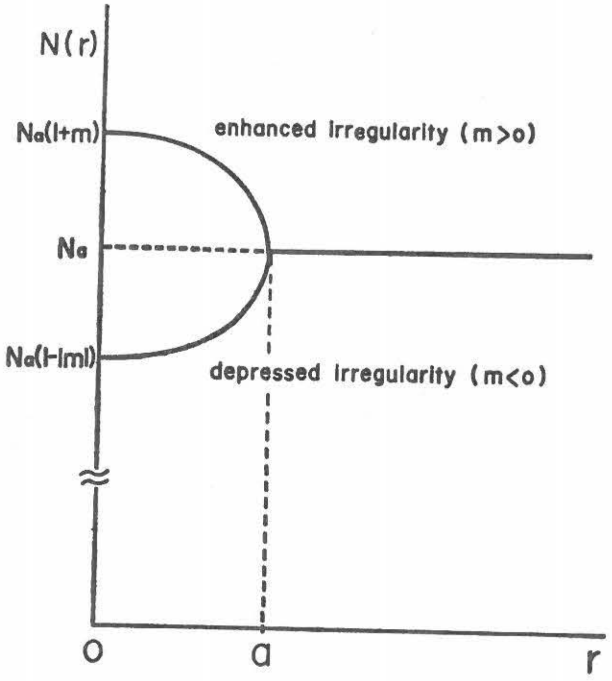

Fig. 2 Radial distribution of electron density.

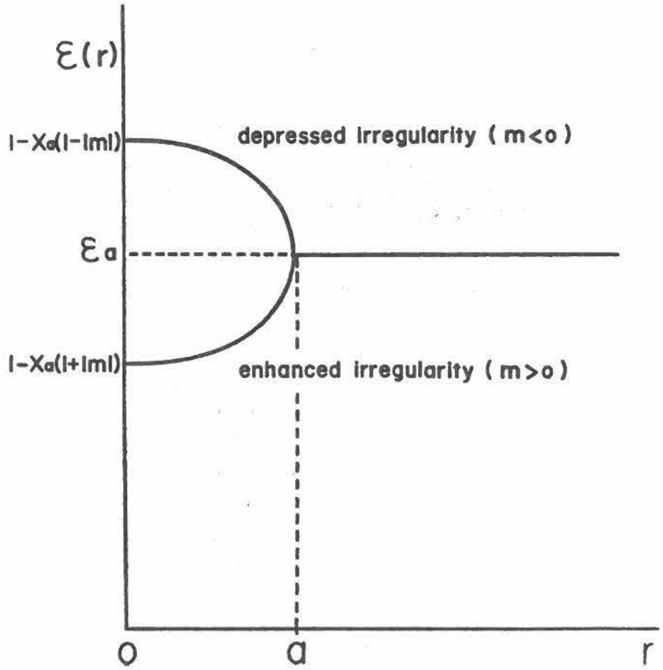

Fig. 3 Radial distribution of relative dielectric constant.

This density distribution is chosen for the following two reasons. First of all, this variation is a relatively good approximation to the observed profile. Secondly with this variation the solutions are expressed explicitly in terms of well known functions.

Then the relative dielectric constant is obtained using eq. (4) as,

$$\varepsilon(r)=\begin{cases}1-x_a(\equiv\varepsilon_a) & ;\ r\geq a\\ 1-x_a\{1+m(1-r^2/a^2)\} & ;\ r\leq a\end{cases} \tag{5}$$

where $x_a=\omega_{pa}^2/\omega^2$. $\omega_{pa}=\sqrt{e^2\mathrm{N}_a/m_e\varepsilon_0}$ is the ambient angular plasma frequency. The radial distribution of the density and relative dielectric constant are shown in Figs. 2 and 3.

3. Scattering by an inhomogeneous plasma column

3.1 Electromagnetic fields of incident, scattered and internal waves

There are two different cases of wave polarization, i.e., TE and TM modes. TM modes correspond to the case where the wave magnetic field vector is transverse to the irregularity axis (z-axis), and we consider only TM mode as the incident wave in the followings. As shown in Fig. 1, the wave vector of the incident wave lies in the plane $\phi=0$, i.e., x-z plane and makes an angle θ with the negative z-axis. And the electric field of the incident wave is polarized to be parallel to the plane $\phi=0$. Then the z-component of electric field of the incident wave is given as,

$$\mathrm{E}_z^{inc}=\mathrm{E}_0\sin\theta\exp(ik_\perp r\cos\phi)\cdot\exp(-ik_\parallel z) \tag{6}$$

where E_0 is the amplitude of the incident wave. $k_\perp=k_a\sin\theta$ and $k_\parallel=k_a\cos\theta$. k_a is the propagation constant in the ambient plasma. Using an addition theorem for Bessel functions, eq. (6) is rewritten

$$\mathrm{E}_z^{inc}=\mathrm{E}_0\sin\theta\sum_{n=-\infty}^{\infty}i^n\mathrm{J}_n(k_\perp r)\mathrm{F}_n \tag{7}$$

$$\mathrm{F}_n=\exp(i\omega t-in\phi-ik_\parallel z) \tag{8}$$

And as the result of the assumed polarization, we obtain

$$\mathrm{H}_z^{inc}=0 \tag{9}$$

Then we get the other components of electric and magnetic fields of the incident wave as follows.

$$\left.\begin{aligned} E_\phi^{inc} &= \sum_n \frac{-nk_\parallel}{k_\perp^2 r} E_0 \sin\theta\, i^n J_n(k_\perp r) F_n \\ H_\phi^{inc} &= \sum_n \frac{-k_a}{\zeta k_\perp} E_0 \sin\theta\, i^{n+1} J_n'(k_\perp r) F_n \end{aligned}\right\} \tag{10}$$

Here $\zeta=\sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_a}}$ is the wave impedance of the ambient plasma.

Next we seek for solutions of the internal waves. The internal field is obtained as the solution of eq. (3). Since the cylindrical irregularity is assumed to be of infinite length, the fields must vary periodically $\exp(-ik_\parallel z)$ in z-didection. Taking into account this, the wave equation inside the irregularity reduces to the following form,

$$\frac{\partial^2 \Pi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \Pi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 \Pi}{\partial \phi^2}+k_0^2\Big[\varepsilon(r)-\varepsilon_a\cos^2\theta\Big]\Pi=0 \tag{11}$$

where Π stands for either E_z or H_z. E_z is associated with TM modes and H_z with TE modes. $\varepsilon(r)$ is defined by eq. (5). As a solution we assume the following form.

$$E_z^{int}=\sum_{n=-\infty}^{\infty} A_n^{int} K_n(r) F_n \tag{12}$$

Substitution of eq. (12) into eq. (11) yields the equation for $K_n(r)$.

$$\frac{d^2K_n}{dr^2}+\frac{1}{r}\cdot\frac{dK_n}{dr}+\Big[k_0^2(\varepsilon(r)-\varepsilon_a\cos^2\theta)-\frac{n^2}{r^2}\Big]K_n=0 \tag{3}$$

After the suitable variable transformation, the solution of $K_n(r)$ which is finite at $r=0$, can be written as

$$K_n(r)=\frac{1}{r}M_{k,\,n/2}(\sqrt{-m}\sqrt{\xi}\ r^2) \tag{14}$$

where $\kappa=k_0^2(\varepsilon_a\sin^2\theta-mx_a)/4\sqrt{-m}\sqrt{\xi}$, $\xi=k_0^2x_a/a^2$ and $M_{\kappa,\,n/2}$ means the Whittaker function.

Therefore eq. (12) is rewritten as

$$E_z^{int}=\sum_{n=-\infty}^{\infty} A_n^{int}\frac{1}{r}M_{\kappa,\,n/2}(\sqrt{-m}\sqrt{\xi}\ r^2)\,F_n \tag{15}$$

By replacing A_n^{int} by B_n^{int}, the z-component of the magnetic field of the internal wave is given by

$$H_z^{int}=\sum_{n=-\infty}^{\infty} B_n^{int}\frac{1}{r}M_{\kappa,\,n/2}(\sqrt{-m}\sqrt{\xi}\ r^2)F_n \tag{16}$$

A_n^{int} and B_n^{int} are unknown coefficients specifying the magnitude of TM and TE modes, respectively.

The other components of the internal wave are obtained

$$\left.\begin{aligned} E_\phi^{int} &= -\frac{1}{p^2(r)r}\sum_n\left\{nk_\parallel A_n^{int}M - i\omega\mu_0 B_n^{int}\left[(2\sqrt{-m}\sqrt{\xi}\,r^2)M' - M\right]\right\}F_n \\ H_\phi^{int} &= -\frac{1}{p^2(r)r}\sum_n\left\{nk_\parallel B_n^{int}M + i\omega\varepsilon_0\,\varepsilon(r)\,A_n^{int}\left[(2\sqrt{-m}\sqrt{\xi}\,r^2)\,M' - M\right]\right\}F_n \end{aligned}\right\} \quad (17)$$

where $p^2(r)=k_0^2\,\varepsilon(r)-k_\parallel^2$ and $p^2(a)=k_\perp^2$.

The formulas (15) to (17) contain sufficient informations on the internal electromagnetic fields after the determination of the amplitude coefficients by introducing the boundary conditions on the surface of the cylinder.

Finally we write the field components of the scattered wave $(r \geq a)$ in terms of the Hankel functions by setting $\varepsilon(r)=\varepsilon_a$ in wave equation (11).

TM modes

$$\left.\begin{aligned} E_z^{sc} &= \sum_n a_n^{sc} H_n^{(2)}(k_\perp r) F_n \\ E_\phi^{sc} &= \sum_n -\frac{nk_\parallel}{k_\perp^2 r} a_n^{sc} H_n^{(2)}(k_\perp r)\, F_n \\ H_\phi^{sc} &= \sum_n -\frac{ik_a}{k_\perp \zeta} a_n^{sc} H_n^{(2)\prime}(k_\perp r) F_n \end{aligned}\right\} \quad (18)$$

TE modes

$$\left.\begin{aligned} H_z^{sc} &= \sum_n b_n^{sc} H_n^{(2)}(k_\perp r) F_n \\ E_\phi^{sc} &= \sum_n \frac{ik_a}{k_\perp} \zeta b_n^{sc} H_n^{(2)\prime}(k_\perp r)\, F_n \\ H_\phi^{sc} &= \sum_n -\frac{nk_\parallel}{k_\perp^2 r} b_n^{sc} H_n^{(2)}(k_\perp r) F_n \end{aligned}\right\} \quad (19)$$

In relations (18) and (19), a_n^{sc} and b_n^{sc} are the amplitude coefficients of the scattered TM and TE modes, respectively. General solution of the scattered wave is a superposition of TM and TE modes.

Total field in the external region $(r>a)$ consists of the incident wave and scattered wave.

3.2 Boundary conditions and amplitude coefficients

The amplitude coefficients a_n^{sc}, b_n^{sc}, A_n^{int} and B_n^{int} are determined by matching the tangential components of the electric and magnetic fields inside and outside the irregularity at the surface of the cylinder $r=a$. We then get,

$$\left.\begin{aligned} a_n^{sc} &= \frac{1}{\Delta}\begin{vmatrix} \eta_4\,\eta_8/\eta_2-\eta_5 & \eta_6-\eta_1\,\eta_7/\eta_2 \\ \eta_8\,\eta_{11}/\eta_2-\eta_9 & \eta_3-\eta_1\,\eta_4/\eta_2 \end{vmatrix} \\ b_n^{sc} &= \frac{1}{\Delta}\begin{vmatrix} \eta_3-\eta_1\,\eta_4/\eta_2 & \eta_4\,\eta_8/\eta_2-\eta_5 \\ \eta_{10}-\eta_1\,\eta_{11}/\eta_2 & \eta_8\eta_{11}/\eta_2-\eta_9 \end{vmatrix} \end{aligned}\right\} \tag{20}$$

with

$$\Delta=(\eta_3-\eta_1\,\eta_4/\eta_2)^2-(\eta_5-\eta_1\,\eta_7/\eta_2)(\eta_{10}-\eta_1\,\eta_{11}/\eta_2)$$

where the values of η_i's are given as follows.

$$\eta_1=\mathrm{H}_n,\qquad \eta_2=\mathrm{M}/a,\qquad \eta_3=-nk_{/\!/}\mathrm{H}_n/k_\perp{}^2a,$$

$$\eta_4=-nk_{/\!/}\mathrm{M}/k_\perp{}^2a^2,\qquad \eta_5=-nk_{/\!/}\eta_8/k_\perp{}^2a,\qquad \eta_6=-\zeta^2\eta_{10},$$

$$\eta_7=-\zeta^2\eta_{11},\qquad \eta_8=\mathrm{E}_0\sin\theta\, i^n\mathrm{J}_n,\qquad \eta_9=-\mathrm{E}_0 i^{n+1}\mathrm{J}_n'/\zeta,$$

$$\eta_{10}=-ik_a\mathrm{H}_n'/\zeta k_\perp \quad\text{and}\quad \eta_{11}=-i\omega\varepsilon_0\varepsilon_a[(2\sqrt{-m}\sqrt{\xi}\,a^2)\mathrm{M}'-\mathrm{M}]/k_\perp{}^2a^2 \tag{21}$$

J_n, H_n and M mean $\mathrm{J}_n(k_\perp a)$, $\mathrm{H}_n(k_\perp a)$ and $\mathrm{M}_{\varepsilon,n/2}(\sqrt{-m}\sqrt{\xi}\,a^2)$, respectively. The arguments of the primed functions are given by putting $r=a$.

Then we can obtain the remaining amplitude coefficients of the internal wave A_n^{int} and B_n^{int} from

$$\left.\begin{aligned} \mathrm{A}_n^{int} &= \frac{1}{\eta_2}(\eta_8+\eta_1\,a_n^{sc}) \\ \mathrm{B}_n^{int} &= \frac{\eta_1}{\eta_2}b_n^{sc} \end{aligned}\right\} \tag{22}$$

Substituting eqs. (20) and (21), exact expressions of the scattered fields are obtained as functions of frequency, irregularity width, incident angle and modulation factor.

Therefore eqs. (6) to (22) constitute the complete solution of the general problem.

4. Scattering cross section

For the purpose of numerical computations of scattering patterns, we now define the differential scattering cross section as follows,

$$\sigma(\phi)=\frac{r\frac{1}{2}Re(\mathbf{E}^{sc}\times\mathbf{H}^{sc*})\cdot\boldsymbol{i}_r}{\frac{1}{2}Re(\mathbf{E}^{inc}\times\mathbf{H}^{inc*})\cdot(-\boldsymbol{i}_r)} \tag{23}$$

where $\boldsymbol{i}_r$ is the unit vector in radial direction. The scattered fields in eq. (23) are far-zone fields using the asymptotic form of the Hankel function for large r.

Then we get the differential scattering cross section explicitly in terms of the amplitude coefficients of the scattered wave.

$$\sigma(\phi)=\frac{\zeta}{\pi k_1\sin\theta}\left[\left|\sum_n a_n^{sc}\exp\left\{-in\left(\phi-\frac{\pi}{2}\right)\right\}\right|^2+\left|\sum_n b_n^{sc}\exp\left\{-in\left(\phi-\frac{\pi}{2}\right)\right\}\right|^2\right] \tag{24}$$

Therefore the differential cross section is given as a function of frequency, irregularity width, incident angle and modulation factor.

5. Scattering by a magnetic field-aligned irregularity

As a special case, the scattering for normal incidence will be discussed. For the case of normal incidence ($\theta=90°$), the amplitude coefficients of the scattered fields, eq. (20) reduces to

$$\left.\begin{aligned} a_n^{sc}&=(\eta_8\,\eta_{11}/\eta_2-\eta_9)(\eta_{10}-\eta_1\,\eta_{11}/\eta_2)^{-1}\\ b_n^{sc}&=0\end{aligned}\right\} \tag{25}$$

where $\eta_i(i=3,\ 4,\ 5)=0$ in the relations (21) are used and the values of any other η_i's are obtained by putting $\theta=90°$ in the same relations (21). $b_n^{sc}=0$ indicates that the scattered wave is also purely TM mode for normal incidence provided that the incident wave is TM mode.

So far we have neglected the effect of the Earth's magnetic field. But the ionospheric irregularities are known to be elongated along the magnetic field. When we consider only TM modes, our theory is exactly applicable to the scattering by a magnetic field-aligned irregularity at normal incidence using eq. (25).

6. Scattering by inhomogeneous plasma columns arranged in regular space arrays

We now discuss the scattering from the irregularities which form regular space arrays. After Sato and Maeda (1966), the scattered electric field from the regularly distributed plasma columns can be expressed as the product of the field scattered from a single plasma column and the so-called array factor which is due to the spatial arrangement of the irregularities. Then using our analytical results and such an array factor as was given by Sato and Maeda, the total field scattered from inhomogeneous plasma columns forming regular space arrays can be obtained.

7. Conclusions

We have derived the analytical expressions of the scattered wave fields produced by radio waves incident at any angle with the irregularity, whose radial density distribution is of parabolic form and also the analytical expressions of the wave field distribution inside the irregularity. Formulas of the scattering patterns are given explicitly as functions of frequency, irregularity width, incident angle and modulation factor. As special cases, scattering by a magnetic field aligned irregularity at normal incidence and scattering by the columnar irregularities with regular spatial distribution are also discussed.

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