The Existence of a Pure Nash Equilibrium in the Two-player Competitive Diffusion Game on Graphs having Chordality[⋆,⋆⋆]

Naoka Fukuzono[a,1], Tesshu Hanaka[b], Hironori Kiya[c], Hirotaka Ono[a]

[a] *Department of Mathematical Informatics, Graduate School of Informatics, Nagoya University, Furo-cho, Chikusa-ku,, Nagoya, 464-8601, Aichi, Japan*
[b] *Department of Informatics, Faculty of Information Science and Electrical Engineering, Kyushu University, 744 Motooka, Nishi-ku,, Fukuoka, 819-0395, Fukuoka, Japan*
[c] *Department of Economic Engineering, Faculty of Economics, Kyushu University, 744 Motooka, Nishi-ku,, Fukuoka, 819-0395, Fukuoka, Japan*

Abstract

The competitive diffusion game is a game-theoretic model of information spreading on a graph proposed by Alon et al. (2010). It models the diffusion process of information in social networks where several competitive companies want to spread their information, for example. The nature of this game strongly depends on the graph topology, and the relationship is studied from several aspects. In this paper, we investigate the existence of a pure Nash equilibrium of the two-player competitive diffusion game on chordal and its related graphs. We show that a pure Nash equilibrium always exists on split graphs, block graphs, and interval graphs, all of which are well-known subclasses of chordal graphs. On the other hand, we show that a pure Nash equilibrium does not always exist on (strongly) chordal graphs; the boundary of the existence of a pure Nash equilibrium is found.

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Email addresses: `naoka0912f@gmail.com` (Naoka Fukuzono), `hanaka@inf.kyushu-u.jp` (Tesshu Hanaka), `h-kiya@econ.kyushu-u.ac.jp` (Hironori Kiya), `ono@nagoya-u.jp` (Hirotaka Ono)

[1] This work is done when she belongs to Nagoya University as a master course student. She is currently working at Nippon Telegraph and Telephone West Corporation.

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1. Introduction

The competitive diffusion game is a game-theoretic model of information spreading on a graph proposed by Alon et al. [1]. It is introduced in order to study information diffusion phenomena on social network services (SNS), such as Facebook and Twitter. For example, viral marketing is a typical commercial activity utilizing information diffusion phenomena on a social network. A game-theoretical setting happens when several companies want to sell interoperable products via viral marketing.

In the model, each player has its own information, and their objective is to spread it to as many vertices as possible; the score of a player is the number of vertices that eventually receives the player's information. Initially, all vertices are inactive. A player's strategy is just to choose a vertex of a given graph as a source, from which their information automatically spreads to other vertices along edges in a step-by-step manner. Once an inactive vertex receives first information from a player, the vertex gets to believe the information, that is, it joins the player's side and newly diffuses the information to its adjacent vertices. Even if a vertex of a player's side newly receives information from another player, it does not change its mind and remains in the current player's side. If an inactive vertex simultaneously receives information from more than one player, the vertex gets confused and does not join any player's side from then on. The scores of the players are determined when the diffusion stops.

This game models the following situation: The graph is a social network, where each vertex represents a person and each edge indicates that two persons (i.e., two endpoints of the edge) are friends with each other in an SNS. The players are commercial companies that want to sell interoperable products via viral marketing. Each company asks a person on the SNS to advertise its own product by paying some amount of money. The person receiving money recommends the product of the company to his/her friends. After a person receives a recommendation of a product from a friend, he/she decides to buy the product and newly recommends the product of the company to his/her friends. Sometimes a person simultaneously receives two types of recommendations. Then he/she gets confused, and he/she does not buy any of the products and recommend anything.

In analyses of game-theoretic models, one of the typical approaches is to focus on Nash equilibria, which is because finding a Nash equilibrium might help to predict the behaviour of rational players. It is known that every finite game always has a mixed-strategy Nash equilibrium, though a pure Nash equilibrium does not always exist. In fact, there is a graph of the two-player competitive diffusion game that has no pure Nash equilibrium [1], though a pure Nash equilibrium always exists for the competitive diffusion game with any number of players under some restricted graph classes such as cycles [3]. If a game has a pure Nash equilibrium, it implies that it is relatively easy to analyze. From such a viewpoint, several studies try to find reasonable classes of graphs under which a pure Nash equilibrium always exists. For more details, see the following subsection.

1.1. Related work

There are many studies that focus on the existence of a pure Nash equilibrium of the two-player competitive diffusion game. For example, Alon et al. give a graph with diameter 3 that has no pure Nash equilibrium [1]. Takehara et al. give a stronger example, a graph with diameter 2 which has no pure Nash equilibrium [19]. On the other hand, Small and Mason show that a pure Nash equilibrium always exists on trees [17]. Roshanbin shows that a pure Nash equilibrium always exists on cycles and grid graphs [16], and Sukenari et al. show that a pure Nash equilibrium always exists on torus grid graphs [18]. These results are about the two-player competitive diffusion game. For three or more players, the situation is different. For example, in most of the cases, a path always has a pure Nash equilibrium. The exception is the case where the number of players is 3 and the number of vertices is at least 6. On the other hand, a cycle always has a pure Nash equilibrium for the case where the number of players and the number of vertices are arbitrary [3]. Li and Shigeno investigate the existence of pure Nash equilibria on weighted paths and cycles with an arbitrary number of players [14], where the score of a player is defined as the total weight of the vertices influenced by the player. Note that their model allows negative weights.

If the number k of players is bounded by a constant, it can be done in polynomial time to check whether a given graph has a pure Nash equilibrium or not, because the number of combinations of strategies is $O(n^k)$, where n is the number of vertices. On the other hand, it is not trivial to check the existence of a pure Nash equilibrium for general k. Etesami and Basar show that the decision problem of the existence of a pure Nash equilibrium

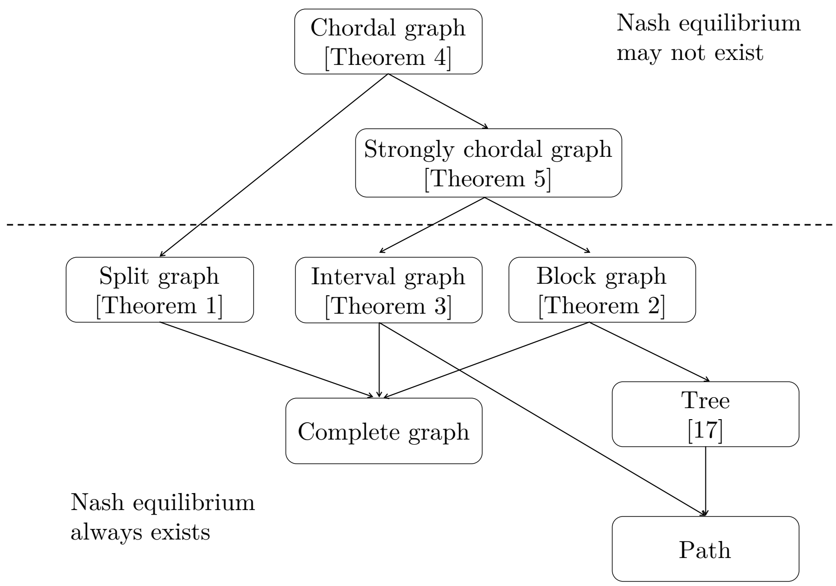


Figure 1: Graph classes and the existence of a pure Nash equilibrium. Connections between two graph classes imply that the above one is a super class of the below one.

for general k is NP-complete [6]. Furthermore, Ito et al. show that the decision problem of the existence of a pure Nash equilibrium is W[1]-hard when parameterized by k [12].

1.2. Our results

In this paper, we investigate the existence of a pure Nash equilibrium of the two-player competitive diffusion game on chordal and its related graphs. A graph is called *chordal* if every induced cycle in the graph has exactly three vertices. The class of chordal graphs is well studied in many research fields, and they are also called rigid circuit graphs or triangulated graphs. Particularly in algorithm theory, it is considered very important, because many NP-hard graph optimization problems become tractable if the input graph is restricted to be chordal. Due to the tractability, chordal approximation (i.e., modifying an input graph to make chordal) is used in various research fields, such as graphical modeling in statistics and numerical computation. Furthermore, the notion of clustering coefficient, which is a well-used measure for social network analysis (e.g., [11, 20]), is related to triangulated structures; a graph with a high clustering coefficient tends to be locally triangulated, that is, the subgraph induced by a vertex and its closed neighborhood tends to be chordal. In other words, social networks might be considered to satisfy a relaxed notion of chordality [4, 9]. These are motivations to focus on chordal graphs.

We obtain the following results: We show that a pure Nash equilibrium always exists on split graphs, block graphs, and interval graphs, all of which are well-known subclasses of chordal graphs. On the other hand, we show that there is a (strongly) chordal graph that has no pure Nash equilibrium; the boundary of the existence of a pure Nash equilibrium is found. The results are summarized in Figure 1.

The rest of the paper is organized as follows. In Section 2, we define several notations and terminologies and introduce graph classes. Section 3 is the main part of this paper. We show that a pure Nash equilibrium always exists on block graphs, split graphs, and interval graphs. In Section 4, we give a (strongly) chordal graph that has no pure Nash equilibrium. Section 5 concludes the paper by giving some remarks for future work.

2. Preliminaries

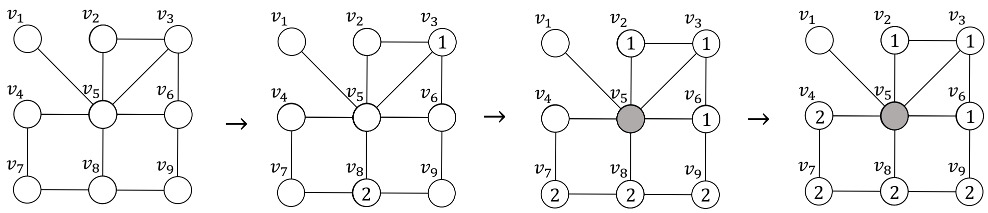
In this paper, we use the standard graph notation. Let $G = (V, E)$ be an undirected connected graph where $|V| = n$ and $|E| = m$. For a graph G', the vertex set (resp., edge set) of G' is denoted by $V(G')$ (resp., $E(G')$). If $\{u, v\} \in E(G)$, we say that u (resp., v) is a *neighbor* of v (resp., u), or vertices u and v are *adjacent* in G. The set of neighbors of v in G is denoted by $N_G(v)$, or simply by $N(v)$. Namely, $N_G(v) = \{u \in V(G) \mid \{v, u\} \in E(G)\}$. Similarly, the set of closed neighbors of v in G is denoted by $N_G[v]$ or $N[v]$, that is, $N_G[v] = N_G(v) \cup \{v\}$. For $V' \subseteq V$, let $G[V']$ denote the subgraph induced by V'. A graph G is called *complete* if every vertex pair is adjacent in G. The complete graph on n vertices is denoted by K_n.

A vertex set C is called a *clique* if $G[C]$ is a complete graph. Moreover, a clique C of G is called *maximal* if G has no clique C' such that $C \subsetneq C'$.

2.1. Competitive diffusion game

Let p_1 and p_2 be players 1 and 2, respectively. Also, let $G = (V, E)$ be an undirected connected graph. Then the two-player competitive diffusion game on G proceeds as follows (see also Figure 2).

Time 0. All the vertices are set *inactive*.



(a) Time 0 (b) Time 1 (c) Time 2 (d) Time 3

Figure 2: An example of a two-player competitive diffusion game. White vertices with "1" and "2" stand for vertices dominated by p_1 and p_2, respectively. A white vertex with no number is inactive and a grey vertex is neutral. (a) At Time 0, the graph is in the initial state where all the vertices are inactive. (b) At time 1, p_1 chooses v_3 and p_2 chooses v_8. (c) At time 2, v_2 and v_6 are dominated by p_1 and v_7 and v_9 are dominated by p_2. Vertex v_5 becomes neutral. (d) At time 3, v_4 is dominated by p_2. Since no player can dominate a vertex any more, the game ends. In the end of the game, v_1 is an inactive vertex. The utility of p_1 is $U_1(v_3, v_8) = 3$ and the utility of p_2 is $U_2(v_3, v_8) = 4$.

Time 1**.** Player p_1 and p_2 choose arbitrary vertices, respectively say v_{p_1} and v_{p_2} in V. These are called *initial vertices*. If a vertex v is chosen by only one player p, we say v is *dominated* by p. If both players choose a vertex v, the vertex v becomes *neutral*. Let $V_1 := \{v_{p_1}\}$ and $V_2 := \{v_{p_2}\}$ if $v_{p_1} \neq v_{p_2}$, and let $V_0 := \{v\}$ as the set of neutral vertices if $v_{p_1} = v_{p_2}$. Once a vertex v is set into V_1, V_2, or V_0, v is never removed from then on.

Time t **(**$t \geq 2$**).** For every inactive vertex $v \in V$, check $N(v) \cap V_{p_1}$ and $N(v) \cap V_{p_2}$. If both are nonempty, v becomes neutral. Update $V_0 := V_0 \cup \{v\}$. If only $N(v) \cap V_{p_1}$ (resp., $N(v) \cap V_{p_2}$) is nonempty, p_1 (resp., p_2) dominates v. Update $V_{p_1} := V_{p_1} \cup \{v\}$ (resp., $V_{p_2} := V_{p_2} \cup \{v\}$). If no inactive vertex changes the status, the process ends. The utilities (or scores) of p_1 and p_2 are respectively determined as $|V_{p_1}|$ and $|V_{p_2}|$.

The vertex s chosen by player p at Time 1 is called the *strategy* of p. For two players p_1 and p_2, a *strategy profile* $\mathbf{s} = (s_1, s_2)$ is a pair of strategies of p_1 and p_2. For a strategy profile $\mathbf{s}$, the utility $U_i(\mathbf{s})$ of p_i is $|V_{p_i}|$, that is, the number of vertices dominated by p_i at the end of a game. In Figure 2, the utility of p_1 is $U_1(v_3, v_8) = 3$ and the utility of p_2 is $U_2(v_3, v_8) = 4$.

For a strategy s of player p_1 (resp., p_2), a strategy s^* of player p_2 (resp., p_1) is called a *best response* if it satisfies that $U_2(s, s^*) = \max_{s' \in V} U_2(s, s')$ (resp.,

$U_1(s^*, s) = \max_{s' \in V} U_1(s', s)$). Then we define a pure Nash equilibrium in the two-player competitive diffusion game.

Definition 1. *A strategy profile* $\mathbf{s} = (s_1, s_2)$ *is called a* pure Nash equilibrium *if there is no vertex* $v \in V$ *such that* $U_1(v, s_2) > U_1(s_1, s_2)$ *or* $U_2(s_1, v) > U_1(s_1, s_2)$*, that is, if no player can increase their utility by changing their own strategy.*

In other words, a strategy profile $\mathbf{s} = (s_1, s_2)$ is a pure Nash equilibrium if s_1 and s_2 are best responses to each other.

We call the two-player competitive diffusion game *2-CDG* for short. Also, we simply use the term "Nash equilibrium" instead of "pure Nash equilibrium" hereafter. Note that if two players choose an identical initial vertex, their utilities are both 0, which cannot be a Nash equilibrium if $|V| \geq 2$. Thus, in the arguments of this paper, we assume that the vertices chosen by the players are distinct and the utilities are at least 1.

Before concluding this subsection, we give a small remark about the usage of the word "changing a strategy" or something like that, to avoid confusion. To argue that a strategy profile $\mathbf{s} = (s_1, s_2)$ is a Nash equilibrium, we sometimes say that p_1 and p_2 has no incentive to *change* their strategies. Or, to show that a strategy profile $\mathbf{s} = (s_1, s_2)$ is not a Nash equilibrium, we may say that player p_1 *changes* their strategy s_1 to another strategy (typically, a best response) s'_1 to increase their utility. These words "change" are used to compare two strategies and are not used for explaining the players' behaviors inside of the game process; the change is done before the game really starts.

2.2. Graph classes

In this subsection, we define several graph classes. A graph $G = (V, E)$ is a *chordal graph* if every cycle of length at least 4 has a chord, or equivalently every induced cycle has exactly 3 vertices [5]. A graph $G = (V, E)$ is a *strongly chordal graph* if it is a chordal graph and every cycle of even length (≥ 6) has an odd chord, that is, an edge that connects two vertices that are an odd distance apart from each other in the cycle. Equivalently, a strongly chordal graph is a chordal graph that includes no n-sun (for $n \geq 3$) as an induced subgraph [7]. Here, an n-sun forms a graph of $2n$ vertices that consist of a central K_n with vertices $\{v_1, v_2, \ldots, v_n\}$ and outer vertices $\{u_1, u_2, \ldots, u_n\}$ with edges $\{u_i, v_i\}$ and $\{u_i, v_{i+1}\}$ for $i = 1, \ldots, n$ and $n + 1 \equiv 1$. Examples of chordal and strongly chordal graphs are shown in Figures 10 and 11 in

Section 4; it is easy to see that both graphs have no chordless cycle with length at least 4, and the graph in Figure 10 is not a strongly chordal graph but chordal graph, because it contains a 3-sun.

A graph $G = (V, E)$ is a *block graph* if every maximal 2-connected component is a clique [10]. Intuitively, a block graph can be considered a tree by regarding each of its maximal cliques as a meta-vertex or a meta-edge. By the definition of a block graph, a tree is also a block graph.

A graph $G = (V, E)$ is a *split graph* if V can be partitioned into an independent set I and a clique C (see Figure 3) [15]. By definition, an n-sun is a split graph. A graph $G = (V, E)$ is an *interval graph* if there is a set of intervals on the real line where the intervals correspond to the vertices such that G has edge $\{u, v\}$ if and only if two intervals corresponding to u and v intersect. We call such a set of intervals an *interval representation of* G [13]. An example of an interval graph with an interval representation is shown in Figure 5.

Note that strongly chordal, block, split, and interval graphs are all chordal. Figure 1 also shows the relations among these graphs. For example, the class of strongly chordal graphs includes that of block graphs, and they include that of trees. For more information about graph classes, see [2].

3. The Existence of a Nash equilibrium

Before starting this section, we give a general and basic observation.

Proposition 1. *Suppose that u, u' and v $(\neq u)$ are vertices in a graph G, and $N(u') \subseteq N(u)$. In a 2-CDG on G, if u' is a best response for v, then u is also a best response for v.*

This proposition clearly holds, because a vertex dominated by p_1 under strategy profile (u', v) is also dominated by p_1 under (u, v).

A typical way to show that a given strategy profile (u, v) is a Nash equilibrium is by contradiction. If it is not a Nash equilibrium, u (resp., v) is not a best response for v (resp., u), which implies that there is a better strategy for v than u. By Proposition 1, candidates of a better strategy can be restricted to $\{v' \in V \mid \nexists v'' \in V : N(v') \subsetneq N(v'')\}$. Namely, what we show is that the existence of a better strategy $u' \in \{v' \in V \mid \nexists v'' \in V : N(v') \subsetneq N(v'')\}$ for v than u leads to a contradiction.

In the following subsections, we investigate the existence of a Nash equilibrium for subclasses of chordal graphs.

3.1. Split graph

Theorem 1. *In any 2-CDG on split graphs, a Nash equilibrium always exists.*

To show Theorem 1, we prove the following three lemmas.

Lemma 1. *Let $G = (C \cup I, E)$ be a split graph, where C forms a clique and I is an independent set. If the strategy profile of p_1 and p_2 is (u, v) with $u, v \in C$, the utilities of p_1 and p_2 are $U_1(u, v) = |N(u) \setminus N(v)| = |N(u)| - |N(u) \cap N(v)|$ and $U_2(u, v) = |N(v) \setminus N(u)| = |N(v)| - |N(u) \cap N(v)|$, respectively.*

Proof. We can observe that every $w \in C \setminus \{u, v\}$ becomes neutral because C forms a clique. For a vertex w in I, we consider the following cases: (a) w is adjacent to both u and v, which also becomes neutral, (b) w is adjacent to u but not adjacent to v, which is dominated by p_1, (c) w is not adjacent to u but adjacent to v, which is dominated by p_2, and (d) w is adjacent to neither vertex u nor v, which remains inactive because all the neighboring vertices are neutral in C. Thus we count the number of the vertices of (b), which is $|(N(u) \setminus N(v)) \cap I|$. By adding 1 for u itself, we obtain $U_1(u, v) = |(N(u) \setminus N(v)) \cap I| + 1$. By $1 = |\{u\}| = |\{v\}| = |(N(u) \setminus N(v)) \cap C|$, we have $U_1(u, v) = |(N(u) \setminus N(v)) \cap I| + |\{u\}| = |(N(u) \setminus N(v)) \cap I| + |(N(u) \setminus N(v)) \cap C| = |N(u) \setminus N(v)|$. Similarly, we have $U_2(u, v) = |N(v) \setminus N(u)|$. □

Lemma 2. *On any split graph $G = (C \cup I, E)$, if both p_1 and p_2 choose strategies (i.e., vertices) in C, neither p_1 nor p_2 can increase their own utility by changing their strategy to a vertex in I.*

Proof. Suppose that p_1 changes the strategy from $u \in C$ to $x \in I$. Then, $U_1(x, v) = 1$, that is, the least score, because the vertices in $N(x)$ $(\subseteq N(v))$ are neutral or v itself. The same argument holds for p_2. □

By a similar argument, we can see that if the game on a split graph has a Nash equilibrium, its strategy profile must consist of two distinct vertices in C.

From Lemmas 1 and 2, we obtain Lemma 3, which concludes Theorem 1.

Lemma 3. *There is a Nash equilibrium (u, v) for $u, v \in C$.*

Proof. We prove this by contradiction; if there is no Nash equilibrium (u, v) for any $u, v \in C$, a contradiction arises as we see below. Consider that p_1 and

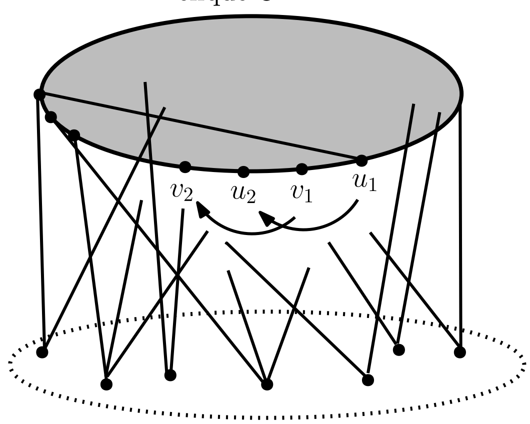


Figure 3: Nash dynamics on a split graph, where clique C is colored in gray

p_2 choose vertices two distinct vertices $u, v \in C$, respectively. Since it is not a Nash equilibrium, at least one player can increase their utility by changing their strategy to a new strategy, which is also a vertex in C by Lemma 2. This new strategy profile again consists of two distinct vertices in C, and the same arguments perpetually continue.

The argument yields an infinite sequence of strategy profiles on C. It contains a cyclic subsequence $\langle \mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \ldots, \mathbf{s}^{(k)} \rangle$ where $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(k)}$ are equivalent, because otherwise it contradicts the finiteness of C. See Figure 3. Let $\mathbf{s}^{(j)} = (u^{(j)}, v^{(j)})$ for $j = 1, \ldots, k$, where $k \geq 3$ by definition. Since p_1 and p_2 alternatively changes their strategies (otherwise, we can ignore intermediate changes), we can assume that player p_1 change the strategy $u^{(j)}$ to $u^{(j+1)}$ and p_2 stays at the strategy $v^{(j+1)} := v^{(j)}$ for odd $j < k$ and player p_2 changes the strategy $v^{(j)}$ to $v^{(j+1)}$ and p_1 stays at the strategy $u^{(j+1)} := u^{(j)}$ for even $j < k$, without loss of generality. If k is odd, $u^{(k)} = u^{(1)}$ and $v^{(k)} = v^{(1)}$ holds by the equivalence of $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(k)}$. If k is even, $u^{(k)} = v^{(1)}$ and $v^{(k)} = u^{(1)}$ holds and the positions of p_1 and p_2 are exchanged. Since the case of even k can be reduced to the odd case by duplicating the sequence, we show that a contradiction arises only for odd k in the following.

The above sequence implies that $U_1(\mathbf{s}^{(1)}) < U_1(\mathbf{s}^{(2)})$, $U_2(\mathbf{s}^{(2)}) < U_2(\mathbf{s}^{(3)})$, $\ldots, U_2(\mathbf{s}^{(k-1)}) < U_2(\mathbf{s}^{(k)}) = U_2(\mathbf{s}^{(1)})$. By summing up these inequalities, we obtain

$$\sum_{i=1}^{(k-1)/2} (U_1(\mathbf{s}^{(2i-1)}) + U_2(\mathbf{s}^{(2i)})) < \sum_{i=1}^{(k-1)/2} (U_1(\mathbf{s}^{(2i)}) + U_2(\mathbf{s}^{(2i+1)})). \tag{1}$$

By Lemma 1, the left side is transformed as follows:

$$\begin{aligned}
&\sum_{i=1}^{(k-1)/2} (|N(u^{(2i-1)}) \setminus N(v^{(2i-1)})| + |N(v^{(2i)}) \setminus N(u^{(2i)})|) \\
=& \sum_{i=1}^{(k-1)/2} (|N(u^{(2i-1)})| + |N(v^{(2i)})|) - \sum_{j=1}^{k-1} |N(u^{(j)}) \cap N(v^{(j)})|
\end{aligned}$$

On the other hand, the right side is transformed as follows:

$$\begin{aligned}
&\sum_{i=1}^{(k-1)/2} (|N(u^{(2i)}) \setminus N(v^{(2i)})| + |N(v^{(2i+1)}) \setminus N(u^{(2i+1)})|) \\
=& \sum_{i=1}^{(k-1)/2} (|N(u^{(2i)})| + |N(v^{(2i+1)})|) - \sum_{j=2}^{k} |N(u^{(j)}) \cap N(v^{(j)})|. \qquad (2)
\end{aligned}$$

Here, recall that $\mathbf{s}^{(1)} = \mathbf{s}^{(k)}$, $v^{(j+1)} = v^{(j)}$ for odd j, and $u^{(j+1)} = u^{(j)}$ for even j hold, which also implies $u^{(k)} = u^{(1)}$ and $v^{(k)} = v^{(1)} = v^{(2)}$. Thus, we have

$$(2) = \sum_{i=1}^{(k-1)/2} (|N(u^{(2i-1)})| + |N(v^{(2i)})|) - \sum_{j=1}^{k-1} |N(u^{(j)}) \cap N(v^{(j)})|,$$

which is equal to the left side and contradicts the strict inequality of (1). This completes the proof. □

3.2. Block graph

Theorem 2. *In any 2-CDG on block graphs, a Nash equilibrium always exists.*

For a block graph G, let $\mathcal{C}$ denote the set of all maximal cliques, i.e., maximal 2-connected components. A vertex u is called a *cut vertex* if $G[V \setminus \{u\}]$ is disconnected. If v is not a cut vertex, v belongs to a unique maximal 2-connected component (i.e., maximal clique) C, and its neighbors are all in

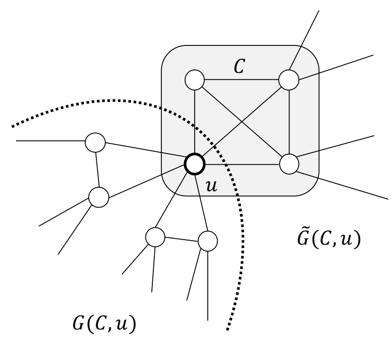


Figure 4: The figure of $G(C,u)$ and $\tilde{G}(C,u)$ for a maximal clique C and a cut vertex u on a block graph.

C. For a cut vertex u in a maximal clique C, let $G(C,u)$ be the connected component in $G[V \setminus C \cup \{u\}]$ that contains u, $\tilde{G}(C,u)$ be the component consisting of the remaining vertices. In other words, $\tilde{G}(C,u)$ is the connected component in $G[V \setminus \{u\}]$ that contains $C \setminus \{u\}$ (see also Figure 4). Let $\nu(C,u) = |V(G(C,u))|$, which is equal to $n - |V(\tilde{G}(C,u))|$. For a pair of $C \in \mathcal{C}$ and a vertex $x \in C$, we define $w(C,x)$ as follows:

$$w(C,x) = \begin{cases} \nu(C,x) & x \text{ is a cut vertex in } C, \\ 1 & \text{otherwise.} \end{cases}$$

For a maximal clique C, we sort the values $w(C,u)$'s of all u's in the descending order as $w_1(C)$, $w_2(C), \ldots, w_{|C|}(C)$, and let the corresponding u's be $u_1^C, u_2^C, \ldots, u_{|C|}^C$, respectively. Then a Nash equilibrium of 2-CDG on a block graph G is characterized as follows.

Lemma 4. *Let C^* be a maximal clique satisfying $w_2(C^*) = \max_{C \in \mathcal{C}} w_2(C)$. Then, the strategy profile $(u_1^{C^*}, u_2^{C^*})$ is a Nash equilibrium, and the utilities of p_1 and p_2 are $w_1(C^*)$ and $w_2(C^*)$, respectively.*

Proof. We first remark that the utilities of p_1 choosing $u \in C$ and p_2 choosing $v \in C$ are $w(C,u)$ and $w(C,v)$, respectively. In fact, if p_1 and p_2 choose u and v in C respectively, the other vertices in C become neutral. Thus, the vertices which can be dominated by p_1 or p_2 are in $G[V \setminus C]$. Since p_1 (resp., p_2) can dominate the vertices in $G(C,u)$ (resp., $G(C,v)$) and cannot dominate the vertices in $\tilde{G}(C,u)$ (resp., $\tilde{G}(C,v)$), the utility is $|V(G(C,u))| = w(C,u)$ (resp., $|V(G(C,v))| = w(C,v)$). Notice that the utility $w(C,u)$ of p_1 choosing u does not depend on p_2's choice as long as p_2 chooses a vertex in C.

We now show that strategy profile $(u_1^{C^*}, u_2^{C^*})$ is a Nash equilibrium. We see neither p_1 nor p_2 has an incentive to change their strategy. We first consider p_1. If p_1 may have an incentive to change their strategy, a better one than $u_1^{C^*}$ must be outside of C^*, because $w(C^*, u_1^{C^*})$ is the largest among $w(C^*, u)$'s for $u \in C^*$. Let v be the vertex in $V \setminus C^*$ that p_1 might move to, and v' be the nearest vertex in C^* from v. Note that such v' is a cut vertex and uniquely determined due to the property of block graphs. Since p_2 is in C^*, the vertices that p_1 at v can dominate are in $G(C^*, v')$, which implies that $U_1(v, u_2^{C^*}) \le |V(G(C^*, v'))| = \nu(C^*, v') = w(C^*, v') \le w(C^*)_1 = U_1(u_1^{C^*}, u_2^{C^*})$ holds. Namely, p_1 does not have an incentive to change the strategy from $u_1^{C^*}$ to such v.

We next show by contradiction that p_2 does not have an incentive to change their strategy; we assume that p_2 has. Then, a better strategy than $u_2^{C^*}$ must be outside of C^* again, because the unique candidate $u_1^{C^*}$ in C^* has been already occupied by p_1. Thus we consider the case where p_2 moves to $v \in V \setminus C^*$, and let v' be the nearest vertex in C^* from v. By a similar argument as above, p_2 has no incentive to move to v if the corresponding v' is in $C^* \setminus \{u_1^{C^*}\}$. Only the possible case is $v' = u_1^{C^*}$. Here, we can assume that v is adjacent to $v' (= u_1^{C^*})$, because otherwise p_2 can increase their utility by approaching to $u_1^{C^*}$, which reduces a neutral vertex. Since v and $u_1^{C^*}$ are adjacent, there is a maximal clique C' such that $v, u_1^{C^*} \in C'$. Then, the utilities of p_1 and p_2 for strategy profile $(u_1^{C^*}, v)$ are $w(C', u_1^{C^*})$ and $w(C', v)$, respectively. Since p_2 moves to v in order to increase their utility, $w(C', v) > w(C^*, u_2^{C^*})$ holds. By p_2 moving to v, $u_2^{C^*} \in C^*$ becomes vacant, which implies that p_1 gets to dominate at least $u_2^{C^*}$ and the vertices in $V(G(C^*, u_2^{C^*}))$, that is, $w(C', u_1^{C^*}) > w(C^*, u_2^{C^*})$. These imply that $w(C^*, u_2^{C^*}) = w_2(C^*) < w_2(C') \le \max_{C \in \mathcal{C}} w_2(C)$, which contradicts the definition of C^*. This completes the proof. □

Clearly the strategy profile of Lemma 4 always exists, and hence a Nash equilibrium always exists on a block graph. This concludes the proof of Theorem 2.

3.3. Interval graph

Theorem 3. *In any 2-CDG on interval graphs, a Nash equilibrium always exists.*

Before starting the proof of Theorem 3, we introduce new notation and basic concepts concerning interval graphs. We assume that an interval graph

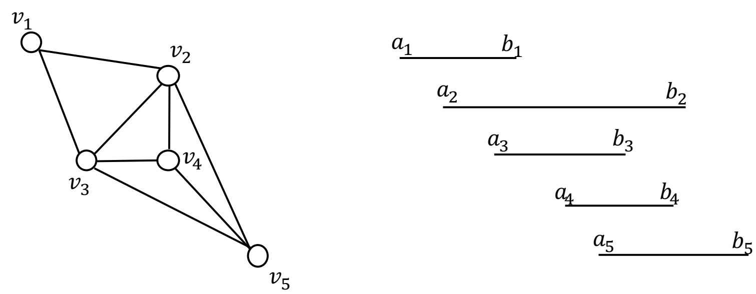


Figure 5: An example of an interval graph (left) and its interval representation sorted in increasing order of the initial endpoints a_i's (right). In this example, $\mathcal{I} = \{[a_1, b_1], \ldots, [a_5, b_5]\}$ and $\mathcal{I}_{\max} = \{[a_1, b_1], [a_2, b_2], [a_5, b_5]\}$.

$G = (V, E)$ is given by intervals $\mathcal{I} = \{I_1, \ldots I_n\}$, where each interval $I_i = [a_i, b_i]$ $(i = 1, \ldots n)$ of two integers $a_i \leq b_i$ corresponds to vertex i. The endpoint a_i of I_i is called the *initial endpoint* and the other endpoint b_i is called *terminal endpoint*. We assume that $\{I_1, \ldots I_n\}$ are sorted in increasing order of the initial endpoints a_i's (see Figure 5).

We also assume that a_i's and b_i's are all distinct without loss of generality. Recall that two vertices i and j are adjacent on an interval graph if and only if the corresponding intervals I_i and I_j intersect. In this section, the arguments are described mainly in interval representations. For example, we say that a player p chooses an interval I_i (or simply i) instead that p chooses vertex i. If interval I_i contains interval I_j, that is, $a_i \leq a_j \leq b_j \leq b_i$, it is denoted by $I_j \subseteq I_i$, and $N(j) \subseteq N(i)$ holds in G. On the other hand, if two intervals I_i and I_j intersect where $i < j$ and $I_j \not\subseteq I_i$, we say that I_i and I_j *properly* intersect.

Let $\mathcal{I}_{\max}$ denote $\{I_i \in \mathcal{I} \mid \nexists I_j \in \mathcal{I} : I_i \subsetneq I_j\}$, that is, $\mathcal{I}_{\max}$ is the set of intervals in $\mathcal{I}$ that are not contained in any other interval. We call $\mathcal{I}_{\max}$ the *maximal set* of $\mathcal{I}$. By this definition, for $I_i, I_j \in \mathcal{I}_{\max}$ with $i < j$, $a_i < a_j$ and $b_i < b_j$ hold. Here, we consider the case $|\mathcal{I}_{\max}| = 1$. In such a case, the vertex u^* corresponding to the interval in $\mathcal{I}_{\max}$ is adjacent to any other vertex in G. It implies that 2-CDG on G always has a Nash equilibrium, which forms (u^*, v) for $\forall v \neq u^*$. Thus, we assume that $|\mathcal{I}_{\max}| > 1$ hereafter. Note that strategies for Nash equilibria in the 2-CDG on $\mathcal{I}$ can be restricted to vertices in $\mathcal{I}_{\max}$ by Proposition 1 and $|\mathcal{I}_{\max}| > 1$. For a vertex set V' corresponding to $\mathcal{I}'$, the subgraph of G induced by V' is denoted also by $G[\mathcal{I}']$. Note that $G[\mathcal{I}_{\max}]$ is also connected. In general, $G[\mathcal{I}']$

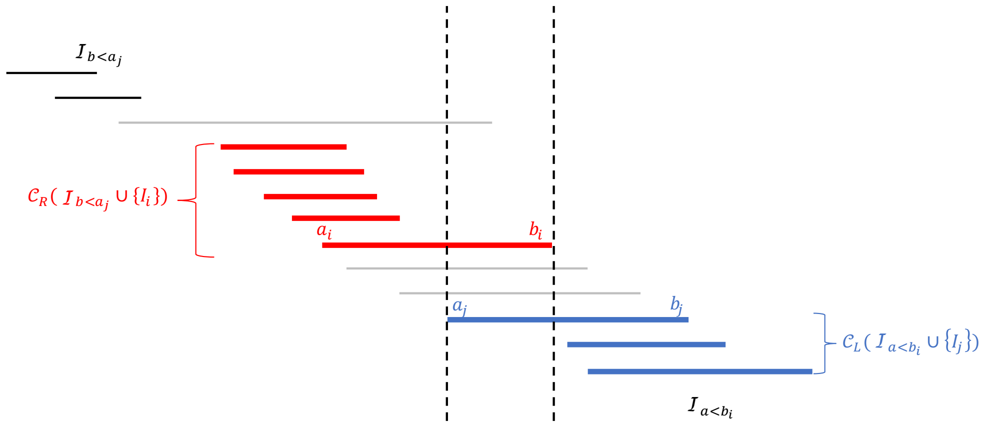


Figure 6: This figure shows the case that p_1 and p_2 choose strategies i and j, respectively. Red (resp., blue) bold intervals are dominated by p_1 (resp., p_2). Grey intervals become neutral. Black intervals remain to be inactive.

can be disconnected. We say that vertex u is *reachable* from v if there is a path between u and v. In terms of intervals, an interval I_u is *reachable* from I_v if there is a sequence of intervals $I_u = I_{u_1}, I_{u_2}, \ldots, I_{u_k} = I_v$, where I_{u_i} and $I_{u_{i+1}}$ intersect for $i = 1, \ldots, k-1$. For $\mathcal{I}' \subseteq \mathcal{I}$, let $R(\mathcal{I}')$ and $L(\mathcal{I}')$ denote the right-most interval and the left-most interval, respectively, which means that $b_r = \max\{b_i \mid [a_i, b_i] \in \mathcal{I}'\}$ holds for $[a_r, b_r] := R(\mathcal{I}')$ and $a_l = \min\{a_i \mid [a_i, b_i] \in \mathcal{I}'\}$ holds for $[a_l, b_l] := L(\mathcal{I}')$. Also, let $\mathcal{C}(\mathcal{I}', I)$ denote the set of intervals in $\mathcal{I}'$ that are reachable from I in $G[\mathcal{I}']$. That is, $\mathcal{C}(\mathcal{I}', I)$ is the set of intervals corresponding to the connected component of $G[\mathcal{I}']$ containing I. Thus, $\mathcal{C}(\mathcal{I}', R(\mathcal{I}'))$ (resp., $\mathcal{C}(\mathcal{I}', L(\mathcal{I}'))$) are the set of intervals in $\mathcal{I}'$ that are reachable from the right-most (resp., left-most) interval. We simply write $\mathcal{C}_R(\mathcal{I}')$ and $\mathcal{C}_L(\mathcal{I}')$ instead of $\mathcal{C}(\mathcal{I}', R(\mathcal{I}'))$ and $\mathcal{C}(\mathcal{I}', L(\mathcal{I}'))$, respectively.

We further introduce new notation. For $x \in \mathbb{R}$, define $\mathcal{I}_{b<x} = \{I_i = [a_i, b_i] \in \mathcal{I} \mid b_i < x\}$ and $\mathcal{I}_{a>x} = \{I_i = [a_i, b_i] \in \mathcal{I} \mid a_i > x\}$, or something like that. Let us focus on a pair (i, j) of intervals in $\mathcal{I}_{\max}$ such that $i \neq j$ and $I_i \cap I_j \neq \emptyset$, that is, (i, j) forms an edge e in G. We call such a pair of intervals *neighboring* or *adjacent*. Note that $a_i < a_j$ implies $b_i < b_j$ by the property of $\mathcal{I}_{\max}$. By using the notation, we can explicitly express the players' utilities when they choose neigboring intervals in $\mathcal{I}_{\max}$.

Lemma 5. *Suppose that G is an interval graph defined by $\mathcal{I}$, and players p_1 and p_2 choose neighboring I_i and I_j with $i < j$ in $\mathcal{I}_{\max}$ in 2-CDG on G,* *respectively. Then, the utilities of p_1 and p_2 are $U_1(i,j) = |\mathcal{C}_R(\mathcal{I}_{b<a_j} \cup \{I_i\})|$ and $U_2(i,j) = |\mathcal{C}_L(\mathcal{I}_{a>b_i} \cup \{I_j\})|$, respectively.*

Proof. Intervals in $\mathcal{I}$ intersecting $[a_j, b_i]$ intersect both I_i and I_j, and thus such intervals (vertices) become neutral at Time 2 (see Figure 6). Since strategy profile (i,j) forms an edge $\{i,j\}$, no other vertex becomes neutral after Time 2, because intervals that are newly dominated by p_1 and p_2 respectively spread to the left-hand direction and the right-hand direction on the line. We can see that the set of such neutral intervals are $\mathcal{I} \setminus (\mathcal{I}_{b<a_j} \cup \mathcal{I}_{a>b_i} \cup \{I_i, I_j\})$. By eliminating these from $\mathcal{I}$, we obtain $G[\mathcal{I}_{b<a_j} \cup \mathcal{I}_{a>b_i} \cup \{I_i, I_j\}]$. If we further remove I_j from $G[\mathcal{I}_{b<a_j} \cup \mathcal{I}_{a>b_i} \cup \{I_i, I_j\}]$, the graph consists of at least two connected components, and the connected component containing I_i is $G[\mathcal{C}_R(\mathcal{I}_{b<a_j} \cup \{I_i\})]$; $\mathcal{C}_R(\mathcal{I}_{b<a_j} \cup \{I_i\})$ is the set of intervals (vertices) that are eventually dominated by p_1. Similarly, $\mathcal{C}_L(\mathcal{I}_{a>b_i} \cup \{I_j\})$ is the set of intervals that are eventually dominated by p_2. From these, we have $U_1(i,j) = |\mathcal{C}_R(\mathcal{I}_{b<a_j} \cup \{I_i\})|$ and $U_2(i,j) = |\mathcal{C}_L(\mathcal{I}_{a>b_i} \cup \{I_j\})|$. $\square$

By this lemma, we define the following for neighboring $I_i, I_j \in \mathcal{I}_{\max}$ with $i < j$:

$$w_{\max}(i,j) = \max\{|\mathcal{C}_R(\mathcal{I}_{b<a_j} \cup \{I_i\})|, |\mathcal{C}_L(\mathcal{I}_{a>b_i} \cup \{I_j\})|\},$$
$$w_{\min}(i,j) = \min\{|\mathcal{C}_R(\mathcal{I}_{b<a_j} \cup \{I_i\})|, |\mathcal{C}_L(\mathcal{I}_{a>b_i} \cup \{I_j\})|\}.$$

These values represent the utilities of two players choosing the vertices of $e = \{i,j\}$ with $i < j$, where $w_{\max}(i,j) \geq w_{\min}(i,j)$.

We now show that a Nash equilibrium always exists in any 2-CDG on interval graphs.

Lemma 6. *Suppose that G is an interval graph defined by $\mathcal{I}$ with $|\mathcal{I}_{\max}| > 1$, and I_α is the strategy of p_1 (resp., p_2) in $\mathcal{I}_{\max}$. Then, there is a best response I_β of p_2 (resp., p_1) in $\mathcal{I}_{\max}$ such that I_β intersects I_α.*

Proof. Suppose that strategy $I_v \in \mathcal{I}_{\max}$ of player p_2 is a best response for strategy α of p_1. If $I_v \cap I_\alpha \neq \emptyset$, the statement of the lemma holds. Thus, we assume otherwise, where $\alpha < v$ without loss of generality; $a_\alpha < b_\alpha < a_v < b_v$ holds. We then claim that there exists a strategy $v' \in \mathcal{I}_{\max}$ of player p_2 which is a best response for strategy α of p_1 such that $\alpha < v' < v$. If it is true, we can prove the lemma by repeatedly applying the argument.

To show the claim, we first show that the graph induced by $\{I_v\} \cup \mathcal{I}_{a>a_v} \setminus (\mathcal{I}_{(\alpha,v,0)} \cup \mathcal{I}_{(\alpha,v,1)})$ is connected and dominated by p_2 under strategy profile (α, v), where $\mathcal{I}_{(\alpha,v,0)}$ and $\mathcal{I}_{(\alpha,v,1)}$ are respectively the set of the neutral intervals under (α, v) and the set of the vertices dominated by p_1 under (α, v). Actually, $\mathcal{I}_{a>a_v} \cap (\mathcal{I}_{(\alpha,v,0)} \cup \mathcal{I}_{(\alpha,v,1)}) = \emptyset$ holds as follows: Otherwise, there exists $I_u = [a_u, b_u] \in \mathcal{I}_{a>a_v} \cap (\mathcal{I}_{(\alpha,v,0)} \cup \mathcal{I}_{(\alpha,v,1)})$, where $v < u$. Since no interval in $\mathcal{I}_{a>a_v}$ intersects I_α by $b_\alpha < a_v$, the timing that I' becomes neutral or dominated by p_1 is after Time 2. That is, there is a sequence of intervals $(I_\alpha, I_{\alpha_1}, \ldots, I_{\alpha_k}, I_u)$ such that $I_{\alpha_i} \cap I_{\alpha_{i+1}} \neq \emptyset$ for $i = 1, \ldots, k-1$, $I_\alpha \cap I_{\alpha_1} \neq \emptyset$, $I_k \cap I_u \neq \emptyset$, and I_{α_i}'s are dominated by p_2. Since $b_\alpha < a_v < b_v < a_u$, some I_{α_i} satisfies $I_{\alpha_i} \cap I_v \neq \emptyset$; this means I_{α_i} is dominated by p_2 or becomes neutral at Time 2, which contradicts the assumption. Thus, it follows that $\{I_v\} \cup \mathcal{I}_{a>a_v} \setminus (\mathcal{I}_{(\alpha,v,0)} \cup \mathcal{I}_{(\alpha,v,1)}) = \{I_v\} \cup \mathcal{I}_{a>a_v}$. Here, to show that the graph induced by $\{I_v\} \cup \mathcal{I}_{a>a_v}$ is connected, we assume otherwise; $\{I_v\} \cup \mathcal{I}_{a>a_v}$ is partitioned into two or more connected sets of intervals. Let $\mathcal{I}'$ and $\mathcal{I}''$ be the left-most and second left-most connected intervals, respectively, and let a'' be the left-end of the intervals in $\mathcal{I}''$. Note that $I_v = [a_v, b_v]$ belongs to $\mathcal{I}'$, and thus $b_v < a''$ holds. Since graph G is connected, $\mathcal{I}'$ and $\mathcal{I}''$ have a common neighboring interval $[a', b']$ outside $\{I_v\} \cup \mathcal{I}_{a>a_v}$. Then, $[a', b']$ should satisfy that $a' < a_v$ and $a'' < b'$, but it contradicts $I_v = [a_v, b_v] \in \mathcal{I}_{\max}$. By these, the graph induced by $\{I_v\} \cup \mathcal{I}_{a>a_v} \setminus (\mathcal{I}_{(\alpha,v,0)} \cup \mathcal{I}_{(\alpha,v,1)}) (= \{I_v\} \cup \mathcal{I}_{a>a_v})$ is connected.

We now see which intervals are dominated by p_2 under strategy profile (α, v). As seen above, the intervals in $\{I_v\} \cup \mathcal{I}_{a>a_v}$ are eventually dominated by p_2. Additionally, some intervals (including I_v) in $\mathcal{I}_{a>b_\alpha} \setminus \mathcal{I}_{a>a_v}$ are dominated by p_2, which we call $\tilde{\mathcal{I}}$ as a set; we have $U_2(\alpha, v) = |\tilde{\mathcal{I}} \cup \{I_v\} \cup \mathcal{I}_{a>a_v}| = |\tilde{\mathcal{I}} \cup \mathcal{I}_{a>a_v}|$. Here, we consider to change p_2's initial vertex from v to a v', where $I_{v'} \in \mathcal{I}_{\max}$ satisfies $I_{v'} \cap I_v \neq \emptyset$ and $v' < v$. Since I_v does not intersect I_α, I_v is dominated by p_2 at Time 2 under strategy profile (α, v'), which implies that p_2 dominates the intervals in $\{I_v\} \cup \mathcal{I}_{a>a_v}$ again. Furthermore, the intervals in $\tilde{\mathcal{I}}$ are also dominated by p_2 under strategy profile (α, v'), because they are nearer to I_v than I_α and nearer also to $I_{v'}$ than I_α. Thus, the intervals in $\tilde{\mathcal{I}} \cup \mathcal{I}_{a>a_j}$ are dominated also by p_2 under strategy profile (α, v'), which implies that v' is also a best response for strategy α of p_1. By repeatedly applying the same argument, we can find a best response α for β where β is adjacent to α. $\square$

We are ready to show the following lemma, which completes the proof of

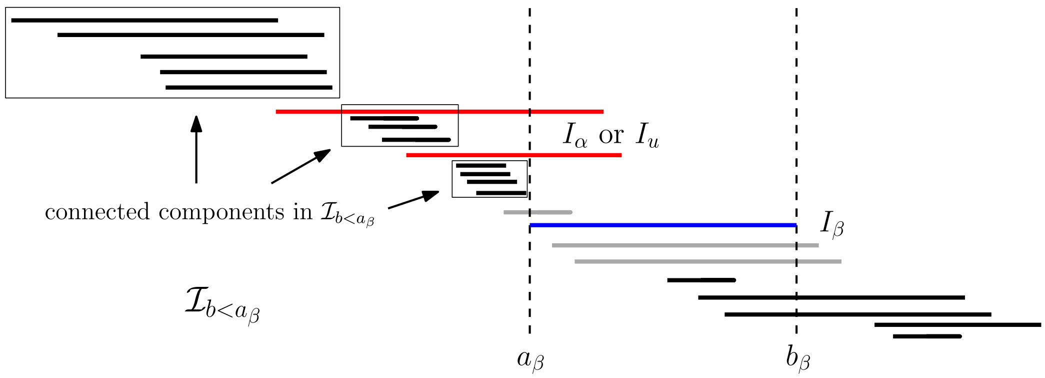


Figure 7: In the proof of Claim 1, consider the case when p_1 changes the initial interval from I_α to I_u ($u < \beta$). The blue interval represents I_β. One of the red intervals is I_α and the other is I_u.

Theorem 3.

Lemma 7. *Suppose that $G = (V, E)$ is an interval graph of interval representation $\mathcal{I}$, and let $\tilde{E} = \{\{i,j\} \in E \mid i,j \in \mathcal{I}_{\max}\}$, $E^* = \{\{i,j\} \in \tilde{E} \mid w_{\min}(i,j) = \max_{\{i,j\}\in\tilde{E}} w_{\min}(i,j)\}$ and $E^{**} = \{\{i,j\} \in E^* \mid w_{\max}(i,j) = \max_{\{i,j\}\in E^*} w_{\max}(i,j)\}$. Let $e^* = \{\alpha, \beta\}$ be an edge maximizing $|j - i|$ among $\{i,j\} \in E^{**}$. Then, strategy profile (α, β) is a Nash equilibrium, and the utilities of p_1 and p_2 are $w_{\max}(\alpha,\beta)$ and $w_{\min}(\alpha,\beta)$, respectively.*

Proof. Without loss of generality, we assume that p_1 and p_2 take strategy profile (α, β), where $\alpha < \beta$ and $U_1(\alpha,\beta) \geq U_2(\alpha,\beta)$. Then, the utilities of p_1 and p_2 are $U_1(\alpha,\beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_\alpha\})|$ and $U_2(\alpha,\beta) = |\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_\beta\})|$, respectively. We show that neither p_1 nor p_2 has an incentive to change their initial vertices (α, β).

Claim 1. Player p_1 has no incentive to change the initial vertex α.

We prove this by contradiction. Suppose that p_1 has an incentive to move. Then, we can restrict candidates of alternative initial vertex u of p_1 as intervals in $\mathcal{I}_{\max}$ that are adjacent to β by Lemma 6; the utilities change to $U_1(u,\beta)$ and $U_2(u,\beta)$. There are two cases: (1) $u < \beta$ and (2) $\beta < u$. For case (1), we can see that I_α and I_u intersect by $I_\beta \cap I_\alpha \neq \emptyset$, $I_\beta \cap I_u \neq \emptyset$, and $I_\beta, I_\alpha, I_u \in \mathcal{I}_{\max}$ as seen in Figure 7. The new utility of p_1 is $U_1(u,\beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_u\})|$. Here, we compare $U_1(\alpha,\beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_\alpha\})|$ and $U_1(u,\beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_u\})|$. Note that $\mathcal{I}_{b<a_\beta}$ may have several connected

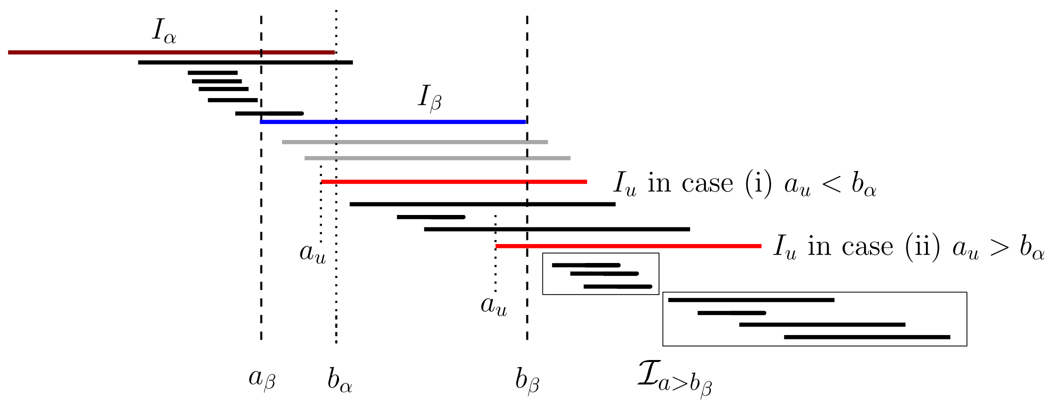


Figure 8: In the proof of Claim 1, consider the case when p_1 changes the initial interval from I_α to I_u ($u > \beta$). The blue interval represents I_β. Player p_2 changes the strategy from I_α (dark red interval) to I_u (a red interval).

components, and $R(\mathcal{I}_{b<a_\beta} \cup \{I_\alpha\}) = I_\alpha$ and $R(\mathcal{I}_{b<a_\beta} \cup \{I_u\}) = I_u$ hold because both I_α and I_u intersect I_β. Here, I_α or I_u may connect several components in $\mathcal{I}_{b<a_\beta}$. Since I_α and I_u properly intersect, the left of the two intervals can connect more connected components in $\mathcal{I}_{b<a_\beta}$ than the right one. There are two cases, $\alpha > u$ or $\alpha < u$. In case of $\alpha < u$, $U_1(\alpha, \beta)$ is not smaller than $U_1(u, \beta)$. Then, p_1 has no incentive to move to u. In case of $\alpha > u$, we have

$$U_1(\alpha, \beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_\alpha\})| \leq |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_u\})| = U_1(u, \beta).$$

Here, the inequality should be strict, because otherwise p_1 has no incentive to move to u again. The utility of p_2 changes from $U_2(\alpha, \beta) = |\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_\beta\})|$ to $U_2(u, \beta) = |\mathcal{C}_L(\mathcal{I}_{a>b_u} \cup \{I_\beta\})|$. Since $\mathcal{I}_{a>b_\alpha} \subseteq \mathcal{I}_{a>b_u}$, $U_2(\alpha, \beta) \leq U_2(u, \beta)$ holds; we have $U_1(\alpha, \beta) < U_1(u, \beta)$ and $U_2(\alpha, \beta) \leq U_2(u, \beta)$. This contradicts $\alpha, \beta\} \in E^{**} \subseteq E^*$ in any case.

For case (2), i.e., $\beta < u$, we have $a_\beta < a_u < b_\beta < b_u$ by $I_\beta, I_u \in \mathcal{I}_{\max}$. See Figure 8. The utilities of p_1 and p_2 change from $U_1(\alpha, \beta)$ and $U_2(\alpha, \beta)$ to $U_1(u, \beta) = |\mathcal{C}_L(\mathcal{I}_{a>b_\beta} \cup \{I_u\})|$ and $U_2(u, \beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_u} \cup \{I_\beta\})|$. Here, we see whether such a change is possible by case analysis: (i) $a_u < b_\alpha$ and (ii) $a_u > b_\alpha$. In case (i), I_α and I_u intersect. We now consider strategy profile (α, u). Then, the utilities of p_1 and p_2 are $U_1(\alpha, u) = |\mathcal{C}_R(\mathcal{I}_{b<a_u} \cup \{I_\alpha\})|$ and $U_2(\alpha, u) = |\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_u\})|$, respectively. Then, we have

$$U_2(\alpha, u) = |\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_u\})| \geq |\mathcal{C}_L(\mathcal{I}_{a>b_\beta} \cup \{I_u\})| = U_1(u, \beta) > U_1(\alpha, \beta),$$

since p_1 have an incentive to move to I_u under (α, β) by the assumption. We also have

$$U_1(\alpha, u) = |\mathcal{C}_R(\mathcal{I}_{b<a_u} \cup \{I_\alpha\})| \geq |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_\alpha\})| = U_1(\alpha, \beta).$$

Namely, the utilities of both players under (α, u) are at least $U_1(\alpha, \beta)$, and the greater one is strictly greater than $U_1(\alpha, \beta)$. This contradicts either $\{\alpha, \beta\} \in E^{**}$. We next consider case (ii) $a_u > b_\alpha$. In this case, $I_u \in \mathcal{I}_{a>b_\alpha}$ holds; I_u is not neutral under (α, β). This together with $I_u \cap I_\beta \neq \emptyset$ and $\mathcal{I}_{a>b_\beta} \subseteq \mathcal{I}_{a>b_\alpha}$ implies that an interval in $\mathcal{I}_{a>b_\beta}$ reachable from I_u is also reachable from I_β in $G[\mathcal{I}_{a>b_\alpha}]$ via I_u; $\mathcal{C}_L(\mathcal{I}_{a>b_\beta} \cup \{I_u\}) \subseteq \mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_\beta\})$ holds, which implies

$$U_1(u, \beta) \leq U_1(\beta, \alpha) = U_2(\alpha, \beta) \leq U_1(\alpha, \beta).$$

This contradicts that p_1 has an incentive to change initial vertex α to u.

Claim 2. Player p_2 has no incentive to change the initial vertex β.

We prove this by contradiction again. Suppose that p_2 has an incentive to move to $v \in \mathcal{I}_{\max}$, which intersects α (by Lemma 6). Then, $U_2(\alpha, v) > U_2(\alpha, \beta)$ holds. Furthermore, after the strategy is changed, p_2 gets a utility not smaller than p_1's utility, that is, $U_2(\alpha, v) = w_{\max}(\{\alpha, v\}) > w_{\min}(\{\alpha, v\}) = U_1(\alpha, v)$ holds because otherwise it contradicts $\{\alpha, \beta\} \in E^*$. By $\{\alpha, \beta\} \in E^*$, we also have $U_1(\alpha, v) = w_{\min}(\{\alpha, v\}) \leq w_{\min}(\{\alpha, \beta\}) \leq w_{\max}(\{\alpha, \beta\}) = U_1(\alpha, \beta)$. Here, actually $U_1(\alpha, v) < U_1(\alpha, \beta)$ holds as follows: if all the inequalities hold with equality, then we have $U_1(\alpha, \beta) = U_2(\alpha, \beta) = U_1(\alpha, v) < U_2(\alpha, v)$; the existence of strategy profile (α, v) contradicts $\{\alpha, \beta\} \in E^{**}$.

We now focus on the order relation of v and α. There are two cases: (1) $v > \alpha$ and (2) $v < \alpha$. In case (1), the utility of p_1 changes from $U_1(\alpha, \beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_\alpha\})|$ to $U_1(\alpha, v) = |\mathcal{C}_R(\mathcal{I}_{b<a_v} \cup \{I_\alpha\})|$, which must be smaller than $U_1(\alpha, \beta)$ by the above argument. This implies $v < \beta$. Then, the utility of p_2 changes from $U_2(\alpha, \beta) = |\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_\beta\})|$ to $U_2(\alpha, v) = |\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_v\})|$, which is not greater than $|\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_\beta\})|$ by $v < \beta$. This contradicts that p_2 has an incentive to move to v.

In case (2) (i.e., $v < \alpha$), the utility of p_1 changes from $U_1(\alpha, \beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_\alpha\})|$ to $U_1(\alpha, v) = |\mathcal{C}_L(\mathcal{I}_{a>b_v} \cup \{I_\alpha\})|$. See Figure 9. We now compare this $U_1(\alpha, v)$ and $U_2(\alpha, \beta) = |\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_\beta\})|$. By $v < \alpha$, we have $\mathcal{I}_{a>b_\alpha} \subseteq \mathcal{I}_{a>b_v}$. If I_v and I_β do not intersect, then $I_\beta \in \mathcal{I}_{a>b_v}$. This

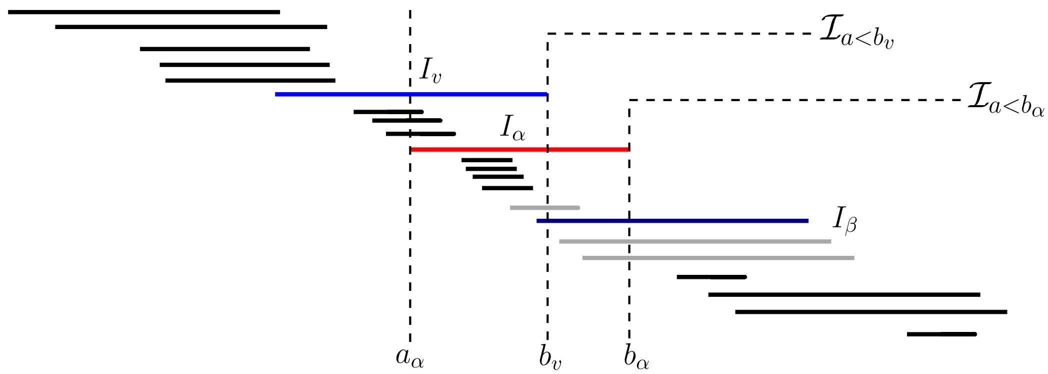


Figure 9: Case (2) (i.e., $v < \alpha$) in the proof of Claim 2. Player p_2 changes the intial vertex from I_β (dark blue interval) to I_v (blue interval).

and $I_\alpha \cap I_\beta \neq \emptyset$ imply that $\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_\beta\}) \subsetneq \mathcal{C}_L(\mathcal{I}_{a>b_v} \cup \{I_\alpha\})$, that is, $U_2(\alpha, \beta) < U_1(\alpha, v) = w_{\min}(\{\alpha, v\})$, which contradicts $\{\alpha, \beta\} \in E^*$. Thus I_v and I_β intersect. We here consider strategy profile (v, β). The utilities of p_1 and p_2 are $U_1(v, \beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_v\})|$ and $U_2(v, \beta) = |\mathcal{C}_L(\mathcal{I}_{a>b_v} \cup \{I_\beta\})|$, respectively. Here, $v < \alpha$ implies

$$U_2(v, \beta) = |\mathcal{C}_L(\mathcal{I}_{a>b_v} \cup \{I_\beta\})| \geq |\mathcal{C}_L(\mathcal{I}_{a>b_\alpha} \cup \{I_\beta\})| = U_2(\alpha, \beta).$$

Furthermore, we have

$$U_1(v, \beta) = |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_v\})| \geq |\mathcal{C}_R(\mathcal{I}_{b<a_\beta} \cup \{I_\alpha\})| = U_1(\alpha, \beta)$$

by $v < \alpha$. Then, $U_2(v, \beta) = U_2(\alpha, \beta)$ and $U_1(v, \beta) = U_1(\alpha, \beta)$ hold, that is, $\{v, \beta\} \in E^{**}$, because otherwise it contradicts $\{\alpha, \beta\} \in E^{**}$. Due to $v < \alpha < \beta$, $|\beta - v|$ is greater than $|\beta - \alpha|$. This again contradicts the choice of (α, β), that is, $\beta - \alpha$ is maximum among $|j - i|$'s of $\{i, j\} \in E^{**}$.

Thus, neither p_1 nor p_2 has incentive to move, which implies that (α, β) is a Nash equilibrium. $\square$

4. The Non-existence of a Nash equilibrium

In this section, we give a chordal graph that has no Nash equilibrium of 2-CDG. We also give a strongly chordal graph that has no Nash equilibrium of 2-CDG. Since a strongly chordal graph is also chordal, it might be sufficient to give the latter graph, although the size of the former graph is smaller as a chordal graph.

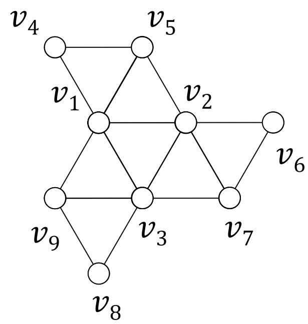


Figure 10: A chordal graph with no Nash equilibrium.

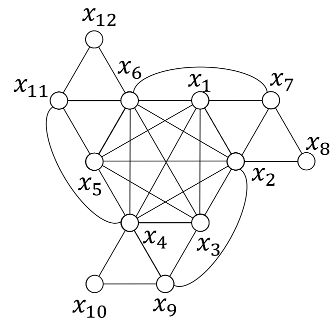


Figure 11: A strongly chordal graph with no Nash equilibrium.

Theorem 4. *There is a chordal graph with* 9 *vertices and diameter* 3 *that has no Nash equilibrium of 2-CDG.*

Theorem 5. *There is a strongly chordal graph with* 12 *vertices and diameter* 3 *that has no Nash equilibrium of 2-CDG.*

Figures 10 and 11 show concrete instances of these theorems, and Table 1 shows the payoff matrix for the instance in Figure 10, though the one for Figure 11 is put in an appendix. In the table, each element (v_i, v_j) in the payoff matrix represents $(U_1(v_i, v_j), U_2(v_i, v_j))$, and we leave the elements of the lower triangle of the matrix empty for legibility. We can get the values (α, β) for (v_i, v_j) with $i > j$ by referring (β, α) at (v_j, v_i). By using Table 1, we can verify that one player has an incentive of changing the strategy for every strategy profile. For example, we start at (v_1, v_4), whose element is $(7, 1)$. Here, p_2 has an incentive to move to v_3, and then the utilities at (v_1, v_3) are $(3, 4)$. Then, p_1 has an incentive to move to v_2, and the utilities at (v_2, v_3) are $(4, 3)$. This procedure continues as (v_2, v_1) and (v_3, v_1), which is essentially equivalent to (v_1, v_3); it is an endless loop.

Proof of Theorem 4. We first confirm that the graph of Figure 10 satisfies the properties of Theorem 4. Clearly, the number of vertices and the diameter are 9 and 3, respectively. It is also a chordal graph, because every induced cycle has exactly 3 vertices. We then prove this theorem by showing any strategy profile of the 2-CDG on the graph is not a Nash equilibrium. Thus one way for proving this is to check all the cells in Table 1, but it is time-consuming, and we reduce the number of checks by symmetry. We first focus

Table 1: The payoff matrix in the chordal graph of Figure 10

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	$(0,0)$	$(4,3)$	$(3,4)$	$(7,1)$	$(6,1)$	$(6,2)$	$(5,2)$	$(6,1)$	$(6,2)$
v_2		$(0,0)$	$(4,3)$	$(6,1)$	$(6,2)$	$(7,1)$	$(6,1)$	$(6,2)$	$(5,2)$
v_3			$(0,0)$	$(6,2)$	$(5,2)$	$(6,1)$	$(6,2)$	$(7,1)$	$(6,1)$
v_4				$(0,0)$	$(1,7)$	$(5,3)$	$(3,5)$	$(3,5)$	$(2,5)$
v_5					$(0,0)$	$(5,2)$	$(3,4)$	$(5,3)$	$(4,3)$
v_6						$(0,0)$	$(1,7)$	$(5,3)$	$(3,5)$
v_7							$(0,0)$	$(5,2)$	$(5,2)$
v_8								$(0,0)$	$(1,7)$
v_9									$(0,0)$

on v_1, v_2 and v_3, which are symmetric. The best response for v_1 is v_3. By the symmetry, that for v_3 is v_2 and that for v_2 is v_1; these form cyclic relations of best responses. Thus any strategy profile containing one of v_1, v_2 and v_3 is not a Nash equilibrium. We next consider v_5, v_7 and v_9. For v_5, there are two best responses, which are v_1 and v_2 with utility 6. This implies that any strategy profile containing one of v_5, v_7 and v_9 is not a Nash equilibrium. Finally, we consider v_4, v_6 and v_8. For v_4, there are two best responses, which are v_1 and v_5 with utility 7, and it follows that any strategy profile containing v_4, v_6 and v_8 cannot be a Nash equilibrium. Thus, this graph has no Nash equilibrium. $\square$

The graph of Figure 10 is not a strongly chordal graph, because an even cycles has no odd chord. In fact, cycle $(v_1, v_9, v_3, v_7, v_2, v_5, v_1)$ does not have odd chord (v_1, v_7) (or, $(v_9, v_2), (v_3, v_5)$). For a stronger result, we need another example.

Proof of Theorem 5. We first confirm that the graph of Figure 11 satisfies the properties of Theorem 4. Clearly, the number of vertices and the diameter are 12 and 3, respectively. It is also a chordal graph, because every induced cycle has exactly 3 vertices. Furthermore, we can see that it does not have any sun as an induced graph, by checking whether every maximal clique satisfies the condition. This graph has three types of maximal cliques, e.g., $\{x_1, \ldots, x_6\}$, $\{x_4, x_5, x_6, x_{11}\}$ and $\{x_2, x_7, x_8\}$, and none of them can be extended to a sun. We then prove this theorem by showing any strategy profile of the 2-CDG on the graph is not a Nash equilibrium. See also Table A.3. We first focus on x_2, x_4 and x_6, which are symmetric to each other. The best response for x_2 is x_4. By symmetry, that for x_4 is x_6 and that for x_6 is x_2. Thus, any strategy profile containing one of x_2, x_4 and x_6 is not a Nash equilibrium. We next see x_1, x_3 and x_5, which are also symmetric to each other. The best response for x_1 is x_4 with utility 5, which implies that any strategy profile containing one of x_1, x_3 and x_5 is also not a Nash equilibrium. We then see x_7 (that is, x_9 and x_{11}). The best responses for x_7 are x_2 and x_6 with utility 8, which implies that any strategy profile containing one of x_7, x_9 and x_{11} is also not a Nash equilibrium. Finally, we consider x_8 (that is, x_{10} and x_{12}). The best responses for x_8 are x_2 and x_7 with utility 10, which again implies that any strategy profile containing one of x_8, x_{10} and x_{12} is also not a Nash equilibrium. Overall, the graph does not have a Nash equilibrium. $\square$

5. Concluding remarks

In this paper, we studied the existence of a Nash equilibrium in 2-CDG. We showed that a Nash equilibrium always exists on a split graph, a block graph, and an interval graph. In particular, the proofs for block graphs and interval graphs give an idea to find a Nash equilibrium efficiently; a Nash equilibrium is found by computing utilities for only $O(n)$ strategy profiles. On the other hand, we gave instances with no Nash equilibrium on (strongly) chordal graphs. These results show the boundary of the existence of a Nash equilibrium in 2-CDG on chordal graphs.

In the proofs for the existence of Nash equilibrium in split, interval, and block graphs, we implicitly or explicitly use the property that for a strategy s_1 there is a best response s_2 such that s_1 and s_2 are adjacent, which is useful to restrict possible strategies for the other player's improvement. Also it gives the observation that a graph in such a graph class has a Nash equilibrium of (s_1, s_2) where s_1 and s_2 are adjacent. We here call such a property of Nash equilibria *adjacency*. A natural question arises: does the above property hold in general? Or does every graph having a Nash equilibrium also have a Nash equilibrium with adjacency?

The answer is no, because there is a graph whose unique Nash equilibrium does not satisfy adjacency. For example, the graph in Figure 12 has a unique Nash equilibrium (v_3, v_5), which are not adjacent. Table 2 shows the pay-off matrix of this graph. By checking the table, it is easy to see that (v_3, v_5) is a Nash equilibrium. Note that v_3 (resp., v_5) is the best response for v_5 (v_3).

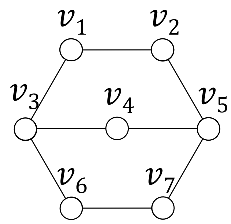


Figure 12: A graph having the unique Nash equilibrium (v_3, v_5) without adjacency.

Also the best responses for v_1 are v_3 and v_4. The best responses for v_4 are v_3 and v_5. By the symmetricity of v_1, v_2, v_6, v_7 and that of v_3, and v_5, we can check that (v_3, v_5) is the unique Nash equilibrium.

Table 2: The pay-off matrix of Figure 12.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	$(0,0)$	$(3,3)$	$(2,4)$	$(2,4)$	$(2,3)$	$(2,2)$	$(3,3)$
v_2		$(0,0)$	$(2,3)$	$(2,4)$	$(2,4)$	$(3,3)$	$(2,2)$
v_3			$(0,0)$	$(3,2)$	$(3,3)$	$(4,2)$	$(3,2)$
v_4				$(0,0)$	$(2,3)$	$(4,2)$	$(4,2)$
v_5					$(0,0)$	$(3,2)$	$(4,2)$
v_6						$(0,0)$	$(3,3)$
v_7							$(0,0)$

Then, the next question arises: are there other natural classes of graphs in which every graph having a Nash equilibrium also has a Nash equilibrium with adjacency? Notice that the graph in Figure 12 is not chordal. Maybe an interesting question is to find such a class of graphs and to investigate the relationship between the class and subclasses of chordal graphs.

Acknowledgement

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Appendix A. The pay-off matrix in the strongly chordal graph of Fig. 11.

Table A.3 is the pay-off matrix in the strongly chordal graph of Fig. 11.

Table A.3: The pay-off matrix in the strongly chordal graph of Figure 11.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
v_1	$(0,0)$	$(1,4)$	$(3,3)$	$(3,5)$	$(3,3)$	$(1,3)$	$(7,2)$	$(9,1)$	$(7,2)$	$(9,2)$	$(7,2)$	$(9,2)$
v_2		$(0,0)$	$(3,1)$	$(3,4)$	$(5,3)$	$(4,3)$	$(8,1)$	$(10,1)$	$(8,2)$	$(8,1)$	$(7,2)$	$(9,2)$
v_3			$(0,0)$	$(1,4)$	$(3,3)$	$(3,5)$	$(7,2)$	$(9,1)$	$(7,2)$	$(9,1)$	$(7,2)$	$(9,2)$
v_4				$(0,0)$	$(3,1)$	$(3,4)$	$(7,2)$	$(8,2)$	$(8,1)$	$(10,1)$	$(8,2)$	$(9,1)$
v_5					$(0,0)$	$(1,4)$	$(7,2)$	$(9,1)$	$(7,2)$	$(9,2)$	$(7,2)$	$(9,1)$
v_6						$(0,0)$	$(8,2)$	$(8,1)$	$(7,2)$	$(9,2)$	$(8,1)$	$(10,1)$
v_7							$(0,0)$	$(10,1)$	$(5,4)$	$(6,3)$	$(4,5)$	$(7,2)$
v_8								$(0,0)$	$(2,7)$	$(3,5)$	$(3,6)$	$(5,3)$
v_9									$(0,0)$	$(10,1)$	$(5,4)$	$(6,3)$
v_{10}										$(0,0)$	$(2,7)$	$(3,5)$
v_{11}											$(0,0)$	$(10,1)$
v_{12}												$(0,0)$