

DUCTED PROPAGATION OF HYDROMAGNETIC WHISTLERS IN THE MAGNETOSPHERE

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Abstract

The purpose of this paper is to investigate the guiding mechanism of hydromagnetic whistlers along the Earth's magnetic field-aligned plasma column in the magnetosphere by use of wave theory. The radial density distribution perpendicular to the magnetic field is assumed to be Gaussian and we use the method in which the fields are expanded into power series of the radial distance from the column axis. And the condition for trapping in a duct is discussed for enhanced, and depressed, ducts. The existence of trapping frequency region, which is closely related to the duct modulation factor, is demonstrated and the duct width necessary for trapping is also estimated.

1. Introduction

Guidance of hydromagnetic waves (Alfvén waves) along the lines of flux of the Earth's magnetic field has been discussed by some workers (Jacobs and Watanabe 1964; Obayashi 1965) in order to explain the micropulsation whistlers. Since then it has been assumed that hydromagnetic whistlers propagate almost along the magnetic lines of force. Fejer and Lee (1967) discussed the guidance of Alfvén waves in a cold plasma from several points of view. They also briefly mentioned the guidance by field-aligned ducts and demonstrated by a numerical example the fact that ducted propagation could occur in a field-aligned duct only a few kilometers wide, in which plasma density was about 15% greater than that in the surrounding medium. Recently Kitamura (1967) made calculations of the propagation paths of hydromagnetic whistlers in the magnetosphere to investigate the deviation of ray paths from the magnetic field line. He showed that the waves cannot reach the ionosphere in almost all cases, since the wave normal angle becomes very large in the vicinity of the bottom of the magnetosphere. To overcome this discrepancy of computational results from the fact that micropulsation whistlers are really observed on the ground, we propose the ducted propagation of Alfvén waves in the magnetosphere (Hayakawa, Ohtsu and Iwai 1968).

2. Electromagnetic wave fields in a plasma

A duct aligned with the Earth's magnetic field is assumed to be a column with non-sharp boundary as shown in Fig. 1. The radial distribution of the plasma density in the duct is taken to be Gaussian as follows and shown in Fig. 2.

$$N(r) = N_a + (N_c - N_a) \exp(-r^2/a^2) \quad (1)$$

where r is the radial distance from axis, and a is a measure of the duct width. N_a is the ambient plasma density and N_c is the central plasma density of the duct. The Earth's magnetic field is assumed to be homogeneous in the whole region and parallel to the duct axis, i.e., z -axis. The cylindrical coordinate system (r, ϕ, z) used are also shown in Fig. 1.

The electric field of electromagnetic wave propagating in an inhomogeneous, and anisotropic, plasma is governed by the following wave equation,

$$\nabla \times \nabla \times \mathbf{E} - k_0^2(\epsilon) \mathbf{E} = 0 \quad (2)$$

where time factor is assumed to be $\exp(i\omega t)$ and k_0 is a free space wave number. Plasma tensor (ϵ) is written as follows.

$$(\epsilon) = \begin{pmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \quad (3)$$

Assuming that the plasma is composed of electrons and protons, the elements of (ϵ) can be given as,

$$\left. \begin{aligned} \epsilon_1 &= 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} \sim \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} \\ \epsilon_2 &= \frac{\omega_{ce}\omega_{pe}^2}{\omega(\omega^2 - \omega_{ce}^2)} - \frac{\omega_{ci}\omega_{pi}^2}{\omega(\omega^2 - \omega_{ci}^2)} \sim \frac{\omega\omega_{pi}^2}{\omega_{ci}(\omega_{ci}^2 - \omega^2)} \\ \epsilon_3 &= 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} \sim -\frac{\omega_{pe}^2}{\omega^2} \end{aligned} \right\} \quad (4)$$

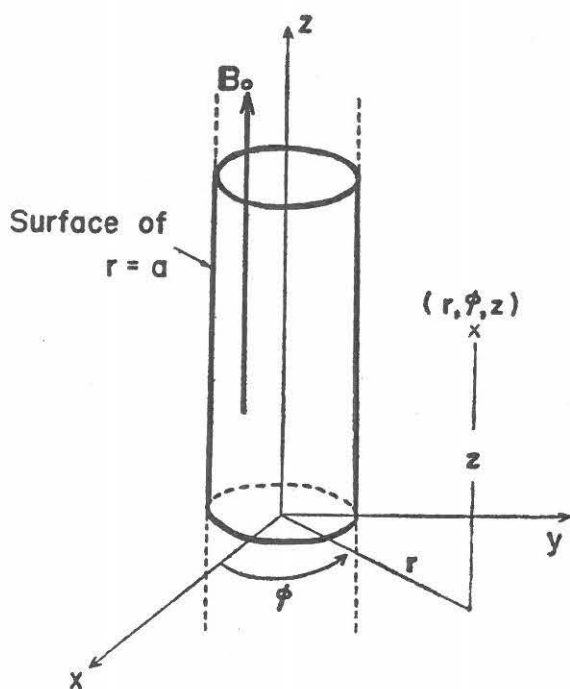
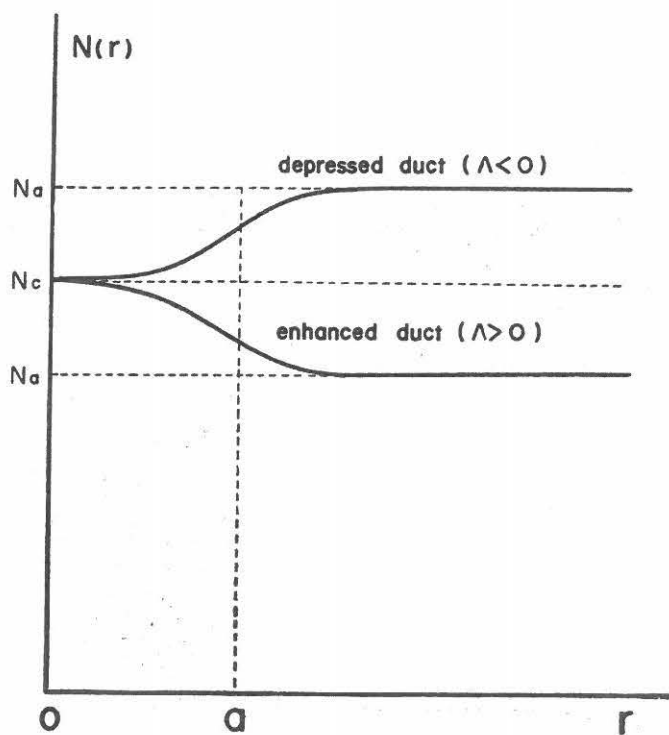


Fig. 1 Duct model and cylindrical coordinates.

Fig. 2 Radial density distribution of the duct.



where ω_{pj} ($j=i, e$) and ω_{cj} ($j=i, e$) are respectively angular plasma frequency and cyclotron frequency of the j particle. The expressions of ϵ_i 's to the right in eq. (4) are approximations under such a condition as that $\omega_{pi} \gg \omega_{ci}$, $\omega_{pe} \gg \omega$ and $m_i \gg m_e$. ϵ_i 's are functions of r due to the radial density inhomogeneity.

Since hydromagnetic whistlers in the magnetosphere are left hand polarized waves, the wave polarization is given by $iE_x/E_y=1$. If $a \rightarrow \infty$, then $N(r)$ must tend to N_e and this means that the plasma is homogeneous. The simple travelling left hand polarized wave along z axis in this homogeneous plasma is given as follows in cartesian and cylindrical coordinates,

$$\mathbf{E}(a \rightarrow \infty) = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \exp(i\omega t - ik_0 n z) = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \exp(i\omega t + i\phi - ik_0 n z) \quad (5)$$

where n (>0) is the longitudinal refractive index of hydromagnetic whistlers in a homogeneous plasma ($N=N_e$). Incidentally the expression of n in terms of ϵ_i 's is

$$n^2 = \epsilon_1^c + \epsilon_2^c \quad (6)$$

where the superscript c indicates the value at the center of a duct. Substituting eq. (4) into eq. (6), we can get the following familiar expression for L-waves.

$$n^2 = 1 - \frac{(\omega_{pe}^c)^2}{\omega(\omega + \omega_{ce})} - \frac{(\omega_{pi}^c)^2}{\omega(\omega - \omega_{ci})} \quad (7)$$

Now we will seek for a solution of the following form, which tends to the asymptotic solution given by eq. (5) when $a \rightarrow \infty$ for fixed r ,

$$\mathbf{E} = \mathbf{F}(r) \exp(i\omega t + i\phi - ik_0 n z) \quad (8)$$

where n takes the value given by eq. (6), and the cylindrical components of \mathbf{F} are functions of r only.

Here we consider the electron density defined by eq. (1) as a function of $\tau=r/a$, and not a function of r . Then we can write down eq. (2) in components using the new variable τ .

3. Trapping conditions

For hydromagnetic whistlers to be trapped in a field-aligned duct, the following two conditions are needed (Northover 1959).

- (1) The amplitude of electromagnetic wave fields decreases radially at a rate larger than what the radial scattering factor $r^{-1/2}$ represents.

(2) Most of the energy flow is confined to the central region of a duct.

To investigate the trapping condition (1), it is necessary to study the behavior of electromagnetic fields at far distances from the axis. When τ is very large, the asymptotic expansions for $\mathbf{F}(\tau)$ are given as follows,

$$\mathbf{F}(\tau) = \exp(-isk_0 a \tau) \sum_{n=0}^{\infty} \mathbf{f}_n \tau^{-\frac{1}{2}-2n} \quad (9)$$

where $\mathbf{f}_n = (f_{nr}, f_{n\phi}, f_{nz})$. s is a complex quantity representing the radial damping rate of wave field.

Substitution of eq. (9) into component expressions of eq. (2) and picking up the most predominant terms in τ yields the following equations,

$$\left. \begin{aligned} (n^2 - \varepsilon_1^a) f_{0r} + i\varepsilon_2^a f_{0\phi} - snf_{0z} &= 0 \\ -i\varepsilon_2^a f_{0r} + [s^2 + (n^2 - \varepsilon_1^a)] f_{0\phi} &= 0 \\ -snf_{0r} + (s^2 - \varepsilon_3^a) f_{0z} &= 0 \end{aligned} \right\} \quad (10)$$

where the superscript a indicates the value in the ambient plasma. For the solution $(f_{0r}, f_{0\phi}, f_{0z})$ to be nontrivial, the determinant composed of the coefficients of $(f_{0r}, f_{0\phi}, f_{0z})$ must be zero and this yields the following equation for s using eq. (4),

$$g^2 y^2 (1 - y^2) s^4 + g[(y + A)\{y^2 - R(1 - y^2)\} + y^4(1 - A)] s^2 + R[y^2(1 - A)^2 - (y + A)^2] = 0 \quad (11)$$

where g , y , A and R are defined as,

$$g = \frac{\omega_{ci}^2}{(\omega_{pi}^c)^2}, \quad y = \frac{\omega}{\omega_{ci}} \quad (0 < y < 1), \quad A = \frac{N_c - N_a}{N_c}, \quad R = \frac{m_i}{m_e}. \quad (12)$$

Among the solutions, those with $\text{Im}(s) < 0$ are physically admissible. For the solutions with $\text{Im}(s) < 0$ to exist, there are two cases, taking eq. (11) as a quadratic equation for s^2 .

(1) Two solutions s_1^2 and s_2^2 are both negative.

(2) s_1^2 and s_2^2 are both imaginary.

Case (2) corresponds to the leaky waves. This case is satisfied only when the longitudinal propagation constant is complex. So we don't consider this case.

Case (1) requires that the discriminant D of eq. (11) is positive, i.e.

$$D > 0 \quad (13)$$

$$\text{and } g[(y + A)\{y^2 - R(1 - y^2)\} + y^4(1 - A)] > 0 \quad (14)$$

$$R[y^2(1 - A)^2 - (y + A)^2] > 0 \quad (15)$$

For the relation (15) to be satisfied, it requires that

$$y > \frac{-A}{2-A} \quad (16)$$

only for $A < 0$. If $A > 0$, it appears that the relation (15) can never be satisfied. Next the relation (14) requires the following relation taking into account the largeness of R ,

$$y + A < 0. \quad (17)$$

Therefore, relations (16) and (17) are combined to yield

$$\frac{-A}{2-A} < y < -A \quad (-A > 0). \quad (18)$$

With relation (18) satisfied, the discriminant of eq. (11) is always positive for reasonable values of modulation factor.

Therefore trapping of waves is possible only for depressed ducts and impossible for enhanced ducts. And the trapping frequency region is given by eq. (18) and shown in Fig. 3. The upper and lower cut-offs of trapping frequency region are found to be

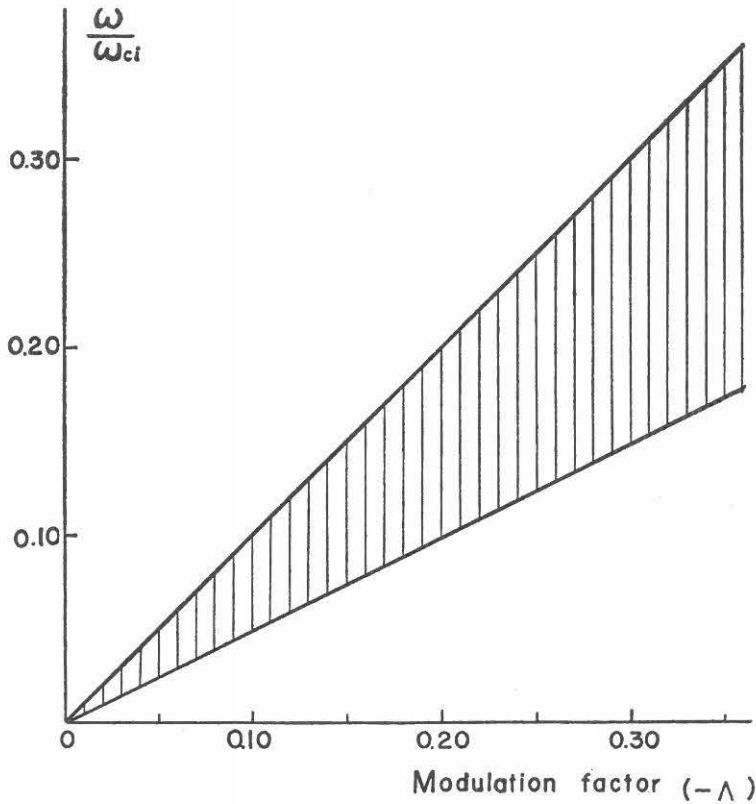


Fig. 3 Trapping frequency region vs. duct modulation factor.
Trapping is possible in shaded region.

closely related to the duct modulation factor. From Fig. 3, we can show the following facts.

- (1) For a hydromagnetic whistler to be trapped in a field-aligned duct, the duct must be depressed.
- (2) If the modulation factor of the depressed duct is relatively small, the trapping frequency width is small, and accordingly the upper, and the lower, cut-off frequencies take relatively low values.
- (3) For higher frequency waves to be trapped, it is necessary that a duct exists where the modulation is deeply depressed.

Suppose that the trapping condition (1) is satisfied, then we must estimate the electromagnetic fields in the axial region of a duct to investigate the trapping condition (2). Using an expansion form of the density distribution given by eq. (1) in a series of τ^2 near the axis and taking into account the fact that $F_r \rightarrow 1$, $F_\phi \rightarrow i$, and $F_z \rightarrow 0$ for $a \rightarrow \infty$, we can write the electric field components in the axial region of a duct as follows.

$$\left. \begin{aligned} F_r &= 1 + \sum_{n=1}^{\infty} \lambda_n \tau^{2n} \\ F_\phi &= i + \sum_{n=1}^{\infty} \mu_n \tau^{2n} \\ F_z &= \sum_{n=0}^{\infty} \nu_n \tau^{2n+1} \end{aligned} \right\} \quad (19)$$

Substituting the above series expansions into the component expressions of eq. (2), we obtain the difference equations for the unknown coefficients. After some manipulations, we will show only the relevant coefficients,

$$\left. \begin{aligned} \nu_0 &= 0 \\ \lambda_1 &= -3i\mu_1 \\ \nu_1 &= -k_0 a n \mu_1 \\ \mu_1 &= \frac{i(\delta + \xi)}{n^2 + 2\varepsilon_2^c} \end{aligned} \right\} \quad (20)$$

where $\delta = \varepsilon_1^c - \varepsilon_1^a$ and $\xi = \varepsilon_2^c - \varepsilon_2^a$. Higher order terms in eq. (19) can be neglected. So we obtain the magnetic field components by the use of Maxwell's equations.

Then we can estimate the energy flow in the axial region of a duct. Using the complex Poynting vector, we can show that the energy flow is directed throughly parallel to the axis of a duct.

Therefore the total energy flow along a duct is given by

$$P = \int_0^\infty \int_0^{2\pi} R_z(S_z) a^2 \tau d\tau d\phi \quad (21)$$

where S_z is the z -component of the Poynting vector. Then we get

$$P = 2\pi a^2 \int_0^\infty \frac{n}{\zeta} \left[1 + \left(\lambda_1 - i\mu_1 - \frac{2i\nu_1}{k_0 n a} \right) \tau^2 + O(\tau^4) \right] \tau d\tau \quad (22)$$

To estimate the order of magnitude of the energy flow, we must define the upper limit of the integral τ_m . Also τ_m is the measure of the confinement of energy flow in the axial region ($\tau < \tau_m$) and defined by

$$k_0 a |Im(s)| \tau_m = \pi \quad (23)$$

Due to the exponential decrease of the field at large radial distances, we can neglect the contribution of the range from τ_m to infinity to the integral. Therefore eq. (22) is integrated to yield

$$P = \frac{2\pi a^2 n}{\zeta} \left[\frac{\tau_m^2}{2} + \frac{1}{4} \left(\lambda_1 - i\mu_1 - \frac{2i\nu_1}{k_0 n a} \right) \tau_m^4 + O(\tau_m^6) \right] \quad (24)$$

where ζ is wave impedance of free space.

For the field to be fairly well localized near the axis of a duct we require

$$\tau_m < 1 \quad (25)$$

in the trapping frequency region. This condition gives the measure of a duct width.

$$k_0 a > \frac{\pi}{|Im(s)|} \quad (26)$$

As the value of $Im(s)$ in eq. (23) and eq. (26), we take the one with smaller absolute value of the two solutions of eq. (11). From the considerations of eq. (11), the minimum duct width necessary for wave trapping appears in the form $k_0 a / \sqrt{g}$ and shown in Fig. 4.

4. From this figure, we can show that

- (1) Provided that g is kept constant, the higher the wave frequency becomes, the thinner does the duct width necessary for trapping, become.
- (2) As the minimum duct width appears in the form $k_0 a / \sqrt{g}$, the quantity g , the ratio of ion gyrofrequency to ion plasma frequency is a very important quantity. We can see that the minimum duct width $k_0 a / \sqrt{g}$ is determined by the use of plots in Fig. 4 for a given value of wave frequency. Then if g is small, even thin ducts can support ducted propagation of hydromagnetic whistlers.

4. Conclusions

We derived the trapping conditions, i.e., trapping frequency region and minimum trapping duct width, which enable the hydromagnetic whistlers to be trapped in a field-

aligned duct. The important point is that only depressed ducts can support the ducted propagation of hydromagnetic whistlers in the magnetosphere. The trapping frequency region, whose upper and lower cut-offs are closely related to the duct modulation factor, is demonstrated, and the minimum trapping duct width is also estimated.

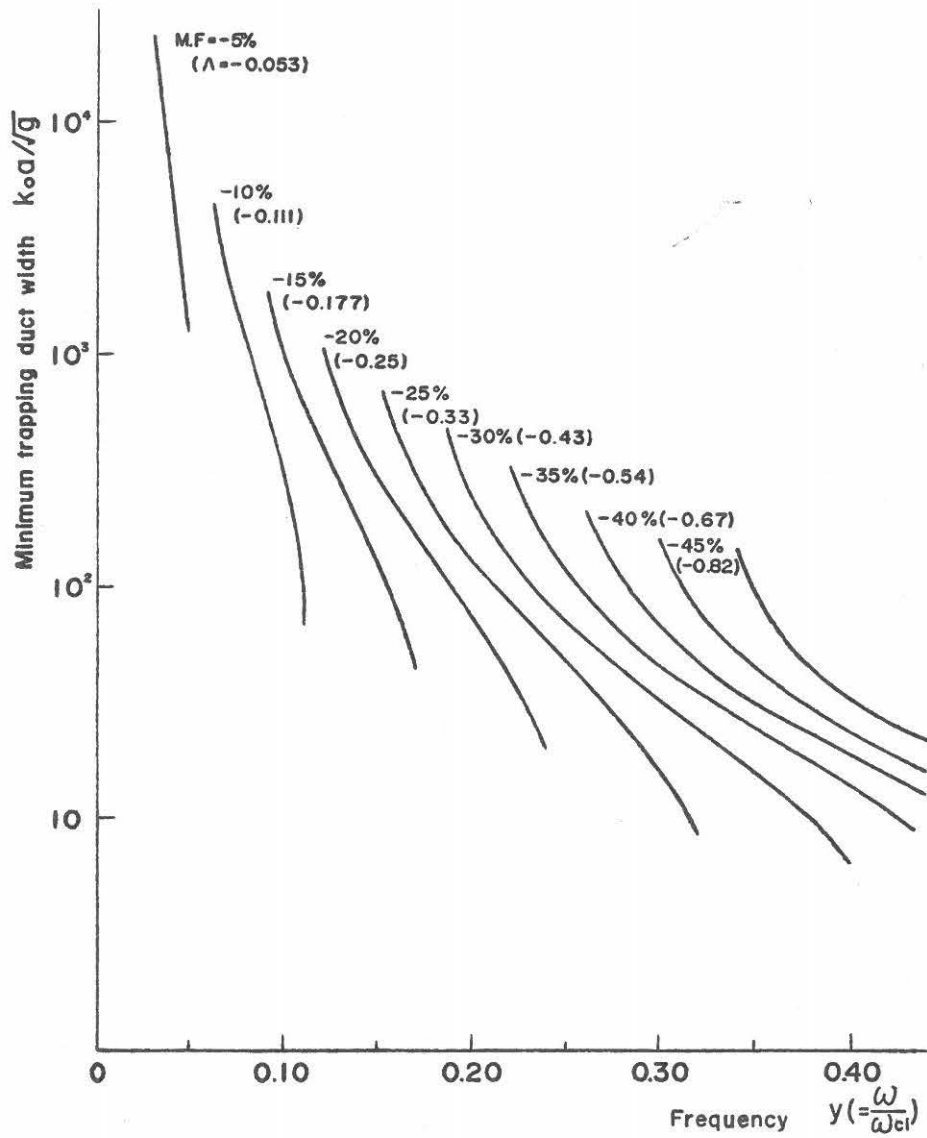


Fig. 4 Minimum trapping duct width vs. wave frequency.

$$M.F. = (N_c - N_a) / N_a$$

As a simple numerical example, we consider the ducted propagation of Pc 1 waves in the magnetosphere. If we assume the modulation factor of a depressed duct to be about -0.1 , the minimum duct width necessary for trapping of 1 Hz waves is about a few hundred kilometers in the magnetosphere ($L \sim 3.5$).

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