PENETRATION OF WHISTLERS THROUGH THE IONOSPHERE

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1. Scope of the study

Among many kinds of VLF natural noises, whistlers, and VLF emissions propagate along the geomagnetic field line in the magnetosphere in whistler modes and penetrate through the ionosphere. So the penetration of VLF waves through the ionosphere is an important problem. Maeda and Oya (1963) have discussed the penetration of VLF radio waves through a sharp ionospheric boundary under a constant magnetic field. They discussed the case where the waves propagate downwards from above, and showed that VLF waves can penetrate down to the Earth's surface, only when the wave normal comes to fall in a narrow angle band nearly normal to the boundary surface. In this case the ray direction of the penetrating waves is closely parallel to the magaetic field.

The investigation of radio wave propagation in the lower ionosphere needs the use of wave theory, because the variation of electron density with height is very sharp. Using wave theory, Piggott et al. (1965) numerically discussed the penetration of VLF waves through the slowly varying lower ionosphere. They dealt with the case where radio waves penetrate from free space in whistler modes, and showed that absorption is more important rather than reflection. The conclusion they got is inconsistent with the results of Maeda and Oya (1963), as well as of Helliwell et al. (1962), which were obtained under the assumption of a sharp bounded ionosphere. Recently Pitteway and Jespersen (1966) made numerical calculations of VLF wave penetration through the ionosphere, which are applicable to any angle of incidence, azimuth and dip, and to any height variation of the model ionosphere.

In the followings, we will discuss the case where a whistler wave penetrates through the inhomogeneous ionosphere in the vertical direction in the presence of a vertical magnetic field and calculate the transmission and reflection coefficients of down-going waves.

2. Electromagnetic wave equation in a plasma

Assuming the time factor exp $(i\omega t)$, the electric and magnetic field vectors in an inhomogeneous and anisotropic plasma are given by the following Maxwell's equations.

$$\left.\begin{aligned} \nabla \times \mathbf{E} &= -\,i\omega\mu_0\,\mathbf{H} \\ \nabla \times \mathbf{H} &= i\omega\varepsilon_0\,(\varepsilon)\,\mathbf{E} \end{aligned}\right\} \tag{1}$$

The propagation direction of whistlers is assumed to be parallel to z axis and further the electron density in the ionosphere is inhomogeneous in z-direction but uniform in x- and y-direction. The magnetic field is taken homogeneous and directed downwards making an angle θ with z-axis. These configurations are shown in Fig. 1.

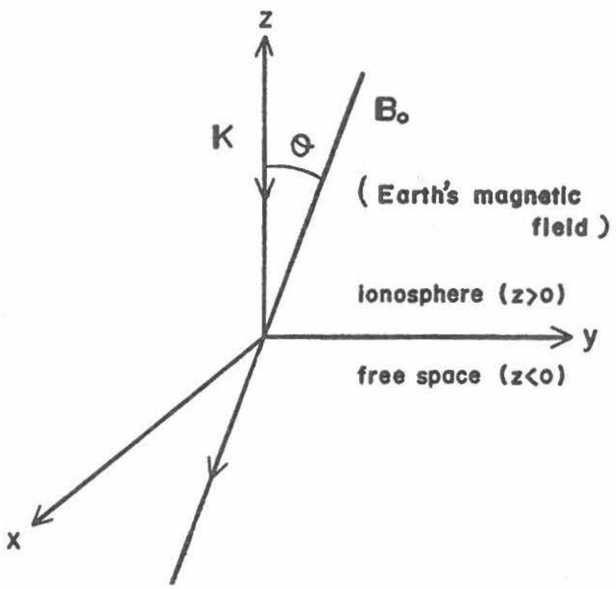

Fig. 1. Configuration of the wave penetration problem.

Then the dielectric plasma tensor (ε) can be written as

$$(\varepsilon) = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \tag{2}$$

where

$$\varepsilon_{xx}=1-\frac{x(z)}{1-y^2},\quad \varepsilon_{xy}\ (=\varepsilon_{yx})=-ix(z)\frac{y\cos\theta}{1-y^2},$$

$$\varepsilon_{xz}\ (=-\varepsilon_{zx})=-ix\ (z)\frac{y\sin\theta}{1-y^2},\quad \varepsilon_{yy}=1-x(z)\frac{1-y^2\sin^2\theta}{1-y^2},$$

$$\varepsilon_{yz}\ (=\varepsilon_{zy})=x(z)\frac{-y^2\sin\theta\cos\theta}{1-y^2},\quad \varepsilon_{zz}=1-x(z)\frac{1-y^2\cos^2\theta}{1-y^2},\ x(z)=\frac{e^2\mathrm{N}(z)}{m\varepsilon_0\omega^2},$$

$$y=\frac{\omega_H}{\omega}\ \text{and}\ \omega_H=\frac{e\mathrm{B}_0}{m} \tag{3}$$

From eqs. (1) and (2), we can get the following coupled wave equations.

$$\left.\begin{aligned}\frac{\partial^2\mathrm{E}_x}{\partial z^2}&=-k_0^2\left[\left(\varepsilon_{xx}+\frac{\varepsilon_{xz}^2}{\varepsilon_{zz}}\right)\mathrm{E}_x+\left(\varepsilon_{xy}-\frac{\varepsilon_{xz}\,\varepsilon_{yz}}{\varepsilon_{zz}}\right)\mathrm{E}_y\right]\\ \frac{\partial^2\mathrm{E}_y}{\partial z^2}&=-k_0^2\left[\left(\varepsilon_{yy}-\frac{\varepsilon_{yz}^2}{\varepsilon_{zz}}\right)\mathrm{E}_y-\left(\varepsilon_{xy}-\frac{\varepsilon_{yz}\varepsilon_{xz}}{\varepsilon_{zz}}\right)\mathrm{E}_x\right]\end{aligned}\right\} \tag{4}$$

For the sake of simplicity of the mathematical treatments, we only consider the case $\theta=0$. This corresponds to the case of polar propagation.

For the case $\theta=0$, R- and L-waves are decoupled as follows,

$$\left.\begin{aligned}&\frac{d^2\mathrm{F}}{dz^2}+k_0^2(\varepsilon_{xx}-i\varepsilon_{xy})\mathrm{F}=0\\ &\frac{d^2\mathrm{G}}{dz^2}+k_0^2(\varepsilon_{xx}+i\varepsilon_{xy})\mathrm{G}=0\end{aligned}\right\} \tag{5}$$

where $\mathrm{F}=\mathrm{E}_x+i\mathrm{E}_y$ and $\mathrm{G}=\mathrm{E}_x-i\mathrm{E}_y$.

As we deal only with R-waves, *i. e.*, whistler waves, the relevant wave equation is the first one of the two in eq. (5), which is rewritten using the relations (3).

$$\frac{d^2\mathrm{F}}{dz^2}+k_0^2\left[1-\frac{x(z)}{1-y^2}\right]\mathrm{F}=0 \tag{6}$$

This wave equation is the basis for the whistler mode propagation in the inhomogeneous ionosphere.

3. Model exosphere

We take the model exosphere as follows,

Region 1 ($z \leq 0$): free space
Region 2 ($0 \leq z \leq z_{23}$): parabolic electron distribution

$$N_2(z) = N_m \left\{ 1 - \frac{(z-z_m)^2}{z_m^2} \right\} \tag{7}$$

where N_m is the maximum density of F_2 layer, and z_m is the height of maximum density.
Region 3 ($z_{23} \leq z \leq z_{34}$): exponential electron distribution

$$N_3(z) = N_3 \exp(-2\alpha z) \quad (\alpha > 0) \tag{8}$$

Region 4 ($z \geq z_{34}$): constant electron distribution

$$N_4(z) = N_4(\text{constant}) \tag{9}$$

In the above notations, various constants are determined by the continuity of electron density distributions between the two neighboring regions. Above model exosphere is a little modification of the result derived by Ohtsu and Iwai (1967) using low latitude nose whistlers, and is shown in Fig. 2.

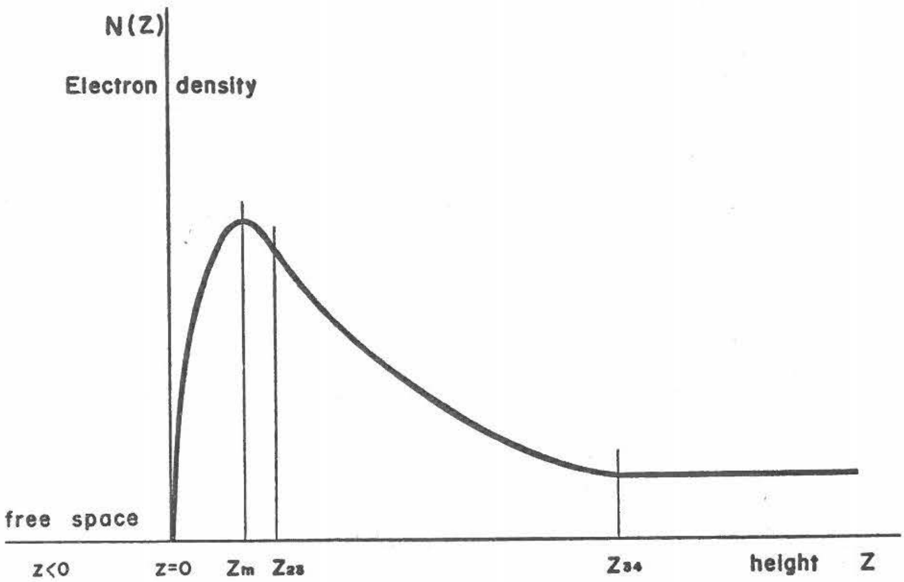

Fig. 2. Model exosphere.

4. Electromagnetic fields in each region of the ionosphere

We only discuss the case in which whistler mode waves penetrate through the ionosphere from above.

So the electric field of whistler mode waves in region 4 is obtained by substituting eq. (9) into eq. (6),

$$\mathrm{F}=\mathrm{E}_i \exp(ik_4 z) + \mathrm{E}_r \exp(-ik_4 z) \tag{10}$$

where E_i and E_r are the amplitudes of the incident and reflected waves, respectively. And $k_4=k_0\sqrt{1-x_4/(1-y)}$.

Next putting eq. (8) into eq. (6) and using the variable transformation $t=\exp(-\alpha z)$, the wave equation in region 3 reduces to

$$\frac{d^2\mathrm{F}}{dt^2}+\frac{1}{t}\frac{d\mathrm{F}}{dt}+\left[-\frac{x_3}{1-y}\cdot\frac{k_0^2}{\alpha^2}+\frac{k_0^2}{\alpha^2 t^2}\right]\mathrm{F}=0. \tag{11}$$

The solution of this equation is expressed in terms of Hankel functions. Therefore the electric field solution is given as,

$$\mathrm{F}=\mathrm{AH}_{i\nu}^{(1)}\left(\sqrt{\frac{x_3}{y-1}}\,\nu\exp(-\alpha z)\right)+\mathrm{BH}_{i\nu}^{(2)}\left(\sqrt{\frac{x_3}{y-1}}\,\nu\exp(-\alpha z)\right) \tag{12}$$

where A and B are unknown coefficients. And $\nu=k_0/\alpha$.

In region 2, the wave equation is obtained by substituting eq. (7) into eq. (6) and by making the variable transformation $s=a(z-z_m)$,

$$\frac{d^2F}{ds^2}+\left[\frac{k_0^2(1+x_2/(y-1))}{a^2}-\frac{s^2}{4}\right]\mathrm{F}=0 \tag{13}$$

where $\quad a=\left(\frac{x_2}{y-1}\right)^{\frac{1}{4}}\sqrt{\frac{2k_0}{z_m}}$ and $x_2=e^2\mathrm{N}_m/m\varepsilon_0\omega^2$.

This is the equation satisfied by Weber functions. Therefore the electric field in region 2 is written as follows,

$$\mathrm{F}=\mathrm{CD}_\lambda(a(z-z_m))+\mathrm{ED}_{-\lambda-1}(ia(z-z_m)) \tag{14}$$

where $\quad \lambda=\left(1+\frac{x_2}{y-1}\right)\frac{k_0^2}{a^2}-\frac{1}{2}$. C and E are unknown coefficients.

Last we take the solution for electric field in region 1 (free space) as

$$F=E_t \exp(ik_0 z). \tag{15}$$

This wave expresses the transmitted wave and E_t is its amplitude. So the electric fields are determined completely except the unknown coefficients.

5. Transmission and reflection coefficients

Since the electric fields are all derived, it is necessary to determine the unknown coefficients by matching the fields at the boundaries $z=0$, z_{23} and z_{34}, respectively. The boundary conditions are the continuities of the fields and, of their derivatives with z at the boundaries.

After some manipulations, we get the transmission and reflection coefficients as follows.

$$t=\frac{\Delta_t}{\Delta} \tag{16}$$

and

$$r=\frac{\Delta_r}{\Delta} \tag{17}$$

The determinant in eqs. (16) and (17) is given by,

$$
\varDelta=\begin{vmatrix}
e^{-ik_4z_{34}} & -\mathrm{H}_{i\nu}^{(1)}(\varepsilon e^{-\alpha z_{34}}) & -\mathrm{H}_{i\nu}^{(2)}(\varepsilon e^{-\alpha z_{34}}) & 0 & 0 & 0\\
-ik_4e^{-ik_4z_{34}} & \varepsilon\alpha e^{-\alpha z_{34}}\mathrm{H}_{i\nu}^{(1)\prime}(\varepsilon e^{-\alpha z_{34}}) & \varepsilon\alpha e^{-\alpha z_{34}}\mathrm{H}_{i\nu}^{(2)\prime}(\varepsilon e^{-\alpha z_{34}}) & 0 & 0 & 0\\
0 & \mathrm{H}_{i\nu}^{(1)}(\varepsilon e^{-\alpha z_{23}}) & \mathrm{H}_{i\nu}^{(2)}(\varepsilon e^{-\alpha z_{22}}) & -\mathrm{D}_\lambda\{a(z_{23}-z_m)\} & -\mathrm{D}_{-\lambda-1}\{ia(z_{23}-z_m)\} & 0\\
0 & \varepsilon\alpha e^{-\alpha z_{23}}\mathrm{H}_{i\nu}^{(1)\prime}(\varepsilon e^{-\alpha z_{23}}) & \varepsilon\alpha e^{-\alpha z_{23}}\mathrm{H}_{i\nu}^{(2)\prime}(\varepsilon e^{-\alpha z_{23}}) & a\mathrm{D}_\lambda'\{a(z_{23}-z_m)\} & ia\mathrm{D}_{-\lambda-1}'\{ia(z_{23}-z_m)\} & 0\\
0 & 0 & 0 & \mathrm{D}_\lambda(-az_m) & \mathrm{D}_{-\lambda-1}(-iaz_m) & -1\\
0 & 0 & 0 & a\mathrm{D}_\lambda'(-az_m) & ia\mathrm{D}_{-\lambda-1}'(-iaz_m) & -ik_0
\end{vmatrix}
$$

where $\quad \varepsilon = \sqrt{\frac{x_3}{y-1}}\,\nu.$

Determinants Δ_t and Δ_r are respectively obtained by replacing the last, and the first, column of Δ with the column matrix $(-\exp(-ik_4 z_{34}), -ik_4 \exp(-ik_4 z_{34}), 0, 0, 0, 0)$.

Therefore the transmission, and the reflection, coefficients of the down-going whistler mode waves are obtained analytically in terms of various parameters, for example, the height and maximum density of F_2 layer and so on.

Using the transmission and reflection coefficients, we can discuss the frequency dependencies of these coefficients and further can see which region has the most predominant influence on whistler penetration. These detailed considerations are now in progress.

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