Optimal constants of smoothing estimates for the 2D and massless 3D Dirac equation
(2次元および質量0の3次元ディラック方程式の平滑化評価式の最良定数)

生駒 真

Contents

Abstract
In this paper, we discuss optimal constants and extremisers of Kato-smoothing estimates for the 2D and massless 3D Dirac equation. Smoothing estimates are inequalities that express the smoothing effect of dispersive equations, and detailed information regarding optimal constants and extremisers for wide classes of Kato-smoothing estimates were given in the last several year by Bez-Saito-Sugimoto [9] and Bez-Sugimoto [10]. This paper is a trial to generalize these previous results to include the Dirac equation which is outside the framework of them.

1 Introduction

The estimates of the form

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left|\psi(D)e^{it\phi(D)}f(x)\right|^2 w(x)dxdt \le C\|f\|^2_{L^2(\mathbb{R}^d)} \tag{1.1}$$

for solutions to dispersive equations are often called smoothing estimates (Kato-smoothing estimates). Kato [19] first deduced local version of them. This motivated the works of Constantin and Saut [12], Sjölin [32] and Vega [36] followed by established local smoothing estimates for Schrödinger equations including Ben-Artzi and Devinatz [3, 4], Hoshiro [15, 16], Kenig, Ponce and Vega [22, 23, 24, 25, 26, 27], Linares and Ponce [28], Ruzhansky and Sugimoto [30], Sugimoto [34] and Walther [38], various other works on the spacetime approach, including Ben-Artzi and Nemirovsky [6] for the generalized relativistic Schrödinger equation and Ben-Artzi [2] for the generalized classical wave equation. Recently, a general approach to the global spacetime and smoothing estimates of evolution groups generated by self-adjoint operators has been developed by Ben-Artzi, Ruzhansky and Sugimoto [7]. This approach leads to similar estimates for various classes of pseudodifferential operators.

There are conventional representative studies on smoothing estimates (1.1). If $\phi(D) = -\Delta$, then $e^{it\phi(D)}$ is the free Schrödinger propagator, in which case it is known that (1.1) holds in the following cases :

(A)	$\psi(D) = (1+\|D\|^2)^{1/4}$,	$w(x) = (1+\|x\|^2)^{-1}$	$(d \ge 3)$
(B)	$\psi(D) = \|D\|^a$,	$w(x) = \|x\|^{2(a-1)}$	$(1-\frac{d}{2} < a < \frac{1}{2}, d \ge 2)$
(C)	$\psi(D) = \|D\|^{1/2}$,	$w(x) = (1+\|x\|^2)^{-s}$	$(s > \frac{1}{2}, d \ge 2)$.

Case (A) is due to Kato and Yajima [21]. See also Ben-Artzi and Klainerman [5]. Case (B) is due to Kato and Yajima [21] for $0 \le a < \frac{1}{2}$ whenever $d \ge 3$, and $0 < a < \frac{1}{2}$ for $d = 2$. See also Ben-Artzi and Klainerman [5] for a different approach, and Sugimoto [35], Vilela [37], Watanabe [40] for the full range $1 - \frac{d}{2} <$

$a < \frac{1}{2}$ whenever $d \geq 2$. Case (C) is due to Kenig, Ponce and Vega [22]. See also Ben-Artzi and Klainerman [5] and Chihara [11].

The main concern of this paper is the optimal constant of smoothing estimates. As for the case (B), it is known that

$$\pi 2^{2a-1} \frac{\Gamma(1-2a)\Gamma(\frac{d}{2}+a-1)}{\Gamma(1-a)^2\Gamma(\frac{d}{2}-a)}$$

is the optimal constant, and the radial function is its extremiser. This result with $a = 0$ and $d \geq 3$ was first established by Simon [31], and Watanabe [40] extended it to the full range of a and d. As for the case (C), Simon [31] showed that the optimal constant is $\frac{\pi}{2}$ when $s = 1$ and $d \geq 3$. As for the rest cases, the optimal constant has remained open, but significant progress has been made in recent years which are described below.

In estimate (1.1), the spatial weight w, the smoothing function ψ, and the dispersion relation ϕ are assumed to be radial, and we reserve the notation

$$C_d(w,\psi,\phi) := \sup_{\substack{f \in L^2(\mathbb{R}^d) \\ f \neq 0}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left|\psi(|D|)e^{it\phi(|D|)}f(x)\right|^2 w(|x|)dxdt/\|f\|^2_{L^2(\mathbb{R}^d)} \tag{1.2}$$

for the optimal constant. Then we know the following result :

Theorem 1.1. *([9, Theorem 1.1]) Let $d \geq 2$. We have*

$$C_d(w,\psi,\phi) = \frac{1}{(2\pi)^{d-1}} \sup_{k\in\mathbb{N}_0} \sup_{r>0} \lambda_k(r), \tag{1.3}$$

where

$$\lambda_k(r) := \left|\mathbb{S}^{d-2}\right| \frac{r^{d-1}\psi(r)^2}{|\phi'(r)|} \int_{-1}^{1} F_w(r^2(1-t))p_{d,k}(t)(1-t^2)^{\frac{d-3}{2}}dt. \tag{1.4}$$

Here F_w is defined by the relation

$$\widehat{w(|\cdot|)}(\xi) = F_w(\tfrac{1}{2}|\xi|^2)$$

and $p_{d,k}$ is the Legendre polynomial of degree k in d dimensions, which may be defined in a number of ways, for example, via the Rodrigues formula,

$$(1-t^2)^{\frac{d-3}{2}}p_{d,k}(t) = (-1)^k \frac{\Gamma(\frac{d-1}{2})}{2^k\Gamma(k+\frac{d-1}{2})} \frac{d^k}{dt^k}(1-t^2)^{k+\frac{d-3}{2}}. \tag{1.5}$$

As a corollary of this result, we obtain that the optimal constant for the case (C) with $s > \frac{1}{2}$ and $d \geq 3$ is

$$\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{2\Gamma(s)}$$

(See [9, Corollary 1.5]). The main purpose of this paper is to extend these results on optimal constants to those for the free Dirac equation which is outside the framework of Theorem 1.1.

We exhibit known facts on smoothing estimates for the Dirac equation. Let $d \geq 2$, and let $N = 2^{[(d+1)/2]}$. The Dirac equation with mass m perturbed by the potential V is expressed by

$$\begin{cases} (i\partial_t - H)\, u(x,t) = 0, \\ u(x,0) = f(x) \in L^2(\mathbb{R}^d, \mathbb{C}^N) \end{cases} \tag{1.6}$$

where $H = H_m + V$, $H_m = \alpha \cdot D + m\beta = \sum_{j=1}^{d} \alpha_j D_j + m\beta$ and $\alpha_1, \alpha_2, \cdots, \alpha_d$, $\alpha_{d+1} = \beta$ are $N \times N$ Hermitian matrices satisfying $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} I_N$. Then the solution is expressed as $u(x,t) = e^{-itH} f(x)$. The case when the potential V vanishes is called the free Dirac equation. We express smoothing estimates for equation (1.6) in the following form as (1.1) :

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left|\psi(D) e^{-itH} f(x)\right|^2 w(x)\, dx dt \leq C \|f\|^2_{L^2(\mathbb{R}^d, \mathbb{C}^N)}. \tag{1.7}$$

It is known that (1.7) holds in the following cases when $\psi(D) = 1$:

(D)	$H = H_m$,	$w(x) = (1+\lvert x\rvert^2)^{-s}$	$(s > 1, m > 0, d = 3)$
(E)	$H = H_0 + V$,	$w(x) = \{\lvert x\rvert(1+\lvert\log\lvert x\rvert\rvert)^{\sigma}\}^{-1}$	$(\sigma > 1, d = 3)$
(F)	$H = H_1 + V$,	$w(x) = (\lvert x\rvert^{\frac{1}{2}-\varepsilon} + \lvert x\rvert)^{-2}$	$(\frac{1}{2} \gg \varepsilon > 0, d = 3)$
(G)	$H = H_0$,	$w(x) = (1+\lvert x\rvert^2)^{-s}\lvert x\rvert^{-2a}$	$(0 \leq a < \frac{1}{2}, a + s > \frac{1}{2}, d \geq 2)$
(H)	$H = H_m$,	$w(x) = (1+\lvert x\rvert^2)^{-1+a}\lvert x\rvert^{-2a}$	$(0 \leq a < \frac{1}{2}, m > 0, d \geq 3)$.

Case (D) is due to Ben-Artzi and Umeda [8]. Cases (E) and (F) are due to D'Ancona and Fanelli [13], where $V = V(x)$ is a singular 4×4 complex-valued Hermitian matrix. Cases (G) and (H) are derived from the following uniform resolvent estimates for Dirac operator :

$$\sup_{z \in \mathbb{C}\setminus\mathbb{R}} \left\|(1+|\cdot|^2)^{-\frac{s}{2}} |\cdot|^{-a} (H_0 - z)^{-1} |\cdot|^{-a} (1+|\cdot|^2)^{-\frac{s}{2}}\right\| \leq C_{s,a},$$

where $0 \leq a < \frac{1}{2}, a + s > \frac{1}{2}, d \geq 2$, and

$$\sup_{z \in \mathbb{C}\setminus\mathbb{R}} \left\|(1+|\cdot|^2)^{\frac{-1+a}{2}} |\cdot|^{-a} (H_m - z)^{-1} |\cdot|^{-a} (1+|\cdot|^2)^{\frac{-1+a}{2}}\right\| \leq C_{a,d},$$

where $0 \leq a < \frac{1}{2}, m > 0, d \geq 3$ (See [18, Theorems 4.2 and 4.3]). By these estimates and the work of Kato [20, Lemma 3.6 and Theorem 5.1], we have cases (G) and (H).

We should mention a new comparison principle established by Ben-Artzi, Ruzhansky and Sugimoto [7] and Ruzhansky and Sugimoto [30], which relate smoothing estimates for different type of equations. For example, by the argument in [30, Section 8], we know that smoothing estimate (1.1) with $\phi(D) = -\Delta$ (Schrödinger eq.) and that with $\phi(D) = \sqrt{-\Delta + m^2}$ (relativistic Schrödinger eq.) are equivalent to each other. Furthermore, by the spectral comparison principle in [7, Corollary 4.5], we know that smoothing estimate (1.7) with $H = H_m$ (free Dirac eq.) is derived from smoothing estimate (1.1) with $\phi(D) = \sqrt{-\Delta + m^2}$. Note that the Dirac operator H_m is diagonalized as

$$U(D)H_mU(D)^{-1} = \begin{pmatrix} \sigma(D)I_{\frac{N}{2}} & O \\ O & -\sigma(D)I_{\frac{N}{2}} \end{pmatrix},$$

by the unitary operator

$$U(D) := \frac{1}{\sqrt{2}}\left(\sqrt{1 + \frac{m}{\sigma(D)}}I_N + \sqrt{\frac{1}{\sigma(D)(\sigma(D)+m)}}\beta\alpha \cdot D\right),$$

where $\sigma(D) = \sqrt{-\Delta + m^2}$.

The objective of this paper is to investigate the optimal constant for (1.7). We note that this problem remains open even for special cases. We consider the free Dirac equation, i.e. $H = H_m$. The major difference between the smoothing estimates (1.1) and (1.7) is that (1.1) is on $L^2(\mathbb{R}^d)$ while (1.7) on $L^2(\mathbb{R}^d, \mathbb{C}^N)$. Also, the dispersion relation $H = H_m$ in (1.7) is a matrix exponential that is not a radial function, hence Theorem 1.1 cannot be directly applied to (1.7). Our natural expectation is that $C_d(w, \psi, \phi)$ in (1.2) with $\phi(r) = \sqrt{r^2 + m^2}$ is the optimal constant of estimate (1.7) since the square of the Dirac operator H_m is $(-\Delta + m^2)I_N$ (See also Remark 2). Our first result (Theorem 2.1) says that it is indeed the upper bound. But it is still unclear whether this upper bound is equal to the optimal constant. Our second and third result (Theorems 2.2 and 2.3) says that, it is incorrect in the case 2D and massless 3D. These main results will be exhibited in Section 2 and proved in Section 3.

2 Main results

In this section, we discuss the main results. The Fourier transform of f, represented by $\hat{f}$, is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx.$$

We assume that the spatial weight w and the smoothing function ψ in the smoothing estimates (1.7) of free Dirac equation are as follows :

Assumption 1. *$w(|\cdot|) : \mathbb{R}^d \backslash \{0\} \to (0, \infty)$ is assumed to be a positive and radial function whose Fourier transform is well-defined on $\mathbb{R}^d \backslash \{0\}$, in which case we write*

$$\widehat{w(|\cdot|)}(\xi) = F_w(\tfrac{1}{2}|\xi|^2) \tag{2.1}$$

for some $F_w : (0, \infty) \to \mathbb{C}$. It is assumed that F_w is locally bounded on $(0, \infty)$ away from zero. Next, let $\psi : [0, \infty) \to [0, \infty)$. Furthermore, (w, ψ) are assumed to be sufficiently regular to guarantee the continuity of $\lambda_k : (0, \infty) \to \mathbb{R}$ for each fixed $k \in \mathbb{N}_0$, where

$$\lambda_k(r) := \left|\mathbb{S}^{d-2}\right| r^{d-2} \sqrt{r^2 + m^2}\, \psi(r)^2 \int_{-1}^{1} F_w(r^2(1-t)) p_{d,k}(t) (1-t^2)^{\frac{d-3}{2}} dt. \tag{2.2}$$

Here, $p_{d,k}$ is the Legendre polynomial of degree k in d dimensions defined in (1.5). Also, we reserve the notation $C_d(w, \psi, m)$ for the optimal constant in the smoothing estimates (1.7) with $H = H_m = \alpha \cdot D + m\beta$:

$$C_d(w, \psi, m) := \sup_{\substack{f \in L^2(\mathbb{R}^d, \mathbb{C}^N) \\ f \neq 0}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left|\psi(|D|) e^{-it(\alpha\cdot D + m\beta)} f(x)\right|^2 w(|x|) dx dt / \|f\|^2_{L^2(\mathbb{R}^d, \mathbb{C}^N)}. \tag{2.3}$$

Now we are in a position to state our main results. The first one gives the upper estimate for the optimal constant.

Theorem 2.1. *([17, Theorem 2.1]) Under Assumption 1, we have*

$$C_d(w, \psi, m) \leq \frac{1}{(2\pi)^{d-1}} \sup_{k \in \mathbb{N}_0} \sup_{r > 0} \lambda_k(r). \tag{2.4}$$

There still remain the problem if the inequality (2.4) is indeed equality or not. We gives a negative answer in the case 2D and massless 3D :

Theorem 2.2. *([17, Theorem 2.2]) Let $d = 2$ and*

$$\widetilde{\lambda}_k(r) := \frac{1}{2}\left\{\lambda_{|k|}(r) + \lambda_{|k+1|}(r) + \frac{m}{\sqrt{r^2+m^2}}\left|\lambda_{|k|}(r) - \lambda_{|k+1|}(r)\right|\right\}. \tag{2.5}$$

Under Assumption 1, we have

$$C_2(w,\psi,m) = \frac{1}{2\pi}\sup_{k\in\mathbb{Z}}\sup_{r>0}\widetilde{\lambda}_k(r). \tag{2.6}$$

Define

$$\widetilde{S}_k := \left\{r > 0 \;\middle|\; \widetilde{\lambda}_k(r) = \sup_{\ell\in\mathbb{Z}}\sup_{s>0}\widetilde{\lambda}_\ell(s)\right\},$$

and

$$\widetilde{K} := \left\{k \in \mathbb{Z} \;\middle|\; \left|\widetilde{S}_k\right| > 0\right\}.$$

Then an extremiser exists if and only if $\widetilde{K}$ is nonempty, in which case, all extremisers can be expressed as

$$\widehat{f} = \sum_{k\in\widetilde{K}} f_k$$

with each f_k of the form

$$f_k(\xi) = \frac{r^{-\frac{1}{2}}e^{ik\theta}}{\sqrt{2\pi}}\begin{pmatrix}\widetilde{f}_1^{(k)}(r)\\ 0\end{pmatrix} + \frac{r^{-\frac{1}{2}}e^{i(k+1)\theta}}{\sqrt{2\pi}}\begin{pmatrix}0\\ \widetilde{f}_2^{(k+1)}(r)\end{pmatrix}, \tag{2.7}$$

where $\widetilde{f}_1^{(k)}(r)$ and $\widetilde{f}_2^{(k+1)}(r)$ are functions contained in $L^2(\mathbb{R}_+)$ such that

$$\operatorname{supp}\left\{\left|\widetilde{f}_1^{(k)}\right|^2 + \left|\widetilde{f}_2^{(k+1)}\right|^2\right\} \subseteq \widetilde{S}_k$$

and they are expressed in the following form, where $\phi(r) = \sqrt{r^2+m^2}$:

(i) *In the case $m = 0$: $\widetilde{f}_1^{(k)}(r)$ and $\widetilde{f}_2^{(k+1)}(r)$ are any functions.*

(ii) *In the case $m > 0$:*

$\widetilde{f}_2^{(k+1)}(r) = \frac{r}{\phi(r)+m}\widetilde{f}_1^{(k)}(r)$ *for* $\lambda_{|k|}(r) > \lambda_{|k+1|}(r)$,

$\widetilde{f}_1^{(k)}(r)$ *and* $\widetilde{f}_2^{(k+1)}(r)$ *are any functions for* $\lambda_{|k|}(r) = \lambda_{|k+1|}(r)$,

$\widetilde{f}_1^{(k)}(r) = -\frac{r}{\phi(r)+m}\widetilde{f}_2^{(k+1)}(r)$ *for* $\lambda_{|k|}(r) < \lambda_{|k+1|}(r)$.

Theorem 2.3. *Let $d = 3$ and $m = 0$. Under Assumption 1, we have*

$$C_3(w, \psi, 0) = \frac{1}{(2\pi)^2} \sup_{k \in \mathbb{N}_0} \sup_{r>0} \frac{\lambda_k(r) + \lambda_{k+1}(r)}{2}. \tag{2.8}$$

Define

$$S_k^* := \left\{ r > 0 \;\middle|\; \frac{\lambda_k(r) + \lambda_{k+1}(r)}{2} = \sup_{\ell \in \mathbb{N}_0} \sup_{r>0} \frac{\lambda_\ell(r) + \lambda_{\ell+1}(r)}{2} \right\},$$

and

$$K^* := \{ k \in \mathbb{N}_0 \mid |S_k^*| > 0 \}.$$

Then an extremiser exists if and only if K^* is nonempty, in which case, all extremisers can be expressed as

$$\widehat{f} = \sum_{k \in K^*} f_k$$

with each f_k of the form

$$f_k(\xi) = r^{-1} \sum_{n=-k}^{k} Y_k^n(\theta, \varphi) \begin{pmatrix} f_1^{(k,n)}(r) \\ f_2^{(k,n)}(r) \\ f_3^{(k,n)}(r) \\ f_4^{(k,n)}(r) \end{pmatrix} + r^{-1} \sum_{n=-k-1}^{k+1} Y_{k+1}^n(\theta, \varphi) \begin{pmatrix} f_1^{(k+1,n)}(r) \\ f_2^{(k+1,n)}(r) \\ f_3^{(k+1,n)}(r) \\ f_4^{(k+1,n)}(r) \end{pmatrix} \tag{2.9}$$

where Y_k^n are the spherical harmonics with $d = 3$ (See (3.21)), $f_j^{(k,n)}(r)$ and $f_j^{(k+1,n)}(r)$ $(j = 1, 2, 3, 4)$ are functions contained in $L^2(\mathbb{R}_+)$ such that

$$\operatorname{supp} \left\{ \sum_{n=-k}^{k} \sum_{j=1}^{4} \left| f_j^{(k,n)} \right|^2 + \sum_{n=-k-1}^{k+1} \sum_{j=1}^{4} \left| f_j^{(k+1,n)} \right|^2 \right\} \subset S_k^*$$

and they are expressed in the following form :

(i) *In the case $\lambda_{k-1}(r) < \lambda_{k+1}(r)$: $f_{j+1}^{(k,n+1)} = \sqrt{\frac{k-n}{k+n+1}} f_j^{(k,n)}$ $(j = 1, 3,\ -k \le n \le k-1)$, $f_j^{(k,k)}$ and $f_{j+1}^{(k,-k)}$ $(j = 1, 3)$ are any functions.*

(ii) *In the case $\lambda_{k-1}(r) = \lambda_{k+1}(r)$: $f_j^{(k,n)}$ $(j = 1, 2, 3, 4,\ -k \le n \le k)$ are any functions.*

(iii) *In the case $\lambda_k(r) > \lambda_{k+2}(r)$: $f_j^{(k+1,n)} = -\sqrt{\frac{k-n+1}{k+n+2}} f_{j+1}^{(k+1,n+1)}$ $(j = 1, 3,\ -k-1 \le n \le k)$, $f_j^{(k+1,k+1)} = f_{j+1}^{(k+1,-k-1)} = 0$ $(j = 1, 3)$.*

(iv) *In the case $\lambda_k(r) = \lambda_{k+2}(r)$: $f_j^{(k+1,n)}(r)$ $(j = 1, 2, 3, 4,\ -k-1 \le n \le k+1)$ are any functions.*

Remark 1. In the case $2D$ and massless $3D$, Theorems 2.2 and 2.3 say that inequality (2.4) is not the equality in general. In fact, the massless 3D case is obvious, and the 2D case is also easily seen from the fact that

$$\widetilde{\lambda}_k(r) < \frac{1}{2}\left\{\lambda_{|k|}(r) + \lambda_{|k+1|}(r) + \left|\lambda_{|k|}(r) - \lambda_{|k+1|}(r)\right|\right\} = \max\left\{\lambda_{|k|}(r), \lambda_{|k+1|}(r)\right\}.$$

Remark 2. As a matter of fact, Assumption 1 is exactly the same one as in Theorem 1.1 with $\phi(r) = \sqrt{r^2+m^2}$. Since the square of the free Dirac operator $\alpha \cdot D + m\beta$ gives $(-\Delta + m^2)I_N$, it is natural that $\phi(r) = \sqrt{r^2+m^2}$ appears in the smoothing estimates (1.7). If we put the dispersion relation $\phi(r) = \sqrt{r^2+m^2}$ in the smoothing estimates (1.1), this is the case of the Klein-Gordon equation. The Klein-Gordon equation, like the Dirac equation, is one of the relativistic Schrödinger equations. However, as can be seen from Theorems 1.1, 2.2 and 2.3, the optimal constant of the Klein-Gordon equation is different from that of the Dirac equation.

Remark 3. For $\lambda_k(r)$ defined by (1.4), we know that for each $k \in \mathbb{N}$ and $r > 0$, the following equation holds :

$$\lambda_k(r) = (2\pi)^d \frac{r\psi(r)^2}{|\phi'(r)|}\int_0^\infty J_{\nu(k)}(rt)^2 t w(t)dt,$$

where $J_{\nu(k)}$ is the Bessel function of the first kind of order ν, and $\nu = k + \frac{d-2}{2}$ (See [9]). From this fact, we know that $\lambda_k(r)$, as defined by (1.4) and (2.2), is always positive.

3 Proof of Main results

In this section, we prove Theorems 2.1, 2.2 and 2.3.

3.1 Preliminary

First, we present theorems which is necessary in this section. Let $\xi \in \mathbb{R}^d$, and let $P^{(k,j)}(\xi)$ ($k \in \mathbb{N}_0$, $j \in \{1, 2, \cdots, a_k\}$) be the homogeneous harmonic polynomials of the degree k normalized on $L^2(\mathbb{S}^{d-1})$. The restriction of $P^{(k,j)}$ on $\mathbb{S}^{d-1}$ is the spherical harmonics of the degree k. Then we have the following decomposition :

Theorem 3.1. *([38, 3.6. Theorem]) Let*

$$N_k(\mathbb{R}^d) = \left\{ \sum_{j=1}^{a_k} P^{(k,j)}(\xi) f_0^{(k,j)}(|\xi|)|\xi|^{-k-\frac{d-1}{2}} \;\middle|\; f_0^{(k,j)} \in L^2(\mathbb{R}_+) \right\}.$$

Then the complete orthogonal decomposition

$$L^2(\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} N_k(\mathbb{R}^d)$$

holds in the sense that

(a) *each subspace $N_k(\mathbb{R}^d)$ is closed,*

(b) *$N_{k_1}(\mathbb{R}^d)$ is orthogonal to $N_{k_2}(\mathbb{R}^d)$ if $k_1 \neq k_2$;*

(c) *every $f \in L^2(\mathbb{R}^d)$ can be written as a sum*

$$f = \sum_{k=0}^{\infty} f_k \ , \quad f_k \in N_k(\mathbb{R}^d),$$

with convergence in $L^2(\mathbb{R}^d)$.

From this theorem, any $f \in L^2(\mathbb{R}^d)$ can be expressed as

$$f(\xi) = \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} P^{(k,j)}(\tfrac{\xi}{|\xi|}) f_0^{(k,j)}(|\xi|)|\xi|^{-\frac{d-1}{2}} \ , \quad f_0^{(k,j)} \in L^2(\mathbb{R}_+). \tag{3.1}$$

We also use the following theorem, where $p_{d,k}$ is the Legendre polynomial of degree k in d dimensions defined in (1.5).

Theorem 3.2. *(Funk-Hecke ([9, Theorem 2.1] etc.)) Let $d \geq 2$, $k \in \mathbb{N}_0$ and P be a spherical harmonic of degree k. Then*

$$\int_{\mathbb{S}^{d-1}} F(\theta \cdot \omega) P(\omega) d\sigma(\omega) = \mu_k P(\theta)$$

for any $\theta \in \mathbb{S}^{d-1}$ and any function $F \in L^1([-1,1],(1-t^2)^{\frac{d-3}{2}})$. Here, the constant μ_k given by

$$\mu_k = \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} F(t) p_{d,k}(t)(1-t^2)^{\frac{d-3}{2}} dt.$$

3.2 Proof of Theorem 2.1

Let $\lambda = \sup_{k\in\mathbb{N}_0} \sup_{r>0} \lambda_k(r)$, $A_\xi = \alpha \cdot \xi + m\beta$, and $\xi \in \mathbb{R}^d$. Consider the linear operator given by

$$(Sf)(x,t) = w(|x|)^{\frac{1}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \psi(|\xi|) e^{-itA_\xi} f(\xi) d\xi$$

for a $\mathbb{C}^N$-valued function $f:\mathbb{R}^d\to\mathbb{C}^N$, where e^{-itA_ξ} is a matrix exponential

$$e^{-itA_\xi}=\sum_{k=0}^{\infty}\frac{(-it)^k}{k!}A_\xi^k.$$

Note that S is an operator from $L^2(\mathbb{R}^d,\mathbb{C}^N)$ to $L^2(\mathbb{R}^{d+1},\mathbb{C}^N)$. By Plancherel's theorem we have

$$\|S\|^2=(2\pi)^d C_d(w,\psi,m). \tag{3.2}$$

Hence, it is enough to show that $\|S\|^2\le 2\pi\lambda$.

From the definitions of α and β, A_ξ^k is an $N\times N$ Hermitian matrix, hence we have the expression of the adjoint operator S^* :

$$(S^*g)(\xi)=\psi(|\xi|)\int_{\mathbb{R}^{d+1}} w(|x|)^{\frac{1}{2}}e^{-ix\cdot\xi}e^{itA_\xi}g(x,t)dxdt.$$

Then by calculation, we have

$$(S^*Sf)(\xi)=\psi(|\xi|)\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}} e^{itA_\xi}e^{-itA_\eta}g_\xi(\eta)dt\right)d\eta, \tag{3.3}$$

where $g_\xi(\eta)=\widehat{w(|\cdot|)}(\xi-\eta)\psi(|\eta|)f(\eta)$. We set

$$\mathrm{I}_\pm(\xi):=\frac{1}{2}\left(I_N\pm\frac{1}{\sqrt{|\xi|^2+m^2}}A_\xi\right). \tag{3.4}$$

Then $\mathrm{I}_\pm(\xi)$ is an orthogonal projection, and any $f\in L^2(\mathbb{R}^d,\mathbb{C}^N)$ is orthogonally decomposed as

$$f(\xi)=\mathrm{I}_+(\xi)f(\xi)+\mathrm{I}_-(\xi)f(\xi),\quad \mathrm{I}_+f\perp\mathrm{I}_-f.$$

We note that $\mathrm{I}_+(\xi)$ and $\mathrm{I}_-(\xi)$ are commutative. Since $A_\xi=\alpha\cdot\xi+m\beta=\sqrt{|\xi|^2+m^2}\{\mathrm{I}_+(\xi)-\mathrm{I}_-(\xi)\}$, we have

$$e^{-itA_\eta}=e^{-it\phi(|\eta|)\{\mathrm{I}_+(\eta)-\mathrm{I}_-(\eta)\}}=e^{-it\phi(|\eta|)\mathrm{I}_+(\eta)}e^{it\phi(|\eta|)\mathrm{I}_-(\eta)},$$

with $\phi(r)=\sqrt{r^2+m^2}$, hence, we have

$$e^{-itA_\eta}g_\xi(\eta)=e^{-it\phi(|\eta|)\mathrm{I}_+(\eta)}e^{it\phi(|\eta|)\mathrm{I}_-(\eta)}\left\{\mathrm{I}_+(\eta)g_\xi(\eta)+\mathrm{I}_-(\eta)g_\xi(\eta)\right\}.$$

Furthermore, since

$$e^{it\phi(|\eta|)\mathrm{I}_-(\eta)}\mathrm{I}_+(\eta)=\sum_{k=0}^{\infty}\frac{\{it\phi(|\eta|)\}^k}{k!}\mathrm{I}_-(\eta)^k\mathrm{I}_+(\eta)=\mathrm{I}_+(\eta),$$
$$e^{it\phi(|\eta|)\mathrm{I}_-(\eta)}\mathrm{I}_-(\eta)=\sum_{k=0}^{\infty}\frac{\{it\phi(|\eta|)\}^k}{k!}\mathrm{I}_-(\eta)^{k+1}=\sum_{k=0}^{\infty}\frac{\{it\phi(|\eta|)\}^k}{k!}\mathrm{I}_-(\eta)=e^{it\phi(|\eta|)}\mathrm{I}_-(\eta),$$

we have

$$e^{-itA_\eta} g_\xi(\eta) = e^{-it\phi(|\eta|)\mathrm{I}_+(\eta)} \left\{ \mathrm{I}_+(\eta) + e^{it\phi(|\eta|)} \mathrm{I}_-(\eta) \right\} g_\xi(\eta).$$

Similarly, we have :

$$\begin{aligned} e^{itA_\xi} e^{-itA_\eta} g_\xi(\eta) = \big[& e^{-it\{\phi(|\eta|)-\phi(|\xi|)\}} \mathrm{I}_+(\xi)\mathrm{I}_+(\eta) + e^{-it\{\phi(|\eta|)+\phi(|\xi|)\}} \mathrm{I}_-(\xi)\mathrm{I}_+(\eta) \\ & + e^{it\{\phi(|\eta|)+\phi(|\xi|)\}} \mathrm{I}_+(\xi)\mathrm{I}_-(\eta) + e^{it\{\phi(|\eta|)-\phi(|\xi|)\}} \mathrm{I}_-(\xi)\mathrm{I}_-(\eta) \big] g_\xi(\eta). \end{aligned}$$

Plugging it into (3.3) and by polar coordinate transformation $\eta = r\theta$ $(r = |\eta|,\ \theta = \frac{\eta}{|\eta|})$, we have

$$\begin{aligned} (S^*Sf)(\xi) = 2\pi\psi(|\xi|) \int_{\mathbb{S}^{d-1}} \Bigg[& \int_0^\infty \delta(\phi(r) - \phi(|\xi|)) r^{d-1} \mathrm{I}_+(\xi)\mathrm{I}_+(r\theta) g_\xi(r\theta) dr \\ & + \int_0^\infty \delta(\phi(r) + \phi(|\xi|)) r^{d-1} \mathrm{I}_-(\xi)\mathrm{I}_+(r\theta) g_\xi(r\theta) dr \\ & + \int_0^\infty \delta(\phi(r) + \phi(|\xi|)) r^{d-1} \mathrm{I}_+(\xi)\mathrm{I}_-(r\theta) g_\xi(r\theta) dr \\ & + \int_0^\infty \delta(\phi(r) - \phi(|\xi|)) r^{d-1} \mathrm{I}_-(\xi)\mathrm{I}_-(r\theta) g_\xi(r\theta) dr \Bigg] d\sigma(\theta), \end{aligned} \tag{3.5}$$

where we have used the fact

$$\int_{\mathbb{R}} e^{-its} dt = \int_{\mathbb{R}} e^{its} dt = 2\pi\delta(s).$$

Here, $\phi(r) + \phi(|\xi|) > 0$ ($r \geq 0$, $|\xi| > 0$), if $|\xi| > 0$ then $\delta(\phi(r) + \phi(|\xi|)) = 0$ for any $r \geq 0$. Therefore, the second and third terms on the right hand side of (3.5) disappear and we have

$$(S^*Sf)(\xi) = 2\pi\psi(|\xi|) \int_{\mathbb{S}^{d-1}} \int_0^\infty \delta(\phi(r) - \phi(|\xi|)) h_{\xi\theta}(r) dr d\sigma(\theta),$$

where

$$h_{\xi\theta}(r) := r^{d-1} \{ \mathrm{I}_+(\xi)\mathrm{I}_+(r\theta) + \mathrm{I}_-(\xi)\mathrm{I}_-(r\theta) \} g_\xi(r\theta).$$

Since $\phi(r) = \sqrt{r^2 + m^2}$ is injective and differentiable on $r \in \mathbb{R}_+$, we have

$$\begin{aligned} & \int_0^\infty \delta(\phi(r) - \phi(|\xi|)) h_{\xi\theta}(r) dr = \frac{h_{\xi\theta}(|\xi|)}{|\phi'(|\xi|)|} \\ & = |\xi|^{d-2} \sqrt{|\xi|^2 + m^2} \big\{ \mathrm{I}_+(\xi)\mathrm{I}_+(|\xi|\theta) + \mathrm{I}_-(\xi)\mathrm{I}_-(|\xi|\theta) \big\} \widehat{w(|\cdot|)}(\xi - |\xi|\theta) \psi(|\xi|) f(|\xi|\theta). \end{aligned}$$

Using it and the fact $\widehat{w(|\cdot|)}(\xi - |\xi|\theta) = F_w(\frac{1}{2}|\xi - |\xi|\theta|^2) = F_w(|\xi|^2(1 - \frac{\xi}{|\xi|} \cdot \theta))$, we have

$$\begin{aligned} (S^*Sf)(\xi) = & \ 2\pi|\xi|^{d-2} \sqrt{|\xi|^2 + m^2} \psi(|\xi|)^2 \times \\ & \int_{\mathbb{S}^{d-1}} F_w(|\xi|^2(1 - \tfrac{\xi}{|\xi|} \cdot \theta)) \big\{ \mathrm{I}_+(\xi)\mathrm{I}_+(|\xi|\theta) + \mathrm{I}_-(\xi)\mathrm{I}_-(|\xi|\theta) \big\} f(|\xi|\theta) d\sigma(\theta). \end{aligned}$$

Let $J(r)=2\pi r^{d-2}\sqrt{r^2+m^2}\psi(r)^2$. Since $\mathrm{I}_{\pm}(\xi)$ is Hermitian, we have

$$
\begin{aligned}
&(S^*Sf,f)_{L^2(\mathbb{R}^d,\mathbb{C}^N)}\\
&=\int_{\mathbb{R}^d} J(|\xi|)\left(\int_{\mathbb{S}^{d-1}} F_w(|\xi|^2(1-\tfrac{\xi}{|\xi|}\cdot\theta))\mathrm{I}_+(|\xi|\theta)f(|\xi|\theta)d\sigma(\theta)\right)\overline{\mathrm{I}_+(\xi)f(\xi)}d\xi\\
&+\int_{\mathbb{R}^d} J(|\xi|)\left(\int_{\mathbb{S}^{d-1}} F_w(|\xi|^2(1-\tfrac{\xi}{|\xi|}\cdot\theta))\mathrm{I}_-(|\xi|\theta)f(|\xi|\theta)d\sigma(\theta)\right)\overline{\mathrm{I}_-(\xi)f(\xi)}d\xi. \quad (3.6)
\end{aligned}
$$

Applying spherical harmonic composition (3.1) to them, we have

$$
\mathrm{I}_{\pm}(\xi)f(\xi)=\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}P^{(k,j)}(\tfrac{\xi}{|\xi|})|\xi|^{-\frac{d-1}{2}}f_{\pm}^{(k,j)}(|\xi|),\qquad f_{\pm}^{(k,j)}\in L^2(\mathbb{R}_+,\mathbb{C}^N). \quad (3.7)
$$

Then we have

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}} F_w(|\xi|^2(1-\tfrac{\xi}{|\xi|}\cdot\theta))\mathrm{I}_{\pm}(|\xi|\theta)f(|\xi|\theta)d\sigma(\theta)\\
&\qquad=\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}|\xi|^{-\frac{d-1}{2}}\left(\int_{\mathbb{S}^{d-1}}F_w(|\xi|^2(1-\tfrac{\xi}{|\xi|}\cdot\theta))P^{(k,j)}(\theta)d\sigma(\theta)\right)f_{\pm}^{(k,j)}(|\xi|)\\
&\qquad=\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}|\xi|^{-\frac{d-1}{2}}\mu_k(|\xi|)P^{(k,j)}(\tfrac{\xi}{|\xi|})f_{\pm}^{(k,j)}(|\xi|) \quad (3.8)
\end{aligned}
$$

by Funk-Hecke theorem (Theorem 3.2), where

$$
\mu_k(r)=\left|\mathbb{S}^{d-2}\right|\int_{-1}^{1}F_w(r^2(1-t))p_{d,k}(t)(1-t^2)^{\frac{d-3}{2}}dt.
$$

Since $2\pi\lambda_k(r)=J(r)\mu_k(r)$ by (2.2), we have

$$
\begin{aligned}
&J(|\xi|)\int_{\mathbb{S}^{d-1}} F_w(|\xi|^2(1-\tfrac{\xi}{|\xi|}\cdot\theta))\mathrm{I}_{\pm}(|\xi|\theta)f(|\xi|\theta)d\sigma(\theta)\\
&\qquad\qquad=2\pi\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}|\xi|^{-\frac{d-1}{2}}\lambda_k(|\xi|)P^{(k,j)}(\tfrac{\xi}{|\xi|})f_{\pm}(|\xi|)
\end{aligned}
$$

by (3.8). Therefore, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} J(|\xi|)\left(\int_{\mathbb{S}^{d-1}} F_w(|\xi|^2(1-\tfrac{\xi}{|\xi|}\cdot\theta))\mathrm{I}_{\pm}(|\xi|\theta)f(|\xi|\theta)d\sigma(\theta)\right)\overline{\mathrm{I}_{\pm}(\xi)f(\xi)}d\xi\\
&=2\pi\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\int_{\mathbb{R}^d}|\xi|^{-(d-1)}\lambda_k(|\xi|)P^{(k,j)}(\tfrac{\xi}{|\xi|})f_{\pm}^{(k,j)}(|\xi|)\sum_{\ell=0}^{\infty}\sum_{n=1}^{a_\ell}\overline{P^{(\ell,n)}(\tfrac{\xi}{|\xi|})f_{\pm}^{(\ell,n)}(|\xi|)}d\xi\\
&=2\pi\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\sum_{\ell=0}^{\infty}\sum_{n=1}^{a_\ell}\int_0^{\infty}\lambda_k(r)\left(\int_{\mathbb{S}^{d-1}}P^{(k,j)}(\theta)\overline{P^{(\ell,n)}(\theta)}d\sigma(\theta)\right)f_{\pm}^{(k,j)}(r)\overline{f_{\pm}^{(\ell,n)}(r)}dr\\
&=2\pi\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\int_0^{\infty}\lambda_k(r)\left|f_{\pm}^{(k,j)}(r)\right|^2dr.
\end{aligned}
$$

Combining it with (3.6), we have

$$\begin{aligned}\|Sf\|^2_{L^2(\mathbb{R}^{d+1},\mathbb{C}^N)} &= (S^*Sf,f)_{L^2(\mathbb{R}^d,\mathbb{C}^N)}\\ &= 2\pi\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\int_0^\infty \lambda_k(r)\left(\left|f_+^{(k,j)}(r)\right|^2+\left|f_-^{(k,j)}(r)\right|^2\right)dr \qquad (3.9)\\ &\le 2\pi\lambda\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\int_0^\infty \left(\left|f_+^{(k,j)}(r)\right|^2+\left|f_-^{(k,j)}(r)\right|^2\right)dr\\ &= 2\pi\lambda\left(\|\mathrm{I}_+f\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^N)}+\|\mathrm{I}_-f\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^N)}\right)\\ &= 2\pi\lambda\|f\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^N)}.\end{aligned}$$

The proof is complete. □

3.3 Proof of Theorem 2.2

Let $\widetilde{\lambda}=\sup_{k\in\mathbb{Z}}\sup_{r>0}\widetilde{\lambda}_k(r)$. By (3.2), it is enough to show that $\|S\|^2=2\pi\widetilde{\lambda}$ for the optimal constant (2.6). Since the spherical harmonics with $d=2$ are expressed as the Fourier series, (3.7) and (3.9) are expressed as follows :

$$\mathrm{I}_\pm(\xi)f(\xi)=r^{-\frac{1}{2}}\sum_{k\in\mathbb{Z}}\frac{e^{ik\theta}}{\sqrt{2\pi}}f_\pm^{(k)}(r),\quad f_\pm^{(k)}\in L^2(\mathbb{R}_+,\mathbb{C}^2), \qquad (3.10)$$

$$\|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)}=2\pi\sum_{k\in\mathbb{Z}}\int_0^\infty\lambda_{|k|}(r)\left(\left|f_+^{(k)}(r)\right|^2+\left|f_-^{(k)}(r)\right|^2\right)dr, \qquad (3.11)$$

where $\xi=(r\sin\theta,r\cos\theta)$. Now, we set

$$f(\xi)=\begin{pmatrix}f_1(\xi)\\ f_2(\xi)\end{pmatrix}=r^{-\frac{1}{2}}\sum_{k\in\mathbb{Z}}\frac{e^{ik\theta}}{\sqrt{2\pi}}\begin{pmatrix}\widetilde{f}_1^{(k)}(r)\\ \widetilde{f}_2^{(k)}(r)\end{pmatrix} \qquad (3.12)$$

and rewrite (3.10) in terms of $\widetilde{f}_1^{(k)}(r)$ and $\widetilde{f}_2^{(k)}(r)$. In the case $d=2$, $A_\xi=\alpha\cdot\xi+m\beta$ is represented by the Pauli matrices

$$\alpha_1=\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \alpha_2=\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \beta=\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \qquad (3.13)$$

and by (3.4), we have

$$\mathrm{I}_\pm(\xi)f(\xi)=\frac{1}{2}\begin{pmatrix}(1\pm\frac{m}{\phi(r)})f_1(\xi)\pm\frac{r}{\phi(r)}e^{-i\theta}f_2(\xi)\\ (1\mp\frac{m}{\phi(r)})f_2(\xi)\pm\frac{r}{\phi(r)}e^{i\theta}f_1(\xi)\end{pmatrix},$$

where $\phi(r) = \sqrt{r^2 + m^2}$. Plugging (3.12) into it, we have

$$\mathrm{I}_\pm(\xi)f(\xi) = \frac{1}{2}r^{-\frac{1}{2}}\sum_{k\in\mathbb{Z}}\frac{e^{ik\theta}}{\sqrt{2\pi}}\begin{pmatrix}(1 \pm \frac{m}{\phi(r)})\widetilde{f}_1^{(k)}(r) \pm \frac{r}{\phi(r)}\widetilde{f}_2^{(k+1)}(r)\\ (1 \mp \frac{m}{\phi(r)})\widetilde{f}_2^{(k)}(r) \pm \frac{r}{\phi(r)}\widetilde{f}_1^{(k-1)}(r)\end{pmatrix}.$$

From it and (3.10), we obtain

$$f_\pm^{(k)}(r) = \frac{1}{2}\begin{pmatrix}(1 \pm \frac{m}{\phi(r)})\widetilde{f}_1^{(k)}(r) \pm \frac{r}{\phi(r)}\widetilde{f}_2^{(k+1)}(r)\\ (1 \mp \frac{m}{\phi(r)})\widetilde{f}_2^{(k)}(r) \pm \frac{r}{\phi(r)}\widetilde{f}_1^{(k-1)}(r)\end{pmatrix}. \tag{3.14}$$

Plugging it into (3.11), we have

$$\begin{aligned}\|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)} = 2\pi\sum_{k\in\mathbb{Z}}\int_0^\infty \Bigg[\frac{1}{2}\left\{\left(1+\frac{m^2}{\phi(r)^2}\right)\lambda_{|k|}(r) + \frac{r^2}{\phi(r)^2}\lambda_{|k+1|}(r)\right\}\left|\widetilde{f}_1^{(k)}(r)\right|^2 \\ + \frac{rm}{\phi(r)^2}\left(\lambda_{|k|}(r) - \lambda_{|k+1|}(r)\right)\operatorname{Re}\widetilde{f}_1^{(k)}(r)\overline{\widetilde{f}_2^{(k+1)}(r)} \\ + \frac{1}{2}\left\{\left(1+\frac{m^2}{\phi(r)^2}\right)\lambda_{|k+1|}(r) + \frac{r^2}{\phi(r)^2}\lambda_{|k|}(r)\right\}\left|\widetilde{f}_2^{(k+1)}(r)\right|^2\Bigg]dr. \end{aligned}\tag{3.15}$$

For simplicity, let $x = \widetilde{f}_1^{(k)}(r)$, $y = \widetilde{f}_2^{(k+1)}(r)$,

$$\begin{aligned} a = a_k(r) &= \frac{1}{2}\left\{\left(1+\frac{m^2}{\phi(r)^2}\right)\lambda_{|k|}(r) + \frac{r^2}{\phi(r)^2}\lambda_{|k+1|}(r)\right\}, \\ c = c_k(r) &= \frac{rm}{2\phi(r)^2}\left(\lambda_{|k|}(r) - \lambda_{|k+1|}(r)\right), \\ b = b_k(r) &= \frac{1}{2}\left\{\left(1+\frac{m^2}{\phi(r)^2}\right)\lambda_{|k+1|}(r) + \frac{r^2}{\phi(r)^2}\lambda_{|k|}(r)\right\} \end{aligned}$$

and replace the integrand on the right hand side of (3.15) by $a|x|^2 + 2c\operatorname{Re} x\overline{y} + b|y|^2$. We claim the following, where $\widetilde{\lambda}_k = \widetilde{\lambda}_k(r)$ is defined by (2.5) :

Claim 1. (i) *In the case $c = 0$, we have the identity*

$$a|x|^2 + 2c\operatorname{Re} x\overline{y} + b|y|^2 = \widetilde{\lambda}_k(|x|^2 + |y|^2). \tag{3.16}$$

(ii) *In the case $c \neq 0$, we have the inequality*

$$a|x|^2 + 2c\operatorname{Re} x\overline{y} + b|y|^2 \le \widetilde{\lambda}_k(|x|^2 + |y|^2), \tag{3.17}$$

where we have the equality if and only if $y = \frac{r}{\phi(r)+m}x$ for $\lambda_{|k|}(r) > \lambda_{|k+1|}(r)$ and $x = -\frac{r}{\phi(r)+m}y$ for $\lambda_{|k|}(r) < \lambda_{|k+1|}(r)$.

In fact, (3.16) is straightforward. As for (3.17), we set $st = c$ $(s, t \in \mathbb{R})$, then

$$\begin{aligned} a|x|^2 + 2c \operatorname{Re} x\overline{y} + b|y|^2 &= a|x|^2 + 2\operatorname{Re}(sx)(\overline{ty}) + b|y|^2 \\ &\le (a+s^2)|x|^2 + (b+t^2)|y|^2, \end{aligned}$$

where we have the equality if and only if $sx = ty$. By taking s, t such that $a + s^2 = b + t^2$ and $st = c$, we have Claim 1 (ii).

Applying Claim 1 to (3.15) :

$$\begin{aligned} &\|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)} = \\ &2\pi \sum_{k\in\mathbb{Z}} \int_0^\infty \left\{ a_k(r) \left| \widetilde{f}_1^{(k)}(r) \right|^2 + 2c_k(r) \operatorname{Re} \widetilde{f}_1^{(k)}(r) \overline{\widetilde{f}_2^{(k+1)}(r)} + b_k(r) \left| \widetilde{f}_2^{(k+1)}(r) \right|^2 \right\} dr \\ &\le 2\pi \sum_{k\in\mathbb{Z}} \int_0^\infty \widetilde{\lambda}_k(r) \left\{ \left| \widetilde{f}_1^{(k)}(r) \right|^2 + \left| \widetilde{f}_2^{(k+1)}(r) \right|^2 \right\} dr \qquad (3.18) \\ &\le 2\pi \widetilde{\lambda} \sum_{k\in\mathbb{Z}} \int_0^\infty \left\{ \left| \widetilde{f}_1^{(k)}(r) \right|^2 + \left| \widetilde{f}_2^{(k+1)}(r) \right|^2 \right\} dr \\ &= 2\pi \widetilde{\lambda} \|f\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^2)}. \end{aligned}$$

Therefore, $\|S\|^2 \le 2\pi\widetilde{\lambda}$. Next, by the Assumption 1 of $\lambda_k(r)$ and the definition of $\widetilde{\lambda}_k(r)$, we have

$$^\forall\varepsilon > 0,\ ^\exists k_0 \in \mathbb{Z},\ ^\exists r_0 > 0,\ ^\exists\delta > 0,\ |r - r_0| < \delta \Rightarrow \widetilde{\lambda}_{k_0}(r) \ge \widetilde{\lambda} - \varepsilon.$$

Therefore, we choose $f \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ in the form of

$$f(\xi) = \frac{r^{-\frac{1}{2}} e^{ik_0\theta}}{\sqrt{2\pi}} \begin{pmatrix} \widetilde{f}_1^{(k_0)}(r) \\ 0 \end{pmatrix} + \frac{r^{-\frac{1}{2}} e^{i(k_0+1)\theta}}{\sqrt{2\pi}} \begin{pmatrix} 0 \\ \widetilde{f}_2^{(k_0+1)}(r) \end{pmatrix},$$

where $\widetilde{f}_1^{(k_0)}(r)$ and $\widetilde{f}_2^{(k_0+1)}(r)$ are such that

$$\left| \widetilde{f}_1^{(k_0)} \right|^2 + \left| \widetilde{f}_2^{(k_0+1)} \right|^2 \not\equiv 0,\ \operatorname{supp}\left(\left| \widetilde{f}_1^{(k_0)} \right|^2 + \left| \widetilde{f}_2^{(k_0+1)} \right|^2 \right) \subset (r_0 - \delta,\ r_0 + \delta),$$

$$\widetilde{f}_2^{(k_0+1)}(r) = \tfrac{r}{\phi(r)+m} \widetilde{f}_1^{(k_0)}(r) \text{ for } \lambda_{|k|}(r) \ge \lambda_{|k+1|}(r),$$

$$\widetilde{f}_1^{(k_0)}(r) = -\tfrac{r}{\phi(r)+m} \widetilde{f}_2^{(k_0+1)}(r) \text{ for } \lambda_{|k|}(r) < \lambda_{|k+1|}(r).$$

Then, by Claim 1 and (3.18), we have

$$\begin{aligned} \|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)} &= 2\pi \int_{r_0-\delta}^{r_0+\delta} \widetilde{\lambda}_{k_0}(r) \left\{ \left| \widetilde{f}_1^{(k_0)}(r) \right|^2 + \left| \widetilde{f}_2^{(k_0+1)}(r) \right|^2 \right\} dr \\ &\ge 2\pi(\widetilde{\lambda} - \varepsilon) \|f\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^2)}. \end{aligned}$$

Since $\|S\|^2 = 2\pi\widetilde{\lambda}$. We have (2.6).

Regarding extremisers, it suffices to determine functions f such that $\|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)} = \|S\|^2\|f\|^2_{L^2(\mathbb{R}^2\mathbb{C}^2)}$. Denote f by (3.12). Then, we have

$$\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^2)} = \sum_{k\in\mathbb{Z}}\int_0^\infty F_k(r)dr, \tag{3.19}$$

$$\|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)} = 2\pi\widetilde{\lambda}\sum_{k\in\mathbb{Z}}\int_0^\infty F_k(r)dr,$$

where $F_k(r) = \left|\widetilde{f}_1^{(k)}(r)\right|^2 + \left|\widetilde{f}_2^{(k+1)}(r)\right|^2$. On the other hand, by (3.18) we have

$$\|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)} \le 2\pi\sum_{k\in\mathbb{Z}}\int_0^\infty \widetilde{\lambda}_k(r)F_k(r)dr.$$

Therefore, we have

$$0 \ge 2\pi\sum_{k\in\mathbb{Z}}\int_0^\infty (\widetilde{\lambda} - \widetilde{\lambda}_k(r))F_k(r)dr.$$

Since the integrand is nonnegative, for any $k \in \mathbb{Z}$ we have

$$(\widetilde{\lambda} - \widetilde{\lambda}_k(r))F_k(r) = 0 \quad \text{a.e. } r > 0. \tag{3.20}$$

If $k \notin \widetilde{K}$, then by definition of $\widetilde{K}$, we have $\widetilde{\lambda} > \widetilde{\lambda}_k(r) = 0$ a.e. $r > 0$, and hence $F_k(r) = 0$ a.e. $r > 0$. Therefore, if $\widetilde{K}$ is empty, then $F_k(r) = 0$ a.e. $r > 0$ for any $k \in \mathbb{Z}$. Hence, $f \equiv 0$ by (3.19), so there is no extremiser. Therefore, it is a necessary condition for the existence of an extremiser that $\widetilde{K}$ is nonempty set. Furthermore, if $\operatorname{supp} F_k \not\subseteq \widetilde{S}_k$, (3.20) does not hold, so $\operatorname{supp} F_k \subseteq \widetilde{S}_k$ must hold for an extremiser to exist. Also, if Theorem 2.2 (ii) is not satisfied, then, by Claim 1 (ii) and (3.18), we have

$$\|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)} < \|S\|^2\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^2)},$$

so f is not extremiser.

Conversely, if $\widetilde{K}$ is nonempty, $\operatorname{supp} F_k \subseteq \widetilde{S}_k$ and Theorem 2.2 (ii) is satisfied, if $f = \sum_{k\in\widetilde{K}} f_k$, with f_k as in (2.7), then, by Claim 1 and (3.18), we have

$$\begin{aligned}
\|Sf\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^2)} &= 2\pi\sum_{k\in\mathbb{Z}}\int_0^\infty \widetilde{\lambda}_k(r)F_k(r)dr \\
&= 2\pi\sum_{k\in\widetilde{K}}\int_{\widetilde{S}_k} \widetilde{\lambda}_k(r)F_k(r)dr \\
&= 2\pi\widetilde{\lambda}\sum_{k\in\widetilde{K}}\int_{\widetilde{S}_k} F_k(r)dr \\
&= \|S\|^2\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^2)}.
\end{aligned}$$

Therefore, the extremiser exists and it is expressed by $\hat{f} = \sum_{k\in\widetilde{K}} f_k$. □

3.4 Proof of Theorem 2.3

Let $\lambda^* = \frac{1}{2}\sup_{k\in\mathbb{Z}}\sup_{r>0}\{\lambda_k(r)+\lambda_{k+1}(r)\}$. By (3.2), it is enough to show that $\|S\|^2 = 2\pi\lambda^*$ for the optimal constant (2.8). The spherical harmonics with $d=3$ are expressed as $\mathbb{R}^3 \ni \xi = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$ as follows (See [1]):

$$Y_k^n = Y_k^n(\theta,\varphi) = \sqrt{\frac{(2k+1)(k-n)!}{4\pi(k+n)!}}e^{in\varphi}P_k^n(\cos\theta), \tag{3.21}$$

where $k \in \mathbb{N}_0$, $n \in \{-k, -k+1, \cdots, k-1, k\}$, P_k^n is the associated Legendre polynomial. P_k^n is expressed as follows :

$$P_k^n(x) = \frac{(-1)^n}{2^k k!}(1-x^2)^{n/2}\frac{d^{k+n}}{dx^{k+n}}(x^2-1)^k.$$

By Y_k^n, (3.7) and (3.9) are expressed as follows :

$$\mathrm{I}_\pm(\xi)f(\xi) = r^{-1}\sum_{k=0}^{\infty}\sum_{n=-k}^{k} Y_k^n f_\pm^{(k,n)}(r), \quad f_\pm^{(k,n)} \in L^2(\mathbb{R}_+, \mathbb{C}^4), \tag{3.22}$$

$$\|Sf\|^2_{L^2(\mathbb{R}^4,\mathbb{C}^4)} = 2\pi\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\int_0^\infty \lambda_k(r)\left(\left|f_+^{(k,n)}(r)\right|^2 + \left|f_-^{(k,n)}(r)\right|^2\right)dr. \tag{3.23}$$

Now, we set

$$f(\xi) = \begin{pmatrix} f_1(\xi) \\ f_2(\xi) \\ f_3(\xi) \\ f_4(\xi) \end{pmatrix} = r^{-1}\sum_{k=0}^{\infty}\sum_{n=-k}^{k} Y_k^n \begin{pmatrix} f_1^{(k,n)}(r) \\ f_2^{(k,n)}(r) \\ f_3^{(k,n)}(r) \\ f_4^{(k,n)}(r) \end{pmatrix} \tag{3.24}$$

and rewrite $f_\pm^{(k,n)} = f_\pm^{(k,n)}(r)$ of (3.22) in terms of $f_j^{(k,n)} = f_j^{(k,n)}(r)$ $(j=1,2,3,4)$. In the case massless 3D, $A_\xi = \alpha\cdot\xi$ is expressed as follows :

$$\alpha_j = \begin{pmatrix} O_2 & \sigma_j \\ \sigma_j & O_2 \end{pmatrix} \quad (j=1,2,3),$$

where σ_j are the Pauli matrices (3.13), and by (3.4) with $m=0$, we have

$$\mathrm{I}_\pm(\xi)f(\xi) = \begin{pmatrix} f_{\pm 1}(\xi) \\ f_{\pm 2}(\xi) \\ f_{\pm 3}(\xi) \\ f_{\pm 4}(\xi) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f_1(\xi) \pm (\sin\theta e^{-i\varphi}f_4(\xi) + \cos\theta f_3(\xi)) \\ f_2(\xi) \pm (\sin\theta e^{i\varphi}f_3(\xi) - \cos\theta f_4(\xi)) \\ f_3(\xi) \pm (\sin\theta e^{-i\varphi}f_2(\xi) + \cos\theta f_1(\xi)) \\ f_4(\xi) \pm (\sin\theta e^{i\varphi}f_1(\xi) - \cos\theta f_2(\xi)) \end{pmatrix}.$$

Plugging (3.24) into it, we have

$$\mathrm{I}_{\pm}(\xi)f(\xi)=\frac{r^{-1}}{2}\sum_{k=0}^{\infty}\sum_{n=-k}^{k}Y_k^n\begin{pmatrix}f_1^{(k,n)}(r)\pm(\sin\theta e^{-i\varphi}f_4^{(k,n)}(r)+\cos\theta f_3^{(k,n)}(r))\\ f_2^{(k,n)}(r)\pm(\sin\theta e^{i\varphi}f_3^{(k,n)}(r)-\cos\theta f_4^{(k,n)}(r))\\ f_3^{(k,n)}(r)\pm(\sin\theta e^{-i\varphi}f_2^{(k,n)}(r)+\cos\theta f_1^{(k,n)}(r))\\ f_4^{(k,n)}(r)\pm(\sin\theta e^{i\varphi}f_1^{(k,n)}(r)-\cos\theta f_2^{(k,n)}(r))\end{pmatrix}. \tag{3.25}$$

To rewrite the right hand side of (3.25) into the form like (3.22), we write $Y_k^n\sin\theta e^{\pm i\varphi}$ and $Y_k^n\cos\theta$ in the form of a linear combination of Y_k^n. For this purpose, we use the following recurrence properties for the associated Legendre polynomial P_k^n (See [14]):

$$\begin{cases}xP_k^n(x)=\frac{1}{2k+1}\{(k-n+1)P_{k+1}^n(x)+(k+n)P_{k-1}^n(x)\},\\ \sqrt{1-x^2}P_k^n(x)\\ \qquad=\frac{1}{2k+1}\{(k-n+1)(k-n+2)P_{k+1}^{n-1}(x)-(k+n-1)(k+n)P_{k-1}^{n-1}(x)\},\\ \sqrt{1-x^2}P_k^n(x)=\frac{-1}{2k+1}\{P_{k+1}^{n+1}(x)-P_{k-1}^{n+1}(x)\},\end{cases} \tag{3.26}$$

where $k\in\mathbb{N}_0$, $n\in\{-k,-k+1,\cdots,k-1,k\}$ and $P_{-1}^0=0$. Plugging (3.26) into $Y_k^n\sin\theta e^{\pm i\varphi}$ and $Y_k^n\cos\theta$ with $x=\cos\theta$, we have

$$\begin{cases}Y_k^n\sin\theta e^{-i\varphi}=a_k^{-n}Y_{k+1}^{n-1}-a_{k-1}^{n-1}Y_{k-1}^{n-1},\\ Y_k^n\sin\theta e^{i\varphi}=-a_k^nY_{k+1}^{n+1}+a_{k-1}^{-n-1}Y_{k-1}^{n+1},\\ Y_k^n\cos\theta=b_k^nY_{k+1}^n+b_{k-1}^nY_{k-1}^n,\end{cases} \tag{3.27}$$

where

$$\begin{cases}a_k^n=a^{(k,n)}=\sqrt{\frac{(k+n+1)(k+n+2)}{(2k+1)(2k+3)}},\\ b_k^n=b^{(k,n)}=\sqrt{\frac{(k+n+1)(k-n+1)}{(2k+1)(2k+3)}}.\end{cases}$$

Plugging it into (3.25), we have

$$f_{\pm}^{(k,n)}=\frac{1}{2}\begin{pmatrix}f_1^{(k,n)}\pm\{a_{k-1}^{-n-1}f_4^{(k-1,n+1)}-a_k^nf_4^{(k+1,n+1)}+b_{k-1}^nf_3^{(k-1,n)}+b_k^nf_3^{(k+1,n)}\}\\ f_2^{(k,n)}\mp\{a_{k-1}^{n-1}f_3^{(k-1,n-1)}-a_k^{-n}f_3^{(k+1,n-1)}+b_{k-1}^nf_4^{(k-1,n)}+b_k^nf_4^{(k+1,n)}\}\\ f_3^{(k,n)}\pm\{a_{k-1}^{-n-1}f_2^{(k-1,n+1)}-a_k^nf_2^{(k+1,n+1)}+b_{k-1}^nf_1^{(k-1,n)}+b_k^nf_1^{(k+1,n)}\}\\ f_4^{(k,n)}\mp\{a_{k-1}^{n-1}f_1^{(k-1,n-1)}-a_k^{-n}f_1^{(k+1,n-1)}+b_{k-1}^nf_2^{(k-1,n)}+b_k^nf_2^{(k+1,n)}\}\end{pmatrix}. \tag{3.28}$$

Plugging (3.28) into (3.23), we have

$$
\begin{aligned}
\|Sf\|^2_{L^2(\mathbb{R}^4,\mathbb{C}^4)} &= 2\pi \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \int_0^\infty \frac{1}{2} \Bigg[\\
&\quad \left\{ C^{(k,n)} \left| f_1^{(k,n)} \right|^2 + 2D^{(k,n)} \operatorname{Re} f_1^{(k,n)} \overline{f_2^{(k,n+1)}} + C^{(k,-n)} \left| f_2^{(k,n)} \right|^2 \right\} \\
&\quad + \left\{ C^{(k,n)} \left| f_3^{(k,n)} \right|^2 + 2D^{(k,n)} \operatorname{Re} f_3^{(k,n)} \overline{f_4^{(k,n+1)}} + C^{(k,-n)} \left| f_4^{(k,n)} \right|^2 \right\} \Bigg] dr \\
&= 2\pi \sum_{k=0}^{\infty} \int_0^\infty \frac{\lambda_k + \lambda_{k+1}}{2} \left\{ \left| f_1^{(k,k)} \right|^2 + \left| f_2^{(k,-k)} \right|^2 + \left| f_3^{(k,k)} \right|^2 + \left| f_4^{(k,-k)} \right|^2 \right\} dr \\
&\quad + 2\pi \sum_{k=1}^{\infty} \sum_{n=-k}^{k-1} \int_0^\infty \frac{1}{2} \left\{ \operatorname{Re} {}^t\overline{f_u^{(k,n)}} A^{(k,n)} f_u^{(k,n)} + \operatorname{Re} {}^t\overline{f_\ell^{(k,n)}} A^{(k,n)} f_\ell^{(k,n)} \right\} dr, \quad (3.29)
\end{aligned}
$$

where

$$
\begin{aligned}
C^{(k,n)} &= C^{(k,n)}(r) = \lambda_k(r) + \frac{k+n+1}{2k+1}\lambda_{k+1}(r) + \frac{k-n}{2k+1}\lambda_{k-1}(r), \\
D^{(k,n)} &= D^{(k,n)}(r) = \frac{\sqrt{(k-n)(k+n+1)}}{2k+1} \left(\lambda_{k+1}(r) - \lambda_{k-1}(r) \right), \\
f_u^{(k,n)} &= \begin{pmatrix} f_1^{(k,n)} \\ f_2^{(k,n+1)} \end{pmatrix}, \quad f_\ell^{(k,n)} = \begin{pmatrix} f_3^{(k,n)} \\ f_4^{(k,n+1)} \end{pmatrix}, \\
A^{(k,n)} &= \begin{pmatrix} C^{(k,n)} & D^{(k,n)} \\ D^{(k,n)} & C^{(k,-n-1)} \end{pmatrix}.
\end{aligned}
$$

(a) In the case $\lambda_{k+1} = \lambda_{k-1}$: we have the identity

$$
\operatorname{Re} {}^t\overline{f_u^{(k,n)}} A^{(k,n)} f_u^{(k,n)} = (\lambda_k + \lambda_{k+1}) \left| f_u^{(k,n)} \right|^2.
$$

(b) In the case $\lambda_{k+1} \neq \lambda_{k-1}$: The eigenvalues of $A^{(k,n)}$ are $\lambda_k + \lambda_{k\pm1}$. The eigenvectors for $\lambda_k + \lambda_{k+1}$ are

$$
\begin{pmatrix} \sqrt{k+n+1} \\ \sqrt{k-n} \end{pmatrix} t, \quad (t \in \mathbb{C})
$$

and those for $\lambda_k + \lambda_{k-1}$ are

$$
\begin{pmatrix} \sqrt{k-n} \\ -\sqrt{k+n+1} \end{pmatrix} t, \quad (t \in \mathbb{C}).
$$

Therefore, in the case $\lambda_{k+1} > \lambda_{k-1}$, we have the inequality

$$
\operatorname{Re} {}^t\overline{f_u^{(k,n)}} A^{(k,n)} f_u^{(k,n)} \leq (\lambda_k + \lambda_{k+1}) \left| f_u^{(k,n)} \right|^2,
$$

where we have the equality if and only if $f_2^{(k,n+1)} = \sqrt{\frac{k-n}{k+n+1}} f_1^{(k,n)}$, and in the case $\lambda_{k+1} < \lambda_{k-1}$, we have the inequality

$$\mathrm{Re}\, {}^t\overline{f_u^{(k,n)}} A^{(k,n)} f_u^{(k,n)} \le (\lambda_k + \lambda_{k-1}) \big| f_u^{(k,n)} \big|^2,$$

where we have the equality if and only if $f_1^{(k,n)} = -\sqrt{\frac{k-n}{k+n+1}} f_2^{(k,n+1)}$.

By (a) and (b), We claim the following :

Claim 2. (i) *In the case $\lambda_{k+1} > \lambda_{k-1}$, we have the inequality*

$$\sum_{n=-k}^{k-1} \frac{1}{2} \mathrm{Re}\, {}^t\overline{f_u^{(k,n)}} A^{(k,n)} f_u^{(k,n)} \le \frac{\lambda_k + \lambda_{k+1}}{2} \sum_{n=-k}^{k-1} \big| f_u^{(k,n)} \big|^2 = \frac{\lambda_k + \lambda_{k+1}}{2} \sum_{n=-k}^{k-1} \left\{ \left| f_1^{(k,n)} \right|^2 + \left| f_2^{(k,n+1)} \right|^2 \right\},$$

where we have the equality if and only if $f_2^{(k,n+1)} = \sqrt{\frac{k-n}{k+n+1}} f_1^{(k,n)}$ $(-k \le n \le k-1)$.

(ii) *In the case $\lambda_{k+1} = \lambda_{k-1}$, we have the identity*

$$\sum_{n=-k}^{k-1} \frac{1}{2} \mathrm{Re}\, {}^t\overline{f_u^{(k,n)}} A^{(k,n)} f_u^{(k,n)} = \frac{\lambda_k + \lambda_{k+1}}{2} \sum_{n=-k}^{k-1} \left\{ \left| f_1^{(k,n)} \right|^2 + \left| f_2^{(k,n+1)} \right|^2 \right\},$$

(iii) *In the case $\lambda_{k+1} < \lambda_{k-1}$, we have the inequality*

$$\sum_{n=-k}^{k-1} \frac{1}{2} \mathrm{Re}\, {}^t\overline{f_u^{(k,n)}} A^{(k,n)} f_u^{(k,n)} \le \frac{\lambda_k + \lambda_{k-1}}{2} \sum_{n=-k}^{k-1} \left\{ \left| f_1^{(k,n)} \right|^2 + \left| f_2^{(k,n+1)} \right|^2 \right\},$$

where we have the equality if and only if $f_1^{(k,n)} = -\sqrt{\frac{k-n}{k+n+1}} f_2^{(k,n+1)}$ $(-k \le n \le k-1)$.

The same is true for $\displaystyle\sum_{n=-k}^{k-1} \frac{1}{2} \mathrm{Re}\, {}^t\overline{f_\ell^{(k,n)}} A^{(k,n)} f_\ell^{(k,n)}$.

We set $\lambda_k^*(r) = \frac{1}{2}\left\{\lambda_k(r) + \max\{\lambda_{k+1}(r), \lambda_{k-1}(r)\}\right\}$, applying Claim 2 to (3.29) :

$$
\begin{aligned}
&\|Sf\|^2_{L^2(\mathbb{R}^4,\mathbb{C}^4)} \\
&= 2\pi \sum_{k=0}^{\infty} \int_0^{\infty} \frac{\lambda_k + \lambda_{k+1}}{2} \left\{ \left| f_1^{(k,k)} \right|^2 + \left| f_2^{(k,-k)} \right|^2 + \left| f_3^{(k,k)} \right|^2 + \left| f_4^{(k,-k)} \right|^2 \right\} dr \\
&\quad + 2\pi \sum_{k=1}^{\infty} \sum_{n=-k}^{k-1} \int_0^{\infty} \frac{1}{2} \left\{ \mathrm{Re}\, {}^t\overline{f_u^{(k,n)}} A^{(k,n)} f_u^{(k,n)} + \mathrm{Re}\, {}^t\overline{f_\ell^{(k,n)}} A^{(k,n)} f_\ell^{(k,n)} \right\} dr \\
&\le 2\pi \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \int_0^{\infty} \lambda_k^*(r) \left\{ \sum_{j=1}^{4} \left| f_j^{(k,n)}(r) \right|^2 \right\} dr \qquad (3.30) \\
&\le 2\pi\lambda^* \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \int_0^{\infty} \left\{ \sum_{j=1}^{4} \left| f_j^{(k,n)}(r) \right|^2 \right\} dr \\
&= 2\pi\lambda^* \|f\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}.
\end{aligned}
$$

Therefore, $\|S\| \le 2\pi\lambda^*$. Next, by the Assumption 1 of $\lambda_k(r)$, we have

$$
{}^\forall\varepsilon > 0,\ {}^\exists k_0 \in \mathbb{Z},\ {}^\exists r_0 > 0,\ {}^\exists\delta > 0,\ |r - r_0| < \delta \Rightarrow \frac{\lambda_{k_0}(r) + \lambda_{k_0+1}(r)}{2} \ge \lambda^* - \varepsilon.
$$

Therefore, we choose $f \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ in the form of

$$
f(\xi) = r^{-1} Y^{(k_0,k_0)}(\theta,\varphi) \begin{pmatrix} f_1^{(k_0,k_0)}(r) \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

are such that

$$
f_1^{(k_0,k_0)}(r) \not\equiv 0, \quad \mathrm{supp}\, f_1^{(k_0,k_0)} \subset (r_0 - \delta, r_0 + \delta).
$$

Then, by (3.29), we have

$$
\begin{aligned}
\|Sf\|^2_{L^2(\mathbb{R}^4,\mathbb{C}^4)} &= 2\pi \int_{r_0-\delta}^{r_0+\delta} \frac{\lambda_{k_0}(r) + \lambda_{k_0+1}(r)}{2} \left| f_1^{(k_0,k_0)}(r) \right|^2 \\
&\ge 2\pi(\lambda^* - \varepsilon) \|f\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}.
\end{aligned}
$$

Since $\|S\|^2 = 2\pi\lambda^*$. Regarding extremisers, $A^{(k_0,n)}$ and $A^{(k_0+1,n)}$ have $\lambda_{k_0} + \lambda_{k_0+1}$ as an eigenvalue, Claim 2 and (3.29), we have extremisers of the Theorem 2.3, as we discussed in the same way as Theorem 2.2. □

Acknowledgement. The author would like to thank Professor Mitsuru Sugimoto for valuable discussions.

References

[1] A.A. Balinsky and W.D. Evans, *Spectral analysis of relativistic operators*, London Imperial College Press, c2011.

[2] M. Ben-Artzi, *Divergence-type operators: spectral theory and spacetime estimates, In: M. Ruzhansky and J. Wirth (Eds.) "Progress in Mathematics"*, pp. 1-39 (vol. 301) Springer Basel, 2012.

[3] M. Ben-Artzi and A. Devinatz, *The limiting absorption principle for partial differential operators*, Mem. Amer. Math. Soc. 66 (1987), no. 364.

[4] M. Ben-Artzi and A. Devinatz, *Local smoothing and convergence properties of Schrödinger type equations*, J. Funct. Anal. 101 (1991), 231–254.

[5] M. Ben-Artzi and S. Klainerman, *Decay and regularity for the Schrödinger equation*, J. Anal. Math 58 (1992), 25–37.

[6] M. Ben-Artzi and J. Nemirovsky, *Remarks on relativistic Schrödinger operators and their extensions*, Ann. Inst. H. Poincare-Phys. Theorique 67 (1997), 29-39.

[7] M. Ben-Artzi, M. Ruzhansky and M. Sugimoto, *Spectral identities and smoothing estimates for evolution operators*, Adv. Diff. Eqs. 25 (2020), 627-650.

[8] M. Ben-Artzi and T. Umeda, *Spectral theory of first-order elliptic systems: from crystal optics to Dirac operators*, Rev. Math. Phys. 33 (2021)

[9] N. Bez, H. Saito and M. Sugimoto, *Applications of the Funk-Hecke theorem to smoothing and trace estimates*, Adv. Math. 285 (2015), 1767-1795.

[10] N. Bez, M. Sugimoto, *Optimal constants and extremisers for some smoothing estimates*, J. Anal. Math. 131 (2017), 159-187.

[11] H. Chihara, *Smoothing effects of dispersive pseudodifferential equations*, Comm. Partial Differential Equations 27 (2002), 1953–2005.

[12] P. Constantin and J. C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. 1 (1988), 413–439.

[13] P. D'Ancona and L. Fanelli, *Strichartz and smoothing for dispersive equations with magnetic potentials*, Comm. Partial Differential Equations 33 (2008), 1082-1112.

[14] A. R. Edmonds, *Angular momentum in quantum mechanics*, Princeton University Press, c1960

[15] T. Hoshiro, *Mourre's method and smoothing properties of dispersive equations*, Comm. Math. Phys. 202 (1999), 255–265.

[16] T. Hoshiro, *Decay and regularity for dispersive equations with constant coefficients*, J. Anal. Math. 91 (2003), 211–230.

[17] M. Ikoma, *Optimal constants of smoothing estimates for the 2D Dirac equation*, J. Fourier Anal. Appl. 28 (2022), Article 57

[18] H. Kalf, T. Okaji and O. Yamada, *Explicit uniform bounds on integrals of Bessel functions and trace theorems for Fourier transforms*, Math. Nachr. 292 (2019), 106-120.

[19] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Studies in Applied Mathematics*, Academic Press, New York (1983), 93–128.

[20] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann. 162 (1965/1966), 258–279.

[21] T. Kato and K. Yajima, *Some examples of smooth operators and the associated smoothing effect*, Rev. Math. Phys. 1 (1989), 481-496.

[22] C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. 40 (1991), 33-69.

[23] C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. 46 (1993), 527–620.

[24] C. E. Kenig, G. Ponce and L. Vega, *Small solutions to nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (1993), 255–288.

[25] C. E. Kenig, G. Ponce and L. Vega, *On the generalized Benjamin-Ono equation*, Trans. Amer. Math. Soc. 342 (1994), 155–172.

[26] C. E. Kenig, G. Ponce and L. Vega, *On the Zakharov and Zakharov-Schulman systems*, J. Funct. Anal. 127 (1995), 204–234.

[27] C. E. Kenig, G. Ponce and L. Vega, *Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations*, Invent. Math. 134 (1998), 489–545.

[28] F. Linares and G. Ponce, *On the Davey-Stewartson systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (1993), 523–548.

[29] C. Morawetz, *Time decay for the nonlinear Klein-Gordon equations*, Proc. Roy. Soc. London Ser. A 306 (1968), 291–296.

[30] M. Ruzhansky and M. Sugimoto, *Smoothing properties of evolution equations via canonical transforms and comparison principle*, Proc. London Math. Soc. (3) 105 (2012), 393–423.

[31] B. Simon, *Best constants in some operator smoothness estimates*, J. Funct. Anal. 107 (1992), 66-71.

[32] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. 55 (1987), 699–715.

[33] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N. J., (1971).

[34] M. Sugimoto, *A smoothing property of Schrödinger equations along the sphere*, J. Anal. Math. 89 (2003), 15–30.

[35] M. Sugimoto, *Global smoothing properties of generalized Schrölinger equations*, J. Anal. Math. 76 (1998), 191-204.

[36] L. Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. 102 (1988), 874–878.

[37] M. C. Vilela, *Regularity of solutions to the free Schrödinger equation with radial initial data*, Illinois J. Math. 45 (2001), 361–370.

[38] B. Walther, *A sharp weighted L^2-estimate for the solution to the time-dependent Schrödinger equation*, Ark. Mat. 37 (1989), 381-393.

[39] B. Walther, *Regularity, decay, and best constants for dispersive equations*, J. Funct. Anal. 189 (2002) 325–335.

[40] K. Watanabe, *Smooth perturbations of the self-adjoint operator* $|\Delta|^{\alpha/2}$, Tokyo J. Math. 14 (1991), 239-250.