A sharp sparse domination of pseudodifferential operators
(擬微分作用素の端点スパース評価について)

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Contents

Abstract

In this paper, we give a sharp sparse domination of pseudodifferential operators associated with symbols belonging to the Hörmander class, and fundamental solutions of dispersive equations. Furthermore, we give boundedness results of these operators on weighted Besov spaces by using the sparse domination.

1 Introduction and results

For any $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$, the Hörmander class $S^m_{\rho,\delta}$ is defined as the set of all $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \lesssim (1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for any $(x,\xi) \in \mathbb{R}^{2n}$. Here, $A \lesssim B$ means $A \leq CB$ with a positive constant $C > 0$. For given $a \in S^m_{\rho,\delta}$, we define the pseudodifferential operator $a(x,D)$ by

$$a(x,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} a(x,\xi) \hat{f}(\xi) d\xi,$$

where $f \in \mathscr{S}$ and $\hat{f}$ denotes the Fourier transform of f. Pseudodifferential operator is a useful tool for study of partial differential equations, and the many boundedness results are known. The most basic result is the L^p-boundedness given by Hörmander [19] and Fefferman [18]. Hörmander [19] showed that $m \leq -n(1-\rho)|1/2-1/p|$ is necessary for $a(x,D)$ with $a \in S^m_{\rho,\delta}$ to be L^p- bounded. Conversely, the L^p-boundedness of $a(x,D)$ with $a \in S^m_{\rho,\delta}$ and $m = -n(1-\rho)|1/2-1/p|$ was established by Fefferman [18]. As for the boundedness on Lebesgue spaces weighted by $\omega \in A_p$ which is so called Muckenhoupt weight, Miller [32] established the $L^p(\omega)$-boundedness of $a(x,D)$ with $a \in S^0_{1,0}$. For general $a \in S^m_{\rho,\delta}$, Michalowski, Rule and Staubach [34] showed the $L^p(\omega)$-boundedness of $a(x,D)$ with $a \in S^{-n(1-\rho)}_{\rho,\delta}$ and $\omega \in A_p$. Chanillo and Torchinsky [13] showed it for a larger class $a \in S^{-n(1-\rho)/2}_{\rho,\delta}$ $(0 \leq \delta < \rho \leq 1)$ and a smaller class $\omega \in A_{p/2}$, and Michalowski, Rule and Staubach [33] showed the same result for $0 < \delta = \rho < 1$. It should be mentioned here that Beltran [3] showed it for $a \in S^m_{\rho,\rho}$ with $-n(1-\rho)/2 < m < -n(1-\rho)|1/2-1/p|$ and $\omega \in A_{p/2} \cap RH_{(2t'/p)'}$, where $2 \leq p < 2t'$ and t' is the conjugate exponent of $t = -n(1-\rho)/(2m)$. We remark that there is no such p that satisfies $2 \leq p < 2t'$ for the critical exponent $m = -n(1-\rho)|1/2-1/p|$. An important idea to deduce weighted estimates is to show pointwise estimates. For example, Chanillo and Torchinsky [13] established pointwise estimate

$$|(a(x,D)f)^*(x)| \lesssim M_2 f(x)$$

for $a \in S^{-n(1-\rho)/2}_{\rho,\delta}$ $(0 \leq \delta < \rho \leq 1)$, where $M_2 f(x) := \sup_{Q \ni x} |Q|^{-1/2} ||f||_{L^2(Q)}$ and $(a(x,D)f)^*$ denotes the sharp maximal function of $a(x,D)f$.

Recently as a refinement of pointwise estimates, the theory of sparse domination of operators was developed by Lerner [27]. For operators T on function spaces, the sparse domination means the inequalities:

$$|Tf(x)| \lesssim \Lambda_{\mathcal{S},r} f(x) \quad and \quad |\langle Tf, g\rangle| \lesssim \Lambda_{\mathcal{S},r,s'}(f,g).$$

In particular, we call the first one *sparse bounds* and the second one *sparse form bounds*. See below for the definition of $\Lambda_{\mathcal{S},r}$ and $\Lambda_{\mathcal{S},r,s'}$.

Definition 1.1. *Let $\eta \in (0,1)$. A collection $\mathcal{S}$ of cubes in $\mathbb{R}^n$ is an η-sparse family if there are pairwise disjoint subsets $\{E_Q\}_{Q\in\mathcal{S}}$ such that $E_Q \subset Q$, and $|E_Q| > \eta|Q|$.*

We often just say *sparse* instead of *η-sparse* whenever there is no confusion. For any cube Q and $p \in [1,\infty)$, we define $\langle f\rangle_{p,Q} := |Q|^{-\frac{1}{p}} ||f||_{L^p(Q)}$. For a sparse collection $\mathcal{S}$ and $r, s \in [1,\infty)$, the (r,s)-sparse form operator $\Lambda_{\mathcal{S},r,s}$ and r-sparse operator $\Lambda_{\mathcal{S},r}$ are defined by

$$\Lambda_{\mathcal{S},r} f(x) := \sum_{Q\in\mathcal{S}} \langle f\rangle_{r,Q} 1_Q(x) \quad , \quad \Lambda_{\mathcal{S},r,s}(f,g) := \sum_{Q\in\mathcal{S}} |Q| \langle f\rangle_{r,Q} \langle g\rangle_{s,Q}$$

for all $f, g \in L^1_{loc}$. If $r < p < s$, we have

$$\Lambda_{\mathcal{S},r,s'}(f,g) \lesssim ||f||_p ||g||_{p'}.$$

This inequality is easily obtained from the L^p-boundedness of r-Hardy Littlewood maximal operator M_r which is defined by $M_r f(x) = \sup_{Q\ni x} \langle f\rangle_{r,Q}$. Furthermore, weighted inequality with Muckenhoupt weights is deduced from sparse domination. Bernicot, Frey and Petermichl [5] showed

$$\Lambda_{\mathcal{S},r,s'}(f,g) \lesssim ([\omega]_{A_{p/r}} [\omega]_{RH_{(s/p)'}})^\alpha ||f||_{L^p(\omega)} ||g||_{L^{p'}(\omega^{1-p'})},$$

where $p \in (r,s)$ and $\alpha = \max(\frac{1}{p-r}, \frac{s-1}{s-p})$, $[\omega]_{A_q} = \sup_Q \langle \omega \rangle_{1,Q} \langle \omega^{1-q'} \rangle_{1,Q}^{q-1}$ and $[\omega]_{RH_q} = \sup_Q \langle \omega \rangle_{1,Q}^{-1} \langle \omega \rangle_{q,Q}$ for any $1 < q < \infty$. From these observations, sparse domination is used to study the weighted boundedness of operators, and Lerner [27] gave the simple proof of A_2 conjecture which means

$$||Tf||_{L^2(\omega)} \lesssim [\omega]_{A_2} ||f||_{L^2(\omega)},$$

where T denotes the Calderón-Zygmund operators. The A_2 conjecture was studied by many researchers. For example, Petermichl [37], [38] solved the A_2 conjecture for Hilbert transform and Riesz transform, and Perez, Treil and Volberg [36] gave

$$||Tf||_{L^2(\omega)} \lesssim [\omega]_{A_2} \log(1 + [\omega]_{A_2}) ||f||_{L^2(\omega)}$$

for general Calderón-Zygmund operators. Finally, A_2 conjecture was completely solved by Hytönen [21]. Lerner [27] gave another proof by establishing

$$||Tf||_X \lesssim \sup_{\mathcal{S}} ||\Lambda_{\mathcal{S},1} f||_X$$

for any Banach function space X, and it was improved to the pointwise estimate

$$|Tf(x)| \lesssim \Lambda_{\mathcal{S},1} f(x)$$

by Lerner [28], Lerner and Nazarov [30]. There are also results of sparse domination with other operators. Sparse form bounds of rough singular integral operators and Bochner-Riesz multipliers were shown by Conde-Alonso, Culic, Plinio and Ou [11], and Lacey, Mena and Reguera [26] respectively.

Beltran and Cladek [4] discussed the sparse domination of pseudodifferential operators with symbols in $S^m_{\rho,\delta}$, and they established

$$|a(x,D)f(x)| \lesssim \Lambda_{\mathcal{S},r} f(x),$$

with $a \in S^{-n(1-\rho)}_{\rho,\delta}$ and $1 < r < \infty$ which implies the weighted boundedness result of [34], that is the $L^p(\omega)$-boundedness with $\omega \in A_p$. We establish a pointwise estimate of $a(x,D)$ with larger class $a \in S^{-n(1-\rho)/2}_{\rho.\rho}$ than $S^{-n(1-\rho)}_{\rho,\rho}$ by introducing another type of sparse bounds:

Theorem 1.1. *Let $a \in S^m_{\rho,\rho}$ with $0 < \rho < 1$ and $m \in \mathbb{R}$. Then, for any $f \in L^\infty_c$, there exists a collection of finitely many sparse families $\{\mathscr{S}_j\}_{j=1}$ such that*

$$|a(x,D)f(x)| \lesssim \sum_j \sum_{Q \in \mathscr{S}_j} \langle f \rangle_{2,Q} \sum_{R \subset Q, R \in \mathscr{S}_j} 1_R(x)$$

if and only if

$$m \leq -n(1-\rho)/2.$$

Then as a corollary, we recover the weighted boundedness result which was showed by Michalowski, Rule and Staubach [33], that is the $L^p(\omega)$-boundedness with $a \in S^{-n(1-\rho)/2}_{\rho,\rho}$ and $\omega \in A_{p/2}$. Furthermore, as a benefit of our new sparse bounds, we have the boundedness of pseudodifferential operators and also the time evolution $e^{it(-\Delta)^{\alpha/2}}$ with $0 < \alpha \leq 2$ of dispersive equations on weighted Besov spaces (Theorem 3.1, Theorem 3.2, Corollary 3.3, Theorem 3.3, Corollary 3.4). We have also the following Coifman-Fefferman estimate for $a(x,D)$ by the same argument used in the proof of Theorem 1.1.

Theorem 1.2. *Let $a \in S^{-n(1-\rho)/2}_{\rho,\rho}$ with $0 < \rho < 1$. Then, for any $\omega \in A_\infty$ and $0 < p < \infty$, we have*

$$||a(x,D)f||_{L^p(\omega)} \lesssim [\omega]_{A_\infty} ||M_2 f||_{L^p(\omega)}.$$

This paper is organized as follows. In the next section, we prove Thoorem1.1 and Theorem 1.2 by using Lerner and Nazarov's method. The Section 3 is devoted to establishing a sparse form bounds and the boundedness on weighted Besov spaces for $a(x,D)$ and $e^{it(-\Delta)^{\alpha/2}}$, Furthemore, we give some results about the sharpness of weighted boundedness of these operators.

2 Sparse bounds for pseudodifferential operators

2.1 The pointwise estimate for pseudodifferential operators

To establish Theorem 1.1, we use the following definition of dyadic lattice and sparse decomposition of measurable functions given by Lerner and Nazarov [30].

Definition 2.1. *A Dyadic lattice $\mathscr{D}$ in $\mathbb{R}^n$ is any collection of cubes such that*

(D-1) if $Q \in \mathscr{D}$, then each child of Q is in $\mathscr{D}$,

(D-2) every two cubes in $\mathscr{D}$ have a common ancestor in $\mathscr{D}$,

(D-3) $\mathscr{D}$ is regular, i.e., for any compact set K in $\mathbb{R}^n$, there exists $Q \in \mathscr{D}$ such that $K \subset Q$.

Theorem 2.1 ([30])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be any measurable almost everywhere finite function such that for every $\varepsilon > 0$,*

$$\lim_{R\to\infty} R^{-n}|\{x \in [-R, R]^n \ ; \ |f(x)| > \varepsilon\} = 0.$$

Then, for any dyadic lattice $\mathscr{D}$ and any $\lambda \in (0, 2^{-n-2}]$, there exists a sparse family $\mathcal{S} \subset \mathscr{D}$ such that

$$|f(x)| \le \sum_{Q\in\mathcal{S}} \omega_\lambda(f; Q)1_Q(x),$$

where

$$\omega_\lambda(f; Q) = \inf_{\substack{E\subset Q\\ |E|>(1-\lambda)|Q|}} \sup_{x,x'\in E} |f(x) - f(x')|.$$

By using Theorem 2.1, we have a pointwise estimate of $a(x, D)$ with $a \in S^{-n(1-\rho)/2}_{\rho,\rho}$:

Lemma 2.1. *Let $a \in S^{-n(1-\rho)/2}_{\rho,\rho}$ with $0 < \rho < 1$. Then, for any $f \in L^\infty_c$, there exists a sparse family $\mathcal{S}$ so that*

$$|a(x, D)f(x)| \lesssim \sum_{k\ge0} 2^{-\varepsilon k} \sum_{Q\in\mathcal{S},|Q|\ge3^{-\frac{2n}{1-\rho}}} \langle f\rangle_{2,2^{k+1}Q}1_Q(x) + \sum_{k\ge0} 2^{-\varepsilon k} \sum_{Q\in\mathcal{S},|Q|<3^{-\frac{2n}{1-\rho}}} \langle f\rangle_{2,2^{k+1}Q^\rho}1_Q(x),$$

where $\varepsilon = \lfloor n/2\rfloor - n/2 + 1$. Here, Q^ρ denotes the cube such that $|Q^\rho| = |Q|^\rho$ whose center is the same as that of Q.

To prove the lemma, we give a partition of unity. Take $\hat\psi \in C^\infty_0(\mathbb{R}^n)$ such that supp $\hat\psi \subset B(0, 2)$, $\hat\psi = 1$ on $B(0, 1)$ and $\psi \ge 0$, and denote $\hat\psi_j(\xi) := \hat\psi(2^{-j}\xi) - \hat\psi(2^{-j+1}\xi)$ for $j \in \mathbb{Z}$,

$$\phi_j = \begin{cases} \psi_j & j \in \mathbb{N} \\ \sum_{i\le0} \psi_i & j = 0 \end{cases}.$$

Then, $a(x, D)$ is decomposed as

$$a(x, D) = \sum_{j=0}^{\infty} a_j(x, D),$$

where $a(x, \xi) = a(x, \xi)\hat\phi_j(\xi)$. Furthermore, we use these notations in the following sections. Let us prove Lemma 2.1.

Proof. From Theorem 2.1, we have

$$|a(x, D)f(x)| \le \sum_{Q\in\mathcal{S}} \omega_\lambda(|a(x, D)f(x)|; Q)1_Q(x).$$

First, we consider the case $|Q| < 3^{-\frac{2n}{1-\rho}}$. Let $\alpha > 0$ and

$$E = \{x \in Q \ ; \ |a(x, D)(f1_{2Q^\rho})| \le \alpha\}.$$

Then, $L^2 \to L^{2/\rho}$ boundness of $a(x, D)$ yields

$$\begin{aligned} |E^c|^{\rho/2} &\le \alpha^{-1}||a(x, D)(f1_{2Q^\rho})||_{L^{2/\rho}} \\ &\le \alpha^{-1}||a(x, D)||_{L^2\to L^{2/\rho}}||f||_{L^2(2Q^\rho)}. \end{aligned}$$

By taking $\alpha = 2^n \lambda^{-\rho/2} ||a(x,D)||_{L^2 \to L^{2/\rho}} \langle f \rangle_{2,2Q^\rho}$, one has $|E^c| \le \lambda |Q|$ and $|E| \ge (1-\lambda)|Q|$. Therefore, we have

$$|a(x,D)f(x) - a(x,D)f(x')| \lesssim \langle f \rangle_{2,2Q^\rho} + |a(x,D)(f1_{(2Q^\rho)^c})(x) - a(x,D)(f1_{(2Q^\rho)^c})(x')|$$

for any $x, x' \in E$. We estimate the second term. Let $a_j(x,\xi) := a(x,\xi)\hat{\phi}_j(\xi)$ and

$$K_j(x,y) = \int e^{i(x-y)\xi} a_j(x,\xi) d\xi.$$

We integrate by parts in ξ to obtain

$$|K(x,y)| \lesssim |x-y|^{-N} \sum_{|\alpha|=N} \left| \int e^{i(x-y)\xi} \partial_\xi^\alpha a_j(x,\xi) d\xi \right|$$

for any $n \in \mathbb{N}$. Hence, we have

$$\begin{aligned}
|a(x,D)(f1_{(2Q^\rho)^c})(x)| &\le \sum_{|\alpha|=N} \int |x-y|^{-N} |f(y)| 1_{(2Q^\rho)^c}(y) \left| \int e^{i(x-y)\xi} \partial_\xi^\alpha a_j(x,\xi) d\xi \right| dy \\
&= \sum_{|\alpha|=N} \sup_{||g||_{L^\infty}=1} \left| \int |x-y|^{-N} f(y) 1_{(2Q^\rho)^c}(y) g(y) \int e^{i(x-y)\xi} \partial_\xi^\alpha a_j(x,\xi) d\xi dy \right| \\
&\le \sum_{|\alpha|=N} \sup_{||g||_{L^\infty}=1} \left(\int |\partial_\xi^\alpha a_j(x,\xi)|^2 \right)^{1/2} ||\mathcal{F}[|x-\cdot|^{-N} f1_{(2Q^\rho)^c} g||_{L^2} \\
&\lesssim 2^{j\rho n/2 - j\rho N} \left(\int_{(2Q^\rho)^c} |x-y|^{-2N} |f(y)|^2 dy \right)^{1/2} \\
&\lesssim 2^{j\rho n/2 - j\rho N} \sum_{k \ge 1} \left(\int_{2^{k+1}Q^\rho \setminus 2^k Q^\rho} |x-y|^{-2N} |f(y)|^2 dy \right)^{1/2} \\
&\lesssim 2^{j\rho n/2 - j\rho N} \ell(Q)^{-\rho N + \rho n/2} \sum_{k \ge 1} 2^{-kN + kn/2} \langle f \rangle_{2,2^{k+1}Q^\rho}.
\end{aligned}$$

By taking $N > n/2$, one has

$$\sum_{2^{-j} \le \ell(Q)} |a(x,D)(f1_{(2Q^\rho)^c})(x)| \lesssim \sum_{k \ge 1} 2^{-kN + kn/2} \langle f \rangle_{2,2^{k+1}Q^\rho}.$$

On the other hands, it holds that

$$\begin{aligned}
& (x-y)^\alpha \{K_j(x,y) - K_j(x',y)\} \\
= \; & (x-y)^\alpha \int e^{i(x-y)\xi} (1 - e^{-i(x-x')\xi}) a_j(x,\xi) d\xi + (x-y)^\alpha \int e^{i(x'-y)\xi} (a_j(x,\xi) - a_j(x',\xi)) d\xi \\
= \; & \int e^{i(x-y)\xi} \partial_\xi^\alpha \{(1 - e^{-i(x-x')\xi}) a_j(x,\xi)\} d\xi + \int e^{i(x'-y)\xi} \partial_\xi^\alpha (a_j(x,\xi) - a_j(x',\xi)) d\xi.
\end{aligned}$$

For any j such that $2^{-j} > \ell(Q)$, Taylor's formula yields

$$|\partial_\xi^\alpha \{(1 - e^{-i(x-x')\xi}) a_j(x,\xi)\}| \lesssim \ell(Q) 2^{-jn(1-\rho)/2 + j - j\rho|\alpha|},$$

and

$$\begin{aligned}
|\partial_\xi^\alpha (a_j(x,\xi) - a_j(x',\xi))| &= \left| \partial_\xi^\alpha \int_0^1 (x-x') \cdot (\nabla_x a_j)(x' + t(x-x'), \xi) dt \right| \\
&\lesssim \ell(Q) 2^{-jn(1-\rho)/2 - j\rho|\alpha| + j\rho}.
\end{aligned}$$

From these results, we obtain

$$\begin{aligned}
& \sum_{2^{-j} > \ell(Q)} |a_j(x,D)(f1_{(2Q^\rho)^c})(x) - a_j(x,D)(f1_{(2Q^\rho)^c})(x')| \\
\lesssim \; & \sum_{2^{-j} > \ell(Q)} 2^{j\rho n/2 + j - j\rho N} \ell(Q)^{1 - \rho N + \rho n/2} \sum_{k \ge 1} 2^{-kN + kn/2} \langle f \rangle_{2,2^{k+1}Q^\rho} \\
\lesssim \; & \sum_{k \ge 1} 2^{-kN + kn/2} \langle f \rangle_{2,2^{k+1}Q^\rho}
\end{aligned}$$

by taking $N = \lfloor n/2 \rfloor + 1$. In the case $|Q| \geq 3^{-\frac{2n}{1-\rho}}$, the desired estimate is easily checked in the same way as above by setting

$$E = \{x \in Q \ ; \ |a(x,D)(f 1_{2Q})| \leq \alpha\} \ , \ \alpha = 2^n \lambda^{-1/2} ||a(x,D)||_{L^2 \to L^2} \langle f \rangle_{2,2Q}.$$

□

2.2 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1 by using Lemma 2.1 and Lerner and Nazarov's technique [30].

Definition 2.2. *Let $\mathcal{P}$ denote a map from $\{(Q,Q') \in \mathscr{D} \times \mathscr{D} \ ; \ Q' \subset Q\}$ to $\{true, false\}$ such that $\mathcal{P}(Q,Q) = true$ for any $Q \in \mathcal{D}$. Then, we call that (Q,Q') is one step if $\mathcal{P}(Q,Q') = false$ and $\mathcal{P}(Q,R) = true$ for any $Q' \subsetneq R \subset Q$, and we call that (Q,Q') is finite step if there exist $m \in \mathbb{N}$ and sequence $Q' = Q_0 \subset Q_1 \subset \cdots \subset Q_m = Q$ such that each (Q_{j+1}, Q_j) is one step. Furthermore, we set*

$$stop(Q,\mathcal{P}) = \{Q' \in \mathscr{D} \ ; \ (Q,Q') \ is \ finite \ step, \ or \ Q' = Q\}.$$

Let $\mathcal{S} \subset \mathscr{D}$ denote a sparse family and that with every cube $Q \in \mathcal{S}$ some family $\mathscr{F}(Q) \subset \mathscr{D}$ of child of Q is associated so that $Q \in \mathscr{F}(Q)$. Then, we define the family of cubes $\tilde{\mathcal{S}}$ by

$$\tilde{\mathcal{S}} := \bigcup_{Q \in \mathcal{S}} \tilde{\mathscr{F}}(Q),$$

$$\tilde{\mathscr{F}}(Q) := \{P \in \mathscr{F}(Q) \ ; \ P \notin \mathscr{F}(R) \ for \ any \ Q \subsetneq R\},$$

and we call $\tilde{\mathcal{S}}$ the argumentation of $\mathcal{S}$ by $\mathscr{F}(Q)$. In [30], Lerner and Nazalov proved $\tilde{\mathcal{S}}$ is a sparse family if $\mathscr{F}(Q)$ are sparse families. In particular, they proved the following result in the same paper.

Proposition 2.1. *Let $\mathcal{S}$ be a sparse family and assume that*

$$\sum_j |Q_j| < \frac{1}{2}|Q|$$

for any $Q \in \mathcal{S}$ and finitely pairwise disjoint cubes $\{Q_j\}_j$ included in Q such that $\mathcal{P}(Q,Q_j) = false$. Then, the augmentation of $\mathcal{S}$ by $stop(Q,\mathcal{P})$ is a sparse family.

Let us prove Theorem 1.1.

Proof. In view of the three lattice theorem in [30], there exists a family of dyadic lattices $\{\mathscr{D}_j\}_{j=1,2,\cdots,3^{2n}}$ such that

- For any cube $Q \in \mathscr{D}$ and j, there exists an unique cube $\overline{Q} \in \mathscr{D}_j$ such that $Q \subset \overline{Q}$ and $|\overline{Q}| = 3^{2n}|Q|$,
- For each pair of cubes (Q,Q') in $\mathbb{R}^n$, there exist j and $(\overline{Q}, \overline{Q'}) \in \mathscr{D}_j \times \mathscr{D}_j$ such that $Q \subset \overline{Q}$, $Q' \subset \overline{Q'}$, and $|Q| \sim |\overline{Q}|$, $|Q'| \sim |\overline{Q'}|$.

For any cubes Q^ρ and $k \in \mathbb{Z}_{\geq 0}$, we can take j and $R_Q, R \in \mathscr{D}_j$ such that $Q^\rho \subset R_Q$, $2^k Q^\rho \subset R$ and $|Q^\rho| \sim |R_Q|$, $|2^k Q^\rho| \sim |R|$. Therefore, we have

$$\sum_{k \geq 0} 2^{-\varepsilon k} \sum_{\substack{Q \in \mathcal{S} \\ |Q| < 3^{-\frac{2n}{1-\rho}}}} \langle f \rangle_{2,2^{k+1}Q^\rho} 1_Q(x) \lesssim \sum_j \sum_{\substack{Q \in \mathcal{S} \\ |Q| < 3^{-\frac{2n}{1-\rho}}}} \sum_{\substack{R \in \mathscr{D}_j \\ R_Q \subset R}} \left(\frac{|R_Q|}{|R|} \right)^\varepsilon \langle f \rangle_{2,R} 1_Q(x).$$

Furthermore, we take $\overline{Q} \in \mathscr{D}_j$ such that $Q \subset \overline{Q}$ and $|\overline{Q}| = 3^{2n}|Q|$ for any j, and define $\mathcal{S}_j = \{\overline{Q} \ ; \ Q \in \mathcal{S}\}$, $\mathcal{S}'_j = \{\overline{Q} \ ; \ Q \in \mathcal{S}, \ |Q| < 3^{-\frac{2n}{1-\rho}}\}$, of course $\mathcal{S}_j$ is a regular sparse collection. Since $Q \to \overline{Q}$ is a injective map, we can define the map $R_{\overline{Q}} := R_Q$. Here, the assumption $|Q| < 3^{-\frac{2n}{1-\rho}}$ gives

$$|\overline{Q}| = 3^{2n}|Q| < |Q|^\rho \leq |R_Q|,$$

which yields $\overline{Q} \subset R_{\overline{Q}}$. From these results, for any regular sparse family $\overline{\mathcal{S}_j}$ so that $\mathcal{S}_j \subset \overline{\mathcal{S}_j} \subset \mathscr{D}_j$, we obtain

$$\begin{aligned}\sum_j \sum_{\substack{Q\in\mathcal{S}\\ |Q|<3^{-\frac{2n}{1-\rho}}}} \sum_{\substack{R\in\mathscr{D}_j\\ R_Q\subset R}} \left(\frac{|R_Q|}{|R|}\right)^{\varepsilon} \langle f\rangle_{2,R} 1_Q(x) &\lesssim \sum_j \sum_{Q\in\mathcal{S}'_j} \sum_{\substack{R\in\mathscr{D}_j\\ R_Q\subset R}} \left(\frac{|R_Q|}{|R|}\right)^{\varepsilon} \langle f\rangle_{2,R} 1_Q(x)\\ &= \sum_j \sum_{U\in\overline{\mathcal{S}_j}} \sum_{Q\in\mathcal{S}'_j} \sum_{\substack{R\in H_{\overline{\mathcal{S}_j}}(U)\\ R_Q\subset R}} \left(\frac{|R_Q|}{|R|}\right)^{\varepsilon} \langle f\rangle_{2,R} 1_Q(x)\\ &\le \sum_j \sum_{U\in\overline{\mathcal{S}_j}} \sup_{R\in H_{\overline{\mathcal{S}_j}}(U)} \langle f\rangle_{2,R} \sum_{\substack{Q\in\mathcal{S}'_j\\ Q\subset U}} \sum_{\substack{R\in H_{\overline{\mathcal{S}_j}}(U)\\ R_Q\subset R}} \left(\frac{|R_Q|}{|R|}\right)^{\varepsilon} 1_Q(x),\end{aligned}$$

where

$$H_{\overline{\mathcal{S}_j}}(U) := \{R\in\mathscr{D}_j\ ;\ R\subset U,\ \textit{there is no cube } P\in\overline{\mathcal{S}_j} \textit{ so that } R\subset P\subsetneq U\}.$$

Here, we claim that

$$\sum_{\substack{R\in H_{\overline{\mathcal{S}_j}}(U)\\ R_Q\subset R}} \left(\frac{|R_Q|}{|R|}\right)^{\varepsilon} \lesssim 1.$$

In fact, since $R, R_Q \in \mathscr{D}_j$, R and R_Q have a common ancestor in $\mathscr{D}_j$. Therefore, if $R\subset R_Q$, there exists $k\in\mathbb{N}$ such that

$$|R| = 2^{kn}|R_Q|.$$

Furthemore, if $R, R' \in \mathscr{D}_j$ satisfy $R_Q \subset R, R'$, we have $R\subset R'$ or $R'\subset R$ by $R\cap R' \supset R_Q \neq \emptyset$. Hence, the map

$$R\ \mapsto |R_Q|/|R| \quad , \quad \{R\in\mathscr{D}_j\ ;\ R_Q\subset R\}\ \to\ \{2^{-kn}\ ;\ k\in\mathbb{N}\}$$

is a injective map, and we have

$$\sum_{\substack{R\in H_{\overline{\mathcal{S}_j}}(U)\\ R_Q\subset R}} \left(\frac{|R_Q|}{|R|}\right)^{\varepsilon} \le \sum_{\substack{R\in\mathscr{D}_j\\ R_Q\subset R}} \left(\frac{|R_Q|}{|R|}\right)^{\varepsilon} \le \sum_{k\ge0} 2^{-kn\varepsilon} \lesssim 1.$$

Therefore, one has

$$\sum_{k\ge0} 2^{-\varepsilon k} \sum_{\substack{Q\in\mathcal{S}\\ |Q|<3^{-\frac{2n}{1-\rho}}}} \langle f\rangle_{2,2^{k+1}Q^{\rho}} 1_Q(x) \lesssim \sum_j \sum_{U\in\overline{\mathcal{S}_j}} \sup_{R\in H_{\overline{\mathcal{S}_j}}(U)} \langle f\rangle_{2,R} \sum_{\substack{Q\in\overline{\mathcal{S}_j}\\ Q\subset U}} 1_Q(x).$$

To establish deisired inequality, we give a regular sparse families $\overline{\mathcal{S}_j}$ such that

$$\sup_{R\in H_{\overline{\mathcal{S}_j}}(U)} \langle f\rangle_{2,R} \lesssim \langle f\rangle_{2,U}.$$

To this end, we define the map $\mathcal{P}$ by

$$\mathcal{P}(U,R) = \begin{cases} \textit{true} & \langle f\rangle_{2,R} \le \sqrt{2}\langle f\rangle_{2,U} \\ \textit{false} & \textit{other} \end{cases}.$$

Let $\{R_j\}_j$ be a pairwise disjoint dyadic children of U such that $\mathcal{P}(U,R_j) = \textit{false}$, then we have

$$\sum_j |R_j| \le \frac{1}{2}|U| \sum_j ||f||^2_{L^2(R_j)} ||f||^{-2}_{L^2(U)} \le \frac{1}{2}|U|.$$

Hence, the argumentation of $\mathcal{S}_j$ by $\textit{stop}(U,\mathcal{P})$ is a regular sparse family and put it as $\overline{\mathcal{S}_j}$. We assume that there exists a $R\in H_{\overline{\mathcal{S}_j}}(U)$ so that $\mathcal{P}(U,R) = \textit{false}$. From the definition of $H_{\overline{\mathcal{S}_j}}(U)$, we obtain $R\notin \overline{\mathcal{S}_j}$ which yields $R\notin \textit{stop}(U,\mathcal{P})$. We take $R\subsetneq R_1\subset U$ such that $\mathcal{P}(U,R_1) = \textit{false}$. If $R_1\neq U$, we can take $R_1\subsetneq R_2\subset U$ such that $\mathcal{P}(U,R_2)=false$ again. By repeating this work, we have $\mathcal{P}(U,U)=false$ which contradict the definition of $\mathcal{P}$. Hence, we have $\mathcal{P}(Q,R)=true$ and

$$\sup_{R\in H_{\overline{\mathcal{S}_j}(U)}}\langle f\rangle_{2,R}\lesssim\langle f\rangle_{2,U}.$$

On the other hands, by Lerner and Nazarov's result [30], we can find $\{\tilde{\mathcal{S}}_j\}_{j=1,\cdots 3^n}$ such that

$$\sum_{k\geq 0}2^{-\varepsilon k}\sum_{Q\in\mathcal{S},|Q|\geq 3^{-\frac{2n}{1-\rho}}}\langle f\rangle_{2,2^{k+1}Q}1_Q(x)\lesssim\sum_j\sum_{Q\in\tilde{\mathcal{S}}_j}\langle f\rangle_{2,Q}1_Q(x).$$

From these estimates, we obtain

$$\begin{aligned}|a(x,D)f(x)| &\lesssim \sum_j\sum_{Q\in\tilde{\mathcal{S}}_j}\langle f\rangle_{2,Q}1_Q(x)+\sum_j\sum_{Q\in\overline{\mathcal{S}_j}}\langle f\rangle_{2,Q}\sum_{\substack{R\in\overline{\mathcal{S}_j}\\ R\subset Q}}1_R(x)\\ &\leq \sum_j\sum_{Q\in\tilde{\mathcal{S}}_j}\langle f\rangle_{2,Q}\sum_{\substack{R\in\tilde{\mathcal{S}}_j\\ R\subset Q}}1_R(x)+\sum_j\sum_{Q\in\overline{\mathcal{S}_j}}\langle f\rangle_{2,Q}\sum_{\substack{R\in\overline{\mathcal{S}_j}\\ R\subset Q}}1_R(x)\\ &\lesssim \sum_j\sum_{Q\in\mathscr{S}_j}\langle f\rangle_{2,Q}\sum_{\substack{R\in\mathscr{S}_j\\ R\subset Q}}1_R(x),\end{aligned}$$

where $\{\mathscr{S}_j\}_j:=\{\tilde{\mathcal{S}}_1\cdots\tilde{\mathcal{S}}_{3^n},\overline{\mathcal{S}_1},\cdots\overline{\mathcal{S}_{3^{2n}}}\}$.

Conversely, we assume

$$|a(x,D)f(x)| \lesssim \sum_j\sum_{Q\in\mathscr{S}_j}\langle f\rangle_{2,Q}\sum_{\substack{R\in\mathscr{S}_j\\ R\subset Q}}1_R(x).$$

For any $s\in(1,\infty)$, we have

$$\begin{aligned}|\langle a(x,D)f,g\rangle| &\lesssim \sum_j\sum_{Q\in\mathscr{S}_j}\langle f\rangle_{2,Q}\sum_{R\subset Q,R\in\mathscr{S}_j}|R|\langle g\rangle_{1,R}\\ &\lesssim \sum_j\sum_{Q\in\mathscr{S}_j}\langle f\rangle_{2,Q}\sum_{R\subset Q,R\in\mathscr{S}_j}\int_{E_R}\langle g\rangle_{1,R}dx\\ &\lesssim \sum_j\sum_{Q\in\mathscr{S}_j}\langle f\rangle_{2,Q}\sum_{R\subset Q,R\in\mathscr{S}_j}\int_{E_R}M(g1_Q)(x)dx\\ &\lesssim \sum_j\sum_{Q\in\mathscr{S}_j}|Q|\langle f\rangle_{2,Q}\langle M(g1_Q)\rangle_{1,Q}\\ &\lesssim \sum_j\Lambda_{\mathcal{S}_j2,s'}(f,g).\end{aligned}$$

Hence, we have the L^p-boundedness of $a(x,D)$ for any $2<p<\infty$. On the other hands, Hörmander [19] showed that $m\leq -n(1-\rho)|1/2-1/p|$ is necessary for $a(x,D)$ to be L^p- bounded. Therefore, we have $m\leq -n(1-\rho)/2$. □

2.3 Weighted L^p bounds for pseudodifferential operators

This subsection is devoted to prove Theorem 1.2. The class A_∞ denotes the set of all nonnegative locally integrable function ω such that

$$[\omega]_{A_\infty}:=\sup_Q\frac{1}{\omega(Q)}\int_Q M(\omega 1_Q)<\infty.$$

The sharp reverse Hölder inequality of A_∞ weights was shown by Hytönen and Pérez [23] .

Theorem 2.2 ([23]). *Let $\omega\in A_\infty$. Then, there exists a constant c_n depending on dimension n such that*

$$\left(\frac{1}{|Q|}\int_Q\omega^\delta\right)^{1/\delta}\leq\frac{2}{|Q|}\omega(Q)$$

for any Q where $\delta=1+c_n[\omega]_{A_\infty}^{-1}$.

From this theorem, we remark that

$$\begin{aligned}\int_Q |f|\omega &\leq \left(\int_Q |f|^{\delta'}\right)^{1/\delta'}\left(\int_Q \omega^{\delta}\right)^{1/\delta} \\ &\leq \frac{2}{|Q|}\omega(Q)\left(\int_Q |f|^{\delta'}\right)^{1/\delta'}\end{aligned}$$

for each nonnegative locally integrable function f. In particular, for any measurable subset $E \subset Q$, we have

$$\omega(E) \leq 2\left(\frac{|E|}{|Q|}\right)^{1/\delta'}\omega(Q)$$

by taking $f = 1_E$. To establish Theorem 1.2, we establish the following lemma which is shown by using Cejas, Li, Pérez and Rivera-Ríos's idea in subsection 5.2.3 in [12].

Lemma 2.2. *Let $X : \{cube\} \to \{cube\}$ be a map such that $Q \subset X(Q)$ for any cube Q, and let*

$$\Lambda_{\mathcal{S},r,X} f(x) := \sum_{Q\in\mathcal{S}} \langle f\rangle_{r,X(Q)} 1_Q(x)$$

for any sparse family $\mathcal{S}$ and $1 \leq r < \infty$. Then, for any $\omega \in A_\infty$ and $p \in (0,\infty)$, one has

$$||\Lambda_{\mathcal{S},r,X} f||_{L^p(\omega)} \lesssim [\omega]_{A_\infty} ||M_r f||_{L^p(\omega)}$$

for any $f \in L^\infty_c$.

Proof. Let $\gamma > 0$ and we have

$$\begin{aligned}||\Lambda_{\mathcal{S},r,X} f||^p_{L^p(\omega)} &\lesssim \sum_{k\in\mathbb{Z}} 2^{kp}\omega(\{\Lambda_{\mathcal{S},r,X} f > 2^k\}) \\ &\leq \sum_{k\in\mathbb{Z}} 2^{kp}\omega(\{\Lambda_{\mathcal{S},r,X} f > 2^k,\ M_r f \leq \gamma 2^k\}) + \sum_{k\in\mathbb{Z}} 2^{kp}\omega(\{M_r f > \gamma 2^k\}) \\ &\lesssim \sum_{k\in\mathbb{Z}} 2^{kp}\omega(\{\Lambda_{\mathcal{S},r,X} f > 2^k,\ M_r f \leq \gamma 2^k\}) + \gamma^{-p}||M_r f||^p_{L^p(\omega)}.\end{aligned}$$

Here, we set

$$\begin{aligned}\mathcal{S}_m &= \{Q \in \mathcal{S}\ ;\ 2^m \leq \langle f\rangle_{r,X(Q)} < 2^{m+1}\}, \\ \mathcal{S}^*_m &= \{Q \in \mathcal{S}_m\ ;\ Q \textit{ is maximal with inclusion}\}\end{aligned}$$

for any $m \in \mathbb{Z}$. If $2^m > \gamma 2^k$, we obtain $M_r f(x) > \gamma 2^k$ for any $x \in Q \in \mathcal{S}_m$ from the assumption $Q \subset X(Q)$. Hence, one obtains

$$\begin{aligned}\omega(\{\Lambda_{\mathcal{S},r,X} f > 2^k, M_r f \leq 2^k\}) &= \omega\left(\left\{\sum_{2^m\leq\gamma 2^k} \Lambda_{\mathcal{S}_m,r,X} f > 2^k,\ M_r f \leq \gamma 2^k\right\}\right) \\ &\leq \sum_{2^m\leq\gamma 2^k} \omega(\{\Lambda_{\mathcal{S}_m,r,X} f > \gamma^{-1/2}2^{(m+k)/2-1}\}) \\ &\leq \sum_{2^m\leq\gamma 2^k} \omega\left(\left\{\sum_{Q\in\mathcal{S}_m} 1_Q > \gamma^{-1/2}2^{(-m+k)/2-2}\right\}\right) \\ &\leq \sum_{2^m\leq\gamma 2^k}\sum_{U\in\mathcal{S}^*_m} \omega\left(\left\{x \in U\ ;\ \sum_{Q\in\mathcal{S}_m, Q\subset U} 1_Q(x) > \gamma^{-1/2}2^{(-m+k)/2-2}\right\}\right) \\ &=: \sum_{2^m\leq\gamma 2^k}\sum_{U\in\mathcal{S}^*_m} \omega(E).\end{aligned}$$

For any $s \in (1,\infty)$, the sparseness of $\mathcal{S}_m$ gives

$$\begin{aligned}\left\|\sum_{Q\in\mathcal{S}_m, Q\subset U} 1_Q\right\|_{L^s} &\le \sup_{||g||_{L^{s'}}=1} \sum_{Q\in\mathcal{S}_m, Q\subset U}\int_Q g \\ &\lesssim \sup_{||g||_{L^{s'}}=1}\int_Q Mg \\ &\le \sup_{||g||_{L^{s'}}=1} |Q|^{1/s}||Mg||_{L^{s'}} \\ &\le s|Q|^{1/s},\end{aligned}$$

which yields

$$|E| \le 2^{(m-k)s/2+2s}\gamma^{-s/2}s^s|U|.$$

From this and Theorem 2.2, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^{kp}\omega(\{\Lambda_{\mathcal{S},r,X}f > 2^k,\ M_rf \le 2^k\}) &\lesssim \sum_{k\in\mathbb{Z}} 2^{kp}\sum_{2^m\le\gamma 2^k} 2^{(m-k)s/(2\delta')+2s/\delta'}\gamma^{-s/(2\delta')}s^{s/\delta'}\sum_{U\in\mathcal{S}_m^*}\omega(U) \\ &\le 2^{2s/\delta'}\gamma^{s/(2\delta')}s^{s/\delta'}\sum_{m\in\mathbb{Z}} 2^{ms/(2\delta')}\omega(\{M_rf > 2^m\})\sum_{2^k\ge\gamma^{-1}2^m} 2^{kp-ks/(2\delta')} \\ &\lesssim 2^{2s/\delta'}\gamma^{-p+s/\delta'}s^{s/\delta'}||M_rf||^p_{L^p(\omega)}\end{aligned}$$

for any $s/(2\delta') > p$. Since

$$\delta' = \frac{1+c_n[\omega]^{-1}_{A_\infty}}{c_n[\omega]^{-1}_{A_\infty}} \sim [\omega]_{A_\infty},$$

we obtain the desired inequality by taking $\gamma = [\omega]^{-1}_{A_\infty}$ and $s = c[\omega]_{A_\infty}$ with some large constant $c > 0$ depending on only n and p.

□

Let us prove Theorem1.2.

Proof. For any $k \in \mathbb{Z}_{\ge 0}$, we define the map $X_k : \{cube\} \to \{cube\}$ by

$$X_k(Q) = \begin{cases} 2^{k+1}Q & |Q| \ge -2n/(1-\rho) \\ 2^{k+1}Q^\rho & |Q| < -2n/(1-\rho)\end{cases}.$$

From Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}||a(x,D)f||_{L^p(\omega)} &\le \sum_{k\ge 0} 2^{-k\varepsilon}||\Lambda_{\mathcal{S},2,X_k}f||_{L^p(\omega)} \\ &\lesssim \sum_{k\ge 0} 2^{-k\varepsilon}[\omega]_{A_\infty}||M_2f||_{L^p(\omega)} \\ &\lesssim [\omega]_{A_\infty}||M_2f||_{L^p(\omega)}\end{aligned}$$

for any $1 \le p < \infty$. The result with $0 < p < 1$ is proved in a same manner by using p-triangle inequality on $L^p(\omega)$. □

3 Sparse form bounds for Pseudodifferential operators

3.1 Besov-type sparse form bounds

Beltran and Cladek [4] established sparse form bounds of pseudodifferential operators

$$|\langle a(x,D)f, g\rangle| \lesssim \Lambda_{r,s'}(f,g)$$

with $a \in S^m_{\rho,\rho}$ and $m < m(r,s)$ where

$$m(r,s) = \begin{cases} -n(1-\rho)(1/r-1/2) & 1\le r\le s\le 2 \\ -n(1-\rho)(1/r-1/s) & 1\le r\le 2\le s\le r'\end{cases}.$$

It is natural to ask whether such bounds hold or not when $m = m(r,s)$. However, we do not know how to settle this problem. Therefore, we treat the case $m = m(r,s)$ by using Besov type sparse form bounds

$$|\langle a(x,D)f, g\rangle| \lesssim \sum_{j\geq 0} 2^{j\kappa} \Lambda_{\mathcal{S}_j, r, s'}(\phi_j * f, g)$$

with suitable $\kappa \in \mathbb{R}$. As for the definition of $\{\phi_j\}$, we refer to subsection 2.1. By using Beltran and Cladek's idea, it is not hard to see

$$|\langle a(x,D)f, g\rangle| \lesssim \sum_{j\geq 0} 2^{jm - jm(r,s) + j\varepsilon} \Lambda_{\mathcal{S}_j, r, s'}(\phi_j * f, g)$$

for any $\varepsilon > 0$. Our purpose is to eliminate ε in the above inequality. More generally, we use

$$\Lambda^{\alpha}_{\mathcal{S},r,s}(f,g) := \left(\sum_{Q\in\mathcal{S}} |Q| \langle f\rangle^{\alpha}_{r,Q} \langle g\rangle^{\alpha}_{s,Q} \right)^{1/\alpha}$$

to obtain the following results:

Theorem 3.1. *Let $2 \leq s \leq \infty$ and $1 \leq 1/\alpha < 3/2 - 1/s$, and $a \in S^m_{\rho,\rho}$ with $m \leq 0$, $0 < \rho < 1$. Then for any $f, g \in \mathscr{S}$, there exists a sequence of sparse families $\{\mathcal{S}_j\}_{j=0,1,\cdots}$ such that*

$$|\langle a(x,D)f, g\rangle| \lesssim \liminf_{R\to\infty} \sum_{j\geq 0} 2^{j\kappa_1} \Lambda^{\alpha}_{\mathcal{S}_j,2,s'}((\phi_j * f)1_{Q_R}, g),$$

where $\kappa_1 = m + n(1-\rho)(1/2 - 1/s) + \rho n(1/\alpha - 1)$. Here, Q_R denotes the cube whose center is origin and side length is R.

Theorem 3.2. (i) *Let $2 \leq s \leq \infty$ and $1 \leq 1/\alpha < 2/s'$, and $a \in S^m_{\rho,\rho}$ with $m \leq 0$, $0 < \rho < 1$. Then for any $f, g \in \mathscr{S}$, there exists a sequence of sparse families $\{\mathcal{S}_j\}_{j=0,1,\cdots}$ such that*

$$|\langle a(x,D)f, g\rangle| \lesssim \liminf_{R\to\infty} \sum_{j\geq 0} 2^{j\kappa_2} \Lambda^{\alpha}_{\mathcal{S}_j,s',s'}((\phi_j * f)1_{Q_R}, g),$$

where $\kappa_2 = m + n(1-\rho)(1 - 2/s) + \rho n(1/\alpha - 1)$.
(ii) *Let $1 \leq s' \leq r \leq 2 \leq s \leq \infty$ and $a \in S^m_{\rho,\rho}$ with $m \leq 0$, $0 < \rho < 1$. Then for any $f, g \in \mathscr{S}$, there exists a sequence of sparse families $\{\mathcal{S}_j\}_{j=0,1,\cdots}$ such that*

$$|\langle a(x,D)f, g\rangle| \lesssim \liminf_{R\to\infty} \sum_{j\geq 0} 2^{j\kappa_3} \Lambda_{\mathcal{S}_j,r,s'}((\phi_j * f)1_{Q_R}, g),$$

where $\kappa_3 = m + n(1-\rho)(1/r - 1/s)$.

To prove Theorem 3.1 and Theorem 3.2, we introduce the maximal operators $M_{T,s}$ defined by

$$M_{T,s}f(x) := \sup_{Q\ni x} |Q|^{-1/s} ||T(f 1_{(3Q)^c})||_{L^s(Q)}$$

for each linear operators T and $s \in [1,\infty]$.

Proposition 3.1. *Let $1 \leq r < s \leq \infty$ and $0 < \alpha \leq 1$, and T denote a linear operator on L^2. We assume weak-type (r,p) of T and $M_{T,s}$ with*

$$\frac{1}{p} = \frac{1}{r} - \frac{1}{\alpha} + 1.$$

Then, for any $f \in L^\infty_c$ and $g \in \mathscr{S}$, there exists a sparse family $\mathcal{S}$ such that

$$|\langle Tf, g\rangle| \lesssim (||T||_{L^r\to L^{p,\infty}} + ||M_{T,s}||_{L^r\to L^{p,\infty}}) \Lambda^{\alpha}_{\mathcal{S},r,s'}(f,g).$$

Proposition 3.1 with $\alpha = 1$ was proved by Lerner in [28], [29]. The proposition with general α is proved in a similar manner, but we give the proof for reader's convenience.

Lemma 3.1. *.Let $1 \le r < s \le \infty$ and $0 < \alpha \le 1$, $f \in L^\infty_c$ and $g \in \mathscr{S}$, and T denote a linear operators on L^2. We assume that for any cubes $Q \subset \mathbb{R}^n$ there exists some family $\mathcal{F}_Q$ of dyadic child of Q such that*

(F-1) $\mathcal{F}_Q$ *is a pairwise disjoint collection, i.e.,* $P \cap P' = \emptyset$ *for any* $P, P' \in \mathcal{F}_Q$ *such that* $P \neq P'$,

(F-2) $\displaystyle\sum_{P \in \mathcal{F}_Q} |P| \le \frac{1}{2}|Q|$,

(F-3) $\displaystyle\left|\int_Q T(f1_{3Q}) g dx\right| \le^{\exists} C|Q|^{1/\alpha} \langle f\rangle_{r,3Q} \langle g\rangle_{s',Q} + \sum_{P\in\mathcal{F}_Q} \left|\int_P T(f1_{3P}) g dx\right|.$

Then, there exists a sparse family $\mathcal{S}$ such that

$$|\langle Tf, g\rangle| \le C\Lambda^\alpha_{\mathcal{S},r,s'}(f,g).$$

Proof. Pick up a cube Q_0 in $\mathbb{R}^n$ containing supports of f. Then, we construct $\{\mathcal{F}_k\}_{k=0,1,2,\cdots}$ by

$$\mathcal{F}_0 = \{Q_0\} \quad , \quad \mathcal{F}_{k+1} = \bigcup_{P\in\mathcal{F}_k} \mathcal{F}_P,$$

and set $\mathcal{S}_k(Q_0) := \mathcal{S}_k := \bigcup_{i=0}^k \mathcal{F}_i$, $\mathcal{S}(Q_0) := \mathcal{S} = \bigcup_k \mathcal{S}_k$. From the assumption (F-1), $\mathcal{F}_k$ is a pairwise disjoint family. The assumption (F-3) gives

$$\begin{aligned}
\left|\int_{Q_0} T(f1_{3Q_0}) g dx\right| &\le C\sum_{P\in\mathcal{S}_k} |P|^{1/\alpha}\langle f\rangle_{r,3P}\langle g\rangle_{s,P} + \sum_{P\in\mathcal{F}_{k+1}} \left|\int_P T(f1_{3P}) g dx\right| \\
&\le C\sum_{P\in\mathcal{S}} |P|^{1/\alpha}\langle f\rangle_{r,3P}\langle g\rangle_{s,P} + \sum_{P\in\mathcal{F}_{k+1}} \left|\int_P T(f1_{3P}) g dx\right|
\end{aligned}$$

for any $k \in \mathbb{N}$. From

$$\sum_{P\in\mathcal{F}_{k+1}} |P| \le \sum_{L\in\mathcal{F}_k}\sum_{P\in\mathcal{F}_L} |P| \le \frac{1}{2}\sum_{L\in\mathcal{F}_k} |L| \le \cdots \le 2^{-k-1}|Q_0|,$$

we have

$$\sum_{P\in\mathcal{F}_{k+1}} \left|\int_P T(f1_{3P}) g dx\right| \to 0 \ \ as\ k \to \infty.$$

Therefore, one obtains

$$\left|\int_{Q_0} T(f1_{3Q_0}) g dx\right| \le C\Lambda^\alpha_{\mathcal{S},r,s}(f,g).$$

We prove the sparseness of $\mathcal{S}$. Let Q be any dyadic child of Q_0. For any k, we have

$$\begin{aligned}
\sum_{P\in\mathcal{F}_{k+1}, P\subset Q} |P| &\le \sum_{L\in\mathcal{F}_k}\sum_{\substack{P\in\mathcal{F}_L\\ P\subset Q}} |P| \\
&\le \sum_{\substack{L\in\mathcal{F}_k\\ L\subset Q}}\sum_{\substack{P\in\mathcal{F}_L\\ P\subset Q}} |P| + \sum_{\substack{L\in\mathcal{F}_k\\ L\supset Q}}\sum_{\substack{P\in\mathcal{F}_L\\ P\subset Q}} |P| \\
&\le \frac{1}{2}\sum_{L\in\mathcal{F}_k, L\subset Q} |L| + \sum_{\substack{L\in\mathcal{F}_k\\ L\supset Q}}\sum_{\substack{P\in\mathcal{F}_L\\ P\subset Q}} |P| \\
&=: a_k + b_k.
\end{aligned}$$

Here, if $b_k \neq 0$ for some k, it holds that $b_i = 0$ for any $i > k$. Actually, $b_k \neq 0$ means that there are $L \in \mathcal{F}_k$ and $P \in \mathcal{F}_L \subset \mathcal{F}_{k+1}$ so that $L \supset Q$ and $P \subset Q$. From the pairwise disjointness of $\mathcal{F}_{k+1}$, any cube in $\bigcup_{i>k}\mathcal{F}_i$ do not contain Q. Hence, we have $b_i = 0$ with $i > k$, and

$$\sum_{k\ge 0} b_k \le |Q|.$$

From these results, one has

$$\sum_{k\geq 0} a_k \leq \frac{1}{2}\sum_{k\geq 0} a_k + |Q|$$
$$\sum_{k\geq 0} a_k \leq 2|Q|,$$

which means $\mathcal{S}$ is a Carleson family, and therefore $\mathcal{S}$ is a sparse family. To complete the proof, we take the pairwise disjoint family of cubes $\{Q_j\}_{j=0,1,2\cdots}$ so that any $3Q_j$ contain the support of f and the union of Q_j coincides $\mathbb{R}^n$. Then, $\mathcal{S} := \cup_{j=0}^{\infty}\mathcal{S}(Q_j)$ is a sparse family, and we obtain the desired sparse form bound. □

Let us prove Proposition 3.1.

Proof. For any cube Q in $\mathbb{R}^n$ and $\lambda > 0$, set

$$E = \{x \in Q\ ;\ T(f1_{3Q}) > \lambda|Q|^{1/\alpha-1}\langle f\rangle_{r,3Q}\} \cup \{x \in Q\ ;\ M_{T,s}(f1_{3Q}) > \lambda|Q|^{1/\alpha-1}\langle f\rangle_{r,3Q}\}.$$

From weak-type boundedness of T and $M_{T,s}$, we obtain

$$\begin{aligned}|\{x \in Q\ ;\ T(f1_{3Q}) > \lambda|Q|^{1/\alpha-1}\langle f\rangle_{r,3Q}\}|^{1/p} &\leq \lambda^{-1}|Q|^{1-1/\alpha}\langle f\rangle_{r,3Q}^{-1}||T||_{L^r\to L^{p,\infty}}||f||_{L^r(3Q)}\\ &\lesssim \lambda^{-1}||T||_{L^r\to L^{p,\infty}}|Q|^{1/p},\end{aligned}$$

and

$$\begin{aligned}|\{x \in Q\ ;\ M_{T,s}(f1_{3Q}) > \lambda|Q|^{1/\alpha-1}\langle f\rangle_{r,3Q}\}|^{1/p} &\leq \lambda^{-1}|Q|^{1-1/\alpha}\langle f\rangle_{r,3Q}^{-1}||M_{T,s}||_{L^r\to L^{p,\infty}}||f||_{L^r(3Q)}\\ &\lesssim \lambda^{-1}||M_{T,s}||_{L^r\to L^{p,\infty}}|Q|^{1/p}.\end{aligned}$$

We apply the Calderon-Zygmund decomposition to 1_E to construct the family $\{P_j\}_j$ of pairwise disjoint dyadic child of Q so that

$$\begin{cases} 2^{-n-1}|P_j| < |P_j \cap E| \leq 2^{-1}|P_j|,\\ |E\setminus P| = 0,\end{cases}$$

where $P = \bigcup P_j$. Here, the pairwise disjointness of $\{P_j\}_j$ gives

$$\begin{aligned}\left|\int_Q T(f1_{3Q})g dx\right| &\leq \left|\int_{Q\setminus P} T(f1_{3Q})gdx\right| + \sum_j\left|\int_{P_j} T(f1_{3Q\setminus 3P_j})gdx\right| + \sum_j\left|\int_{P_j} T(f1_{3P_j})gdx\right|\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Since $|E\setminus P| = 0$, one obtains

$$I_1 \leq \int_{Q\setminus E}|T(f1_{3Q})||g|dx \lesssim \lambda|Q|^{1/\alpha-1}\langle f\rangle_{r,3Q}\int_Q |g| \leq \lambda|Q|^{1/\alpha}\langle f\rangle_{r,3Q}\langle g\rangle_{s',Q}.$$

On the other hands, since $P_j \cap E^c \neq \emptyset$, we have

$$\begin{aligned} I_2 &\leq \sum_j ||T(f1_{3Q\setminus 3P_j})||_{L^s(P_j)}||g||_{L^{s'}(P_j)}\\ &\leq \left(\sum_j ||T(f1_{3Q\setminus 3P_j})||_{L^s(P_j)}^s\right)^{1/s}||g||_{L^{s'}(Q)}\\ &\leq \left(\sum_j |P_j|\left(\inf_{x\in P_j} M_{T,s}(f1_{3Q})(x)\right)^s\right)^{1/s}||g||_{L^{s'}(Q)}\\ &\lesssim \left(\sum_j |P_j|\right)^{1/s}\lambda|Q|^{1/\alpha-1}\langle f\rangle_{r,3Q}||g||_{L^{s'}(Q)}\\ &\leq \lambda|Q|^{1/\alpha}\langle f\rangle_{r,3Q}\langle g\rangle_{s',Q}.\end{aligned}$$

From these results with sufficient large $\lambda \sim (||T||_{L^r\to L^{p,\infty}} + ||M_{T,s}||_{L^r\to L^{p,\infty}})$ and Lemma 3.1, we obtain $\sum|P_j| < 2^{-1}|Q|$ and complete the proof. □

Remark 3.1. *From Lebesgue's differentiation theorem, we obtain*

$$\begin{aligned}|Tf(x)| &= \lim_{\substack{|Q|\to 0\\ Q\ni x}} \left(\frac{1}{|Q|}\int_Q |Tf|^s\right)^{1/s} \\ &\le M_{T,s}f(x) + \liminf_{\substack{|Q|\to 0\\ Q\ni x}} \left(\frac{1}{|Q|}\int_Q |T(f1_{(3Q)})|^s\right)^{1/s}.\end{aligned}$$

If T is a bounded operator from $L^{s-\varepsilon}$ to L^s with some $\varepsilon > 0$, then we have

$$\begin{aligned}\liminf_{\substack{|Q|\to 0\\ Q\ni x}} \left(\frac{1}{|Q|}\int_Q |T(f1_{(3Q)})|^s\right)^{1/s} &\lesssim \liminf_{\substack{|Q|\to 0\\ Q\ni x}} \frac{1}{|Q|^{1/s}}\left(\int_{3Q} |f|^{s-\varepsilon}\right)^{1/(s-\varepsilon)} \\ &\lesssim \liminf_{\substack{|Q|\to 0\\ Q\ni x}} |Q|^{1/(s-\varepsilon)-1/s}\left(\frac{1}{|3Q|}\int_{3Q} |f|^{s-\varepsilon}\right)^{1/(s-\varepsilon)} \\ &= 0.\end{aligned}$$

Hence, we have $|Tf(x)| \le M_{T,s}f(x)$ and $||T||_{L^r\to L^{p,\infty}} \le ||M_{T,s}||_{L^r\to L^{p,\infty}}$.

The Proposition 3.1 gives some interpolation theorem.

Corollary 3.1. *Let $1 \le r \le s_0, s_1, p_0, p_1 \le \infty$. We assume a linear operator T satisfies*

$$\begin{aligned}||M_{T,s_0}f||_{L^{p_0,\infty}} &\le C_0||f||_{L^r}, \\ ||M_{T,s_1}f||_{L^{p_1,\infty}} &\le C_1||f||_{L^r}.\end{aligned}$$

Then, for any $\theta \in (0,1)$, we have

$$||M_{T,s}||_{L^r\to L^{p,\infty}} \lesssim {C_0}^{1-\theta}{C_1}^{\theta}$$

where $1/s = (1-\theta)/s_0 + \theta/s_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. In particular, we have

$$|\langle Tf, g\rangle| \lesssim (||T||_{L^r\to L^{p,\infty}} + {C_0}^{1-\theta}{C_1}^{\theta})\Lambda^{\alpha}_{\mathcal{S},r,s'}(f,g),$$

where

$$\frac{1}{\alpha} = \frac{1}{r} + \frac{1}{p'}.$$

Proof. Let $Q \subset \mathbb{R}^n$ be any cube and $x \in Q$. For any simple functions f, g so that $||g||_{s'} = 1$, we define the analytic function F on the open strip $\{z \in \mathbb{C} \;\; 0 < Rez < 1\}$ by

$$F(z) = \int_Q T(f1_{(3Q)^c})(x)g_z(x)dx,$$

where

$$g_z = sgn(g)|g|^{s'\{(1-z)/s_0' + z/s_1'\}}.$$

Then, it holds that

$$\begin{aligned}|F(iy)| &\le \int_Q |T(f1_{(3Q)^c})||g|^{s'/s_0'} \\ &\le ||T(f1_{(3Q)^c})||_{L^{s_0}(Q)} \\ &\le |Q|^{1/s_0}M_{T,s_0}f(x).\end{aligned}$$

On the other hands, one has

$$\begin{aligned}|F(1+iy)| &\le \int_Q |T(f1_{(3Q)^c})||g|^{s'/s_1'} \\ &\le ||T(f1_{(3Q)^c})||_{L^{s_1}(Q)} \\ &\le |Q|^{1/s_1}M_{T,s_1}f(x).\end{aligned}$$

By using Hadamard's three lines lemma, we have

$$\begin{aligned}|F(\theta)| &\leq |Q|^{(1-\theta)/s_0+\theta/s_1} M_{T,s_0}f(x)^{1-\theta} M_{T,s_1}f(x)^{\theta}\\ &= |Q|^{1/s} M_{T,s_0}f(x)^{1-\theta} M_{T,s_1}f(x)^{\theta},\end{aligned}$$

which yields

$$M_{T,s}f(x) \leq M_{T,s_0}f(x)^{1-\theta} M_{T,s_1}f(x)^{\theta}.$$

By Hölder's inequality, we have

$$\begin{aligned}||(M_{T,s_0}f)^{1-\theta}(M_{T,s_1}f)^{\theta}||_{L^{p,\infty}} &\lesssim ||M_{T,s_0}f||_{L^{p_0,\infty}}^{1-\theta} ||M_{T,s_1}f||_{L^{p_1,\infty}}^{\theta}\\ &\leq {C_0}^{1-\theta}{C_1}^{\theta}||f||_{L^r}\end{aligned}$$

for $1/p = (1-\theta)/p_0 + \theta/p_1$. By this and Proposition 3.1, we have $||M_{T,s}||_{L^r\to L^{p,\infty}} \lesssim {C_0}^{1-\theta}{C_1}^{\theta}$ and the desired sparse form bounds for T. □

Corollary 3.2. *Let $1 \leq r_0, r_1 \leq s_0, s_1, p_0, p_1 \leq \infty$. We assume a linear operator T satisfies*

$$\begin{aligned}||M_{T,s_0}f||_{L^{p_0,\infty}} &\leq C_0||f||_{L^{r_0}},\\ ||M_{T,s_1}f||_{L^{p_1,\infty}} &\leq C_1||f||_{L^{r_1}}.\end{aligned}$$

and

$$|Tf(x)| \leq T(|f|)(x) \quad a.e.\ x \in \mathbb{R}^n.$$

Then, for any $\theta \in (0,1)$, we have

$$||M_{T,s}||_{L^r\to L^{p,\infty}} \lesssim {C_0}^{1-\theta}{C_1}^{\theta},$$

where $1/s = (1-\theta)/s_0 + \theta/s_1$ and $1/q = (1-\theta)/p_0 + \theta/p_1$. In particular, we have

$$|\langle Tf, g\rangle| \lesssim (||T||_{L^r\to L^{p,\infty}} + {C_0}^{1-\theta}{C_1}^{\theta})\Lambda^{\alpha}_{\mathcal{S},r,s'}(f,g),$$

where

$$\frac{1}{\alpha} = \frac{1}{r} + \frac{1}{p'}.$$

Proof. The proof is similar to that of Corollary 3.1. Let $Q \subset \mathbb{R}^n$ be any cube and $x \in Q$. For any simple functions f, g so that $||g||_{s'} = 1$, we define the analytic function F on the open strip $\{z \in \mathbb{C}\ \ 0 < Rez < 1\}$ by

$$F(z) = \int_Q T(f_z 1_{(3Q)^c})(x) g_z(x)dx,$$

where

$$\begin{aligned}f_z &= sgn(f)|f|^{r\{(1-z)/r_0+z/r_1\}},\\ g_z &= sgn(g)|g|^{s'\{(1-z)/s_0'+z/s_1'\}}.\end{aligned}$$

From $|Tf| \leq T(|f|)$, we have

$$\begin{aligned}|F(iy)| &\leq \int_Q |T(f1_{(3Q)^c})||g|^{s'/s_0'}\\ &\leq ||T(f_z 1_{(3Q)^c})||_{L^{s_0}(Q)}\\ &\leq |Q|^{1/s_0} M_{T,s_0}(|f|^{r/r_0})(x),\end{aligned}$$

and

$$\begin{aligned}|F(1+iy)| &\leq \int_Q |T(f1_{(3Q)^c})||g|^{s'/s_1'}\\ &\leq ||T(f_z 1_{(3Q)^c})||_{L^{s_1}(Q)}\\ &\leq |Q|^{1/s_1} M_{T,s_1}(|f|^{r/r_1})(x).\end{aligned}$$

Hence, one obtains

$$M_{T,s}f(x) \le M_{T,s_0}(|f|^{r/r_0})(x)^{1-\theta} M_{T,s_1}(|f|^{r/r_1})(x)^{\theta}.$$

By using Hölder's inequality, we have

$$\begin{aligned} ||(M_{T,s_0}(|f|^{r/r_0}))^{1-\theta}(M_{T,s_1}(|f|^{r/r_1}))^{\theta}||_{L^{p,\infty}} &\lesssim ||M_{T,s_0}(|f|^{r/r_0})||_{L^{p_0,\infty}}^{1-\theta}||M_{T,s_1}(|f|^{r/r_1})||_{L^{p_1,\infty}}^{\theta} \\ &\le {C_0}^{1-\theta}{C_1}^{\theta}||f||_{L^r}. \end{aligned}$$

□

We give a proof of Theorem 3.1.

Proof. We recall the dyadic decomposition in subsection 2.1. Since $\phi_j * f = (\phi_{j-1}+\phi_j+\phi_{j+1}) * \phi_j * f$, we have

$$\begin{aligned} |\langle a(x,D)f,g\rangle| &= \sum_{j\ge 0}\sum_{i=j-1}^{j+1} |\langle a_i(x,D)(\phi_j * f), g\rangle| \\ &= \sum_{j\ge 0}\sum_{i=j-1}^{j+1} |\langle a_i(x,D)(\lim_{R\to\infty} 1_{Q_R}\phi_j * f), g\rangle| \\ &\le \liminf_{R\to\infty}\sum_{j\ge 0}\sum_{i=j-1}^{j+1} |\langle a_i(x,D)((\phi_j * f)1_{Q_R}), g\rangle|. \end{aligned}$$

Therefore, it is enough to prove

$$|\langle a_j(x,D)f,g\rangle| \lesssim 2^{j\kappa_1}\Lambda^{\alpha}_{\mathcal{S}_j,2,s'}(f,g)$$

for any $f\in L^\infty_c$ and $g\in\mathscr{S}$. For any $x,z\in Q$ and $\gamma\in[0,1)$, we integrate by parts $N\in\mathbb{N}$ times to obtain

$$\begin{aligned} |a_j(x,D)(f1_{(3Q)^c})(z)| &\lesssim 2^{jm+jn/2}\left\{\int_{(3Q)^c}(1+2^{2j\rho N}|z-y|^{2N})^{-2}|f(y)|^2dy\right\}^{1/2} \\ &\lesssim 2^{jm+jn/2}\left\{\int (1+2^{2j\rho N}|x-y|^{2N})^{-2}|f(y)|^2dy\right\}^{1/2} \\ &\lesssim 2^{jm+jn/2}\sum_{k\in\mathbb{Z}}\left\{\int_{|x-y|\sim 2^{-j\rho}2^k}(1+2^{2kN})^{-2}|f(y)|^2dy\right\}^{1/2} \\ &\lesssim 2^{jm+jn(1-\rho(1-\gamma))/2}M^{\gamma}(|f|^2)(x)^{1/2}, \end{aligned}$$

where $M^{\gamma}h(x) := \sup_{Q\ni x}|Q|^{\gamma}\langle h\rangle_Q$. Hence, we obtain

$$M_{a_j(x,D),\infty} \lesssim 2^{jm+jn(1-\rho(1-\gamma))/2}M^{\gamma}(|f|^2)(x)^{1/2}.$$

By weak-type boundedness of M^{γ} (see, for example, Remark 2.10 in Lacey, Moen, P'erez and Torres [25]), for any $p_0\ge 2$, one has

$$||M_{a_j(x,D),\infty}||_{L^{p_0,\infty}} \lesssim 2^{jm+jn(1-\rho)/2+j\rho n(1/2-1/p_0)}||f||_{L^2}$$

by taking $\gamma = 1-2/p_0$. On the other hands, we have

$$||a_j(x,D)f||_{L^{p_1}} \lesssim 2^{jm+jn(1/2-1/p_1)}||f||_{L^2}$$

for any $p_1\ge 2$. From this and

$$M_{a_j(x,D),p_1}f(x) \lesssim M_{p_1}(a_j(x,D)f)(x) + 2^{jm+jn(1/2-1/p_1)}M^{1-2/p_1}(|f|^2)(x)^{1/2},$$

we obtain

$$||M_{a_j(x,D),p_1}f||_{L^{p_1,\infty}} \lesssim 2^{jm+jn(1/2-1/p_1)}||f||_{L^2}.$$

Therefore, Corollary 3.1 gives

$$|\langle a_j(x,D)f,g\rangle| \lesssim 2^{jm}2^{jn(1-\theta)(1-\rho)/2+j\rho n(1-\theta)(1/2-1/p_0)}2^{jn\theta(1/2-1/p_1)}\Lambda^{\alpha}_{\mathcal{S},2,s'}(f,g),$$

with $1/s=\theta/p_1$ and $1/\alpha = 1/2-(1-\theta)/p_0-\theta/p_1+1 < 3/2-1/s$. By simple calculation as following,

$$\begin{aligned}(1-\theta)(1-\rho)/2+\rho(1-\theta)(1/2-1/p_0)+\theta(1/2-1/p_1) &= -\rho(1-\theta)/p_0+1/2-\theta/p_1\\ &= -\rho(1-1/\alpha+1/2-1/s)+1/2-1/s\\ &= (1-\rho)(1/2-1/s)+\rho(1/\alpha-1),\end{aligned}$$

we have the desired sparse bounds. □

To establish Theorem 3.2 by the interpolation argument as Corollary 3.2, we need the condition $|a(x,D)f| \le a(x,D)(|f|)$. Unfortunately, it fails in general and we need the following alternative argument:

Lemma 3.2. *Let $0\le\gamma<1$. We assume a linear operator T satisfies*

$$\begin{aligned}||T||_{L^2\to L^2} &\le C_0,\\ M_{T,\infty}f(x) &\le C_1M^{\gamma}f(x) \quad a.e.\ x\in\mathbb{R}^n.\end{aligned}$$

Then, for any $\theta\in(0,1)$, we have

$$||M_{T,r'}||_{L^r\to L^{p,\infty}} \lesssim {C_0}^{1-\theta}{C_1}^{\theta},$$

where $1/r=(1-\theta)/2+\theta$ and $1/p=(1-\gamma)\theta+1/r'$. In particular, we have

$$|\langle Tf,g\rangle| \lesssim (||T||_{L^r\to L^{p,\infty}}+{C_0}^{1-\theta}{C_1}^{\theta})\Lambda^{\alpha}_{\mathcal{S},r,r}(f,g),$$

where

$$\frac{1}{\alpha}=1+\left(\frac{2}{r}-1\right)\gamma.$$

Proof. We put $E=\{M_{T,r'}f>\lambda\}$ for any $\lambda>0$. For each $\delta>0$, we have

$$\begin{aligned}|E| &\le |\{M_{T,r'}f>\lambda,\ M_r^{\gamma}f\le\delta\lambda\}|+|\{M_r^{\gamma}f>\delta\lambda\}|\\ &=: |E_0|+|E_1|,\end{aligned}$$

where $M_r^{\gamma}f=M^{\gamma}(|f|^r)^{1/r}$. Weak-type boundedness of M_r^{γ} gives

$$|E_1|^{1/q} \lesssim \delta^{-1}\lambda^{-1}||f||_{L^r},$$

with $1/q=1/r-\gamma/r$. We need to estimate $|E_0|$. For any $x\in E_0$, there exists a cube Q_x such that

$$|Q_x| < \lambda^{r'}||T(f1_{(3Q_x)^c})||^{r'}_{L^{r'}(Q_x)}.$$

Let $K\subset E_0$ be any compact set, then we can select a finite pairwise disjoint subcollection $\{3Q_j\}_j\subset\{3Q_x\}_{x\in E}$ such that

$$|K|\lesssim\sum_j|Q_j|.$$

From the duality of $\ell^{r'}(\mathbb{N};L^{r'})$, we obtain

$$\begin{aligned}|K|^{1/r'} &\le \lambda^{-1}\left(\sum_j||T(f1_{(3Q_j)^c})||^{r'}_{L^{r'}(Q_j)}\right)^{1/r'}\\ &= \lambda^{-1}\sup_{\{g_j\}_j}\left|\sum_j\int_{Q_j}T(f1_{(3Q_j)^c})g_j\right|.\end{aligned}$$

Here, the supremum is taken all over $g=\{g_j\}_j$ such that $||g||_{\ell^r(\mathbb{N};L^r)}\le1$. We define the analytic function F on the open strip $\{z\in\mathbb{C}\ \ 0<Re z<1\}$ by

$$F(z)=\sum_j\int_{Q_j}T(f_z1_{(3Q_j)^c})(x)g_{z,j}(x)dx,$$

where

$$\begin{aligned} f_z &= sgn(f)|f|^{r\{(1-z)/2+z\}}, \\ g_{z,j} &= sgn(g_j)|g_j|^{r\{(1-z)/2+z\}}. \end{aligned}$$

By L^2 boundness of T, one has

$$\begin{aligned} |F(iy)| &\leq \sum_j ||Tf_z||_{L^2(Q_j)}||g_j||_{L^r}^{r/2} + C_0 \sum_j ||f_z||_{L^2(3Q_j)}||g_j||_{L^r}^{r/2} \\ &\leq ||Tf_z||_{L^2} + C_0||f_z||_{L^2} \\ &\lesssim C_0||f||_{L^r}^{r/2}. \end{aligned}$$

Since $Q_j \cap E_0 \neq \emptyset$, we obtain

$$\begin{aligned} |F(1+iy)| &\leq \sum_j \inf_{x\in Q_j} M_{T,\infty}f_z(x)||g_j||_{L^r}^r \\ &\leq C_1 \sum_j \inf_{x\in Q_j} M_r^\gamma f(x)^r ||g_j||_{L^r}^r \\ &\leq C_1\delta^r\lambda^r. \end{aligned}$$

By these results, we have

$$\begin{aligned} |K| &\leq (C_0^{1-\theta}C_1^\theta)^{r'} \delta^{rr'\theta}\lambda^{rr'\theta-r'}||f||_{L^r}^{rr'(1-\theta)/2} \\ &= (C_0^{1-\theta}C_1^\theta)^{r'} \delta^{rr'\theta}\lambda^{-r}||f||_{L^r}^r, \end{aligned}$$

and

$$|E| \leq \delta^{-q}\lambda^{-q}||f||_{L^r}^q + (C_0^{1-\theta}C_1^\theta)^{r'}\delta^{rr'\theta}\lambda^{-r}||f||_{L^r}^r.$$

Here, we optimize for δ to obtain

$$|E|^{1/p} \leq \lambda^{-1}C_0^{1-\theta}C_1^\theta||f||_{L^r},$$

where $1/p = r\theta/q + 1/r' = (1-\gamma)\theta + 1/r'$. Hence, $M_{T,r'}$ be a weak-type (r,p) operator which yields

$$|\langle Tf, g\rangle| \lesssim (||T||_{L^r\to L^{p,\infty}} + C_0^{1-\theta}C_1^\theta)\Lambda_{\mathcal{S},r,r}^\alpha(f,g).$$

□

Let us prove the Theorem 3.2.

Proof. The theorem follows from the pointwise estimate

$$M_{a_j(x,D),\infty}f(x) \lesssim 2^{jm+jn(1-\rho(1-\gamma))}M^\gamma f(x),$$

Lemma 3.2 and Marchinkiewicz interpolation theorem. Indeed, this estimate and Lemma 3.2 yield

$$||M_{a_j(x,D),s}||_{L^{s'}\to L^{s',\infty}} \lesssim 2^{jm+jn(1-\rho)(1-2/s)+j\rho n(1/\alpha-1)}$$

by taking $1/\alpha = 1 + (2/r-1)\gamma$. Moreover, by interpolating this with $\alpha = 1$ and $||M_{a_j(x,D),s}||_{L^2\to L^{2,\infty}} \lesssim 2^{jm+jn(1-\rho)(1/2-1/s)}$, we have

$$||M_{a_j(x,D),s}||_{L^r\to L^{r,\infty}} \lesssim 2^{jm+jn(1-\rho)(1/r-1/s)}.$$

Thus, we obtain the desired sparse form bounds. Now, we prove the above pointwise estimate. For any $x, z \in Q$ and $\gamma \in [0,1)$, we integrate by parts $N \in \mathbb{N}$ times to obtain

$$\begin{aligned} |a_j(x,D)(f1_{(3Q)^c})(z)| &\lesssim 2^{jm+jn}\int_{(3Q)^c} (1+2^{2j\rho N}|z-y|^{2N})^{-1}|f(y)|dy \\ &\lesssim 2^{jm+jn}\int (1+2^{2j\rho N}|x-y|^{2N})^{-1}|f(y)|dy \\ &\lesssim 2^{jm+jn(1-\rho(1-\gamma))}M^\gamma f(x). \end{aligned}$$

Hence, we obtain

$$M_{a_j(x,D),\infty}(x) \lesssim 2^{jm+jn(1-\rho(1-\gamma))}M^\gamma f(x),$$

and complete the proof. □

3.2 Application to the boundedness on weighted Besov spaces

This section is devoted to obtain the boundedness on weighted Besov space of pseudodifferential operators. To do this, we establish the weighted bounds for $\Lambda^{\alpha}_{\mathcal{S},r,s'}$ by using Bernicot, Frey and Petermichl's idea in [5].

Proposition 3.2. *Let $1 \le r < q \le p < s \le \infty$ and $1/\alpha = 1/p' + 1/q$. We assume a weight ω satisfies $\omega^q \in A_{q/r} \cap RH_{(p/q)(s/p)'}$. Then, for any sparse family $\mathcal{S} \subset \mathscr{D}$ with some dyadic lattice $\mathscr{D}$, we have*

$$\Lambda^{\alpha}_{\mathcal{S},r,s'}(f,g) \lesssim ([\omega^q]_{A_{q/r}}[\omega^q]_{RH_{(p/q)(s/p)'}})^{\delta} ||f||_{L^q(\omega^q)} ||g||_{L^{p'}(\omega^{-p'})},$$

where

$$\delta = \max\left\{\frac{1}{q-r}, \frac{p(s-1)}{q(s-p)}\right\}.$$

Proof. We set

$$\mu = \omega^{-rq/(q-r)} \quad and \quad \nu = \omega^{p's'/(p'-s')}.$$

Furthermore, let us define

$$F_Q = \left(\frac{1}{\mu(Q)}\int_Q |f|^r\right)^{1/r} \quad and \quad G_Q = \left(\frac{1}{\nu(Q)}\int_Q |g|^{s'}\right)^{1/s'}.$$

Then, we have

$$\Lambda^{\alpha}_{\mathcal{S},r,s'}(f,g) \le \left(\sum_{Q\in\mathcal{S}} |Q|\langle\mu\rangle_Q^{\alpha/r}\langle\nu\rangle_Q^{\alpha/s'} {F_Q}^{\alpha}{G_Q}^{\alpha}\right)^{1/\alpha}.$$

We estimate $|Q|\langle\mu\rangle_Q^{\alpha/r}\langle\nu\rangle_Q^{\alpha/s'}$. By taking

$$\beta = 1 + \frac{1/r - 1/q}{1/p - 1/s},$$

we obtain $\mu = \nu^{1-\beta'}$ and $\langle\nu\rangle_Q\langle\mu\rangle_Q^{\beta-1} \le [\nu]_{A_\beta}$. Here, we assume

$$\frac{1}{q-r} \le \frac{p(s-1)}{q(s-p)},$$

which gives $\gamma := 1/r - (\beta-1)/s' \le 0$. From this assumption and the sparseness of $\mathcal{S}$, one obtains

$$\begin{aligned} |Q|\langle\mu\rangle_Q^{\alpha/r}\langle\nu\rangle_Q^{\alpha/s'} &\le [\nu]_{A_\beta}^{\alpha/s'}|Q|\langle\mu\rangle_Q^{\alpha\gamma} \\ &\le [\nu]_{A_\beta}^{\alpha/s'}|E_Q|^{1-\alpha\gamma}\left(\int_{E_Q}\mu\right)^{\alpha\gamma}. \end{aligned}$$

On the other hands, it holds that $\mu^{-\gamma}\mu^{1/q}\nu^{1/p'} = 1$ since $\nu = \mu^{1-\beta}$. Hence, by setting $1/t = 1/q + 1/p' - \gamma = 1/\alpha - \gamma$ and using Hölder's inequality, we have

$$\begin{aligned} |E_Q|^{1/t} &= ||\mu^{-\gamma}\mu^{1/q}\nu^{1/p'}||_{L^t(E_Q)} \\ &\le \mu(E_Q)^{-\gamma}\mu(E_Q)^{1/q}\nu(Q)^{1/p'}, \end{aligned}$$

which yields

$$|Q|\langle\mu\rangle_Q^{\alpha/r}\langle\nu\rangle_Q^{\alpha/s'} \le [\nu]_{A_\beta}^{\alpha/s'}\mu(E_Q)^{\alpha/q}\nu(Q)^{\alpha/p'}.$$

From these results, we obtain

$$\begin{aligned} \Lambda^{\alpha}_{\mathcal{S},r,s'}(f,g) &\le [\nu]_{A_\beta}^{1/s'}\left(\sum_{Q\in\mathcal{S}}\left(F_Q\mu(E_Q)^{1/q}G_Q\nu(E_Q)^{1/p'}\right)^{\alpha}\right)^{1/\alpha} \\ &\le [\nu]_{A_\beta}^{1/s'}\left(\sum_{Q\in\mathcal{S}}{F_Q}^q\mu(E_Q)\right)^{1/q}\left(\sum_{Q\in\mathcal{S}}{G_Q}^{p'}\nu(E_Q)\right)^{1/p'} \\ &\le [\nu]_{A_\beta}^{1/s'}\left(\int |M^{\mathscr{D}}_{r,\mu}(f\mu^{-1/r})|^q d\mu\right)^{1/q}\left(\int |M^{\mathscr{D}}_{s',\nu}(g\nu^{-1/s'})|^{p'} d\nu\right)^{1/p'} \\ &\lesssim [\nu]_{A_\beta}^{1/s'}||f||_{L^q(\omega^q)}||g||_{L^{p'}(\omega^{-p'})}. \end{aligned}$$

In another case, by using

$$\begin{aligned}|Q|\langle\mu\rangle_Q^{\alpha/r}\langle\nu\rangle_Q^{\alpha/s'} &\le [\nu]_{A_\beta}^{\alpha/\{r(\beta-1)\}}|Q|\langle\nu\rangle_Q^{\alpha/s'-\alpha/\{r(\beta-1)\}}\\ &\le [\nu]_{A_\beta}^{\alpha/\{r(\beta-1)\}}|E_Q|^{1-\alpha\gamma}\left(\int_{E_Q}\mu\right)^{\alpha\gamma},\end{aligned}$$

and the same discussion as above, we have

$$\Lambda^\alpha_{\mathcal{S},r,s'}(f,g)\lesssim[\nu]_{A_\beta}^{1/\{r(\beta-1)\}}||f||_{L^q(\omega^q)}||g||_{L^{p'}(\omega^{-p'})}.$$

Concluding these results, we have

$$\Lambda^\alpha_{\mathcal{S},r,s'}(f,g)\lesssim[\nu]_{A_\beta}^{\delta}||f||_{L^q(\omega^q)}||g||_{L^{p'}(\omega^{-p'})},$$

where

$$\delta=\max\left\{\frac{q(s-p)}{ps(q-r)},\frac{s-1}{s}\right\}.$$

To complete the proof, we need to estimate $[\nu]_{A_\beta}$. However, it is deduced from a simple calculation. The detail is the following:

$$\begin{aligned}\langle\nu\rangle_Q\langle\mu\rangle_Q^{\beta-1} &= \langle\omega^{q\cdot p(s/p)'/q}\rangle_Q\langle\omega^{q\cdot(-r/(q-r))}\rangle_Q^{p(s/p)'/q\cdot(q/r-1)}\\ &\le \left([\omega^q]_{RH_{(p/q)(s/q)'}}\langle\omega^q\rangle_Q\langle\omega^{q\cdot(-r/(q-r))}\rangle_Q^{q/r-1}\right)^{p(s/p)'/q}\\ &\le \left([\omega^q]_{RH_{(p/q)(s/p)'}}[\omega^q]_{A_{q/r}}\right)^{ps/\{q(s-p)\}}.\end{aligned}$$

□

We define the weighted Besov spaces according to Bui [6]. Suppose $0<p,\sigma<\infty$ and $\kappa\in\mathbb{R}$, then weighted Besov spaces $B^\kappa_{p,q}(\omega)$ are defined by

$$\begin{aligned}B^\kappa_{p,\sigma}(\omega) &= \{f\in\mathscr{S}'\ ;\ ||f||_{B^\kappa_{p,\sigma}(\omega)}<\infty\},\\ ||f||_{B^\kappa_{p,\sigma}(\omega)} &= \left(\sum_{j\ge0}2^{j\kappa\sigma}||\phi_j*f||^\sigma_{L^p(\omega)}\right)^{1/\sigma}\end{aligned}$$

for any $\omega\in A_\infty$. Bui showed that $\mathscr{S}$ is a dense subset of $B^\kappa_{p,\sigma}(\omega)$. Hence, Theorem 3.1 and Theorem 3.2, and Proposition 3.2 give the following results about boundedness of pseudodifferential operators on weighted Besov spaces.

Corollary 3.3. *Let $a\in S^m_{\rho,\delta}$ with $m\in\mathbb{R}$ and $0\le\delta<\rho\le1$. Then, we have the following bounds.*
(i) *Let $2<q\le p<\infty$ and $\omega^q\in A_{q/2}\cap RH_{(p/q)(s/p)'}$ with some $s\in(p,\infty]$. Then, for any $\kappa\in\mathbb{R}$ and $0<\sigma<\infty$, $a(x,D)$ is a bounded operator from $B^{\kappa+\tilde\kappa_1}_{q,\sigma}(\omega^q)$ to $B^\kappa_{p,\sigma}(\omega^p)$ where*

$$\tilde\kappa_1=m+n(1-\rho)\left(\frac12-\frac1s\right)+\rho n\left(\frac1q-\frac1p\right).$$

(ii) *Let $1<q\le p<\infty$ and $\omega^q\in A_{q/r}\cap RH_{(p/q)(r'/p)'}$ with some $r\in[1,\min\{p',q\})$. Then, for any $\kappa\in\mathbb{R}$ and $0<\sigma\le\infty$, $a(x,D)$ is a bounded operator from $B^{\kappa+\tilde\kappa_2}_{q,\sigma}(\omega^q)$ to $B^\kappa_{p,\sigma}(\omega^p)$ where*

$$\tilde\kappa_2=m+n(1-\rho)\left(\frac2r-1\right)+\rho n\left(\frac1q-\frac1p\right).$$

Remark 3.2. *The Corollary 3.3 contains the following known boundedness results of $a(x,D)$ with $a\in S^m_{\rho,\delta}$.*
(i) *By taking $\omega=1$, $p=q$ and suitable s in neighborhood of p in* (i) *of Corollary 3.3, we have the L^p-boundedness with $m<-n(1-\rho)|1/p-1/2|$ which was established by Fefferman* [18].
(ii) *By taking $p=q$ and sufficiently large s in* (i) *of Corollary 3.3, we have the $L^p(\omega)$-boundedness with $m=-n(1-\rho)/2$ and $\omega\in A_{p/2}$ which was established by Chanillo and Torchinsky* [13].
(iii) *By taking $r=1$ and $p=q$ in* (ii) *of Corollary 3.3, we have the $L^p(\omega)$-boundedness with $m=-n(1-\rho)$ and $\omega\in A_p$ which was established by Michalowski, Rule and Staubach* [34].

Proof. First, we assume $1 \leq \sigma < \infty$. For any $\ell \in \mathbb{Z}$, there exists $b_\ell \in S^{m+\ell}_{\rho,\delta}$ so that $b(x,D) = \langle D\rangle^\ell a(x,D)$ since $\delta < \rho$. From this, for any $f \in \mathscr{S}$ and $g \in C_0^\infty$ suth that $||g||_{L^{p'}(\omega^{-p'})} \leq 1$, we have

$$\begin{aligned} |\langle \phi_k * a(x,D)(\phi_j * f), g\rangle| &= |\langle \langle D\rangle^{-\ell}\phi_k * \langle D\rangle^{\ell} a(x,D)(\phi_j * f), g\rangle| \\ &= |\langle b_\ell(x,D)f, \langle D\rangle^{-\ell}\phi_k(-\cdot) * g\rangle| \\ &\lesssim 2^{j\ell + j\tilde{\kappa}(p,q)} \liminf_{|R|\to\infty} \Lambda^{\alpha}_{r(p,q),s(p,q)}((\phi_j * f)1_{Q_R}, \langle D\rangle^{-\ell}\phi_k(-\cdot) * g) \\ &\lesssim 2^{j\ell + j\tilde{\kappa}(p,q)} ||\phi_j * f||_{L^q(\omega^q)} ||\langle D\rangle^{-\ell}\phi_k(-\cdot) * g||_{L^{p'}(\omega^{-p'})}, \end{aligned}$$

where

$$(\tilde{\kappa}(p,q), r(p,q), s(p,q)) = \begin{cases} (\tilde{\kappa}_1, 2, s') & 2 < q \leq p < \infty \\ (\tilde{\kappa}_2, r, r) & 1 < q \leq 2 \leq p \leq q' < \infty \end{cases},$$

and $\alpha = 1/p' + 1/q$. After this, we write $\tilde{\kappa} = \tilde{\kappa}(p,q)$. Now, we obtain

$$\begin{aligned} |\langle D\rangle^{-\ell}\phi_k(z)| &= \left|\int e^{iz\xi}\langle \xi\rangle^{-\ell}\hat{\phi}_k(\xi)d\xi\right| \\ &= (1 + 2^{2kN}|z|^{2N})^{-1}\left|\int (1 + 2^{2kN}(-\Delta)^N)(e^{iz\xi})\langle \xi\rangle^{-\ell}\hat{\phi}_k(\xi)d\xi\right| \\ &\leq (1 + 2^{2kN}|z|^{2N})^{-1}\left|\int |(1 + 2^{2kN}(-\Delta)^N)(\langle \xi\rangle^{-\ell}\hat{\phi}(2^{-k}\xi))|d\xi\right| \\ &\lesssim 2^{-k\ell + kn}(1 + 2^{2kN}|z|^{2N})^{-1} \end{aligned}$$

for any $z \in \mathbb{R}^n$ and $N \in \mathbb{N}$, that means

$$\begin{aligned} |\langle D\rangle^{-\ell}\phi_k(-\cdot) * g(x)| &\lesssim 2^{-k\ell + kn}\int (1 + 2^{2kN}|x-y|^{2N})^{-1}|g(y)|dy \\ &\leq 2^{-k\ell + kn}\sum_{j\in\mathbb{Z}}\int_{|x-y|\sim 2^{-k}2^j} (1 + 2^{2jN})^{-1}|g(y)|dy \\ &\leq 2^{-k\ell}\sum_{j\in\mathbb{Z}} \frac{2^{jn}}{1 + 2^{2jN}}\frac{1}{2^{-kn}2^{jn}}\int_{|x-y|\sim 2^{-k}2^j} |g(y)|dy \\ &\lesssim 2^{-k\ell}Mg(x) \end{aligned}$$

Combining this and $\omega^{-p'} \in A_{p'}$, we obtain

$$||\phi_k * a(x,D)(\phi_j * f)||_{L^p(\omega^p)} \lesssim 2^{-k\ell}2^{j\ell + j\tilde{\kappa}}||\phi_j * f||_{L^q(\omega^q)}.$$

Here, $\omega^{-p'} \in A_{p'}$ follows from

$$\begin{aligned} \sup_Q\left(\frac{1}{|Q|}\int_Q \omega^{-p'}\right)\left(\frac{1}{|Q|}\int_Q \omega^{p'/(p'-1)}\right)^{p'-1} &= \sup_Q\left(\frac{1}{|Q|}\int_Q \omega^{-p/(p-1)}\right)\left(\frac{1}{|Q|}\int_Q \omega^{p}\right)^{1/(p-1)} \\ &\leq \sup_Q\left(\frac{1}{|Q|}\int_Q \omega^{-q/(q-1)}\right)^{p(q-1)/q(p-1)}\left(\frac{1}{|Q|}\int_Q \omega^{p(s/p)'}\right)^{1/(s/p)'(p-1)} \\ &\leq [\omega^q]^{p/q(p-1)}_{RH_{(p/q)(s/p)'}}\sup_Q\left(\frac{1}{|Q|}\int_Q \omega^{-q/(q-1)}\right)^{p(q-1)/q(p-1)}\left(\frac{1}{|Q|}\int_Q \omega^{q}\right)^{p/q(p-1)} \\ &\leq [\omega^q]^{p/q(p-1)}_{RH_{(p/q)(s/p)'}}[\omega^q]^{p/q(p-1)}_{A_q} \\ &\leq [\omega^q]^{p/q(p-1)}_{RH_{(p/q)(s/p)'}}[\omega^q]^{p/q(p-1)}_{A_{q/r}}. \end{aligned}$$

The Besov norm of $a(x,D)$ is handled by

$$\begin{aligned} & I_1 + I_2 \\ := & \left(\sum_{k\geq 0} 2^{k\kappa\sigma}\left(\sum_{0\leq j\leq k}||\phi_k * a(x,D)(\phi_j * f)||_{L^p(\omega^p)}\right)^{\sigma}\right)^{1/\sigma} + \left(\sum_{k\geq 0} 2^{k\kappa\sigma}\left(\sum_{k<j}||\phi_k * a(x,D)(\phi_j * f)||_{L^p(\omega^p)}\right)^{\sigma}\right)^{1/\sigma}. \end{aligned}$$

Our purpose is to control I_1 and I_2 by $||f||_{B^{\kappa+\tilde{\kappa}}_{q,\sigma}(\omega^q)}$. First, we give an estimation of I_1. From the observation above, we obtain

$$\begin{aligned} I_1 &\lesssim \left(\sum_{k\geq 0} 2^{k\kappa\sigma - k\ell\sigma} \left(\sum_{0\leq j\leq k} 2^{j\ell + j\tilde{\kappa}} ||\phi_j * f||_{L^q(\omega^q)} \right)^{\sigma} \right)^{1/\sigma} \\ &\leq \left(\sum_{k\geq 0} 2^{k\kappa\sigma - k\ell\sigma} \left(\sum_{-k\leq j\leq 0} 2^{(j+k)\ell + (j+k)\tilde{\kappa}} ||\phi_{j+k} * f||_{L^q(\omega^q)} \right)^{\sigma} \right)^{1/\sigma} \\ &\leq \sum_{j\leq 0} 2^{j\ell + j\tilde{\kappa}} \left(\sum_{k\geq -j} 2^{k(\kappa+\tilde{\kappa})\sigma} ||\phi_{j+k} * f||^{\sigma}_{L^q(\omega^q)} \right)^{1/\sigma} \\ &= \sum_{j\leq 0} 2^{j\ell + j\tilde{\kappa} - j(\kappa+\tilde{\kappa})} \left(\sum_{k\geq 0} 2^{k(\kappa+\tilde{\kappa})\sigma} ||\phi_k * f||^{\sigma}_{L^q(\omega^q)} \right)^{1/\sigma} \\ &\lesssim ||f||_{B^{\kappa+\tilde{\kappa}}_{q,\sigma}(\omega^q)} \end{aligned}$$

by taking sufficiently large ℓ. On the other hands, the same calculation gives

$$\begin{aligned} I_2 &\lesssim \left(\sum_{k\geq 0} 2^{k\kappa\sigma - k\ell\sigma} \left(\sum_{k< j} 2^{j\ell + j\tilde{\kappa}} ||\phi_j * f||_{L^q(\omega^q)} \right)^{\sigma} \right)^{1/\sigma} \\ &\leq \left(\sum_{k\geq 0} 2^{k\kappa\sigma - k\ell\sigma} \left(\sum_{j>0} 2^{(j+k)\ell + (j+k)\tilde{\kappa}} ||\phi_{j+k} * f||_{L^q(\omega^q)} \right)^{\sigma} \right)^{1/\sigma} \\ &\leq \sum_{j>0} 2^{j\ell + j\tilde{\kappa}} \left(\sum_{k\geq 0} 2^{k(\kappa+\tilde{\kappa})\sigma} ||\phi_{j+k} * f||^{\sigma}_{L^q(\omega^q)} \right)^{1/\sigma} \\ &= \sum_{j>0} 2^{j\ell + j\tilde{\kappa} - j(\kappa+\tilde{\kappa})} \left(\sum_{k\geq 0} 2^{k(\kappa+\tilde{\kappa})\sigma} ||\phi_k * f||^{\sigma}_{L^q(\omega^q)} \right)^{1/\sigma} \\ &\lesssim ||f||_{B^{\kappa+\tilde{\kappa}}_{q,\sigma}(\omega^q)} \end{aligned}$$

by taking $\ell \ll -1$. Hence, we complete the proof in the case of $1 \leq \sigma < \infty$. To complete the proof, we treat the case of $0 < \sigma < 1$. However, the result with this case is proved in a same manner by using σ-triangle inequality on ℓ^σ. □

3.3 The special case of pseudodifferential operators

For a given $-1 \leq \rho < 1$, $U_\rho f$ denotes the solution of

$$\begin{cases} i\partial_t u + (-\Delta)^{(1-\rho)/2} u = 0, \\ u(0) = f. \end{cases}$$

U_ρ with $0 \leq \rho < 1$ is a pseudodifferential operators belonging to $S^0_{\rho,0}$ (for large frequency), and therefore gives sparse bounds in Theorem 3.1 and Theorem 3.2. However, we can improve the above results:

Theorem 3.3. *Let $1 \leq r \leq 2$ and $-1 \leq \rho < 1$.*
(i) *Given $\rho \neq 0$, $1/r + 1/2 < 1/\alpha < 2/r$ and $f, g \in \mathscr{S}$, there exists a sequence of sparse families $\{\mathcal{S}_j\}_{j=0,1,\cdots}$ such that*

$$|\langle U_\rho f(t), g\rangle| \lesssim t^{n(1/r + /2 - 1/\alpha)} \liminf_{R\to\infty} \sum_{j\geq 0} 2^{j\kappa_4} \Lambda^{\alpha}_{\mathcal{S}_j, r, r}((\phi_j * f)1_{Q_R}, g),$$

where $\kappa_4 = n(1-\rho)(1/r - 1/2) + \rho n(1/\alpha - 1)$.
(ii) *Given $\alpha \in \mathbb{R}$ such that*

$$\frac{n+1}{rn} + \frac{n-1}{2n} < \frac{1}{\alpha} < \frac{2}{r},$$

and $f, g \in \mathscr{S}$, there exists a sequence of sparse families $\{\mathcal{S}_j\}_{j=0,1,\cdots}$ such that

$$|\langle U_0 f(t), g\rangle| \lesssim t^{(n+1)(1/r-1/2)-n(1/\alpha-1)} \liminf_{R\to\infty} \sum_{j\geq 0} 2^{j\kappa_5} \Lambda^{\alpha}_{\mathcal{S}_j,r,r}((\phi_j * f)1_{Q_R}, g),$$

where $\kappa_5 = (n+1)(1/r - 1/2)$.

To prove the theorem, we use the following stationary phase type estimate which was proved by Domar [15] (cf. also [9], [31]).

Lemma 3.3. *Suppose that Φ is a real valued smooth funcion on $\mathbb{R}^n \setminus \{0\}$, and Ψ is a smooth function supported in the set $\{\xi\ ;\ 1/2 \leq |\xi| \leq 2\}$. Assume that the rank of $(\partial_i\partial_j\Phi(\xi))_{i,j}$ is at least $\Sigma(\geq 0)$ on the set $\{\xi\ ;\ 1/2 \leq |\xi| \leq 2\}$. Then, there exists $M > 0$ such that*

$$\left|\int e^{ix\xi+i\lambda\Phi(\xi)}\Psi(\xi)d\xi\right| \lesssim \lambda^{-\Sigma/2}||\Psi||_{C^M}$$

for any $\lambda > 0$.

Let us prove Theorem 3.3.

Proof. (i) It suffices to prove the pointwise estimate

$$M_{U_{\rho,j},\infty} f(t,x) \lesssim t^{-n(\gamma-1/2)} 2^{jn(1-\rho)/2+jn\rho\gamma} M^{\gamma} f(x)$$

for any $1/2 < \gamma < 1$, where $U_{\rho,j} f = U_\rho(\phi_j * f)$. Take any cube Q and any $x, z \in Q$. First, we consider the case $j \geq 1$ and $2^{-j(1-\rho)} \leq t$. We integrate by parts $N \in \mathbb{N}$ times to obtain

$$\begin{aligned}
&|U_{\rho,j}(f 1_{(3Q)^c})(t,z)| \\
\lesssim\ & \int_{(3Q)^c} \left(1 + 2^{2j\rho N} t^{-2N}|z-y|^{2N}\right)^{-1} |f(y)| \left|\int e^{i(z-y)\xi}(1 + 2^{2j\rho N}t^{-2N}(-\Delta)^N)(e^{it|\xi|^{1-\rho}}\hat{\phi}_j(\xi))d\xi\right| dy \\
\lesssim\ & 2^{-j\rho n(1-\gamma)} t^{n(1-\gamma)} M^{\gamma} f(x) \sup_{w\in\mathbb{R}^n}\left|\int e^{iw\xi}(1 + 2^{2j\rho N}t^{-2N}(-\Delta)^N)(e^{it|\xi|^{1-\rho}}\hat{\phi}_j(\xi))d\xi\right|.
\end{aligned}$$

To obtain desired pointwise estimate, we need to prove

$$\sup_{w\in\mathbb{R}^n}\left|\int e^{iw\xi}(1 + 2^{2j\rho N}t^{-2N}(-\Delta)^N)(e^{it|\xi|^{1-\rho}}\phi_j(\xi))d\xi dy\right| \lesssim 2^{jn(1+\rho)/2} t^{-n/2}.$$

By the Leibniz formula, we have

$$\begin{aligned}
\Delta^N((e^{it|\xi|^{1-\rho}}\hat{\phi}_j(\xi)) &= \sum_{|\alpha|=2N}\sum_{\beta\leq\alpha}(\partial^{\alpha-\beta}e^{it|\xi|^{1-\rho}})(\partial^{\beta}\hat{\phi}_j(\xi)) \\
&= e^{it|\xi|^{1-\rho}} \sum_{|\alpha|=2N}\sum_{\beta\leq\alpha}(P_{\alpha,\beta}(\xi))(\partial^{\beta}\hat{\phi}_j(\xi)),
\end{aligned}$$

where $P_{\alpha,\beta}$ denote a functions such that

$$||\partial^{\sigma} P_{\alpha,\beta}(2^j\cdot)||_{L^\infty} \lesssim (2^{-j} + t2^{-j\rho})^{2N-|\beta|}$$

on support of $\hat{\psi}$ for any $\sigma \in \mathbb{N}^n$. By using lemma 3.3, we have

$$\begin{aligned}
&\left|\int e^{iw\xi}(1 + 2^{2j\rho N}t^{-2N}(-\Delta)^N)(e^{it|\xi|^{1-\rho}}\hat{\phi}_j(\xi))d\xi\right| \\
\lesssim\ & \left|\int e^{iw\xi+it|\xi|^{1-\rho}}\hat{\phi}_j(\xi)d\xi\right| + 2^{2j\rho N}t^{-2N}\sum_{|\alpha|=2N}\sum_{\beta\leq\alpha}\left|\int e^{iw\xi+it|\xi|^{1-\rho}}P_{\alpha.\beta}(\xi)\partial^{\beta}\hat{\phi}_j(\xi)d\xi\right| \\
\lesssim\ & 2^{jn}\left|\int e^{iw\xi+it2^{j(1-\rho)}|\xi|^{1-\rho}}\hat{\psi}(\xi)d\xi\right| + 2^{jn+2j\rho N}t^{-2N}\sum_{|\alpha|=2N}\left|\sum_{\beta\leq\alpha}2^{-j|\beta|}\int e^{iw\xi+it2^{j(1-\rho)}|\xi|^{1-\rho}}P_{\alpha.\beta}(2^j\xi)\partial^{\beta}\hat{\psi}(\xi)d\xi\right| \\
\lesssim\ & 2^{jn(1+\rho)/2}t^{-n/2} + 2^{jn(1+\rho)/2+2j\rho N}t^{-n/2-2N}\sum_{|\alpha|=2N}\sum_{\beta\leq\alpha}2^{-j|\beta|}(2^{-j}+t2^{-j\rho})^{2N-|\beta|} \\
\lesssim\ & 2^{jn(1+\rho)/2}t^{-n/2}.
\end{aligned}$$

Here, the last inequality follows from

$$\sum_{|\alpha|=2N}\sum_{\beta\le\alpha}2^{-j|\beta|}(2^{-j}+t2^{-j\rho})^{2N-|\beta|}\lesssim t^{2N}2^{-2j\rho N}\sum_{|\alpha|=2N}\sum_{\beta\le\alpha}t^{-|\beta|}2^{-j|\beta|+j\rho|\beta|}\lesssim t^{2N}2^{-2j\rho N}.$$

The desired estimate with $j\ge 1$ and $2^{j(1-\rho)}\le t^{-1}$ is obtained from

$$\begin{aligned}
&|U_{\rho,j}(f1_{(3Q)^c})(t,z)|\\
\lesssim\;& \int_{(3Q)^c}\left(1+2^{2jN}|z-y|^{2N}\right)^{-1}|f(y)|\left|\int e^{i(z-y)\xi}(1+2^{2jN}(-\Delta)^N)(e^{it|\xi|^{1-\rho}}\hat{\phi}_j(\xi))d\xi\right|dy\\
\lesssim\;& 2^{-jn(1-\gamma)}M^\gamma f(x)\sup_{w\in\mathbb{R}^n}\left|\int e^{iw\xi}(1+2^{2jN}(-\Delta)^N)(e^{it|\xi|^{1-\rho}}\hat{\phi}_j(\xi))d\xi\right|\\
\lesssim\;& 2^{jn\gamma}M^\gamma f(x)\\
\le\;& t^{-n(\gamma-1/2)}2^{jn(1-\rho)/2+jn\rho\gamma}M^\gamma f(x).
\end{aligned}$$

Here, we use the condition $\gamma\ge 1/2$ to obtain

$$2^{jn\gamma}=2^{jn(1-\rho)/2+jn\rho\gamma}2^{jn(1-\rho)(\gamma-1/2)}\le t^{-n(\gamma-1/2)}2^{jn(1-\rho)/2+jn\rho\gamma}.$$

When $j=0$, we recall $\phi_0=\sum_{\ell\le 0}\psi_\ell$ and obtain

$$\begin{aligned}
&|U_{\rho,0}(f1_{(3Q)^c})(t,z)|\\
\le\;& \sum_{\ell\le 0}\left|\int e^{i(z-y)\xi}\hat{\psi}_\ell(\xi)f(y)1_{(3Q)^c}(y)dyd\xi\right|\\
\lesssim\;& \sum_{\ell\le 0}\int_{(3Q)^c}\left(1+2^{2\ell N}\tau^{-2N}|z-y|^{2N}\right)^{-1}|f(y)|\left|\int e^{i(z-y)\xi}(1+2^{2\ell N}\tau^{-2N}(-\Delta)^N)(e^{it|\xi|^{1-\rho}}\hat{\psi}_\ell(\xi))d\xi\right|dy\\
\lesssim\;& \sum_{\ell\le 0}2^{-\ell n(1-\gamma)}\tau^{n(1-\gamma)}M^\gamma f(x)\sup_{w\in\mathbb{R}^n}\left|\int e^{iw\xi}(1+2^{2\ell N}\tau^{-2N}(-\Delta)^N)(e^{it|\xi|^{1-\rho}}\hat{\psi}_\ell(\xi))d\xi\right|,
\end{aligned}$$

where $\tau=\max\{1,t\}$. Since $||\partial^\sigma P_{\alpha,\beta}(2^\ell\cdot)||_{L^\infty}\lesssim\tau^{|\alpha|-|\beta|}2^{-\ell(|\alpha|-|\beta|)}$, one has

$$\begin{aligned}
|U_{\rho,0}(f1_{(3Q)^c})(t,z)| &\lesssim \left(\sum_{\ell\le 0}2^{\ell n(\gamma-1/2)}\right)\tau^{n(1-\gamma)-n/2}M^\gamma f(x)\\
&\lesssim t^{-n(\gamma-1/2)}M^\gamma f(x).
\end{aligned}$$

(ii) It suffices to prove the pointwise estimate

$$M_{U_{0,j},\infty}f(t,x)\lesssim t^{-n(\gamma-1/2)+1/2}2^{j(n+1)/2}M^\gamma f(x)$$

for any $(n+1)/2n<\gamma<1$. Take any cube Q and any $x,z\in Q$. First, we consider the case $j\ge 1$ and $2^{-j}\le t$. We integrate by parts $N\in\mathbb{N}$ times to obtain

$$|U_{0,j}(f1_{(3Q)^c})(t,z)|\lesssim t^{n(1-\gamma)}M^\gamma f(x)\sup_{w\in\mathbb{R}^n}\left|\int e^{iw\xi}(1+t^{-2N}(-\Delta)^N)(e^{it|\xi|}\hat{\phi}_j(\xi))d\xi\right|.$$

By using Lemma 3.3, we have

$$\begin{aligned}
&\left|\int e^{iw\xi}(1+t^{-2N}(-\Delta)^N)(e^{it|\xi|}\hat{\phi}_j(\xi))d\xi\right|\\
\lesssim\;& \left|\int e^{iw\xi+it|\xi|}\hat{\phi}_j(\xi)d\xi\right|+t^{-2N}\sum_{|\alpha|=2N}\sum_{\beta\le\alpha}\left|\int e^{iw\xi+it|\xi|}P_{\alpha.\beta}(\xi)\partial^\beta\hat{\phi}_j(\xi)d\xi\right|\\
\lesssim\;& 2^{jn}\left|\int e^{iw\xi+it2^j|\xi|}\hat{\psi}(\xi)d\xi\right|+2^{jn}t^{-2N}\sum_{|\alpha|=2N}\sum_{\beta\le\alpha}2^{-j|\beta|}\left|\int e^{iw\xi+it2^j|\xi|}P_{\alpha.\beta}(2^j\xi)\partial^\beta\hat{\psi}(\xi)d\xi\right|\\
\lesssim\;& 2^{j(n+1)/2}t^{-(n-1)/2}+2^{j(n+1)/2}t^{-(n-1)/2-2N}\sum_{|\alpha|=2N}\sum_{\beta\le\alpha}2^{-j|\beta|}(2^{-j}+t)^{2N-|\beta|}\\
\lesssim\;& 2^{j(n+1)/2}t^{-(n-1)/2}
\end{aligned}$$

for any $w \in \mathbb{R}^n$. The desired estimate with $j \geq 1$ and $2^j \leq t^{-1}$ is obtained from

$$\begin{aligned}
&|U_{0,j}(f 1_{(3Q)^c})(t,z)| \\
\lesssim\ & \int_{(3Q)^c} (1+2^{2jN}|z-y|^{2N})^{-1}|f(y)| \left| \int e^{i(z-y)\xi}(1+2^{2jN}(-\Delta)^N)(e^{it|\xi|}\hat{\phi}_j(\xi))d\xi \right| dy \\
\lesssim\ & 2^{-jn(1-\gamma)} M^\gamma f(x) \sup_{w\in\mathbb{R}^n} \left| \int e^{iw\xi}(1+2^{2jN}(-\Delta)^N)(e^{it|\xi|}\hat{\phi}_j(\xi))d\xi \right| \\
\lesssim\ & 2^{jn\gamma} M^\gamma f(x) \\
\leq\ & t^{-n(\gamma-1/2)+1/2} 2^{j(n+1)/2} M^\gamma f(x).
\end{aligned}$$

Here, we use the condition $\gamma \geq (n+1)/2n$ to obtain

$$2^{jn\gamma} = 2^{-j(n+1)/2+jn\gamma} 2^{j(n+1)/2} \leq t^{-n(\gamma-1/2)+1/2} 2^{j(n+1)/2}.$$

When $j=0$, we recall $\phi_0 = \sum_{\ell \leq 0} \psi_\ell$ and obtain

$$\begin{aligned}
&|U_{0,0}(f 1_{(3Q)^c})(t,z)| \\
\leq\ & \sum_{\ell\leq 0} \left| \int e^{i(z-y)\xi} a(z,\xi)\hat{\psi}_\ell(\xi) f(y) 1_{(3Q)^c}(y) dy d\xi \right| \\
\lesssim\ & \sum_{\ell\leq 0} \int_{(3Q)^c} (1+2^{2\ell N}\tau^{-2N}|z-y|^{2N})^{-1}|f(y)| \left| \int e^{i(z-y)\xi}(1+2^{2\ell N}\tau^{-2N}(-\Delta)^N)(e^{it|\xi|}\hat{\psi}_\ell(\xi))d\xi \right| dy \\
\lesssim\ & \sum_{\ell\leq 0} 2^{-\ell n(1-\gamma)}\tau^{n(1-\gamma)} M^\gamma f(x) \sup_{w\in\mathbb{R}^n} \left| \int e^{iw\xi}(1+2^{2\ell N}\tau^{-2N}(-\Delta)^N)(e^{it|\xi|}\hat{\psi}_\ell(\xi))d\xi \right|,
\end{aligned}$$

where $\tau = \max\{1,t\}$. Since $||\partial^\sigma P_{\alpha,\beta}(2^\ell \cdot)||_{L^\infty} \lesssim \tau^{|\alpha|-|\beta|} 2^{-\ell(|\alpha|-|\beta|)}$, one has

$$\begin{aligned}
|U_{0,0}(f 1_{(3Q)^c})(t,z)| &\lesssim \left(\sum_{\ell\leq 0} 2^{\ell n(\gamma-(n+1)/2n)} \right) \tau^{n(1-\gamma)-(n+1)/2} M^\gamma f(x) \\
&\lesssim t^{-n(\gamma-1/2)+1/2} M^\gamma f(x).
\end{aligned}$$

□

Theorem 3.3 and Proposition 3.2 give a boundness of U_ρ on weighted Besov spaces.

Corollary 3.4. *Let $-1 \leq \rho < 1$, $1 < q \leq 2 \leq p \leq q' < \infty$ and $\omega^q \in A_{q/r} \cap RH_{(p/q)(r'/p)'}$ with some $r \in [1,q)$.*
(i) *If $\rho \neq 0$ and*

$$\frac{1}{r} - \frac{1}{2} \leq \frac{1}{q} - \frac{1}{p},$$

then for any $\kappa \in \mathbb{R}$ and $0 < \sigma \leq \infty$, $U_\rho(t)$ is a bounded operator from $B^{\kappa+\tilde{\kappa}_4}_{q,\sigma}(\omega^q)$ to $B^\kappa_{p,\sigma}(\omega^p)$ where

$$\tilde{\kappa}_4 = n(1-\rho)\left(\frac{1}{r}-\frac{1}{2}\right) + \rho n\left(\frac{1}{q}-\frac{1}{p}\right).$$

Furthermore, we have

$$||U_\rho(t)||_{B^{\kappa+\tilde{\kappa}_4}_{q,\sigma}(\omega^q)\to B^\kappa_{p,\sigma}(\omega^p)} \lesssim t^{-n((1/q-1/p)-(1/r-1/2))}([\omega^q]_{A_{q/r}}[\omega^q]_{RH_{(p/q)(r'/p)'}})^\delta,$$

with δ in Proposition 3.2.
(ii) *If*

$$\frac{n+1}{n}\left(\frac{1}{r}-\frac{1}{2}\right) \leq \frac{1}{q} - \frac{1}{p},$$

then for any $\kappa \in \mathbb{R}$ and $0 < \sigma \leq \infty$, $U_\rho(t)$ is a bounded operator from $B^{\kappa+\tilde{\kappa}_5}_{q,\sigma}(\omega^q)$ to $B^\kappa_{p,\sigma}(\omega^p)$ where

$$\tilde{\kappa}_5 = (n+1)\left(\frac{1}{r}-\frac{1}{2}\right).$$

Furthermore, we have

$$||U_\rho(t)||_{B^{\kappa+\tilde{\kappa}_5}_{q,\sigma}(\omega^q)\to B^\kappa_{p,\sigma}(\omega^p)} \lesssim t^{-\{n(1/q-1/p)-(n+1)(1/r-1/2)\}}([\omega^q]_{A_{q/r}}[\omega^q]_{RH_{(p/q)(r'/p)'}})^\delta.$$

3.4 A sharpness of weighted boundedness of pseudodifferential operators

In previous sections and subsections, we obtain some weighted inequalities for pseudodifferential operators and the time evolution $U_\rho(t)$ of dispersive equations. In this subsection, we insure a sharpness of some of these inequalities as follows:

Proposition 3.3. *Let $1 < q \le p \le q' < \infty$ and $\gamma \in [1,\infty)$, and $a(\xi) = e^{i|\xi|^{1-\rho}}|\xi|^m$ with $m \in \mathbb{R}$ and $0 < \rho \le 1$. If we have $L^q(|\cdot|^{qs})$-$L^p(|\cdot|^{ps})$ boundedness of $a(D)$ for any $s \in (-n/\gamma, 0)$, then we have*

$$m \le -n(1-\rho)\left(\frac{1}{2}-\frac{1}{p}\right) - \rho n\left(\frac{1}{q}-\frac{1}{p}\right) - \frac{n(1-\rho)}{\gamma}.$$

In particular, if we have $L^q(\omega^q)$-$L^p(\omega^p)$ boundedness of $a(D)$ with any $\omega^q \in RH_{(p/q)(r'/p)'}$ for some $r \in [1,q)$, then we have

$$m \le -n(1-\rho)\left(\frac{1}{r}-\frac{1}{2}\right) - \rho n\left(\frac{1}{q}-\frac{1}{p}\right).$$

Proof. Our assumption gives

$$|\langle a(D)f, g\rangle| \lesssim ||f||_{L^q(|\cdot|^{qs})}||g||_{L^p(|\cdot|^{-p's})}$$

for any $s \in (-n/\gamma, 0)$. We take a nonnegative function $\phi \in C_0^\infty$ such that *supp* $\phi \subset \{1/4 \le |\xi| \le 2\}$ and $\phi = 1$ *on* $\{1/2 \le |\xi| \le 1\}$, and let

$$\hat{f}(\xi) = e^{-i|\xi|^{1-\rho}}\phi(\xi/R),$$

and

$$\check{g}(\xi) = \phi(\xi/R)$$

for any $R > 0$. Then, we have

$$\begin{aligned} |\langle a(D)f, g\rangle| &= \left|\int |\xi|^m \phi(\xi/R)\phi(\xi/R)d\xi\right| \\ &\sim R^{m+n}. \end{aligned}$$

On the other hands, we have

$$|f(x)| \lesssim \min\{R^{n(1+\rho)/2}, R^{n(1+\rho)/2-2\rho N}|x|^{-2N}\}$$

for any $N \in \mathbb{N}$. In fact, Lemma 3.3 gives

$$\begin{aligned} |f(x)| &= \left|\int e^{ix\xi - i|\xi|^{1-\rho}}\phi(\xi/R)d\xi\right| \\ &\le R^n \sup_z \left|\int e^{iz\xi + iR^{1-\rho}|\xi|^{1-\rho}}\phi(\xi)d\xi\right| \\ &\lesssim R^n R^{-n(1-\rho)/2} \\ &= R^{n(1+\rho)/2}. \end{aligned}$$

As for the second estimates, we have

$$\begin{aligned} |f(x)| &= |x|^{-2N}\left|\int \Delta_\xi^N e^{ix\xi - i|\xi|^{1-\rho}}\phi(\xi/R)d\xi\right| \\ &= |x|^{-2N}\left|\int e^{ix\xi}\Delta_\xi^N(e^{-i|\xi|^{1-\rho}}\phi(\xi/R))d\xi\right| \\ &\le |x|^{-2N}\sum_{|\alpha|=2N}\sum_{\beta\le\alpha} R^{-|\beta|}\left|\int e^{ix\xi - it|\xi|^{1-\rho}}(P_{\alpha,\beta}(\xi))(\partial^\beta\phi)(\xi/R)d\xi\right| \end{aligned}$$

where $P_{\alpha,\beta}$ denote a functions such that

$$||\partial^\sigma P_{\alpha,\beta}(R\cdot)||_{L^\infty} \lesssim R^{-2\rho N+\rho|\beta|}$$

on support of ϕ for any $\sigma \in \mathbb{N}^n$. From this and Lemma 3.3, we obtain the desired estimate. Therefore, we have

$$
\begin{aligned}
||f||_{L^q(|\cdot|^{qs})} &\leq \left(\int_{|x|\leq R^{-\rho}} |f(x)|^q |x|^{qs} dx\right)^{1/q} + \left(\int_{|x|\geq R^{-\rho}} |f(x)|^q |x|^{qs} dx\right)^{1/q} \\
&\leq R^{n(1+\rho)/2}\left(\int_{|x|\leq R^{-\rho}} |x|^{qs} dx\right)^{1/q} + R^{n(1+\rho)/2-2\rho N}\left(\int_{|x|\geq R^{-\rho}} |x|^{qs-2qN} dx\right)^{1/q} \\
&\lesssim R^{n(1+\rho)/2-\rho n/q-\rho s},
\end{aligned}
$$

and

$$
\begin{aligned}
||g||_{L^{p'}(|\cdot|^{-p's})} &= R^n\left(\int |\hat{\phi}(Rx)|^{p'} |x|^{-p's} dx\right)^{1/p'} \\
&\lesssim R^{n-n/p'+s} \\
&= R^{n/p+s}.
\end{aligned}
$$

From these observations, we obtain

$$
\begin{aligned}
R^{m+n} &\lesssim R^{n(1+\rho)/2-\rho n/q-\rho s} R^{n/p+s} \\
R^m &\lesssim R^{-n(1/2-1/p)+n\rho(1/2-1/q)+s(1-\rho)} \\
R^m &\lesssim R^{-n(1-\rho)(1/2-1/p)-\rho n(1/q-1/p)+s(1-\rho)} \\
m &\leq -n(1-\rho)(1/2-1/p)-\rho n(1/q-1/p)-n(1-\rho)/\gamma
\end{aligned}
$$

Here, we take the infimum all over $s \in (-n/\gamma, 0)$ to obtain the final inequality. In particular, we have $|\cdot|^{qs} \in RH_{(p/q)(r'/p)'}$ with $s \in (-n/p(r'/p)', 0)$, that means

$$
\begin{aligned}
m &\leq -n(1-\rho)(1/2-1/p)-\rho n(1/q-1/p)-n(1-\rho)/p(r'/p)' \\
&= -n(1-\rho)(1/2-1/p)-\rho n(1/q-1/p)-n(1-\rho)(1-p/r')/p \\
&= -n(1-\rho)(1/r-1/2)-\rho n(1/q-1/p).
\end{aligned}
$$

by taking $\gamma = p(r'/p)'$. □

Remark 3.3. *Since* $e^{i|\xi|^{1-\rho}}|\xi|^m \notin S^m_{\rho,0}$, *Proposition 3.3 cannot be applied to the pseudodifferential operators associated with symbols belonging to the Hörmander class directly. However, by the same proof of the proposition, it holds with* $a \in S^m_{\rho,0}$ *such that* $a(\xi) = e^{i|\xi|^{1-\rho}}|\xi|^m$ *for any* $|\xi| > 1$, *that means a sharpness of weighted inequalities in Theorem 1.2 and* (i) *of Corollary 3.3.*

A Appendix A

To see the proof of Corollary 3.3, the operator norms of $a(x, D)$ on weighted Besov spaces are controlled by

$$
([\omega^q]_{A_{q/r}}[\omega^q]_{RH_{(p/q)(r'/p)'}})^{\delta}[\omega^{-p'}]_{A_{p'}}.
$$

However, we can eliminate the factor $[\omega^{-p'}]_{A_{p'}}$ by having the sparse form bounds $\phi_k * a(x, D)(\phi_j * \cdot)$ directly.

Proposition A.1. (i) *Let* $2 \leq s \leq \infty$ *and* $1 \leq 1/\alpha < 3/2 - 1/s$, *and* $a \in S^m_{\rho,\delta}$ *with* $m \leq 0$, $0 \leq \delta < \rho \leq 1$. *Then for any* $f, g \in \mathscr{S}$ *and* $j, k \in \mathbb{Z}_{\geq 0}$, *there exists a sequence of sparse families* $\{\mathcal{S}_j\}_{j=0,1,\cdots}$ *such that*

$$
|\langle \phi_k * a(x, D)(\phi_j * f), g\rangle| \lesssim 2^{-k\ell}2^{j\ell+j\kappa_1} \liminf_{R\to\infty} \Lambda^{\alpha}_{\mathcal{S},2,s'}((\phi_j * f)1_{Q_R}, g).
$$

(ii) *Let* $2 \leq s \leq \infty$ *and* $1 \leq 1/\alpha < 2/s'$, *and* $a \in S^m_{\rho,\delta}$ *with* $m \leq 0$, $0 \leq \delta < \rho \leq 1$. *Then for any* $f, g \in \mathscr{S}$, *there exists a sequence of sparse families* $\{\mathcal{S}_j\}_{j=0,1,\cdots}$ *such that*

$$
|\langle \phi_k * a(x, D)(\phi_j * f), g\rangle| \lesssim 2^{-k\ell}2^{j\ell+j\kappa_2} \liminf_{R\to\infty} \Lambda^{\alpha}_{\mathcal{S},s',s'}((\phi_j * f)1_{Q_R}, g).
$$

(iii) *Let* $1 \leq s' \leq r \leq 2 \leq s \leq \infty$ *and* $a \in S^m_{\rho,\delta}$ *with* $m \leq 0$, $0 \leq \delta < \rho \leq 1$. *Then for any* $f, g \in \mathscr{S}$, *there exists a sequence of sparse families* $\{\mathcal{S}_j\}_{j=0,1,\cdots}$ *such that*

$$
|\langle \phi_k * a(x, D)(\phi_j * f), g\rangle| \lesssim 2^{-k\ell}2^{j\ell+j\kappa_3} \liminf_{R\to\infty} \Lambda_{\mathcal{S},r,s'}((\phi_j * f)1_{Q_R}, g).
$$

Proof. We put

$$T_{j,k}f := \phi_k * a(x,D)(\phi_j * f).$$

Here, we remark that

$$\begin{aligned} T_{j,k}f &= (\langle D\rangle^{-\ell}\phi_k) * (\langle D\rangle^{\ell} a(x,D)(\phi_j * f)) \\ &= (\langle D\rangle^{-\ell}\phi_k) * b_\ell(x,D)(\phi_j * f), \end{aligned}$$

with some $b_\ell \in S^{m+\ell}_{\rho,\delta}$. For any cube Q and $x \in Q$, one has

$$\begin{aligned} & ||T_{j,k}(f1_{(3Q)^c})||_{L^\infty(Q)} \\ \leq\; & ||(\langle D\rangle^{-\ell}\phi_k) * [1_{(2Q)^c} b_\ell(x,D)(\phi_j * (f1_{(3Q)^c}))||_{L^\infty(Q)} + 2^{-k\ell}||b_\ell(x,D)(\phi_j * (f1_{(3Q)^c}))||_{L^\infty(2Q)} \\ \leq\; & f_0(x) + f_1(x), \end{aligned}$$

where

$$\begin{aligned} f_0(x) &:= \sup_{Q\in x} ||(\langle D\rangle^{-\ell}\phi_k) * [1_{(2Q)^c} b_\ell(x,D)(\phi_j * (f1_{(3Q)^c}))||_{L^\infty(Q)}, \\ f_1(x) &:= \sup_{Q\ni x} 2^{-k\ell}||b_\ell(x,D)(\phi_j * (f1_{(3Q)^c}))||_{L^\infty(2Q)}. \end{aligned}$$

(i) Now, we have

$$\begin{aligned} f_0(x) &\lesssim 2^{-k\ell}\sup_{Q\ni x} M[b_\ell(x,D)(\phi_j * (f1_{(3Q)^c}))](x) \\ &\lesssim 2^{-k\ell}2^{jm+j\ell+jn(1-\rho(1-\gamma))/2} MM_2^\gamma f(x) \end{aligned}$$

for any $0 \leq \gamma < 1$. By using the argument in the proof of Theorem 3.1, it is not hard to see that

$$f_1(x) \lesssim 2^{-k\ell}2^{jm+j\ell+jn(1-\rho(1-\gamma))/2} M_2^\gamma f(x).$$

Therefore, we obtain

$$||M_{T_{j,k},\infty} f||_{L^2\to L^{p_0,\infty}} \lesssim 2^{-k\ell}2^{jm+j\ell+jn(1-\rho)/2+jn\rho(1/2-1/p_0)}$$

for any $p_0 \geq 2$. On the other hands, we have

$$||T_{j,k}||_{L^2\to L^{p_1}} \lesssim 2^{-k\ell}2^{jm+j\ell+jn(1/2-1/p)} \quad \textit{and} \quad ||M_{T_{j,k},p_1}||_{L^2\to L^{p_1,\infty}} \lesssim 2^{-k\ell}2^{jm+j\ell+jn(1/2-1/p_1)}$$

for any $p_1 \geq 2$. By interpolating them, we have the desired sparse bounds.
(ii), (iii) It suffices to prove the pointwise estimate

$$f_0(x) + f_1(x) \lesssim 2^{-k\ell}2^{jm+j\ell+jn(1-\rho(1-\gamma))} M^\gamma f(x).$$

We just handle the $f_0(x)$ since the estimate of $f_1(x)$ be obtained immediately from the proof of Theorem 3.2. For any $N \in \mathbb{N}$ and $h \in L^1$, we have

$$\begin{aligned} &(\langle D\rangle^{-\ell}\phi_k) * h(z) \lesssim 2^{-k\ell+kn}\int \left(1 + 2^{2kN}|z-y|^{2N}\right)^{-1}|h(y)|dy, \\ &|b_\ell(y,D)(\phi_j * (f1_{(3Q)^c}))(y)| \lesssim 2^{j\ell+jm+jn}\int \left(1 + 2^{2j\rho N}|y-w|^{2N}\right)^{-1}|f(w)|1_{(3Q)^c}(w)dw. \end{aligned}$$

Hence, we obtain

$$f_0(x) \lesssim 2^{-k\ell+kn}2^{j\ell+jm+jn}\sup_{Q\ni x}||\Phi * (|f|1_{(3Q)^c})||_{L^\infty(Q)},$$

where Φ denotes the radial function

$$\Phi(z) = \int \frac{1}{1+2^{2j\rho N}|z-y|^{2N}} \cdot \frac{1}{1+2^{2kN}|y|^{2N}}dy.$$

To complete the proof, we decompose the integral region:

$$\Phi(z) = \int_{2|y|<|z|} + \int_{2|z|\leq|y|} + \int_{|z|/2\leq|y|<2|z|}.$$

Since $|z-y| \gtrsim |z|$ under the $2|y|<|z|$ or $2|z| \le |y|$, one has

$$\int_{2|y|<|z|} + \int_{2|z|\le|y|} \lesssim \frac{2^{-kn}}{1+2^{2j\rho N}|z|^{2N}}.$$

Furthermore, it is not hard to see that

$$\int_{|z|/2\le|y|<2|z|} \lesssim \min\left\{\frac{2^{-j\rho n}}{1+2^{2kN}|z|^{2N}}, \frac{|z|^n}{1+2^{2kN}|z|^{2N}}\right\}.$$

From them, for any $k \le j\rho$, we have

$$\begin{aligned} f_0(x) &\lesssim 2^{-k\ell}2^{j\ell+jm+jn(1-\rho)}(2^{j\rho n\gamma}+2^{kn\gamma})M^\gamma f(x) \\ &\le 2^{-k\ell}2^{jm+j\ell+jn(1-\rho(1-\gamma))}M^\gamma f(x). \end{aligned}$$

We assume $k > j\rho$. Then, we have

$$\begin{aligned} \sup_{z\in Q}\int_{(3Q)^c}\frac{|z-w|^n}{1+2^{2kN}|z-w|^{2N}}|f(w)|dw &\lesssim \int \frac{|x-w|^n}{1+2^{2kN}|x-w|^{2N}}|f(w)|dw \\ &\le \sum_{i\in\mathbb{Z}}\int_{|x-w|\sim 2^{-k}2^i}\frac{|x-w|^n}{1+2^{2kN}|x-w|^{2N}}|f(w)|dw \\ &\lesssim 2^{-kn}2^{-kn(1-\gamma)}M^\gamma f(x) \\ &\lesssim 2^{-kn}2^{-j\rho n(1-\gamma)}M^\gamma f(x), \end{aligned}$$

which completes the proof. □

B Appendix B

From Proposition 3.1, the weak-type boundedness of $M_{T,s}$ is a sufficient condition to have the sparse domination. It is natural to ask whether such condition be a necessary condition or not. However, the answer of this question appears to be negative since it is not true for the sparse operator $T = \Lambda_{\mathcal{S},r}$

Proposition B.1. (i) *Let $1 \le r < \infty$. Then, there exist $f \in L^\infty_c$ and collction of sparse families $\{\mathcal{S}(Q)\}_{Q:cube}$, and measurable set K which has a non-zero measure, such that*

$$\sup_{Q\in x}\|\Lambda_{\mathcal{S}(Q),r}(f1_{(3Q)^c})\|_{L^\infty(Q)} = \infty$$

for any $x \in K$.
(ii) *Let $1 \le r < s \le \infty$. Then, there exist $f \in L^\infty_c$ and collction of sparse families $\{\mathcal{S}(Q)\}_{Q:cube}$, and measurable set K which has a non-zero measure, such that*

$$\sup_{Q\ni x}\sup_{\|g\|_{L^{s'}(Q)}=1}|Q|^{1/s}\Lambda_{\mathcal{S}(Q),r,s'}(f1_{\mathbb{R}^n\setminus 3Q}, g) = \infty$$

for any $x \in K$.

Proof. (i) Fix a cube Q_0, and let $f = 1_{Q_0}$. Furthermore, we define the sparse collection $\mathcal{S}(Q)$ for any cube Q by

$$\mathcal{S}(Q) = \{3^kQ\ ;\ k = 1,2,3,\cdots\}.$$

For any cube $3Q \subset Q_0$ and $z \in Q$, we choose $N \in \mathbb{N}$ such that $3^{N+1}Q \cap Q_0^c \ne \emptyset$ and $3^NQ \subset Q_0$. Then, we have

$$\begin{aligned} \Lambda_{\mathcal{S}(Q),r}(f1_{\mathbb{R}^n\setminus 3Q})(z)1_Q(z) &= \sum_{k=1}^\infty \langle 1_{Q_0\setminus 3Q}\rangle_{r,3^kQ}1_Q(z) \\ &\ge \sum_{k=1}^N\left(\frac{|3^kQ\setminus 3Q|}{|3^kQ|}\right)^{1/r}1_Q(z) \\ &\gtrsim N1_Q(z), \end{aligned}$$

which yields

$$\|\Lambda_{\mathcal{S}(Q),r}(f1_{\mathbb{R}^n\setminus 3Q})\|_{L^\infty(Q)} \gtrsim N.$$

Since $N\to\infty$ at $|Q|\to 0$, we have

$$\sup_{Q\in x}\|\Lambda_{\mathcal{S}(Q),r}(f1_{\mathbb{R}^n\setminus 3Q})\|_{L^\infty(Q)} = \infty$$

for any $x\in Q_0$.

(ii) By taking f and $\mathcal{S}(Q)$ as above, we have

$$\begin{aligned}\sup_{\|g\|_{L^{s'}(Q)}=1}\Lambda_{\mathcal{S}(Q),r,s'}(f1_{\mathbb{R}^n\setminus 3Q},g) &= \sup_{\|g\|_{L^{s'}(Q)}=1}\sum_{k=1}^{\infty}|3^kQ|\langle 1_{Q_0\setminus 3Q}\rangle_{r,3^kQ}\langle g\rangle_{s',3^kQ}\\ &\geq \sum_{k=1}^{N}|3^kQ|\left(\frac{|3^kQ\setminus 3Q|}{|3^kQ|}\right)^{1/r}|Q|^{-1/s'}\left(\frac{|Q|}{|3^kQ|}\right)^{1/s'}\\ &\gtrsim |Q|^{1/s}3^{Nn/s},\end{aligned}$$

which complete the proof. □

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