ON DERIVED EQUIVALENCES OF NAKAYAMA ALGEBRAS

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Abstract. In this paper, we investigate the derived category of the Nakayama algebra $N(n,\ell) = K\mathbb{A}_n/\operatorname{rad}(K\mathbb{A}_n)^\ell$. We construct a derived equivalence between Nakayama algebras $N(n,\ell)$ and $N(n,\ell+1)$ where $n = p(p+1)q + p(p-1)r$ and $\ell = (p+1)q + pr$ for each triple of integers $p \geqslant 2$, $q \geqslant 1$, $r \geqslant 0$. To achieve it, we introduce families of idempotent subalgebras of $K\mathbb{A}_s \otimes K\mathbb{A}_t$ and characterize their derived categories by the existence of a certain family of objects called S-families.

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1. Introduction

Two rings A and B are said to be *derived equivalent* if there exists a triangle equivalence $\operatorname{per} A \to \operatorname{per} B$. In the representation theory of rings, it is an important problem to determine whether given two rings are derived equivalent or not. An object X in the perfect derived category $\operatorname{per} A$ of a ring A is said to be *tilting* if $\operatorname{Hom}_{\operatorname{per} A}(X, X[n]) \simeq 0$ for any integer $n \neq 0$ and the thick subcategory generated by X coincides with $\operatorname{per} A$. The following result is known as Rickard's theorem:

Theorem 1.1. [20, Thm. 6.4] *For any two rings A and B, the following conditions are equivalent:*

(i) *There exists a tilting object X in $\operatorname{per} A$ such that $\operatorname{End}_{\operatorname{per} A}(X) \simeq B$.*
(ii) *There exists a triangle equivalence $\operatorname{per} A \to \operatorname{per} B$.*
(iii) *There exists a triangle equivalence $\mathsf{D}^{\mathrm{b}}(\operatorname{Mod} A) \to \mathsf{D}^{\mathrm{b}}(\operatorname{Mod} B)$.*
(iv) *There exists a triangle equivalence $\mathsf{D}(\operatorname{Mod} A) \to \mathsf{D}(\operatorname{Mod} B)$.*

However, in general, it is a difficult problem to determine whether two given rings A and B satisfy the condition in Theorem 1.1.

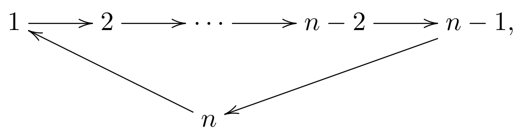
In this paper, we study this problem for a certain class of Nakayama algebras. Let K be a field. Throughout this paper, by a K-algebra we mean a finite dimensional associative K-algebra with an identity element. A K-algebra is said to be *right serial* (resp. *left serial*) if any indecomposable projective A-module (resp. A^{op}-module) has a unique composition series. A right and left serial algebra is called a *Nakayama algebra* [18]. A K-algebra A is said to be *connected* if $A \simeq A_0 \times A_1$ as K-algebras, then $A_0 \simeq 0$ or $A_1 \simeq 0$. Any K-algebra is isomorphic to a finite product of connected K-algebras. A K-algebra A is said to be *basic* if there exists a complete set of primitive orthogonal idempotents $(e_i)_{i\in[1,n]}$ such that for any $i, j \in [1, n]$, the relation $i \neq j$ implies the relation $e_iA \not\simeq e_jA$. Any K-algebra is Morita equivalent to a basic K-algebra. If K is an algebraically closed field, any basic K-algebra is isomorphic to a K-algebra KQ/I where Q is a finite quiver and I is an admissible ideal of the path algebra KQ (i.e. an ideal of KQ satisfying $(\operatorname{rad} A)^q \subset I \subset (\operatorname{rad} A)^2$ for an integer $q \geqslant 2$). The following two results for Nakayama algebras are well-known:

Theorem 1.2. [1, Thm. 3.2] *Let A be a basic and connected algebra over an algebraically closed field K. Then A is a Nakayama algebra if and only if A is isomorphic to KQ/I as K-algebras where Q is one of the following quivers:*

(a) $\mathbb{A}_n$:

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n,$$

(b) $\widetilde{\mathbb{A}}_{n-1}$:



and I is an admissible ideal of KQ.

Theorem 1.3. [1, Thm. 3.5] *Let A be a basic and connected Nakayama algebra over an algebraically closed field K. Then for any indecomposable A-module X, there exists an indecomposable projective A-module P and an integer k such that $X \simeq P/\operatorname{rad}^k P$. In particular, A is representation-finite (i.e. the number of isomorphism classes of indecomposable A-module is finite).*

By Theorem 1.3, the structure of the module categories over Nakayama algebras is well understood. But little is known about the structure of their derived categories. Let

$$N(n, \ell) = K\mathbb{A}_n/(\operatorname{rad} K\mathbb{A}_n)^{\ell}.$$

Then the algebra $N(n, \ell)$ is a Nakayama algebra of finite global dimension and a natural embedding $\operatorname{per} N(n, \ell) \to \mathsf{D}^{\mathrm{b}}(\operatorname{mod} N(n, \ell))$ is a triangle equivalence. An abelian category $\mathcal{H}$ is said to be *hereditary* if $\operatorname{Ext}^n_{\mathcal{H}}(X, Y) \simeq 0$ for any two objects $X, Y \in \mathcal{H}$ and any integer $n \geqslant 2$. A K-algebra A is said to be *piecewise hereditary* if there exists a hereditary abelian category $\mathcal{H}$ and a triangle equivalence $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathsf{D}^{\mathrm{b}}(\mathcal{H})$. In [9], Happel-Seidel classified Nakayama algebras $N(n, \ell)$ which are piecewise hereditary. One of the consequences of their results is the following:

Theorem 1.4. [9, Prop. 2.3] *There exists a triangle equivalence*

$$\operatorname{per} N(s + 6, s + 4) \to \operatorname{per} N(s + 6, s + 3) \quad \textit{for any integer } s \geqslant 0.$$

In [16], Lenzing-Meltzer-Ruan classified Nakayama algebras $N(n, \ell)$ whose bounded derived categories are triangle equivalent to the stable categories of vector bundles over the weighted projective lines $\mathbb{X}(a, b, c)$, and obtained the following similar result as a special case:

Theorem 1.5. [16, Thm. 6.3] *There exists a triangle equivalence*

$$\operatorname{per} N(s + 12, s + 7) \to \operatorname{per} N(s + 12, s + 6) \quad \textit{for any integer } s \geqslant 0.$$

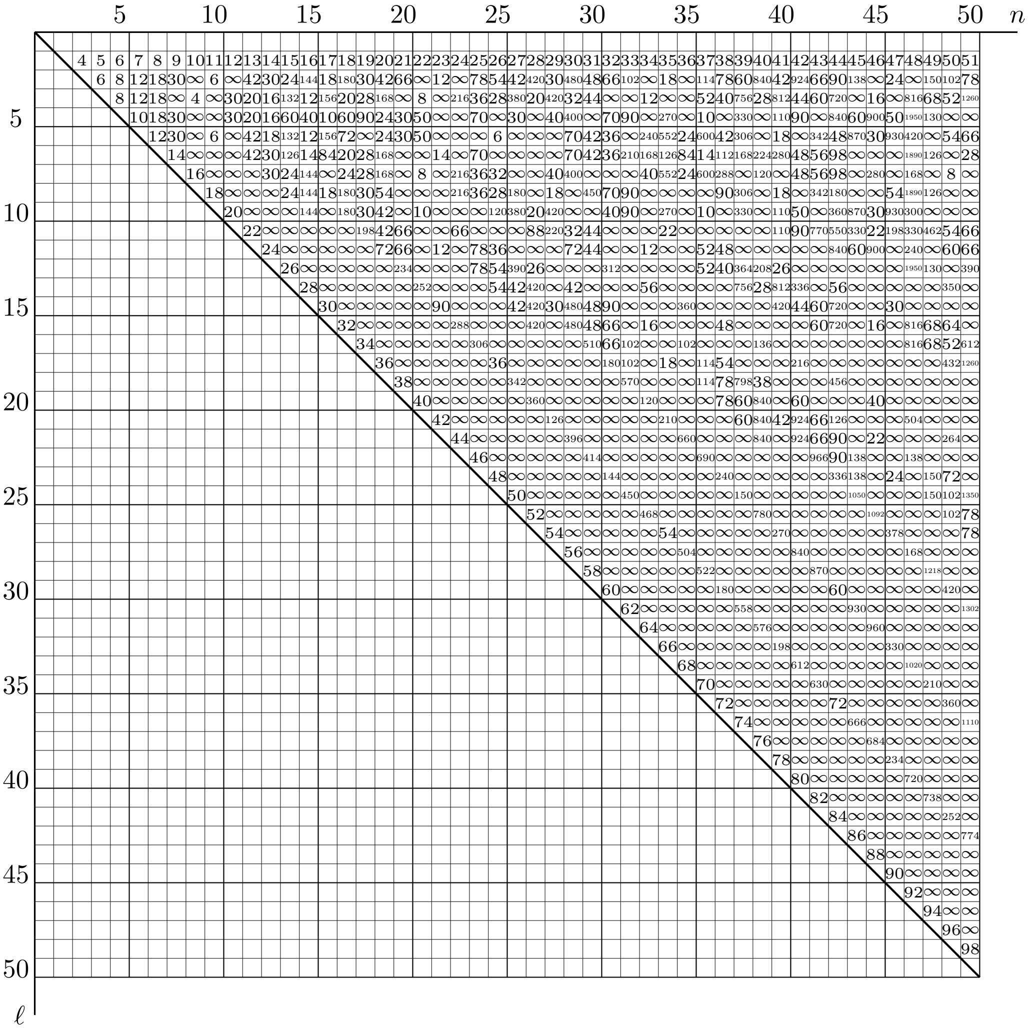


FIGURE 1. In the above graph due to Lenzing [15], each number shows the Coxeter number (i.e. the order of the Coxeter matrix) of $N(n,\ell)$. If two Nakayama algebras $N(n,\ell)$ and $N(n',\ell')$ are derived equivalent, their Coxeter numbers are equal.

In this paper, we prove the following result which is a far reaching generalization of the above two results:

Theorem 1.6 (Corollary 5.15)**.** *Let p, q be two integers such that $p \geqslant 2$, $q \geqslant 1$. Suppose that one of the following conditions is satisfied.*

(a) $r \in \mathbb{Z}_{\geqslant 0}$.

(b) $p = 2$ *and* $r \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$.

Then there exists a triangle equivalence

$$\operatorname{per} N(n, \ell+1) \to \operatorname{per} N(n,\ell) \quad \textit{where } n = p(p+1)q + p(p-1)r,\ \ell = (p+1)q + pr.$$

For $p = 2$, $r = \frac{s}{2}$, we obtain a triangle equivalence

(1.1) $\operatorname{per} N(s+6q, s+3q+1) \to \operatorname{per} N(s+6q, s+3q)$ for any integers $q \geqslant 1$, $s \geqslant 0$.

For $q = 1$, (1.1) is a triangle equivalence in Theorem 1.4 due to Happel-Seidel where they proved the above two derived categories are triangle equivalent to the derived category of the path algebra

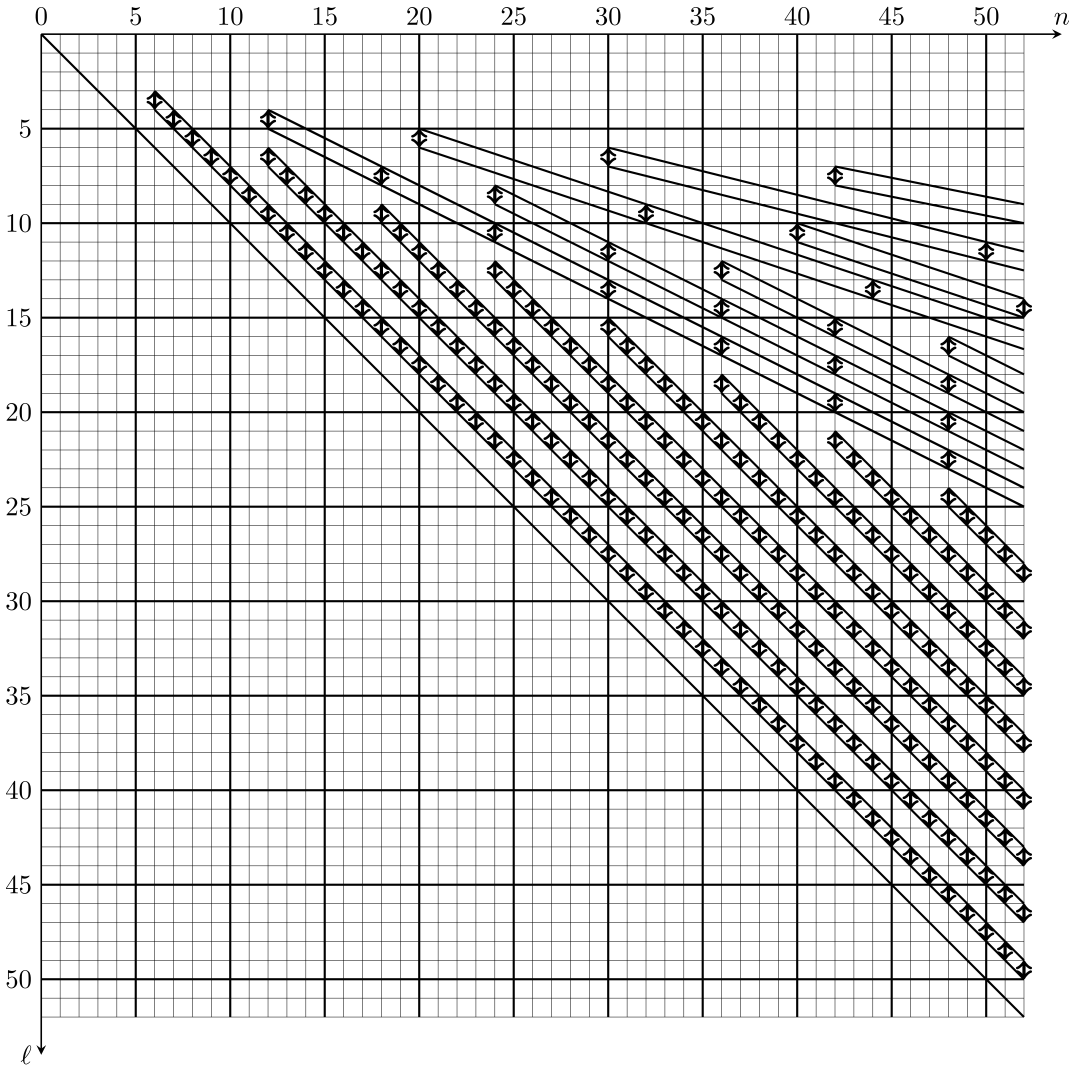


FIGURE 2. In the above graph, the arrow $\leftrightarrow$ shows the pairs of Nakayama algebras in Theorem 1.6.

of the star quiver with three branches of length $2, 3, s+3$ respectively.
For $q = 2$, (1.1) is a triangle equivalence in Theorem 1.5 due to Lenzing-Meltzer-Ruan where they proved the above two derived categories are triangle equivalent to the stable category of vector bundles over the weighted projective line $\mathbb{X}(2, 3, s+7)$.

Notice that the proofs of Theorems 1.4 and 1.5 were quite different. In this paper, we develop a systematic method to prove our main Theorem 1.6 by using the tensor products of two Nakayama algebras. Let

$$L(s,t,u) = (N(s,2)^{\mathrm{op}} \otimes N(t,2)^{\mathrm{op}})/\langle \sum_{i=0}^{u-1} e_s \otimes e_{t-i} \rangle,$$

$$L^!(s,t,u) = (K\mathbb{A}_s \otimes K\mathbb{A}_t)/\langle \sum_{i=0}^{u-1} e_s^! \otimes e_{t-i}^! \rangle$$

where $\otimes = \otimes_K$ and $e_i \otimes e_j$ (resp. $e^!_i \otimes e^!_j$) is the idempotent of $N(s,2)^{\mathrm{op}} \otimes N(t,2)^{\mathrm{op}}$ (resp. $K\mathbb{A}_s \otimes K\mathbb{A}_t$) corresponding the vertex (i,j). Then we show the following result:

Theorem 1.7 (Proposition 4.2, Theorem 4.7)**.** *There exist triangle equivalences*

$$\operatorname{per} N(st-u, t+1) \to \operatorname{per} L(s,t,u) \to \operatorname{per} L^!(s,t,u)$$

for any integers $s \geqslant 1$, $t \geqslant 1$, $0 \leqslant u \leqslant t$. In particular, $N(st-u, t+1)$ is derived equivalent to a K-algebra with global dimension $\leqslant 2$.

For $u = 0$, the above result gives the following due to Ladkani since $L^!(s,t,0) \simeq K\mathbb{A}_s \otimes K\mathbb{A}_t$ as K-algebras:

Theorem 1.8. [14, Cor. 1.2] *There exists a triangle equivalence*

$$\operatorname{per} N(st, t+1) \to \operatorname{per} K\mathbb{A}_s \otimes K\mathbb{A}_t \quad \textit{for any integers } s \geqslant 1,\ t \geqslant 1 \textit{ such that } st > t+1.$$

In particular there exists a triangle equivalence

$$\operatorname{per} N(st, s+1) \to \operatorname{per} N(st, t+1) \quad \textit{for any integers } s \geqslant 2,\ t \geqslant 2.$$

By Theorem 1.7, Theorem 1.6 is equivalent to the following:

Theorem 1.9 (Theorem 5.14)**.** *Let s, t, u be three positive integers such that $1 \leqslant u \leqslant t$. Suppose that one of the following conditions is satisfied.*
(a) *$u \in \mathbb{Z}s$ and $t - u \in \mathbb{Z}(s+1)$.*
(b) *$s = 2$ and $t - u \in 3\mathbb{Z}$.*
Then there exists a triangle equivalence $\operatorname{per} L(s,t,u) \to \operatorname{per} L(s, t-1, u-s)$.

A central role in our proof above is played by the notion of S-families (Definition 3.8). By using the following result, we can construct a triangle equivalence between a given triangulated category and $\operatorname{per} L(S)$.

Theorem 1.10 (Theorem 3.11)**.** *Let $\mathcal{D}$ be an algebraic, idempotent complete, and* Ext*-finite triangulated category with a Serre functor. If there exists a full S-family $(X_{i,j})_{(i,j)\in S}$, then there exists a triangle equivalence $F : \mathcal{D} \to \operatorname{per} L(S)$ such that $F(X_{i,j}) \simeq P(i,j)$ for any $(i,j) \in S$.*

The following figure presents the relationship of definitions of S-families.

weak S-family Definition 3.2 (L1), (L2)	$\Leftarrow$	S-family Definition 3.8 (S1)-(S3)	$\Leftarrow$	$\mathsf{Y}(\mathsf{p};\mathsf{q})$-family Proposition 4.3 (Y1)-(Y4)

As another application of Theorem 1.10, we have the following result related to a triangle equivalence in Theorem 1.8.

Theorem 1.11 (Theorem 5.16)**.** *There exists a triangle equivalence*

$$\operatorname{per} L(p+1, q, q-1) \to \operatorname{per} L(q+1, p, p-1) \quad \textit{for any integers } p \geqslant 2,\ q \geqslant 2.$$

By Theorem 1.7, the above result gives the following triangle equivalence due to Lenzing-Meltzer-Ruan [16, Prop. 4.1].

Corollary 1.12 (Corollary 5.17)**.** *There exists a triangle equivalence*

$$\operatorname{per} N(pq+1, q+1) \to \operatorname{per} N(pq+1, p+1) \quad \textit{for any integers } p \geqslant 2,\ q \geqslant 2.$$

In the rest, we describe the summary of each chapter of this paper. In Chapter 2, we give basic results for Serre functors, semi-orthogonal decompositions, admissible subcategories, tilting objects, and exceptional sequences. In Chapter 3, we characterize the perfect derived categories of the algebras $L(S)$ by using the terminology of S-families which are families of objects satisfying some axioms (Definition 3.8). By showing the existence of an S-family, we can construct a triangle equivalence between a triangulated category satisfying some conditions and the perfect derived category of the algebra $L(S)$ (Theorem 3.11). In Chapter 4, we study S-families when S is a

Young diagram. And we show that if S is the Young diagram $Y(s,t,u)$, the algebra $L(S)$ is derived equivalent to the Nakayama algebra $N(st-u,t+1)$ (Theorem 4.7). In Chapter 5, we introduce mutations of S-families under some assumption for S which are mutations as exceptional sequences (Theorem 5.6, 5.12). By using results for mutations of S-families, we prove main Theorem 5.14.

Conventions In this paper, K is a field and all modules over K-linear categories are right modules. We denote by $\otimes$ the tensor product over K. For any K-vector space V, we denote by V^* the dual of V. For any set $\{X_i \mid i \in J\}$ of objects in a triangulated category $\mathcal{D}$, we denote by $\langle X_i \mid i \in J\rangle$ the thick subcategory of $\mathcal{D}$ generated by $\{X_i \mid i \in J\}$. For any K-linear category $\mathcal{C}$, we denote by $\mathsf{D}(\mathcal{C})$ the derived category of $\mathcal{C}$, and $\operatorname{per}\mathcal{C}$ the perfect derived category of $\mathcal{C}$ i.e. the thick subcategory $\langle P_{\mathcal{C}}(i) \mid i \in \mathcal{C}\rangle$ of $\mathsf{D}(\mathcal{C})$ where $P_{\mathcal{C}}(i) := \operatorname{Hom}_{\mathcal{C}}(-,i)$.

For any two arrows $\alpha : i \to j$ and $\beta : j \to k$ in a quiver Q, we denote the their composition by $\beta\alpha : i \to k$. For any admissible ideal I of KQ, we denote by $S_{KQ/I}(i)$ (resp. $P_{KQ/I}(i)$, $I_{KQ/I}(i)$), the simple (resp. projective, injective) KQ/I-module corresponding a vertex i of Q. In our conventions, for any source i in Q, $S_{KQ/I}(i) \simeq P_{KQ/I}(i)$. We often simply denote by $S(i)$ (resp. $P(i)$, $I(i)$) instead of $S_A(i)$ (resp. $P_A(i)$, $I_A(i)$).

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2. Preliminaries

2.1. ***K*-linear categories and modules.** We refer to [6] for the representation theory of K-linear categories. A category $\mathcal{C}$ is called a *K-linear category* if each hom-set is a K-vector space and each composition map

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \times \operatorname{Hom}_{\mathcal{C}}(Y,Z) \to \operatorname{Hom}_{\mathcal{C}}(X,Z); (f,g) \mapsto g \circ f$$

is a bilinear map. Let $\mathcal{C}$ and $\mathcal{C}'$ be two K-linear categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is called a *K-linear functor* if each mapping

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}'}(F(X),F(X')); f \mapsto F(f)$$

is a K-linear map for any two objects X and X' of $\mathcal{C}$.

Let $\operatorname{Mod} K$ be the K-linear category consisting of K-vector spaces. A *(right) $\mathcal{C}$-module* is a K-linear functor $X : \mathcal{C}^{\mathrm{op}} \to \operatorname{Mod} K$. For any small K-linear category $\mathcal{C}$, we denote by $\operatorname{Mod}\mathcal{C}$ the K-linear category consisting of $\mathcal{C}$-modules. Then we define $\mathcal{C}$-module $P_{\mathcal{C}}(i)$ as

$$P_{\mathcal{C}}(i) = P(i) := \operatorname{Hom}_{\mathcal{C}}(-,i).$$

Then $P_{\mathcal{C}}(i)$ is a projective module for any $i \in \mathcal{C}$. For any $\mathcal{C}$-module X, there exists a surjective morphism

$$\bigoplus_{i\in\mathcal{C}} P_{\mathcal{C}}(i)^{\oplus E_i} \to X$$

where E_i is a basis of $\operatorname{Hom}_{\operatorname{Mod}\mathcal{C}}(P_{\mathcal{C}}(i),X)$. In particular, any projective $\mathcal{C}$-module P is a direct summand of a direct sum of projective modules $P_{\mathcal{C}}(i)$.

A $\mathcal{C}$-module X is *finitely generated* if there exist a family $(n_i)_{i\in I}$ of nonnegative integers n_i indexed by a finite set I of objects of $\mathcal{C}$ and a surjective morphism $p : \bigoplus_{i\in I} P_{\mathcal{C}}(i)^{\oplus n_i} \to X$. We denote by $\operatorname{mod}\mathcal{C}$, the category of finitely generated modules. A K-linear category $\mathcal{C}$ is said to be *Hom-finite* over K if $\dim_K \operatorname{Hom}_{\mathcal{C}}(i,j) < \infty$ for any i, $j \in \mathcal{C}$. Let $\mathcal{C}$ be a Hom-finite K-linear category. For any $i \in \mathcal{C}$, define

$$S_{\mathcal{C}}(i) := P_{\mathcal{C}}(i)/\operatorname{rad} P_{\mathcal{C}}(i).$$

Then $S_{\mathcal{C}}(i)$ is a finitely generated semi-simple $\mathcal{C}$-module.

A category $\mathcal{C}$ is said to be *svelte* if for any objects $i, j \in \mathcal{C}$, the relation $i \simeq j$ implies the relation $i = j$. A svelte K-linear category $\mathcal{C}$ is said to be *locally bounded* if for any $i \in \mathcal{C}$, the set

$$\{j \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(i,j) \neq 0 \text{ or } \mathrm{Hom}_{\mathcal{C}}(j,i) \neq 0\}$$

is a finite set. We define the support of a $\mathcal{C}$-module X as the following:

$$\mathrm{supp}\, X = \{i \in \mathcal{C} \mid X(i) \neq 0\}.$$

A svelte K-linear category $\mathcal{C}$ is locally bounded if and only if $\mathrm{supp}\, P_{\mathcal{C}}(i)$ is a finite set for any object i.

Let $\mathcal{C}$ be a Hom-finite and locally bounded K-linear category, $X \in \mathrm{Mod}\,\mathcal{C}$. It is elementary that the following conditions are equivalent:

(i) X is a finitely generated module.
(ii) $\mathrm{supp}\, X$ is a finite set and $\dim_K X(i) < \infty$ for any $i \in \mathcal{C}$.

If $\mathcal{C}$ is a Hom-finite and locally bounded K-linear category, then $\mathrm{mod}\,\mathcal{C}$ is a Hom-finite abelian category. We define $\mathcal{C}$-module $I_{\mathcal{C}}(i)$ by

$$I_{\mathcal{C}}(i) = I(i) := \mathrm{Hom}_{\mathcal{C}}(i, -)^*.$$

We denote by $\mathrm{proj}\,\mathcal{C}$ (resp. $\mathrm{inj}\,\mathcal{C}$), the full subcategory of $\mathrm{mod}\,\mathcal{C}$ consisting of finitely generated projective (resp. injective) $\mathcal{C}$-modules.

Let $\mathcal{C}$ be a Hom-finite and locally bounded K-linear category. Then the K-linear functor $D = (-)^* : (\mathrm{mod}\,\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{mod}\,\mathcal{C}$ is an equivalence of K-linear categories and the restriction functor $D|_{(\mathrm{proj}\,\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}} : (\mathrm{proj}\,\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{inj}\,\mathcal{C}$ is an equivalence of K-linear categories. Let

$$\nu = (-) \otimes_{\mathcal{C}} \mathcal{C}^* : \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}, \quad \nu^- = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}^*, -) : \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}$$

be Nakayama functors. More precisely, they are defined as

$$\nu(X)(i) = \mathrm{Coker}\Big(\bigoplus_{j,k \in \mathcal{C}} X_j \otimes \mathrm{Hom}_{\mathcal{C}}(k,j) \otimes \mathrm{Hom}_{\mathcal{C}}(k,i)^* \xrightarrow{f} \bigoplus_{k \in \mathcal{C}} X_k \otimes \mathrm{Hom}_{\mathcal{C}}(k,i)^*\Big),$$

$$\nu^-(X)(i) = \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{C}}(\mathrm{Hom}_{\mathcal{C}}(i,-)^*, X)$$

for any $X \in \mathrm{Mod}\,\mathcal{C}$ and $i \in \mathcal{C}$ where

$$f : x \otimes \alpha \otimes y \mapsto x\alpha \otimes y - x \otimes \alpha y.$$

By the definitions of ν and ν^-, (ν^-, ν) is a pair of adjoint functors.

Proposition 2.1. *Let $\mathcal{C}$ be a* Hom*-finite and locally bounded K-linear category. Then the functor*

$$\nu|_{\mathrm{proj}\,\mathcal{C}} : \mathrm{proj}\,\mathcal{C} \to \mathrm{inj}\,\mathcal{C}$$

is an equivalence of K-linear categories satisfying $\nu(P_{\mathcal{C}}(i)) \simeq I_{\mathcal{C}}(i)$ and there exists a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{mod}\,\mathcal{C}}(P, \nu(Q)) \simeq \mathrm{Hom}_{\mathrm{mod}\,\mathcal{C}}(Q, P)^* \text{ for any } P, Q \in \mathrm{proj}\,\mathcal{C}.$$

2.2. **Serre functors.** Let $\mathcal{D}$ be a triangulated category. Recall that a *Serre functor* of $\mathcal{D}$ is a K-linear autoequivalence $\mathbb{S} : \mathcal{D} \to \mathcal{D}$ such that there exists a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(X, \mathbb{S}(Y)) \tilde{\to} \mathrm{Hom}_{\mathcal{D}}(Y, X)^*$$

for any two objects $X, Y \in \mathcal{D}$. By the definition of a Serre functor, any two Serre functors are isomorphic.

Proposition 2.2. [4] *Let $\mathcal{D}$ be a triangulated category with a Serre functor $\mathbb{S} : \mathcal{D} \to \mathcal{D}$.*

(a) *There exists a natural isomorphism $\alpha : \mathbb{S}[1] \to [1]\mathbb{S}$ such that $(\mathbb{S}, \alpha) : \mathcal{D} \to \mathcal{D}$ is a triangle autoequivalence.*
(b) *Any Serre functor of $\mathcal{D}$ is isomorphic to $\mathbb{S}$.*
(c) *For any triangle equivalence $F : \mathcal{D} \to \mathcal{D}'$, $F\mathbb{S}F^{-1} : \mathcal{D}' \to \mathcal{D}'$ is a Serre functor of $\mathcal{D}'$.*

A K-linear category $\mathcal{C}$ is called an *Iwanaga-Gorenstein category* if $\mathcal{C}$ is Hom-finite and for any $i \in \mathcal{C}$, $I_{\mathcal{C}}(i) \in \operatorname{per}\mathcal{C}$ and $I_{\mathcal{C}^{\mathrm{op}}}(i) \in \operatorname{per}\mathcal{C}^{\mathrm{op}}$. The following result is well-known [8]:

Proposition 2.3. *For any* Hom*-finite K-linear category $\mathcal{C}$, the following conditions are equivalent:*
(i) *$\mathcal{C}$ is an Iwanaga-Gorenstein category.*
(ii) *$\operatorname{per}\mathcal{C}$ has a Serre functor $\mathbb{S} : \operatorname{per}\mathcal{C} \to \operatorname{per}\mathcal{C}$.*

If the above conditions are satisfied, the functor $\nu = (-) \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} \mathcal{C}^* : \operatorname{per}\mathcal{C} \to \operatorname{per}\mathcal{C}$ is a Serre functor of $\operatorname{per}\mathcal{C}$ and $\nu^{-1} = \mathbf{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}^*, -) : \operatorname{per}\mathcal{C} \to \operatorname{per}\mathcal{C}$ is an inverse of ν.

Proof. (i)⇒(ii): It follows from [12, 10.4].
(ii)⇒(i): Let $\mathcal{D} = \operatorname{per}\mathcal{C}$. Since

$$\mathrm{Hom}_{\mathcal{D}}(P_{\mathcal{C}}(j), \mathbb{S}(P_{\mathcal{C}}(i))[n]) \simeq \mathrm{Hom}_{\mathcal{D}}(P_{\mathcal{C}}(i), P_{\mathcal{C}}(j)[-n])^* \simeq \begin{cases} \mathrm{Hom}_{\mathcal{D}}(P_{\mathcal{C}}(j), I_{\mathcal{C}}(i)) & n = 0, \\ 0 & n \neq 0, \end{cases}$$

for any $j \in \mathcal{C}$, we have

$$H^n\mathbb{S}(P_{\mathcal{C}}(i)) \simeq \begin{cases} I_{\mathcal{C}}(i) & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Thus $I_{\mathcal{C}}(i) \simeq \mathbb{S}(P_{\mathcal{C}}(i)) \in \mathcal{D}$. Dually, we see that $I_{\mathcal{C}^{\mathrm{op}}}(i) \in \operatorname{per}\mathcal{C}^{\mathrm{op}}$. Thus the assertion follows. □

Lemma 2.4. *For any positive integer p, the category $\operatorname{per} K\mathbb{A}_p$ is a fractional Calabi-Yau category of dimension $\frac{p-1}{p+1}$ i.e. there exists an isomorphism $\nu_{K\mathbb{A}_p}^{p+1} \xrightarrow{\sim} [p-1]$ of functors.*

2.3. **Semi-orthogonal decompositions and Admissible subcategories.** Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be full subcategories of $\mathcal{D}$. We denote by $\mathcal{D}_1 * \mathcal{D}_2$ the full subcategory consisting of objects X in $\mathcal{D}$ such that there exist objects $X_1 \in \mathcal{D}_1$, $X_2 \in \mathcal{D}_2$ and a triangle

$$X_1 \to X \to X_2 \to X_1[1] \quad \text{in } \mathcal{D}.$$

Then the operation $*$ is associative and we define the full subcategory $\mathcal{D}_1 * \mathcal{D}_2 * \cdots * \mathcal{D}_n$ for full subcategories $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ inductively.

The sequence $(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)$ of thick subcategories $\mathcal{D}_i$ of $\mathcal{D}$ is called a *semi-orthogonal decomposition* of $\mathcal{D}$ if $\mathcal{D} = \mathcal{D}_1 * \mathcal{D}_2 * \cdots * \mathcal{D}_n$ and $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_k, \mathcal{D}_{k'}) = 0$ for any integers $k, k' \in [1, n]$ such that $k < k'$. In this case, we denote $\mathcal{D}_1 * \mathcal{D}_2 * \cdots * \mathcal{D}_n$ by

$$\mathcal{D}_1 \perp \mathcal{D}_2 \perp \cdots \perp \mathcal{D}_n.$$

The properties of admissible subcategories are detailed in [10]. Let $\mathcal{D}$ be a triangulated category. A thick subcategory $\mathcal{E}$ of $\mathcal{D}$ is said to be *right admissible* (resp. *left admissible*) if the natural embedding $\mathbb{I}_{\mathcal{E}}^{\mathcal{D}} : \mathcal{E} \to \mathcal{D}$ has a right adjoint functor $\mathbb{T}_{\mathcal{E}}^{\mathcal{D}} : \mathcal{D} \to \mathcal{E}$ (resp. a left adjoint functor $\mathbb{F}_{\mathcal{E}}^{\mathcal{D}} : \mathcal{D} \to \mathcal{E}$). If a thick subcategory $\mathcal{E}$ of $\mathcal{D}$ is right admissible and left admissible, $\mathcal{E}$ is said to be *admissible*. We often simply denote $\mathbb{I}_{\mathcal{E}}^{\mathcal{D}}$ (resp. $\mathbb{T}_{\mathcal{E}}^{\mathcal{D}}$, $\mathbb{F}_{\mathcal{E}}^{\mathcal{D}}$) by $\mathbb{I}_{\mathcal{E}}$ (resp. $\mathbb{T}_{\mathcal{E}}$, $\mathbb{F}_{\mathcal{E}}$). By the definitions of $\mathbb{T}_{\mathcal{E}}^{\mathcal{D}}$ and $\mathbb{F}_{\mathcal{E}}^{\mathcal{D}}$, we have

$$\mathrm{Hom}_{\mathcal{D}}(X, \mathbb{I}_{\mathcal{E}}^{\mathcal{D}}(X')) \simeq \mathrm{Hom}_{\mathcal{E}}(\mathbb{F}_{\mathcal{E}}^{\mathcal{D}}(X), X') \text{ for any objects } X \in \mathcal{D},\ X' \in \mathcal{E}, \tag{2.1}$$

$$\mathrm{Hom}_{\mathcal{D}}(\mathbb{I}_{\mathcal{E}}^{\mathcal{D}}(Y'), Y) \simeq \mathrm{Hom}_{\mathcal{E}}(Y', \mathbb{T}_{\mathcal{E}}^{\mathcal{D}}(Y)) \text{ for any objects } Y \in \mathcal{D},\ Y' \in \mathcal{E}. \tag{2.2}$$

A thick subcategory $\mathcal{E}$ of $\mathcal{D}$ is admissible if and only if

$$\mathcal{D} = \mathcal{E} \perp \mathcal{E}^{\perp} = {}^{\perp}\mathcal{E} \perp \mathcal{E} \tag{2.3}$$

where

$$\mathcal{E}^{\perp_{\mathcal{D}}} = \mathcal{E}^{\perp} := \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(Y, X) = 0 \text{ for any } Y \in \mathcal{E}\},$$

$${}^{\perp_{\mathcal{D}}}\mathcal{E} = {}^{\perp}\mathcal{E} := \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(X, Y) = 0 \text{ for any } Y \in \mathcal{E}\}.$$

In particular, the unit and counit morphisms induce triangles

$$\mathbb{T}^{\mathcal{D}}_{\mathcal{E}}(X) \to X \to \mathbb{F}^{\mathcal{D}}_{\mathcal{E}^{\perp}}(X) \to \mathbb{T}^{\mathcal{D}}_{\mathcal{E}}(X)[1], \tag{2.4}$$

$$\mathbb{T}^{\mathcal{D}}_{{}^{\perp}\mathcal{E}}(X) \to X \to \mathbb{F}^{\mathcal{D}}_{\mathcal{E}}(X) \to \mathbb{T}^{\mathcal{D}}_{{}^{\perp}\mathcal{E}}(X)[1]. \tag{2.5}$$

The following three facts are elementary:

Lemma 2.5. [17] *Let $\mathcal{D}$ be a triangulated category, $\mathcal{E}$ an admissible subcategory of $\mathcal{D}$, and $\mathcal{F}$ a thick subcategory of $\mathcal{E}$.*

(a) *$\mathcal{F}$ is an admissible subcategory of $\mathcal{E}$ if and only if $\mathcal{F}$ is an admissible subcategory of $\mathcal{D}$.*

(b) *If the condition in* (a) *is satisfied, then*

$$\mathbb{T}^{\mathcal{D}}_{\mathcal{F}} \simeq \mathbb{T}^{\mathcal{E}}_{\mathcal{F}}\mathbb{T}^{\mathcal{D}}_{\mathcal{E}}, \quad \mathbb{F}^{\mathcal{D}}_{\mathcal{F}} \simeq \mathbb{F}^{\mathcal{E}}_{\mathcal{F}}\mathbb{F}^{\mathcal{D}}_{\mathcal{E}}, \quad \mathbb{T}^{\mathcal{E}}_{\mathcal{F}} \simeq \mathbb{T}^{\mathcal{D}}_{\mathcal{F}}|_{\mathcal{E}}, \quad \mathbb{F}^{\mathcal{E}}_{\mathcal{F}} \simeq \mathbb{F}^{\mathcal{D}}_{\mathcal{F}}|_{\mathcal{E}}.$$

Lemma 2.6. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{E}$ a thick subcategory of $\mathcal{D}$. If there exist admissible subcategories $\mathcal{F}$ and $\mathcal{F}'$ of $\mathcal{D}$ such that $\mathcal{E} = \mathcal{F} \perp \mathcal{F}'$, then $\mathcal{E}$ is an admissible subcategory of $\mathcal{D}$.*

Proof. Since $\mathcal{F}$ is an admissible subcategory of $\mathcal{D}$, we have $\mathcal{D} = \mathcal{F} \perp \mathcal{G}$ where $\mathcal{G} = \mathcal{F}^{\perp_{\mathcal{D}}}$. Since $\mathcal{F}'$ is an admissible subcategory of $\mathcal{D}$ and $\mathcal{F}' \subset \mathcal{G}$, it follows from Lemma 2.5 that $\mathcal{F}'$ is an admissible subcategory of $\mathcal{G}$. So $\mathcal{G} = \mathcal{F}' \perp \mathcal{G}'$ where $\mathcal{G}' = \mathcal{F}'^{\perp_{\mathcal{G}}}$. Since

$$\mathcal{D} = \mathcal{F} \perp \mathcal{G} = \mathcal{F} \perp \mathcal{F}' \perp \mathcal{G}' = \mathcal{E} \perp \mathcal{G}',$$

$\mathcal{E}$ is a right admissible subcategory of $\mathcal{D}$. Dually, $\mathcal{E}$ is a left admissible subcategory of $\mathcal{D}$. □

Lemma 2.7. *Let $\mathcal{D}$ be a triangulated category and let $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{F}$, $\mathcal{F}'$ be four admissible subcategories of $\mathcal{D}$ such that $\mathcal{D} = \mathcal{E} \perp \mathcal{F}$, $\mathcal{E}' \subset \mathcal{E}$, $\mathcal{F}' \subset \mathcal{F}$. Then for $\mathcal{D}' = \mathcal{E}' \perp \mathcal{F}'$, we have*

$$\mathbb{T}^{\mathcal{D}}_{\mathcal{E}}|_{\mathcal{D}'} \simeq \mathbb{T}^{\mathcal{D}'}_{\mathcal{E}'}, \quad \mathbb{F}^{\mathcal{D}}_{\mathcal{F}}|_{\mathcal{D}'} \simeq \mathbb{F}^{\mathcal{D}'}_{\mathcal{F}'}.$$

Proof. For any $X \in \mathcal{D}'$, there exists a triangle

$$\mathbb{T}^{\mathcal{D}'}_{\mathcal{E}'}(X) \to X \to \mathbb{F}^{\mathcal{D}'}_{\mathcal{F}'}(X) \to \mathbb{T}^{\mathcal{D}'}_{\mathcal{E}'}(X)[1].$$

Since $\mathbb{T}^{\mathcal{D}'}_{\mathcal{E}'}(X) \in \mathcal{E}$ and $\mathbb{F}^{\mathcal{D}'}_{\mathcal{F}'}(X) \in \mathcal{F}$, the assertion follows. □

The following observation should be well-known but we could not find a reference. We give a proof for the convenience of the reader.

Proposition 2.8. *Let $\mathcal{D}$ be a triangulated category with a Serre functor $\mathbb{S} = \mathbb{S}_{\mathcal{D}}$, $\mathcal{E}$ a left or right admissible subcategory of $\mathcal{D}$.*

(a) *$\mathcal{E}$ is an admissible subcategory of $\mathcal{D}$ if and only if $\mathcal{E}$ has a Serre functor $\mathbb{S}_{\mathcal{E}}$.*

If the conditions in (a) *are satisfied, then the following two assertions hold:*

(b) *The functor $\mathbb{S}_{\mathcal{E}}$ in* (a) satisfies $\mathbb{S}_{\mathcal{E}} \simeq \mathbb{T}_{\mathcal{E}}\mathbb{S}|_{\mathcal{E}}$ and $\mathbb{S}_{\mathcal{E}}^{-1} \simeq \mathbb{F}_{\mathcal{E}}\mathbb{S}^{-1}|_{\mathcal{E}}$.

(c) If X, $\mathbb{S}(X) \in \mathcal{E}$, then $\mathbb{S}_{\mathcal{E}}(X) \simeq \mathbb{S}(X)$.

Proof. (a) We prove "only if" part. We show that $\mathbb{T}_{\mathcal{E}}\mathbb{S}|_{\mathcal{E}}$ is a Serre functor. Since for any $X, Y \in \mathcal{E}$, there exist functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{E}}(X, \mathbb{T}_{\mathcal{E}}\mathbb{S}(Y)) = \mathrm{Hom}_{\mathcal{D}}(X, \mathbb{S}(Y)) \simeq \mathrm{Hom}_{\mathcal{D}}(Y, X)^* \simeq \mathrm{Hom}_{\mathcal{E}}(Y, X)^*, \tag{2.6}$$

$$\mathrm{Hom}_{\mathcal{E}}(\mathbb{F}_{\mathcal{E}}\mathbb{S}^{-1}(X), Y) = \mathrm{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}(X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(Y, X)^* \simeq \mathrm{Hom}_{\mathcal{E}}(Y, X)^*, \tag{2.7}$$

it follows from [19, Lem. 1.1.5], that $\mathbb{T}_{\mathcal{E}}\mathbb{S}|_{\mathcal{E}}$ is a Serre functor of $\mathcal{E}$ with an inverse $\mathbb{F}_{\mathcal{E}}\mathbb{S}^{-1}|_{\mathcal{E}}$.
We prove "if" part. Let $\mathbb{T}_{\mathcal{E}}$ be a right adjoint functor of $\mathbb{I}_{\mathcal{E}}$, and $\mathbb{S}_{\mathcal{E}}$ a Serre functor of $\mathcal{E}$. For any $X, Y \in \mathcal{E}$, there exist functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{E}}(\mathbb{S}_{\mathcal{E}}^{-1}\mathbb{T}_{\mathcal{E}}\mathbb{S}_{\mathcal{D}}(Y), X) \simeq \mathrm{Hom}_{\mathcal{E}}(X, \mathbb{T}_{\mathcal{E}}\mathbb{S}_{\mathcal{D}}(Y))^* \simeq \mathrm{Hom}_{\mathcal{D}}(X, \mathbb{S}_{\mathcal{D}}(Y))^* \simeq \mathrm{Hom}_{\mathcal{D}}(Y, X).$$

Thus $\mathbb{S}_{\mathcal{E}}^{-1}\mathbb{T}_{\mathcal{E}}\mathbb{S}_{\mathcal{D}}$ is a left adjoint functor of $\mathbb{I}_{\mathcal{E}}$.
(b) In the proof of (a), we proved that $\mathbb{T}_{\mathcal{E}}\mathbb{S}|_{\mathcal{E}}$ is a Serre functor. By the uniqueness of Serre functor, the assertion follows.
(c) This is clear by (b). □

The proof of the following result is clear from Proposition 2.8.

Lemma 2.9. *Let $\mathcal{D}$ be a triangulated category with a Serre functor $\mathbb{S} = \mathbb{S}_{\mathcal{D}}$, and let $\mathcal{E}$ and $\mathcal{F}$ be two admissible subcategories of $\mathcal{D}$.*
(a) *The functor $\mathbb{F}^{\mathcal{D}}_{\mathcal{F}}|_{\mathcal{E}} : \mathcal{E} \to \mathcal{F}$ has a right adjoint functor $\mathbb{T}^{\mathcal{D}}_{\mathcal{E}}|_{\mathcal{F}} : \mathcal{F} \to \mathcal{E}$.*
(b) *If $\mathbb{S}^{-1}(\mathcal{F}) \subset \mathcal{E}$, we have an isomorphism $\mathbb{T}^{\mathcal{D}}_{\mathcal{E}}|_{\mathcal{F}} \simeq \mathbb{S}_{\mathcal{E}}\mathbb{S}^{-1}|_{\mathcal{F}} : \mathcal{F} \to \mathcal{E}$ of functors.*
(c) *If $\mathbb{S}(\mathcal{E}) \subset \mathcal{F}$, we have an isomorphism $\mathbb{F}^{\mathcal{D}}_{\mathcal{F}}|_{\mathcal{E}} \simeq \mathbb{S}^{-1}_{\mathcal{F}}\mathbb{S}|_{\mathcal{E}} : \mathcal{E} \to \mathcal{F}$ of functors.*
(d) *If $\mathbb{S}(\mathcal{E}) = \mathcal{F}$, then the functors $\mathbb{F}^{\mathcal{D}}_{\mathcal{F}}|_{\mathcal{E}} : \mathcal{E} \to \mathcal{F}$ and $\mathbb{T}^{\mathcal{D}}_{\mathcal{E}}|_{\mathcal{F}} : \mathcal{F} \to \mathcal{E}$ are mutually inverse triangle equivalences.*

Proof. (a) For any $X \in \mathcal{E}$ and $Y \in \mathcal{F}$, we have functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{F}}(\mathbb{F}^{\mathcal{D}}_{\mathcal{F}}(X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{E}}(X, \mathbb{T}^{\mathcal{D}}_{\mathcal{E}}(Y)).$$

Thus the assertion follows.
(b) Since $\mathcal{E}$ is an admissible subcategory, $\mathcal{E}$ has a Serre functor $\mathbb{S}_{\mathcal{E}} = \mathbb{T}^{\mathcal{D}}_{\mathcal{E}}\mathbb{S}|_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ by Proposition 2.8. Since $\mathbb{S}^{-1}(\mathcal{F}) \subset \mathcal{E}$, Lemma 2.8 (b) implies $\mathbb{S}_{\mathcal{E}}\mathbb{S}^{-1}|_{\mathcal{F}} \simeq \mathbb{T}^{\mathcal{D}}_{\mathcal{E}}\mathbb{S}\mathbb{S}^{-1}|_{\mathcal{F}} \simeq \mathbb{T}^{\mathcal{D}}_{\mathcal{E}}|_{\mathcal{F}}$.
(c) This is the dual of (b).
(d) Since the functor $\mathbb{F}^{\mathcal{D}}_{\mathcal{F}}|_{\mathcal{E}}$ is the composition of $\mathbb{S}|_{\mathcal{E}} : \mathcal{E} \to \mathcal{F}$ and $\mathbb{S}^{-1}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ by (c), we have that it is a triangle equivalence. By (a), $\mathbb{T}^{\mathcal{D}}_{\mathcal{E}}|_{\mathcal{F}}$ is an inverse of $\mathbb{F}^{\mathcal{D}}_{\mathcal{F}}|_{\mathcal{E}}$. □

2.4. **Tilting objects and Exceptional sequences.** Let $\mathcal{D}$ be a triangulated category. An object $T \in \mathcal{D}$ is called a *pretilting object* if $\mathrm{Hom}_{\mathcal{D}}(T, T[n]) \simeq 0$ for any integer $n \neq 0$. A pretilting object T is called a *tilting object* if $\mathcal{D} = \langle T \rangle$. An additive category $\mathcal{D}$ is said to be *idempotent complete* if any idempotent morphism $e : X \to X$ (i.e. endomorphism $e : X \to X$ satisfying $e^2 = e$) in $\mathcal{D}$, there exist two morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{D}$ such that $fg = 1_Y$ and $gf = e$. A triangulated category $\mathcal{D}$ is said to be *algebraic* if $\mathcal{D}$ is triangle equivalent to the stable category of a Frobenius category.

Proposition 2.10. [12] *Let $\mathcal{D}$ be an algebraic triangulated category, T a pretilting object in $\mathcal{D}$, and $A = \mathrm{End}_{\mathcal{D}}(T)$. If $\mathcal{D}$ is idempotent complete, there exists a triangle equivalence $F : \langle T \rangle \to \operatorname{per} A$ such that $F(T) \simeq A$.*

The following result is clear by Proposition 2.10 and F is directly given.

Example 2.11. *Let A be a K-algebra such that $\operatorname{gl.dim} A < \infty$, and let $(e_k)_{k\in[1,n]}$ be a complete set of primitive orthogonal idempotents of A, and $e = \sum_{k\in[1,m]} e_k$, $B = eAe$. Let*

$$P = eA = \bigoplus_{k\in[1,m]} P_A(k), \quad P' = (1-e)A = \bigoplus_{k\in[m+1,n]} P_A(k),$$

$$S = \bigoplus_{k\in[1,m]} S_A(k), \quad S' = \bigoplus_{k\in[m+1,n]} S_A(k).$$

(a) *The sequence $(e_k)_{k\in[1,m]}$ is a complete set of primitive orthogonal idempotents of B, and there exists a triangle equivalence $F = \mathbf{R}\mathrm{Hom}_A(P, -) : \langle P \rangle \to \operatorname{per} B$ such that*

(2.8) $\quad F(P_A(i)) \simeq P_B(i)$ *for any $i \in [1, m]$.*

(b) *If $\mathrm{Hom}_{\operatorname{per} A}(P', P) = 0$, then the following conditions are satisfied:*

(2.9) $\quad \operatorname{per} A = \langle P' \rangle \perp \langle P \rangle = \langle S \rangle \perp \langle S' \rangle, \ \ \langle P \rangle = \langle S \rangle.$

(2.10) $\quad F(S_A(i)) \simeq S_B(i)$ *for any $i \in [1, m]$.*

Proof. (a) The functor $F = (-) \overset{\mathbf{L}}{\otimes}_A Ae : \operatorname{per} A \to \operatorname{per} B$ has a left adjoint functor $G = (-) \overset{\mathbf{L}}{\otimes}_B eA :$ $\operatorname{per} B \to \operatorname{per} A$. Since $eA \overset{\mathbf{L}}{\otimes}_A Ae \simeq B$ as (B, B)-bimodules, we have $FG \simeq \mathrm{Id}$. Thus $F : \langle P \rangle \to \operatorname{per} B$ is a triangle equivalence such that

$$F(P_A(i)) \simeq \mathbf{R}\mathrm{Hom}_A(P, P_A(i)) \simeq \mathrm{Hom}_{\operatorname{per} A}(P, P_A(i)) \simeq e_i Ae \simeq P_B(i).$$

(b) Since $\mathrm{Hom}_{\mathrm{per}\,A}(P, P'[n]) \simeq 0$ for any integer n, we have $\langle P\rangle = \langle S\rangle$. Since $\mathrm{Hom}_{\mathrm{per}\,A}(P, S'[n]) \simeq 0$ for any integer n, we have that $\mathrm{Hom}_{\mathrm{per}\,A}(S, S'[n]) \simeq 0$ for any integer n. Thus

$$\mathrm{per}\,A = \langle S \oplus S'\rangle = \langle S\rangle \perp \langle S'\rangle.$$

Since

$$F(S_A(i)) \simeq \mathbf{R}\mathrm{Hom}_A(P, S_A(i)) \simeq \mathrm{Hom}_{\mathrm{per}\,A}(P, S_A(i)) \simeq (e_iA/A(1-e_i)A)e \simeq S_B(i)$$

as B-modules, we have $F(S_A(i)) \simeq S_B(i)$. □

A pretilting object E is called an *exceptional object* in $\mathcal{D}$ if $\mathrm{End}_{\mathcal{D}}(E) \simeq K$.

Lemma 2.12. [10, Lem. 1.58] *Let E be an exceptional object in $\mathcal{D}$. Then for any object $X \in \langle E\rangle$,*

$$X \simeq \bigoplus_{n\in\mathbb{Z}} E^{\oplus d_n}[n]$$

where $d_n = \dim_K \mathrm{Hom}_{\mathcal{D}}(E, X[-n]) = \dim_K \mathrm{Hom}_{\mathcal{D}}(X, E[n])$.

We recall the notion of exceptional sequences which is slightly modified for later use in this paper. In fact we allow the index set to be a finite totally ordered set.

Definition 2.13. [10] *Let $\mathcal{D}$ be a triangulated category, S a finite totally ordered set. A family $(E_k)_{k\in S}$ of objects in $\mathcal{D}$ indexed by S is called an* exceptional sequence *if the following conditions are satisfied:*

(E1) *E_k is an exceptional object for any $k \in S$.*
(E2) *If $k < k'$, then $\mathrm{Hom}_{\mathcal{D}}(E_k, E_{k'}[n]) = 0$ for any integer n.*
(E3) *$\langle E_k \mid k \in S\rangle$ is an admissible subcategory of $\mathcal{D}$.*

An exceptional sequence $(E_k)_{k\in S}$ is said to be full if

$$\langle E_k \mid k \in S\rangle = \mathcal{D}.$$

The properties of exceptional sequences are detailed in [10]. In general, for any finite totally orderd set S, by using the unique ordered isomorphism $s : [1, n] \to S$, we can identify $(E_k)_{k\in S}$ with $(E_{s(k)})_{k\in[1,n]}$. If $S = [1, n]$, we denote the family $(E_k)_{k\in S}$ by $(E_1, E_2, \ldots, E_n)$.

A triangulated category $\mathcal{D}$ is said to be *Ext-finite* over K if $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, Y[n])$ has a finite dimension for any objects $X, Y \in \mathcal{D}$.

Lemma 2.14. [10, Lem. 1.58] *Let $\mathcal{D}$ be an* Ext*-finite triangulated category, E an exceptional object in $\mathcal{D}$. Then* (E3) *is satisfied.*

Let $(E_k)_{k\in S}$ be an exceptional sequence in $\mathcal{D}$. For any $T \subset S$, define

$$E_T := \bigoplus_{k\in T} E_k.$$

Lemma 2.15. *Let $\mathcal{D}$ be an* Ext*-finite triangulated category, $(E_k)_{k\in S}$ an exceptional sequence in $\mathcal{D}$. For any $T \subset S$, $\langle E_T\rangle$ is an admissible subcategory of $\mathcal{D}$.*

Proof. Without loss of generality, we can assume $T = [1, n]$. Since

$$\langle E_T\rangle = \langle E_1\rangle \perp \langle E_2\rangle \perp \cdots \perp \langle E_n\rangle$$

and $\langle E_k\rangle$ are admissible subcategories of $\mathcal{D}$ by Lemma 2.14, $\langle E_T\rangle$ is also an admissible subcategory of $\mathcal{D}$ by Lemma 2.6. □

Lemma 2.16. *Let $\mathcal{D}$ be an* Ext*-finite triangulated category with a Serre functor $\mathbb{S} : \mathcal{D} \to \mathcal{D}$, and let*

$$(E_1, E_2, \ldots, E_n)$$

be an exceptional sequence in $\mathcal{D}$.

(a) *The sequence*

$$(E_2, \dots, E_n, \mathbb{S}_{\langle E_{[1,n]}\rangle}(E_1))$$

is an exceptional sequence and $\langle E_{[1,n]}\rangle = \langle E_{[2,n]}\rangle \perp \langle \mathbb{S}_{\langle E_{[1,n]}\rangle}(E_1)\rangle$.

(b) *The sequence*

$$(\mathbb{S}^{-1}_{\langle E_{[1,n]}\rangle}(E_n), E_1, \dots, E_{n-1})$$

is an exceptional sequence and $\langle E_{[1,n]}\rangle = \langle \mathbb{S}^{-1}_{\langle E_{[1,n]}\rangle}(E_n)\rangle \perp \langle E_{[1,n-1]}\rangle$.

Proof. (a) Since $\mathbb{S}_{\langle E_{[1,n]}\rangle} : \langle E_{[1,n]}\rangle \to \langle E_{[1,n]}\rangle$ is a triangle equivalence and E_1 is an exceptional object, $\mathbb{S}_{\langle E_{[1,n]}\rangle}(E_1)$ is also an exceptional object. If $k \in [2,n]$, then

$$\mathrm{Hom}_{\mathcal{D}}(E_k, \mathbb{S}_{\langle E_{[1,n]}\rangle}(E_1)[n]) \simeq \mathrm{Hom}_{\mathcal{D}}(E_1, E_k[-n])^* = 0$$

for any integer n. Thus the sequence

$$(E_2, \dots, E_n, \mathbb{S}_{\langle E_{[1,n]}\rangle}(E_1))$$

is an exceptional sequence. Since

$$\mathbb{T}_{\langle E_1\rangle}\mathbb{S}_{\langle E_{[1,n]}\rangle}(E_1) \overset{2.8(\mathrm{b})}{=} \mathbb{S}_{\langle E_1\rangle}(E_1) \overset{2.12}{=} E_1,$$

there exists a triangle

$$E_1 \to \mathbb{S}_{\langle E_{[1,n]}\rangle}(E_1) \to \mathbb{F}_{\langle E_{[2,n]}\rangle}\mathbb{S}_{\langle E_{[1,n]}\rangle}(E_1) \to E_1[1],$$

thus the last assertion follows.
(b) This is the dual of (a). □

Lemma 2.17. *Let $\mathcal{D}$ be an* Ext*-finite triangulated category with a Serre functor $\mathbb{S} : \mathcal{D} \to \mathcal{D}$, and let $(E_k)_{k\in[1,n]}$ be a full exceptional sequence in $\mathcal{D}$. If there exist integers $p < q$ such that*

(2.11) $\mathrm{Hom}_{\mathcal{D}}(E_{[q+1,n]}, \mathbb{S}_{\langle E_{[p,q]}\rangle}(E_p)[n]) = 0$ *for any integer n,*

then

$$\mathbb{T}^{\mathcal{D}}_{\langle E_{[1,n]\setminus\{p\}}\rangle}(E_p) \simeq \mathbb{T}^{\langle E_{[p,q]}\rangle}_{\langle E_{[p+1,q]}\rangle}(E_p).$$

Proof. Let $\mathcal{D}' = \langle E_{[p,q]}\rangle$, $\mathcal{E} = \langle E_{[1,n]\setminus\{p\}}\rangle$, $\mathcal{F} = \langle \mathbb{S}_{\langle E_{[p,q]}\rangle}(E_p)\rangle$, $\mathcal{E}' = \langle E_{[p+1,q]}\rangle$. Then we have semi-orthogonal decompositions

$$\mathcal{D}' = \langle E_p\rangle \perp \langle E_{[p+1,q]}\rangle \overset{2.16(a)}{=} \mathcal{E}' \perp \mathcal{F} \quad \text{and}$$

$$\begin{aligned}\mathcal{D} = \langle E_{[1,p-1]}\rangle \perp \langle E_p\rangle \perp \mathcal{E}' \perp \langle E_{[q+1,n]}\rangle &\overset{2.16(a)}{=} \langle E_{[1,p-1]}\rangle \perp \mathcal{E}' \perp \mathcal{F} \perp \langle E_{[q+1,n]}\rangle \\ &\overset{(2.11)}{=} \langle E_{[1,p-1]}\rangle \perp \mathcal{E}' \perp \langle E_{[q+1,n]}\rangle \perp \mathcal{F} = \mathcal{E} \perp \mathcal{F}.\end{aligned}$$

Thus $\mathbb{T}^{\mathcal{D}}_{\mathcal{E}}(E_p) \overset{2.7}{=} \mathbb{T}^{\mathcal{D}'}_{\mathcal{E}'}(E_p)$ and the assertion follows. □

Lemma 2.18. *Let A, B be two K-algebras, E an exceptional object in* $\mathrm{per}\,A$*. Then*

$$F = E \otimes (-) : \mathrm{per}\,B \to \mathrm{per}(A \otimes B)$$

is a fully faithful triangle functor and induces a triangle equivalence $F : \mathrm{per}\,B \to \langle E \otimes B\rangle$.

Proof. Since $E \otimes B$ is a pretilting object and the morphism $f : B \to \mathbf{R}\mathrm{End}_{A\otimes B}(E \otimes B); b \to 1 \otimes b$ is a quasi-isomorphism of dg algebras, the functor

$$F' = \mathbf{R}\mathrm{Hom}_{A\otimes B}(E \otimes B, -) : \langle E \otimes B\rangle \to \mathrm{per}\,B$$

is a triangle equivalence. Since $F'F \simeq \mathrm{Id}$, the assertion follows. □

Lemma 2.19. *Let A be a K-algebra, B an Iwanaga-Gorenstein K-algebra, $C = A \otimes B$. Then $G = (-) \overset{\mathbf{L}}{\otimes}_C (A \otimes B^*) : \operatorname{per} C \to \operatorname{per} C$ is a triangle autoequivalence, and for any exceptional object $E \in \operatorname{per} A$, $G|_{\langle E \otimes B \rangle} : \langle E \otimes B \rangle \to \langle E \otimes B \rangle$ is a Serre functor.*

Proof. Since B is an Iwanaga-Gorenstein algebra, $G = (-) \overset{\mathbf{L}}{\otimes}_C (A \otimes B^*) : \operatorname{per} C \to \operatorname{per} C$ is a triangle equivalence. By Lemma 2.18, for any object $X \in \langle E \otimes B \rangle$, there exists an object $Y \in \operatorname{per} B$ such that $X \simeq E \otimes Y$. Since there exist functorial isomorphisms

$$\operatorname{Hom}_{\operatorname{per} C}(E \otimes Y, G(E \otimes Y')) \simeq \operatorname{Hom}_{\operatorname{per} C}(E \otimes Y, E \otimes \nu_B(Y')) \simeq \operatorname{Hom}_{\operatorname{per} C}(E \otimes Y', E \otimes Y)^*,$$

we have that $G|_{\langle E \otimes B \rangle}$ is a Serre functor. □

3. S-families

3.1. **Weak S-families.** In this section, let $\mathcal{D}$ be a triangulated category satisfying the following conditions:

(3.1) $\mathcal{D}$ is algebraic, idempotent complete, Ext-finite and has a Serre functor $\mathbb{S}$.

Let S be a finite subset of $\mathbb{Z}^2$. For any element $(i, j) \in S$,

$$S_{i,j} := ([i-1, i] \times [j-1, j]) \cap S.$$

Example 3.1. *Let $I_1 = [1, 5]$, $I_2 = I_3 = [1, 3]$, $I_4 = \{1\}$ be intervals of $\mathbb{Z}$. The figure of $S = \bigsqcup_{i \in [1,4]} \{i\} \times I_i$ is the following:*

$$\begin{array}{lllll} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) \\ (2,1) & \multicolumn{1}{|l}{(2,2)} & \multicolumn{1}{l|}{(2,3)} & & \\ (3,1) & \multicolumn{1}{|l}{(3,2)} & \multicolumn{1}{l|}{(3,3)} & & \\ (4,1) & & & & \end{array}$$

In the above figure, the square shows the subset $S_{3,3}$.

Definition 3.2. *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1)*. A family $(X_{i,j})_{(i,j) \in S}$ of objects in $\mathcal{D}$ indexed by a finite subset S of $\mathbb{Z}^2$ is called a* weak S-family *if the following conditions are satisfied:*
(L1) *$X_{i,j}$ is an exceptional object for any $(i, j) \in S$.*
(L2) $\operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}[n]) = 0$ *for any integer n unless $(i', j') \in S_{i,j}$.*
A weak S-family $(X_{i,j})_{(i,j) \in S}$ is said to be full if

$$\langle X_{i,j} \mid (i, j) \in S \rangle = \mathcal{D}.$$

The name of S-family comes from lattices.

Remark 3.3. *The condition* (L2) *is satisfied if and only if the following conditions are satisfied:*
(L2.1) $\operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}[n]) = 0$ *for any integer n unless $i' \in [i-1, i]$.*
(L2.2) $\operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}[n]) = 0$ *for any integer n unless $j' \in [j-1, j]$.*

If $(X_{i,j})_{(i,j) \in S}$ is a weak S-family, then for any $T \subset S$, a family $(X_{i,j})_{(i,j) \in T}$ is a weak T-family. Let $(X_{i,j})_{(i,j) \in S}$ be a family of objects in $\mathcal{D}$. For any finite subset T of S, let

(3.2) $$X_T := \bigoplus_{(i,j) \in T} X_{i,j} \in \mathcal{D}.$$

In particular, X_k and X^k are defined as

(3.3) $$X_k := X_{\{k\} \times S_k} = \bigoplus_{j \in S_k} X_{k,j}, \quad X^k := X_{S^k \times \{k\}} = \bigoplus_{i \in S^k} X_{i,k}$$

where $S_k = \{j \in \mathbb{Z} \mid (k,j) \in S\}$, $S^k = \{i \in \mathbb{Z} \mid (i,k) \in S\}$.

Lemma 3.4. *Let $(X_{i,j})_{(i,j)\in S}$ be a weak S-family in $\mathcal{D}$. Then $\langle X_S \rangle$ is an admissible subcategory of $\mathcal{D}$. In particular, $\langle X_i \rangle$ and $\langle X^j \rangle$ are admissible subcategories.*

Proof. By restricting the lexicographic order $\preccurlyeq$ of $\mathbb{Z}^2$, we regard S as a totally ordered set. The lexicographic order of S in Example 3.1 is as the following.

$$
\begin{array}{l}
(1,1) \to (1,2) \to (1,3) \to (1,4) \to (1,5) \\
\swarrow \\
(2,1) \to (2,2) \to (2,3) \\
\swarrow \\
(3,1) \to (3,2) \to (3,3) \\
\swarrow \\
(4,1)
\end{array}
\tag{3.4}
$$

The condition (L2) implies the following one:

(E2) If $(i,j) \prec (i',j')$, then $\mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}[n]) = 0$ for any integer n.

So we can regard a weak S-family $(X_{i,j})_{(i,j)\in S}$ as an exceptional sequence. Thus the assertion follows from Lemma 2.15. □

The following observation is clear from (L2).

Lemma 3.5. *Let $(X_{i,j})_{(i,j)\in S}$ be a weak S-family in $\mathcal{D}$. Then X_S is a pretilting object if and only if $X_{S_{i,j}}$ is a pretilting object for any $(i,j) \in S$.*

Proof. It suffices to show that $\mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}[n]) = 0$ for each (i,j), $(i',j') \in S$ and $n \neq 0$. If $(i',j') \notin S_{i,j}$, then this holds by (2.9). Otherwise, this holds since $X_{S_{i,j}}$ is a pretilting object. □

3.2. **Algebras $L(S)$.** Let $\mathcal{N}$ be the K-linear category defined as

$$\mathcal{N} := (K\mathbb{A}_\infty^\infty)^{\mathrm{op}}/(\mathrm{rad}(K\mathbb{A}_\infty^\infty)^{\mathrm{op}})^2$$

where $\mathbb{A}_\infty^\infty$ is the quiver

$$\mathbb{A}_\infty^\infty = [\cdots \xrightarrow{a_{-3}} -2 \xrightarrow{a_{-2}} -1 \xrightarrow{a_{-1}} 0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots],$$

and $\mathrm{rad}\, K\mathbb{A}_\infty^\infty = \langle a_n \mid n \in \mathbb{Z}\rangle$. For any subset I of $\mathbb{Z}$, let Q_I be the full subquiver of $\mathbb{A}_\infty^\infty$ with the set I of vertices. We define $N(I)$ as

$$N(I) := KQ_I^{\mathrm{op}}/(\mathrm{rad}\, KQ_I^{\mathrm{op}})^2.$$

In particular, we define $N(k)$ as

$$N(k) := N([1,k]).$$

The typical example of a weak S-family is given by the following:

Example 3.6. *Let $\mathcal{L} = \mathcal{N} \otimes \mathcal{N}$. The category $\mathrm{per}\,\mathcal{L}$ is a triangulated category satisfying (3.1). For any finite subset S of $\mathbb{Z}^2$, the family $(P_{\mathcal{L}}(i,j))_{(i,j)\in S}$ is a weak S-family in $\mathrm{per}\,\mathcal{L}$.*

For any finite subset S of $\mathbb{Z}^2$, we define the algebra $L(S)$ as

$$L(S) = \mathrm{End}_{\mathrm{per}\,\mathcal{L}}\Big(\bigoplus_{(i,j)\in S} P_{\mathcal{L}}(i,j)\Big).$$

Then there exists a triangle equivalence

$$\langle P_{\mathcal{L}}(i,j) \mid (i,j) \in S \rangle_{\mathrm{per}\,\mathcal{L}} \to \mathrm{per}\, L(S).$$

Lemma 3.7. *The K-linear category $\mathcal{L}$ is self-injective, and the category $\mathrm{per}\,\mathcal{L}$ is a triangulated category satisfying (3.1).*

Proof. Since

$$I_{\mathcal{L}}(i,j) = P_{\mathcal{L}}(i-1,j-1)$$

for any $(i,j) \in \mathbb{Z}^2$, $\mathcal{L}$ is self-injective. By Proposition 2.3, the assertion follows. □

3.3. **S-families.**

Definition 3.8. *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1). *Let S be a finite subset of $\mathbb{Z}^2$. A weak S-family $(X_{i,j})_{(i,j)\in S}$ in $\mathcal{D}$ is called an* S-family *if the following conditions are satisfied:*

(S1) *If $(i,j),(i,j-1) \in S$, then $\mathbb{S}_{\langle X_i\rangle}(X_{i,j}) \simeq X_{i,j-1}$.*
(S2) *If $(i,j),(i-1,j) \in S$, then $\mathbb{S}_{\langle X^j\rangle}(X_{i,j}) \simeq X_{i-1,j}$.*
(S3) *If $(i,j),(i-1,j-1) \in S$, then $\mathbb{S}_{\langle X_S\rangle}(X_{i,j}) \simeq X_{i-1,j-1}$.*

An S-family $(X_{i,j})_{(i,j)\in S}$ is said to be full if

$$\langle X_{i,j} \mid (i,j) \in S\rangle = \mathcal{D}.$$

A typical example of an S-family is obtained by the following result:

Example 3.9. *For any finite subset $S \subset \mathbb{Z}^2$, the family $(P_{\mathcal{L}}(i,j))_{(i,j)\in S}$ is an S-family in* $\operatorname{per}\mathcal{L}$.

Proof. Let $X_{i,j} = P_{\mathcal{L}}(i,j)$. The family $(X_{i,j})_{(i,j)\in S}$ is a weak S-family in $\operatorname{per}\mathcal{L}$ by Example 3.6. If (i,j), $(i,j-1) \in S$, since

$$\nu_{\langle P_{\mathcal{L}}(i,j')\mid j'\in\mathbb{Z}\rangle}(X_{i,j}) = X_{i,j-1} \in \langle X_i\rangle,$$

we have $\nu_{\langle X_i\rangle}(X_{i,j}) \overset{2.8(c)}{=} X_{i,j-1}$. Dually, we have that if (i,j), $(i-1,j) \in S$, then $\nu_{\langle X^j\rangle}(X_{i,j}) \simeq X_{i-1,j}$. Thus (S1) and (S2) are satisfied. If (i,j), $(i-1,j-1) \in S$,

$$\nu(X_{i,j}) \simeq X_{i-1,j-1} \in \langle X_S\rangle.$$

So we have $\nu_{\langle X_S\rangle}(X_{i,j}) \overset{2.8(c)}{=} X_{i-1,j-1}$ and (S3) is satisfied. Thus the assertion follows. □

Example 3.10. *Let $S = [1,p] \times [1,q]$. The family $(S(i,j)[-i-j])_{(i,j)\in S}$ is a full S-family in* $\operatorname{per}(K\mathbb{A}_p \otimes K\mathbb{A}_q)$. *There exists a triangle equivalence $F : \operatorname{per}(K\mathbb{A}_p \otimes K\mathbb{A}_q) \to \operatorname{per}(N(p) \otimes N(q))$ such that $F(S(i,j)[-i-j]) \simeq P(i,j)$.*

Proof. Let $A = K\mathbb{A}_p \otimes K\mathbb{A}_q$, $B = N(p) \otimes N(q)$,

$$X_{i,j} = S_A(i,j)[-i-j].$$

We construct a triangle equivalence $G : \operatorname{per} A \to \operatorname{per} B$ such that $G(X_{i,j}) = P_B(i,j)$. Since

$$\operatorname{Hom}_{\operatorname{per} K\mathbb{A}_m}(S_{K\mathbb{A}_m}(j)[-j], S_{K\mathbb{A}_m}(i)[-i+k]) \simeq \begin{cases} K & j \in \{i, i+1\},\ k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

the object $T_m = \bigoplus_{i\in[1,m]} S_{K\mathbb{A}_m}(i)[-i]$ is a tilting object in $\operatorname{per} K\mathbb{A}_m$ such that $\operatorname{End}_{\operatorname{per} K\mathbb{A}_m}(T_m) \simeq N(m)$. Then there exists a triangle equivalence $F : \operatorname{per} K\mathbb{A}_m \to \operatorname{per} N(m)$ such that $F(S_{K\mathbb{A}_m}(i)[-i]) \simeq P_{N(m)}(i)$. Thus $T = T_p \otimes T_q$ is a tilting object such that $\operatorname{End}_{\operatorname{per} A}(T) \simeq B$, and there exists a triangle exivalence $G : \operatorname{per} A \to \operatorname{per} B$ such that $G(X_{i,j}) \simeq P_B(i,j)$. By Example 3.9, the assertion follows. □

In general, by using the following result, if there exists a full S-family $(X_{i,j})_{(i,j)\in S}$ in $\mathcal{D}$, then there exists a triangle equivalence $F : \mathcal{D} \to \operatorname{per} L(S)$ and F sends $(X_{i,j})_{(i,j)\in S}$ to $(P(i,j))_{(i,j)\in S}$ in Example 3.9.

Theorem 3.11. *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1), *S a finite subset of $\mathbb{Z}^2$. For any S-family $(X_{i,j})_{(i,j)\in S}$ in $\mathcal{D}$, X_S is a pretilting object such that* $\operatorname{End}_{\mathcal{D}}(X_S) \simeq L(S)$, *and there exists a triangle equivalence*

$$F : \langle X_S\rangle \to \operatorname{per} L(S)$$

such that $F(X_{i,j}) \simeq P_{\mathcal{L}}(i,j)$ *for any* $(i,j) \in S$. *In particular, there exists a fully faithful triangle functor* $G : \langle X_S \rangle \to \operatorname{per} \mathcal{L}$ *such that* $G(X_{i,j}) = P_{\mathcal{L}}(i,j)$ *for any* $(i,j) \in S$.

The following result is a key step of the proof of Theorem 3.11:

Proposition 3.12. *Let* $(X_{i,j})_{(i,j)\in S}$ *be a weak* S*-family satisfying* (S3).

(a) *The condition* (S1) *is equivalent to the following one:*
  (S1′) *If* $(i,j),(i,j-1) \in S$, *then* $\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}) = X_{i,j-1}$.

(b) *The condition* (S2) *is equivalent to the following one:*
  (S2′) *If* $(i,j),(i-1,j) \in S$, *then* $\mathbb{F}_{\langle X_{i-1}\rangle}(X_{i,j}) = X_{i-1,j}$.

To prove Proposition 3.12, we prove the following result:

Lemma 3.13. *Let* S *be a finite subset of* $\mathbb{Z}^2$, i *an integer such that* $S_i \neq \emptyset$. *Let* $(X_{i,j})_{(i,j)\in S}$ *be a weak* S*-family satisfying* (S1′). *Then* X_i *is a pretilting object such that* $\operatorname{End}_{\mathcal{D}}(X_i) \simeq N(S_i)$, *and there exists a triangle equivalence* $F : \langle X_i \rangle \to \operatorname{per} N(S_i)$ *such that* $F(X_{i,j}) \simeq P(j)$.

Proof. If $j, j-1 \in S_i$, then

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i,j-1}[n]) &\overset{(2.1)}{=} \operatorname{Hom}_{\mathcal{D}}(\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}), X_{i,j-1}[n]) \\ &\overset{(\mathrm{S1}')}{=} \operatorname{Hom}_{\mathcal{D}}(X_{i,j-1}, X_{i,j-1}[n]) \overset{(\mathrm{L1})}{=} \begin{cases} K & n = 0, \\ 0 & n \neq 0. \end{cases}\end{aligned}$$

By Lemma 3.5, X_i is a pretilting oblect. If $j' \notin [j-1, j]$, since $\operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i,j'}) \overset{(\mathrm{L2})}{=} 0$, we have $\operatorname{End}_{\mathcal{D}}(X_i) \simeq N(S_i)$. □

By symmetry, we have the following result:

Lemma 3.14. *Let* S *be a finite subset of* $\mathbb{Z}^2$, *and* j *an integer such that* $S^j \neq \emptyset$. *Let* $(X_{i,j})_{(i,j)\in S}$ *be a weak* S*-family satisfying* (S2′). *Then* X^j *is a pretilting object such that* $\operatorname{End}_{\mathcal{D}}(X^j) \simeq N(S^j)$, *and there exists a triangle equivalence* $G : \langle X^j \rangle \to \operatorname{per} N(S^j)$ *such that* $G(X_{i,j}) \simeq P(i)$.

Now we are ready to prove Proposition 3.12.

Proof of Proposition 3.12. (a) We prove "if" part. Since (S1)′ is satisfied, by Lemma 3.13, there exists a triangle equivalence $F : \langle X_i \rangle \to \operatorname{per} A$ such that $F(X_{i,j}) \simeq P_A(j)$ where $A = N(S_i)$. If $(i,j),(i,j-1) \in S$, then

$$\mathbb{S}_{\langle X_i\rangle}(X_{i,j}) \simeq \mathbb{S}_{\langle X_i\rangle} F^{-1}(P_A(j)) \overset{2.2}{=} F^{-1}(\nu_A(P_A(j))) \simeq F^{-1}(P_A(j-1)) \simeq X_{i,j-1}.$$

Thus (S1) is satisfied.

We prove "only if" part. Suppose that $(i,j),(i,j-1) \in S$. If $(i-1,j-1) \notin S$, then

$$\begin{aligned}\langle X^{j-1} \rangle &= \langle X_{\leqslant i-2,j-1} \rangle \perp \langle X_{i,j-1} \rangle \perp \langle X_{\geqslant i+1,j-1} \rangle \\ &\overset{(\mathrm{L2})}{=} \langle X_{i,j-1} \rangle \perp \langle X_{\leqslant i-2,j-1} \rangle \perp \langle X_{\geqslant i+1,j-1} \rangle.\end{aligned}$$

Since

$$\operatorname{Hom}_{\mathcal{D}}(\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}), X_{\geqslant i+1,j-1}[n]) \simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{\geqslant i+1,j-1}[n]) \overset{(\mathrm{L2})}{=} 0 \quad \text{and}$$

$$\operatorname{Hom}_{\mathcal{D}}(\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}), X_{\leqslant i-2,j-1}[n]) \simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{\leqslant i-2,j-1}[n]) \overset{(\mathrm{L2})}{=} 0$$

for any integer n, we have $\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}) \in \langle X_{i,j-1} \rangle$. Since

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}), X_{i,j-1}[n]) &\simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i,j-1}[n]) \overset{(\mathrm{S1})}{=} \operatorname{Hom}_{\mathcal{D}}(X_{i,j}, \mathbb{S}_{\langle X_i\rangle}(X_{i,j})) \\ &\simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i,j}[n])^* \overset{(\mathrm{L1})}{=} \begin{cases} K & n = 0, \\ 0 & n \neq 0, \end{cases}\end{aligned}$$

we have $\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}) \overset{2.12}{=} X_{i,j-1}$. If $(i-1,j-1)\in S$, then

$$\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}) \overset{\text{(S3)}}{=} \mathbb{F}_{\langle X^{j-1}\rangle}\mathbb{S}^{-1}(X_{i-1,j-1}) \overset{2.8(b)}{=} \mathbb{S}^{-1}_{\langle X^{j-1}\rangle}(X_{i-1,j-1}). \tag{3.5}$$

Since

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}_{\langle X^{j-1}\rangle}(X_{i-1,j-1}), X_{\geqslant i+1,j-1}[n]) &\overset{(3.5)}{=} \operatorname{Hom}_{\mathcal{D}}(\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}), X_{\geqslant i+1,j-1}[n])\\ &\simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{\geqslant i+1,j-1}[n]) \simeq 0\end{aligned}$$

for any integer n, we have $\mathbb{S}^{-1}_{\langle X^{j-1}\rangle}(X_{i-1,j-1}) \in \langle X_{\leqslant i,j-1}\rangle$, and so

$$\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}) \overset{(3.5)}{=} \mathbb{S}^{-1}_{\langle X^{j-1}\rangle}(X_{i-1,j-1}) \overset{2.8(c)}{=} \mathbb{S}^{-1}_{\langle X_{\leqslant i,j-1}\rangle}(X_{i-1,j-1}). \tag{3.6}$$

Let $T = X_{i,j-1}\oplus X_{i-1,j-1}$. Since

$$\mathbb{T}_{\langle X_i\rangle}(X_{i-1,j-1}) \overset{\text{(S3)}}{=} \mathbb{T}_{\langle X_i\rangle}\mathbb{S}_{\langle X_S\rangle}(X_{i,j}) \overset{\text{(S1)}}{=} X_{i,j-1},$$

we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(X_{i,j-1}, X_{i-1,j-1}[n]) &\simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j-1}, \mathbb{T}_{\langle X_i\rangle}(X_{i-1,j-1})[n])\\ &\simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j-1}, X_{i,j-1}[n]) \overset{\text{(L1)}}{=} \begin{cases} K & n=0,\\ 0 & n\neq 0.\end{cases}\end{aligned} \tag{3.7}$$

So T is a pretilting object such that $\operatorname{End}_{\mathcal{D}}(T)\simeq N(2)$, and so there exists a triangle equivalence $F:\langle T\rangle \to \operatorname{per} B$ such that $F(P_B(1))\simeq X_{i-1,j-1}$ and $F(P_B(2))\simeq X_{i,j-1}$ where $B=N(2)$. Let $\mathcal{E}=\langle X_{\leqslant i,j-1}\rangle$, $\mathcal{F}=\langle X_{\leqslant i-2,j-1}\rangle$ and $\mathcal{F}'=\langle \mathbb{S}^{-1}_{\langle T\rangle}(X_{i,j-1})\rangle$. Then

$$\begin{aligned}\mathcal{E} &\overset{\text{(L2)}}{=} \mathcal{F}\perp\langle X_{i-1,j-1}\rangle\perp\langle X_{i,j-1}\rangle \overset{2.16(b)}{=} \mathcal{F}\perp\mathcal{F}'\perp\langle X_{i-1,j-1}\rangle\\ &\overset{2.16(b)}{=} \langle \mathbb{S}^{-1}_{\mathcal{E}}(X_{i-1,j-1})\rangle\perp\mathcal{F}\perp\mathcal{F}'.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{E} &\overset{\text{(L2)}}{=} \mathcal{F}\perp\langle X_{i-1,j-1}\rangle\perp\langle X_{i,j-1}\rangle \overset{2.16(a)}{=} \mathcal{F}\perp\langle X_{i,j-1}\rangle\perp\langle \mathbb{S}_{\langle T\rangle}(X_{i-1,j-1})\rangle\\ &\overset{\text{(L2)}}{=} \langle X_{i,j-1}\rangle\perp\mathcal{F}\perp\langle \mathbb{S}_{\langle T\rangle}(X_{i-1,j-1})\rangle \overset{2.4}{=} \langle X_{i,j-1}\rangle\perp\mathcal{F}\perp\mathcal{F}'.\end{aligned}$$

Thus $\langle \mathbb{S}^{-1}_{\mathcal{E}}(X_{i-1,j-1})\rangle = \langle X_{i,j-1}\rangle$. Since

$$\operatorname{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}_{\mathcal{E}}(X_{i-1,j-1}), X_{i,j-1}[n]) \simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j-1}, X_{i-1,j-1}[-n])^* \overset{(3.7)}{=} \begin{cases} K & n=0,\\ 0 & n\neq 0,\end{cases}$$

we have $\mathbb{S}^{-1}_{\mathcal{E}}(X_{i-1,j-1}) \overset{2.12}{=} X_{i,j-1}$, and so $\mathbb{F}_{\langle X^{j-1}\rangle}(X_{i,j}) \overset{(3.6)}{=} \mathbb{S}^{-1}_{\mathcal{E}}(X_{i-1,j-1}) \simeq X_{i,j-1}$. Thus (S1)$'$ is satisfied.

(b) This is the dual of (a). □

The following result is clear by the definition of $L(S)$. The proof is left to the reader:

Lemma 3.15. *Let $(X_{i,j})_{(i,j)\in S}$ be a family of objects $X_{i,j}\in\mathcal{D}$. Suppose that there exists a family $(\upsilon^{i,j}_{i',j'}: X_{i,j}\to X_{i',j'})_{(i,j),(i',j')\in S}$ of (possibly zero) morphisms and families $(c_{i,j})_{(i,j)\in S}$, $(c'_{i,j})_{(i,j)\in S}$ of non-zero scalars $c_{i,j}, c'_{i,j}\in K^\times$ satisfying the following conditions:*

$$\operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}) = K\upsilon^{i,j}_{i',j'}. \tag{3.8}$$

(3.9) $(i',j')\in S_{i,j}$ *if and only if* $\upsilon^{i,j}_{i',j'}\neq 0$.

(3.10) *If* (i,j), $(i,j-1)$, $(i-1,j-1)\in S$, *then* $c_{i,j}\upsilon^{i,j-1}_{i-1,j-1}\upsilon^{i,j}_{i,j-1} = \upsilon^{i,j}_{i-1,j-1}$.

(3.11) *If* (i,j), $(i-1,j)$, $(i-1,j-1)\in S$, *then* $c'_{i,j}\upsilon^{i-1,j}_{i-1,j-1}\upsilon^{i,j}_{i-1,j} = \upsilon^{i,j}_{i-1,j-1}$.

Then there exists an isomorphism $f : \mathrm{End}_{\mathcal{D}}(X_S) \to L(S)$ *of* K*-algebras such that* $P(i,j)_f \simeq \mathrm{Hom}_{\mathcal{D}}(X_S, X_{i,j})$ *as* $\mathrm{End}_{\mathcal{D}}(X_S)$*-modules where* $P(i,j)_f$ *is an* $\mathrm{End}_{\mathcal{D}}(X_S)$*-module associated with* f *and* $P(i,j)$.

Now we are ready to prove Theorem 3.11.

Proof of Theorem 3.11. Since $(X_{i,j})_{(i,j)\in S}$ is a weak S-family, we have

$$\dim_K \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i-1,j}[n]) \overset{3.13}{=} \begin{cases} 1 & n = 0, \\ 0 & n \neq 0, \end{cases}$$

$$\dim_K \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i,j-1}[n]) \overset{3.14}{=} \begin{cases} 1 & n = 0, \\ 0 & n \neq 0, \end{cases}$$

$$\dim_K \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i-1,j-1}[n]) \overset{\text{(S3)}}{=} \dim_K \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, \mathbb{S}_{\langle X_S\rangle}(X_{i,j})[n])$$

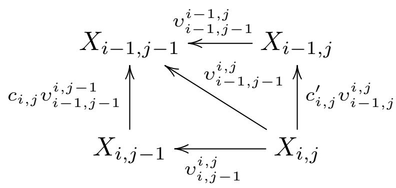
$$= \dim_K \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i,j}[-n])^* \overset{\text{(L1)}}{=} \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}$$

In particular, $X_{S_{i,j}}$ is a pretilting object. Thus X_S is a tilting object by Lemma 3.5.

Thanks to the above calculation, the following condition is satisfied:

(3.12) If $(i',j') \in S_{i,j}$, then $\dim_K \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}) = 1$.

By the conditions (L2) and (3.12), there exists a family $(v^{i,j}_{i',j'} : X_{i,j} \to X_{i',j'})_{(i,j),(i',j')\in S}$ of morphisms satisfying (3.8) and (3.9). By (S1′) and (S2′), the conditions (3.10) and (3.11) are satisfied.



Thus the assertion follows from Lemma 3.15. □

3.4. **A property of S-families.** Let $(X_{i,j})_{(i,j)\in S}$ be an S-family. For any $J \subset \mathbb{Z}$, define

$$X_{i,J} = \bigoplus_{j \in J \cap S_i} X_{i,j}.$$

In particular

$$X_{i,>j} = X_{(j,\infty)}, \quad X_{i,\geqslant j} = X_{[j,\infty)}, \quad X_{i,<j} = X_{(-\infty,j)}, \quad X_{i,\leqslant j} = X_{(-\infty,j]}.$$

In this section, we prove the following result which is the key step of the proof of Theorem 4.3 in the next section. Let $(X_{i,j})_{(i,j)\in S}$ be an S-family and $(i,j) \in S$. If $(i-1,j),(i-1,j-1) \in S$, by (S1) and (S3), we have $\mathbb{S}_{\langle X_S\rangle}(X_{i,j}) \simeq X_{i-1,j-1} \simeq \mathbb{S}_{\langle X_{i-1}\rangle}(X_{i-1,j})$. The following shows that $\mathbb{S}_{\langle X_S\rangle}(X_{i,j}) \simeq \mathbb{S}_{\langle X_{i-1}\rangle}(X_{i-1,j})$ holds without assuming $(i-1,j-1) \in S$ if $S_i \subset S_{i-1}$.

Proposition 3.16. *Let* $(X_{i,j})_{(i,j)\in S}$ *be an* S*-family in* $\mathcal{D}$*, and* $(i,j) \in S$*. If* $S_i \subset S_{i-1}$*, then* $\mathbb{S}_{\langle X_S\rangle}(X_{i,j}) \simeq \mathbb{S}_{\langle X_{i-1}\rangle}(X_{i-1,j}) \in \langle X_{i-1}\rangle$.

To prove Proposition 3.16, we prove the following result:

Lemma 3.17. *Let* $(X_{i,j})_{(i,j)\in S}$ *be an* S*-family in* $\mathcal{D}$ *and* (i,j), $(i,j-1) \in S$*. Then*

$$\mathbb{T}^{\langle X_i\rangle}_{\langle X_{i,\geqslant j}\rangle}(X_{i,j-1}) = \mathbb{S}_{\langle X_{i,S_i\setminus\{j-1\}}\rangle}(X_{i,j}).$$

Proof. Let $Y = \mathbb{T}^{\langle X_i \rangle}_{\langle X_{i,\geqslant j} \rangle}(X_{i,j-1})$. Since $(X_{i,j})_{j \in S_i}$ is regarded as the exceptional sequence,

$$\langle X_{i,\geqslant j} \rangle = \langle X_{i,j} \rangle \perp \langle X_{i,>j} \rangle \overset{2.16(a)}{=} \langle X_{i,>j} \rangle \perp \langle E \rangle$$

where $E = \mathbb{S}_{\langle X_{i,\geqslant j} \rangle}(X_{i,j})$. Then

$$\mathrm{Hom}_{\mathcal{D}}(X_{i,>j}, Y[n]) \simeq \mathrm{Hom}_{\mathcal{D}}(X_{i,>j}, X_{i,j-1}[n]) = 0$$

for any integer n. We have $Y \in \langle E \rangle$. Since E is an exceptional object and

$$\mathrm{Hom}_{\mathcal{D}}(Y, E[n]) \simeq \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, Y[-n])^* \simeq \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i,j-1}[-n])^* \overset{(\mathrm{S1})}{=} \begin{cases} K & n = 0, \\ 0 & \text{otherwise}, \end{cases}$$

we have $Y \overset{2.12}{=} E$. Since

$$\langle X_{i,S_i \setminus \{j-1\}} \rangle = \langle X_{i,<j-1} \rangle \perp \langle X_{i,\geqslant j} \rangle \overset{(\mathrm{L2})}{=} \langle X_{i,\geqslant j} \rangle \perp \langle X_{i,<j-1} \rangle,$$

we have

$$\mathbb{S}_{\langle X_{i,S_i \setminus \{j-1\}} \rangle}(X_{i,j}) \simeq \mathbb{T}^{\langle X_i \rangle}_{\langle X_{i,S_i \setminus \{j-1\}} \rangle} \mathbb{S}_{\langle X_i \rangle}(X_{i,j}) \simeq \mathbb{T}^{\langle X_i \rangle}_{\langle X_{i,S_i \setminus \{j-1\}} \rangle}(X_{i,j-1}) \simeq Y \simeq E.$$

Thus the assertion follows. □

Now we are ready to prove Proposition 3.16.

Proof of Proposition 3.16. By Theorem 3.11, we can assume that $\mathcal{D} = \mathrm{per}\,\mathcal{L}$, $X_{i,j} = P_{\mathcal{L}}(i,j) \in \mathrm{per}\,\mathcal{L}$. If $(i-1, j-1) \in S$,

$$\mathbb{S}_{\langle X_S \rangle}(X_{i,j}) \simeq X_{i-1,j-1} \simeq \mathbb{S}_{\langle X_{i-1} \rangle}(X_{i-1,j}).$$

If $(i-1, j-1) \notin S$, let $T := S \cup \{(i-1, j-1)\}$. Then $(X_{i,j})_{(i,j) \in T}$ is a T-family in $\mathcal{D}$. By restricting the lexicographic order $\preccurlyeq$ of $\mathbb{Z}^2$, we regard T as a totally ordered set (see the figure (3.4)). Let $s : [1, n] \to T$ be the ordered isomorphism, and let $E_k = X_{s(k)}$. Let p, q be two integers such that

$$p = s^{-1}(i-1, j-1),\ q = s^{-1}(i-1, j_0) \text{ where } j_0 = \sup\{j' \in \mathbb{Z} \mid (i-1, j') \in T\}, \tag{3.13}$$

and let $Y = \mathbb{S}_{\langle X_{i-1,\geqslant j-1} \rangle}(X_{i-1,j-1}) = \mathbb{S}_{\langle E_{[p,q]} \rangle}(E_p)$.
If $(i', j') \in S$ and $i' > i$,

$$\mathrm{Hom}_{\mathcal{D}}(X_{i',j'}, Y[n]) \overset{(\mathrm{L2})}{=} 0 \text{ for any integer } n.$$

If $(i', j') \in S$, $i' = i$, and $j' < j - 1$,

$$\mathrm{Hom}_{\mathcal{D}}(X_{i',j'}, Y[n]) \overset{(\mathrm{L2})}{=} 0 \text{ for any integer } n.$$

If $(i', j') \in S$, $i' = i$, and $j' > j - 1$,

$$\mathrm{Hom}_{\mathcal{D}}(X_{i',j'}, Y[n]) \overset{(\mathrm{S2}')}{=} \mathrm{Hom}_{\mathcal{D}}(X_{i-1,j'}, Y[n]) \simeq \mathrm{Hom}_{\mathcal{D}}(X_{i-1,j-1}, X_{i-1,j'}[-n])^* \overset{(\mathrm{L2})}{=} 0$$

for any integer n. So we have that

$$\mathrm{Hom}_{\mathcal{D}}(E_{[q+1,n]}, \mathbb{S}_{\langle E_{[p,q]} \rangle}(E_p)[n]) = 0 \text{ for any integer } n.$$

Thus

$$\begin{aligned} \mathbb{S}_{\langle X_S \rangle}(X_{i,j}) &\overset{2.8(b)}{=} \mathbb{T}_{\langle X_S \rangle} \mathbb{S}_{\langle X_T \rangle}(X_{i,j}) \simeq \mathbb{T}_{\langle X_S \rangle}(X_{i-1,j-1}) = \mathbb{T}_{\langle E_{[1,n] \setminus \{p\}} \rangle}(E_p) \\ &\overset{2.17}{=} \mathbb{T}^{\langle E_{[p,q]} \rangle}_{\langle E_{[p+1,q]} \rangle}(E_p) \overset{(3.13)}{=} \mathbb{T}^{\langle X_{i-1,J} \rangle}_{\langle X_{i-1,J \setminus \{j-1\}} \rangle}(X_{i-1,j-1}) \overset{3.17}{=} \mathbb{S}_{\langle X_{i-1} \rangle}(X_{i-1,j}) \end{aligned}$$

where $J = [j-1, j_0] \cap S_{i-1}$. □

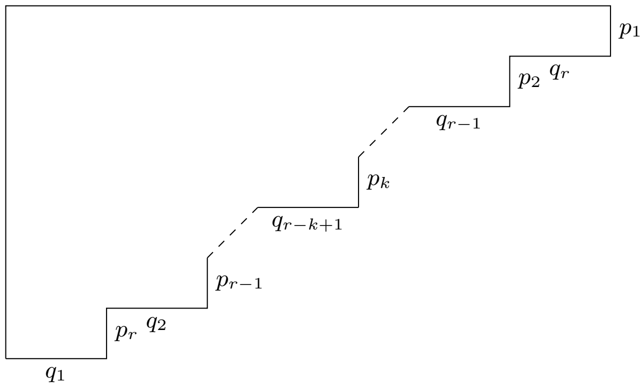


FIGURE 3. The shape of the Young diagram $\mathrm{Y}(\mathsf{p};\mathsf{q})$

4. S-families in derived categories of Nakayama algebras

4.1. The algebras $L(\mathsf{p};\mathsf{q})$ and $L^!(\mathsf{p};\mathsf{q})$. Let $\mathsf{p}=(p_k)_{k\in[1,r]}$ and $\mathsf{q}=(q_k)_{k\in[1,r]}$ be two sequences of positive integers. For any integer $s\in[0,r]$, define

$$\bar{p}_s := \sum_{k=1}^{s} p_k,\quad \bar{q}_s := \sum_{k=1}^{s} q_k,\quad \bar{p} := \bar{p}_r,\quad \bar{q} := \bar{q}_r.$$

Consider the Young diagram

$$\mathrm{Y}(\mathsf{p};\mathsf{q}) := \bigcup_{k\in[0,r-1]} [1+\bar{p}_k, \bar{p}_{k+1}]\times[1,\bar{q}_{r-k}] = \bigcup_{k\in[0,r-1]} [1,\bar{p}_{r-k}]\times[1+\bar{q}_k,\bar{q}_{k+1}]$$

(see Figure 3). Let

$$e(\mathsf{p};\mathsf{q}) := \sum_{(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})} e_i\otimes e_j,\quad e^!(\mathsf{p};\mathsf{q}) := \sum_{(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})} e^!_i\otimes e^!_j$$

where $e_i\otimes e_j$ (resp. $e^!_i\otimes e^!_j$) is the primitive idempotent in $N(\bar{p})\otimes N(\bar{q})$ (resp. $K\mathbb{A}_{\bar{p}}\otimes K\mathbb{A}_{\bar{q}}$) corresponding to the vertex $(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})$. Then the algebras $L(\mathsf{p};\mathsf{q})$ and $L^!(\mathsf{p};\mathsf{q})$ are defined as

$$\begin{aligned} L(\mathsf{p};\mathsf{q}) &= L(p_1,\dots,p_r;q_1,\dots,q_r) := L(\mathrm{Y}(\mathsf{p};\mathsf{q})) = e(\mathsf{p};\mathsf{q})(N(\bar{p})\otimes N(\bar{q}))e(\mathsf{p};\mathsf{q}) \\ &\simeq (N(\bar{p})\otimes N(\bar{q}))/\langle 1-e(\mathsf{p};\mathsf{q})\rangle, \end{aligned} \tag{4.1}$$

$$\begin{aligned} L^!(\mathsf{p};\mathsf{q}) &= L^!(p_1,\dots,p_r;q_1,\dots,q_r) := e^!(\mathsf{p};\mathsf{q})(K\mathbb{A}_{\bar{p}}\otimes K\mathbb{A}_{\bar{q}})e^!(\mathsf{p};\mathsf{q}) \\ &\simeq (K\mathbb{A}_{\bar{p}}\otimes K\mathbb{A}_{\bar{q}})/\langle 1-e^!(\mathsf{p};\mathsf{q})\rangle. \end{aligned} \tag{4.2}$$

By the definition of $L(\mathsf{p};\mathsf{q})$ and $L^!(\mathsf{p};\mathsf{q})$, there exist natural isomorphisms $L(\mathsf{p};\mathsf{q})\to L(\mathsf{q};\mathsf{p})$ and $L^!(\mathsf{p};\mathsf{q})\to L^!(\mathsf{q};\mathsf{p})$.

Example 4.1. *The quivers of $L(3;4)$ and $L^!(3;4)$ are the following respectively:*

$$\begin{array}{ccccccc}
(1,1) & \xleftarrow{u_{1,1}} & (1,2) & \xleftarrow{u_{1,2}} & (1,3) & \xleftarrow{u_{1,3}} & (1,4) \\
\uparrow v_{1,1} & & \uparrow v_{1,2} & & \uparrow v_{1,3} & & \uparrow v_{1,4} \\
(2,1) & \xleftarrow{u_{2,1}} & (2,2) & \xleftarrow{u_{2,2}} & (2,3) & \xleftarrow{u_{2,3}} & (2,4) \\
\uparrow v_{2,1} & & \uparrow v_{2,2} & & \uparrow v_{2,3} & & \uparrow v_{2,4} \\
(3,1) & \xleftarrow{u_{3,1}} & (3,2) & \xleftarrow{u_{3,2}} & (3,3) & \xleftarrow{u_{3,3}} & (3,4)
\end{array}
\qquad
\begin{array}{ccccccc}
(1,1) & \xrightarrow{u^!_{1,1}} & (1,2) & \xrightarrow{u^!_{1,2}} & (1,3) & \xrightarrow{u^!_{1,3}} & (1,4) \\
\downarrow v^!_{1,1} & & \downarrow v^!_{1,2} & & \downarrow v^!_{1,3} & & \downarrow v^!_{1,4} \\
(2,1) & \xrightarrow{u^!_{2,1}} & (2,2) & \xrightarrow{u^!_{2,2}} & (2,3) & \xrightarrow{u^!_{2,3}} & (2,4) \\
\downarrow v^!_{2,1} & & \downarrow v^!_{2,2} & & \downarrow v^!_{2,3} & & \downarrow v^!_{2,4} \\
(3,1) & \xrightarrow{u^!_{3,1}} & (3,2) & \xrightarrow{u^!_{3,2}} & (3,3) & \xrightarrow{u^!_{3,3}} & (3,4)
\end{array}$$

The quivers of $L(1,2,1;1,2,2)$ *and* $L^!(1,2,1;1,2,2)$ *are the following respectively:*

$$\begin{array}{ccccccccc} (1,1) & \xleftarrow{u_{1,1}} & (1,2) & \xleftarrow{u_{1,2}} & (1,3) & \xleftarrow{u_{1,3}} & (1,4) & \xleftarrow{u_{1,4}} & (1,5) \\ \uparrow{\scriptstyle v_{1,1}} & & \uparrow{\scriptstyle v_{1,2}} & & \uparrow{\scriptstyle v_{1,3}} & & & & \\ (2,1) & \xleftarrow{u_{2,1}} & (2,2) & \xleftarrow{u_{2,2}} & (2,3) & & & & \\ \uparrow{\scriptstyle v_{2,1}} & & \uparrow{\scriptstyle v_{2,2}} & & \uparrow{\scriptstyle v_{2,3}} & & & & \\ (3,1) & \xleftarrow{u_{3,1}} & (3,2) & \xleftarrow{u_{3,2}} & (3,3) & & & & \\ \uparrow{\scriptstyle v_{3,1}} & & & & & & & & \\ (4,1) & & & & & & & & \end{array} \qquad \begin{array}{ccccccccc} (1,1) & \xrightarrow{u^!_{1,1}} & (1,2) & \xrightarrow{u^!_{1,2}} & (1,3) & \xrightarrow{u^!_{1,3}} & (1,4) & \xrightarrow{u^!_{1,4}} & (1,5) \\ \downarrow{\scriptstyle v^!_{1,1}} & & \downarrow{\scriptstyle v^!_{1,2}} & & \downarrow{\scriptstyle v^!_{1,3}} & & & & \\ (2,1) & \xrightarrow{u^!_{2,1}} & (2,2) & \xrightarrow{u^!_{2,2}} & (2,3) & & & & \\ \downarrow{\scriptstyle v^!_{2,1}} & & \downarrow{\scriptstyle v^!_{2,2}} & & \downarrow{\scriptstyle v^!_{2,3}} & & & & \\ (3,1) & \xrightarrow{u^!_{3,1}} & (3,2) & \xrightarrow{u^!_{3,2}} & (3,3) & & & & \\ \downarrow{\scriptstyle v^!_{3,1}} & & & & & & & & \\ (4,1) & & & & & & & & \end{array}$$

and the relations are the following respectively:

$$u_{i,j}u_{i,j+1}=0, \quad v_{i,j}v_{i+1,j}=0, \quad u_{i,j}v_{i,j+1}-v_{i,j}u_{i+1,j}=0, \quad v^!_{i,j+1}u^!_{i,j}-u^!_{i+1,j}v^!_{i,j}=0.$$

From the following result, $L(\mathsf{p};\mathsf{q})$ is derived equivalent to $L^!(\mathsf{p};\mathsf{q})$.

Proposition 4.2. *The family* $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ *of objects*

$$X_{i,j}=S_{L^!(\mathsf{p};\mathsf{q})}(i,j)[-i-j]$$

is a full $\mathrm{Y}(\mathsf{p};\mathsf{q})$*-family in* $\operatorname{per} L^!(\mathsf{p};\mathsf{q})$*. Then the object* $X_{\mathrm{Y}(\mathsf{p};\mathsf{q})}=\bigoplus_{(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})} X_{i,j}$ *is a tilting object in* $\operatorname{per} L^!(\mathsf{p};\mathsf{q})$ *such that* $\operatorname{End}_{\operatorname{per} L^!(\mathsf{p};\mathsf{q})}(X_{\mathrm{Y}(\mathsf{p};\mathsf{q})})\simeq L(\mathsf{p};\mathsf{q})$*, and there exists a triangle equivalence*

$$F:\operatorname{per} L^!(\mathsf{p};\mathsf{q})\to \operatorname{per} L(\mathsf{p};\mathsf{q})$$

such that $F(X_{i,j})\simeq P(i,j)$ *for any* $(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})$*.*

Proof. Let $A=L^!(\bar{p};\bar{q})=K\mathbb{A}_{\bar{p}}\otimes K\mathbb{A}_{\bar{q}}$, $B=N(\bar{p})\otimes N(\bar{q})$. By Example 3.10, there exists a triangle equivalence $G:\operatorname{per} A\to\operatorname{per} B$ such that $G(S_A(i,j)[-i-j])\simeq P_B(i,j)$ for any $(i,j)\in\mathrm{Y}(\bar{p},\bar{q})$. Let

$$P=e^!(\mathsf{p};\mathsf{q})A,\ \ P'=(1-e^!(\mathsf{p};\mathsf{q}))A,\ \ T=\bigoplus_{(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})} S_A(i,j)[-i-j],\ \ R=G(T)\simeq e(\mathsf{p};\mathsf{q})B.$$

Since $\operatorname{End}_{\operatorname{per} B}(R)\simeq L(\mathsf{p};\mathsf{q})$, there exists a triangle equivalence $F_1:\langle R\rangle\to\operatorname{per} L(\mathsf{p};\mathsf{q})$ such that $F_1(P_B(i,j))\simeq P_{L(\mathsf{p};\mathsf{q})}(i,j)$ for any $(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})$ by Example 2.11.

Since $\operatorname{End}_{\operatorname{per} A}(P)\simeq L^!(\mathsf{p};\mathsf{q})$ and $\operatorname{Hom}_{\operatorname{per} A}(P',P)\simeq 0$, there exists a triangle equivalence $F_2:\langle P\rangle\to\operatorname{per} L^!(\mathsf{p};\mathsf{q})$ such that $F_2(P_A(i,j))\simeq P_{L^!(\mathsf{p};\mathsf{q})}(i,j)$ and $F_2(S_A(i,j))\simeq S_{L^!(\mathsf{p};\mathsf{q})}(i,j)$ for any $(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})$ by Example 2.11. Since

$$F_1GF_2^{-1}(X_{i,j})\simeq P_{L(\mathsf{p};\mathsf{q})}(i,j),$$

the triangle equivalence $F_1GF_2^{-1}:\operatorname{per} L^!(\mathsf{p};\mathsf{q})\to\operatorname{per} L(\mathsf{p};\mathsf{q})$ sends the family $(X_{i,j})_{(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})}$ to the full $\mathrm{Y}(\mathsf{p};\mathsf{q})$-family $(P_{L(\mathsf{p};\mathsf{q})}(i,j))_{(i,j)\in\mathrm{Y}(\mathsf{p};\mathsf{q})}$ in $\operatorname{per} L(\mathsf{p};\mathsf{q})$ given by Example 3.9. Thus the assertion follows.

$$\begin{array}{ccccccccc} \operatorname{per} L^!(\mathsf{p};\mathsf{q}) & \xleftarrow[\simeq]{F_2} & \langle P\rangle & = & \langle T\rangle & \xrightarrow[\simeq]{G} & \langle R\rangle & \xrightarrow[\simeq]{F_1} & \operatorname{per} L(\mathsf{p};\mathsf{q}) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \operatorname{per} A & = & \operatorname{per} A & \xrightarrow[G]{\simeq} & \operatorname{per} B. & & \end{array} \tag{4.3}$$

□

Recall that $\mathcal{D}$ satisfies the condition (3.1) if

$\mathcal{D}$ is algebraic, idempotent complete, Ext-finite and has a Serre functor $\mathbb{S}$.

For any two sequences of positive integers $\mathsf{p}=(p_k)_{k\in[1,r]}$ and $\mathsf{q}=(q_k)_{k\in[1,r]}$, define

$$\lambda_{\mathsf{p};\mathsf{q}}(i)=\lambda(i):=\sup\{j'\in\mathbb{Z}\mid (i,j')\in\mathrm{Y}(\mathsf{p};\mathsf{q})\}.$$

When $S=\mathrm{Y}(\mathsf{p};\mathsf{q})$, the definition of S-families are characterized by the following much simpler conditons than (L1), (L2), (S1)-(S3).

Theorem 4.3. *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1), $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ *a family of exceptional objects in $\mathcal{D}$. Then $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ is a full $\mathrm{Y}(\mathsf{p};\mathsf{q})$-family in $\mathcal{D}$ if and only if $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ satisfies the following conditions:*

(Y1) $\mathcal{D} = \langle X_1\rangle \perp \langle X_2\rangle \perp \cdots \perp \langle X_{\bar{p}}\rangle$.
(Y2) $\langle X_1\rangle = \langle X_{1,1}\rangle \perp \langle X_{1,2}\rangle \perp \cdots \perp \langle X_{1,\lambda(1)}\rangle$.
(Y3) $\mathbb{S}_{\langle X_1\rangle}(X_{1,j}) \simeq X_{1,j-1}$ *for any integer* $j \in (1, \lambda(i)]$.
(Y4) $\mathbb{S}(X_{i,j}) \simeq \mathbb{S}_{\langle X_{i-1}\rangle}(X_{i-1,j})$ *for any integers* $i \in (1, \bar{p}]$ *and* $j \in [1, \lambda(i)]$.

In this case, there exists a triangle equivalence

$$F : \mathcal{D} \to \operatorname{per} L(\mathsf{p};\mathsf{q})$$

such that $F(X_{i,j}) \simeq P(i,j)$ *for any* $(i,j) \in \mathrm{Y}(\mathsf{p};\mathsf{q})$.

To prove Theorem 4.3, we prepare the following two results.

Lemma 4.4. *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1), $(X_i)_{i\in[1,n]}$ *a family of objects in $\mathcal{D}$. If the conditons*

(4.4) $\mathcal{D} = \langle X_1\rangle \perp \langle X_2\rangle \perp \cdots \perp \langle X_n\rangle$

(4.5) $\mathbb{S}(X_i) \in \langle X_{i-1}\rangle$ *for any integer* $i \in (1, n]$

are satisfied, then

(4.6) $\operatorname{Hom}_{\mathcal{D}}(X_i, X_{i'}[n]) = 0$ *for any integer n unless* $i' \in [i-1, i]$.

Proof. If $i < i'$, $\operatorname{Hom}_{\mathcal{D}}(X_i, X_{i'}[n]) \overset{(4.4)}{=} 0$ for any integer n. If $i - i' \geqslant 2$, then $\mathbb{S}(X_i) \in \langle X_{i-1}\rangle$ by (4.5) and $\operatorname{Hom}_{\mathcal{D}}(X_i, X_{i'}[n]) \simeq \operatorname{Hom}_{\mathcal{D}}(X_{i'}, \mathbb{S}(X_i)[-n])^* \overset{(4.4)}{=} 0$ for any integer n. Thus the assertion follows. □

Lemma 4.5. *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1), $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ *a family of exceptional objects in $\mathcal{D}$ satisfying* (Y1)-(Y4). *Then $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ satisfies the following conditions:*

(Y2′) $\langle X_i\rangle = \langle X_{i,1}\rangle \perp \langle X_{i,2}\rangle \perp \cdots \perp \langle X_{i,\lambda(i)}\rangle$ *for any integer* $i \in [1, \bar{p}]$.

(S1) $\mathbb{S}_{\langle X_i\rangle}(X_{i,j}) \simeq X_{i,j-1}$ *for any integers* $i \in [1, \bar{p}]$, $j \in (1, \lambda(i)]$.

Proof. When $i = 1$, (Y2′) is nothing but (Y2). Assume that (Y2′) holds for i. If $j < j'$,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(X_{i+1,j}, X_{i+1,j'}[n]) &\overset{(\mathrm{Y4})}{=} \operatorname{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}\mathbb{S}_{\langle X_i\rangle}(X_{i,j}), \mathbb{S}^{-1}\mathbb{S}_{\langle X_i\rangle}(X_{i,j'})[n]) \\ &\simeq \operatorname{Hom}_{\mathcal{D}}(X_{i,j}, X_{i,j'}[n]) = 0\end{aligned}$$

for any integer n. So (Y2′) holds for $i+1$.

When $i = 1$, (S1) is nothing but (Y3). Assume that (S1) holds for i. Then $F = \mathbb{S}^{-1}\mathbb{S}_{\langle X_i\rangle} : \langle X_{i,\leqslant\lambda(i+1)}\rangle \to \langle X_{i+1}\rangle$ is a triangle equivalence by (Y4) and

$$\begin{aligned}\mathbb{S}_{\langle X_{i+1}\rangle}(X_{i+1,j}) &\overset{(\mathrm{Y4})}{=} \mathbb{S}_{\langle X_{i+1}\rangle}F(X_{i,j}) \overset{2.2(b)(c)}{=} F\mathbb{S}_{\langle X_{i,\leqslant\lambda(i+1)}\rangle}(X_{i,j}) \\ &\overset{2.8(c)}{=} F\mathbb{S}_{\langle X_i\rangle}(X_{i,j}) \overset{(\mathrm{S1})}{=} F(X_{i,j-1}) \overset{(\mathrm{Y4})}{=} X_{i+1,j-1}.\end{aligned}$$

Thus (S1) holds for $i+1$.

$$\begin{array}{ccccc}
\langle X_i\rangle & \hookleftarrow & \langle X_{i,\leqslant\lambda(i+1)}\rangle & \xrightarrow{F} & \langle X_{i+1}\rangle \\
\downarrow{\scriptstyle \mathbb{S}_{\langle X_i\rangle}} & & \downarrow{\scriptstyle \mathbb{S}_{\langle X_{i,\leqslant\lambda(i+1)}\rangle}} & & \downarrow{\scriptstyle \mathbb{S}_{\langle X_{i+1}\rangle}} \\
\langle X_i\rangle & \xrightarrow[\mathbb{T}_{\langle X_{i,\leqslant\lambda(i+1)}\rangle}]{} & \langle X_{i,\leqslant\lambda(i+1)}\rangle & \xrightarrow[F]{} & \langle X_{i+1}\rangle
\end{array}$$

□

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. If $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ is a full $\mathrm{Y}(\mathsf{p};\mathsf{q})$-family, there exists a triangle equivalence $F:\mathcal{D}\to \operatorname{per} L(\mathsf{p};\mathsf{q})$ by Theorem 3.11. So we will prove the equivalence of two conditions.

We prove "only if" part. Since $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ is a full weak $\mathrm{Y}(\mathsf{p};\mathsf{q})$-family, (Y1) and (Y2) are satisfied. By (S2), we have that (Y3) is satisfied. By Proposition 3.16, (Y4) is satisfied. Thus the assertion follows.

We prove "if" part. By the assumption, (L1) is satisfied. By (Y1) and (Y4), we see that (L2.1) is satisfied by Lemma 4.4. Since

$$\mathbb{F}_{\langle X_{i-1}\rangle}(X_{i,j}) \simeq \mathbb{S}^{-1}_{\langle X_{i-1}\rangle}\mathbb{S}(X_{i,j}) \overset{(\mathrm{Y4})}{=} \mathbb{S}^{-1}_{\langle X_{i-1}\rangle}\mathbb{S}_{\langle X_{i-1}\rangle}(X_{i-1,j}) \simeq X_{i-1,j},$$

(S2)$'$ is satisfied. If $j-j'\geqslant 2$,

$$\operatorname{Hom}_{\mathcal{D}}(X_{i,j},X_{i-1,j'}[n]) \simeq \operatorname{Hom}_{\mathcal{D}}(\mathbb{F}_{\langle X_{i-1}\rangle}(X_{i,j}),X_{i-1,j'}[n]) \overset{(\mathrm{S2})'}{=} \operatorname{Hom}(X_{i-1,j},X_{i,j'}) \overset{(\mathrm{L2.1})}{=} 0.$$

So (L2.2) is satisfied. Thus $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ is a weak S-family satisfying (S2)$'$.

In the last, we prove that $(X_{i,j})_{(i,j)\in \mathrm{Y}(\mathsf{p};\mathsf{q})}$ is an S-family. By Lemma 4.5, (S1) is satisfied. By (S1) and (Y4), we see that (S3) is satisfied. Thus the assertion follows. □

4.2. $\mathrm{Y}(p,q,r)$**-families.** The aim of this section is to prove Theorem 4.7. Let p, q, and r be three positive integers such that $pq>q+1$, $0\leqslant r\leqslant q$. Let

$$\mathrm{Y}(p,q,r)=\begin{cases}\mathrm{Y}(p-1,1;q-r,r) & r<q,\\ \mathrm{Y}(p-1;q) & r=q,\end{cases}$$

$$L(p,q,r)=L(\mathrm{Y}(p,q,r)).$$

The following application of Theorem 4.3 is useful to construct a $\mathrm{Y}(p;q)$-family:

Proposition 4.6. *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1), $(E_k)_{k\in[1,q]}$ *an exceptional sequece in $\mathcal{D}$, and $E=\bigoplus_{k\in[1,q]}E_k$. Suppose that the following conditions are satisfied:*

(4.7) $\mathcal{D}=\langle \mathbb{S}^{p-1}(E)\rangle \perp \langle \mathbb{S}^{p-2}(E)\rangle \perp \cdots \perp \langle E\rangle.$

(4.8) $\mathbb{S}_{\langle E\rangle}(E_i)\simeq E_{i-1}$ *for any integer* $i\in(1,p]$.

Then the family $(X_{i,j})_{(i,j)\in \mathrm{Y}(p;q)}$ of objects

$$X_{i,j}:=\mathbb{F}_{\langle \mathbb{S}^{p-i}(E)\rangle}\mathbb{F}_{\langle \mathbb{S}^{p-i-1}(E)\rangle}\cdots \mathbb{F}_{\langle \mathbb{S}(E)\rangle}(E_j)$$

is a full $\mathrm{Y}(p;q)$-family in $\mathcal{D}$. In particular, there exists a triangle equivalence

$$F:\mathcal{D}\to \operatorname{per} L(p;q)$$

such that $F(X_{i,j})\simeq P(i,j)$ for any $(i,j)\in \mathrm{Y}(p;q)$.

Proof. It suffices to check the conditions (Y1)-(Y4). Since $\mathbb{F}_{\langle \mathbb{S}^{i+1}(E)\rangle}:\langle \mathbb{S}^{i}(E)\rangle\to\langle \mathbb{S}^{i+1}(E)\rangle$ is a triangle equivalence for each $i\in\mathbb{Z}$ by Lemma 2.9(d), $\langle X_i\rangle=\langle \mathbb{S}^{p-i}(E)\rangle$. Since (4.7) is satisfied, (Y1) is satisfied. Then we see that $\mathbb{S}(\langle X_i\rangle)=\langle X_{i-1}\rangle$. Since

$$\mathbb{S}^{-1}_{\langle X_{i-1}\rangle}\mathbb{S}(X_{i,j}) \overset{2.9(c)}{=} \mathbb{F}_{\langle X_{i-1}\rangle}(X_{i,j})=X_{i-1,j},$$

(Y4) is satisfied. Since $\mathbb{F}_{\langle \mathbb{S}^{i+1}(E)\rangle}:\langle \mathbb{S}^{i}(E)\rangle\to\langle \mathbb{S}^{i+1}(E)\rangle$ is a triangle equivalence for each $i\in\mathbb{Z}$ by Lemma 2.9(d),

$$G=\mathbb{F}_{\langle \mathbb{S}^{p-1}(E)\rangle}\mathbb{F}_{\langle \mathbb{S}^{p-2}(E)\rangle}\cdots \mathbb{F}_{\langle \mathbb{S}(E)\rangle}:\langle E\rangle\to\langle X_1\rangle$$

is a triangle equivalence. Since G sends the exceptional sequence $(E_i)_{i\in[1,q]}$ to the family $(X_{1,j})_{j\in[1,q]}$, it follows that (Y2) is satisfied. If $j\in(1,q]$,

$$\mathbb{S}_{\langle X_1\rangle}(X_{1,j})\simeq \mathbb{S}_{\langle X_1\rangle}G(E_j)\simeq G\mathbb{S}_{\langle E\rangle}(E_j) \overset{(4.8)}{=} G(E_{j-1})\simeq X_{1,j-1}.$$

So (Y3) is satisfied. Thus the assertion follows from Theorem 4.3. □

The following result is one of our main results:

Theorem 4.7. *Let p, q, and r be three positive integers such that* $pq > q+1$, $0 \leqslant r \leqslant q-1$. *Then there exists a full* $\Upsilon(p,q,r)$*-family* $(X_{i,j})_{(i,j)\in\Upsilon(p,q,r)}$ *in* $\operatorname{per} N(pq-r, q+1)$. *In particular, there exists a triangle equivalence*

$$F : \operatorname{per} N(pq-r, q+1) \to \operatorname{per} L(p,q,r)$$

such that $F(X_{i,j}) = P_{L(p,q,r)}(i,j)$ *for any* $(i,j) \in \Upsilon(p,q,r)$.

We first prove the case $r = 0$ of Theorem 4.7 by applying Proposition 4.6.

Proposition 4.8. *Let p and q be two positive integers such that* $pq > q+1$, $A = N(pq, q+1)$.

(a) *Let* $(X_{i,j})_{(i,j)\in\Upsilon(p;q)}$ *be a family of objects* $X_{i,j} \in \operatorname{per} A$ *defined as*

$$X_{i,j} = \mathbb{F}_{\langle \nu^{p-i}(P)\rangle}\mathbb{F}_{\langle \nu^{p-i-1}(P)\rangle}\cdots\mathbb{F}_{\langle \nu(P)\rangle}(S(j))[-j]$$

where $P = \bigoplus_{k\in[1,q]} P(k)$. *Then* $(X_{i,j})_{(i,j)\in\Upsilon(p;q)}$ *is a full* $\Upsilon(p;q)$*-family in* $\operatorname{per} A$.

(b) *Let* $(X'_{i,j})_{(i,j)\in\Upsilon(p;q)}$ *be a family of objects* $X'_{i,j} \in \operatorname{per} A$ *defined as*

$$X'_{i,j} = \mathbb{F}_{\langle \nu^{p-i}(S)\rangle}\mathbb{F}_{\langle \nu^{p-i-1}(S)\rangle}\cdots\mathbb{F}_{\langle \nu(S)\rangle}(S((p-1)q+j))[-j]$$

where $S = \bigoplus_{k\in[1,q]} S((p-1)q+k)$. *Then* $(X'_{i,j})_{(i,j)\in\Upsilon(p;q)}$ *is a full* $\Upsilon(p;q)$*-family in* $\operatorname{per} A$.

Proof. (a) It suffices to check the conditions (4.7)-(4.8) by Proposition 4.6. Let $E_i = S(i)[-i]$, $E = \bigoplus_{i\in[1,q]} E_i$. Since $\langle E\rangle = \langle P\rangle$ and $\nu^k(P(i)) \simeq P(pk+i)$, we have

$$\mathcal{D} = \langle \nu^{p-1}(P)\rangle \perp \langle \nu^{p-2}(P)\rangle \perp \cdots \perp \langle P\rangle = \langle \nu^{p-1}(E)\rangle \perp \langle \nu^{p-2}(E)\rangle \perp \cdots \perp \langle E\rangle,$$

and so (4.7) is satisfied.

Since $E = E_{[1,q]}$ is a pretilting object such that $\operatorname{End}_{\mathcal{D}}(E) \simeq N(q)$, there exists a triangle functor $F : \langle E\rangle \to \operatorname{per} N(q)$ such that $F(E_i) \simeq P(i)$, and so (4.8) is satisfied. Thus the assertion follows.

(b) Let $S' = \bigoplus_{k\in[1,q]} S(k)$. Since $\nu^{p-1}(S(k)) \simeq S((p-1)q+k)$ for any $k \in [1,q]$, we have $\langle \nu^{p-1}(P)\rangle = \langle \nu^{p-1}(S')\rangle = \langle S\rangle$, and so the assertion follows from (a). □

Now we are ready to prove Theorem 4.7.

Proof of Theorem 4.7. Let $A = N(pq, q+1)$, $B = N(pq-r, q+1)$ and let

$$P = \bigoplus_{k\in I} P(k), \quad P' = \bigoplus_{k\in[1,pq]\setminus I} P(k), \quad S = \bigoplus_{k\in I} S(k), \quad S' = \bigoplus_{k\in[1,pq]\setminus I} S(k)$$

where $I = [1, pq-r]$. Since $\operatorname{Hom}_{\operatorname{per} A}(P', P) \simeq 0$ and $\operatorname{End}_{\operatorname{per} A}(P) \simeq B$, we have $\langle P\rangle = \langle S\rangle$ and there exists a triangle equivalence $F : \langle P\rangle \to \operatorname{per} B$ such that $F(P_A(i)) = P_B(i)$, $F(S_A(i)) \simeq S_B(i)$ by Example 2.11. Thus it suffices to show that there exists a full $\Upsilon(p,q,r)$-family in $\langle S\rangle$.

Let $(X'_{i,j})_{(i,j)\in\Upsilon(p;q)}$ be a full $\Upsilon(p;q)$-family in $\operatorname{per} N(pq, q+1)$ given by Proposition 4.8(b). Since $Y_{p,q-j} = S(pq-j)[-q+j]$, we have $\langle S'\rangle = \langle X'_{p,[q-r+1,q]}\rangle$, and so $\langle X'_{\Upsilon(p,q,r)}\rangle = \langle S'\rangle^{\perp_{\operatorname{per} A}} = \langle S\rangle$ by Example 2.11. Thus the subfamily $(X'_{i,j})_{(i,j)\in\Upsilon(p,q,r)}$ is a full $\Upsilon(p,q,r)$-family in $\langle S\rangle$. □

5. Mutations of S-families

In this section, let $\mathcal{D}$ be a triangulated category satisfying (3.1). The purpose of this section is to introduce mutations of S-families on some assumption for S, and to prove Theorems 5.14 and 5.16 by using those results. Let

$$\mathfrak{P}_{\mathrm{fin}}(\mathbb{Z}^2) := \{S \subset \mathbb{Z}^2 \mid |S| < \infty\}.$$

In $\mathfrak{P}_{\rm fin}(\mathbb{Z}^2)$, the relation $S \equiv S'$ defined as

"there exists an element $v \in \mathbb{Z}^2$ such that $S' = S + v$"

is an equivalence relation on $\mathfrak{P}_{\rm fin}(\mathbb{Z}^2)$. Clearly, if a family $(X_{i,j})_{(i,j)\in S}$ is an S-family, then for $S' = S+(a,b)$ with $(a,b) \in \mathbb{Z}^2$, the family $(X_{i-a,j-b})_{(i,j)\in S'}$ is an S'-family. For any $S \in \mathfrak{P}_{\rm fin}(\mathbb{Z}^2)$, define

$$^tS := \{(i,j) \in \mathbb{Z}^2 \mid (j,i) \in S\}.$$

Clearly, a family $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ is an S-family if and only if the family $^t\mathsf{X} = (X_{i,j})_{(i,j)\in {}^tS}$ is an ^tS-family.

5.1. **Gluings of S-families.** In this section, we introduce gluings of S-families (Proposition 5.2, Proposition 5.3). For any interval I of $\mathbb{Z}$,

$$S_I := \{(i,j) \in S \mid i \in I\}, \quad S^I := \{(i,j) \in S \mid j \in I\}.$$

In particular,

$$S_{\leqslant k} := S_{(\infty,k]}, \quad S_{\geqslant k} := S_{[k,\infty)}, \quad S^{\leqslant k} := S^{(\infty,k]}, \quad S^{\geqslant k} := S^{[k,\infty)}.$$

Let $\mathsf{X} = (X_{i,j})$ be a family of objects in $\mathcal{D}$ indexed by $S \in \mathfrak{P}_{\rm fin}(\mathbb{Z}^2)$. For any integer k, define

$$\mathsf{X}_{\leqslant k} = (X_{i,j})_{(i,j)\in S_{\leqslant k}}, \quad \mathsf{X}_{\geqslant k} = (X_{i,j})_{(i,j)\in S_{\geqslant k}},$$
$$\mathsf{X}^{\leqslant k} = (X_{i,j})_{(i,j)\in S^{\leqslant k}}, \quad \mathsf{X}^{\geqslant k} = (X_{i,j})_{(i,j)\in S^{\geqslant k}}.$$

Lemma 5.1. *Let $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ be a family of objects in $\mathcal{D}$ indexed by $S \in \mathfrak{P}_{\rm fin}(\mathbb{Z}^2)$ satisfying* (L2.1)*. Let k be an integer.*

(a) *If $(i,j) \in S_{\geqslant k+1}$, then $\mathbb{S}_{\langle X_S\rangle}(X_{i,j}) \simeq \mathbb{S}_{\langle X_{S_{\geqslant k}}\rangle}(X_{i,j})$ and $\mathbb{S}_{\langle X^j\rangle}(X_{i,j}) \simeq \mathbb{S}_{\langle X_{\geqslant k,j}\rangle}(X_{i,j})$.*

(b) *If $(i,j) \in S_{\leqslant k-1}$, then $\mathbb{S}^{-1}_{\langle X_S\rangle}(X_{i,j}) \simeq \mathbb{S}^{-1}_{\langle X_{S_{\leqslant k}}\rangle}(X_{i,j})$ and $\mathbb{S}^{-1}_{\langle X^j\rangle}(X_{i,j}) \simeq \mathbb{S}^{-1}_{\langle X_{\leqslant k,j}\rangle}(X_{i,j})$.*

Proof. (a) From (L2.1), we have $\langle X_S\rangle = \langle X_{S_{\leqslant k-1}}\rangle \perp \langle X_{S_{\geqslant k}}\rangle$. Since

$$\mathrm{Hom}_{\mathcal{D}}(X_{S_{\leqslant k-1}}, \mathbb{S}_{\langle X_S\rangle}(X_{i,j})[n]) \simeq \mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{S_{\leqslant k-1}}[-n])^* \overset{\text{(L2.1)}}{=} 0,$$

we have $\mathbb{S}_{\langle X_S\rangle}(X_{i,j}) \in \langle X_{S_{\geqslant k}}\rangle$. Thus the assertion follows from Proposition 2.8(c). Since the subfamily $(X_{i,j})_{(i,j)\in S^j\times\{j\}}$ satisfies (L2.1), the assertion follows.
(b) This is the dual of (a). □

Proposition 5.2 (Gluing I)**.** *Let $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ be a family of objects in $\mathcal{D}$ indexed by $S \in \mathfrak{P}_{\rm fin}(\mathbb{Z}^2)$ satisfying* (L2.1)*. For any integer k, the following conditions are equivalent:*

(i) *X is an S-family.*
(ii) *$\mathsf{X}_{\leqslant k}$ is an $S_{\leqslant k}$-family and $\mathsf{X}_{\geqslant k}$ is an $S_{\geqslant k}$-family.*

Proof. (i)⇒ (ii): This is clear.
(ii)⇒ (i): We need to show that X satisfies the conditions (L1), (L2.2) and (S1)-(S3). Clearly (L1) is satisfied. We show that (L2.2) is satisfied. Let (i,j) and (i',j') be two elements in S such that $|j-j'| > 1$. If $(i,j) \in S_{\leqslant k}\backslash S_{\{k\}}$ and $(i',j') \in S_{\geqslant k}\backslash S_{\{k\}}$, we have $\mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}[n]) \simeq 0$ and $\mathrm{Hom}_{\mathcal{D}}(X_{i',j'}, X_{i,j}[n]) \simeq 0$ from (L2.1). If $(i,j),(i',j') \in S_{\leqslant k}$ or $(i,j),(i',j') \in S_{\geqslant k}$, then $\mathrm{Hom}_{\mathcal{D}}(X_{i,j}, X_{i',j'}[n]) \simeq 0$ since $\mathsf{X}_{\leqslant k}$ is an $S_{\leqslant k}$-family and $\mathsf{X}_{\geqslant k}$ is an $S_{\geqslant k}$-family. Thus (L2.2) is satisfied, and so X is a weak S-family.

Since $\mathsf{X}_{\leqslant k}$ is an $S_{\leqslant k}$-family and $\mathsf{X}_{\geqslant k}$ is an $S_{\geqslant k}$-family, (S1) is satisfied.

Suppose that $(i,j),(i-1,j) \in S$. If $(i,j) \in S_{\geqslant k+1}$, then $\mathbb{S}_{\langle X^j\rangle}(X_{i,j}) \overset{5.1(a)}{=} \mathbb{S}_{\langle X_{\geqslant k,j}\rangle}(X_{i,j}) \overset{\text{(S2)}}{=} X_{i-1,j}$. If $(i,j) \in S_{\leqslant k}$, then $(i-1,j) \in S_{\leqslant k-1}$ and $\mathbb{S}^{-1}_{\langle X^j\rangle}(X_{i-1,j}) \overset{5.1(b)}{=} \mathbb{S}^{-1}_{\langle X_{\leqslant k,j}\rangle}(X_{i-1,j}) \overset{\text{(S2)}}{=} X_{i,j}$. Thus (S2) is satisfied.

Suppose that $(i,j),(i-1,j-1) \in S$. If $(i,j) \in S_{\geqslant k+1}$, then $(i-1,j-1) \in S_{\geqslant k}$ and $\mathbb{S}_{\langle X_S\rangle}(X_{i,j}) \overset{5.1(a)}{=}$

$\mathbb{S}_{\langle X_{S_{\geqslant k}}\rangle}(X_{i,j}) \overset{\text{(S3)}}{=} X_{i-1,j-1}$. If $(i,j) \in S_{\leqslant k}$, then $(i-1,j-1) \in S_{\leqslant k-1}$ and $\mathbb{S}^{-1}_{\langle X_S\rangle}(X_{i-1,j-1}) \overset{5.1(b)}{=}$ $\mathbb{S}^{-1}_{\langle X_{S_{\leqslant k}}\rangle}(X_{i-1,j-1}) \overset{\text{(S3)}}{=} X_{i,j}$. Thus (S3) is satisfied. □

By transposing, we have the following result.

Proposition 5.3 (Gluing II). *Let $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ be a family of objects in $\mathcal{D}$ indexed by $S \in \mathfrak{P}_{\rm fin}(\mathbb{Z}^2)$ satisfying* (L2.2). *For any integer k, the following conditions are equivalent:*

(i) X *is an S-family.*
(ii) $\mathsf{X}^{\leqslant k}$ *is an $S^{\leqslant k}$-family and $\mathsf{X}^{\geqslant k}$ is an $S^{\geqslant k}$-family.*

5.2. **Mutations of S-families.** In this subsection, we prove results for mutations of S-families (Theorem 5.6, Theorem 5.12). Let $S \in \mathfrak{P}_{\rm fin}(\mathbb{Z}^2)$. Recall that

$$S_k := \{j \in \mathbb{Z} \mid (k,j) \in S\}, \quad S^k := \{i \in \mathbb{Z} \mid (i,k) \in S\}$$

for any integer k. Let $\sigma_{\leqslant k}, \sigma_{\geqslant k}, \rho_{\leqslant k}, \rho_{\geqslant k}$ be permutaions of $\mathbb{Z}^2$ such that

$$\sigma_{\leqslant k}(i,j) = \begin{cases} (i,j-1) & i \leqslant k, \\ (i,j) & k < i, \end{cases} \quad \sigma_{\geqslant k}(i,j) = \begin{cases} (i,j-1) & i \geqslant k, \\ (i,j) & k > i, \end{cases}$$

$$\rho_{\leqslant k}(i,j) = \begin{cases} (i-1,j) & j \leqslant k, \\ (i,j) & k < j, \end{cases} \quad \rho_{\geqslant k}(i,j) = \begin{cases} (i-1,j) & j \geqslant k, \\ (i,j) & k > j. \end{cases}$$

For any subset I of $\mathbb{Z}$ and any integer $n \in \mathbb{Z}$, we denote by $I + n$ the subset

$$I + n = \{j \in \mathbb{Z} \mid \exists i \in I; j = i + n\}.$$

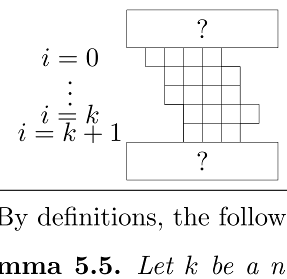
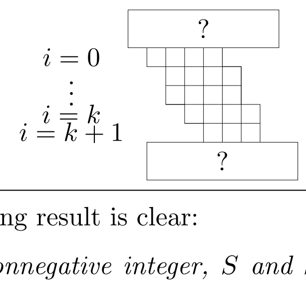
Definition 5.4. *Let k be a nonnegative integer. A finite subset S of $\mathbb{Z}^2$ is called an M_k^+*-subset *if the following conditions are satisfied:*

(M_k^+1) *If $i \in [0,k+1]$, then S_i is an interval of $\mathbb{Z}$.*
(M_k^+2) *If $i \in [1,k]$, then $S_i = S_{i-1}$ or $S_i = S_{i-1}+1$.*
(M_k^+3) *$S_{k+1} \subset S_k$.*

A finite subset S' of $\mathbb{Z}^2$ is called an M_k^--subset *if the following conditions are satisfied:*

(M_k^-1) *If $i \in [0,k+1]$, then S'_i is an interval of $\mathbb{Z}$.*
(M_k^-2) *If $i \in [1,k]$, then $S'_i = S'_{i-1}$ or $S'_i = S'_{i-1}+1$.*
(M_k^-3) *$S'_{k+1} \subset S'_k + 1$.*

For any $(i,j) \in S$, if $S - (i,j)$ is an M_k^+-subset (resp. M_k^--subset), S is called an $M_k^+(i,j)$-subset *(resp. $M_k^-(i,j)$*-subset*).*

<table>
<tr><th>S</th><th>S′</th></tr>
<tr><td>
</td><td>
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By definitions, the following result is clear:

Lemma 5.5. *Let k be a nonnegative integer, S and S' two finite subsets of $\mathbb{Z}^2$ such that $S' = \sigma_{\leqslant k}(S)$. Then S is an M_k^+-subset if and only if S' is an M_k^--subset.*

The following result is the one of main results in this subsection:

Theorem 5.6 (Mutation I). *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1), *S an M_k^+-subset satisfying $S_{\leqslant -1} = \emptyset$, and let $S' = \sigma_{\leqslant k}(S)$.*

(a) *If* $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ *is an* S*-family, then the family* $\mathsf{X}' = (X'_{i,j})_{(i,j)\in S'}$ *is an* S'*-family where*

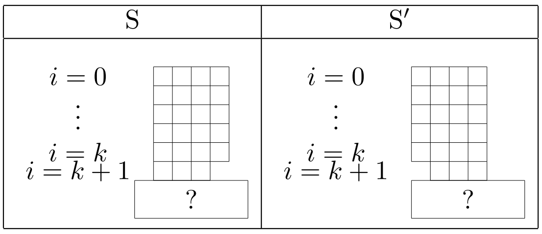
$$X'_{i,j} = \begin{cases} \mathbb{S}_{\langle X_i\rangle}(X_{i,j+1}) & i \leqslant k, \\ X_{i,j} & k < i. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S) \to \operatorname{per} L(S')$.

(b) *If* $\mathsf{Y} = (Y_{i,j})_{(i,j)\in S'}$ *is an* S'*-family, then the family* $\mathsf{Y}' = (Y'_{i,j})_{(i,j)\in S}$ *is an* S*-family where*

$$Y'_{i,j} = \begin{cases} \mathbb{S}^{-1}_{\langle Y_i\rangle}(Y_{i,j-1}) & i \leqslant k, \\ Y_{i,j} & k < i. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S') \to \operatorname{per} L(S)$.



In Theorem 5.6 (a), by the definitions of X and X', there exists a sequence of mutations of exceptional sequences from X to X'.

Remark 5.7. *Let* $I_{-1} = \{2\}$, $I_0 = [1,2]$, $I_1 = \{2\}$ *and*

$$S = \bigsqcup_{k\in[-1,1]} \{k\} \times I_k, \quad S' = \sigma_{\leqslant 0}(S).$$

Then S *is an* M_0^+*-subset such that* $S_{\leqslant -1} \neq \emptyset$. *Since there exist triangle equivalences*

$$\operatorname{per} L(S) \to \operatorname{per} K\mathbb{A}_4 \text{ and } \operatorname{per} L(S') \to \operatorname{per} K\mathbb{D}_4,$$

we have that $\operatorname{per} L(S)$ *and* $\operatorname{per} L(S')$ *are not triangle equivalent to each other.*

To prove Theorem 5.6, we prepare the following three results.

Lemma 5.8. *Let* $\mathcal{D}$ *be a triangulated category satisfying* (3.1), $(X_{i,j})_{(i,j)\in \mathrm{Y}(p;q)}$ *a* $\mathrm{Y}(p;q)$*-family in* $\mathcal{D}$. *Then there exists a triangle autoequivalence* $G : \langle X_{\mathrm{Y}(p;q)}\rangle \to \langle X_{\mathrm{Y}(p;q)}\rangle$ *such that*

$$G|_{\langle X_i\rangle} \simeq \mathbb{S}_{\langle X_i\rangle} : \langle X_i\rangle \to \langle X_i\rangle \text{ for any } i \in [1,p]. \tag{5.1}$$

Proof. Let $A = N(p)\otimes N(q)$. By Theorem 3.11, there exists a triangle equivalence $F : \langle X_{\mathrm{Y}(p;q)}\rangle \to \operatorname{per} A$ such that $F(X_{i,j}) \simeq P(i,j)$ for any $(i,j) \in \mathrm{Y}(p;q)$. Then $F(X_i) \simeq P(i)\otimes N(q)$ for any $i \in [1,p]$. By Lemma 2.19,

$$G' = (-) \overset{\mathbf{L}}{\otimes}_A (N(p)\otimes N(q)^*) : \operatorname{per} A \to \operatorname{per} A$$

is a triangle autoequivalence such that

$$G'|_{\langle P(i)\otimes N(q)\rangle} : \langle P(i)\otimes N(q)\rangle \to \langle P(i)\otimes N(q)\rangle$$

is a Serre functor for any $i \in [1,p]$. Thus $G = F^{-1}G'F$ satisfies (5.1). □

By the following result, if S is an M_k^+ subset satisfying $S_{\leqslant -1} = \emptyset$ and $S_{\geqslant k+1} = \emptyset$, any S-family is a mutation of a $\mathrm{Y}(k+1;h)$-family.

Lemma 5.9. *Let* $\mathcal{D}$ *be a triangulated category satisfying* (3.1), S *an* M_k^+*-subset satisfying* $S_{\geqslant k+1} = \emptyset$. *Let* $(X_{i,j})_{(i,j)\in S}$ *be an* S*-family in* $\mathcal{D}$ *and* $h = |S_0|$.

(a) *There exists a* $\mathrm{Y}(k+1;h)$*-family* $(Y_{i,j})_{(i,j)\in \mathrm{Y}(k+1;h)}$ *such that* $\langle X_i\rangle = \langle Y_i\rangle$ *for any* $i \in [0,k]$, *and there exists a triangle equivalence* $\langle X_S\rangle \to \operatorname{per} L(k+1;h)$ *such that* $F(Y_{i,j}) \simeq P(i,j)$.

(b) *The family* $(X'_{i,j})_{(i,j)\in S}$ *of objects* $X'_{i,j} := \mathbb{S}_{\langle X_i\rangle}(X_{i,j})$ *is an* S*-family such that* $\langle X_i\rangle = \langle X'_i\rangle$ *for any* $i \in [0,k]$.
(c) *The family* $(X''_{i,j})_{(i,j)\in S}$ *of objects* $X''_{i,j} := \mathbb{S}^{-1}_{\langle X_i\rangle}(X_{i,j})$ *is an* S*-family such that* $\langle X_i\rangle = \langle X''_i\rangle$ *for any* $i \in [0,k]$.

Proof. (a) We show that $(X_{k,j})_{j\in[0,h]}$ is an exceptional sequence satisfying (4.7)-(4.8) in Proposition 4.6. From (L2), $(X_{k,j})_{j\in[0,h]}$ is an exceptional sequence. From (S1), we have that (4.8) is satisfied. From (L2),

$$\langle X_S\rangle = \langle X_0\rangle \perp \langle X_2\rangle \perp \cdots \perp \langle X_k\rangle.$$

Let $i \in [1,k]$. If $S_i = S_{i-1}$, then $\mathbb{S}(X_{i,j}) \overset{3.16}{=} \mathbb{S}_{\langle X_{i-1}\rangle}(X_{i-1,j})$, and so we have $\langle \mathbb{S}(X_i)\rangle = \langle X_{i-1}\rangle$. If $S_i = S_{i-1}+1$, then $\mathbb{S}(X_{i,j}) \overset{(S3)}{=} X_{i-1,j-1}$, and so we have $\langle \mathbb{S}(X_i)\rangle = \langle X_{i-1}\rangle$. So (4.7) is satisfied. Thus the assertion follows from Proposition 4.6.
(b) By Lemma 5.8, there exists a triangle autoequivalence $G : \langle X_S\rangle \to \langle X_S\rangle$ such that $G(X_{i,j}) = \mathbb{S}_{\langle X_i\rangle}(X_{i,j})$. Thus the assertion follows.
(c) This is the dual of (b). □

The following result is equivalent to Theorem 5.6 in the case $k = 0$ and $S_{>k+1} = \emptyset$.

Lemma 5.10. *Let* $\mathcal{D}$ *be a triangulated category satisfying* (3.1)*, and let* I_0 *and* I_1 *be two intervals of* $\mathbb{Z}$ *such that* $I_1 \subset I_0$*,* S *and* S' *two finite subsets of* $\mathbb{Z}^2$ *such that*

$$S = \bigsqcup_{k\in[0,1]} \{k\} \times I_k, \quad S' = \sigma_{\leqslant 0}(S).$$

(a) *If* $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ *is an* S*-family, then the family* $\mathsf{X}' = (X'_{i,j})_{(i,j)\in S'}$ *is an* S'*-family where*

$$X'_{i,j} = \begin{cases} \mathbb{S}_{\langle X_0\rangle}(X_{0,j+1}) & i = 0, \\ X_{1,j} & i = 1. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S) \to \operatorname{per} L(S')$.
(b) *If* $\mathsf{Y} = (Y_{i,j})_{(i,j)\in S'}$ *is an* S'*-family, then the family* $\mathsf{Y}' = (Y'_{i,j})_{(i,j)\in S}$ *is an* S*-family where*

$$Y'_{i,j} = \begin{cases} \mathbb{S}^{-1}_{\langle Y_0\rangle}(Y_{0,j-1}) & i = 0, \\ Y_{1,j} & i = 1. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S') \to \operatorname{per} L(S)$.

Proof. Without loss of generality, we can assume that $I_0 = [p_1, p_2]$ and $I_1 = [1, q]$.
(a) Since $\langle X'_i\rangle = \langle \mathbb{S}_{\langle X_i\rangle}(X_i)\rangle = \langle X_i\rangle$ for any $i \in [0,1]$, we have that X' satisfies (L2.1). Let j and j' be two integers such that $|j - j'| > 1$. Then

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(X'_{0,j}, X'_{0,j'}) &= \operatorname{Hom}_{\mathcal{D}}(\mathbb{S}_{\langle X_0\rangle}(X_{0,j+1}), \mathbb{S}_{\langle X_0\rangle}(X_{0,j'+1})) \\ &\simeq \operatorname{Hom}_{\mathcal{D}}(X_{0,j+1}, X_{0,j'+1}) \overset{\text{(L2) for } \mathsf{X}}{=} 0,\end{aligned}$$

$$\operatorname{Hom}_{\mathcal{D}}(X'_{1,j}, X'_{1,j'}) = \operatorname{Hom}_{\mathcal{D}}(X_{1,j}, X_{1,j'}) \overset{\text{(L2) for } \mathsf{X}}{=} 0.$$

If $j' \neq p_1 - 1$,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(X'_{1,j}, X'_{0,j'}) &= \operatorname{Hom}_{\mathcal{D}}(X_{1,j}, \mathbb{S}_{\langle X_0\rangle}(X_{0,j'+1})) \overset{\text{(S1) for } \mathsf{X}}{=} \operatorname{Hom}_{\mathcal{D}}(X_{1,j}, X_{0,j'}) \\ &\overset{\text{(S2)' for } \mathsf{X}}{=} \operatorname{Hom}_{\mathcal{D}}(X_{0,j}, X_{0,j'}) \overset{\text{(L2) for } \mathsf{X}}{=} 0.\end{aligned}$$

If $j' = p_1 - 1$,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(X'_{1,j}, X'_{0,p_1-1}) &= \operatorname{Hom}_{\mathcal{D}}(X_{1,j}, \mathbb{S}_{\langle X_0\rangle}(X_{0,p_1})) \overset{\text{(S2)' for } \mathsf{X}}{=} \operatorname{Hom}_{\mathcal{D}}(X_{0,j}, \mathbb{S}_{\langle X_0\rangle}(X_{0,p_1})) \\ &\simeq \operatorname{Hom}_{\mathcal{D}}(X_{0,p_1}, X_{0,j})^* \overset{\text{(L2) for } \mathsf{X}}{=} 0.\end{aligned}$$

Thus X' satisfies (L2.2), and so X' is a weak S'-family.

Clearly X' satisfies (S1). By Proposition 3.16, X' satisfies (S3). If $(0,j),(1,j) \in S'$, then $(0,j+1) \in S$ and

$$\mathbb{F}_{\langle X_0'\rangle}(X'_{1,j}) = \mathbb{F}_{\langle X_0\rangle}(X_{1,j}) \overset{\text{(S2)}' \text{ for } \mathsf{X}}{=} X_{0,j} \overset{\text{(S1) for } \mathsf{X}}{=} \mathbb{S}_{\langle X_0\rangle}(X_{0,j+1}) = X'_{0,j}.$$

So X' satisfies (S2)$'$. Thus the assertion follows.

(b) Since $\langle Y_i'\rangle = \langle \mathbb{S}_{\langle Y_i\rangle}(Y_i)\rangle = \langle Y_i\rangle$ for any $i \in [0,1]$, we have that Y satisfies (L2.1). Let j and j' be two integers such that $|j-j'|>1$. Then

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(Y'_{0,j}, Y'_{0,j'}) &= \operatorname{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}_{\langle Y_0\rangle}(Y_{0,j-1}), \mathbb{S}^{-1}_{\langle Y_0\rangle}(Y_{0,j'-1})) \\ &\simeq \operatorname{Hom}_{\mathcal{D}}(Y_{0,j-1}, Y_{0,j'-1}) \overset{\text{(L2) for } \mathsf{Y}}{=} 0,\end{aligned}$$

$$\operatorname{Hom}_{\mathcal{D}}(Y'_{1,j}, Y'_{1,j'}) = \operatorname{Hom}_{\mathcal{D}}(Y_{1,j}, Y_{1,j'}) \overset{\text{(L2) for } \mathsf{Y}}{=} 0.$$

Since $S_1' = S_1 \subset S_0 = S_0' + 1$,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(Y'_{1,j}, Y'_{0,j'}) &= \operatorname{Hom}_{\mathcal{D}}(Y_{1,j}, \mathbb{S}^{-1}_{\langle Y_0\rangle}(Y_{0,j'-1})) \overset{\text{(S3) for } \mathsf{Y}}{=} \operatorname{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}(Y_{0,j-1}), \mathbb{S}^{-1}_{\langle Y_0\rangle}(Y_{0,j'-1})) \\ &\simeq \operatorname{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}_{\langle Y_0\rangle}(Y_{0,j'-1}), Y_{0,j-1})^* \simeq \operatorname{Hom}_{\mathcal{D}}(Y_{0,j-1}, Y_{0,j'-1}) \overset{\text{(L2) for } \mathsf{Y}}{=} 0.\end{aligned}$$

Thus Y satisfies (L2.2), and so Y is a weak S'-family.

Since Y satisfies (S1), we have

$$Y_{i,j} = \begin{cases} \mathbb{S}^{-1}_{\langle Y_0\rangle}(Y_{0,p_2-1}) & (i,j) = (1,p_2), \\ Y_{i,j} & \text{otherwise.} \end{cases} \tag{5.2}$$

From (5.2), Y satisfies (S1) and (S3). If $(1,j),(2,j) \in S'$, then

$$\mathbb{F}_{\langle Y_0'\rangle}(Y'_{1,j}) = \mathbb{F}_{\langle Y_0\rangle}(Y_{1,j}) \overset{\text{(S3) for } \mathsf{Y}}{=} \mathbb{F}_{\langle Y_0\rangle}\mathbb{S}^{-1}(Y_{0,j-1}) \overset{2.8\text{(b)}}{=} \mathbb{S}^{-1}_{\langle Y_0\rangle}(Y_{0,j-1}) = Y'_{0,j},$$

and so Y satisfies (S2)$'$. Thus the assertion follows. □

Now we are ready to prove Theorem 5.6.

Proof of Theorem 5.6. (a) Let $\mathsf{X}' = (X'_{i,j})_{(i,j)\in S'}$. By Lemma 5.9, $\mathsf{X}'_{\leqslant k}$ is an $S_{\leqslant k}$-family in $\mathcal{D}$. By Lemma 5.10, $\mathsf{X}'_{\geqslant k}$ is an $S_{\geqslant k}$-family in $\mathcal{D}$. Thus the assertion follows from Proposition 5.2.
(b) Let $\mathsf{X}' = (X'_{i,j})_{(i,j)\in S'}$. By Lemma 5.9, $\mathsf{X}'_{\leqslant k}$ is an $S_{\leqslant k}$-family in $\mathcal{D}$. By Lemma 5.10, $\mathsf{X}'_{\geqslant k}$ is an $S_{\geqslant k}$-family in $\mathcal{D}$. Thus the assertion follows from Proposition 5.2. □

A subset S of $\mathbb{Z}^2$ is called a *${}^tM_k^+$-subset* if ${}^tS$ is an M_k^+-subset. By Theorem 5.6, we obtain the following result:

Theorem 5.11 (Mutation ^{t}I)**.** *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1)*, S a ${}^tM_k^+$-subset satisfying $S^{\leqslant -1} = \emptyset$, and let $S' = \rho_{\leqslant k}(S)$.*

(a) *If $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ is an S-family, then the family $\mathsf{X}' = (X'_{i,j})_{(i,j)\in S'}$ is an S'-family where*

$$X'_{i,j} = \begin{cases} \mathbb{S}_{\langle X^j\rangle}(X_{i+1,j}) & j \leqslant k, \\ X_{i,j} & k < j. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S) \to \operatorname{per} L(S')$.

(b) *If $\mathsf{Y} = (Y_{i,j})_{(i,j)\in S'}$ is an S'-family, then the family $\mathsf{Y}' = (Y'_{i,j})_{(i,j)\in S}$ is an S-family where*

$$Y'_{i,j} = \begin{cases} \mathbb{S}^{-1}_{\langle Y^j\rangle}(Y_{i-1,j}) & j \leqslant k, \\ Y_{i,j} & k < j. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S') \to \operatorname{per} L(S)$.

The following result is the one of main results in this subsection.

Theorem 5.12 (Mutation II)**.** *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1)*, and let k, h be two positive integers such that $k \in (h+1)\mathbb{Z}$, and let $s = \frac{(h-1)k}{h+1}$. Let S be an M_{k-1}^{+}-subset of $\mathbb{Z}^2$ satisfying $|S_0| = h$ and $S_i = S_{i-1}$ for any $i \in [1, k-1]$, and let $S' = \sigma_{\leqslant 0}\sigma_{\leqslant 1}\dots\sigma_{\leqslant k-1}(S)$.*

(a) *If $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ is an S-family, then the family $\mathsf{X}' = (X'_{i,j})_{(i,j)\in S'}$ is an S'-family such that $\langle X_S \rangle = \langle X'_{S'} \rangle$ where*

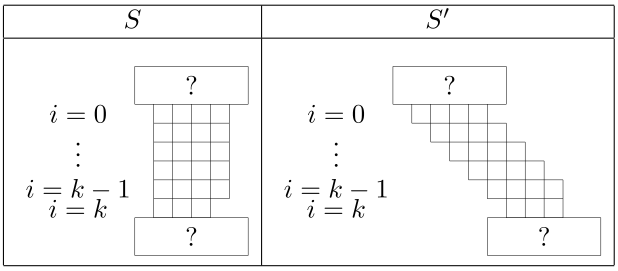
$$X'_{i,j} = \begin{cases} X_{i,j}[s] & i < 0, \\ \mathbb{S}^{k-i}_{\langle X_i \rangle}(X_{i,j}) & 0 \leqslant i \leqslant k-1, \\ X_{i,j} & i > k-1. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S) \to \operatorname{per} L(S')$.

(b) *If $\mathsf{Y} = (Y_{i,j})_{(i,j)\in S'}$ is an S'-family, then the family $\mathsf{Y}' = (Y'_{i,j})_{(i,j)\in S}$ is an S-family such that $\langle Y_S \rangle = \langle Y'_{S'} \rangle$ where*

$$Y'_{i,j} = \begin{cases} Y_{i,j}[-s] & i < 0, \\ \mathbb{S}^{-k+i}_{\langle Y_i \rangle}(Y_{i,j}) & 0 \leqslant i \leqslant k-1, \\ Y_{i,j} & i > k-1. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S') \to \operatorname{per} L(S)$.



Proof. (a) Since $S_{\geqslant 0}$ is an M_{k-1}^{+}-subset satisfying $(S_{\geqslant 0})_{\leqslant -1} = \emptyset$, the subfamily $\mathsf{X}'_{\geqslant 0}$ is an $S'_{\geqslant 0}$-family by Theorem 5.6(a). Since there exists a triangle equivalence $F : \langle X_0 \rangle \to \operatorname{per} N(S_0)$ such that $F(X_{0,j}) \simeq P(j)$ for any $j \in S_0$ by Lemma 3.13, we have

$$X'_{0,j} = \mathbb{S}^{k}_{\langle X_0 \rangle}(X_{0,j}) \overset{2.4}{=} X_{0,j}[s],$$

and so we have that $\mathsf{X}'_{\leqslant 0} = (X_{i,j}[s])_{(i,j)\in S'_{\leqslant 0}}$ is an $S'_{\leqslant 0}$-family. Thus the assertion follows from Proposition 5.2.

(b) This is the dual of (a). □

By transposing, we obtain the following result.

Theorem 5.13 (Mutation ^{t}II)**.** *Let $\mathcal{D}$ be a triangulated category satisfying* (3.1)*, and let k, h be two positive integers such that $k \in (h+1)\mathbb{Z}$, and let $s = \frac{(h-1)k}{h+1}$. Let S be a ${}^tM_{k-1}^{+}$-subset of $\mathbb{Z}^2$ satisfying $|S^0| = h$ and $S^i = S^{i-1}$ for any $i \in [1, k-1]$, and let $S' = \rho_{\leqslant 0}\rho_{\leqslant 1}\dots\rho_{\leqslant k-1}(S)$.*

(a) *If $\mathsf{X} = (X_{i,j})_{(i,j)\in S}$ is an S-family, then the family $\mathsf{X}' = (X'_{i,j})_{(i,j)\in S'}$ is an S'-family such that $\langle X_S \rangle = \langle X'_{S'} \rangle$ where*

$$X'_{i,j} = \begin{cases} X_{i,j}[s] & j < 0, \\ \mathbb{S}^{k-j}_{\langle X^j \rangle}(X_{i,j}) & 0 \leqslant j \leqslant k-1, \\ X_{i,j} & j > k-1. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S) \to \operatorname{per} L(S')$.

(b) *If* $\mathsf{Y} = (Y_{i,j})_{(i,j)\in S'}$ *is an* S'*-family, then the family* $\mathsf{Y}' = (Y'_{i,j})_{(i,j)\in S}$ *is an* S*-family such that* $\langle Y_S \rangle = \langle Y'_{S'} \rangle$ *where*

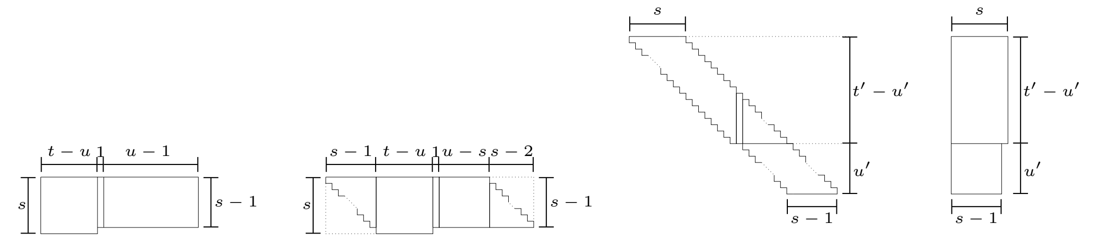
$$Y'_{i,j} = \begin{cases} Y_{i,j}[-s] & j < 0, \\ \mathbb{S}^{-k+j}_{\langle Y^j \rangle}(Y_{i,j}) & 0 \leqslant j \leqslant k-1, \\ Y_{i,j} & j > k-1. \end{cases}$$

In particular, there exists a triangle equivalence $\operatorname{per} L(S') \to \operatorname{per} L(S)$.

5.3. **Applications of mutations of S-families.** We are ready to prove one of our main results by applying Theorems 5.6 and 5.13 as follows where

$$t' = t-1, \quad u' = u-s, \quad Y = Y(s,t,u), \quad Y' = Y(s,t',u').$$

$$Y \xrightarrow{\text{Mutation I}} S \xrightarrow{\text{Mutation } {}^t\text{II}} S' \xrightarrow{\text{Mutation I}} {}^tY'$$



Theorem 5.14. *Let s, t, u be three positive integers such that $1 \leqslant u \leqslant t$. Suppose that one of the following conditions is satisfied.*

(a) $u \in \mathbb{Z}s$ *and* $t-u \in \mathbb{Z}(s+1)$.

(b) $s = 2$ *and* $t - u \in 3\mathbb{Z}$.

Then there exist triangle equivalences

$$\operatorname{per} L(s,t,u) \to \operatorname{per} L(S) \to \operatorname{per} L(S') \to \operatorname{per} L(s,t-1,u-s)$$

where

$$S = \sigma_{\leqslant 1}\sigma_{\leqslant 2}\dots\sigma_{\leqslant s-1}(\mathrm{Y}(s,t,u)),$$

$$S' = (\rho_{\leqslant 1}\rho_{\leqslant 2}\dots\rho_{\leqslant t-u})(\rho^{-1}_{\geqslant t-s+1}\rho^{-1}_{\geqslant t-s}\dots\rho^{-1}_{\geqslant t-u+2})(S).$$

Proof. (a) There exists a triangle equivalence $\operatorname{per} L(s,t,u) \to \operatorname{per} L(S)$ by Theorem 5.6.

Let $I = [1, t-u]$, $J = [t-u+2, t-s+1]$. Since $S^I \equiv \mathrm{Y}(s; t-u)$ and $S^{t-u+1} \subset S^{t-u}$, we have that S is a ${}^tM^+_{t-u-1}(1,1)$-subset satisfying $t-u \in \mathbb{Z}(s+1)$. Since $S^J \equiv \mathrm{Y}(s-1; u-s)$ and $S^{t-u+2} = S^{t-u+1}$, we have that $-S$ is a ${}^tM^+_{u-s-1}(-1, -t+s-1)$-subset satisfying $u-s \in \mathbb{Z}s$. Thus there exists a triangle equivalence $\operatorname{per} L(S) \to \operatorname{per} L(S')$ by Theorem 5.13.

Since

$${}^t\mathrm{Y}(s,t-1,u-s) \equiv (\sigma^{-1}_{\leqslant s-t+u}\sigma^{-1}_{\leqslant s-t+u+1}\dots\sigma^{-2}_{\leqslant s-1})(\sigma_{\geqslant u}\sigma_{\geqslant u-1}\dots\sigma_{\geqslant s+1})(S'),$$

there exists a triangle equivalence $\operatorname{per} L(S') \to \operatorname{per} L(s,t-1,u-s)$ by Theorem 5.6.

(b) There exists a triangle equivalence $\operatorname{per} L(2,t,u) \to \operatorname{per} L(S)$ by Theorem 5.6.

Let $I = [1, t-u]$, $J = [t-u+2, t-1]$. Since $S^I \equiv \mathrm{Y}(2; t-u)$ and $S^{t-u+1} \subset S^{t-u}$, we have that S is a ${}^tM^+_{t-u-1}(1,1)$-subset satisfying $t-u \in 3\mathbb{Z}$. Since $S^J \equiv \mathrm{Y}(1; u-s)$, we have that $-S$ is a ${}^tM^+_{u-3}(-1, -t+1)$-subset. Thus there exists a triangle equivalence $\operatorname{per} L(S) \to \operatorname{per} L(S')$ by Theorems 5.11 and 5.13.

Since

$${}^t\mathrm{Y}(2,t-1,u-2) \equiv (\sigma^{-1}_{\leqslant -t+u+2}\sigma^{-1}_{\leqslant -t+u+3}\dots\sigma^{-2}_{\leqslant 1})(\sigma_{\geqslant u}\sigma_{\geqslant u-1}\dots\sigma_{\geqslant 3})(S'),$$

there exists a triangle equivalence $\operatorname{per} L(S') \to \operatorname{per} L(2,t-1,u-2)$ by Theorem 5.6. $\square$

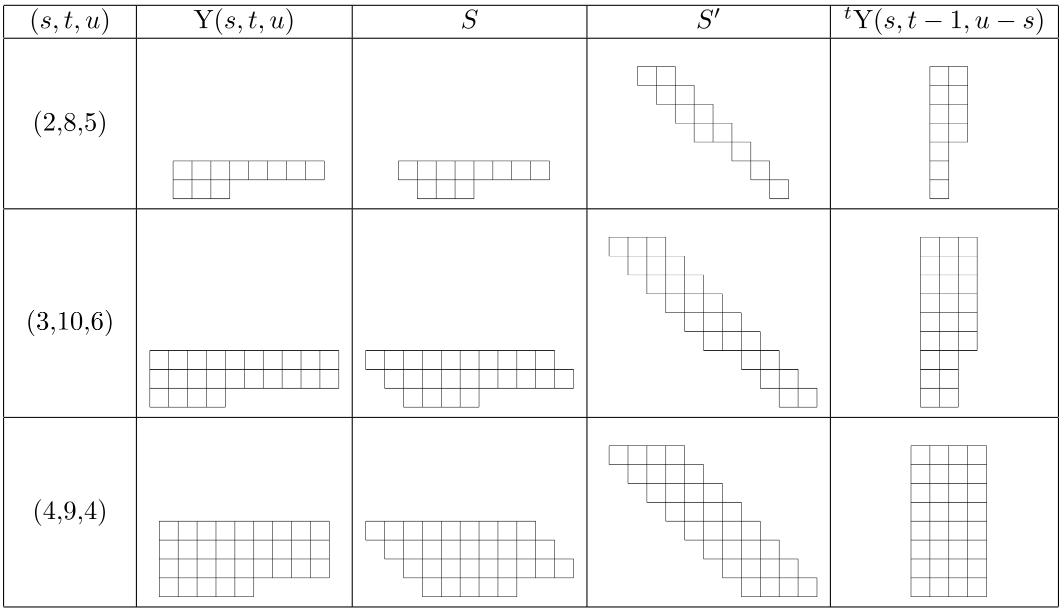


FIGURE 4. Mutations from $\mathrm{Y}(s,t,u)$ to ${}^t\mathrm{Y}(s,t-1,u-s)$

By Theorem 4.7, we have the following result.

Corollary 5.15. *Let p, q be two integers such that $p \geqslant 2$, $q \geqslant 1$. Suppose that one of the following conditions is satisfied.*

(a) $r \in \mathbb{Z}_{\geqslant 0}$.
(b) $p = 2$ *and* $r \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$.

Then there exists a triangle equivalence

$$\operatorname{per} N(n, \ell+1) \to \operatorname{per} N(n,\ell) \text{ where } n = p(p+1)q + p(p-1)r,\ \ell = (p+1)q + pr.$$

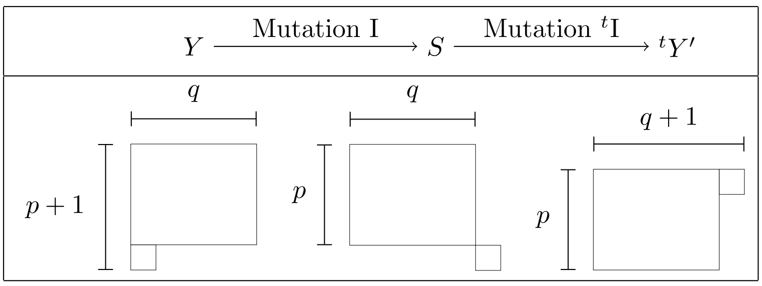
Proof. Let $(s,t,u) = (p, (p+1)q + pr, pr)$. By Theorem 4.7, there exist triangle equivalences

$$\operatorname{per} N(n, \ell+1) \to \operatorname{per} L(s,t,u), \quad \operatorname{per} N(n,\ell) \to \operatorname{per} L(s, t-1, u-s).$$

Thus the assertion follows from Theorem 5.14. $\square$

The following result is proved by applying Theorems 5.6 and 5.11 as follows where

$$Y = \mathrm{Y}(p+1, q, q-1), \quad Y' = \mathrm{Y}(q+1, p, p-1).$$



Theorem 5.16. *Let p, q be two integers such that $p \geqslant 2$, $q \geqslant 2$. Then there exists a full $\mathrm{Y}(q+1, p, p-1)$-family in $\operatorname{per} L(p+1, q, q-1)$. In particular, there exists a triangle equivalence*

$$\operatorname{per} L(p+1, q, q-1) \to \operatorname{per} L(q+1, p, p-1).$$

Proof. Let S be a subset of $\mathbb{Z}^2$ such that $S = \sigma^q_{\leqslant p}(Y(p+1, q, q-1))$. Since

$$^tY(q+1, p, p-1) \equiv \rho^{-p}_{\leqslant 0}(S),$$

the assertion follows from Theorems 5.6 and 5.11. □

Corollary 5.17. *There exists a triangle equivalence*

$$\operatorname{per} N(pq+1, q+1) \to \operatorname{per} N(pq+1, p+1) \quad \textit{for any integers } p \geqslant 2,\ q \geqslant 2.$$

Proof. By Theorem 4.7, there exist triangle equivalences

$$\operatorname{per} N(pq+1, q+1) \to \operatorname{per} L(p+1, q, q-1), \quad \operatorname{per} N(pq+1, p+1) \to \operatorname{per} L(q+1, p, p-1).$$

Thus the assertion follows from Theorem 5.16. □

References

[1] I. Assem, D. Simson, A. Skowroński, Elements of the Representation theory of associative algebras Vol. 1., Techniques of Representation Theory, London Mathematical Society Student Texts vol 65. , Cambridge University Press, Cambridge (2006).

[2] M. Auslander, I. Reiten, S. Smalø, Representation theory of Artin algebras, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge (1995).

[3] A. I. Bondal, Representations of associative algebras and coherent sheaves, Math. USSR Izvestia 34 (1990), 23-42.

[4] A. I. Bondal, M. M. Kapranov, Representable functors, Serre functors, and reconstructions, Math. USSR Izvestia 35 (1990), 519-541.

[5] S. Brenner, M. C. R. Butler, Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors, Lecture Notes in Math. No. 832, Springer Verlag, Berlin, Heidelberg, New York (1980), 103-169.

[6] P. Gabriel, A. V. Roiter, Representations of finite-dimensional algebras, Encyclopaedia of Mathematical Sciences vol. 73, Algebra VIII, Springer-Velrag, Berlin, Heidelberg, New York (1992).

[7] D. Happel, On Gorenstein algebras, Representation theory of finite groups and finite-dimensional algebras (Bielfeld, 1991), Progr. Math., 95, Birkhäuser, Basel (1991), 389-404.

[8] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Mathematical Society Lecture Note Series 119, Cambridge University Press, Cambridge (1988).

[9] D. Happel, U. Seidel, Piecewise Hereditary Nakayama Algebras, Algebr. Represent. Theory 13, no. 6 (2010), 693-704.

[10] D. Huybrechts, Fourier-Mukai Transforms in Algebraic Geometry, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford (2006).

[11] L. A. Hügel, D. Happel, H. Krause, Handbook of tilting theory, London Mathematical Society Lecture Note Series 332, Cambridge University Press, Cambridge (2007).

[12] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. (4) 27 no. 1 (1994), 63-102.

[13] D. Kussin, H. Lenzing, H. Meltzer, Triangle singularities, ADE-chains, and weighted projective lines, Adv. Math. 237 (2013), 194-251.

[14] S. Ladkani, On derived equivalences of lines, rectangles and triangles, J. Lond. Math. Soc. (2) 87 no. 1 (2013), 157-176.

[15] H. Lenzing, The $\mathbb{E}$-series, Happel-Seidel symmetry, and Orlov's theorem (2012), https://www.math.uni-bielefeld.de/icra2012/ presentations/icra2012_lenzing.pdf.

[16] H. Lenzing, H. Meltzer, S. Ruan, Nakayama algebras and fuchsian singularities (2022), arXiv 2112.15587v2.

[17] S. Mac Lane, Categories for the working mathematician (2nd ed), Graduate Texts in Mathematics Vol. 5, Springer-Verlag, New York (1998).

[18] T. Nakayama, Note on uni-serial and generalized uni-serial rings, Proceedings of the Imperial Academy. Tokyo, 16, (1940), 285-289.

[19] I. Reiten, M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15, no. 2 (2002), 295-366.

[20] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) 39 no. 3, (1989), 436-546.

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