On a relation between leafwise cohomology theory and representation theory

（葉向コホモロジー理論と表現論の関係について）

Shota Mori

Contents

1 Introduction

A leafwise complex is a quotient of a de Rham complex. The complex is incidental to a C^∞ foliation on a C^∞ manifold. It has been studied in connection with parameter rigidity theory of group actions whose orbits compose a C^∞ foliation. First, we describe a part of the history of these objects.

Leafwise cohomology theory was likely introduced in the 1950s. For example, it appears in a paper [31] by Reinhart. At this time, a leafwise cohomology was not called by that name. It was called a d' cohomology, with d' used as the notation for the leafwise exterior derivative. Reinhart studied harmonic integrals in a leafwise complex under the existence of a metric compatible with leaves. In particular, Reinhart computed leafwise cohomology groups of all 1-dimensional linear foliations on a 2-torus. This was done to compare the dimensions of the leafwise cohomology group and the space of leafwise harmonic forms. See [31, Chapters 3-4].

Parameter rigidity theory was likely introduced in celestial mechanics. For example, it appears in [35, Sections 3.1-3.3] by Sternberg. At this time, parameter rigidity was not called by that name either. The problem addressed here was the reparametrization of a foliation on a 2-torus whose orbits are straight lines. This problem is called linearizing the flow. A torus plays an important role in the study of periodic orbits and quasi-periodic orbits in celestial mechanics. In connection with this, dynamical systems on a torus have also been considered. See also [34, Section 36].

An application of leafwise cohomology theory to parameter rigidity theory was studied in a paper [4] by Arraut and dos Santos. At this time, a leafwise cohomology is called a foliated cohomology. Also, parameter rigidity is called reparametrization. The interest so far had been in flows as $\mathbb{R}$-actions. The topic was generalized to orbit foliations from $\mathbb{R}^N$ actions. In particular, divided results were obtained depending on whether the matrix giving linear foliations on a torus is diophantine or Liouville. To be precise, this application is for 1-codimensional linear foliations in a torus. This difference in phenomena due to the number theoretic conditions was also pointed out in [31, Chapters 3-4] and [35, Sections 3.1-3.3]. A later paper [5] proved that it is correct for any linear foliations in a torus. In these verifications, the computation results of leafwise cohomology groups play essential roles. Furthermore, a sufficient condition for parameter rigidity continues to be generalized in the form of using leafwise cohomology groups. See [24, Proposition 2.6], [21, Section 2], [22, Theorem 2.2.5], etc.

We make an additional summary of what leafwise cohomology theory and parameter rigidity are called. In the above history, they have not yet been called by that name. Leafwise cohomology theory has been called by that name since [3] and [1]. In the former, it is listed along with the name "foliated cohomology theory". However, in the latter, only the name "leafwise cohomology theory" is described. Also, parameter rigidity theory has been called by that name since [24]. These names are adopted in later papers, surveys, and lecture notes. See [25], [8], [7], etc.

What we have described above is a part of the history of leafwise cohomology theory and parameter rigidity theory. Starting from the above paper [4], it has become important to compute leafwise cohomology groups and rings. Next, we describe previous research on the computational results of those computations. Because of the wide variety of these results, we will only present those that are relevant to our discussion. We list [25], [8], and [7] as good surveys and a lecture note with other examples.

The most important computation result is a linear foliation on a torus in [5, Theorem 2.2]. Let p, q, and n be natural numbers satisfying $p+q=n$. A linear foliation is defined on $\mathbb{T}^n$, where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. Let A be an $\mathbb{R}$-valued $q\times p$-matrix. Then each orbit of a linear foliation $\mathcal{F}_A$ is a projection from a graph $y=Ax+b$ in $\mathbb{R}^n$ for some $b\in\mathbb{R}^q$. We treat $\mathcal{F}_A$ as an orbit foliation obtained from an $\mathbb{R}^p$ action. If A is diophantine, then the following isomorphism is proven to exist:

$$H^*(\mathcal{F}_A)\cong H^*_{\mathrm{dR}}(\mathbb{T}^p).$$

By a method of proof in [5], this isomorphism is an isomorphism as a ring. On the other hand, if A is Liouville, then the formula

$$\dim H^{p'}(\mathcal{F}_A)=\infty$$

is proved as a contrasting result for each $1\leq p'\leq p$. In both of these proofs, the method is a developed version of the computation of $H^*_{\mathrm{dR}}(\mathbb{T}^n)$ by the Fourier series. However, the idea of solving linear equations efficiently is nontrivial. In fact, only partial results are proved in [31, Section 4], [16, Theorem 3.5], [10, Théorème V.3.1.1.2], [4, Lemma 2.6], etc. It was not until [5, Theorem 2.2], 30 years after [31], that $H^*(\mathcal{F}_A)$ was fully determined.

Except for this computation, many of the results had been of first leafwise cohomology groups. An example includes the following. Set $G=SL(2,\mathbb{R})$. Let $P\subset G$ be the subgroup of all upper triangular matrices, $\mathfrak{p}$ be the Lie algebra of P and $\Gamma\subset G$ be a cocompact lattice. Also, set $M_\Gamma=\Gamma\backslash G$. Let $\mathcal{F}_P$ be the orbit foliation induced from the natural action of P on M_Γ. Then the following isomorphism is proven to exist in [24, Theorem 1]:

$$H^1(\mathcal{F}_P)\cong H^1_{\mathrm{Lie}}(\mathfrak{p})\oplus H^1_{\mathrm{dR}}(M_\Gamma). \tag{1.1}$$

This is defined as an extension map of the natural map $H^1_{\mathrm{dR}}(M_\Gamma)\to H^1(\mathcal{F}_P)$. The method of the proof is based on a geometry of M_Γ. For example, in the proof of injectivity of a natural map $H^1_{\mathrm{dR}}(M_\Gamma)\to H^1(\mathcal{F}_P)$, simple closed curves which span $H_1(M_\Gamma;\mathbb{R})$ are used. Also, in the proof of surjectivity of (1.1), an ergodicity of horocycle flows is used.

In 2021, progress was made in the research for higher leafwise cohomology groups. Let G, P, Γ, M_Γ, and $\mathcal{F}_P$ be the same notation as in the previous example. The author obtained a dimensional formula of $H^2(\mathcal{F}_P)$ in [26]. Under the G-invariant measure, $L^2(M_\Gamma)$ decomposes into a countable sum of irreducible unitary representations by [14, Theorem 1.2.3]. Let $g\in\mathbb{N}$ be the multiplicity of the irreducible unitary representation U^{-1} in $L^2(M_\Gamma)$, where the lowest weight of U^{-1} is 1. The finiteness is ensured by the duality theorem [14, Theorem 1.4.2]. This theorem says that the multiplicity of U^1 is also g, where the highest weight of U^1 is -1. Then the following holds:

$$\dim H^2(\mathcal{F}_P) = 2g. \tag{1.2}$$

In general, leafwise cohomology groups are not necessarily finite-dimensional. From [2], we obtain several sufficient conditions for leafwise cohomology groups to be infinite-dimensional. On the other hand, the sufficient conditions for them to be finite-dimensional are not known. The method of the proof in [26] is an application of irreducible unitary representation theory of G. In particular, we exploit non-abelian harmonic analysis. Our method is an extension of method in [5] to $\mathcal{F}_P$. In other words, it is analogous to the computation of $H^*_{\mathrm{dR}}(\mathbb{T}^n)$ by the Fourier series. The key point of the proof is also how to solve linear equations efficiently.

Twelves days after [26], Maruhashi and Tsutaya obtain the following result in [23, Theorem 66]:

$$H^2(\mathcal{F}_P) \cong H^2_{\mathrm{dR}}(M_\Gamma), \tag{1.3}$$

where Γ is torsion-free. Their focus is to construct something like the Hodge decomposition on a leafwise complexes. This is similar to the idea in [31]. Their primary tools are cohomology theory of Lie algebras with twisted coefficients and Hochschild-Serre spectral sequence. A representation space of G appears as a twisted coefficient. The computation of the cohomology groups of such coefficient is known in [11]. In this setting, they take a homological algebraic approach.

The methods in [26] and [23] are completely independent. Also, the method in our thesis is based on [26] and [27]. Thus, it is also completely independent of [23]. On the other hand, there is a common point that representation theory is exploited in some forms. In studies of rigidity of group actions, representation theory has been used in the past. See [15] and [18]. The studies of linear foliations on $\mathbb{T}^n$, which uses Fourier series, are also interpreted as indirectly exploiting representation theory. These facts make us expect that representation theory has remained useful in the study of leafwise cohomology theory and parameter rigidity theory even today.

The above is a guide to previous research on the computational result of leafwise cohomology groups. Now, let us state our main result. Set also $G = SL(2,\mathbb{R})$. Recall that $P \subset G$ is the subgroup of all upper triangular matrices, and $\Gamma \subset G$ is any cocompact lattice. We combine the results (1.1), (1.2), and (1.3) by exploiting non-abelian harmonic analysis on G. The method is based on [26] and [27]. Furthermore, we determine the ring structure of $H^*(\mathcal{F}_P)$. This is essentially the second study to determine the ring structure following [5]. It is also the first result of a non-abelian group action. Our main result reads as follows:

Theorem 1.1. *(Main Theorem)*
Let g be the multiplicity of the irreducible unitary representation U^{-1} in $L^2(M_\Gamma)$, where the lowest weight of U^{-1} is 1. Then there exist an isomorphism as graded rings

$$H^*(\mathcal{F}_P) \cong \bigwedge[X, Y_1, \cdots, Y_{2g}]/\big(\{Y_i \wedge Y_j\}_{1\le i,j\le 2g}\big),$$

where $X, Y_1, \cdots, Y_{2g}$ are indeterminate variables.

Theorem 1.1 is stated in [27, Theorem 1.1]. We further characterize the parameter g geometrically. By using (1.1) and Theorem 1.1, we obtain

$$\dim H^1_{\mathrm{dR}}(M_\Gamma) = 2g. \tag{1.4}$$

Set $K = SO(2)$ and $\Sigma_\Gamma = \Gamma\backslash G/K$. Under the dimension formula (1.4), the following theorem is well-known to experts.

Theorem 1.2. *The following properties hold.*

(i) The space Σ_Γ is homeomorphic to a closed orientable surface.

(ii) Let g_Γ be the genus of Σ_Γ. Then $g = g_\Gamma$.

The thesis is organized as follows. Chapters 2-4 are surveys. Chapters 5 and 6 are proofs of our main results.

In Chapter 2, we give definitions to some of the notions that appear in Introduction. In Section 2.1, we define foliations and leafwise cohomology rings. In Section 2.2, we define cohomology groups of Lie algebras. In Section 2.3, we define parameter rigidity. We also introduce the study [15] by Ghys. In that paper, another rigidity of group actions, which is related to parameter rigidity, is researched. The objects of this research overlap with our interests.

In Chapter 3, we survey the computation of leafwise cohomology for the most important example by Arraut and dos Santos [5]. The aim is to prove Theorem 3.14. In Section 3.1, we define notions for computation after. In Section 3.2, we characterize the minimality of linear foliation algebraically. The tools of proof are Kronecker's theorem (Theorem 3.6), linear algebra over $\mathbb{R}$ and $\mathbb{Q}$. In Section 3.3, we prove the computational result of Arraut and dos Santos [5, Theorem 2.2] (see Theorem 3.14). The proof is provided in a numerical and straightforward method by Fourier series on $\mathbb{T}^n$.

In Chapter 4, we survey the classification of irreducible unitary representations on $SL(2,\mathbb{R})$. The way is according to [37, Chapter 5, Section 6.6]. We clarify the property which differential representations of irreducible unitary representations should satisfy. Conversely, we assume the existence of such irreducible unitary representations. See [37, Chapter 5, Section 6.2, 6.3, 6.4]. In Section 4.1, we introduce basics of representation theory of semisimple Lie groups. In Section 4.2, we introduce basics of functional analysis for infinite-dimensional representations. In Section 4.3, we complete the classification of irreducible unitary representations of $SL(2,\mathbb{R})$.

In Chapter 5, we prove the main theorem (Theorem 1.1). First, in Section 5.1, we provide tools for discussion. Some facts are cited from representation theory and Sobolev space theory. We also do preliminary computations. Second, in Section 5.2, we confirm that $H^0(\mathcal{F}_P)$ is one-dimensional. It is assured by the fact of the geodesic flow or the horocycle flow. Third, in Section 5.3 and 5.4, we compute all explicit generators of $H^*(\mathcal{F}_P)$ in terms of representation theory. We also make a explicit estimations of L^2-Soborev norms. All these calculations are provided by a simple and direct method. Fourth, we determine the ring structure by computing eigenvalues of cocycles in section 5.5. The contents of this chapter are based on [27].

In the final chapter, Chapter 6, we prove Theorem 1.2 to characterize the parameter g in Theorem 1.1. Through the linear fractional action, G/K is identified with the upper half plane $\mathbb{H}^2$. In Section 6.1, we consider the quotient space of $\mathbb{H}^2$ divided by a cocompact lattice Γ. We confirm that the space is a closed orientable surface Σ_Γ topologically. Let g_Γ be the genus of Σ_Γ. In Section 6.2, we prepare basics for Seifert bundles. It is known that the canonical projection $M_\Gamma \to \Sigma_\Gamma$ is a Seifert bundle (see [Scott, Section "$\widetilde{SL}(2,\mathbb{R})$"]). In Section 6.3, we prove $H^1(M_\Gamma;\mathbb{Z}) \cong \mathbb{Z}^{2g_\Gamma}$. In particular, we obtain $g = g_\Gamma$ by (1.4). The author was given this formula as a question by Professor Hitoshi Moriyoshi. The contents of Chapter 6 are well-known to experts. However, they are included in the thesis to clarify the meaning of our main result. In addition, Sections 6.2 and 6.3 are based on personal discussions with Dr. Shuhei Maruyama. Specifically, he suggested to use the Seifert bundle structure and its Euler class to compute the cohomology group of M_Γ. The author complemented it.

Finally, the author acknowledges Professor Hideyuki Ishi for supporting his study of representation theory. The author also acknowledges helpful comments from Dr. Hirokazu Maruhashi, Dr. Shuhei Maruyama, and Professor Hitoshi Moriyoshi. Especially, the author thanks Dr. Shuhei Maruyama for suggesting key ideas in Sections 6.2 and 6.3. Moreover, the author acknowledges worthwhile points from the anonymous referee at the time of the paper submission to Tokyo Journal of Mathematics.

2 Cohomology theory and the rigidity of group actions

2.1 Foliations and leafwise cohomology

In this section, we summarize the basics for foliations and leafwise cohomology. Let M be a C^∞ manifold of dimension n.

Definition 2.1. *Let $\mathcal{F} = \{L_\lambda\}_{\lambda\in\Lambda}$ be a family of C^∞ immersed sub-manifolds. This is a C^∞ foliation of codimension q if the following two properties are satisfied:*

(i) $M = \sqcup_{\lambda\in\Lambda} L_\lambda$.

(ii) *There exists an atlas $\{(U_i, \phi_i : U_i \to \mathbb{R}^n)\}_{i\in I}$ such that*

$$\phi_i^{-1}(\mathbb{R}^{n-q} \times \{y\})$$

is a connected component of some $U \cap L_\lambda$ for each $i \in I$ and $y \in \mathbb{R}^q$.

Each L_λ is called a leaf.

Example 2.2. *Let $M \times N$ be a product manifold. Then,*

$$\mathcal{F}_M = \{M \times \{y\}\}_{y\in N}$$

is a C^∞ foliation.

Example 2.3. *Fix $a \in \mathbb{R}$. Let $\mathcal{L}_a$ be the family of lines*

$$y = ax + b$$

in $\mathbb{R}^2$ for each $b \in \mathbb{R}$. Set $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let $\mathcal{F}_a$ be the projection of $\mathcal{L}_a$. Then, $\mathcal{F}_a$ is a C^∞ foliation. This is called a 1-dimensional linear foliation.

If $\mathcal{F}$ is a C^∞ foliation, it defines the vector sub-bundle $T\mathcal{F} \subset TM$ whose direction is along leaves. Conversely, let $E \subset TM$ be a vector sub-bundle.

Definition 2.4. *We call E involutive if $\Gamma(E) \subset \mathfrak{X}(M)$ is a Lie sub-algebra.*

Definition 2.5. *An p-form $\omega \in \Omega^p_{\mathrm{dR}}(M)$ is an annihilator in E if $\omega(v) = 0$ for each p-alternative form v on $\Gamma(E)$. Let $I^*(E)$ be the ideal consisting of all annihilators in E.*

Theorem 2.6 (Frobenius)**.** *Let $E \subset TM$ be a vector sub-bundle. Then, the following are equivalent:*

(i) *There exists a C^∞ foliation $\mathcal{F}$ such that $E = T\mathcal{F}$.*

(ii) *E is involutive.*

(iii) $dI^*(E) \subset I^*(E)$.

See [9, Theorem 1.3.8.] for more statements and a part of the proof. See also [40, Theorem 1.60.] for the completion of the proof. If one of the conditions in Frobenius Theorem is satisfied, we put

$$I^*(\mathcal{F}) = I^*(E).$$

Example 2.7. *Let $\mathcal{F}_a$ be a 1-dimensional linear foliation on $\mathbb{T}^2$. Then, we obtain*

$$\Gamma(T\mathcal{F}_a) = \left\{ f \cdot \left(\frac{\partial}{\partial x} + a \frac{\partial}{\partial y} \right) \middle| f \in C^\infty(\mathbb{T}^2) \right\},$$

$$I^1(\mathcal{F}_a) = \left\{ f \cdot (a\, dx - dy) \middle| f \in C^\infty(\mathbb{T}^2) \right\}.$$

Example 2.8 (Our targets)**.** *Let G be a Lie group, $P \subset G$ be a Lie sub-group, and $\Gamma \subset G$ be a discrete subgroup. The quotient space $M_\Gamma = \Gamma \backslash G$ is a C^∞ manifold since Γ is a closed sub-group. Then the family $\mathcal{F}_P$ consisting of all homogeneous P-orbits in M_Γ is a C^∞ foliation.*

Definition 2.9. *Let $\mathcal{F}$ be a C^∞ foliation. We define the leafwise complex $\Omega^*(\mathcal{F})$ by the short exact sequence*

$$0 \to I^*(\mathcal{F}) \to \Omega^*_{\mathrm{dR}}(M) \to \Omega^*(\mathcal{F}) \to 0.$$

We call $(\Omega^*(\mathcal{F}), d_\mathcal{F})$ a leafwise complex. Its cohomology $H^*(\mathcal{F})$ is a leafwise cohomology.

Example 2.10. *Let $M \times N$ be a product manifold. Then,*

$$H^*(\mathcal{F}_M) \cong H^*_{\mathrm{dR}}(M) \otimes C^\infty(N)$$

over $\mathbb{C}$.

This is perhaps the only trivial example.

2.2 Cohomology of Lie algebras

We describe a definition and examples of cohomology groups for Lie algebras in accordance with [19, Section 4.3]. Let $\mathfrak{p}$ be a finite dimensional Lie algebra over $\mathbb{R}$. For each $n \in \mathbb{N}$, set

$$C^n_{\mathrm{Lie}}(\mathfrak{p}) = \mathrm{Hom}_\mathbb{R}\left(\bigwedge^n \mathfrak{p}, \mathbb{R} \right).$$

Let $d : C^n_{\mathrm{Lie}}(\mathfrak{p}) \to C^{n+1}_{\mathrm{Lie}}(\mathfrak{p})$ be the coboundary operator defined by

$$d\omega(X_1, \cdots, X_{n+1}) = \sum_{j<k} (-1)^{j+k} \omega([X_j, X_k], X_1, \cdots, \widehat{X}_j, \cdots, \widehat{X}_k, \cdots, X_{n+1})$$

for each $X_1, \cdots, X_{n+1} \in \mathfrak{p}$, and $\omega \in C^n_{\mathrm{Lie}}(\mathfrak{p})$. Then $(C^*_{\mathrm{Lie}}(\mathfrak{p}), d)$ is a cochain complex. We denote its cocycle group, coboundary group, and cohomology group by $Z^*_{\mathrm{Lie}}(\mathfrak{p})$, $B^*_{\mathrm{Lie}}(\mathfrak{p})$, and $H^*_{\mathrm{Lie}}(\mathfrak{p})$, respectively.

The 0-th coboundary operator $d : C^0_{\mathrm{Lie}}(\mathfrak{p}) \to C^1_{\mathrm{Lie}}(\mathfrak{p})$ is the 0-map. Thus we have $H^0_{\mathrm{Lie}}(\mathfrak{p}) \cong \mathbb{R}$ and $H^1_{\mathrm{Lie}}(\mathfrak{p}) \cong Z^1_{\mathrm{Lie}}(\mathfrak{p})$. The first coboundary operator $d : C^1_{\mathrm{Lie}}(\mathfrak{p}) \to C^2_{\mathrm{Lie}}(\mathfrak{p})$ has the following form:

$$d\omega(X_1, X_2) = -\omega([X_1, X_2])$$

for each $X_1, X_2 \in \mathfrak{p}$, and $\omega \in C^1_{\mathrm{Lie}}(\mathfrak{p})$. From this formula we obtain the statement below.

Proposition 2.11. *The following natural isomorphism exists.*

$$H^1_{\mathrm{Lie}}(\mathfrak{p}) \cong \mathrm{Hom}_{\mathbb{R}}\left(\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}], \mathbb{R}\right).$$

We see some examples.

Example 2.12. *Set $\mathfrak{p} = \mathbb{R}^N$. All coboundary operators are 0-maps. Thus we obtain*

$$H^n_{\mathrm{Lie}}(\mathfrak{p}) \cong \bigwedge^n \mathbb{R}^N.$$

Example 2.13. *Let $\mathfrak{p} \subset \mathfrak{sl}(N+1, \mathbb{R})$ be the Lie sub-algebra consisting of entire upper triangular matrices. Then $[\mathfrak{p}, \mathfrak{p}]$ is the Lie sub-algebra consisting of entire upper triangular matrices whose all diagonal elements are 0. Thus we obtain*

$$H^1_{\mathrm{Lie}}(\mathfrak{p}) \cong \mathbb{R}^N$$

by Proposition 2.11.

Example 2.14. *Let $\mathfrak{p} \subset \mathfrak{sl}(2, \mathbb{R})$ be the Lie sub-algebra consisting of entire upper triangular matrices. Set*

$$X_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\{X_1, X_2\}$ is a basis of $\mathfrak{p}$. Let $\{\omega_1, \omega_2\} \subset C^1_{\mathrm{Lie}}(\mathfrak{p})$ be the dual basis. We obtain

$$d\omega_1 = 0, \quad d\omega_2 = -\omega_1 \wedge \omega_2.$$

In particular, we have $B^2_{\mathrm{Lie}}(\mathfrak{p}) = Z^2_{\mathrm{Lie}}(\mathfrak{p}) = C^2_{\mathrm{Lie}}(\mathfrak{p})$ and $H^2_{\mathrm{Lie}}(\mathfrak{p}) = \{0\}$.

2.3 Parameter rigidity of group actions

In [15], significant results on the rigidity of C^∞ locally free group actions were obtained. To begin with, we describe two important examples of group actions in these.

Example 2.15. *Let $P \subset SL(2,\mathbb{R})$ be the subgroup as defined below:*

$$P = \left\{ \begin{pmatrix} e^{\frac{t}{4}} & e^{-\frac{t}{4}}x \\ 0 & e^{-\frac{t}{4}} \end{pmatrix} \middle| t, x \in \mathbb{R} \right\}.$$

This is the identity component of the subgroup consisting of all upper triangular matrices in $SL(2,\mathbb{R})$. Let $\widetilde{P} \subset \widetilde{SL}(2,\mathbb{R})$ be the lift of P, which $\widetilde{SL}(2,\mathbb{R})$ is the universal covering group of $SL(2,\mathbb{R})$. In this case, P is isomorphic to $\widetilde{P}$ naturally. We identify P with $\widetilde{P}$. Let $\Gamma \subset \widetilde{SL}(2,\mathbb{R})$ be a cocompact lattice. Then P acts on $\Gamma\backslash\widetilde{SL}(2,\mathbb{R})$ by right multiplication.

Example 2.16. *Let $Solv$ and $P \subset Solv$ be the groups as defined below:*

$$Solv = \left\{ \begin{pmatrix} e^{\frac{t}{2}} & 0 & x \\ 0 & e^{-\frac{t}{2}} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| t, x, y \in \mathbb{R} \right\},$$

$$P = \left\{ \begin{pmatrix} e^{\frac{t}{2}} & 0 & x \\ 0 & e^{-\frac{t}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| t, x \in \mathbb{R} \right\}.$$

This P is isomorphic to the one in Example 2.15. Let $\Gamma \subset Solv$ be a cocompact lattice. Then P acts on $\Gamma\backslash Solv$ by right multiplication.

Definition 2.17. *We call the two group actions in Examples 2.15 and 2.16 as* 3*-homogeneous actions.*

We define the relations between C^∞ locally free group actions. For each C^∞ manifolds M_1 and M_2, let $\mathrm{Diffeo}(M_1, M_2)$ be the set consisting of all C^∞ diffeomorphisms between M_1 and M_2. If $M = M_1 = M_2$, we denote it as $\mathrm{Diffeo}(M)$.

Definition 2.18. *Let P be a Lie group and M_ν $(\nu = 1, 2)$ be C^∞ closed manifolds. For each $\nu = 1, 2$, let ρ_ν be a C^∞ locally free P-action on M_ν. Then ρ_1 and ρ_2 are C^∞ conjugate if and only if there exist $h \in \mathrm{Diffeo}(M_1, M_2)$ and $\theta \in \mathrm{Aut}(P)$ such that the following diagram is commute:*

$$\begin{tikzcd} M_1 \arrow[r, "h"] \arrow[d, "\rho_1(p)"'] & M_2 \arrow[d, "\rho_2(\theta(p))"] \\ M_1 \arrow[r, "h"'] & M_2 \end{tikzcd}$$

for each $p \in P$.

The following is a global rigidity theorem of C^∞ locally free group actions proved by Ghys.

Theorem 2.19. *([15, Théorème D]) Let $P \subset SL(2,\mathbb{R})$ be the identity component of the subgroup consisting of all upper triangular matrices in $SL(2,\mathbb{R})$. Let M be a closed orientable 3-manifold with $H^1_{\mathrm{dR}}(M) = \{0\}$. Then any C^∞ locally free P-action on M is C^∞ conjugate with a 3-homogeneous action.*

We see another global rigidity theorem. It is based on personal communications between Asaoka and Ghys.

Theorem 2.20. *([6, Theorem 1.4]) Let $P \subset SL(2,\mathbb{R})$ be the identity component of the subgroup consisting of all upper triangular matrices in $SL(2,\mathbb{R})$. Let $\Gamma \subset Solv$ be a cocompact lattice. Then any C^∞ locally free P-action on $\Gamma\backslash Solv$ is C^∞ conjugate with a 3-homogeneous action.*

On the other hand, the rigidity of reparametrizations of orbits is also investigated. Let M be a manifold and $\mathcal{F}$ be a foliation on M. Let $\mathrm{Diffeo}(\mathcal{F}) \subset \mathrm{Diffeo}(M)$ be the subgroup consisting of all diffeomorphisms which preserve each leaf of $\mathcal{F}$. Let $\mathrm{Diffeo}(\mathcal{F})_0 \subset \mathrm{Diffeo}(\mathcal{F})$ be the path-connected identity component under the Whitney C^∞-topology.

Definition 2.21. *Let P be a Lie group, M be a manifold, ρ be a C^∞ locally free P-action on M, and $\mathcal{F}_\rho$ is the orbit foliation on M. Then ρ has parameter rigidity if and only if the following property holds: for each C^∞ locally free P-action ρ' on M with $\mathcal{F}_\rho = \mathcal{F}_{\rho'}$, there exist $h \in \mathrm{Diffeo}(\mathcal{F}_\rho)_0$ and $\theta \in \mathrm{Aut}(P)$ such that the following diagram is commute:*

$$\begin{array}{ccc} M & \xrightarrow{h} & M \\ {\scriptstyle \rho(p)}\downarrow & & \downarrow{\scriptstyle \rho'(\theta(p))} \\ M & \xrightarrow[h]{} & M \end{array}$$

for each $p \in P$.

It is known that ρ has local rigidity if ρ has parameter rigidity and $\mathcal{F}_\rho$ has some stability (see [8, Proposition 10.2]). See also [8], [24], and [25] for explanations of the relevance between these properties.

The following result contrasts with the global rigidity theorems above.

Theorem 2.22. *([6, Main Theorem]) Let $P \subset SL(2,\mathbb{R})$ be the identity component of the subgroup consisting of all upper triangular matrices in $SL(2,\mathbb{R})$. Let $\Gamma \subset \widetilde{SL}(2,\mathbb{R})$ be a cocompact lattice with $H^1_{\mathrm{dR}}(\Gamma\backslash\widetilde{SL}(2,\mathbb{R})) \neq \{0\}$. Then the P-action by right multiplication on $\Gamma\backslash\widetilde{SL}(2,\mathbb{R})$ does not have parameter rigidity.*

Next, we explain sufficient conditions for when a C^∞ locally free group action has parameter rigidity. Let P be a Lie group and M be a manifold. Fix a C^∞ locally free P-action ρ on M. We define the map

$$\iota_\rho : \mathfrak{p} \to \Gamma^\infty(T\mathcal{F}_\rho)$$

by the derivative of ρ, where $\mathfrak{p}$ is the Lie algebra of P. This is injective since ρ is locally free. The map ι_ρ induces the natural isomorphism

$$C^\infty(M) \otimes \mathrm{Im}(\rho) \cong \Gamma^\infty(T\mathcal{F}_\rho).$$

Using this formula, we define the same notation map

$$\iota_\rho : C^n_{\mathrm{Lie}}(\mathfrak{p}) \to \Omega^n(\mathcal{F}_\rho)$$

by the condition below:

$$(\iota_\rho \omega)(\iota_\rho X_1, \cdots, \iota_\rho X_n) = \omega(X_1, \cdots, X_n)$$

for each $X_1, \cdots, X_n \in \mathfrak{p}$, and $\omega \in C^n_{\mathrm{Lie}}(\mathfrak{p})$. This map induces the following map between cohomology groups:

$$(\iota_\rho)_* : H^*_{\mathrm{Lie}}(\mathfrak{p}) \to H^*(\mathcal{F}_\rho).$$

The following is known for this map.

Proposition 2.23. *(See [24, Proposition 2.5], [25, Proposition 3.3], and [22, Proposition 1.0.2]) The map $(\iota_\rho)_1 : H^1_{\mathrm{Lie}}(\mathfrak{p}) \to H^1(\mathcal{F}_\rho)$ is injective. In particular, we obtain*

$$\dim H^1_{\mathrm{Lie}}(\mathfrak{p}) \leq \dim H^1(\mathcal{F}_\rho).$$

In connection with this inequality, the following question has been asked.

Question 2.24. *The equality*

$$\dim H^1_{\mathrm{Lie}}(\mathfrak{p}) = \dim H^1(\mathcal{F}_\rho)$$

is a sufficient condition for parameter rigidity of ρ.

To this question, several answers are known in the form of conditions imposed on a Lie group P. Let us enumerate these.

- When $P = \mathbb{R}^N$, Question 2.24 is true. Moreover, the equality is also a necessary condition. (See [4, Theorem 2.4] and [24, Proposition 2.6].)
- When P is a connected simply-connected nilpotent, Question 2.24 is true. ([21, Section 2])
- When P is a connected simply-connected solvable, Question 2.24 is true under the additional condition regarding leafwise cohomology theory with twisted coefficients. ([22, Theorem 2.2.5])

Finally, we give an example of using this condition.

Example 2.25. 1*-dimentional linear foliation $\mathcal{F}_a$ on $\mathbb{T}^2$ is obtained by the* $\mathbb{R}$*-action. We denote its action as ρ_a. Then a necessary and sufficient condition for ρ_a to have parameter rigidity is that the number a is Diophantine. In general, this fact holds true for linear foliations on $\mathbb{T}^N$, which is discussed in the next chapter. The above is originally stated in [5, Section 4].*

3 Linear foliations as classical examples

3.1 Notions

For each set S, let $M(q,p;S)$ denotes the space of all $(q\times p)$-matrices whose coefficients belong to S. In particular, we denote sometimes $M(q,1;S)$ as S^q. Let $p,q\in\mathbb{N}$ and put $n=p+q$. We denote each point in $\mathbb{R}^n=\mathbb{R}^p\times\mathbb{R}^q$ as (x,y). Set $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ and also denote each point in $\mathbb{T}^n=\mathbb{T}^p\times\mathbb{T}^q$ as (x,y).

Let $A\in M(q,p;\mathbb{R})$ and put $w\in M(q,1;\Omega^1_{\mathrm{dR}}(\mathbb{T}^n))$ as

$$w = A\,dx - dy.$$

Then

$$\mathscr{D}^* = \{dx_1,\cdots,dx_p,w_1,\cdots,w_q\}$$

is a frame on $T^*\mathbb{T}^n$. Put $X\in M(p,1;\mathfrak{X}(\mathbb{T}^n))$ as

$$X = \frac{\partial}{\partial x} + {}^tA\frac{\partial}{\partial y}, \tag{3.1}$$

then

$$\mathscr{D} = \left\{X_1,\cdots,X_p,-\frac{\partial}{\partial y_1},\cdots,-\frac{\partial}{\partial y_q}\right\}$$

is a frame on $T\mathbb{T}^n$.

Lemma 3.1. *The tuple $\mathscr{D}^*$ is the dual frame of $\mathscr{D}$.*

Proof. Let $1\le\lambda,\lambda'\le p$ and $1\le\mu,\mu'\le q$. First, the following three formulae

$$dx_\lambda(X_{\lambda'}) = \delta_{\lambda\,\lambda'},$$

$$dx_\lambda\left(-\frac{\partial}{\partial y_\mu}\right) = 0,$$

$$w_\mu\left(-\frac{\partial}{\partial y_{\mu'}}\right) = \delta_{\mu\,\mu'}$$

are obvious. Second,

$$\begin{aligned} w_\mu(X_\lambda) &= \left(\sum_{\lambda'} a_{\mu\,\lambda'}dx_{\lambda'} - dy_\mu\right)\left(\frac{\partial}{\partial x_\lambda} + \sum_{\mu'} a_{\mu'\,\lambda}\frac{\partial}{\partial y_{\mu'}}\right) \\ &= \sum_{\lambda'} a_{\mu\,\lambda'}\delta_{\lambda'\,\lambda} - \sum_{\mu'} a_{\mu'\,\lambda}\delta_{\mu\,\mu'} \\ &= a_{\mu\,\lambda} - a_{\mu\,\lambda} \\ &= 0. \end{aligned}$$

□

Let $\mathcal{F}_A$ be the linear foliation corresponding to $\mathrm{Ker}(w) \subset T\mathbb{T}^n$. For each $1 \leq p' \leq p$ and $\omega \in \Omega^{p'}(\mathcal{F}_A)$, there exist smooth functions $f_{\lambda_1,\cdots,\lambda_{p'}} \in C^\infty(\mathbb{T}^n)$ such that

$$\omega = \sum_{\lambda_1,\cdots,\lambda_{p'}} f_{\lambda_1,\cdots,\lambda_{p'}}\, dx_{\lambda_1} \wedge \cdots \wedge dx_{\lambda_{p'}}.$$

From Lemma 3.1, we obtain

$$d_{\mathcal{F}_A}\omega = \sum_{\lambda} \sum_{\lambda_1,\cdots,\lambda_{p'}} X_\lambda(f_{\lambda_1,\cdots,\lambda_{p'}})\, dx_\lambda \wedge dx_{\lambda_1} \wedge \cdots \wedge dx_{\lambda_{p'}}.$$

We define the metric. Let $N \in \mathbb{N}$. We denote each point in $\mathbb{R}^N$ or $\mathbb{T}^N$ as z. Put

$$|z| = \max_{1 \leq \nu \leq N} |z_\nu|,$$

$$||z|| = \min_{l \in \mathbb{Z}^N} |z + l|.$$

Lemma 3.2. *$||z||$ is a metric on $\mathbb{T}^N$.*

Proof. We only need to prove for $N = 1$. The triangular inequality is non-trivial. This is easily shown by a case classification. □

3.2 Minimality

In a 1-dimensional linear foliation $\mathcal{F}_a$, each leaf is dense if and only if a is irrational. We generalize this fact about orientation to general linear foliations.

Theorem 3.3. *The following properties are equivalent:*

(i) *$\mathcal{F}_A$ is minimal.*

(ii) *For each $k \in \mathbb{Q}^q$, if ${}^t k w$ is rational, then $k = 0$.*

The proof of this fact is omitted in the original paper. However, this is an important statement linking minimality and a computation of the metric $|| \cdot ||$. To be complete discussion, we give proof.

We prove this step by step.

Lemma 3.4. *Let $c \in \mathbb{Q}^N \setminus \{0\}$ and set the hyperplane*

$$W = \left\{ z \in \mathbb{R}^N \,\middle|\, {}^t c z = 0 \right\}.$$

Then the image of W in the projection $\mathbb{R}^N \to \mathbb{T}^N$ is not dense.

Proof. Let $(\ ,\ )_{\mathbb{R}^N}$ be the Euclidean inner product and $|\cdot|_{\mathbb{R}^N}$ be the Euclidean norm. For each $z \in \mathbb{R}^N$ and $w \in W$,

$$\begin{aligned} ||z-w|| &= \min_{l\in\mathbb{Z}^N} |(z+l)-w| \\ &\geq \min_{l\in\mathbb{Z}^N} |(z+l)-w|_{\mathbb{R}^N} \\ &\geq \inf_{l\in\mathbb{Z}^N} \left| \frac{(z+l,c)_{\mathbb{R}^N}}{(c,c)_{\mathbb{R}^N}} \right|. \end{aligned}$$

Thus it is enough to estimate a lower bound of $|(z+l,c)_{\mathbb{R}^N}|$ for some $z \in \mathbb{R}^N$. For each $1 \leq \nu \leq N$, we represent c_ν as a fraction

$$c_\nu = \frac{p_\nu}{q_\nu}$$

for some $p_\nu, q_\nu \in \mathbb{Z}$. Let $q_\infty \in \mathbb{Z}$ be a prime number which does not appear in all p_ν and q_ν. Since $c \neq 0$, there exists ν_∞ such that $c_{\nu_\infty} \neq 0$. We define $z \in \mathbb{R}^N$ as

$$z_\nu = \begin{cases} \frac{1}{q_\infty} & \text{if } \nu = \nu_\infty, \\ 0 & \text{if } \nu \neq \nu_\infty. \end{cases}$$

Then we obtain a lower bound

$$|(z+l,c)_{\mathbb{R}^N}| \geq \frac{1}{q_\infty} \prod_{\nu=1}^{N} \frac{1}{q_\nu} > 0.$$

□

Proof of (i) ⇒ (ii) in Theorem 3.3. We assume that there exists $k \in \mathbb{Q}^q$ such that ${}^t k w$ is rational and $k \neq 0$. It is enough to prove that the leaf L including $0 \in \mathbb{T}^n$ is not dense. Set

$$W = \left\{ (x,y) \in \mathbb{R}^n \middle| {}^t k A x + {}^t k y = 0 \right\}.$$

Since $L \subset W$ in $\mathbb{T}^n$, it is enough to prove that W is not dense in $\mathbb{T}^n$. From the rationality of ${}^t k w$, the hyperplane W satisfies the assumption in Lemma 3.4. Thus W is not dense. □

We consider the opposite. We represent A as the sequence of column vectors $(A_1, \cdots, A_p) \in M(q,p;\mathbb{R})$.

Lemma 3.5. *If $\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_p \subset \mathbb{T}^q$ is dense, then $\mathcal{F}_A$ is minimal.*

Proof. It is enough to prove that

$$\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_p + Ax_0 + y_0 \subset \mathbb{T}^q$$

is dense for each $(x_0, y_0) \in \mathbb{T}^n$. From assumption, this is valid. □

From Lemma 3.5, the case for $p = 1$ is attributed to the following.

Theorem 3.6 (Kronecker)**.** *If $1, \alpha_1, \cdots, \alpha_N \in \mathbb{R}$ are $\mathbb{Q}$-linear independent, then $\mathbb{Z}(\alpha_\nu)_{\nu=1}^N \subset \mathbb{T}^N$ is dense.*

See [17, Chapter 23] for some proofs. We use this fact in the proof of the general case as well.

In the inductive step of our proof, the following map is useful.

Definition 3.7. *Let $\mathcal{V} = \{v_1, \cdots, v_q\} \subset \mathbb{Z}^q$ be a $\mathbb{Q}$-basis of $\mathbb{Q}^q$. Then we put $\pi_{\mathcal{V}} : \mathbb{T}^q \to \mathbb{T}^q$ as*

$$\pi(y) = ({}^t v_\mu y)_{\mu=1}^q.$$

This is a finite covering homomorphism. In fact, identify $\mathcal{V}$ with the qth-order matrix: $\mathcal{V} \in M(q, q; \mathbb{Z})$ whose determinant is non-zero. By elementary divisor theory, there exist $P, Q \in GL(q; \mathbb{Z})$ and a diagonal matrix $E \in M(q, q; \mathbb{Z})$ whose each element is non-zero such that

$$P\mathcal{V}Q = E.$$

We obtain the following commutative diagram.

$$\begin{tikzcd}
\mathbb{T}^n \arrow[d, "\pi_{\mathcal{V}}"'] & \mathbb{T}^n \arrow[l, "\pi_P"'] \arrow[d, "\pi_E"] \\
\mathbb{T}^n \arrow[r, "\pi_Q"'] & \mathbb{T}^n
\end{tikzcd}$$

The map π_E is a finite covering homomorphism. Since π_P and π_Q are homeomorphisms and group isomorphisms, the map $\pi_{\mathcal{V}}$ is also a finite covering homomorphism.

We treat $\pi_{\mathcal{V}}$ as "decomposition in direction $\mathcal{V}$". By the following lemma, the density of $\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_p$ can be considered in $\mathrm{Im}(\pi_{\mathcal{V}})$.

Lemma 3.8. *Let $\pi : \mathbb{T}^N \to \mathbb{T}^N$ be a finite covering homomorphism and $H \subset \mathbb{T}^N$ be a sub-group. Set $H' = \pi(H)$. If H' is dense, then H is also dense.*

Proof. Put $d = \deg(\pi) \in \mathbb{N}$. Since π is a covering map, there exists a neighborhood O' of $0 \in \mathbb{T}^N$ such that

$$\pi^{-1}(O') = \bigsqcup_{s=1}^{d} O_s$$

and $\pi|_{O_s}$ is a homeomorphism for each $1 \leq s \leq d$. Assume $0 \in O_1$.

Then take any $z \in \mathbb{T}^N$ and any neighborhood $U \subset O_1$ of $0 \in \mathbb{T}^N$. We prove

$$(z + U) \cap H \neq \emptyset.$$

Set $U' = \pi(U)$. This U' is a neighborhood of $0 \in \mathbb{T}^N$. We take a connected open neighborhood $V' \subset U'$ of $0 \in \mathbb{T}^N$ which satisfies $d \cdot V' \subset U'$. Here, we set

$$d \cdot V' = \{d \cdot v' \in \mathbb{T}^n | v' \in V'\}.$$

We represent $\pi^{-1}(V')$ as

$$\pi^{-1}(V') = \bigsqcup_{s=1}^{d} V_s.$$

For each $1 \leq s \leq d$, there is a $1 \leq s' \leq d$ uniquely such that

$$d \cdot V_s \subset O_{s'}$$

by $d \cdot V' \subset O'$ and connectedness of V'. Since $d \cdot \mathrm{Ker}(\pi) = \{0\}$, we have $s' = 1$:

$$d \cdot V_s \subset O_1.$$

Combine this with $d \cdot V' \subset U'$ to obtain the following:

$$d \cdot V_s \subset U \subset O_1$$

for each $1 \leq s \leq d$. Set $w = \frac{z}{d}$ and $w' = \pi(w)$. Since H' is dense, we have

$$(w' + V') \cap H' \neq \emptyset.$$

Thus there exists $v' \in V'$ such that $w' + v' \in H'$. We write $\pi^{-1}(\{v'\})$ as

$$\pi^{-1}(\{v'\}) = \{v_1, \cdots, v_d\},$$

where $v_s \in V_s$ for each $1 \leq s \leq d$. We have

$$\pi^{-1}(\{w' + v'\}) = \{w + v_1, \cdots, w + v_d\}.$$

Then there exists $1 \leq s \leq d$ such that $w + v_s \in H$. We obtain

$$z + d \cdot v_s = d(w + v_s) \in H.$$

On the other hand, since $d \cdot v_s \in d \cdot V_s \subset U$,

$$z + d \cdot v_s \in (z + U).$$

Therefore we proved $(z + U) \cap H \neq \emptyset$. □

Proof of (ii) ⇒ (i) in Theorem 3.3. From Lemma 3.5, it is enough to prove

$$\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_p \subset \mathbb{T}^q$$

is dense. For each $1 \leq \lambda \leq p$, we define the map $\Phi_\lambda : \mathbb{Q}^q \to \mathbb{R}$ as

$$\Phi_\lambda(k) = {}^t k A_\lambda.$$

The assumption (ii) is the same as

$$\bigcap_{\lambda=1}^{p} \Phi_\lambda^{-1}(\mathbb{Q}) = \{0\}.$$

Thus, by replacing the indexes in $A_1, \cdots, A_p$, we can assume the following: there exists $1 \leq p' \leq p$ such that if we put

$$W_0 = \mathbb{Q}^q,$$

$$W_\lambda = W_{\lambda-1} \cap \Phi_\lambda^{-1}(\mathbb{Q}) \ \ (1 \leq \lambda \leq p'),$$

then

$$W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_{p'} = \{0\}.$$

We take a $\mathbb{Q}$-basis $\mathcal{V} \subset \mathbb{Z}^q$ of $\mathbb{Q}^q$ which satisfies the following: there exists the decomposition

$$\mathcal{V} = \bigsqcup_{\lambda=1}^{p'} \mathcal{V}_\lambda$$

such that $\mathcal{V}_\lambda \sqcup \cdots \sqcup \mathcal{V}_{p'}$ is a basis of $W_{\lambda-1}$ for each $1 \leq \lambda \leq p'$. Since $v_{\lambda'} \in \Phi_\lambda^{-1}(\mathbb{Q})$ for each $1 \leq \lambda < \lambda' \leq p'$, we can assume

$$^t v_{\lambda'} A_\lambda \in \mathbb{Z} \tag{3.2}$$

by multiplying $v_{\lambda'}$ by an integer. Put $q_\lambda = \#\mathcal{V}_\lambda$. Take $\pi_\mathcal{V}$ and set $A'_\lambda = \pi_\mathcal{V}(A_\lambda)$. From Lemma 3.8, it is enough to prove

$$\mathbb{Z}A'_1 + \cdots + \mathbb{Z}A'_{p'} + \cdots + \mathbb{Z}A'_p \subset \mathbb{T}^q$$

is dense. More strongly, we prove that the following is dense:

$$\mathbb{Z}A'_1 + \cdots + \mathbb{Z}A'_{p'} \subset \mathbb{T}^q$$

The $\mathbb{T}$-valued matrix $(A'_1, \cdots, A'_{p'}) \in M(q, p'; \mathbb{T})$ is blocked upper triangular by (3.2), which the number of columns in each block is 1. Then it is enough to prove that $\mathbb{Z}A'_\lambda$ is dense in the image of the projection $\mathbb{T}^q \to \mathbb{T}^{q_\lambda}$ for each $1 \leq \lambda \leq p'$. We represent $\mathcal{V}_\lambda$ as

$$\mathcal{V}_\lambda = (v_{\lambda\, 1}, \cdots, v_{\lambda\, q_\lambda}) \in M(q, q_\lambda; \mathbb{Z}).$$

Claim 3.9. *For each $1 \leq \lambda \leq p'$, the numbers $1, {}^t v_{\lambda\, 1} A_\lambda, \cdots, {}^t v_{\lambda\, q_\lambda} A_\lambda \in \mathbb{R}$ are $\mathbb{Q}$-linear independent.*

Proof. For each $\gamma_0, \gamma_1, \cdots, \gamma_{q_\lambda} \in \mathbb{Q}$, we assume

$$\gamma_0 + \sum_{\mu=1}^{q_\lambda} \gamma_\mu {}^t v_{\lambda\,\mu} A_\lambda = 0.$$

This is transformed into the following equation:

$${}^t\left(\sum_{\mu=1}^{q_\lambda} \gamma_\mu v_{\lambda\,\mu}\right) A_\lambda = -\gamma_0.$$

Set $k = \sum_{\mu=1}^{q_\lambda} \gamma_\mu v_{\lambda\,\mu}$. This equation means that $k \in \Phi_\lambda^{-1}(\mathbb{Q})$. Since $k \in W_{\lambda-1}$ whose basis is $\mathcal{V}_\lambda \sqcup \cdots \sqcup \mathcal{V}_{p'}$, we obtain $k \in W_\lambda$. Then k is a linear sum on $\mathcal{V}_{\lambda+1} \sqcup \cdots \sqcup \mathcal{V}_{p'}$. Thus we have $\gamma_1 = \cdots = \gamma_{q_\lambda} = 0$ and $\gamma_0 = 0$ $\square$

From this claim and Theorem 3.6, $\mathbb{Z}A'_\lambda$ is dense in the image of the projection $\mathbb{T}^q \to \mathbb{T}^{q_\lambda}$. $\square$

3.3 Leafwise cohomology

From now on, we assume $\mathcal{F}_A$ is minimal. From Theorem 3.3, we have

$$||{}^tAk|| > 0$$

for each $k \in \mathbb{Z}^q \setminus \{0\}$. We add a further condition for orientation of $\mathcal{F}_A$.

Definition 3.10. (i) *We call A is diophantine if there exist $\beta \geq 0$ and $c > 0$ such that*

$$||{}^tAk|| > \frac{c}{|k|^{q+\beta}}$$

for each $k \in \mathbb{Z}^q \setminus \{0\}$.

(ii) *We call A is Liouville if A is not diophantine.*

Example 3.11. *Each algebraic irrational number a over $\mathbb{Q}$ is diophantine.*

Proof. Put $d = \deg(a) \geq 2$. We prove the existence of $0 < c < 1$ such that

$$\left|\frac{l}{k} - a\right| > \frac{c}{|k|^d} \tag{3.3}$$

for each $k \in \mathbb{Z} \setminus \{0\}$ and $l \in \mathbb{Z}$ under $|\frac{l}{k} - a| < 1$. Let $P(T) \in \mathbb{Q}[T]$ be the minimal polynomial of a. We have the Taylor expansion of $P(T)$ at a:

$$P(T) = \sum_{s=1}^{d} \frac{P^{(s)}(a)}{s!}(T-a)^s.$$

Under $|T-a|<1$,

$$|P(T)| \leq |T-a| \sum_{s=1}^{d} \left| \frac{P^{(s)}(a)}{s!} \right|. \tag{3.4}$$

Since $P(T)$ is monic, we have $P^{(d)}(a) = d! > 0$. Thus we can set

$$c = \min \left\{ \frac{1}{2}, \frac{1}{\sum_{s=1}^{d} \left| \frac{P^{(s)}(a)}{s!} \right|} \right\}.$$

Then take any $k \in \mathbb{Z} \setminus \{0\}$ and $l \in \mathbb{Z}$ with $|\frac{l}{k} - a| < 1$. Since $P(T)$ is irreducible,

$$\left| P\Big(\frac{l}{k}\Big) \right| \geq \frac{1}{|k|^d}.$$

This estimation and (3.4) prove desired inequality (3.3). □

Example 3.12. *The following number a is Liouville:*

$$a = \sum_{s=1}^{\infty} \frac{1}{2^{s!}}.$$

Proof. Take any $\beta \geq 0$ and $c > 0$. For each $\sigma \in \mathbb{N}$, set

$$k = 2^{\sigma!},$$

$$l = -\sum_{s=1}^{\sigma} \frac{2^{\sigma!}}{2^{s!}}.$$

Then

$$\begin{aligned} ||ak|| &\leq |ak + l| \\ &= \sum_{s=\sigma+1}^{\infty} \frac{2^{\sigma!}}{2^{s!}} \\ &= \sum_{s=0}^{\infty} \frac{1}{2^{(\sigma+1+s)!-\sigma!}} \\ &\leq \sum_{s=0}^{\infty} \frac{1}{2^{(\sigma+1+s)\sigma!-\sigma!}} \\ &= \frac{1}{2^{\sigma\sigma!}} \sum_{s=0}^{\infty} \frac{1}{2^{s\sigma!}} \\ &\leq \frac{2}{2^{\sigma\sigma!}}. \end{aligned}$$

Thus it is enough to prove

$$\frac{2}{2^{\sigma\sigma!}} \leq \frac{c}{(2^{\sigma!})^{1+\beta}}.$$

This is transformed into

$$(2^{\sigma!})^{\sigma-2-\beta} \geq \frac{1}{c}.$$

This is valid for a sufficient large $\sigma \in \mathbb{N}$. $\square$

We rewrite the Liouville condition.

Lemma 3.13. *If A is Liouville, then there exists $\{k(s)\}_{s=1}^{\infty} \subset \mathbb{Z}^q \setminus \{0\}$ which are mutually exclusive such that*

$$||{}^t A k(s)|| < \frac{1}{|k(s)|^s}$$

for each $k \in \mathbb{N}$. Especially, $\lim_{s\to\infty} |k(s)| = \infty$ *and* $\lim_{s\to\infty} ||{}^t A k(s)|| = 0$.

Proof. Put $M(1) = 1$. For each $s \in \mathbb{N}$, assume that $M(s) > 0$ and $\{k(s')\}_{s'=1}^{s-1}$ are given with $|k(s')| \leq M(s)$ for each $s' \leq s-1$. Set $\beta = \min\{0, s-q\}$ and $c = \frac{1}{2}$. From a Liouville condition, we obtain $k(s) \in \mathbb{Z}^q \setminus \{0\}$ with $|k(s)| > M(s)$ which satisfies

$$||{}^t A k(s)|| < \frac{1}{|k(s)|^s}.$$

Put $M(s+1) = |k(s)|$ and repeat the inductive steps. $\square$

Then we describe the computation results of the leafwise cohomology.

Theorem 3.14. *([5, Theorem 2.2])*
Let $1 \leq p' \leq p$.

(i) *If A is diophantine, then $H^{p'}(\mathcal{F}_A) \cong H^{p'}_{\mathrm{dR}}(\mathbb{T}^p)$.*

(ii) *If A is Liouville, then $\dim H^{p'}(\mathcal{F}_A) = \infty$.*

In the original paper, this proof is given when $q = p = 2$ and $p' = 1$. We give the proof for general q, p and $p' = 1$.

Proof of (i) in Theorem 3.14 for $p' = 1$. Fix constants $\beta \geq 0$ and $c > 0$ in the diophantine condition. Take any

$$\eta = \sum_{\lambda=1}^{p} f_\lambda \, dx_\lambda \in Z^1(\mathcal{F}_A) \ \ (f_\lambda \in C^\infty(\mathbb{T}^n)).$$

We have the Fourier expansion

$$f_\lambda(x,y) = \sum_{l\in\mathbb{Z}^p,\, k\in\mathbb{Z}^q} f_{\lambda,\, l\, k} e^{2\pi i({}^t l x + {}^t k y)}.$$

First, $dx_1, \cdots, dx_p \in H^1(\mathcal{F}_A)$ are linear independent. To identify them with a basis of $H^1_{\mathrm{dR}}(\mathbb{T}^p)$, it is enough to prove the following: if $f_{\lambda,\,0\,0} = 0$ for each $1 \leq \lambda \leq p$, then $\eta \in B^1(\mathcal{F}_A)$. For each $h \in C^\infty(\mathbb{T}^n)$, we expand

$$h(x,y) = \sum_{l\, k} h_{l\, k} e^{2\pi i({}^t l x + {}^t k y)}.$$

From partial integrations, we obtain

$$X(h) = \sum_{l\, k} h_{l\, k} 2\pi i(l + {}^t A k) e^{2\pi i({}^t l x + {}^t k y)} \in M(p,1; C^\infty(\mathbb{T}^n)),$$

where $X \in M(p,1;\mathfrak{X}(\mathbb{T}^n))$ is the differential operator defined in (3.1). Here, set

$$K_\lambda = \{k \in \mathbb{Z}^q \,|\, ||{}^t A_\lambda k|| = ||{}^t A k||\},$$

then

$$\mathbb{Z}^q = \bigcup_{\lambda=1}^p K_\lambda.$$

Take any $(l,k) \in \mathbb{Z}^n$. If $k = 0$, we take λ such that $l_\lambda \neq 0$. If $k \neq 0$, we take λ such that $k \in K_\lambda$.

Claim 3.15. *The following holds:*

$$l_\lambda + {}^t A_\lambda k \neq 0.$$

Proof. If $k = 0$, it is obvious. Assume $k \neq 0$. Then

$$|l_\lambda + {}^t A_\lambda k| \geq ||{}^t A_\lambda k|| = ||{}^t A k|| > \frac{c}{|k|^{q+\beta}}.$$

□

We set

$$h_{l\, k} = \frac{f_{\lambda,\, l\, k}}{2\pi i(l_\lambda + {}^t A_\lambda k)}. \tag{3.5}$$

Claim 3.16. *The definition (3.5) is independent of choices λ.*

Proof. We take another λ'. We prove

$$\frac{f_{\lambda,\, l\, k}}{2\pi i(l_\lambda + {}^t A_\lambda k)} = \frac{f_{\lambda',\, l\, k}}{2\pi i(l_{\lambda'} + {}^t A_{\lambda'} k)}. \tag{3.6}$$

Since $\eta \in Z^1(\mathcal{F}_A)$, we have

$$X_{\lambda'}(f_\lambda) = X_\lambda(f_{\lambda'}). \tag{3.7}$$

From partial integrations, we obtain

$$X_{\lambda'}(f_\lambda) = \sum_{l\,k} f_{\lambda,\,l\,k} 2\pi i(l_{\lambda'} + {}^tA_{\lambda'}k)e^{2\pi i({}^tlx + {}^tky)}. \tag{3.8}$$

The formulae (3.7) and (3.8) prove desired (3.6). $\square$

Let h be a formal function whose Fourier coefficients are given above.

Claim 3.17. *The formal function h is C^∞ function.*

Proof. We estimate Fourier coefficients to use Sobolev embedding. For each $(l, k) \in \mathbb{Z}^n$, there exists λ such that

$$\begin{aligned}
|h_{l\,k}| &= \frac{|f_{\lambda,\,l\,k}|}{2\pi|l_\lambda + {}^tA_\lambda k|} \\
&\leq \frac{|f_{\lambda,\,l\,k}|}{2\pi}\frac{1}{||{}^tA_\lambda k||} \\
&= \frac{|f_{\lambda,\,l\,k}|}{2\pi}\frac{1}{||{}^tAk||} \\
&< \frac{|f_{\lambda,\,l\,k}|}{2\pi}\frac{|k|^{q+\beta}}{c}.
\end{aligned}$$

Since f_λ is smooth, the coefficients $|f_{\lambda,\,l\,k}|$ decrease rapidly. This estimation means that $|h_{l\,k}|$ also decrease rapidly. $\square$

Then we obtain $h \in C^\infty(\mathbb{T}^n)$ and $\eta = d_{\mathcal{F}_A}h$. Thus $\eta \in B^1(\mathcal{F}_A)$. $\square$

Proof of (ii) in Theorem 3.14 for $p' = 1$. First, we prepare notations. Take a sequence $\{k(s)\}_{s=1}^\infty \subset \mathbb{Z}^q\backslash\{0\}$ in Lemma 3.13. There exists a sequence $\{l(s)\}_{s=1}^\infty \subset \mathbb{Z}^p$ such that

$$||{}^tAk(s)|| = |l(s) + {}^tAk(s)|.$$

Moreover there exists a sequence $\{\lambda(s)\}_{s=1}^\infty \subset \{1, \cdots, p\}$ such that

$$|l(s) + {}^tAk(s)| = |l(s)_{\lambda(s)} + {}^tA_{\lambda(s)}k(s)|.$$

Then there exists $\lambda_A \in \{1, \cdots, p\}$ such that

$$\lambda(s) = \lambda_A$$

for infinite number of s. We, therefore, proceed assuming

$$\lambda(s) = 1.$$

That is,

$$||{}^tAk(s)|| = |l(s) + {}^tAk(s)| = |l(s)_1 + {}^tA_1k(s)|. \tag{3.9}$$

For each $s \in \mathbb{N}$ and $\sigma \geq s$, we have

$$|l(s)_1 + {}^tA_1k(s)| < \frac{1}{|k(s)|^{\sigma}}. \tag{3.10}$$

Since $||{}^tAk|| > 0$ for each $k \in \mathbb{Z}^q \setminus \{0\}$, stated at the beginning of this section citing Theorem 3.3, we obtain the maximal σ. We denote this σ as $\sigma(s)$. Then we have

$$\frac{1}{|k(s)|^{\sigma(s+1)}} < |l(s)_1 + {}^tA_1k(s)| < \frac{1}{|k(s)|^{\sigma(s)}}$$

and

$$|l(s)_\lambda + {}^tA_\lambda k(s)| < \frac{1}{|k(s)|^{\sigma(s)}}$$

for each $2 \leq \lambda \leq p$.

Second, we construct non-trivial $\eta_S \in Z^1(\mathcal{F}_A)$ for each infinite subset $S \subset \mathbb{N}$. We define $f_{S\,1} \in C^\infty(\mathbb{T}^n)$. For each $s \in S$, set

$$f_{S\,1,\,l(s)\,k(s)} = \frac{1}{(|l(s)_1|+1)^{\frac{\sigma(s)}{3p}} \cdots (|l(s)_p|+1)^{\frac{\sigma(s)}{3p}} |k(s)|^{\frac{\sigma(s)}{3}}}.$$

For other $(l, k) \in \mathbb{Z}^n$, set $f_{S\,1,\,l\,k} = 0$. We put

$$f_{S\,1}(x, y) = \sum_{l\,k} f_{S\,1,\,l\,k} e^{2\pi i({}^tlx + {}^tky)}.$$

This is smooth. Next, we define $f_{S\,\lambda} \in C^\infty(\mathbb{T}^n)$ for each $2 \leq \lambda \leq p$. For each $s \in S$, set

$$f_{S\,\lambda,\,l\,k} = \frac{l(s)_\lambda + {}^tA_\lambda k(s)}{l(s)_1 + {}^tA_1k(s)} f_{S\,1,\,l(s)\,k(s)}.$$

For other $(l, k) \in \mathbb{Z}^n$, set $f_{S\,\lambda,\,l\,k} = 0$. We put

$$f_{S\,\lambda}(x, y) = \sum_{l\,k} f_{S\,\lambda,\,l\,k} e^{2\pi i({}^tlx + {}^tky)}.$$

From (3.9) and (3.10), we obtain

$$|f_{S\,\lambda,\,l\,k}| \leq |k(s)||f_{S\,1,\,l\,k}|$$

for each (l, k). Thus $f_{S\,\lambda}$ is smooth. Then we put

$$\eta_S = \sum_{\lambda=1}^{p} f_{S\,\lambda} dx_\lambda \in Z^1(\mathcal{F}_A).$$

Claim 3.18. *The above η_S is non-trivial.*

Proof. For preparing, we derive the evaluation equations that hold between $l(s)_\lambda$, $k(s)$, and A_λ for each $1 \leq \lambda \leq p$ and sufficient large $s \in \mathbb{N}$. In (3.9) and (3.10), use trigonometric inequality and get

$$\begin{aligned} |l(s)_\lambda| &< |{}^t A_\lambda k(s)| + \frac{1}{|k(s)|^{\sigma(s)}} \\ &\leq |A_\lambda||k(s)| + \frac{1}{|k(s)|^{\sigma(s)}}. \end{aligned}$$

For sufficient large $s \in \mathbb{N}$, we can assume $|k(s)| \geq 2$ and obtain

$$\frac{|l(s)_\lambda| + 1}{|k(s)|} < |A_\lambda| + 1. \tag{3.11}$$

Then assume $\eta_S \in B^1(\mathcal{F}_A)$. There exists $h \in C^\infty(\mathbb{T}^n)$ such that $\eta_S = d_{\mathcal{F}_A} h$. We estimate $|h_{l(s)\,k(s)}|$ by the following.

$$\begin{aligned} &2\pi |h_{l(s)\,k(s)}| \\ &= \frac{|f_{S\,1,\,l(s)\,k(s)}|}{|l(s)_1 + {}^t A_1 k(s)|} \\ &> |k(s)|^{\sigma(s)} |f_{S\,1,\,l(s)\,k(s)}| \quad \text{(by (3.9))} \\ &= \frac{|k(s)|^{\frac{2}{3}\sigma(s)}}{(|l(s)_1| + 1)^{\frac{\sigma(s)}{3p}} \cdots (|l(s)_p| + 1)^{\frac{\sigma(s)}{3p}}} \\ &= \left(\frac{|k(s)|}{|l(s)_1| + 1} \right)^{\frac{\sigma(s)}{3p}} \cdots \left(\frac{|k(s)|}{|l(s)_p| + 1} \right)^{\frac{\sigma(s)}{3p}} |k(s)|^{\frac{\sigma(s)}{3}} \\ &> \left(\frac{1}{|A_1| + 1} \right)^{\frac{\sigma(s)}{3p}} \cdots \left(\frac{1}{|A_1| + 1} \right)^{\frac{\sigma(s)}{3p}} |k(s)|^{\frac{\sigma(s)}{3}} \quad \text{(by (3.11))} \\ &= \left(\frac{|k(s)|}{|A_1| + 1} \right)^{\frac{\sigma(s)}{3p}} \cdots \left(\frac{|k(s)|}{|A_1| + 1} \right)^{\frac{\sigma(s)}{3p}} \to \infty. \end{aligned}$$

This contradicts to $\sum_{l\,k} |h_{l\,k}|^2 < \infty$. □

Finally, we construct an infinite number of linearly independent 1-cocycles. Take a family $\{S_\nu\}_{\nu=1}^\infty \subset 2^{\mathbb{N}}$ which satisfies the following two properties:

- For each $\nu \in \mathbb{N}$, S_ν is infinite.
- For each $\nu, \nu' \in \mathbb{N}$, if $\nu \neq \nu'$, then $S_\nu \cap S_{\nu'} = \emptyset$.

For example, let S_ν be the set of whole numbers that are not divisible by the first through the $(\nu - 1)$-th prime numbers but are divisible by the ν-th prime number. Then $\{\eta_{S_\nu}\}_{\nu=1}^\infty$ is linear independent in $H^1(\mathcal{F}_A)$. The proof is the same as Claim 3.18. Thus we have proved the result. □

4 Irreducible unitary representations for non-abelian harmonic analysis

4.1 Some facts from representation theory

We summarize notions from representation theory for understanding the way to the classification. Especially, we connect representation theories on Lie groups and Lie algebras. Let G be a Lie group, V be a Banach space over $\mathbb{C}$, and $\mathcal{L}(V)$ be the space consisting of all bounded linear operators on V.

Definition 4.1. **(i)** *A pair (π, V) is a Banach representation if $\pi : G \to \mathcal{L}(V)$ is a homomorphism and the following map is continuous:*

$$G \times V \ni (x, v) \mapsto \pi(x)v \in V.$$

(ii) *A pair (π, V) is a Hilbert representation if (π, V) is a Banach representation and V is a Hilbert space.*

(iii) *A pair (π, V) is a unitary representation if (π, V) is a Hilbert representation and $\pi(x)$ is a unitary operator for each $x \in G$.*

Definition 4.2. *Let (π, V) and (σ, W) be Banach representations. We call (π, V) is equivalent to (σ, W) if there exists a bi-continuous bijection linear map $T : V \to W$ such that the following diagram is commutative for each $x \in G$.*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi(x)"'] & W \arrow[d, "\sigma(x)"] \\ V \arrow[r, "T"'] & W \end{tikzcd}$$

Definition 4.3. *Let (π, V) and (σ, W) be unitary representations. We call (π, V) is unitarily equivalent to (σ, W) if there exists a unitary bijection linear map $T : V \to W$ such that the following diagram is commutative for each $x \in G$.*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi(x)"'] & W \arrow[d, "\sigma(x)"] \\ V \arrow[r, "T"'] & W \end{tikzcd}$$

Remark 4.4. *Let (π, V) and (σ, W) be unitary representations. Then the followings are equivalent.*

(i) *(π, V) is equivalent to (σ, W).*

(ii) *(π, V) is unitarily equivalent to (σ, W).*

See [39, Corollary 4.3.1.2] for the proof. We use the term "equivalent" instead of "unitarily equivalent" for unitary representations. Next, we define "atomic units" of representations.

Definition 4.5. *Let (π, V) be a Banach representation.*

(i) *A sub-space $W \subset V$ is invariant if*

$$\pi(x)W \subset W$$

for each $x \in G$.

(ii) *We call (π, V) is irreducible if each closed invariant sub-space $W \subset V$ is trivial.*

Definition 4.6. *Set*

$$\hat{G} = \{\textit{irreducible unitary representation of } G\}/(\textit{equivalent}).$$

Our aim is the classification in $\hat{G}$. We will attribute this problem to the algebraic representation theory of $\mathfrak{sl}(2, \mathbb{R})$.

Definition 4.7. *Let (π, V) be a Banach representation and set $r = \infty$ or ω.*

(i) *A vector $v \in V$ is a C^r-vector if the following map is C^r-map.*

$$G \ni x \mapsto \pi(x)v \in V.$$

(ii) *Let $V_r \subset V$ be the space consisting of all C^r-vectors.*

Remark 4.8. *The subspaces $V_\infty, V_\omega \subset V$ are dense.*

See [39, Proposition 4.4.1.1] and [39, Theorem 4.4.5.7] for the proofs.

Remark 4.9. *The subspaces $V_\infty, V_\omega \subset V$ are invariant.*

See [37, Chapter5, Proposition 2.3] and [37, Chapter 5, Proposition 6.6] for the proofs. Let $\mathfrak{g}$ be the Lie algebra of G. For $\mathfrak{g}$, we define notions "algebraically representation", "algebraically equivalent", "algebraically invariant", and "algebraically irreducible" in the same way as G. The exception is not to use topology theory words.

Definition 4.10. *Let (π, V) be a Banach representation. We define the differential representation $\pi' : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V_\infty)$ as*

$$\pi'(X)v = \lim_{t \to 0} \frac{\pi(e^{tX})v - v}{t}$$

for each $X \in \mathfrak{g}$ and $v \in V_\infty$.

Then (π', V_∞) is an algebraically representation. See [37, Chap 5, Proposition 2.4]. Let $K \subset G$ be a compact sub-group. It is known that $\dim(\tau) < \infty$ for each $\tau \in \hat{K}$. See [37, Chap 1, Theorem 3.1].

Definition 4.11. *Let (π, V) be a Banach representation.*

(i) *For each $\tau \in \hat{K}$, Let $V(\tau) \subset V$ be the isotypic component of type τ.*

(ii) *We call (π, V) is K-finite if $\dim(V(\tau)) < \infty$ for each $\tau \in \hat{K}$.*

The following is fundamental fact.

Theorem 4.12. *[20] Set $(G, K) = (SO(n,1), SO(n))$ or $(SU(n,1), S(U(n) \times U(1)))$. For each $(\pi, V) \in \hat{G}$ and $(\tau, W) \in \hat{K}$, we obtain*

$$V(\tau) \cong \{0\} \text{ or } W.$$

Especially, the multiplicity of W in $V(\tau)$ is at most 1 and (π, V) is K-finite.

Remark 4.13. *In general, the following fact is known. (See [39, Proposition 4.3.1.7] and [39, Theorem 4.5.2.11].) Let G be a connected semisimple Lie group with finite center, $K \subset G$ be a maximal compact sub-group, and $(\pi, V) \in \hat{G}$. Then, for each $(\tau, W) \in \hat{K}$, the multiplicity of W in $V(\tau)$ is at most $\dim(W)$. Especially, (π, V) is K-finite.*

Since

$$(SL(2,\mathbb{R}), SO(2)) \cong (SO(2,1), SO(2)) \cong (SU(1,1), S(U(1) \times U(1))),$$

each $(\pi, V) \in \widehat{SL(2,\mathbb{R})}$ is $SO(2)$-finite.

Definition 4.14. *Let (π, V) be a Banach representation. Set*

$$V_K = \sum_{\tau \in \hat{K}} V(\tau).$$

Remark 4.15. *If (π, V) is a K-finite Banach representation, we have $V_K \subset V_\omega$.*

See [39, Corollary 4.4.5.17] for the proof. This remark and [39, Theorem 4.4.5.16] derive the following.

Remark 4.16. *If (π, V) is a K-finite Banach representation, then $V_K \subset V$ is dense.*

Remark 4.17. *For each K-finite Banach representation, $V_K \subset V_\omega$ is algebraically invariant.*

See [39, Proposition 4.4.5.18] for the proof. For each K-finite Banach representation (π, V), let (π', V_K) be the algebraically representation induced from (π', V_∞).

Definition 4.18. *Let (π, V) and (σ, W) be K-finite Banach representations. We call (π, V) is infinitesimally equivalent to (σ, W) if (π', V_K) is algebraically equivalent to (σ', W_K).*

Finally, we state two important facts.

Theorem 4.19. *Let G be a unimodular Lie group, $K \subset G$ be a connected compact sub-group, and (π, V) and (σ, W) be K-finite unitary representations. Then the followings are equivalent.*

(i) *(π, V) is equivalent to (σ, W).*

(ii) *(π, V) is infinitesimal equivalent to (σ, W).*

See [39, Corollary 4.5.5.3] for the proof.

Theorem 4.20. *Let G be a unimodular Lie group, $K \subset G$ be a connected compact sub-group, and (π, V) be a K-finite unitary representation. Then the followings are equivalent.*

(i) *(π, V) is irreducible.*

(ii) *(π', V_K) is algebraically irreducible.*

See [39, Theorem 4.5.5.4] for the proof. By two facts above, certainly the classification of irreducible unitary representation on $SL(2, \mathbb{R})$ is attributed to the algebraic representation theory of $\mathfrak{sl}(2, \mathbb{R})$.

4.2 Some facts from functional analysis

We summarize notions from functional analysis for understanding the way to the classification. First, we introduce closed symmetric extension and Cayley transform for linear maps. Let V be a Hilbert space over $\mathbb{C}$.

Definition 4.21. *A notation $T : V \to V$ is a linear operator if there exists a sub-space $\mathcal{D}(T) \subset V$ such that $T : \mathcal{D}(T) \to V$ is a linear map.*

Definition 4.22. *A linear operator $T : V \to V$ is closed if $\mathcal{D}(T)$ is complete with respect to the graph norm*

$$||v||_T = ||v||_V + ||Tv||_V,$$

where $||\cdot||_V$ is the norm of V.

Definition 4.23. *Let $T : V \to V$ be a linear operator. Assume that $\mathcal{D}(T) \subset V$ is dense. We define the adjoint operator of T as follows.*

$$\begin{aligned}\mathcal{D}(T^*) = &\{v \in V | \\ &\quad \textit{For each } u \in \mathcal{D}(T), \textit{ there exists } w \in V \textit{ such that} (Tu, v) = (u, w).\}.\end{aligned}$$

$$T^* v = w.$$

Definition 4.24. *Let $T, \tilde{T} : V \to V$ be linear operators. We call T has an extension $\tilde{T}$ if the followings hold.*

$$\mathcal{D}(T) \subset \mathcal{D}(\tilde{T}).$$

$$T = \tilde{T}|_{\mathcal{D}(T)}.$$

Definition 4.25. *Let $T : V \to V$ be a linear operator and assume that $\mathcal{D}(T) \subset V$ is dense. We call T is symmetric if T has an extension T^*.*

Theorem 4.26. *A symmetric operator $H : V \to V$ has a closed symmetric extension H^{**}.*

See [41, Chapter 7, Section 3, Proposition 1] for the proof.

Definition 4.27. *A linear operator $T : V \to V$ is isometric if*

$$(Tv, Tw) = (v, w)$$

for each $v, w \in V$.

Proposition 4.28. *If a linear operator $T : V \to V$ is closed isometric, then $\mathcal{D}(T) \subset V$ is closed.*

Theorem 4.29. *Let $H : V \to V$ be a closed symmetric operator. Then the followings hold.*

(i) *The linear closed isometric operator*

$$U_H = (H - iI)(H + iI)^{-1}$$

exists, where $\mathcal{D}(U_H) = \mathcal{D}((H + iI)^{-1})$.

(ii) *We have*

$$H = i(I + U_H)(I - U_H)^{-1}.$$

The linear operator U_H is called Cayley transform of H. See [41, Chapter 7, Section 4, Theorem 1] for the proof.

Next, we introduce a spectral resolution of a self-adjoint operator according to [32, Chapters 4-5].

Definition 4.30. *Let $T : V \to V$ be a symmetric operator. We call T is self-adjoint if $\mathcal{D}(T) = \mathcal{D}(T^*)$.*

In general, let $(S, \mathcal{B})$ be a measurable space.

Definition 4.31. *A family $E = \{E(\Lambda)\}_{\Lambda \in \mathcal{B}}$ is a resolution of unity on $(S, \mathcal{B})$ if the following four properties hold.*

(i) *For each $\Lambda \in \mathcal{B}$, $E(\Lambda)$ is a projective operator on V.*

(ii) *For each $\Lambda_1, \Lambda_2 \in \mathcal{B}$, if $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $E(\Lambda_1) \perp E(\Lambda_2)$.*

(iii) *If $\Lambda = \sqcup_{n=1}^{\infty} \Lambda_n$, then $E(\Lambda) = \sum_{n=1}^{\infty} E(\Lambda_n)$ in the sense of strong limit (pointwise convergence).*

(iv) *$E(S) = I$.*

Let E be a resolution of unity on $(S, \mathcal{B})$. For each $v \in V$, set

$$\mu_v(\Lambda) = ||E(\Lambda)v||^2.$$

Then μ_v is a measure on $(S, \mathcal{B})$. The measure μ_v is finite:

$$\mu_v(S) = ||v||^2.$$

Let $f : S \to \mathbb{C}$ be a measurable simple function. We write

$$f = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{\Lambda_k},$$

where $\alpha_k \in \mathbb{C}$ and $\Lambda_k \in \mathcal{B}$. Put

$$T_f : V \ni v \mapsto \sum_{k=1}^{n} \alpha_k E(\Lambda_k) v \in V.$$

Then

$$||T_f v||^2 = \sum_{k=1}^{n} |\alpha_k|^2 ||E(\Lambda_k)v||^2 = \int_S |f|^2 d\mu_v.$$

Let $f : S \to \mathbb{C}$ be a measurable function. Set

$$\mathcal{D}(T_f) = \left\{ v \in V \middle| \int_S |f|^2 d\mu_v < \infty \right\}.$$

For each $v \in \mathcal{D}(T_f)$, we take a sequence of measurable simple functions $\{f_n\}$ such that

$$\lim_{n \to \infty} \int_S |f_n - f|^2 d\mu_v = 0.$$

Then

$$||T_{f_m} v - T_{f_n} v||^2 = \int_S |f_m - f_n|^2 d\mu_v \to 0.$$

Thus there exists

$$T_f v = \lim_{n \to \infty} T_{f_n} v.$$

We write

$$T_f = \int_S f(\lambda) dE(\lambda).$$

Theorem 4.32. *Let $H : V \to V$ be a self-adjoint operator. There exists the resolution of unity E on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ uniquely such that*

$$H = \int_{\mathbb{R}} \lambda dE(\lambda). \tag{4.1}$$

See [32, Theorem 5.7] for the proof. If (4.1) is satisfied, we call E is the spectral resolution of H.

Finally, we introduce a von-Neumann algebra and its relation to a spectral resolution of a bounded self-adjoint operator. For a von-Neumann algebra, see also [28, Section 4.1].

Definition 4.33. **(i)** *A sub-algebra $A \subset \mathcal{L}(V)$ is $*$-sub-algebra if A is $*$-closed.*

(ii) *A $*$-sub-algebra $A \subset \mathcal{L}(V)$ is a von-Neumann algebra if A is strongly closed.*

Definition 4.34. *For each $L \subset \mathcal{L}(V)$, let $L' \subset \mathcal{L}(V)$ be the commutant.*

Proposition 4.35. *If $L \subset \mathcal{L}(V)$ is a $*$-closed subset, then $A = L'$ is a von-Neumann algebra.*

If E is a resolution of unity on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, put

$$E(\lambda) = E((-\infty, \lambda])$$

for each $\lambda \in \mathbb{R}$.

Theorem 4.36. *Let $H \in \mathcal{L}(V)$ be a bounded self-adjoint operator, $A \subset \mathcal{L}(V)$ be a von-Neumann algebra such that $H \in A$, and E be the spectral resolution of H:*

$$H = \int_{\mathbb{R}} \lambda dE(\lambda).$$

Then $E(\lambda) \in A$ for each $\lambda \in \mathbb{R}$.

See [37, Appendix E, Theorem] for the proof.

4.3 Classification

Using the preliminary above, we discuss the way to the classification of irreducible unitary representations on $SL(2, \mathbb{R})$. Let G be a Lie group. We assume G is a sub-group in $GL(N, \mathbb{C})$: $G \subset GL(N, \mathbb{C})$. Then the Lie algebra $\mathfrak{g}$ of G is realized as

$$\mathfrak{g} = \{X \in \mathfrak{gl}(N, \mathbb{C}) | e^{tX} \in G \text{ for each } t \in \mathbb{R}.\}.$$

We define the adjoint representation $(\mathrm{Ad}, \mathfrak{g})$ as

$$\mathrm{Ad}(x)X = xXx^{-1}$$

for each $x \in G$ and $X \in \mathfrak{g}$. Then we have

$$\mathrm{Ad}' = \mathrm{ad}.$$

The following theorem is important for non-compact semisimple Lie groups.

Theorem 4.37. *Set $G = SL(2,\mathbb{R})$ and let (π, V) be a irreducible unitary representation of G. If $\dim(V) < \infty$, then (π, V) is trivial.*

Proof. Assume $\dim(V) < \infty$. Then we treat π as $\pi : G \to U(\dim(V))$. This is a continuous map but more strongly C^∞-map. Take elements

$$X_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

from $\mathfrak{g}$. We have

$$\mathrm{ad}(X_1)X_2 = [X_1, X_2] = X_2.$$

Then we obtain

$$\mathrm{Ad}(e^{tX_1})X_2 = e^t X_2$$

and

$$\pi'(\mathrm{Ad}(e^{tX_1})X_2) = e^t \pi'(X_2)$$

for each $t \in \mathbb{R}$. The left hand side is transformed as

$$\begin{aligned}
\pi'(\mathrm{Ad}(e^{tX_1})X_2) &= \pi'(e^{t\mathrm{ad}(X_1)}X_2) \\
&= \pi'\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathrm{ad}(X_1)^n X_2\right) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!}\pi'\left(\mathrm{ad}(X_1)^n X_2\right) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!}\left(\mathrm{ad}(\pi'(X_1))^n \pi'(X_2)\right) \\
&= e^{t\mathrm{ad}(\pi'(X_1))}\pi'(X_2) \\
&= \mathrm{Ad}(e^{t\pi'(X_1)})\pi'(X_2) \\
&= \mathrm{Ad}(\pi(e^{tX_1}))\pi'(X_2).
\end{aligned}$$

Thus we obtain

$$\mathrm{Ad}(\pi(e^{tX_1}))\pi'(X_2) = e^t \pi'(X_2).$$

Since the image of the left hand side is contained in $\mathrm{Ad}(U(\dim(V)))$, it is relative compact. Then we have $\pi'(X_2) = 0$. Especially, $\mathrm{Ker}(\pi') \neq \{0\}$. Since (π', V) is algebraically irreducible, we get $\mathrm{Ker}(\pi') = V$. Thus (π, V) is trivial. □

We prove the fundamental lemma in representation theory.

Theorem 4.38 (Schur's Lemma)**.** *Let G be a group and (π, V) be an irreducible unitary representation of G. Set $L = \{\pi(x)\}_{x \in G} \subset \mathcal{L}(V)$ and $A = L' \subset \mathcal{L}(V)$ (commutant). Then $A = \mathbb{C}I$.*

Proof. Since L is a $*$-closed subset, A is a von-Neumann algebra from Proposition 4.35.

We provide preliminary considerations. For each $T \in \mathcal{L}(V)$, we write

$$T = H_1 + iH_2,$$

$$H_1 = \frac{T + T^*}{2},$$

$$H_2 = \frac{T - T^*}{2i}.$$

Especially, H_1 and H_2 are bounded self-adjoint operators. Since A is a von-Neumann algebra, the following two properties are equivalent.

(i) $T \in A$.

(ii) $H_1, H_2 \in A$.

Thus it is enough to prove the theorem for bounded self-adjoint operators.

Let $H \in A$ be a bounded self-adjoint operator. We apply Theorem 4.32 for H:

$$H = \int_{\mathbb{R}} \lambda dE(\lambda).$$

From Theorem 4.36, we have $E(\lambda) \in A$ for each $\lambda \in \mathbb{R}$. Then $E(\lambda)V \subset V$ is invariant. Moreover $E(\lambda)V$ is closed since $E(\lambda)$ is projective. Thus $E(\lambda)V$ is trivial from irreducibility of (π, V). Since $E(\lambda)V$ is monotone increasing, there exists $\lambda_H \in \mathbb{R}$ such that

$$E(\lambda) = \begin{cases} I & \text{if } \lambda > \lambda_H, \\ 0 & \text{if } \lambda < \lambda_H. \end{cases}$$

In addition, we obtain $E(\lambda_H) = I$ from the right continuous of $E(\lambda)$ in the sense of a strong limit. Thus

$$H = \lambda_H I.$$

□

Corollary 4.39. *Let G be a group and (π, V) be an irreducible representation of G. If G is abelian, then* $\dim(V) = 1$.

Proof. It is enough to prove that $\pi(a)$ is a scaler operator for each $a \in G$. Since G is abelian, we have $\pi(a) \in A$, where $A = L'$ and $L = \{\pi(x)\}_{x \in G}$. From Theorem 4.38, $\pi(a)$ is a scaler operator. □

Set

$$X_0 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

and

$$K = SO(2) = \{e^{tX_0} | t \in \mathbb{R}\}.$$

We define the representation $\tau_m : SO(2) \to GL(1, \mathbb{C})$ as

$$\tau_m(e^{tX_0}) = e^{imt}$$

for each $m \in \frac{1}{2}\mathbb{Z}$.

Proposition 4.40. *The following holds.*

$$\hat{K} = \{\tau_m\}_{m \in \frac{1}{2}\mathbb{Z}}.$$

Proof. Take any $\tau \in \hat{K}$. From Corollary 4.39, assume $\dim(\tau) = 1$. We write

$$\tau(e^{tX_0}) = e^{t\tau'(X_0)}.$$

Since τ is unitary, there exists $\xi \in \mathbb{R}$ such that $\tau'(X_0) = i\xi$. Moreover we obtain $\xi = m \in \frac{1}{2}\mathbb{Z}$ from $\tau(e^{4\pi X_0}) = 1$. Then $\tau = \tau_m$. □

Set $G = SL(2, \mathbb{R})$ and take

$$X_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

from $\mathfrak{g}$. Set

$$Y = -X_0 + X_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Fix $(\pi, V) \in \hat{G}$. We define

$$\mathcal{D}(C) = V_\infty,$$

$$C = \pi'(X_0)^2 - \pi'(X_1) - \pi'(Y)^2.$$

The following theorem is also important for non-compact semisimple Lie groups.

Theorem 4.41. *There exists $q \in \mathbb{R}$ such that*

$$Cv = qv$$

for each $v \in V_\infty$.

We provide preliminary for this theorem.

Lemma 4.42. *For each $x \in G$, we have*

$$\pi(x)C\pi(x^{-1}) = C.$$

Proof. For each $X \in \mathfrak{g}$, we have

$$\pi(x)\pi'(X)\pi(x^{-1}) = \pi'(\mathrm{Ad}(x)X).$$

In fact, for each $t \in \mathbb{R}$,

$$\begin{aligned}
e^{\pi'(\mathrm{Ad}(x)X)} &= \pi(e^{t\mathrm{Ad}(x)X}) \\
&= \pi(e^{txXx^{-1}}) \\
&= \pi(xe^{tX}x^{-1}) \\
&= \pi(x)\pi(e^{tX})\pi(x^{-1}) \\
&= \pi(x)e^{t\pi'(X)}\pi(x^{-1}) \\
&= e^{t\pi(x)\pi'(X)\pi(x^{-1})}.
\end{aligned}$$

Then

$$\begin{aligned}
&\pi(x)C\pi(x^{-1}) \\
&= \pi'(\mathrm{Ad}(x)X_0)^2 - \pi'(\mathrm{Ad}(x)X_1)^2 - \pi'(\mathrm{Ad}(x)Y)^2 \\
&= C
\end{aligned}$$

by direct computation. □

Since π is unitary, for each $X \in \mathfrak{g}$ and $u, v \in V_\infty$,

$$(\pi'(X)u, v) = -(u, \pi'(X)v). \tag{4.2}$$

Then

$$(Cu, v) = (u, Cv).$$

and C is symmetric. Thus C has closed symmetric extension $H = C^{**}$ from Theorem 4.26.

Lemma 4.43. *The following two properties hold.*

(i) *$\mathcal{D}(H)$ is invariant.*

(ii) *For each $x \in G$,*

$$\pi(x)H\pi(x^{-1}) = H.$$

Proof. First, we prove $\mathcal{D}(C^*)$ is invariant. Specially, we prove $\pi(x)v \in \mathcal{D}(C^*)$ for each $x \in G$ and $v \in \mathcal{D}(C^*)$. Take any $u \in \mathcal{D}(C)$. Then

$$\begin{aligned}(Cu, \pi(x)v) &= (\pi(x^{-1})Cu, v)\\ &= (C\pi(x^{-1})u, v) \text{ (Lemma 4.42)}\\ &= (\pi(x^{-1})u, C^*v) \text{ ($\mathcal{D}(C)$ is invariant.)}\\ &= (u, \pi(x)C^*v).\end{aligned}$$

Thus $\mathcal{D}(C^*)$ is invariant and $C^*\pi(x)v = \pi(x)C^*v$ for each $v \in \mathcal{D}(C^*)$. The rest of the discussion can be done by replacing C and C^* with C^* and H, respectively. □

We define U by the following. (Cayley transform in Theorem 4.29)

$$U = (H - iI)(H + iI)^{-1}.$$

Lemma 4.44. *The following two properties hold.*

(i) $\mathcal{D}(U) = \mathcal{D}((H + iI)^{-1})$ *is invariant.*

(ii) *For each* $x \in G$*,*

$$\pi(x)U\pi(x^{-1}) = U.$$

Proof. Since $\mathcal{D}((H + iI)^{-1})$ is the image of $H + iI$, (i) is obeyed from both (i) and (ii) in Lemma 4.43. Also (ii) is obeyed from (ii) in Lemma 4.43. □

Proof of Theorem 4.41. Recall $\mathcal{D}(U)$ is invariant from (i) in Lemma 4.44. Since U is closed isometric, $\mathcal{D}(U)$ is closed by Proposition 4.28. From irreducibility of (π, V), we have $\mathcal{D}(U) = V$ and $U \in \mathcal{L}(V)$. By Theorem 4.38 and (ii) in Lemma 4.44, there exists $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ such that

$$U = e^{i\theta}I.$$

Then we put

$$q = i\frac{1 + e^{i\theta}}{1 - e^{i\theta}} \in \mathbb{R}$$

and get

$$H = qI.$$

Since H is an extension of C, then

$$C = qI_{V_\infty}.$$

□

We define the most important differential operators. Set

$$H = -i\pi'(X_0),\ E = \pi'(X_1) + i\pi'(Y),\ F = -\pi'(X_1) + i\pi'(Y).$$

When we regard π' as the representation of a complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, H, E and F are given by

$$H = \pi'\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix},\ E = \pi'\begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{pmatrix},\ F = \pi'\begin{pmatrix} -\frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}.$$

We get following four formulae by algebraic computation.

$$[H, E] = E,$$

$$[H, F] = -F,$$

$$[E, F] = -2H,$$

$$C = -H^2 + \frac{1}{2}(EF + FE).$$

Moreover (4.2) derives the following.

$$(Eu, v) = (u, Fv) \tag{4.3}$$

for each $u, v \in V_\infty$.

Definition 4.45. *There exists $\mathbb{M} \subset \frac{1}{2}\mathbb{Z}$ uniquely such that*

$$\pi \cong \widehat{\bigoplus_{m\in\mathbb{M}}} \tau_m$$

as the representation of K. We call $m \in \mathbb{M}$ is a K-weight. We call $v \in \tau_m$ is a K-weight m vector.

For each $m_0 \in \frac{1}{2}\mathbb{Z}$ and $\mu \in \mathbb{N} = \{0, 1, 2, \cdots\}$, we put

$$\rho_\mu = q + (m_0 + \mu)(m_0 + \mu + 1),$$

$$\sigma_\mu = q + (m_0 - \mu)(m_0 - \mu - 1).$$

Lemma 4.46. *Let $v \in V_K$ be a K-weight m_0 vector. Then, for each $\mu \in \mathbb{N}$, the following six formulae hold.*

(i) $HE^\mu v = (m_0 + \mu)v$.

(ii) $HF^\mu v = (m_0 - \mu)v$.

(iii) $FE^{\mu+1}v = \rho_\mu E^\mu v$.

(iv) $EF^{\mu+1}v = \sigma_\mu F^\mu v$.

(v) $||E^{\mu+1}v||^2 = \rho_\mu ||E^\mu v||^2$.

(vi) $||F^{\mu+1}v||^2 = \sigma_\mu ||F^\mu v||^2$.

Proof. **(i)** Use $[H, E] = E$ in induction.

(ii) Use $[H, F] = -F$ in induction.

(iii) For each $\mu \in \mathbb{N}$, $E^\mu v$ is a K-weight $(m_0 + \mu)$ vector by (i). From $[E, F] = -2H$, we have

$$(EF - FE)E^\mu v = -2(m_0 + \mu)E^\mu v.$$

From $EF + FE = 2C + 2H^2$, we have

$$(EF + FE)E^\mu v = (2q + 2(m_0 + \mu)^2)E^\mu v.$$

Then we obtain

$$FE^{\mu+1}v = \rho_\mu E^\mu v.$$

(iv) For each $\mu \in \mathbb{N}$, $F^\mu v$ is a K-weight $(m_0 - \mu)$ vector by (ii). From $[E, F] = -2H$, we have

$$(EF - FE)F^\mu v = -2(m_0 - \mu)F^\mu v.$$

From $EF + FE = 2C + 2H^2$, we have

$$(EF + FE)F^\mu v = (2q + 2(m_0 - \mu)^2)F^\mu v.$$

Then we obtain

$$EF^{\mu+1}v = \sigma_\mu F^\mu v.$$

(v) We get the result from the transformation below.

$$\begin{aligned} ||E^{\mu+1}v||^2 &= (E^{\mu+1}v, E^{\mu+1}v) \\ &= (E^\mu v, FE^{\mu+1}v) \ \ \text{(by (4.3))} \\ &= \rho_\mu ||E^\mu v||^2 \ \ \text{(by (iii))}. \end{aligned}$$

(vi) We get the result from the transformation below.

$$\begin{aligned} ||F^{\mu+1}v||^2 &= (F^{\mu+1}v, F^{\mu+1}v) \\ &= (F^\mu v, EF^{\mu+1}v) \ \ \text{(by (4.3))} \\ &= \sigma_\mu ||F^\mu v||^2 \ \ \text{(by (iv))}. \end{aligned}$$

□

The following theorem is the key for using non-abelian harmonic analysis.

Theorem 4.47. *There exists a complete orthonormal system* $\{\phi_m\}_{m\in\mathbb{M}} \subset V_K$ *of* V *such that*

$$H\phi_m = m\phi_m, \tag{4.4}$$

$$E\phi_m = \sqrt{q+m(m+1)}\phi_{m+1}, \tag{4.5}$$

$$F\phi_m = \sqrt{q+m(m-1)}\phi_{m-1} \tag{4.6}$$

for each $m \in \mathbb{M}$.

Proof.

- There exists K-weight $m_0 \in \frac{1}{2}\mathbb{Z}$ such that we can take a K-weight unit vector $v \in V_K$. We fix them.
- If there exists $\mu \in \mathbb{N}$ such that $E^\mu v \neq 0$ and $E^{\mu+1}v = 0$, put $\mu_h = \mu$. If not, put $\mu_h = \infty$.
- If there exists $\mu \in \mathbb{N}$ such that $F^\mu v \neq 0$ and $F^{\mu+1}v = 0$, put $\mu_l = \mu$. If not, put $\mu_l = \infty$.
- Set

$$\mathbb{M}_0 = \{m_0 + \mu | \mu \in \mathbb{N},\, \mu \leq \mu_h\} \cup \{m_0 - \mu | \mu \in \mathbb{N},\, \mu \leq \mu_l\}.$$

- When $m_0 + (\mu+1) \in \mathbb{M}_0$, we have $\rho_\mu > 0$ by Lemma 4.46(v).
- When $m_0 - (\mu+1) \in \mathbb{M}_0$, we have $\sigma_\mu > 0$ by Lemma 4.46(vi).
- Put

$$\phi_{m_0} = v,$$

$$\phi_{m_0+(\mu+1)} = \frac{1}{\sqrt{\rho_\mu}} E\phi_{m_0+\mu} \text{ if } m_0 + (\mu+1) \in \mathbb{M}_0, \tag{4.7}$$

$$\phi_{m_0-(\mu+1)} = \frac{1}{\sqrt{\sigma_\mu}} F\phi_{m_0-\mu} \text{ if } m_0 - (\mu+1) \in \mathbb{M}_0. \tag{4.8}$$

- For each $m \in \mathbb{M}_0$, (4.4) is checked from Lemma 4.46(i) and (ii).
- By (4.7),

$$E\phi_{m_0+\mu} = \sqrt{\rho_\mu}\phi_{m_0+(\mu+1)}.$$

  Then we get (4.5) for $m = m_0 + \mu$.

- We apply E to (4.8) and get

$$E\phi_{m_0-(\mu+1)} = \frac{1}{\sqrt{\sigma_\mu}} EF\phi_{m_0-\mu}.$$

  On the other hand, we have

$$EF\phi_{m_0-\mu} = \sigma_\mu \phi_{m_0-\mu}$$

  from Lemma 4.46(iv). Thus we obtain

$$E\phi_{m_0-(\mu+1)} = \sqrt{\sigma_\mu}\phi_{m_0-\mu}.$$

  Then we get (4.5) for $m = m_0 - (\mu + 1)$.

- By (4.8),

$$F\phi_{m_0-\mu} = \sqrt{\sigma_\mu}\phi_{m_0-(\mu+1)}.$$

  Then we get (4.6) for $m = m_0 - \mu$.

- We apply F to (4.7) and get

$$F\phi_{m_0+(\mu+1)} = \frac{1}{\sqrt{\rho_\mu}} FE\phi_{m_0+\mu}.$$

  On the other hand, we have

$$FE\phi_{m_0+\mu} = \rho_\mu \phi_{m_0+\mu}$$

  from Lemma 4.46(iii). Thus we obtain

$$F\phi_{m_0+(\mu+1)} = \sqrt{\rho_\mu}\phi_{m_0+\mu}.$$

  Then we get (4.6) for $m = m_0 + (\mu + 1)$.

- Orthogonality is satisfied since H is symmetric.

- Normality is satisfied by Lemma 4.46(v) and (vi).

- Set $W = \mathrm{span}\{\phi_m\}_{m\in\mathbb{M}_0}$. From (4.4), (4.5), and (4.6), W is algebraically invariant. Since V_K is algebraically irreducible, we have $W = V_K$ and $\mathbb{M}_0 = \mathbb{M}$.

- Since $V_K \subset V$ is dense, we have $\overline{W} = V$. Thus $\{\phi_m\}_{m\in\mathbb{M}}$ is a complete orthonormal system.

$\square$

Then we obtain the following result. For each $(\pi, V) \in \hat{G}$, let q_π be the number in Theorem 4.41 and $\mathbb{M}_\pi$ be the set in Definition 4.45.

Corollary 4.48. *Take any $(\pi, V), (\sigma, W) \in \hat{G}$. Then the followings are equivalent.*

(i) *π is equivalent to σ.*

(ii) *$q_\pi = q_\sigma$ and $\mathbb{M}_\pi = \mathbb{M}_\sigma$.*

By the corollary above, it is enough to classify q and $\mathbb{M}$. We refine the possible combinations of q and $\mathbb{M}$.

Lemma 4.49. *If $\mathbb{M} = \mathbb{Z}$, then $q > 0$.*

Proof. For each $m \in \mathbb{Z}$, we have $q + m(m+1) > 0$. Put $m = 0$ and get the result. $\square$

We denote q in the lemma as

$$q = \frac{1}{4} + \nu^2 \ \ (\nu \geq 0),$$

$$q = \sigma(1-\sigma) \ \ \left(\frac{1}{2} < \sigma < 1\right).$$

Each of these corresponds to $V^{0, \frac{1}{2}+i\nu}, V^\sigma \in \hat{G}$.

Lemma 4.50. *If $\mathbb{M} = \frac{1}{2} + \mathbb{Z}$, then $q > \frac{1}{4}$.*

Proof. For each $m \in \frac{1}{2} + \mathbb{Z}$, we have $q + m(m+1) > 0$. Put $m = \frac{1}{4}$ and get the result. $\square$

We denote q in the lemma as

$$q = \frac{1}{4} + \nu^2 \ \ (\nu > 0).$$

This corresponds to $V^{\frac{1}{2}, \frac{1}{2}+i\nu}$.

Lemma 4.51. *If $\mathbb{M} = n + \mathbb{N}$ for some $n \in \frac{1}{2}\mathbb{Z}$, then $q = n(1-n)$ and $n > 0$.*

Proof. First, since $q + (n-1)n = 0$, we have $q = n(1-n)$. Next, we get $n > 0$ from $q + n(n+1) > 0$. $\square$

We divide the lemma into cases $n \geq 1$ and $n = \frac{1}{2}$. Each of these corresponds to U^{-n} and $V_+^{\frac{1}{2}, \frac{1}{2}}$.

Lemma 4.52. *If $\mathbb{M} = -n - \mathbb{N}$ for some $n \in \frac{1}{2}\mathbb{Z}$, then $q = n(1-n)$ and $n > 0$.*

Proof. First, since $q + (-n)((-n)+1) = 0$, we have $q = n(1-n)$. Next, we get $n > 0$ from $q + ((-n)-1)(-n) > 0$. $\square$

We divide the lemma into cases $n \geq 1$ and $n = \frac{1}{2}$. Each of these corresponds to U^n and $V_-^{\frac{1}{2}, \frac{1}{2}}$.

Lemma 4.53. *If $\mathbb{M}$ is bounded, then $\mathbb{M} = \{0\}$ and $q = 0$. Specially, π is trivial.*

Proof. Since $\mathbb{M}$ is bounded, π is finite dimensional. Then π is trivial by Theorem 4.37. Thus $\mathbb{M} = \{0\}$ and $q = 0$. □

From the above four lemmata, we have completed the proof of the following classification theorem.

Theorem 4.54. *(see [37, Proposition 6.13, Theorem6.2, 6.4, 6.5])*
For $\pi \in \hat{G}$, let $\mathbb{M} \subset \frac{1}{2}\mathbb{Z}$ and $q \in \mathbb{R}$ be given in the table below. Then, there exists an orthonormal basis $\{\phi_m\}_{m\in\mathbb{M}}$ of π such that

$$C\phi_m = q\phi_m,$$

$$H\phi_m = m\phi_m,$$

$$E\phi_m = \sqrt{q + m(m+1)}\phi_{m+1},$$

$$F\phi_m = \sqrt{q + m(m-1)}\phi_{m-1}$$

for each $m \in \mathbb{M}$.

π	$\mathbb{M}$	q	conditions
$V^{0,\frac{1}{2}+i\nu}$	$\mathbb{Z}$	$\frac{1}{4}+\nu^2$	$\nu \geq 0$
$V^{\frac{1}{2},\frac{1}{2}+i\nu}$	$\frac{1}{2}+\mathbb{Z}$	$\frac{1}{4}+\nu^2$	$\nu > 0$
$V_+^{\frac{1}{2},\frac{1}{2}}$	$\frac{1}{2}+\mathbb{N}$	$\frac{1}{4}$	-
$V_-^{\frac{1}{2},\frac{1}{2}}$	$-\frac{1}{2}-\mathbb{N}$	$\frac{1}{4}$	-
U^n	$-n-\mathbb{N}$	$n(1-n)$	$n \in \frac{1}{2}\mathbb{Z}$ with $n \geq 1$
U^{-n}	$n+\mathbb{N}$	$n(1-n)$	$n \in \frac{1}{2}\mathbb{Z}$ with $n \geq 1$
V^σ	$\mathbb{Z}$	$\sigma(1-\sigma)$	$\frac{1}{2} < \sigma < 1$
I	$\{0\}$	0	-

5 Proof of the main result

We state Theorem 1.1 in its precise form as Theorem 5.1. Set $G = SL(2,\mathbb{R})$. Let $P \subset G$ be the subgroup of all upper triangular matrices, and $\Gamma \subset G$ be a cocompact lattice. Also, set $M_\Gamma = \Gamma\backslash G$. Let $\mathcal{F}_P$ be the orbit foliation induced from the natural action of P on M_Γ. Under the G-invariant measure, $L^2(M_\Gamma)$ decomposes into a countable sum of irreducible unitary representations by [14, Theorem 1.2.3]. Recall that the multiplicity of U^{-1} is equal to the one of U^1 by [14, Theorem 1.4.2]. Then we prove the following theorem in this chapter.

Theorem 5.1. *Let g be the multiplicity of the irreducible unitary representation U^{-1} in $L^2(M_\Gamma)$ with the G-invariant measure, where the lowest weight of U^{-1} is 1. Then there exist 1-cocycles $x, y_1, ..., y_{2g}$ such that*

$$H^*(\mathcal{F}_P) \cong \bigwedge[x, y_1, \cdots, y_{2g}]/\big(\{y_i \wedge y_j\}_{1\le i,j\le 2g}\big).$$

Furthermore, 1-cocycles $x, y_1, ..., y_{2g}$ are treated as indeterminate variables.

5.1 Preliminaries

We summarize computation formulae.

5.1.1 Some facts from representation theory

Let $\widehat{G}$ be the unitary dual of G. It is sufficient for us to compute $d_{\mathcal{F}_P}$ on each $\pi \in \widehat{G}$ by using the differential representation. Indeed, $L^2(M_\Gamma)$ decomposes into a countable sum of irreducible unitary representations (see [14, Theorem 1.2.3]). Take elements

$$X_0 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

from $\mathfrak{sl}(2,\mathbb{R})$. When we regard X_0, X_1, X_2 as vector fields on M_Γ, let $\omega_0, \omega_1, \omega_2 \in \Omega^1(M_\Gamma)$ be the dual forms of them. We put

$$Y = -X_0 + X_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

For $\pi \in \widehat{G}$, letting π' be the derivative of π, we set

$$\begin{aligned} H &= -i\pi'(X_0), \\ E &= \pi'(X_1) + i\pi'(Y), \\ F &= -\pi'(X_1) + i\pi'(Y). \end{aligned}$$

When we regard π' as the representation of a complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, H, E and F are given by

$$H = \pi'\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, \quad E = \pi'\begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{pmatrix}, \quad F = \pi'\begin{pmatrix} -\frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix},$$

respectively. We set

$$C = \pi'(X_0)^2 - \pi'(X_1)^2 - \pi'(Y)^2$$

which is called the Casimir element.

We use Theorem 4.54 in Chapter 4. The notations $\mathbb{M}$ and q are sometimes denoted by $\mathbb{M}_\pi$ and q_π, respectively. Based on Theorem 4.54, it is possible to consider the exterior derivative:

$$d_{\mathcal{F}_P} : \Omega^0(\mathcal{F}_P) \to \Omega^1(\mathcal{F}_P).$$

Let h be a C^∞ vector of some $\pi \in \widehat{G}$. We have the Fourier expansion

$$h = \sum_{m\in\mathbb{M}} h_m \phi_m.$$

Set $f_1\omega_1 + f_2\omega_2 = d_{\mathcal{F}_P} h$. Then Fourier coefficients of f_1 and f_2 are given by

$$f_{1\,m} = \frac{h_{m-1}}{2}\alpha_{q\,m-1} - \frac{h_{m+1}}{2}\beta_{q\,m+1}$$

and

$$f_{2\,m} = -\frac{ih_{m-1}}{2}\alpha_{q\,m-1} + imh_m - \frac{ih_{m+1}}{2}\beta_{q\,m+1},$$

where

$$\alpha_{q\,m} = \sqrt{q + m(m+1)},$$
$$\beta_{q\,m} = \sqrt{q + m(m-1)}.$$

Next, we consider

$$d_{\mathcal{F}_P} : \Omega^1(\mathcal{F}_P) \to \Omega^2(\mathcal{F}_P).$$

Let f_1, f_2 be C^∞ vectors of some $\pi \in \widehat{G}$. Set

$$g\,\omega_1 \wedge \omega_2 = d_{\mathcal{F}_P}(f_1\omega_1 + f_2\omega_2).$$

Then g's Fourier coefficients are given by

$$g_m = \frac{if_{1\,m-1} + f_{2\,m-1}}{2}\alpha_{q\,m-1} - (imf_{1\,m} + f_{2\,m}) + \frac{if_{1\,m+1} - f_{2\,m+1}}{2}\beta_{q\,m+1}.$$

For convenience, we replace h by $-4h$, $f_{1\,m}$ by $2f_{1\,m}$, $f_{2\,m}$ by $2if_{2\,m}$, and g by ig, respectively. Then we always assume that

$$f_{1\,m} = -h_{m-1}\alpha_{q\,m-1} + h_{m+1}\beta_{q\,m+1}, \tag{5.1}$$

$$f_{2\,m} = h_{m-1}\alpha_{q\,m-1} - 2mh_m + h_{m+1}\beta_{q\,m+1}, \tag{5.2}$$

or

$$g_m = (f_{1\,m-1}+f_{2\,m-1})\alpha_{q\,m-1} - 2(mf_{1\,m}+f_{2\,m}) + (f_{1\,m+1}-f_{2\,m+1})\beta_{q\,m+1}. \tag{5.3}$$

5.1.2 Some facts from theory of Sobolev spaces

In this paper, we construct formal functions by using (5.1), (5.2), and (5.3). To ensure their smoothness, we use L^2-Sobolev norms. In general, let M be a compact Riemannian manifold and $(\lambda_s)_{s=0}^{\infty}$ be the sequence consisting of eigenvalues of the Laplace-Beltrami operator:

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_s \leq \cdots \to \infty.$$

Then, for each $k \in \mathbb{N}$, L^2-Sobolev norm of k-th order with respect to the Bessel potential is given by

$$||f||_k^2 = \sum_{s=0}^{\infty} (1+\lambda_s)^k |f_s|^2,$$

where $f \in C^\infty(M)$ and $(f_s)_{s=0}^{\infty}$ is the Fourier coefficients of f. For example, see [36, Definition 4.1]. We apply this fact to our case. We define the Riemannian metric on M_Γ whose orthogonal frame is $\{X_0, X_1, Y\}$. Then the Laplace-Beltrami operator is

$$\Delta = -X_0^2 - X_1^2 - Y^2$$

(see [38, Theorem 1]). Since G is connected unimodular, it follows that

$$\mathrm{Trace}(\mathrm{ad}(\cdot)) = 0.$$

We transform with $\Delta = C - 2X_0^2$. Then we get

$$\Delta \phi_m = (q + 2m^2)\phi_m.$$

Thus L^2-Sobolev norm of k-th order is given by

$$||f||_k^2 = \sum_{\pi \subset L^2(M_\Gamma),\, m \in \mathbb{M}_\pi} (1 + q + 2m^2)^k |f_{\pi\, m}|^2 \tag{5.4}$$

for each $k \in \mathbb{N}$ and $f \in C^\infty(M_\Gamma)$.

We estimate L^2-Sobolev norms (5.4) in Sections 5.3 and 5.4. We provide some constants for this purpose. Observe the behavior of number q:

Lemma 5.2. *The sequence $(q_\pi)_{\pi \subset L^2(M_\Gamma)}$ has no accumulation points, where q_π is the number of π.*

Proof. The non-existence of accumulation points is derived from the fact that eigenvalues of Δ diverge. □

Resorting to Lemma 5.2, we can define a quantity as follows:

$$q_\Gamma = \inf \left\{ |q_\pi| \,\middle|\, \pi \subset L^2(M_\Gamma),\ \pi \neq I, U^{-1}, U^1 \right\} > 0. \tag{5.5}$$

If $q_\Gamma > 1$, then we put $q_\Gamma = 1$. For each $k \in \mathbb{N}$, set

$$C_{\Gamma\, k} = \left(\frac{3!}{q_\Gamma^2} \right)^2 2^{5k+8}. \tag{5.6}$$

The estimation is performed by using the constant $C_{\Gamma\,k}$ under equations (5.1), (5.2), and (5.3). When a special cocycle $\eta \in Z^*(\mathcal{F}_P)$ is given, we construct $\xi \in \Omega^*(\mathcal{F}_P)$ which satisfies $\eta = d_{\mathcal{F}_P}\xi$ formally and prove that

$$||\xi||_k^2 \leq C_{\Gamma\,k+3}||\eta||_{k+3}^2$$

for each $k \in \mathbb{N}$.

5.2 Computation of zeroth cocycles

See [12, Introduction] and cited references in it for what follows. The flows generated by X_1 in M_Γ is called the geodesic flow. It is known that almost all orbits are dense. The flow generated by X_2 in M_Γ is called the horocycle flow. It is also known that all orbits are dense. In particular, all leaves of $\mathcal{F}_P$ are dense. Thus the following holds.

Proposition 5.3. *Any non-zero constant function generates $H^0(\mathcal{F}_P)$.*

5.3 Computation of second cocycles

To prove the lemmata below, we solve a linear equation for all Fourier coefficients on each $\pi \in \widehat{G}$. The symbol $|_\pi$ means "restricted to π".

5.3.1 Trivial representation.

We get the following lemma directly.

Lemma 5.4. $H^2(\mathcal{F}_P)|_I = \{0\}$.

5.3.2 Correspondence to the lowest weight 1.

We characterize the coboundary space.

Lemma 5.5. *The following equation holds*:

$$B^2(\mathcal{F}_P)|_{U^{-1}} = \left\{ g\,\omega_1 \wedge \omega_2 \,\middle|\, \sum_{m=1}^{\infty} \sqrt{m} g_m = 0 \right\}. \tag{5.7}$$

In particular, $Z^2(\mathcal{F}_P)|_{U^{-1}}$ is spanned by $\phi_1\,\omega_1 \wedge \omega_2$ and $B^2(\mathcal{F}_P)|_{U^{-1}}$.

Proof. Proving that the right-hand side of (5.7) contains $B^2(\mathcal{F}_P)|_{U^{-1}}$ is easy. To prove the opposite, let N be a positive integer. Putting

$$f_{1\,m}^{(N+1)} = \frac{N+1-m}{\sqrt{m(N+1)}} \ \ (1 \leq m \leq N),$$
$$f_{1\,N+1}^{(N+1)} = 0,$$
$$f_1^{(N+1)} = \sum_{m=1}^{N} f_{1\,m}^{(N+1)} \phi_m,$$

we have, by using by (5.3),

$$d_{\mathcal{F}_P}(f_1^{(N+1)}\omega_1) = (-\sqrt{N+1}\phi_1 + \phi_{N+1})\,\omega_1 \wedge \omega_2. \tag{5.8}$$

To check (5.8), put

$$g^{(N+1)}\,\omega_1 \wedge \omega_2 = d_{\mathcal{F}_P}(f_1^{(N+1)}\omega_1).$$

For each $2 \le m \le N$, we have

$$\begin{aligned}
&g_m^{(N+1)} = f_{1\,m-1}^{(N+1)}\alpha_{q\,m-1} - 2m f_{1\,m}^{(N+1)} + f_{1\,m+1}^{(N+1)}\beta_{q\,m+1} \\
&= \frac{N+1-(m-1)}{\sqrt{(m-1)(N+1)}}\sqrt{(m-1)m} - 2m\frac{N+1-m}{\sqrt{m(N+1)}} + \frac{N+1-(m+1)}{\sqrt{(m+1)(N+1)}}\sqrt{m(m+1)} \\
&= \sqrt{\frac{m}{N+1}}\Big(N+1-(m-1) - 2(N+1-m) + N+1-(m+1)\Big) \\
&= 0.
\end{aligned}$$

Next, we observe that

$$\begin{aligned}
g_1^{(N+1)} &= -\,2 \cdot 1 \cdot f_{1\,1}^{(N+1)} + f_{1\,2}^{(N+1)}\beta_{q\,2} \\
&= -\,2\frac{N+1-1}{\sqrt{1\cdot(N+1)}} + \frac{N+1-2}{\sqrt{2\cdot(N+1)}}\sqrt{1\cdot 2} \\
&= -\,\sqrt{N+1},
\end{aligned}$$

and

$$\begin{aligned}
g_{N+1}^{(N+1)} &= f_{1\,N}^{(N+1)}\alpha_{q\,N} - 2(N+1)f_{1\,N+1}^{(N+1)} \\
&= \frac{N+1-N}{\sqrt{N(N+1)}}\sqrt{N(N+1)} - 0 \\
&= 1.
\end{aligned}$$

Thus the formula (5.8) is valid.

We put

$$\xi^{(N+1)} = f_1^{(N+1)}\omega_1.$$

Let $\eta = g\,\omega_1 \wedge \omega_2$ be an element from the right-hand side of (5.7). Put

$$\xi = \sum_{N=1}^{\infty} g_{N+1}\,\xi^{(N+1)}. \tag{5.9}$$

We obtain $\eta = d_{\mathcal{F}_P}\xi$ formally. This is determined to be smooth 2-coboundary after the Sobolev estimation from Lemma 5.6 below. □

Lemma 5.6. *Let η be an element from the right-hand side of* (5.7) *from Lemma* 5.5, *and ξ be as* (5.9). *Then, for each $k \in \mathbb{N}$, one has*

$$||\xi||_k^2 \le ||\eta||_{k+3}^2.$$

Since the coefficient of ϕ_m in ξ defined by (5.9) is

$$\sum_{N=m}^{\infty} g_{N+1} \frac{N+1-m}{\sqrt{m(N+1)}},$$

it is sufficient to prove the following claim instead of Lemma 5.6.

Claim 5.7. *Take any sequence* $(g_N)_{N=1}^{\infty}$ *satisfying*

$$\sum_{N=1}^{\infty} N^{2k} |g_N|^2 < \infty$$

for each $k \in \mathbb{N}$. *Then, for each* $k \in \mathbb{N}$, *the following inequality holds*:

$$\sum_{m=1}^{\infty} (1+2m^2)^k \left| \sum_{N=m}^{\infty} g_{N+1} \frac{N+1-m}{\sqrt{m(N+1)}} \right|^2 \leq \sum_{N=1}^{\infty} (1+2(N+1)^2)^{k+3} |g_{N+1}|^2. \tag{5.10}$$

Proof. Fix $k \in \mathbb{N}$. First, we estimate each term in the left-hand side of (5.10). Put

$$g'_{N+1} = (1+2(N+1)^2) g_{N+1}.$$

Then we have

$$\begin{aligned}
&\left| \sum_{N=m}^{\infty} g_{N+1} \frac{N+1-m}{\sqrt{m(N+1)}} \right|^2 \\
&= \left| \sum_{N=m}^{\infty} \frac{g'_{N+1}}{1+2(N+1)^2} \frac{N+1-m}{\sqrt{m(N+1)}} \right|^2 \\
&\leq \left(\sum_{N=m}^{\infty} \frac{1}{(1+2(N+1)^2)^2} \frac{(N+1-m)^2}{m(N+1)} \right) \sum_{N=m}^{\infty} |g'_{N+1}|^2 \quad \text{(Cauchy-Schwartz)} \\
&\leq \left(\sum_{N=m}^{\infty} \frac{1}{1+2(N+1)^2} \right) \sum_{N=m}^{\infty} |g'_{N+1}|^2 \\
&\leq 1 \cdot \sum_{N=m}^{\infty} (1+2(N+1)^2)^2 |g_{N+1}|^2.
\end{aligned}$$

Hence, we estimate the left-hand side of (5.10) as

$$
\begin{aligned}
&\sum_{m=1}^{\infty}(1+2m^2)^k\left|\sum_{N=m}^{\infty} g_{N+1}\frac{N+1-m}{\sqrt{m(N+1)}}\right|^2 \\
&\le \sum_{m=1}^{\infty}(1+2m^2)^k\sum_{N=m}^{\infty}(1+2(N+1)^2)^2|g_{N+1}|^2 \\
&= \sum_{N=1}^{\infty}(1+2(N+1)^2)^2\left(\sum_{m=1}^{N}(1+2m^2)^k\right)|g_{N+1}|^2 \\
&\le \sum_{N=1}^{\infty}(1+2(N+1)^2)^2 N(1+2N^2)^k|g_{N+1}|^2 \\
&\le \sum_{N=1}^{\infty}(1+2(N+1)^2)^{k+3}|g_{N+1}|^2.
\end{aligned}
$$

This proves Claim 5.7. □

5.3.3 Correspondence to the highest weight -1.

A similar argument also holds in this case.

Lemma 5.8. *The following equation holds*:

$$
B^2(\mathcal{F}_P)|_{U^1} = \left\{ g\,\omega_1\wedge\omega_2 \,\middle|\, \sum_{m=-\infty}^{-1}(-1)^{-m}\sqrt{-m}g_m = 0 \right\}. \tag{5.11}
$$

In particular, $Z^2(\mathcal{F}_P)|_{U^1}$ *is spanned by* $\phi_{-1}\,\omega_1\wedge\omega_2$ *and* $B^2(\mathcal{F}_P)|_{U^1}$.

Proof. Proving that the right-hand side of (5.11) contains $B^2(\mathcal{F}_P)|_{U^1}$ is easy. To prove the opposite, let N be a positive integer. We put

$$
\begin{aligned}
f_{1\,m}^{-(N+1)} &= \frac{N+1+m}{\sqrt{-m(N+1)}}\quad(-N\le m\le -1),\\
f_{1\,-(N+1)}^{-(N+1)} &= 0,\\
f_1^{-(N+1)} &= \sum_{m=-N}^{-1}(-1)^{N-m}f_{1\,m}^{-(N+1)}\phi_m.
\end{aligned}
$$

Then we have, by using (5.3),

$$
d_{\mathcal{F}_P}(f_1^{-(N+1)}\omega_1) = (\phi_{-(N+1)} + (-1)^{N+1}\sqrt{N+1}\phi_{-1})\,\omega_1\wedge\omega_2 \tag{5.12}
$$

To check (5.12), put

$$
g^{-(N+1)}\,\omega_1\wedge\omega_2 = d_{\mathcal{F}_P}(f_1^{-(N+1)}\omega_1).
$$

For each $-N \leq m \leq -2$, we have

$$
\begin{aligned}
&(-1)^{N-(m-1)} g_m^{-(N+1)} \\
&= f_{1\,m-1}^{-(N+1)} \alpha_{q\,m-1} + 2m f_{1\,m}^{-(N+1)} + f_{1\,m+1}^{-(N+1)} \beta_{q\,m+1} \\
&= \frac{N+1+(m-1)}{\sqrt{|m-1|(N+1)}} \sqrt{(m-1)m} + \frac{2m(N+1+m)}{\sqrt{|m|(N+1)}} + \frac{N+1+(m+1)}{\sqrt{|m+1|(N+1)}} \sqrt{m(m+1)} \\
&= \sqrt{\frac{|m|}{N+1}} \Big(N+1+(m-1) - 2(N+1+m) + N+1+(m+1) \Big) \\
&= 0.
\end{aligned}
$$

Next, we observe that

$$
\begin{aligned}
g_{-(N+1)}^{-(N+1)} &= -2(-(N+1))(-1)^{N+(N+1)} f_{1\,-(N+1)}^{-(N+1)} + (-1)^{N+N} f_{1\,-N}^{(N+1)} \beta_{q\,-N} \\
&= 0 + \frac{N+1-N}{\sqrt{N\cdot(N+1)}} \sqrt{(-N-1)(-N)} \\
&= 1,
\end{aligned}
$$

and

$$
\begin{aligned}
g_{-1}^{-(N+1)} &= (-1)^{N+2} f_{1\,-2}^{-(N+1)} \alpha_{q\,-2} - 2(-1)(-1)^{N+1} f_{1\,-1}^{-(N+1)} \\
&= (-1)^{N+1} \left(-\frac{N+1-2}{\sqrt{2\cdot(N+1)}} \sqrt{2\cdot 1} + 2\frac{N+1-1}{\sqrt{1\cdot(N+1)}} \right) \\
&= (-1)^{N+1}\sqrt{N+1}.
\end{aligned}
$$

Thus the formula (5.12) is valid. This fact implies the result in the same way as Lemma 5.5. The Sobolev estimation is the same as Lemma 5.6:

$$||\xi||_k^2 \leq ||\eta||_{k+3}^2.$$

This is also proved by Claim 5.7 except for the difference in the sign of m. $\square$

5.3.4 The other cases.

Let $\pi \neq I, U^{-1}, U^1$ and fix $m_\pi \in \mathbb{M}$. To begin with, we observe the behavior of $d_{\mathcal{F}_P}|_\pi$. Put $f_{1\,m} = f_{2\,m} = f_{1\,m+3} = f_{2\,m+3} = 0$ for any $m \in \mathbb{M}$ which satisfies $m \equiv m_\pi \pmod 4$. We consider the linear map

$$(f_{1\,m+1}, f_{2\,m+1}, f_{1\,m+2}, f_{2\,m+2}) \mapsto (g_m, g_{m+1}, g_{m+2}, g_{m+3})$$

for all $m \equiv m_\pi$ by (5.3). The coefficient matrix is a block diagonal matrix whose block is a 4×4 matrix. Each block is represented as

$$
\begin{pmatrix} g_m \\ g_{m+1} \\ g_{m+2} \\ g_{m+3} \end{pmatrix}
=
\begin{pmatrix}
\beta_{q\,m+1} & -\beta_{q\,m+1} & 0 & 0 \\
-2(m+1) & -2 & \beta_{q\,m+2} & -\beta_{q\,m+2} \\
\alpha_{q\,m+1} & \alpha_{q\,m+1} & -2(m+2) & -2 \\
0 & 0 & \alpha_{q\,m+2} & \alpha_{q\,m+2}
\end{pmatrix}
\begin{pmatrix} f_{1\,m+1} \\ f_{2\,m+1} \\ f_{1\,m+2} \\ f_{2\,m+2} \end{pmatrix}. \tag{5.13}
$$

We denote by $A_{q\,m}$ this 4×4 matrix. Its determinant $\gamma_{q\,m} = \det A_{q\,m}$ is written as

$$\gamma_{q\,m} = -4q\sqrt{m^2+m+q}\sqrt{m^2+5m+6+q}. \tag{5.14}$$

This value is always non-vanishing when $m, m+3 \in \mathbb{M}$ by the following lemma.

Lemma 5.9. *The following inequality holds*:

$$|\gamma_{q\,m}| \geq 4\min\{q^2, 1\}.$$

In particular, one has $|\gamma_{q\,m}| \geq 4q_\Gamma^2$.

Proof. When $\pi \neq I, U^{-n}, U^n$, since $\mathbb{M} \subset \frac{1}{2}\mathbb{Z}$, we have

$$|\gamma_{q\,m}| = 4q\sqrt{\left(m+\frac{1}{2}\right)^2 - \frac{1}{4} + q}\sqrt{\left(m+\frac{5}{2}\right)^2 - \frac{1}{4} + q} \geq 4q^2.$$

When $\pi = U^{-n}$ ($n \geq \frac{3}{2}$), $|\gamma_{q\,m}|$ takes the minimum at $m = n$:

$$\begin{aligned}
|\gamma_{q\,m}| &\geq 4|q|\sqrt{n^2+n+n(1-n)}\sqrt{n^2+5n+6+n(1-n)} \\
&= 4|q|\sqrt{2n}\sqrt{6(n+1)} \\
&\geq 4\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{3}\sqrt{15} \\
&\geq 4\cdot 1.
\end{aligned}$$

When $\pi = U^n$ ($n \geq \frac{3}{2}$), $|\gamma_{q\,m}|$ takes the minimum at $m+3 = -n$:

$$\begin{aligned}
|\gamma_{q\,m}| &\geq 4|q|\sqrt{(n+3)^2-(n+3)+n(1-n)}\sqrt{(n+3)^2-5(n+3)+6+n(1-n)} \\
&= 4|q|\sqrt{6(n+1)}\sqrt{2n} \\
&\geq 4\cdot 1.
\end{aligned}$$

The proof of Lemma 5.9 is finished. $\square$

By the above argument, we can determine the values $f_{1\,m+1}, f_{2\,m+1}, f_{1\,m+2}, f_{2\,m+2}$ which satisfy (5.13). Then take any

$$\eta = g\,\omega_1 \wedge \omega_2 \in Z^2(\mathcal{F}_P)|_\pi.$$

Put $f_{1\,m} = f_{2\,m} = f_{1\,m+3} = f_{2\,m+3} = 0$ for any $m \in \mathbb{M}$ which satisfy $m \equiv m_\pi \pmod 4$. We determine $f_{1\,m+1}, f_{2\,m+1}, f_{1\,m+2}, f_{2\,m+2}$ in (5.13) and set

$$\xi = f_1\,\omega_1 + f_2\,\omega_2.$$

We get $\eta = d_{\mathcal{F}_P}\xi$ formally. We continue the Sobolev estimations (see Lemmas 5.10 and 5.12 below).

Lemma 5.10. *When* $\pi \neq I, U^{-n}, U^n$, *one has*

$$||\xi||_k^2 \leq \left(\frac{3!}{q_\Gamma^2}\right)^2 2^{5(k+3)+8}||\eta||_{k+3}^2$$

for each $k \in \mathbb{N}$, *where* q_Γ *is defined in* (5.5).

Proof. Fix $k \in \mathbb{N}$ and $m \equiv m_\pi$. Each entry of $A_{q\,m}$ is bounded by $2\sqrt{q+m^2}$ or $2\sqrt{q+(m+3)^2}$. It is sufficient to consider the latter. Here, a cofactor of $A_{q\,m}$ is degree 3 polynomial of entries of $A_{q\,m}$. Thus any entry of the cofactor matrix of $A_{q\,m}$ is bounded by

$$3!2^3(q+(m+3)^2)^{\frac{3}{2}}.$$

Then, for each $l = 1, 2$ and $m' = m+1, m+2$,

$$\begin{aligned}
|f_{l\,m'}| &\le \frac{1}{|\gamma_{q\,m}|} 3!2^3(q+(m+3)^2)^{\frac{3}{2}} \left(\sum_{m''=m}^{m+3} |g_{m''}| \right) \\
&\le \frac{3!}{4q_\Gamma^2} 2^3(q+(m+3)^2)^{\frac{3}{2}} \left(4\sqrt{\sum_{m''=m}^{m+3} |g_{m''}|^2} \right) \\
&\le \frac{3!}{q_\Gamma^2} 2^3(1+q+2(m+3)^2)^{\frac{3}{2}} \sqrt{\sum_{m''=m}^{m+3} |g_{m''}|^2}
\end{aligned}$$

from Lemma 5.9.

Claim 5.11. *For each* $s', s'' \in \{0, 1, 2, 3\}$, *one has*

$$1+q+2(m+s')^2 \le 2^5(1+q+2(m+s'')^2).$$

Proof of Claim 5.11. We find a constant $c > 1$ satisfying

$$1+q+2(m+s)'^2 \le c(1+q+2(m+s'')^2),$$

which is equivalent to the following:

$$2c(m+s'')^2 - 2(m+s')^2 + (c-1)(1+q) \ge 0.$$

Since $\pi \ne I, U^{-n}, U^n$, it follows that q is positive. Then it is enough to satisfy

$$2c(m+s'')^2 - 2(m+s')^2 + c - 1 \ge 0.$$

In the left-hand side, we can write

$$2c(m+s'')^2 - 2(m+s')^2 = 2(c-1)\left(m + \frac{cs''-s}{c-1}\right)^2 - 2c\frac{(s''-s')^2}{c-1} \ge -\frac{18c}{c-1}.$$

Then it is enough to satisfy

$$-\frac{18c}{c-1} + c - 1 \ge 0$$

or

$$-18c + (c-1)^2 \ge 0.$$

Roughly, this is valid for $c \ge 20$. Thus we can set $c = 2^5$. $\square$

We continue the proof of Lemma 5.10 by using this claim:

$$
\begin{aligned}
&(1+q+2m'^2)^k|f_{l\,m}|^2 \\
&\le \left(\frac{3!}{q_\Gamma^2}\right)^2 2^6(1+q+2m'^2)^k(1+q+2(m+3)^2)^3 \sum_{m''=m}^{m+3} |g_{m''}|^2 \\
&\le \left(\frac{3!}{q_\Gamma^2}\right)^2 2^{5(k+3)+6} \sum_{m''=m}^{m+3} (1+q+2m''^2)^{k+3}|g_{m''}|^2 \ \text{(Claim 5.11)}.
\end{aligned}
$$

Add together for $l = 1, 2$ and $m' = m+1, m+2$. Then we get

$$
\begin{aligned}
&\sum_{l=1,2,\ m'=m+1,m+2} (1+q+2m'^2)^k|f_{l\,m}|^2 \\
&\le \left(\frac{3!}{q_\Gamma^2}\right)^2 2^{5(k+3)+8} \sum_{m''=m}^{m+3} (1+q+2m''^2)^{k+3}|g_{m''}|^2.
\end{aligned}
$$

Finally, the desired inequality is obtained by adding up for $m \equiv m_\pi$. □

Lemma 5.12. *When $\pi = U^{-n}, U^n$ ($n \ge \frac{3}{2}$), one has*

$$
||\xi||_k^2 \le \left(\frac{3!}{q_\Gamma^2}\right)^2 2^{5(k+3)+8}||\eta||_{k+3}^2
$$

for each $k \in \mathbb{N}$.

Proof. It is sufficient to prove the case $\pi = U^{-n}$, since the proof for the case $\pi = U^n$ is similar. Fix $k \in \mathbb{N}$ and $m \equiv m_\pi$. Each entry of $A_{q\,m}$ is bounded by $2(m+3)$. Then any entry of the cofactor matrix of $A_{q\,m}$ is bounded by

$$
3!2^3(m+3)^3.
$$

Then, for each $l = 1, 2$ and $m' = m+1, m+2$, we estimate, by using Lemma 5.9,

$$
\begin{aligned}
|f_{l\,m'}| &\le \frac{1}{|\gamma_{q\,m}|} 3!2^3(m+3)^3 \left(\sum_{m''=m}^{m+3} |g_{m''}|\right) \\
&= \frac{3!}{q_\Gamma^2} 2^3(m+3)^3 \sqrt{\sum_{m''=m}^{m+3} |g_{m''}|^2}.
\end{aligned}
$$

Claim 5.13. *The following equation holds*:

$$
m+3 \le \sqrt{1+n(1-n)+2(m+3)^2}. \tag{5.15}
$$

Proof of Claim 5.13. We write (5.15) as

$$
1+n(1-n)+(m+3)^2 \ge 0.
$$

The left-hand side attains a minimum if $m = n$. Then

$$1 + n(1-n) + (n+3)^2 = 7n + 10.$$

This is non-negative. $\square$

We continue the proof of Lemma 5.12 by assuming this claim:

$$|f_{l\,m'}| \leq \frac{3!}{q_\Gamma^2} 2^3 (1 + n(1-n) + 2(m+3)^2)^{\frac{3}{2}} \sqrt{\sum_{m''=m}^{m+3} |g_{m''}|^2}.$$

Claim 5.14. *For each $s', s'' \in \{0,1,2,3\}$, one has*

$$1 + n(1-n) + 2(m+s')^2 \leq 2^5 (1 + n(1-n) + 2(m+s'')^2).$$

Proof of Claim 5.14. We find a constant $c > 13$ satisfying

$$1 + n(1-n) + 2(m+s')^2 \leq c\left[1 + n(1-n) + 2(m+s'')^2\right],$$

which is equivalent to

$$2c(m+s'')^2 - 2(m+s')^2 + (c-1)(1 + n(1-n)) \geq 0.$$

Since $c > 3$ and $m \geq n \geq \frac{3}{2}$, we have

$$\begin{aligned} 2c(m+s'')^2 - 2(m+s')^2 &\geq 2cm^2 - 2(m+3)^2 \\ &= 2(c-1)\left(m - \frac{3}{c-1}\right)^2 - \frac{18c}{c-1} \\ &\geq 2cn^2 - 2(n+3)^2. \end{aligned}$$

Then it is enough to satisfy

$$2cn^2 - 2(n+3)^2 + (c-1)(1 + n(1-n)) \geq 0.$$

Since $c > 13$ and $n \geq \frac{3}{2}$, the left-hand side is estimated from below:

$$\begin{aligned} &2cn^2 - 2(n+3)^2 + (c-1)(1 + n(1-n)) \\ &= (c-1)n^2 + (c-13)n + (c-19) \\ &\geq (c-1)\left(\frac{3}{2}\right)^2 + (c-13)\frac{3}{2} + (c-19) \\ &= \frac{19}{4}c - \frac{163}{4}. \end{aligned}$$

This is positive under $c > 13$. Thus we can set $c = 2^5$ roughly. $\square$

By this claim, the rest for the proof of Lemma 5.12 is the same as the argument after Claim 5.11 in the proof of Lemma 5.10. This ends the proof of Lemma 5.12. $\square$

Then the following holds.

Lemma 5.15. $H^2(\mathcal{F}_P)|_\pi = \{0\}$.

5.3.5 The whole sum.

For any $\eta \in B^2(\mathcal{F}_P)$ and $\pi \subset L^2(M_\Gamma)$, let η_π be the π-component. Then we get a smooth cochain $\xi_\pi \in \Omega^1(\mathcal{F}_P)|_\pi$ such that

$$||\xi_\pi||_k^2 \leq C_{\Gamma\,k+3}||\eta_\pi||_{k+3}^2$$

for each $k \in \mathbb{N}$. Thus we get $\eta = d_{\mathcal{F}_P}\xi$ and

$$||\xi||_k^2 \leq C_{\Gamma\,k+3}||\eta||_{k+3}^2,$$

where $\xi = \sum_\pi \xi_\pi$.

We summarize the discussion so far. Recall that both the multiplicity of U^{-1} and U^1 in $L^2(M_\Gamma)$ are g. We put

$$\begin{aligned} x &= \omega_1, \\ y_j &= \phi_1(\omega_1 - \omega_2) \quad \text{in } j\text{-th } U^{-1}, \\ y_{g+j} &= \phi_{-1}(\omega_1 + \omega_2) \quad \text{in } j\text{-th } U^1. \end{aligned}$$

These are 1-cocycles. Thus we get the following:

Proposition 5.16. *The set $\{x \wedge y_1, \cdots, x \wedge y_{2g}\}$ is basis for $H^2(\mathcal{F}_P)$, where the number g is the multiplicity of U^{-1} and U^1.*

Remark 5.17. *This recovers the result (1.3) by Maruhashi and Tsutaya [23].*

5.4 Computation of first cocycles

We also solve a linear equation. The method is similar to the previous section, with the addition of vanishing lemmata for special cocycles.

5.4.1 Trivial representation.

We get the following lemma directly.

Lemma 5.18. $H^1(\mathcal{F}_P)|_I = \mathbb{C}\omega_1$.

5.4.2 Correspondence to the lowest weight 1.

We also characterize the coboundary space. Before doing it, we prove that the special 1-cocycles are trivial.

Lemma 5.19. *If $f_1\,\omega_1 \in Z^1(\mathcal{F}_P)|_{U^{-1}}$, then $f_1 = 0$.*

Proof. Put $g\,\omega_1 \wedge \omega_2 = d_{\mathcal{F}_P}(f_1\,\omega_1)$ and $f_{1\,0} = 0$. Using (5.3), for any positive integer $N \geq 3$, we compute

$$\begin{aligned}
&\sum_{m=1}^{N-1} \sqrt{m} g_m \\
&= \sum_{m=1}^{N-1} \sqrt{m} \left(f_{1\,m-1}\sqrt{(m-1)m} - 2m f_{1\,m} + f_{1\,m+1}\sqrt{m(m+1)} \right) \\
&= \sum_{m=0}^{N-2} \sqrt{m+1} f_{1\,m}\sqrt{m(m+1)} + \sum_{m=1}^{N-1} \sqrt{m}(-2m f_{1\,m}) + \sum_{m=2}^{N} \sqrt{m-1} f_{1\,m}\sqrt{(m-1)m} \\
&= \sum_{m=0}^{N-2} (m+1)\sqrt{m} f_{1\,m} + \sum_{m=1}^{N-1} (-2m)\sqrt{m} f_{1\,m} + \sum_{m=2}^{N} (m-1)\sqrt{m} f_{1\,m} \\
&= 0 + \sum_{m=N-1}^{N-1} (-2m)\sqrt{m} f_{1\,m} + \sum_{m=N-1}^{N} (m-1)\sqrt{m} f_{1\,m} \\
&= -N\sqrt{N-1} f_{1\,N-1} + (N-1)\sqrt{N} f_{1\,N}.
\end{aligned}$$

Since $g = 0$, we see that

$$f_{1\,N} = \sqrt{\frac{N}{N-1}} f_{1\,N-1}.$$

Thus we deduce that

$$f_{1\,N} = \sqrt{N} f_{1\,1}.$$

Then $f_1 = 0$, since $\sum_{N=1}^{\infty} |f_{1\,N}|^2 < \infty$. □

Lemma 5.20. *The following equation holds*:

$$B^1(\mathcal{F}_P)|_{U^{-1}} = \left\{ f_1\,\omega_1 + f_2\,\omega_2 \in Z^1(\mathcal{F}_P)|_{U^{-1}} \;\middle|\; \sum_{m=1}^{\infty} \sqrt{m} f_{2\,m} = 0 \right\}.$$

In particular, $Z^1(\mathcal{F}_P)|_{U^{-1}}$ *is spanned by* $\phi_1(\omega_1 - \omega_2)$ *and* $B^1(\mathcal{F}_P)|_{U^{-1}}$.

Proof. Proving that the right-hand side contains $B^1(\mathcal{F}_P)|_{U^{-1}}$ is easy. To prove the opposite, let N be a positive integer. We put

$$h_m^{(N+1)} = \frac{N+1-m}{\sqrt{m(N+1)}} \ \ (1 \leq m \leq N),$$
$$h^{(N+1)} = \sum_{m=1}^{N} h_m^{(N+1)} \phi_m.$$

Then we have

$$d_{\mathcal{F}_P} h^{(N+1)} = (\text{some function})\,\omega_1 + (-\sqrt{N+1}\phi_1 + \phi_{N+1})\,\omega_2$$

by (5.2). Then let $\eta = f_1\,\omega_1 + f_2\,\omega_2$ be an element from the right-hand side. Put

$$\xi = \sum_{N=1}^{\infty} f_{2\,N+1}\, h^{(N+1)}.$$

The 1-cocycle $\eta - d_{\mathcal{F}_P}\xi$ satisfies the assumption of Lemma 5.19 formally. Thus we obtain $\eta = d_{\mathcal{F}_P}\xi$ formally. In the same way as Section 5.3.2, the following Sobolev estimate holds:

$$||\xi||_k^2 \leq ||\eta||_{k+3}^2. \tag{5.16}$$

Replacing g with f_2, we can prove (5.16) by using Claim 5.7. □

5.4.3 Correspondence to the highest weight -1.

A similar argument also holds in this case.

Lemma 5.21. *The following equation holds*:

$$B^1(\mathcal{F}_P)|_{U^1} = \left\{ f_1\,\omega_1 + f_2\,\omega_2 \in Z^1(\mathcal{F}_P)|_{U^1} \,\middle|\, \sum_{m=-\infty}^{-1} (-1)^{-m}\sqrt{-m} f_{2\,m} = 0 \right\}. \tag{5.17}$$

In particular, $Z^1(\mathcal{F}_P)|_{U^1}$ is spanned by $\phi_{-1}(\omega_1 + \omega_2)$ and $B^1(\mathcal{F}_P)|_{U^1}$.

In the same way as Section 5.3.3, the following Sobolev estimate holds:

$$||\xi||_k^2 \leq ||\eta||_{k+3}^2,$$

where $\eta = f_1\,\omega_1 + f_2\,\omega_2$ is an element of the right-hand side of (5.17) in Lemma 5.21 and $\xi = h$ is some 0-cochain.

5.4.4 The other cases.

Let $\pi \neq I, U^{-1}, U^1$ and fix $m_\pi \in \mathbb{M}$. We still start with proving the triviality of the special 1-cocycles.

Lemma 5.22. *Let $f_1\,\omega_1 + f_2\,\omega_2 \in Z^1(\mathcal{F}_P)|_\pi$. Assume $f_{1\,m} = f_{2\,m} = f_{1\,m+3} = f_{2\,m+3} = 0$ for any $m \in \mathbb{M}$ which satisfies $m \equiv m_\pi \pmod 4$. Then $f_1 = f_2 = 0$.*

Proof. Under the assumption, 1-cocycle conditions (5.3) is realized as the kernel of the linear map

$$(f_{1\,m+1}, f_{2\,m+1}, f_{1\,m+2}, f_{2\,m+2}) \mapsto (g_m, g_{m+1}, g_{m+2}, g_{m+3})$$

for each $m \equiv m_\pi$ (see (5.13)). Recall that the determinant $\gamma_{q\,m}$ is not 0. Thus $f_1 = f_2 = 0$. □

We consider the linear map

$$(h_m, h_{m+1}, h_{m+2}, h_{m+3}) \mapsto (f_{1\,m+1}, f_{2\,m+1}, f_{1\,m+2}, f_{2\,m+2})$$

for each $m \equiv m_\pi + 2$ in (5.1) and (5.2). The coefficient matrix is also a block diagonal matrix whose block is a 4×4 matrix. Each block is represented as

$$\begin{pmatrix} f_{1\,m+1} \\ f_{2\,m+1} \\ f_{1\,m+2} \\ f_{2\,m+2} \end{pmatrix} = \begin{pmatrix} -\alpha_{q\,m} & 0 & \beta_{q\,m+2} & 0 \\ \alpha_{q\,m} & -2(m+1) & \beta_{q\,m+2} & 0 \\ 0 & -\alpha_{q\,m+1} & 0 & \beta_{q\,m+3} \\ 0 & \alpha_{q\,m+1} & -2(m+2) & \beta_{q\,m+3} \end{pmatrix} \begin{pmatrix} h_m \\ h_{m+1} \\ h_{m+2} \\ h_{m+3} \end{pmatrix}. \tag{5.18}$$

Its determinant is also $\gamma_{q\,m}$ defined in (5.14).

Lemma 5.23. $H^1(\mathcal{F}_P)|_\pi = \{0\}$.

Proof. Take any $f_1\,\omega_1 + f_2\,\omega_2 \in Z^1(\mathcal{F}_P)|_\pi$. From (5.18), we can construct h such that $f_1\,\omega_1 + f_2\,\omega_2 - d_{\mathcal{F}_P} h$ satisfies the assumption of Lemma 5.22. Then $f_1\,\omega_1 + f_2\,\omega_2 = d_{\mathcal{F}_P} h$. □

In the same way as Section 5.3.4, the following Sobolev estimate holds:

$$||\xi||_k^2 \leq \left(\frac{3!}{q_\Gamma^2}\right)^2 2^{5(k+3)+8} ||\eta||_{k+3}^2,$$

where $\eta = f_1\,\omega_1 + f_2\,\omega_2$ is any 1-cocycle and $\xi = h$ is some 0-cochain. This proof is the same as that of Lemmas 5.10 and 5.12. However, we use the matrix (5.18) instead of (5.13).

5.4.5 The whole sum.

In the same way as Section 5.3.5, for each $\eta \in B^1(\mathcal{F}_P)$, we have $\xi \in \Omega^0(\mathcal{F}_P)$ satisfying

$$||\xi||_k^2 \leq C_{\Gamma\,k+3} ||\eta||_{k+3}^2.$$

The following assertion holds:

Proposition 5.24. *The set $\{x, y_1, \cdots, y_{2g}\}$ is basis for $H^1(\mathcal{F}_P)$, where the number g is the multiplicity of U^{-1} and U^1.*

Remark 5.25. *This recovers the result (1.1) by Matsumoto and Mitsumatsu [24].*

5.5 Determination of the ring structure

We can prove our main theorem by combining the above preparation with the following lemma.

Lemma 5.26. *Let $\phi_1, \phi_1' \in L^2(M_\Gamma)$ be weight vectors of U^{-1}. Here, ϕ_1 and ϕ_1' do not necessarily belong to the same irreducible component. Also, let $\phi_{-1}, \phi_{-1}' \in L^2(M_\Gamma)$ be weight vectors of U^1. Then*

$$\begin{aligned} (X_0^2 - X_1^2 - Y^2)(\phi_1\phi_1') &= -2\phi_1\phi_1', \\ (X_0^2 - X_1^2 - Y^2)(\phi_{-1}\phi_{-1}') &= -2\phi_{-1}\phi_{-1}', \\ X_0(\phi_1\phi_{-1}) &= 0. \end{aligned} \tag{5.19}$$

Especially, $\phi_1\phi_1'$, $\phi_{-1}\phi_{-1}'$ and $\phi_1\phi_{-1}$ orthogonal to U^{-1} and U^1.

Proof. Formulae are proved easily. The first two of them mean that $\phi_1\phi'_1$ and $\phi_{-1}\phi'_{-1}$ are eigenvectors corresponding to -2 of the Casimir element. On the other hand, the Casimir element vanishes on U^{-1} and U^1. Then they are orthogonal to U^{-1} and U^1. Also $\phi_1\phi_{-1}$ is orthogonal to U^{-1} and U^1 by (5.19). Indeed, the set $\mathbb{M}$ of U^{-1} and U^1 does not contain 0. $\square$

Proof of Theorem 5.1. Generators of $H^*(\mathcal{F}_P)$ are given in Propositions 5.3, 5.16, and 5.24. The vanishing of $y_i \wedge y_j$ in $H^2(\mathcal{F}_P)$ follows from Lemma 5.26 for each $1 \leq i, j \leq 2g$. $\square$

6 Characterization of the parameter g

In Introduction, we obtained the dimension formula (1.4) as a consequence of (1.1) and Theorem 1.1. We restate it below:

$$\dim H^1_{\mathrm{dR}}(M_\Gamma) = 2g.$$

We also restate Theorem 1.2 in Introduction as Theorem 6.1 and prove it. Set $G = SL(2, \mathbb{R})$ and $K = SO(2)$. We identify G/K with the upper half plane. through the linear fractional action. Let Γ be a cocompact lattice in G. We set $M_\Gamma = \Gamma \backslash G$ and $\Sigma_\Gamma = \Gamma \backslash G/K$.

Theorem 6.1. *The following properties hold.*

(i) The space Σ_Γ is homeomorphic to a closed orientable surface.

(ii) Let g_Γ be the genus of Σ_Γ. Then $g = g_\Gamma$.

6.1 Closed orientable surfaces from the upper half plane

Let (X, d) be a metric space and Γ be a group. The following property is fundamental when considering quotient spaces in group actions.

Definition 6.2. *A left action of Γ on X is called a discontinuous action if the set*

$$\{\gamma \in \Gamma | C \cap \gamma C \neq \emptyset\}$$

is a finite set for each compact subset $C \subset X$.

If Γ acts on X discontinuously, then each Γ-orbit is discrete in X. In fact, assume that there is a convergent sequence in some Γ-orbit. Let $C \subset X$ be a set consisting of such sequence and its limit point. Then C is a compact infinite set. However, C contradicts the discontinuousness.

First, we will describe how to determine discontinuousness.

Definition 6.3. *A metric space (X, d) is finitely compact if and only if any closed ball in X is compact.*

A finitely compact space is complete. The converse is not true. In fact, infinite-dimensional Banach spaces are complete but not finitely compact.

Let $\mathrm{Isom}(X)$ be the group consisting of all isometries. We regard $\mathrm{Isom}(X)$ as a topological group by the compact-open topology.

Theorem 6.4. *(See [30, Theorem 5.3.5].) Assume that (X, d) is finitely compact and Γ is a subgroup of $\mathrm{Isom}(X)$. Then the action of Γ is discontinuous if and only if Γ is discrete in $\mathrm{Isom}(X)$.*

Next, we discuss the conditions under which a quotient space has a natural metric.

Definition 6.5. *Let (X, d) be a metric space and Γ be a subgroup of $\mathrm{Isom}(X)$. Let $d_\Gamma : \Gamma\backslash X \times \Gamma\backslash X \to \mathbb{R}$ be the function defined below:*

$$d_\Gamma(\Gamma x, \Gamma x') = \inf_{\gamma,\gamma' \in \Gamma} d(\gamma x, \gamma' x').$$

At this time, the following holds.

Theorem 6.6. *(See [30, Theorem 6.6.1]) Let (X, d) be a metric space and Γ be a subgroup of $\mathrm{Isom}(X)$. Then $(\Gamma\backslash X, d_\Gamma)$ is a metric space if and only if each Γ-orbit is closed in X. In particular, if Γ is a discontinuous subgroup, then $(\Gamma\backslash X, d_\Gamma)$ is a metric space.*

The following theorem, considered quantitatively, is important.

Theorem 6.7. *(See [30, Theorem 13.1.1]) Let (X, d) be a metric space, Γ be a discontinuous subgroup of $\mathrm{Isom}(X)$, and $\pi : X \to \Gamma\backslash X$ be the projection. Take any $x \in X$ and $r > 0$ which satisfy $0 < r < \frac{1}{4} d(x, \Gamma x - \{x\})$. Then π induces the isometry from $\Gamma_x\backslash B(x, r)$ to $B(\pi(x), r)$. Here, B means an open ball, and Γ_x means the stabilizer at x.*

A point $x \in X$ is exceptional if and only if Γ_x is non-trivial. The other points in X are called regular. Let X be S^2 or $\mathbb{R}^2$ or $\mathbb{H}^2$. From the discussion in [30, Section 13.2], any exceptional point $x \in X$ can be one of the following three types:

- The group Γ_x is isomorphic to $\mathbb{Z}_2$. It consists of the identity map and a reflection. In this case, x is called a mirror point.
- The group Γ_x is isomorphic to $\mathbb{Z}_\mu$ for some $\mu \geq 2$. It is generated by a rotation whose angle is $\frac{2\pi}{\mu}$. In this case, x is called a cone point.
- The group Γ_x is isomorphic to some dihedral group. In this case, x is called a corner point.

Proof of (i) in Theorem 6.1. Since $\Gamma \subset G$ is cocompact, Σ_Γ is compact. Before continuing the proof, recall that $PSL(2, \mathbb{R})$ is isomorphic to the identity component of $\mathrm{Isom}(\mathbb{H}^2)$ through the linear fractional action. By projecting Γ to $PSL(2, \mathbb{R})$, we apply the preparation above to our discussion. Each element of Γ preserves the orientation of $\mathbb{H}^2$. Therefore any exceptional point is a cone point. In particular, Σ_Γ is a surface without boundary. The fact of orientation preserving also derives that Σ_Γ is orientable. $\square$

6.2 A view point from Seifert bundles

We summarize the facts from Seifert bundles. These are provided as needed for our discussion. Therefore some of statements are in a limited form. See [29] for complete discussions.

Take $(\mu, \nu) \in \mathbb{Z}$ which satisfies $0 < \nu \leq \mu$ and $\gcd(\mu, \nu) = 1$. We treat $\mathbb{Z}_\mu$ as the subgroup of S^1. We define the left action of $\mathbb{Z}_\mu$ on S^1 by the multiplication. Define the left action of $\mathbb{Z}_\mu$ on $\mathbb{D}^2$ below:

$$e^{i\frac{2\pi}{\mu}} \cdot z = e^{i\nu\frac{2\pi}{\mu}} z$$

for each $z \in \mathbb{D}^2$. Let the left action of $\mathbb{Z}_\mu$ on $\mathbb{D}^2 \times S^1$ be the diagonal action. Let $p : \mathbb{D}^2 \times S^1 \to \mathbb{D}^2$ be the first projection. Let $p'_{\mu\,\nu} : \mathbb{Z}_\mu \backslash (\mathbb{D}^2 \times S^1) \to \mathbb{Z}_\mu \backslash \mathbb{D}^2$ denote the induced map, which makes the following diagram commutative:

$$\begin{tikzcd}
\mathbb{D}^2 \times S^1 \arrow[r] \arrow[d, "p"'] & \mathbb{Z}_\mu \backslash (\mathbb{D}^2 \times S^1) \arrow[d, "p'_{\mu\,\nu}"] \\
\mathbb{D}^2 \arrow[r] & \mathbb{Z}_\mu \backslash \mathbb{D}^2
\end{tikzcd}$$

In the diagram above, the horizontally oriented maps are canonical projections.

Definition 6.8. *We call $p'_{\mu\,\nu} : \mathbb{Z}_\mu \backslash (\mathbb{D}^2 \times S^1) \to \mathbb{Z}_\mu \backslash \mathbb{D}^2$ Seifert product bundle.*

Definition 6.9. *Take any $z \in \mathbb{D}^2$.*

(i) When $\mu \geq 2$ and $z = 0$, we call $(p'_{\mu\,\nu})^{-1}(\{0\})$ an exceptional fiber.

(ii) When $\mu = 1$ or $z \neq 0$, we call $(p'_{\mu\,\nu})^{-1}(\{z\})$ a regular fiber.

Example 6.10. *If $\mu = \nu = 1$, then the Seifert product bundle is a trivial S^1-bundle.*

Example 6.11. *Set $\mu = 3$ and $\nu = 1$. We identify $\mathbb{Z}_3 \backslash (\mathbb{D}^2 \times S^1)$ with $\mathbb{D}^2 \times I$, where $I = \{e^{i\theta} \in S^1 | 0 \leq \theta < \frac{2}{3}\pi\}$. Then the following properties hold:*

(i) The exceptional fiber $(p'_{\mu\,\nu})^{-1}(\{0\})$ is $\{0\} \times I$ in $\mathbb{D}^2 \times I$.

(ii) A regular fiber $(p'_{\mu\,\nu})^{-1}(\{\frac{1}{2}\})$ is $\{\frac{1}{2}, \frac{1}{2}e^{i\frac{2}{3}\pi}, \frac{1}{2}e^{i\frac{4}{3}\pi}\} \times I$ in $\mathbb{D}^2 \times I$.

Next, we define the Seifert bundles.

Definition 6.12. *Let M be a closed orientable 3-manifold and Σ be a close orientable surface. A continuous surjection $\pi : M \to \Sigma$ is a Seifert bundle if and only if there exists an open covering $\{O_\lambda\}_{\lambda \in \Lambda}$ of Σ such that the following holds: for each $\lambda \in \Lambda$, there exist $(\mu, \nu) \in \mathbb{Z}^2$ with $0 < \nu \leq \mu$ and $\gcd(\mu, \nu) = 1$, a homeomorphism $h_\lambda : \pi^{-1}(O_\lambda) \to \mathbb{Z}_\mu \backslash (\mathbb{D}^2 \times S^1)$, and a homeomorphism $t_\lambda : O_\lambda \to \mathbb{Z}_\mu \backslash \mathbb{D}^2$ such that the diagram below is commutative:*

$$\begin{tikzcd}
\pi^{-1}(O_\lambda) \arrow[r, "h_\lambda"] \arrow[d, "\pi"'] & \mathbb{Z}_\mu \backslash (\mathbb{D}^2 \times S^1) \arrow[d, "p'_{\mu\,\nu}"] \\
O_\lambda \arrow[r, "t_\lambda"'] & \mathbb{Z}_\mu \backslash \mathbb{D}^2
\end{tikzcd}$$

Exceptional fibers and regular fibers are defined in the same way as in the Seifert product bundles. The number of exceptional fibers is finite since Σ is compact. Suppose that there are r exceptional fibers. Take $\{(\mu_j, \nu_j)\}_{1 \leq j \leq r} \subset \mathbb{Z}^2$ that goes with them. For each $1 \leq j \leq r$, define $(\alpha_j, \beta_j) \in \mathbb{Z}^2$ by the three conditions below:

(i) $0 < \beta_j < \alpha_j$.

(ii) $\alpha_j = \mu_j$.

(iii) $\beta_j \nu_j \equiv 1 \pmod{\mu_j}$.

Then there exists $b \in \mathbb{Z}$ such that M is isomorphic to some uniquely determined canonical form:

$$M \cong S(g_\Sigma; b; (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)),$$

where g_Σ is the genus of Σ. See [29, Theorem 3 in Section 5.2]. The number b is called the obstruction. Set

$$e(\pi) = -\left(b + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j}\right).$$

The number $e(\pi)$ is called the Euler number.

Example 6.13. *Let Γ be a cocompact lattice in G. Then $\pi_\Gamma : M_\Gamma \to \Sigma_\Gamma$ is a Seifert bundle and $e(\pi_\Gamma) \neq 0$. See [33, Section "$\widetilde{SL}(2, \mathbb{R})$"]. These results are summarized in [13, Section 1].*

Finally, we describe the fact about generators of a fundamental group.

Theorem 6.14. *(See [29, Section 5.3].) Let $\pi : M \to \Sigma$ be a Seifert bundle. Then $\pi_1(M)$ has generators*

$$a_1, b_1, \cdots, a_{g_\Sigma}, b_{g_\Sigma}, q_1, \cdots, q_r, h$$

under five relations below:

(i) $[a_k, h] = 1$,

(ii) $[b_k, h] = 1$,

(iii) $[q_j, h] = 1$,

(iv) $q_j^{\alpha_j} h^{\beta_j} = 1$,

(v) $q_1 \cdots q_r [a_1, b_1] \cdots [a_{g_\Sigma}, b_{g_\Sigma}] = h^b$.

6.3 The first cohomology group

We complete characterizing the parameter g by computing the first cohomology group.

Theorem 6.15. *Let $\pi : M \to \Sigma$ be a Seifert bundle. If $e(\pi) \neq 0$, then $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g_\Sigma}$.*

Proof. From the Hurewicz theorem, we obtain

$$\pi_1(M)/[\pi_1(M), \pi_1(M)] \cong H_1(M;\mathbb{Z}).$$

Through the Hurewicz isomorphism, we treat generators

$$a_1, b_1, \cdots, a_{g_\Sigma}, b_{g_\Sigma}, q_1, \cdots, q_r, h$$

of $\pi_1(M)$ as that of $H_1(M;\mathbb{Z})$. Let $U \subset H_1(M;\mathbb{Z})$ be the subgroup generated by $a_1, b_1, \cdots a_{g_\Sigma}, b_{g_\Sigma}$ and $V \subset H_1(M;\mathbb{Z})$ be the subgroup generated by $q_1, \cdots q_r, h$. Then we have $H_1(M;\mathbb{Z}) = U + V$. Relations (i), (ii), and (iii) for generators are trivial in $H_1(M;\mathbb{Z})$. Relations (iv) and (v) lead to the following equations:

(iv)' $\alpha_j q_j + \beta_j h = 0$,

(v)' $\displaystyle\sum_{j=1}^{r} q_j - bh = 0$.

In particular, we obtain $H_1(M;\mathbb{Z}) = U \oplus V$ and $U \cong \mathbb{Z}^{2g_\Sigma}$. The following holds for V.

Claim 6.16. *There exist $e_1, \cdots, e_{r+1} \in \mathbb{Z} - \{0\}$ with $e_j | e_{j+1}$ such that $V \cong \mathbb{Z}_{e_1} \oplus \cdots \oplus \mathbb{Z}_{e_{r+1}}$.*

Proof of Claim 6.16. Let $W \subset \mathbb{Z}^{r+1}$ be the subgroup generated by $r+1$ vectors below:

$$\begin{pmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta_1 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_r \\ \beta_r \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ -b \end{pmatrix}.$$

From (iv)' and (v), we obtain $V \cong \mathbb{Z}^{r+1}/W$. Let A be the $(r+1)$th-order matrix that arranges the generating system in W above. Then we have

$$\det A = \left(\prod_{j=1}^{r} \alpha_j\right) e(\pi) \neq 0.$$

Therefore the required $e_1, \cdots, e_{r+1} \in \mathbb{Z} - \{0\}$ are obtained from elementary divisor theory. $\square$

From Claim 6.16 and the universal coefficient theorem, we have $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g_\Sigma}$. $\square$

Proof of (ii) in Theorem 6.1. From Example 6.13, $e(\pi_\Gamma) \neq 0$. Apply Theorem 6.15 and obtain $H^1(M_\Gamma;\mathbb{Z}) \cong \mathbb{Z}^{2g_\Gamma}$. Therefore $g = g_\Gamma$. $\square$

References

[1] J. A. Álvarez López, G. Hector, Leafwise homologies, leafwise cohomology, and subfoliations (Santiago de Compostela, 1994), World Sci. Publ., River Edge (1995), 1-12.

[2] J. A. Álvarez López, G. Hector, The dimension of the leafwise reduced cohomology, Am. J. Math. **123**(4) (2001), 607-646.

[3] J. A. Álvarez López, S. Hurder, Pure-point spectrum for foliation geometric operators, Preprint, 1994.

[4] J. L. Arraut and N. M. dos Santos, Differentiable conjugation of actions of $\mathbb{R}^p$. Bol. Soc. Brasil. Mat. (N.S.) **19**(1) (1988), 119.

[5] J. L. Arraut, N. M. dos Santos, Linear foliations of $\mathbb{T}^n$, Bol. Soc. Bras. Mat. **21**(2) (1991), 189-204.

[6] M. Asaoka, Non-homogeneous locally free actions of the affine group. Ann. of Math. **175**(1) (2012), 121.

[7] M. Asaoka, Deformation of locally free actions and the leafwise cohomology. preprint,

[8] M. Asaoka, Local rigidity problem of smooth group actions [translation of Sugaku **63**(4) (2013), 337-359], Sugaku Expositions **30**(2) (2017), 207-233.

[9] A. Candel, L. Conlon, Foliations I, Grad. Stud. Math. **23**, Amer. Math. Soc., 2000.

[10] A. El. Kacimi Alaoui, A. Tihami, Cohomologie bigraduée de certains feuilletages, Bull. Soc. Math. Belg. **38** (1986), 144-156.

[11] Flaminio, L., and Forni, G. Invariant distributions and time averages for horocycle

[12] H. Furstenberg, The unique ergodicity of the horocycle flow, Springer L. N. Math. **318** (1973), 95-114.

[13] H. Geiges, J. Gonzalo, Seifert invariants of left-quotients of 3-dimensional simple Lie groups, Topology Appl. **66** (1995), 117-127.

[14] I. M. Gel'fand, M. I. Graev, I. I. Pyatetskii-Shapiro, Representation theory and automorphic functions, W. B. Saunders Company, 1969.

[15] É. Ghys, Actions localement libres du groupe affine. Invent. Math. **82**(3) (1985), 479-526.

[16] J. L. Heitsch, A cohomology for foliated manifolds, Comment. Math. Helvetici **50** (1975), 197-218.

[17] G. H. Hardy, E. M. Wright, An introduction to the theory of numbers, Oxford at the Clarendon press, 1938.

[18] A. Katok and R. J. Spatzier, First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity, Publ. Math. IHES **79** (1994), 131-156.

[19] A. Knapp, Lie Groups, Lie Algebras, and Cohomology, Mathematical Notes **34**, Princeton University Press, Princeton, New Jersey, 1988.

[20] T. H. Koornwinder, A note on the multiplicity free reduction of certain orthogonal and unitary groups, Indag. Math. **44** (1982), 215-218.

[21] H. Maruhashi, Parameter rigid actions of simply connected nilpotent Lie groups, Ergod. Theory Dyn. Syst. **33** (2013), 1864-1875.

[22] H. Maruhashi. Vanishing of cohomology and parameter rigidity of actions of solvable Lie groups. Geom. Topol. **21**(1) (2017), 157191.

[23] H. Maruhashi, M. Tsutaya, De Rham cohomology of the weak stable foliation of the geodesic flow of a hyperbolic surface, arXiv:2103.12403.

[24] S. Matsumoto, Y. Mitsumatsu, Leafwise cohomology and rigidity of certain Lie group actions, Ergod. Th. & Dynam. Sys. **23** (2003), 1839-1866.

[25] S. Matsumoto Rigidity of locally free Lie group actions and leafwise cohomology [translation of Sugaku **58**(1) (2006), 86-101], Sugaku Expositions **22**(1) (2009), 21-36.

[26] S. Mori, A dimensional formula of some leafwise cohomology, arXiv:2103.06661[v1].

[27] S. Mori, Computation of some leafwise cohomology ring, Preprint, 2022.

[28] G. J. Murphy, C^*-algebras and operator theory, Academic press, Academic Press, 1990.

[29] P. Orlik, Seifert Manifolds, Lecture Notes in Mathematics **291**, Springer, 1972.

[30] J. Ratcliffe, Foundations of Hyperbolic Manifolds, Graduate Texts in Mathematics **149**, Springer, 2006.

[31] B. L. Reinhart, Harmonic integrals on almost product manifolds, Trans. Amer. Math. Soc. **88** (1958), 243-276.

[32] K. Schmüdgen, Unbounded self-adjoint operators on Hilbert space, Springer, 2012.

[33] P. Scott, The geometries of 3-manifolds, Bull. London Math. Sot. **15** (1983) 401-487.

[34] C. L. Siegel, J. K. Moser, Lectures on celestial mechanics, Springer, New York, 1971.

[35] S. Sternberg, Celestial mechanics, Part II, W. A. Benjamin Inc., New York, 1969.

[36] R. S. Sstrichartz, Analysis of the Laplacian on a complete Riemannian manifold, J. Funct. Anal. **52** (1983), 48-79.

[37] M. Sugiura, Unitary representations and harmonic analysis, North-Holland, 1990.

[38] H. Urakawa, On the least positive eigenvalue of the Laplacian for compact

[39] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups I, Springer-Verlag, 1972.

[40] F. Warner, Fondations of differentiable manifolds and Lie groups, Springer-Verlag, New York, 1983.

[41] K. Yoshida, Functional analysis, Springer-Verlag, 1980. arXiv:1012.2946. flows, Duke Math. J. **119**(3) (2003), 465526. group manifolds, J. Math. Soc. Japan **31** (1979), 209-226.