Grothendieck monoids and their applications to representation theory and algebraic geometry

(Grothendieck モノイドとその表現論および代数幾何学への応用)

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Chapter 0

Introduction

This thesis aims to study *Grothendieck monoids* and give their applications to representation theory and algebraic geometry. We will explain the background of our study and describe the main results.

0.1 Background

0.1.1 Homological methods in representation theory and algebraic geometry

The notion of *homology* is a tool to translate geometric information to linear algebraic data. It was introduced by Poincaré in 1895 to give a rigorous treatment of the Betti numbers [Poi1895]. Several algebraic techniques were developed to compute homology groups of topological spaces, and these influenced various areas of mathematics, such as algebraic geometry, representation theory and number theory. Cartan and Eilenberg unified these ad hoc techniques using *derived functors* and published the textbook *Homological Algebra* [CE56] in 1956. In the following year, Grothendieck introduced the notion of *abelian categories* to give an appropriate framework for homological algebra [Gro57]. The category $\mathsf{mod}\,\Lambda$ of finitely generated modules over a noetherian ring Λ and the category $\mathsf{coh}\,X$ of coherent sheaves on a noetherian scheme X are basic examples of abelian categories. In 1962, Gabriel, a student of Grothendieck, intensively studied abelian categories and obtained significant results in both algebraic geometry and representation theory (see §0.2.1 and 0.2.3 below). His works promoted abelian categories as an important research object, not just a framework of homological algebra.

On the other hand, Grothendieck attempted to construct a duality theory on arbitrary schemes (or, more generally, morphisms) around 1960 and realized that this required a more serious treatment of complexes of sheaves. Based on the vision of Grothendieck, Verdier developed the theory of *derived categories* and *triangulated categories*, which gives an appropriate framework to formulate a duality theory for arbitrary schemes [Ver67, Har66]. In short, the derived category $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category obtained from the category of chain complexes over $\mathcal{A}$ by formally inverting chain maps which induce isomorphisms on the homology. The derived category has the structure of a triangulated category, in which we can develop a homological algebra similarly as that of abelian categories. As in the case of abelian categories, triangulated categories had become an important research object that is worth more than the framework by various studies, for example, [Bei78, Muk81, Hap88]. Nowadays, the term *homological algebra* means the study of abelian categories and triangulated categories.

This thesis is devoted to *homological representation theory* and *homological algebraic geometry*. Representation theory studies how an algebraic object acts on vector spaces. Homological representation theory studies *categories* of actions of a fixed algebraic object, such as $\mathsf{mod}\,\Lambda$ and its bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ for an algebra Λ. Similarly, homological algebraic geometry studies *categories* related to an algebraic variety X such as $\mathsf{coh}\,X$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\,X)$.

The notion of *Grothendieck groups* is an effective tool to study these categories, which translates categorical information to linear algebraic data. We will give a short review of Grothendieck groups in the next part.

0.1.2 Grothendieck groups

The Grothendieck group $\mathsf{K}_0(\mathcal{A})$ is an abelian group defined for each skeletally small abelian category $\mathcal{A}$ (more generally, skeletally small *exact category*, a generalization of abelian categories which includes the category $\mathsf{vect}\, X$ of vector bundles over a noetherian scheme X as an example). We can associate to an object X of $\mathcal{A}$ an element $[X]$ of $\mathsf{K}_0(\mathcal{A})$, and the equality $[B] = [A] + [C]$ holds for any short exact sequence $0 \to A \to B \to C \to 0$. It was originally introduced by Grothendieck in a seminar at Princeton in 1957 [BS58]. He used it to formulate a relative version of Hirzebruch-Riemann-Roch theorem, which is now called Grothendieck-Riemann-Roch theorem. Atiyah and Hirzebruch mimicked Grothendieck's construction for topological vector bundles on a compact Hausdorff space X and proved a Riemann-Roch theorem for differentiable manifolds [AH59]. This work led many authors to study the structure of Grothendieck groups, and these studies eventually developed into K-theory. Since then, the Grothendieck group has been established as a basic and important invariant.

In homological representation theory, Grothendieck groups are used as a source of *Morita invariants.* Recall that two Artin algebras Λ and Γ are called *Morita equivalent* if $\mathsf{mod}\, \Lambda$ and $\mathsf{mod}\, \Gamma$ are equivalent as categories. The Grothendieck group $\mathsf{K}_0(\mathsf{mod}\, \Lambda)$ is a Morita invariant in the sense that $\mathsf{K}_0(\mathsf{mod}\, \Lambda) \cong \mathsf{K}_0(\mathsf{mod}\, \Gamma)$ as groups if Λ and Γ are Morita equivalent. If Λ is an Artin algebra, then $\mathsf{K}_0(\mathsf{mod}\, \Lambda)$ is a free $\mathbb{Z}$-module whose rank is equal to the number of isomorphism classes of simple Λ-modules. Thus the number of isomorphism classes of simple modules is also a Morita invariant.

Another important notion for Artin algebras is *derived equivalence*, which gives a more flexible framework to study Artin algebras. Recall that two Artin algebras Λ and Γ are derived equivalent if $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \Lambda)$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \Gamma)$ are triangulated equivalent. Since the derived category is constructed from the given abelian category, we see that Morita equivalence implies derived equivalence. In the study of derived equivalence, Grothendieck groups still give us great help, since they are defined for any skeletally small triangulated category $\mathcal{T}$. In fact, in the case $\mathcal{T} = \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ for a skeletally small abelian category $\mathcal{A}$, we have an isomorphism $\mathsf{K}_0(\mathcal{A}) \cong \mathsf{K}_0(\mathsf{D}^{\mathrm{b}}(\mathcal{A}))$. Thus, if two Artin algebras Λ and Γ are derived equivalent, then we have an isomorphism $\mathsf{K}_0(\mathsf{mod}\, \Lambda) \cong \mathsf{K}_0(\mathsf{mod}\, \Gamma)$. The argument so far shows that the Grothendieck group of an Artin algebra is not only a Morita invariant but also a *derived invariant.* In particular, the number of isomorphism classes of simple modules is also a derived invariant.

However, there is some criticism of the above argument. The fact that Grothendieck groups are derived invariants indicates that they do not fully reflect the information of abelian categories. For example, the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, \mathbb{P}^1)$ of the projective line $\mathbb{P}^1$ is triangulated equivalent to the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \Lambda)$ of the path algebra Λ of the Kronecker quiver $1 \leftleftarrows 2$. Thus, the Grothendieck groups cannot distinguish the abelian categories $\mathsf{coh}\, \mathbb{P}^1$ and $\mathsf{mod}\, \Lambda$. However, they are quite different, for instance, $\mathsf{coh}\, \mathbb{P}^1$ has infinitely many simple objects while $\mathsf{mod}\, \Lambda$ has exactly two simple objects up to isomorphisms. The Grothendieck groups lose even such basic information of abelian categories. The *Grothendieck monoids* are invariants that resolve these shortcomings, which will be explained in the following subsection.

0.1.3 Grothendieck monoids

The Grothendieck monoid $\mathsf{M}(\mathcal{E})$ is a natural monoid version of the Grothendieck group, which is defined for each skeletally small exact category $\mathcal{E}$. Several authors studied the Grothendieck monoid and extract information that the Grothendieck group does not contain. As far as the author knows, Grothendieck monoids were first studied in [Bro98] by Brookfield. He defined the Grothendieck monoid $\mathsf{M}(\mathsf{Mod}\, \Lambda)$ of the category $\mathsf{Mod}\, \Lambda$ of modules over a ring Λ to study the direct sum cancellation problem. The Grothendieck monoid of an exact category was first introduced in [BG16] as a monoid which gives a natural grading of the Ringel-Hall algebras of an exact category. Enomoto intensively studied the Grothendieck monoid of an exact category from the viewpoint of homological representation theory [Eno22]. In particular, he observed that the isomorphism classes of simple objects bijectively correspond to some distinguished elements of the Grothendieck monoid. As explained in the previous subsection, this information is lost in the Grothendieck group.

In this thesis, we define and study the Grothendieck monoid of an *extriangulated category*, which was recently introduced in [NP19] as a unification of abelian categories and triangulated categories. It includes extension-closed subcategories of abelian and triangulated categories as examples. We have the notion of *conflations* in an extriangulated category, which generalize both short exact sequences in an abelian category and exact triangles in a triangulated category. The Grothendieck monoid $\mathsf{M}(\mathcal{C})$ of an skeletally small extriangulated category $\mathcal{C}$ is a commutative monoid, and we can associate to an object X of $\mathcal{C}$ an element $[X]$ of $\mathsf{M}(\mathcal{C})$ such that $[B] = [A] + [C]$ for any conflation $A \to B \to C \dashrightarrow$. We can construct the Grothendieck group $\mathsf{K}_0(\mathcal{C})$ from the Grothendieck monoid $\mathsf{M}(\mathcal{C})$ by an operation on monoids called the group completion. Thus $\mathsf{M}(\mathcal{C})$ has more information of $\mathcal{C}$ than $\mathsf{K}_0(\mathcal{C})$.

We use the Grothendieck monoids of extriangulated categories to study *classifying subcategories*, *categorifications of monoid operations*, *reconstruction theorems* and *periodic derived invariants*.

0.2 Main results

In this section, we will explain the main results of this thesis without going into the set-theoretic details. The set-theoretic foundations of this thesis are given in §1.1, to which we refer the interested reader.

0.2.1 Classifying subcategories

Classifying nice subcategories of an abelian category or a triangulated category is quite an active subject in homological representation theory and homological algebraic geometry. It started from the following result of Gabriel:

Fact 0.2.1 ([Gab62, Proposition VI.2.4], see also [Kan12, Theorem 5.5])**.** *Let X be a noetherian scheme. There is an inclusion-preserving bijection between the following sets:*
- *The set of Serre subcategories $\mathcal{S}$ of the category $\mathsf{coh}\, X$ of coherent sheaves on X.*
- *The set of specialization-closed subsets Z of X.*

Here the assignments are given by

$$\mathcal{S} \mapsto \operatorname{Supp} \mathcal{S} := \bigcup_{\mathscr{F} \in \mathcal{S}} \operatorname{Supp} \mathscr{F}, \quad Z \mapsto \mathsf{coh}_Z\, X := \{\mathscr{F} \in \mathsf{coh}\, X \mid \operatorname{Supp} \mathscr{F} \subseteq Z\}.$$

As can be seen from Fact 0.2.1, a solution to the subcategory classification problem is given by constructing a bijection between the subcategories of interest and mathematical objects which are easier to understand. We will establish bijections between several subcategories of an extriangulated category and certain subsets of its Grothendieck monoid in Chapter 3 and 6. Using this, in Chapter 3 and 7, we will address classifications of Serre subcategories of some concrete *exact categories*, while most researchers so far have classified those of *abelian categories* such as the fact above.

We first establish a bijection between Serre subcategories of an extriangulated category and certain subsets of its Grothendieck monoid. An additive subcategory $\mathcal{S}$ of $\mathcal{C}$ is *Serre* if for any conflation $A \to B \to C \dashrightarrow$ in $\mathcal{C}$, we have $B \in \mathcal{S}$ if and only if both $A \in \mathcal{S}$ and $C \in \mathcal{S}$.

Theorem A (= Proposition 6.2.1, cf. [Sai2, Proposition 5.14], [ES, Proposition 3.5])**.** Let $\mathcal{C}$ be a skeletally small extriangulated category. Then there are bijections between the following sets:
1. (1) The set $\operatorname{Serre}(\mathcal{C})$ of Serre subcategories of $\mathcal{C}$.
2. (2) The set $\operatorname{Face} \mathsf{M}(\mathcal{C})$ of faces of $\mathsf{M}(\mathcal{C})$.
3. (3) The set $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$ of prime ideals of $\mathsf{M}(\mathcal{C})$.

Here
- a submonoid F of a monoid M is a *face* if for any $a, b \in M$, we have that $a + b \in F$ if and only if both $a \in F$ and $b \in F$,
- a proper subset $\mathfrak{p}$ of a monoid M is a *prime ideal*[1] if it satisfies (i) $x + a \in \mathfrak{p}$ for any $x \in \mathfrak{p}$ and $a \in M$ and (ii) $a + b \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ for any $a, b \in M$.

The bijection between the first and second sets generalizes [Bro97, Proposition 16.8] for module categories (see Remark 3.3.7). The second set $\operatorname{Face}(\mathsf{M}(\mathcal{C}))$ can be computed purely algebraically, and its computation is much easier than examining the whole structure of the extriangulated category $\mathcal{C}$. The third set $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$ has a topology, which is a natural analogue of the Zariski topology on the spectrum $\operatorname{Spec} R$ of a commutative ring R. These lead us in two directions.

The one direction is the classification of Serre subcategories using faces of the Grothendieck monoid. We propose the following strategy to classify Serre subcategories of an extriangulated category $\mathcal{C}$.

[1] Note that we treat the empty set as a prime ideal since it corresponds to $\mathcal{C}$ itself by the bijection between (1) and (3).

(i) Relate the Grothendieck monoid $\mathsf{M}(\mathcal{C})$ with an abstract monoid M.
(ii) Classify faces of the abstract monoid M.
(iii) Classify Serre subcategories of $\mathcal{C}$ by using (i) and (ii).

We apply this strategy to exact categories related to finite dimensional algebras and noetherian schemes. In particular, we compute the Grothendieck monoids of the following exact categories related to a smooth projective curve C and classify Serre subcategories of them:

- The category $\mathsf{coh}\, C$ of coherent sheaves on C.
- The category $\mathsf{vect}\, C$ of vector bundles on C.
- The category $\mathsf{tor}\, C$ of coherent torsion sheaves on C.

Theorem B (= Propositions 7.1.4, 7.2.4 and 7.3.2, cf. [Sai2, Propositions 4.4, 4.9 and 4.12])**.** *Let C be a smooth projective curve over an algebraically closed field $\Bbbk$.*
(1) $\mathsf{M}(\mathsf{tor}\, C) \cong \operatorname{Div}^+ C$ *holds, where* $\operatorname{Div}^+ C$ *is the monoid of effective divisors on* C.
(2) $\mathsf{M}(\mathsf{vect}\, C) \cong (\operatorname{Pic} C \times \mathbb{N}^+) \cup \{(\mathscr{O}_C, 0)\} \subseteq \operatorname{Pic} C \times \mathbb{Z}$ *holds, where* $\operatorname{Pic}(C)$ *is the Picard group of* C *and* $\mathbb{N}^+$ *is the semigroup of strictly positive integers.*
(3) *We can regard* $\mathsf{M}(\mathsf{tor}\, C)$ *and* $\mathsf{M}(\mathsf{vect}\, C)$ *as submonoids of* $\mathsf{M}(\mathsf{coh}\, C)$*. Then* $\mathsf{M}(\mathsf{coh}\, C)$ *is the disjoint union of* $\mathsf{M}(\mathsf{tor}\, C)$ *and* $\mathsf{M}(\mathsf{vect}\, C)^+ := \mathsf{M}(\mathsf{vect}\, C) \setminus \{0\}$*. See Corollary 7.3.8 for the complete description of* $\mathsf{M}(\mathsf{coh}\, C)$ *as a monoid.*

Theorem C (= Corollary 7.2.5, cf. [Sai2, Corollary 4.10])**.** *Let C be a smooth projective curve over an algebraically closed field $\Bbbk$. Then $\mathsf{vect}\, C$ has no nontrivial Serre subcategories.*

The other direction is the study of the space $\operatorname{Serre}(\mathcal{C})$ whose topology is induced by the topology on $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$. We classify finitely generated Serre subcategories via this topology. A Serre subcategory $\mathcal{S}$ of $\mathcal{C}$ is *finitely generated* if it is generated by some single object.

Theorem D (= Proposition 6.2.6, cf. [Sai2, Proposition 5.19])**.** *Let $\mathcal{C}$ be a skeletally small extriangulated category. Then there is a bijection between the following two sets:*
(1) *The set of finitely generated Serre subcategories of $\mathcal{C}$.*
(2) *The set of nonempty strongly quasi-compact open subsets of $\operatorname{Serre}(\mathcal{C})$.*

Here, a topological space X is *strongly quasi-compact* if for every open covering $\{U_i\}_{i\in I}$ of X, there exists $i \in I$ such that $X = U_i$.

Next, we establish a classification of *dense 2-out-of-3 subcategories* of $\mathcal{C}$, that is, an additive subcategory $\mathcal{D}$ of $\mathcal{C}$ satisfying $\mathsf{add}\, \mathcal{D} = \mathcal{C}$ and the 2-out-of-3 property for conflations.

Theorem E (= Theorem 3.6.6, cf. [ES, Theorem 3.14, Corollary 3.17])**.** *Let $\mathcal{C}$ be a skeletally small extriangulated category. Then there are bijections between the following sets:*
(1) *The set of dense 2-out-of-3 subcategories of $\mathcal{C}$.*
(2) *The set of cofinal subtractive submonoid of $\mathsf{M}(\mathcal{C})$ (see Definition 3.5.1).*
(3) *The set of subgroups of $\mathsf{K}_0(\mathcal{C})$ such that $\rho^{-1}(H)$ is cofinal in $\mathsf{M}(\mathcal{C})$, where $\rho\colon \mathsf{M}(\mathcal{C}) \to \mathsf{K}_0(\mathcal{C})$ is the natural map.*

Applying Theorem E to a triangulated category, we immediately obtain Thomason's classification [Tho97] of dense triangulated subcategories via subgroups of the Grothendieck group (Corollary 3.6.7). In [Mat18], Matsui classified dense resolving subcategories of an exact category *containing a generator* via certain subgroups of the Grothendieck group (see also [ZZ21]). Applying Theorem E to an exact category, we obtain the classification of *all* dense resolving subcategories, which generalizes [Mat18, ZZ21] (Corollary 3.6.10).

0.2.2 Categorifications of monoid operations

A *categorification* is a method of considering a given algebraic object as an invariant of a category and studying the algebraic object at the level of the category. This method enables us to simplify complicated algebraic relations from the eye of category theory and get new insights for the category from operations on the algebraic object. We address a categorification of two basic monoid operations, *quotient* and *localization*, via Grothendieck monoids in Chapter 4 and 5.

We first study the Grothendieck monoid of the *exact localization* of Nakaoka–Ogawa–Sakai [NOS22] to categorify quotient of monoids. An exact localization $\mathcal{C}/\mathcal{N}$ of an extriangulated category $\mathcal{C}$ by an extension-closed subcategory $\mathcal{N}$ of $\mathcal{C}$ is (if exists) the universal extriangulated category which sends $\mathcal{N}$ to 0 (Proposition 4.2.3). Typical examples are Serre quotients of abelian categories and Verdier quotients of triangulated categories. Under certain assumptions, where an exact localization exists, we show that this localization gives a kind of categorification of the quotient of monoids in the following sense.

Theorem F (= Corollary 4.3.12, cf. [ES, Corollary 4.28])**.** Let $\mathcal{C}$ be a skeletally small extriangulated category and $\mathcal{N}$ an extension-closed subcategory of $\mathcal{C}$. Under some conditions (see Corollary 4.3.12), we have the following coequalizer diagram of commutative monoids:

$$\mathsf{M}(\mathcal{N}) \underset{0}{\overset{\mathsf{M}(\iota)}{\rightrightarrows}} \mathsf{M}(\mathcal{C}) \xrightarrow{\mathsf{M}(Q)} \mathsf{M}(\mathcal{C}/\mathcal{N}),$$

where $\iota\colon \mathcal{N} \hookrightarrow \mathcal{C}$ and $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ are the inclusion and the localization functor respectively. In particular, $\mathsf{M}(\mathcal{C}/\mathcal{N})$ is isomorphic to the quotient monoid $\mathsf{M}(\mathcal{C})/\operatorname{Im}\mathsf{M}(\iota)$ (see Definition 4.1.1). This can be applied to the following cases.

- (i) $\mathcal{C}$ is a triangulated category and $\mathcal{N}$ is a thick subcategory of $\mathcal{C}$. In this case, $\mathcal{C}/\mathcal{N}$ is the usual Verdier quotient of a triangulated category.
- (ii) $\mathcal{C}$ is a Frobenius category and $\mathcal{N}$ is the subcategory of all projective objects in $\mathcal{C}$. In this case, $\mathcal{C}/\mathcal{N}$ is the usual (triangulated) stable category.
- (iii) $\mathcal{C}$ is an abelian category and $\mathcal{N}$ is a Serre subcategory of $\mathcal{C}$. In this case, $\mathcal{C}/\mathcal{N}$ is the usual Serre quotient of an abelian category.

Moreover, if $\mathcal{N}$ is a Serre subcategory (e.g. (iii)), then $\mathsf{M}(\iota)\colon \mathsf{M}(\mathcal{N}) \to \mathsf{M}(\mathcal{C})$ is an injection.

The above coequalizer diagram immediately yields the right exact sequence of the Grothendieck group (Corollary 4.3.16):

$$\mathsf{K}_0(\mathcal{N}) \longrightarrow \mathsf{K}_0(\mathcal{C}) \longrightarrow \mathsf{K}_0(\mathcal{C}/\mathcal{N}) \longrightarrow 0.$$

This unifies the well-known results for the abelian case and the triangulated case. Moreover, the injectivity of $\mathsf{M}(\mathcal{N}) \to \mathsf{M}(\mathcal{C})$ for a Serre subcategory $\mathcal{N}$ is quite remarkable, since it fails for the Grothendieck groups (see e.g. Example 3.3.6).

Next, we address a categorification of the *monoid localization*, which makes certain elements of a monoid invertible (see Definition 5.1.1). For this purpose, we study *intermediate subcategories* of the derived category in detail. Let $\mathcal{A}$ be a skeletally small abelian category and $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. Then a subcategory $\mathcal{C}$ of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is called an *intermediate subcategory* if $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{A}[1] * \mathcal{A}$ holds and $\mathcal{C}$ is closed under extensions and direct summands. We show in Theorem 5.2.3 that there is a bijection between torsionfree classes $\mathcal{F}$ in $\mathcal{A}$ and intermediate subcategories $\mathcal{C}$ of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, where the corresponding intermediate subcategory is given by $\mathcal{F}[1] * \mathcal{A}$. This is illustrated as follows.

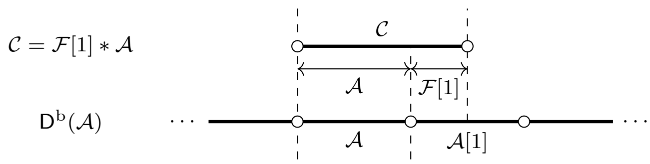


Figure 0.2.1: An intermediate subcategory

We show that the Grothendieck monoid of an intermediate subcategory $\mathcal{F}[1] * \mathcal{A}$ can be described using a monoid localization (see Definition 5.1.1) as follows.

Theorem G (= Theorem 5.3.1, cf. [ES, Theorem 5.4])**.** Let $\mathcal{A}$ be a skeletally small abelian category and $\mathcal{F}$ a torsionfree class in $\mathcal{A}$. Then $\mathsf{M}(\mathcal{F}[1] * \mathcal{A})$ is isomorphic to the monoid localization $\mathsf{M}(\mathcal{A})_{\mathsf{M}_{\mathcal{F}}}$ of $\mathsf{M}(\mathcal{A})$ with respect to a subset $\mathsf{M}_{\mathcal{F}} := \{[F] \mid F \in \mathcal{F}\} \subseteq \mathsf{M}(\mathcal{A})$.

This enables us to compute $\mathsf{M}(\mathcal{F}[1]*\mathcal{A})$ in concrete situations. Also, this can be interpreted as follows: *the inclusion of a subcategory* $\mathcal{A} \hookrightarrow \mathcal{F}[1]*\mathcal{A}$ *categorifies the monoid localization* $\mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A})_{\mathsf{M}_{\mathcal{F}}}$ in the sense that by applying the functor $\mathsf{M}(-)\colon \mathsf{ETCat} \to \mathsf{Mon}$ we obtain the monoid localization. Moreover, we show that *all the monoid localization of* $\mathsf{M}(\mathcal{A})$ *appear in this way* (Remark 5.3.3). This result explains why it is natural to study Grothendieck monoids in the generality of extriangulated categories. The

Grothendieck monoid of an exact category is reduced, that is, 0 is the only invertible element, while that of a triangulated category is a group, that is, every element is invertible. In particular, most of the monoid localizations of the Grothendieck monoid do not appear as the Grothendieck monoid if we only consider exact categories and triangulated categories. Thus, to realize a localization as the Grothendieck monoid, we have to consider extriangulated categories which are neither abelian nor triangulated.

Finally, we consider Serre subcategories of $\mathcal{C} := \mathcal{F}[1] * \mathcal{A}$ and the behavior of an exact localization of $\mathcal{C}$ in detail:

Theorem H (= Proposition 5.4.1, Theorem 5.4.4, cf. [ES, Proposition 5.10, Theorem 5.13])**.** Let $\mathcal{A}$ be a skeletally small abelian category and $\mathcal{F}$ a torsionfree class in $\mathcal{A}$, and put $\mathcal{C} := \mathcal{F}[1] * \mathcal{A}$.

(1) The assignment $\mathcal{S} \mapsto \mathcal{F}[1] * \mathcal{S}$ gives a bijection between the set of Serre subcategories $\mathcal{S}$ of $\mathcal{A}$ containing $\mathcal{F}$ and the set of Serre subcategories of $\mathcal{C}$.

(2) Let $\mathcal{S}$ be a Serre subcategory of $\mathcal{A}$ containing $\mathcal{F}$. Then we can apply Theorem F to an exact localization $\mathcal{C}/(\mathcal{F}[1] * \mathcal{S})$, and we have an exact equivalence $\mathcal{A}/\mathcal{S} \simeq \mathcal{C}/(\mathcal{F}[1] * \mathcal{S})$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A} & \hookrightarrow & \mathcal{F}[1] * \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{A}/\mathcal{S} & \xrightarrow{\sim} & (\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S}). \end{array} \tag{0.2.1}$$

0.2.3 Reconstruction theorems

In 1962, Gabriel proved that any noetherian scheme X can be reconstructed from the category $\mathsf{Qcoh}\, X$ of quasi-coherent sheaves on X [Gab62]. This result indicates that the geometric object X and the abelian category $\mathsf{Qcoh}\, X$ are equivalent. It is a starting point of *noncommutative algebraic geometry* which studies geometric objects through categories. Inspired by the Gabriel's work, several authors addressed the reconstruction problem of schemes from some specified categories. For example, Buan, Krause and Solberg recovered a noetherian scheme X from the abelian category $\mathsf{coh}\, X$ [BKS07]. Bondal and Orlov showed that a smooth projective variety X can be reconstructed from the triangulated category $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, X)$ if X has ample or anti-ample canonical bundle [BO01]. Balmer reconstructed a noetherian scheme X from its perfect derived category $\mathsf{perf}\, X$ with the tensor triangulated structure [Bal05]. Note that $\mathsf{perf}\, X \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, X)$ as triangulated categories when X is regular. We will give a kind of reconstruction of noetherian schemes from the Grothendieck monoids in Chapter 8.

We recover the topology of a noetherian scheme X from the spectrum of Grothendieck monoid $\mathsf{M}(\mathsf{coh}\, X)$ and prove the following theorem.

Theorem I (= Theorem 8.0.4, cf. [Sai2, Theorem 5.27])**.** Consider the following conditions for noetherian schemes X and Y.

(1) $X \cong Y$ as schemes.
(2) $\mathsf{M}(\mathsf{coh}\, X) \cong \mathsf{M}(\mathsf{coh}\, Y)$ as monoids.
(3) $X \cong Y$ as topological spaces.

Then "(1) $\Rightarrow$ (2) $\Rightarrow$ (3)" hold.

The nontrivial part is, of course, the implication "(2) $\Rightarrow$ (3)". It is surprisingly enough because the Grothendieck monoid $\mathsf{M}(\mathsf{coh}\, X)$ loses a lot of information and the Grothendieck group $\mathsf{K}_0(\mathsf{coh}\, X)$ never recovers the topology of X as the following examples show.

Example 0.2.2. Let $\Bbbk$ be an algebraic closed field.

(1) Let R be a finite dimensional commutative $\Bbbk$-algebra. Then $\mathsf{M}(\mathsf{mod}\, R) \cong \mathbb{N}^{\oplus n}$, where n is the number of maximal ideals of R (see Example 3.4.5). In particular, if R is local, then we have $\mathsf{M}(\mathsf{mod}\, R) \cong \mathbb{N}$.
Consider a finite dimensional commutative local $\Bbbk$-algebra $R := \Bbbk[x, y]/(x^2, xy, y^2)$. For any $\lambda \in \Bbbk$, define an $\Bbbk$-algebra homomorphism $\phi_\lambda \colon R \to M_2(\Bbbk)$ to the matrix algebra $M_2(\Bbbk)$ of degree 2 by

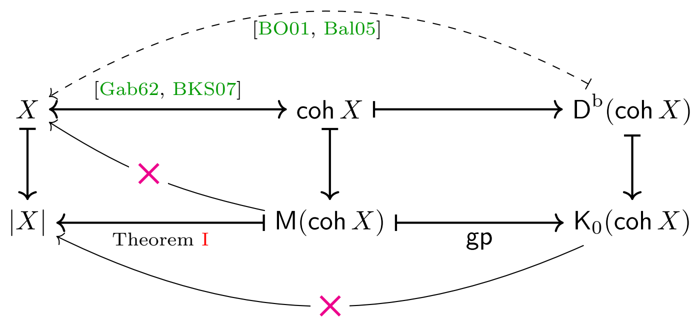
$$\phi_\lambda(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \phi_\lambda(y) = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}.$$

Then ϕ_λ defines a 2-dimensional R-module M_λ. We can easily see that M_λ is indecomposable and $M_\lambda \not\cong M_\mu$ if $\lambda \neq \mu$. Thus, $\mathsf{mod}\, R$ has infinitely many indecomposable objects. On the other hand, $\mathsf{mod}\, \Bbbk$ has exactly one indecomposable object $\Bbbk$. Hence, $\mathsf{mod}\, R$ and $\mathsf{mod}\, \Bbbk$ are very different. Despite this, by the fact above, we have $\mathsf{M}(\mathsf{mod}\, \Bbbk) \cong \mathbb{N} \cong \mathsf{M}(\mathsf{mod}\, R)$. This shows that Grothendieck monoids lose a lot of categorical information.

(2) Let $\mathbb{P}^1$ be the projective line over $\Bbbk$. Then we have $\mathsf{K}_0(\mathsf{coh}\, \mathbb{P}^1) \cong \mathbb{Z}^{\oplus 2}$ (see Example 7.3.5). On the other hand, we have $\mathsf{K}_0(\mathsf{coh}(\operatorname{Spec}(\Bbbk \times \Bbbk))) \cong \mathsf{K}_0(\mathsf{mod}(\Bbbk \times \Bbbk)) \cong \mathbb{Z}^{\oplus 2}$. Hence $\mathbb{P}^1 \not\cong \operatorname{Spec}(\Bbbk \times \Bbbk)$ as topological spaces, but $\mathsf{K}_0(\mathsf{coh}\, \mathbb{P}^1) \cong \mathsf{K}_0(\mathsf{coh}(\operatorname{Spec}(\Bbbk \times \Bbbk)))$ as groups. This shows that "(2) $\Rightarrow$ (3)" of Theorem I becomes false if Grothendieck monoids are replaced by Grothendieck groups.

Example 0.2.2 (1) is also a counter example of "(2) $\Rightarrow$ (1)" of Theorem I. In Example 8.0.5, we will give a counter example of "(3) $\Rightarrow$ (2)" of Theorem I.

The argument so far is illustrated as follows:



Here
- X is a noetherian scheme,
- $|X|$ is the underlying topological space of X,
- $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, X)$ is the bounded derived category of $\mathsf{coh}\, X$,
- gp is the group completion (see Definition 2.1.5 and Remark 2.3.6),
- the thick arrow $A \mapsto B$ indicates that B can be constructed from A,
- the arrow $A \to B$ marked with a cross indicates that B cannot be recovered from A (see Example 0.2.2),
- the dashed arrow indicates that some assumption or additional structure is needed.

0.2.4 Periodic derived invariants

Fix a positive integer m. The *m-periodic derived category* $\mathsf{D}_m(\mathcal{A})$ of an abelian category $\mathcal{A}$ is a natural $\mathbb{Z}/m\mathbb{Z}$-periodic analogue of the usual derived category. Two Artin algebras Λ and Γ are *m-periodic derived equivalence* if $\mathsf{D}_m(\mathsf{mod}\, \Lambda) \simeq \mathsf{D}_m(\mathsf{mod}\, \Gamma)$ as triangulated categories. The notion of 2-periodic derived category was first introduced by Peng and Xiao [PX97] to construct a categorification of the full part of a semisimple Lie algebra via a 2-periodic version of Ringel-Hall algebra, which was based on Ringel's construction of the half of the quantum group via his Hall algebra. Inspired by this work, Bridgeland used the 2-periodic derived category of a hereditary algebra to construct the full quantum group of a symmetric Kac-Moody Lie algebra [Bri13]. Motivated by these studies, several authors analyzed the structure of m-periodic derived categories [Fu12, Gor, Zha14, Sta18]. Recently, the author studied periodic derived categories from the viewpoint of homological representation theory and developed *periodic tilting theory* which describe a way to relate *periodic triangulated categories* with periodic derived categories of Artin algebras [Sai1]. As a sequel of this work, we will study *periodic derived invariants* of Artin algebras in Chapter 9.

We first compute the Grothendieck group of the periodic derived categories. Remark that the Grothendieck monoid of a triangulated category coincides with the Grothendieck group of it (Proposition 2.4.3). Thus, the study of the Grothendieck groups of triangulated categories is a part of that of the Grothendieck monoids of them.

Theorem J (= Theorem 9.3.1, cf. [Sai22, Theorem 1.1])**.** Let $\mathcal{A}$ be a skeletally small abelian category with enough projectives. Suppose that the global dimension of $\mathcal{A}$ is finite. Then we have an isomorphism

$$\mathsf{K}_0(\mathsf{D}_m(\mathcal{A})) \cong \begin{cases} \mathsf{K}_0(\mathcal{A}) & \text{if } m \text{ is even,} \\ \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2} := \mathsf{K}_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \text{if } m \text{ is odd,} \end{cases}$$

which is induced by the natural functor $\mathcal{A} \to \mathsf{D}_m(\mathcal{A})$. Here $\mathbb{F}_2$ is the finite field of two elements.

As an immediate corollary of this, we can conclude that the number of isomorphism classes of simple modules is a periodic derived invariant. That is, if Artin algebras are periodic derived equivalent, then the numbers of isomorphism classes of simple modules over them are coincide. We also obtain the following result which is an original motivation for this study. See §9.3 for precise meaning and significance of this theorem.

Theorem K (= Corollary 9.3.6, cf. [Sai22, Corollary 1.6])**.** Let m be a positive integer, and $\mathcal{T}$ be an idempotent complete algebraic m-periodic triangulated category over a perfect field $\Bbbk$. Suppose that $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is finite dimensional over $\Bbbk$ for all objects $X, Y \in \mathcal{T}$. Then the number of non-isomorphic summands of a strict periodic tilting object is constant.

0.3 Organization

This thesis divides into two parts. In Part I, we develop a general theory of Grothendieck monoids of extriangulated categories (Chapters 2–6). In Part II, we give applications of Grothendieck monoids to representation theory and algebraic geometry (Chapters 7–9). In more detail, this thesis is organized as follows.

In Chapter 1, we confirm the premises of this thesis. In §1.1, we give a precise formulation of the content of this paper by using the Grothendieck universe. In §1.2, we fix the notations and conventions that will be used throughout this thesis.

In Chapter 2, we introduce the main subject in this thesis the *Grothendieck monoid* of an extriangulated category. In §2.1 and 2.2, we recall the basics of commutative monoids and extriangulated categories which will be used throughout this thesis. In §2.3, we define the Grothendieck monoids by universal property and then construct it. We also explain the relationship between Grothendieck monoids and groups. In §2.4, we explain the special future of the Grothendieck monoids of a triangulated category and an exact category, respectively. We especially show that the Grothendieck monoid of a triangulated category coincides with its Grothendieck group.

In Chapter 3, we classify several subcategories of an extriangulated category via its Grothendieck monoid. In §3.1, we introduce the key notion of *c-closed subcategories* which are the largest class of subcategories that can be classified via the Grothendieck monoid. In §3.3 and 3.6, we classify Serre subcategories (Theorem A) and dense 2-out-of-3 subcategories (Theorem E) via the Grothendieck monoid by showing that they are c-closed.

In Chapter 4, we show that the localization of an extriangulated category "commutes with the monoid quotients". In §4.1, we recall the quotient of monoids, which is a natural analogue of that of abelian groups. In §4.2, we review the exact localization of an extriangulated category which was introduced recently in [NOS22]. In §4.3, we prove Theorem F. The saturatedness of the exact localization is the key observation of the proof.

In Chapter 5, we study intermediate subcategories of the derived category in detail, which also gives a concrete example of the theory developed in Chapter 3 and 4 for an extriangulated category which is neither abelian nor triangulated. In §5.1, we give a review on monoid localization, which makes certain elements of a monoid invertible. In §5.2, we introduce intermediate subcategories which are contained in the derived category of an abelian category. We give a bijective correspondence between them and torsionfree classes of the abelian category. In §5.3, we compute the Grothendieck monoid of an intermediate category and obtain a categorification of the monoid localization (Theorem G). In §5.4, we study Serre subcategories of an intermediate subcategory. We classify Serre subcategories of it and describe the behavior of the exact localization of an intermediate subcategory by its Serre subcategory (Theorem H).

In Chapter 6, we study the monoid spectrum $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$ of the Grothendieck monoid of an extriangulated category $\mathcal{C}$. In §6.1, we recall the monoid spectrum of a commutative monoid, which is an analogue of the spectrum of a commutative ring. In §6.2, we classify finitely generated Serre subcategories of an extriangulated category via the topology of the monoid spectrum of its Grothendieck monoid (Theorem D). We also study the sheaf of monoids on the monoid spectrum of the Grothendieck monoid of an abelian category. We describe it by using the abelian quotient category by a Serre subcategory.

In Chapter 7, we compute the Grothendieck monoids of exact categories related to a smooth projective curve and classify Serre subcategories of them (Theorems B and C).

In Chapter 8, we recover the topology of a noetherian scheme from the Grothendieck monoid of the category of coherent sheaves on it by using the topology of the monoid spectrum (Theorem I).

In Chapter 9, we compute the Grothendieck group of the periodic derived category (Theorem J) and explain an application to periodic tilting theorem (Theorem K).

We explain the relationships between this thesis and our other papers. Chapters 2–5 are based on the collaboration paper [ES] with H. Enomoto. Chapters 6–8 are based on the author's preprint [Sai2]. Chapter 9 is based on the author's paper [Sai22].

0.4 Acknowledgement

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Chapter 1

Preliminaries

This chapter aims to confirm the premises of this thesis and fix notations and conventions throughout this thesis.

In §1.1, we first recall that the definition of categories within the ZFC set theory. Next, we introduce the notion of Grothendieck universes and some finiteness conditions for categories. The basic setup of this paper is then described (see Convention 1.1.12). This setup is important to formulate precisely the operation of taking the Grothendieck monoid as a functor from the category of skeletally small extriangulated categories to the category of monoids (see §2.3 in the next chapter). We also discuss that some categories are independent of the choice of Grothendieck universes, up to equivalences (see Propositions 1.1.16 and 1.1.17). It guarantees that the Grothendieck monoids of these categories do not depend on the choice of Grothendieck universes (see Proposition 2.3.5 in the next chapter).

1.1 Foundations for category theory

In this section, we will discuss the foundations for category theory which we use. References for this section are [Kun09, SGA4-1, Mac98, DHKS04, KS06].

In this thesis, we entirely work in the ZFC set theory and assume the Grothendieck's axiom of universe **(UA)**: for every set x, there exists a Grothendieck universe $\mathbb{U}$ such that $x \in \mathbb{U}$. We abbreviate the ZFC set theory with **(UA)** as the *ZFCU set theory*. We use the following notations:

- $\emptyset$ denotes the empty set.
- $\mathsf{P}(x)$ denotes the power set of a set x.
- $\{x, y\}$ denotes the pair of sets x and y.
- $(x_1, \dots, x_n)$ denotes the ordered tuple of sets $x_1, \dots, x_n$.
- $\bigcup x$ denotes the union of elements of a set x.

Categories are always considered within the ZFC set theory. To describe the precise meaning of this, we recall the definition of categories.

Definition 1.1.1. A *category* is an ordered tuple $\mathcal{C} = (\mathsf{Ob}\,\mathcal{C}, \mathsf{Mor}\,\mathcal{C}, \mathsf{dom}, \mathsf{codom}, e, m)$ consisting of the following:

- Sets $\mathsf{Ob}\,\mathcal{C}$ and $\mathsf{Mor}\,\mathcal{C}$. We call an element of $\mathsf{Ob}\,\mathcal{C}$ an *object* and an element of $\mathsf{Mor}\,\mathcal{C}$ a *morphism*.
- Maps $\mathsf{dom}, \mathsf{codom} \colon \mathsf{Mor}\,\mathcal{C} \to \mathsf{Ob}\,\mathcal{C}$. We denote by $f \colon X \to Y$ a morphism f such that $\mathsf{dom}(f) = X$ and $\mathsf{codom}(f) = Y$.
- A map $e \colon \mathsf{Ob}\,\mathcal{C} \to \mathsf{Mor}\,\mathcal{C}$. We write $\mathsf{id}_X := e(X)$ for any object X and call it the *identity morphism* of X.
- A map $m \colon \mathsf{Comp}_2\,\mathcal{C} \to \mathsf{Mor}\,\mathcal{C}$ from $\mathsf{Comp}_2\,\mathcal{C} := \{(f, g) \in \mathsf{Mor}\,\mathcal{C} \times \mathsf{Mor}\,\mathcal{C} \mid \mathsf{codom}(f) = \mathsf{dom}(g)\}$. Morphisms f and g are said to be *composable* if $(f, g) \in \mathsf{Comp}_2\,\mathcal{C}$. In this case, we write $gf := g \circ f := m(f, g)$ and call it the *composition* of f and g.

These are subject to the following conditions:

(C1) For any composable morphisms f and g, we have $\mathsf{dom}(gf) = \mathsf{dom}(f)$ and $\mathsf{codom}(gf) = \mathsf{codom}(g)$.
(C2) For any morphism $f \colon X \to Y$, we have $f \circ \mathsf{id}_X = f = \mathsf{id}_Y \circ f$.
(C3) For any $(f, g), (g, h) \in \mathsf{Comp}_2\,\mathcal{C}$, we have $h(gf) = (hg)f$.

Let $\mathcal{C}$ be a category. We simply write $X \in \mathcal{C}$ instead of $X \in \mathsf{Ob}\,\mathcal{C}$. For a subset S of $\mathsf{Mor}\,\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, we define $S(X,Y) := \{f \in S \mid \mathsf{dom}(f) = X, \mathsf{codom}(f) = Y\}$. We simply write $\mathcal{C}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y) := (\mathsf{Mor}\,\mathcal{C})(X,Y)$.

In our definition of categories, we cannot consider the category of "all sets" since there is no set of all sets. To deal with this problem, we will introduce the notion of Grothendieck universes and consider the category of sets within a fixed Grothendieck universe.

Definition 1.1.2. A set $\mathbb{U}$ is said to be a *Grothendieck universe* if it satisfies the following properties:
(U1) If $x \in \mathbb{U}$ and $y \in x$, then $y \in \mathbb{U}$.
(U2) If $x, y \in \mathbb{U}$, then $\{x, y\} \in \mathbb{U}$.
(U3) If $x \in \mathbb{U}$, then $\mathsf{P}(x) \in \mathbb{U}$.
(U4) If $f \colon I \to \mathbb{U}$ is a map with $I \in \mathbb{U}$, then $\bigcup_{i \in I} f(i) \in \mathbb{U}$.
(U5) The first infinite ordinal ω is an element of $\mathbb{U}$[1].

As described in the beginning of this section, we assume the *Grothendieck's axiom of universe* **(UA)**: for every set x, there exists a universe $\mathbb{U}$ such that $x \in \mathbb{U}$.

Let $\mathbb{U}$ be a Grothendieck universe. An element of $\mathbb{U}$ is called a $\mathbb{U}$-*set*. A set is called $\mathbb{U}$-*small* if it is isomorphic to a $\mathbb{U}$-set. A subset of $\mathbb{U}$ is called a $\mathbb{U}$-*class*.

Remark 1.1.3. Let $\mathbb{U}$ be a Grothendieck universe. The following immediately follow from Definition 1.1.2.

- A subset of a $\mathbb{U}$-set is also a $\mathbb{U}$-set.
- For any sets a and b, the ordered pair (a, b) (resp. the set $\mathsf{Map}(a, b)$ of maps from a to b) is a $\mathbb{U}$-set if and only if both a and b are $\mathbb{U}$-sets.
- For any set x, the power set $\mathsf{P}(x)$ is a $\mathbb{U}$-set if and only if x is a $\mathbb{U}$-set.
- For any family $\{x_i\}_{i \in I}$ of sets, it is a $\mathbb{U}$-set if and only if $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for any $i \in I$.
- For any family $\{x_i\}_{i \in I}$ of sets, if it is a $\mathbb{U}$-set, then so are their product $\prod_{i \in I} x_i$ and their disjoint union $\coprod_{i \in I} x_i$.
- If x is a $\mathbb{U}$-set, then so is the union $\bigcup x$ of elements of x.
- If x is a $\mathbb{U}$-set and R is an equivalence relation on x, then the quotient set x/R is also a $\mathbb{U}$-set.
- If there exists a surjective map $f \colon x \to y$ from a $\mathbb{U}$-set x to a $\mathbb{U}$-class y, then y is also a $\mathbb{U}$-set.
- A $\mathbb{U}$-small $\mathbb{U}$-class is a $\mathbb{U}$-set.

We now introduce some hierarchy of categories using a Grothendieck universe.

Definition 1.1.4. Let $\mathbb{U}$ be a Grothendieck universe, and let $\mathcal{C}$ be a category.

(1) $\mathcal{C}$ is called a $\mathbb{U}$-*category* if $\mathcal{C} = (\mathsf{Ob}\,\mathcal{C}, \mathsf{Mor}\,\mathcal{C}, \mathsf{dom}, \mathsf{codom}, e, m) \in \mathbb{U}$.
(2) $\mathcal{C}$ is called a *locally* $\mathbb{U}$-*set category* if $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathbb{U}$ for any $X, Y \in \mathcal{C}$.
(3) $\mathcal{C}$ is called a $\mathbb{U}$-*small category* if $\mathsf{Ob}\,\mathcal{C}$ and $\mathsf{Mor}\,\mathcal{C}$ are $\mathbb{U}$-small.
(4) $\mathcal{C}$ is called a *locally* $\mathbb{U}$-*small category* if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is $\mathbb{U}$-small for any $X, Y \in \mathcal{C}$.
(5) $\mathcal{C}$ is said to be $\mathbb{U}$-*moderate* if $\mathsf{Ob}\,\mathcal{C} \subseteq \mathbb{U}$.
(6) $\mathcal{C}$ is said to be *skeletally* $\mathbb{U}$-*small* if it is locally $\mathbb{U}$-small and the set $|\mathcal{C}|$ of isomorphism classes of objects of $\mathcal{C}$ is $\mathbb{U}$-small.

Remark 1.1.5. Let $\mathbb{U}$ be a Grothendieck universe. We describe the relationship between our terminology and that of other literature.

[SGA4-1]:
- $\mathbb{U}$-*petite* in [SGA4-1] is $\mathbb{U}$-small in our terminology.
- A $\mathbb{U}$-*catégorie* in [SGA4-1] is a locally $\mathbb{U}$-small category in our terminology.

[Mac98]:
- A *small set* in [Mac98] is a $\mathbb{U}$-set in our terminology.
- A *small category* in [Mac98] is a $\mathbb{U}$-category in our terminology.
- A *category with small hom-sets* in [Mac98] is a locally $\mathbb{U}$-set category in our terminology.

[DHKS04]:
- A $\mathbb{U}$-*category* in [DHKS04] is a $\mathbb{U}$-moderate locally $\mathbb{U}$-set category in our terminology.
- A *small* $\mathbb{U}$-*category* in [DHKS04] is a $\mathbb{U}$-category in our terminology.

[1] Some literatures do not include this condition in the definition of Grothendieck universes. This condition excludes that the first infinite ordinal ω is a Grothendieck universe.

[KS06]:

- A $\mathbb{U}$-*category* in [KS06] is a locally $\mathbb{U}$-small category in our terminology.
- A $\mathbb{U}$-*small category* in [KS06] is a $\mathbb{U}$-small category in our terminology.
- *Essentially* $\mathbb{U}$-*small* in [KS06] is skeletally $\mathbb{U}$-small in our terminology.

We describe the relationship between the concepts introduced in Definition 1.1.4. We omit the proofs since it is straightforward.

Remark 1.1.6. Let $\mathbb{U}$ be a Grothendieck universe.

(1) The following are equivalent for a category $\mathcal{C}$.
  - $\mathcal{C}$ is a $\mathbb{U}$-category.
  - $\mathsf{Ob}\,\mathcal{C}$ and $\mathsf{Mor}\,\mathcal{C}$ are $\mathbb{U}$-sets.
  - $\mathcal{C}$ is a locally $\mathbb{U}$-set category such that $\mathsf{Ob}\,\mathcal{C} \in \mathbb{U}$.
  - $\mathcal{C}$ is a $\mathbb{U}$-moderate locally $\mathbb{U}$-set category such that $\mathsf{Ob}\,\mathcal{C}$ is $\mathbb{U}$-small.

(2) The following are equivalent for a category $\mathcal{C}$.
  - $\mathcal{C}$ is a $\mathbb{U}$-small category.
  - $\mathcal{C}$ is a locally $\mathbb{U}$-small category such that $\mathsf{Ob}\,\mathcal{C}$ is $\mathbb{U}$-small.
  - $\mathcal{C}$ is isomorphic to a $\mathbb{U}$-category.

(3) The following are equivalent for a category $\mathcal{C}$.
  - $\mathcal{C}$ is skeletally $\mathbb{U}$-small.
  - There is a $\mathbb{U}$-small full subcategory $\mathcal{C}'$ such that the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an equivalence of categories.
  - $\mathcal{C}$ is equivalent to a $\mathbb{U}$-category.

(4) The property of being a $\mathbb{U}$-category (resp. locally $\mathbb{U}$-set, resp. $\mathbb{U}$-moderate) is *not* closed under isomorphisms of categories.

(5) The property of being $\mathbb{U}$-small is closed under isomorphisms of categories.

(6) The property of being locally $\mathbb{U}$-small (resp. skeletally $\mathbb{U}$-small) is closed under equivalences of categories.

We will give examples of locally $\mathbb{U}$-small categories.

Example 1.1.7. Let $\mathbb{U}$ be a Grothendieck universe. The following categories are frequently used in this thesis:

- The category $\mathsf{Set}_{\mathbb{U}}$ of $\mathbb{U}$-sets and maps.
- The category $\mathsf{Ab}_{\mathbb{U}}$ of abelian groups belonging to $\mathbb{U}$ and group homomorphisms.
- The category $\mathsf{Mon}_{\mathbb{U}}$ of commutative monoids belonging to $\mathbb{U}$ and monoid homomorphisms.
- For a (unital and associative) ring Λ, the category $\mathsf{Mod}_{\mathbb{U}}\,\Lambda$ of (right) Λ-modules which belong to $\mathbb{U}$ and Λ-linear maps.
- For a scheme X, the category $\mathsf{Qcoh}_{\mathbb{U}}\,X$ of quasi-coherent $\mathscr{O}_X$-modules which belong to $\mathbb{U}$ and $\mathscr{O}_X$-linear maps.

These are $\mathbb{U}$-moderate locally $\mathbb{U}$-set categories.

- For a ring Λ belonging to $\mathbb{U}$, the category $\mathsf{mod}_{\mathbb{U}}\,\Lambda$ of finitely generated Λ-modules belonging to $\mathbb{U}$ and Λ-linear maps.
- Let X be a scheme belonging to $\mathbb{U}$.
  - The category $\mathsf{coh}_{\mathbb{U}}\,X$ of coherent $\mathscr{O}_X$-modules belonging to $\mathbb{U}$ and $\mathscr{O}_X$-linear maps.
  - The full subcategory $\mathsf{tor}_{\mathbb{U}}\,X$ of $\mathsf{coh}_{\mathbb{U}}\,X$ consisting of coherent torsion sheaves.
  - The full subcategory $\mathsf{vect}_{\mathbb{U}}\,X$ of $\mathsf{coh}_{\mathbb{U}}\,X$ consisting of vector bundles.

These are skeletally $\mathbb{U}$-small categories.

Remark 1.1.8. There is another foundation for category theory using the NBG set theory (due to von Neumann, Bernays and Gödel). See [Men15] for a detailed account of the NBG set theory. In the NBG set theory, the notion of *classes* is introduced, and we can consider the products of classes and maps between classes. Defining a category by an ordered pair of the class of objects, the class of morphisms and some maps such as Definition 1.1.1, we can define the category Set of all sets since there is the class of all sets in the NGB set theory.

We describe the relationship between the NBG set theory and the ZFCU set theory. Let $\mathbb{U}$ be a Grothendieck universe. Then the power set $\mathsf{P}(\mathbb{U})$ is a model for the NBG set theory. Thus, replacing sets and classes with $\mathbb{U}$-sets and $\mathbb{U}$-classes respectively, the proposition correct in the NBG set theory gives the proposition correct in the ZFCU set theory. In this sense, the ZFCU set theory is stronger than the NBG set theory.

Let us discuss the reasons why we have chosen the ZFCU set theory rather than the NBG set theory as the foundation for this thesis. In this thesis, we will treat the collection of full subcategories and the Picard group of a scheme. Hence, we have to consider the collection of classes in both cases. Whereas we can consider a set of $\mathbb{U}$-classes in our foundation, we cannot consider a class of (proper) classes in the NBG set theory. For this reason, we use the ZFCU set theory which is stronger than the NBG set theory, as the foundation for this thesis.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We denote by $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations.

Example 1.1.9. Fix a Grothendieck universe $\mathbb{U}$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.
(1) If both $\mathcal{C}$ and $\mathcal{D}$ are $\mathbb{U}$-categories (resp. $\mathbb{U}$-small), then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.
(2) If $\mathcal{C}$ is $\mathbb{U}$-small and $\mathcal{D}$ is locally $\mathbb{U}$-small, then $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is also locally $\mathbb{U}$-small.

Let $\mathcal{C}$ be a category and S a subset of $\mathsf{Mor}\,\mathcal{C}$. The *localization* $\mathcal{C}[S^{-1}]$ of $\mathcal{C}$ with respect to S is a pair of a category $\mathcal{C}[S^{-1}]$ and a functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ satisfying the following:
(i) $Q(s)$ is an isomorphism for any $s \in S$.
(ii) For any category $\mathcal{D}$ and any functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for any $s \in S$, there exists a unique functor $\overline{F}\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ satisfying $F = \overline{F} \circ Q$.

For any category $\mathcal{C}$ and a subset S of $\mathsf{Mor}\,\mathcal{C}$, the localization $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ certainly exists (cf. [DHKS04, Subsection 33.8]). By the construction, $\mathsf{Ob}(\mathcal{C}[S^{-1}]) = \mathsf{Ob}(\mathcal{C})$ and Q is the identity on the objects.

Example 1.1.10. Fix a Grothendieck universe $\mathbb{U}$. Let $\mathcal{C}$ be a category and S a subset of $\mathsf{Mor}\,\mathcal{C}$.
(1) If $\mathcal{C}$ is a $\mathbb{U}$-category (resp. $\mathbb{U}$-small, resp. skeletally $\mathbb{U}$-small), then so is $\mathcal{C}[S^{-1}]$.
(2) Even if $\mathcal{C}$ is locally $\mathbb{U}$-set, the localization $\mathcal{C}[S^{-1}]$ is not necessarily locally $\mathbb{U}$-small (cf. [Kra10, Subsection 4.15]).

We often fix two Grothendieck universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$ and consider the following categories.

Definition 1.1.11. Let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $\mathbb{U} \in \mathbb{V}$, and let $\mathcal{C}_{\mathbb{U}}$ be one of the categories in Example 1.1.7. We define the category $\mathcal{C}_{\widetilde{\mathbb{U}}}$ by the essential image of the natural inclusion functor $\mathcal{C}_{\mathbb{U}} \hookrightarrow \mathcal{C}_{\mathbb{V}}$.

For example, $\mathsf{Set}_{\widetilde{\mathbb{U}}}$ is the category of $\mathbb{U}$-small $\mathbb{V}$-sets. It is clear that $\mathcal{C}_{\widetilde{\mathbb{U}}}$ is equivalent to $\mathcal{C}_{\mathbb{U}}$ and a full subcategory of $\mathcal{C}_{\mathbb{V}}$ closed under isomorphisms.

We often use the following convention:

Convention 1.1.12. Let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $\mathbb{U} \in \mathbb{V}$. We use the following convention:
- A *set* := $\mathbb{U}$-small $\mathbb{V}$-set.
- A *collection* := a $\mathbb{V}$-set.
- A *category* := a $\mathbb{V}$-category.
- small (resp. locally small, resp. skeletally small) := $\mathbb{U}$-small (resp. locally $\mathbb{U}$-small, resp. skeletally $\mathbb{U}$-small).
- Assume that the underlying sets of monoids, abelian groups and topological spaces are $\mathbb{U}$-small $\mathbb{V}$-sets.
- Assume that rings, schemes and modules over them belong to $\mathbb{U}$.
- $\mathsf{Mon} := \mathsf{Mon}_{\widetilde{\mathbb{U}}}$ and $\mathsf{Ab} := \mathsf{Ab}_{\widetilde{\mathbb{U}}}$.
- For a ring Λ belonging to $\mathbb{U}$, $\mathsf{Mod}\,\Lambda := \mathsf{Mod}_{\mathbb{U}}\,\Lambda$ and $\mathsf{mod}\,\Lambda := \mathsf{mod}_{\mathbb{U}}\,\Lambda$.
- For a scheme X belonging to $\mathbb{U}$, $\mathsf{Qcoh}\,X := \mathsf{Qcoh}_{\mathbb{U}}\,X$, $\mathsf{coh}\,X := \mathsf{coh}_{\mathbb{U}}\,X$, $\mathsf{tor}\,X := \mathsf{tor}_{\mathbb{U}}\,X$ and $\mathsf{vect}\,X := \mathsf{vect}_{\mathbb{U}}\,X$.

For example, Fact 0.2.1 is interpreted as follows under this convention.

Fact 1.1.13 (The precise statment of Fact 0.2.1)**.** *Let X be a noetherian scheme belonging to $\mathbb{U}$. There is an inclusion preserving bijection between the following sets:*

- *The $\mathbb{U}$-small $\mathbb{V}$-set of Serre subcategories of* $\mathsf{coh}_{\mathbb{U}} X$.
- *The $\mathbb{U}$-small $\mathbb{V}$-set*[2] *of specialization-closed subsets of X.*

More precisely, in [Gab62], Gabriel fixed a single Grothendieck universe $\mathbb{U}$ and did not mention the size of the sets appearing in Fact 1.1.13. However, it is clear that they are $\mathbb{U}$-small $\mathbb{V}$-sets.

Warning 1.1.14. Let us explain the purpose of Convention 1.1.12. In this thesis, some sets, monoids, and topological spaces are constructed from skeletally small categories. For example, we consider the set of full subcategories closed under isomorphism (cf. Chapter 3), Grothendieck monoids (cf. Chapter 2), and their monoid spectra (cf. Chapter 6). Thus, in Convention 1.1.12, we assume that monoids, abelian groups and topological spaces are $\mathbb{U}$-small $\mathbb{V}$-sets. In this thesis, we do not construct rings, schemes and modules over them from the set of full subcategories closed under isomorphism, Grothendieck monoids, and their monoid spectra. Therefore, this convention does not cause any inconsistency.

Warning 1.1.15. Let us clarify the relationship and differences between our basic setup (Convention 1.1.12) and those of other literature cited in §0.2.

- In [Gab62], the ZFCU set theory was used and only one Grothendieck universe was fixed. Similar to Fact 1.1.13, the content of this paper can be interpreted consistently in our setup.
- In [Tho97], the NBG set theory is referred in the definition of *essentially small categories* (cf. [Tho97, page 4, (1.7)]). Thus, the foundation of this paper can be interpreted as the NBG set theory. As explained in Remark 1.1.8, replacing sets and classes with $\mathbb{U}$-sets and $\mathbb{U}$-classes respectively, we can interpret the statements in [Tho97] as statements in our setup. In particular, under this interpretation, we can easily see that our definition of skeletally $\mathbb{U}$-small coincides with the definition of essentially small in [Tho97] by Remark 1.1.6 (3).
- In [Bro97, Mat18, ZZ21], there is no mention for set-theoretical foundations. However, [Mat18, ZZ21] cite [Tho97] as a previous study, so their foundation can be interpreted as the NBG set theory. Therefore, as with the second item, these papers are also consistent with the current setup.

Finally, we prove that the category $\mathsf{mod}_{\mathbb{U}} \Lambda$ of finitely generated modules over a ring Λ and the category $\mathsf{coh}_{\mathbb{U}} X$ of coherent sheaves over a scheme X do not depend on the Grothendieck universe $\mathbb{U}$ up to equivalences.

Proposition 1.1.16. *Let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $\mathbb{U} \in \mathbb{V}$. For any ring Λ belonging to $\mathbb{U}$, the natural inclusion functor $\mathsf{mod}_{\mathbb{U}} \Lambda \hookrightarrow \mathsf{mod}_{\mathbb{V}} \Lambda$ is an equivalence of categories.*

Proof. For any finitely generated Λ-module M, there are integer $n \geq 0$ and a surjection Λ-linear map $w \colon \Lambda^{\oplus n} \twoheadrightarrow M$. Thus, there is an isomorphism $\Lambda^{\oplus n}/\operatorname{Ker}(w) \xrightarrow{\cong} M$. It is clear that $\Lambda^{\oplus n}/\operatorname{Ker}(w)$ is a $\mathbb{U}$-set. Therefore, the natural inclusion functor $\mathsf{mod}_{\mathbb{U}} \Lambda \hookrightarrow \mathsf{mod}_{\mathbb{V}} \Lambda$ is essentially surjective. $\square$

Proposition 1.1.17. *Let X be a scheme, and let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $X \in \mathbb{U} \in \mathbb{V}$. Then the following hold:*

(1) *The natural inclusion functor $\mathsf{coh}_{\mathbb{U}} X \hookrightarrow \mathsf{coh}_{\mathbb{V}} X$ is an equivalence of categories.*
(2) *The natural inclusion functor $\mathsf{tor}_{\mathbb{U}} X \hookrightarrow \mathsf{tor}_{\mathbb{V}} X$ is an equivalence of categories.*
(3) *The natural inclusion functor $\mathsf{vect}_{\mathbb{U}} X \hookrightarrow \mathsf{vect}_{\mathbb{V}} X$ is an equivalence of categories.*

Proof. It is enough to show that the following claim:

- Every $\mathscr{O}_X$-module of finite type is isomorphic to an $\mathscr{O}_X$-module belonging to $\mathbb{U}$.

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type. Then for any $x \in X$ there exists an open neighborhood U_x of x and an exact sequence of $\mathscr{O}_{U_x}$-modules of the form

$$\mathscr{O}_{U_x}^{\oplus n_x} \xrightarrow{w^{(x)}} \mathscr{F}_{|U_x} \to 0,$$

where $n_x \geq 0$ is an integer. Then $w^{(x)}$ induces an isomorphism $\phi^{(x)} \colon \mathscr{O}_{U_x}^{\oplus n_x}/\operatorname{Ker}\left(w^{(x)}\right) \xrightarrow{\cong} \mathscr{F}_{|U_x}$. It is clear that $\mathscr{O}_{U_x}^{\oplus n_x}/\operatorname{Ker}\left(w^{(x)}\right)$ is a $\mathbb{U}$-set. Gluing $\left\{\mathscr{O}_{U_x}^{\oplus n_x}/\operatorname{Ker}\left(w^{(x)}\right)\right\}_{x\in X}$ and $\left\{\phi^{(x)}\right\}_{x\in X}$, we obtain an $\mathscr{O}_X$-module $\mathscr{G}$ belonging to $\mathbb{U}$ and an isomorphism $\phi \colon \mathscr{G} \to \mathscr{F}$ of $\mathscr{O}_X$-modules. $\square$

Remark 1.1.18. Let Λ be a ring, and let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $\Lambda \in \mathbb{U} \in \mathbb{V}$. Then the natural inclusion functor $\mathsf{Mod}_{\mathbb{U}} \Lambda \hookrightarrow \mathsf{Mod}_{\mathbb{V}} \Lambda$ is *not* essentially surjective since $\Lambda^{\oplus \mathbb{U}} \in \mathsf{Mod}_{\mathbb{V}} \Lambda$ is not isomorphic to an object of $\mathsf{Mod}_{\mathbb{U}} \Lambda$. Thus, the property of being finitely generated is essential in Proposition 1.1.16. The same can be said for Proposition 1.1.17.

[2]It is, in fact, a $\mathbb{U}$-set.

1.2 Global notations and convention

In this section, we fix a Grothendieck universe $\mathbb{U}$.

Let $\mathcal{C}$ be a category. We denote by $|\mathcal{C}|$ the set of isomorphism classes of objects. The isomorphism class of $X \in \mathcal{C}$ is also denoted by X. A subcategory is said to be *strictly full* if it is a full subcategory closed under isomorphisms. The strictly full subcategories of $\mathcal{C}$ are often identified with the subsets of $|\mathcal{C}|$. If $\mathcal{C}$ is skeletally $\mathbb{U}$-small (see Definition 1.1.4), then $|\mathcal{C}|$ is $\mathbb{U}$-small by the definition. Thus, the set of strictly full subcategories of $\mathcal{C}$ is also $\mathbb{U}$-small.

Let $\mathcal{C}$ be an additive category. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called an *additive subcategory* if it is a strictly full subcategory closed under taking finite direct sums. In particular, each additive subcategory of $\mathcal{C}$ contains a zero object. For a non-empty full subcategory $\mathcal{D}$ of $\mathcal{C}$, we denote by $\mathsf{add}\,\mathcal{D}$ the subcategory of $\mathcal{C}$ consisting of direct summands of direct sums of objects in $\mathcal{D}$. For the empty subcategory $\emptyset$, we set $\mathsf{add}(\emptyset) := \{0\}$. Then $\mathsf{add}\,\mathcal{D}$ is the smallest additive subcategory of $\mathcal{C}$ closed under direct summands containing $\mathcal{D}$.

Let Λ be a (unital and associative but not necessarily commutative) ring. For (right) Λ-modules M and N, we denote by $\mathrm{Hom}_\Lambda(M, N)$ the set of Λ-linear maps from M to N.

Let X be a scheme with structure sheaf $\mathscr{O}_X$. A point of X is not necessarily assumed to be closed. For a point $x \in X$, we denote by $\mathfrak{m}_x$ the maximal ideal of $\mathscr{O}_{X,x}$ and $\kappa(x) := \mathscr{O}_{X,x}/\mathfrak{m}_x$ the residue field of x. Let $\mathscr{F}$ and $\mathscr{G}$ be quasi-coherent sheaves on X. We denotes by $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ the set of $\mathscr{O}_X$-linear maps form $\mathscr{F}$ to $\mathscr{G}$. The tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathscr{O}_X$ is denoted by $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$. The sheaf of homomorphisms from $\mathscr{F}$ to $\mathscr{G}$ is denoted by $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$. If no ambiguity can arise, we will often omit the subscript $\mathscr{O}_X$. The support of $\mathscr{F}$ is the subset of X defined by $\mathrm{Supp}\,\mathscr{F} := \{x \in X \mid \mathscr{F}_x \neq 0\}$. A *variety* over a field $\Bbbk$ means a separated integral scheme of finite type over $\Bbbk$. A *curve* is a 1-dimensional variety.

For a commutative ring R, we often identify $\mathsf{Mod}_{\mathbb{U}}\,R$ with $\mathsf{Qcoh}_{\mathbb{U}}(\mathrm{Spec}\,R)$. For a morphism $f\colon X \to Y$ of schemes belonging to $\mathbb{U}$, we denote by $f_*\colon \mathsf{Qcoh}_{\mathbb{U}}\,X \to \mathsf{Qcoh}_{\mathbb{U}}\,Y$ the direct image functor and $f^*\colon \mathsf{Qcoh}_{\mathbb{U}}\,Y \to \mathsf{Qcoh}_{\mathbb{U}}\,X$ the pull-back functor.

A *monoid* means a semigroup with unit. Every monoid is assumed to be *commutative*. We use an additive notation, that is, the operation is denoted by $+$ with its unit 0. We denote by $\mathbb{N}$ the monoid of non-negative integers. For a subset S of a monoid M, we denote by $\langle S \rangle_{\mathbb{N}}$ the smallest submonoid of M containing S.

Part I

General theory

Chapter 2

Grothendieck monoids of extriangulated categories

For a triangulated category or an exact category $\mathcal{C}$, its *Grothendieck group* $\mathsf{K}_0(\mathcal{C})$ is a basic invariant and has been studied and used in various areas. Recently, the *Grothendieck monoid of an exact category*, a natural monoid version of the Grothendieck group, has been introduced by Berenstein–Greenstein in [BG16] to study Hall algebras, and its relation to the categorical property of an exact category was studied in [Eno22].

We can naturally generalize these constructions to an extriangulated category $\mathcal{C}$. The Grothendieck group $\mathsf{K}_0(\mathcal{C})$ was formulated by Zhu–Zhuang [ZZ21] and Haugland [Hua21] and was used to classify certain subcategories of an extriangulated category. The aim of this chapter is to define and investigate the *Grothendieck monoid* $\mathsf{M}(\mathcal{C})$ *of* $\mathcal{C}$.

2.1 Preliminaries: commutative monoids

We collect minimal definitions and properties of commutative monoids to describe our first results. The main reference of this section is [Ogu18].

A *monoid* is a semigroup with a unit. In this paper, every monoid is assumed to be *commutative*. Hence, we use the additive notation, that is, the binary operation is denoted by $+$, and the unit is denoted by 0. A *homomorphism* of monoids is a map $f\colon M \to N$ satisfying $f(x+y) = f(x) + f(y)$ and $f(0_M) = 0_N$. Let $\mathbb{U}$ be a Grothendieck universe. We denote by $\mathsf{Mon}_{\mathbb{U}}$ the category of (commutative) monoids belonging to $\mathbb{U}$ and homomorphisms of them. The category $\mathsf{Mon}_{\mathbb{U}}$ has arbitrary $\mathbb{U}$-small limits and colimits (see [Ogu18, Section I.1.1]). We can define the product $\prod_{i\in I} M_i$ and the direct sum (= coproduct) $\bigoplus_{i\in I} M_i$ of monoids similarly to abelian groups. In particular, finite products and finite direct sums coincide.

A basic example of monoids is the set $\mathbb{N}$ of non-negative integers with the arithmetic addition. A monoid M is said to be *free* if it is isomorphic to $\mathbb{N}^{\oplus I}$ for some index set I. In this case, the cardinality of I is called the *rank* of M. A *basis* of a free monoid is defined by a similar way to abelian groups.

Remark 2.1.1. For a monoid homomorphism $f\colon M \to N$, define a submonoid of M by

$$\operatorname{Ker}(f) := \{x \in M \mid f(x) = 0\}.$$

A caution is that *the condition* $\operatorname{Ker}(f) = 0$ *does not imply* f *is injective.* Indeed, the monoid homomorphism

$$f\colon \mathbb{N}^{\oplus 2} \to \mathbb{N}, \quad (x,y) \mapsto x+y$$

is not injective but $\operatorname{Ker}(f) = 0$.

The notion of quotients of monoids slightly differs from that of abelian groups. We introduce a class of equivalence relations $\sim$ on a monoid M to guarantee that the quotient set $M/{\sim}$ becomes a monoid.

Definition 2.1.2. The equivalence relation $\sim$ on a monoid M is called a *congruence* if $x \sim y$ implies $a + x \sim a + y$ for every $a, x, y \in M$.

We can check easily that the quotient set $M/{\sim}$ of a monoid M by a congruence $\sim$ has the unique monoid structure such that the quotient map $M \to M/{\sim}$ is a monoid homomorphism.

Next, we list the properties of monoids which we will use.

Definition 2.1.3. Let M be a monoid.
(1) M is *sharp* (or *reduced*) if $a + b = 0$ implies $a = b = 0$ for any $a, b \in M$.
(2) M is *cancellative* (or *integral*) if $a + x = a + y$ implies $x = y$ for any $a, x, y \in M$.

Remark 2.1.4. Let M be a monoid. An element $x \in M$ is said to be *unit* if there is $y \in M$ such that $x + y = 0$. We denote by $M^\times$ the set of units in M. Then M is sharp if and only if $M^\times = 0$.

Finally, we discuss the relationship between monoids and groups.

Definition 2.1.5. The *group completion* of a monoid M is a pair $(\mathsf{gp}M, \rho)$ of a group $\mathsf{gp}M$ and a monoid homomorphism $\rho\colon M \to \mathsf{gp}M$ satisfying the following universal property:
- For every monoid homomorphism $f\colon M \to G$ into a group G, there exists a unique group homomorphism $\overline{f}\colon \mathsf{gp}M \to G$ such that $f = \overline{f}\rho$.

The group completion $\mathsf{gp}M$ certainly exists for any monoid M. It is constructed as the localization of M with respect to M itself (see Definition 5.1.1). The group completion has the following properties by the construction.

Proposition 2.1.6 (cf. Definition 5.1.1)**.** *Let M be a monoid and $(\mathsf{gp}M, \rho\colon M \to \mathsf{gp}M)$ its group completion.*
(1) *$\mathsf{gp}M$ is an abelian group.*
(2) *For any $x, y \in M$, the equality $\rho(x) = \rho(y)$ holds in $\mathsf{gp}M$ if and only if $x + s = y + s$ in M for some $s \in M$.*
(3) *If M is a group, then $\mathsf{gp}M = M$.*
(4) *Let $\mathbb{U}$ be a Grothendieck universe. If M is a $\mathbb{U}$-set (resp. $\mathbb{U}$-small), then so is $\mathsf{gp}M$.*

The cancellation property is related to the group completion as follows.

Proposition 2.1.7. *Let M be a monoid and $(\mathsf{gp}M, \rho\colon M \to \mathsf{gp}M)$ its group completion. Then the following are equivalent.*
(1) *M is cancellative.*
(2) *The monoid homomorphism $\rho\colon M \to \mathsf{gp}M$ is injective.*
(3) *There is an injective monoid homomorphism from M to some group.*

Proof. It follows immediately from Proposition 2.1.6 and the fact that any submonoid of an abelian group is cancellative. □

Let $\mathbb{U}$ be a Grothendieck universe. The assignment $M \mapsto \mathsf{gp}M$ gives rise to a functor $\mathsf{gp}\colon \mathsf{Mon}_{\mathbb{U}} \to \mathsf{Ab}_{\mathbb{U}}$ by the universal property. Also, the assignment $M \mapsto M^\times$ (cf. Remark 2.1.4) gives rise to a functor $(-)^\times\colon \mathsf{Mon}_{\mathbb{U}} \to \mathsf{Ab}_{\mathbb{U}}$. Let $\mathbb{V}$ be a Grothendieck universe such that $\mathbb{U} \in \mathbb{V}$. Then the following diagram commutes by the construction:

$$\begin{array}{ccc} \mathsf{Mon}_{\mathbb{V}} & \overset{\mathsf{gp}}{\underset{(-)^\times}{\rightrightarrows}} & \mathsf{Ab}_{\mathbb{V}} \\ \uparrow & & \uparrow \\ \mathsf{Mon}_{\mathbb{U}} & \overset{\mathsf{gp}}{\underset{(-)^\times}{\rightrightarrows}} & \mathsf{Ab}_{\mathbb{U}}\,. \end{array}$$

Moreover, they restrict to $\mathsf{Mon}_{\widetilde{\mathbb{U}}} \to \mathsf{Ab}_{\widetilde{\mathbb{U}}}$ by Proposition 2.1.6.

Proposition 2.1.8. *Let $\mathbb{U}$ be a Grothendieck universe. The forgetful functor $\mathsf{Ab}_{\mathbb{U}} \to \mathsf{Mon}_{\mathbb{U}}$ has both the left adjoint functor $\mathsf{gp}\colon \mathsf{Mon}_{\mathbb{U}} \to \mathsf{Ab}_{\mathbb{U}}$ and the right adjoint functor $(-)^\times\colon \mathsf{Mon}_{\mathbb{U}} \to \mathsf{Ab}_{\mathbb{U}}$. A similar statement also holds for $\mathsf{Mon}_{\widetilde{\mathbb{U}}}$ and $\mathsf{Ab}_{\widetilde{\mathbb{U}}}$ if we fix a Grothendieck universe $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$.*

Proof. This follows from Definition 2.1.5 and the fact that units are preserved by monoid homomorphisms. □

2.2 Preliminaries: extriangulated categories

In this paper, we omit the precise definitions and axioms of extriangulated categories. We refer the reader to [NP19, LN19] for the basics of extriangulated categories.

Let $\mathbb{U}$ be a Grothendieck universe. An *extriangulated* $\mathbb{U}$*-category* is an ordered tuple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ consisting of the following data satisfying certain axioms:

- $\mathcal{C}$ is an additive $\mathbb{U}$-category.
- $\mathbb{E}$ is an additive bifunctor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}_{\mathbb{U}}$.
- $\mathfrak{s} = \{\mathfrak{s}_{Z,X}\}_{(Z,X)\in \mathsf{Ob}\,\mathcal{C}\times\mathsf{Ob}\,\mathcal{C}}$ is a family of functions $\mathfrak{s}_{Z,X} : \mathbb{E}(Z,X) \to \mathsf{E}(Z,X)$. Here the set $\mathsf{E}(Z,X)$ is defined as follows.

A complex V over $\mathcal{C}$ is called a 3*-term complex starting with* A *and ending in* B if $V^0 = A$, $V^2 = B$ and $V^i = 0$ for all $i \neq 0,1,2$. Two 3-term complexes $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{f'} Y' \xrightarrow{g'} Z$ are *equivalent* if there is an isomorphism $y\colon Y \xrightarrow{\cong} Y'$ which makes the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{y} & Z \\ \| & & {\scriptstyle y}\downarrow{\scriptstyle \cong} & & \| \\ X & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z. \end{array}$$

We denote by $\mathsf{E}(Z,X)$ the set of equivalence classes of 3-term complexes starting with X and ending with Z. Note that $\mathsf{E}(Z,X)$ is a $\mathbb{U}$-set. Thus, an extriangulated $\mathbb{U}$-category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is also a $\mathbb{U}$-set.

For an extriangulated $\mathbb{U}$-category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, we call a 3-term complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ a *conflation* if its equivalence class is equal to $\mathfrak{s}(\delta)$ for some $\delta \in \mathbb{E}(Z,X)$. In this case, we call f an *inflation* and g a *deflation*, and say that this complex *realizes* δ. We write this situation as follows:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow} ,$$

and we also call this diagram a conflation in $\mathcal{C}$. In what follows, we often write $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ for an extriangulated $\mathbb{U}$-category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ for simplicity.

Triangulated categories and Quillen's exact categories can be naturally considered as extriangulated categories as follows:

(1) An *exact* $\mathbb{U}$*-category* is an exact category whose underlying category is a $\mathbb{U}$-category. Let $\mathcal{E}$ be an exact $\mathbb{U}$-category. We denote by $\mathrm{Ext}^1_{\mathcal{E}}(Z,X)$ the set of equivalence classes of conflations (in the sense of exact categories) in $\mathcal{E}$ of the form $0 \to X \to Y \to Z \to 0$. Then we have a functor $\mathrm{Ext}^1_{\mathcal{E}}(-,-) : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathsf{Ab}_{\mathbb{U}}$. Then by setting $\mathbb{E} := \mathrm{Ext}^1_{\mathcal{E}}$ and $\mathfrak{s} := \mathsf{id}$, we may regard $\mathcal{E}$ as an extriangulated $\mathbb{U}$-category.

(2) A *triangulated* $\mathbb{U}$*-category* is a triangulated category whose underlying category is a $\mathbb{U}$-category. Let $\mathcal{T}$ be a triangulated $\mathbb{U}$-category with shift functor Σ. Then by setting $\mathbb{E}(Z,X) := \mathcal{T}(Z, \Sigma X)$, we may regard $\mathcal{T}$ as an extriangulated $\mathbb{U}$-category. In this case, for each $h \in \mathbb{E}(Z,X) = \mathcal{T}(Z,\Sigma X)$, its realization $\mathfrak{s}(h)$ is given by $[X \xrightarrow{f} Y \xrightarrow{g} Z]$ which makes $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ a triangle in $\mathcal{T}$.

Throughout this paper, we always regard triangulated $\mathbb{U}$*-categories and exact* $\mathbb{U}$*-categories (in particular, abelian* $\mathbb{U}$*-categories) as extriangulated* $\mathbb{U}$*-categories in this way.* Conversely, exact and triangulated categories can be characterized within extriangulated $\mathbb{U}$-categories as follows.

Fact 2.2.1 ([NP19, Corollary 3.18, 7.6])**.** *Let* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated* $\mathbb{U}$*-category.*

(1) *If any inflation is monic and any deflation is epic, then* $\mathcal{C}$ *admits a natural structure of exact category, in which conflations in the sense of an exact categories are conflations in the sense of extriangulated categories.*

(2) *If any morphism is both an inflation and a deflation, then* $\mathcal{C}$ *admits a natural structure of a triangulated category, in which triangles comes from conflations.*

For later use, we fix some terminologies for extriangulated categories. Let $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated $\mathbb{U}$-category. For any $\delta \in \mathbb{E}(Z,X)$ and any morphisms $x \in \mathcal{C}(X,X')$ and $z \in \mathcal{C}(Z',Z)$, we denote $\mathbb{E}(\mathsf{id}_Z, x)(\delta)$ and $\mathbb{E}(z, \mathsf{id}_X)(\delta)$ briefly by $x_*\delta$ and $z^*\delta$ respectively.

A *morphism of conflations* from $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$ to $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \overset{\delta'}{\dashrightarrow}$ is a triplet $(x\colon X \to X', y\colon Y \to Y', z\colon Z \to Z')$ of morphisms in $\mathcal{C}$ satisfying $yf = f'x$, $zg = g'y$ and $x_*\delta = z^*\delta'$. We often write a morphism of conflations (x, y, z) as follows:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \overset{\delta}{\dashrightarrow} & \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \\ X' & \xrightarrow[f']{} & Y & \xrightarrow[g']{} & Z' & \underset{\delta'}{\dashrightarrow} & . \end{array}$$

For two subcategories $\mathcal{C}_1$ and $\mathcal{C}_2$ of an extriangulated $\mathbb{U}$-category $\mathcal{C}$, we denote by $\mathcal{C}_1 * \mathcal{C}_2$ the subcategory of $\mathcal{C}$ consisting of $X \in \mathcal{C}$ such that there is a conflation

$$C_1 \longrightarrow X \longrightarrow C_2 \dashrightarrow$$

in $\mathcal{C}$ with $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$.

We recall some properties of a subcategory $\mathcal{D}$ of an extriangulated $\mathbb{U}$-category $\mathcal{C}$.

- $\mathcal{D}$ is *closed under finite direct sums* if it contains a zero object of $\mathcal{C}$ and for any objects $X, Y \in \mathcal{D}$, their direct sum $X \oplus Y$ in $\mathcal{C}$ belongs to $\mathcal{D}$. We also say that $\mathcal{D}$ is an *additive subcategory* in this case.
- $\mathcal{D}$ is *closed under direct summands* if $X \oplus Y \in \mathcal{D}$ implies X, $Y \in \mathcal{D}$ for any objects $X, Y \in \mathcal{C}$.
- $\mathcal{D}$ is *closed under extensions* if $\mathcal{D}$ is additive and $\mathcal{D} * \mathcal{D} \subseteq \mathcal{D}$ holds, that is, for any conflation $X \to Y \to Z \dashrightarrow$ in $\mathcal{C}$, we have that $X, Z \in \mathcal{D}$ implies $Y \in \mathcal{D}$. We also say that $\mathcal{D}$ is an *extension-closed subcategory* in this case.

An extension-closed subcategory $\mathcal{D}$ of an extriangulated $\mathbb{U}$-category $\mathcal{C}$ also has the structure of an extriangulated $\mathbb{U}$-category induced from that of $\mathcal{C}$, see [NP19, Remark 2.18]. In this structure, a conflation $X \to Y \to Z \dashrightarrow$ in $\mathcal{D}$ is precisely the conflation in $\mathcal{C}$ with X, Y, and Z in $\mathcal{D}$. When we consider extension-closed subcategories of an extriangulated $\mathbb{U}$-category, we always regard $\mathcal{D}$ as an extriangulated $\mathbb{U}$-category in this way.

We also need the natural notion of functors between extriangulated categories, namely, an *exact functor*:

Definition 2.2.2 ([BS21, Definition 2.32])**.** Let $(\mathcal{C}_i, \mathbb{E}_i, \mathfrak{s}_i)$ be extriangulated $\mathbb{U}$-categories for $i = 1, 2, 3$.

(1) An *exact functor* $(F, \phi)\colon (\mathcal{C}_1, \mathbb{E}_1, \mathfrak{s}_1) \to (\mathcal{C}_2, \mathbb{E}_2, \mathfrak{s}_2)$ is a pair of an additive functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ and a natural transformation $\phi\colon \mathbb{E}_1 \Rightarrow \mathbb{E}_2 \circ (F^{\mathrm{op}} \times F)$ such that for every conflation

$$X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$$

in $\mathcal{C}_1$, the following is a conflation in $\mathcal{C}_2$:

$$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \overset{\phi_{Z,X}(\delta)}{\dashrightarrow} .$$

We often write $F = (F, \phi)\colon \mathcal{C}_1 \to \mathcal{C}_2$ in this case.

(2) For two exact functors $F_1 = (F_1, \phi_1)\colon \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 = (F_2, \phi_2)\colon \mathcal{C}_2 \to \mathcal{C}_3$, their composition is defined by $F_2 \circ F_1 = (F_2 \circ F_1, \phi_2 \cdot \phi_1) : \mathcal{C}_1 \to \mathcal{C}_3$, where

$$(\phi_2 \cdot \phi_1)_{C,A}\colon \mathbb{E}_1(C, A) \xrightarrow{(\phi_1)_{C,A}} \mathbb{E}_2(FC, FA) \xrightarrow{(\phi_2)_{FC,FA}} \mathbb{E}_3(GFC, GFA).$$

(3) Let $F = (F, \phi)$ and $G = (G, \psi)$ be two exact functors $\mathcal{C}_1 \to \mathcal{C}_2$. A *natural transformation* $\eta\colon F \Rightarrow G$ of *exact functors* is a natural transformation of additive functors such that, for any conflation $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ in $\mathcal{C}$, the following is a morphism of conflations in $\mathcal{D}$:

$$\begin{array}{ccccccc} FA & \xrightarrow{Fx} & FB & \xrightarrow{Fy} & FC & \xrightarrow{\phi_{C,A}(\delta)} & \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} & & \downarrow{\scriptstyle \eta_C} & & \\ GA & \xrightarrow[Gx]{} & GB & \xrightarrow[Gy]{} & GC & \underset{\psi_{C,A}(\delta)}{\dashrightarrow} & . \end{array}$$

(4) $\mathsf{ETcat}_{\mathbb{U}}$ is the category of extriangulated $\mathbb{U}$-categories and exact functors.

Horizontal compositions and *vertical compositions* of natural transformations of exact functors are defined by those for natural transformations of additive functors. These compositions are also natural transformations of exact functors. In particular, we can define an *exact equivalence* of extriangulated categories in the same way as that of additive categories.

Remark 2.2.3. Let $F = (F, \phi)\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of extriangulated $\mathbb{U}$-categories, and let (a, b, c) be a morphism of conflations

$$\begin{array}{ccccccc} A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C_1 & \overset{\delta_1}{\dashrightarrow} & \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ A_2 & \xrightarrow[x_2]{} & B_2 & \xrightarrow[y_2]{} & C_2 & \underset{\delta_2}{\dashrightarrow} & \end{array}$$

in $\mathcal{C}$. Then (Fa, Fb, Fc) gives a morphism of conflations

$$\begin{array}{ccccccc} FA_1 & \xrightarrow{Fx_1} & FB_1 & \xrightarrow{Fy_1} & FC_1 & \overset{\phi_{C_1,A_1}(\delta_1)}{\dashrightarrow} & \\ {\scriptstyle Fa}\downarrow & & {\scriptstyle Fb}\downarrow & & {\scriptstyle Fc}\downarrow & & \\ FA_2 & \xrightarrow[Fx_2]{} & FB_2 & \xrightarrow[Fy_2]{} & FC_2 & \underset{\phi_{C_2,A_2}(\delta_2)}{\dashrightarrow} & \end{array}$$

in $\mathcal{D}$ by the naturality of ϕ.

Later, we need the following observation about exact functors to exact categories, which can be easily proved by considering the extriangulated structure on exact categories.

Lemma 2.2.4. *Let $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated $\mathbb{U}$-category, $\mathcal{E}$ an exact $\mathbb{U}$-category, and $F\colon \mathcal{C} \to \mathcal{E}$ an additive functor. Then the following are equivalent.*

(1) *There is some ϕ which makes (F, ϕ) an exact functor between extriangulated categories.*

(2) *For each conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \dashrightarrow$ in $\mathcal{C}$, the following is a conflation in $\mathcal{E}$:*

$$0 \longrightarrow FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \longrightarrow 0.$$

Moreover, in this case, ϕ in (1) *is uniquely determined.*

Due to this fact, we call an additive functor $F\colon \mathcal{C} \to \mathcal{E}$ satisfying (2) above just an *exact functor*.

Let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $\mathbb{U} \in \mathbb{V}$. Then it is clear that the natural inclusion functor $\mathsf{ETcat}_{\mathbb{U}} \to \mathsf{ETcat}_{\mathbb{V}}$ is fully faithful. We consider the following full subcategories of $\mathsf{ETcat}_{\mathbb{V}}$.

- The category $\mathsf{ETcat}_{\widetilde{\mathbb{U}}}$ of $\mathbb{U}$-small extriangulated $\mathbb{V}$-categories.
- The category $\mathsf{ETCat}_{\mathbb{U}}$ of skeletally $\mathbb{U}$-small extriangulated $\mathbb{V}$-categories.

Note that a skeletally $\mathbb{U}$-small category is not necessarily locally $\mathbb{U}$-small in our definition (Definition 1.1.4).

In the remaining of this thesis, we often use the following convention:

Convention 2.2.5. Let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $\mathbb{U} \in \mathbb{V}$.

- An *extriangulated category* := an extriangulated $\mathbb{V}$-category.
- $\mathsf{ETcat} := \mathsf{ETcat}_{\widetilde{\mathbb{U}}}$.
- $\mathsf{ETCat} := \mathsf{ETCat}_{\mathbb{U}}$.
- $\mathsf{ETCAT} := \mathsf{ETcat}_{\mathbb{V}}$.

2.3 Definition and construction of Grothendieck monoids

We define the Grothendieck monoid by the universal property. Recall that all monoids are commutative.

Definition 2.3.1. Let $\mathbb{U}$ be a Grothendieck universe, and let $\mathcal{C}$ be an extriangulated $\mathbb{U}$-category.

(1) An *additive function* on $\mathcal{C}$ with values in a monoid M is a map $f\colon |\mathcal{C}| \to M$ satisfying the following conditions:
   (1) $f(0) = 0$ holds.
   (2) For every conflation
$$X \longrightarrow Y \longrightarrow Z \dashrightarrow$$
   in $\mathcal{C}$, we have $f(Y) = f(X) + f(Z)$ in M.

   We also say that $f\colon |\mathcal{C}| \to M$ *respects conflations.*

(2) A *Grothendieck monoid* $\mathsf{M}(\mathcal{C})$ is a monoid $\mathsf{M}(\mathcal{C})$ together with an additive function $\pi\colon |\mathcal{C}| \to \mathsf{M}(\mathcal{C})$ which satisfies the following universal property:
   - For any additive function $f\colon |\mathcal{C}| \to M$ with values in a monoid M, there exists a unique monoid homomorphism $\overline{f}\colon \mathsf{M}(\mathcal{C}) \to M$ such that $f = \overline{f}\pi$.

$$\begin{array}{ccc} |\mathcal{C}| & \xrightarrow{\ f\ } & M \\ {\scriptstyle\pi}\downarrow & \nearrow{\scriptstyle\overline{f}} & \\ \mathsf{M}(\mathcal{C}) & & \end{array}$$

   We often write $[X] := \pi(X)$ for $X \in |\mathcal{C}|$.

Since the Grothendieck monoid of $\mathcal{C}$ is defined by the universal property, we must show that it certainly exists. For later use, we give an explicit construction of $\mathsf{M}(\mathcal{C})$.

Definition 2.3.2. Let $\mathbb{U}$ be a Grothendieck universe and $\mathcal{C}$ an extriangulated $\mathbb{U}$-category. Let A and B be objects in $\mathcal{C}$.

(1) A and B are *conflation-related*, abbreviated by *c-related*, if there is a conflation
$$X \longrightarrow Y \longrightarrow Z \dashrightarrow$$
   such that either of the following holds in $\mathcal{C}$:
   (1) $Y \cong A$ and $X \oplus Z \cong B$, or
   (2) $Y \cong B$ and $X \oplus Z \cong A$.

   In this case, we write $A \sim_c B$.

(2) A and B in $\mathcal{C}$ are *conflation-equivalent*, abbreviated by *c-equivalent*, if there are objects $A_0, A_1, \dots, A_n$ in $\mathcal{C}$ for some $n \geq 0$ satisfying $A_0 \cong A$, $A_n \cong B$, and $A_i \sim_c A_{i+1}$ for each i. In this case, we write $A \approx_c B$.

It is immediate that $\sim_c$ and $\approx_c$ induce binary relations $\sim_c$ and $\approx_c$ on the set $|\mathcal{C}|$. Then clearly $\approx_c$ is an equivalence relation on $|\mathcal{C}|$ generated by $\sim_c$. On the other hand, $|\mathcal{C}|$ can be regarded as a monoid with the addition given by $A + B := A \oplus B$. Then we can state the following explicit construction of the Grothendieck monoid.

Proposition 2.3.3. *Let $\mathbb{U}$ be a Grothendieck universe and $\mathcal{C}$ an extriangulated $\mathbb{U}$-category. Then the following hold.*

(1) *$\approx_c$ is a monoid congruence on $|\mathcal{C}|$ (see Definition 2.1.2 for a monoid congruence).*

(2) *The quotient monoid $|\mathcal{C}|/{\approx_c}$ together with the natural projection $\pi\colon |\mathcal{C}| \twoheadrightarrow |\mathcal{C}|/{\approx_c}$ gives a Grothendieck monoid of $\mathcal{C}$.*

Proof. (1) We must show that $A \approx_c B$ implies $A \oplus C \approx_c B \oplus C$ holds for every A, B, and C in $\mathcal{C}$. By the definition of $\approx_c$, it clearly suffices to show that $A \sim_c B$ implies $A \oplus C \sim_c B \oplus C$, so suppose that $A \sim_c B$ holds. Then there is a conflation
$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \dashrightarrow$$
such that either (a) $Y \cong A$ and $X \oplus Z \cong B$ or (b) $Y \cong B$ and $X \oplus Z \cong A$. On the other hand, we have the following conflation:
$$C = C \longrightarrow 0 \dashrightarrow$$

since $\mathfrak{s}$ is an additive realization [NP19, Definition 2.10]. Moreover, since conflations are closed under finite direct sums (because $\mathfrak{s}$ is an additive realization), we obtain the following conflation:

$$X \oplus C \xrightarrow{f \oplus \mathsf{id}_C} Y \oplus C \xrightarrow{[g,0]} Z \dashrightarrow$$

Thus, we obtain $Y \oplus C \sim_c (X \oplus C) \oplus Z \cong (X \oplus Z) \oplus C$. In either case, we obtain $A \oplus C \sim_c B \oplus C$.

(2) By (1), we obtain the quotient monoid $|\mathcal{C}|/{\approx_c}$ and the natural monoid homomorphism $\pi\colon |\mathcal{C}| \twoheadrightarrow |\mathcal{C}|/{\approx_c}$. We first show that $\pi\colon |\mathcal{C}| \twoheadrightarrow |\mathcal{C}|/{\approx_c}$ respects conflations. Since π is a monoid homomorphism, we have $\pi(0) = 0$ and $\pi(X \oplus Y) = \pi(X) + \pi(Y)$ for all objects X and Y in $\mathcal{C}$. Let $X \to Y \to Z \dashrightarrow$ be a conflation in $\mathcal{C}$. Then we have $Y \sim_c X \oplus Z$, which implies $\pi(Y) = \pi(X \oplus Z) = \pi(X) + \pi(Y)$ in $|\mathcal{C}|/{\approx_c}$.

Next, suppose that we have additive function $f\colon |\mathcal{C}| \to M$ with values in a monoid M, and we will show that there is a unique monoid homomorphism $\overline{f}\colon |\mathcal{C}|/{\approx_c} \to M$ satisfying $f = \overline{f}\pi$. The uniqueness is clear since π is surjective, so we only show the existence of such $\overline{f}$. To obtain a well-defined map $\overline{f}\colon |\mathcal{C}|/{\approx_c} \to M$ satisfying $f = \overline{f}\pi$, it clearly suffices to show that $A \sim_c B$ implies $f(A) = f(B)$, so suppose $A \sim_c B$. Then there is a conflation $X \to Y \to Z \dashrightarrow$ such that either (a) $Y \cong A$ and $X \oplus Z \cong B$ or (b) $Y \cong B$ and $X \oplus Z \cong A$. On the other hand, since f respects conflations, we have $f(Y) = f(X) + f(Z)$. Moreover, we have the following split conflation

$$X \xrightarrow{\left[\begin{smallmatrix}\mathsf{id}_X\\0\end{smallmatrix}\right]} X \oplus Z \xrightarrow{[0,\,\mathsf{id}_Z]} Z \overset{0}{\dashrightarrow}$$

because $\mathfrak{s}$ is an additive realization. Therefore, $f(X) + f(Z) = f(X \oplus Z)$ holds, and hence $f(Y) = f(X \oplus Z)$. Thus $f(A) = f(B)$ holds in either case. □

Let $\mathbb{U}$ be a Grothendieck universe and $\mathcal{C}$ an extriangulated $\mathbb{U}$-category. In what follows, we often identify $\mathsf{M}(\mathcal{C})$ with $|\mathcal{C}|/{\approx_c}$ and write $[X] \in |\mathcal{C}|/{\approx_c}$ to represent an element for $X \in \mathcal{C}$. Note that $\mathsf{M}(\mathcal{C})$ is a $\mathbb{U}$-set by the construction.

The assignment $\mathcal{C} \mapsto \mathsf{M}(\mathcal{C})$ actually gives a functor:

Proposition 2.3.4. *Let $\mathbb{U}$ be a Grothendieck universe. We have a functor $\mathsf{M}(-)\colon \mathsf{ETcat}_{\mathbb{U}} \to \mathsf{Mon}_{\mathbb{U}}$ defined as follows.*

- *To $\mathcal{C} \in \mathsf{ETcat}_{\mathbb{U}}$, we associate the Grothendieck monoid $\mathsf{M}(\mathcal{C}) \in \mathsf{Mon}_{\mathbb{U}}$.*
- *To an exact functor $F\colon \mathcal{C} \to \mathcal{D}$, we associate a monoid homomorphism $\mathsf{M}(F)\colon \mathsf{M}(\mathcal{C}) \to \mathsf{M}(\mathcal{D})$ defined by $[C] \mapsto [F(C)]$.*

Moreover, if $F\colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ is an exact equivalence, then $\mathsf{M}(F)\colon \mathsf{M}(\mathcal{C}) \to \mathsf{M}(\mathcal{D})$ is a monoid isomorphism.

Since the proof is straightforward by using the defining universal property of the Grothendieck monoid, we omit it.

Let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $\mathbb{U} \in \mathbb{V}$. Then the following diagram commutes by the construction of Grothendieck monoids:

$$\begin{array}{ccc} \mathsf{ETcat}_{\mathbb{V}} & \xrightarrow{\mathsf{M}(-)} & \mathsf{Mon}_{\mathbb{V}} \\ \uparrow & & \uparrow \\ \mathsf{ETcat}_{\mathbb{U}} & \xrightarrow{\mathsf{M}(-)} & \mathsf{Mon}_{\mathbb{U}}\,. \end{array}$$

If $\mathcal{C}$ is a skeletally $\mathbb{U}$-small extriangulated $\mathbb{V}$-category, then $\mathsf{M}(\mathcal{C})$ is clearly a $\mathbb{U}$-small monoid. Thus, the functor $\mathsf{M}(-)\colon \mathsf{ETcat}_{\mathbb{V}} \to \mathsf{Mon}_{\mathbb{V}}$ restricts to a functor $\mathsf{M}(-)\colon \mathsf{ETCat}_{\mathbb{U}} \to \mathsf{Mon}_{\widetilde{\mathbb{U}}}$ (see §1.1 and 2.2 for the notations).

We obtain the following result by the discussion above and Propositions 1.1.16 and 1.1.17.

Proposition 2.3.5. *Let $\mathbb{U}$ and $\mathbb{V}$ be Grothendieck universes such that $\mathbb{U} \in \mathbb{V}$.*

(1) *We have the following isomorphism of monoids for a ring Λ belonging to $\mathbb{U}$.*
   - $\mathsf{M}(\mathsf{mod}_{\mathbb{U}}\,\Lambda) \xrightarrow{\cong} \mathsf{M}(\mathsf{mod}_{\widetilde{\mathbb{U}}}\,\Lambda) \xrightarrow{\cong} \mathsf{M}(\mathsf{mod}_{\mathbb{V}}\,\Lambda)$.

(2) *We have the following isomorphisms of monoids for a scheme X belonging to $\mathbb{U}$.*
   - $\mathsf{M}(\mathsf{coh}_{\mathbb{U}}\,X) \xrightarrow{\cong} \mathsf{M}(\mathsf{coh}_{\widetilde{\mathbb{U}}}\,X) \xrightarrow{\cong} \mathsf{M}(\mathsf{coh}_{\mathbb{V}}\,X)$.
   - $\mathsf{M}(\mathsf{vect}_{\mathbb{U}}\,X) \xrightarrow{\cong} \mathsf{M}(\mathsf{vect}_{\widetilde{\mathbb{U}}}\,X) \xrightarrow{\cong} \mathsf{M}(\mathsf{vect}_{\mathbb{V}}\,X)$.

- $\mathsf{M}(\mathsf{tor}_{\mathbb{U}}\, X) \xrightarrow{\cong} \mathsf{M}(\mathsf{tor}_{\widetilde{\mathbb{U}}}\, X) \xrightarrow{\cong} \mathsf{M}(\mathsf{tor}_{\mathbb{V}}\, X)$.

Finally, we compare the Grothendieck monoid $\mathsf{M}(\mathcal{C})$ with the Grothendieck group $\mathsf{K}_0(\mathcal{C})$.

Remark 2.3.6. Let $\mathbb{U}$ be a Grothendieck universe and $\mathcal{C}$ an extriangulated $\mathbb{U}$-category.

(1) Recall that the *Grothendieck group* $\mathsf{K}_0(\mathcal{C})$ of $\mathcal{C}$ is defined by

$$\mathsf{K}_0(\mathcal{C}) := \bigoplus_{X \in |\mathcal{C}|} \mathbb{Z}X \Big/ \langle A - B + C \mid A \to B \to C \dashrightarrow \text{ is a conflation} \rangle.$$

The image of $X \in |\mathcal{C}|$ in $\mathsf{K}_0(\mathcal{C})$ is denoted by $[X]$. Then there is a natural monoid homomorphism

$$\rho\colon \mathsf{M}(\mathcal{C}) \to \mathsf{K}_0(\mathcal{C}), \quad [X] \mapsto [X].$$

Then the defining properties of $\mathsf{M}(\mathcal{C})$ and $\mathsf{K}_0(\mathcal{C})$ immediately show that $(\mathsf{K}_0(\mathcal{C}), \rho\colon \mathsf{M}(\mathcal{C}) \to \mathsf{K}_0(\mathcal{C}))$ is the group completion of $\mathsf{M}(\mathcal{C})$ (see Definition 2.1.5). Moreover, we have a natural isomorphism of functors $\mathsf{K}_0(-) \simeq \mathsf{gp} \circ \mathsf{M}(-)\colon \mathsf{ETcat}_{\mathbb{U}} \to \mathsf{Ab}_{\mathbb{U}}$.

(2) The natural map ρ is injective if and only if $\mathsf{M}(\mathcal{C})$ is cancellative by Proposition 2.1.7. In this case, the Grothendieck monoid $\mathsf{M}(\mathcal{C})$ can be identified with the positive part

$$\mathsf{K}_0^+(\mathcal{C}) := \{[X] \in \mathsf{K}_0(\mathcal{C}) \mid X \in \mathcal{C}\}$$

of the Grothendieck group. Thus, if $\mathsf{M}(\mathcal{C})$ is cancellative, the computation of $\mathsf{M}(\mathcal{C})$ becomes much easier. However, not much is known about the conditions for an extriangulated category $\mathcal{C}$ under which $\mathsf{M}(\mathcal{C})$ becomes cancellative.

(3) An element of $\mathsf{M}(\mathcal{C})$ can be expressed by $[X]$ for some single object $X \in \mathcal{C}$, while an element of $\mathsf{K}_0(\mathcal{C})$ can only be expressed by $[X] - [Y]$ for some objects $X, Y \in \mathcal{C}$ in general. It is an advantage of the Grothendieck monoid.

2.4 Grothendieck monoids of exact and triangulated categories

In this section, we discuss the Grothendieck monoids of exact and triangulated categories, respectively. We will see that the invertible elements of the Grothendieck monoid of an exact category are only 0, while every element of that of a triangulated category is invertible. Fix a Grothendieck universe $\mathbb{U}$.

We first consider the Grothendieck monoid of an exact category. Let $\mathcal{E}$ be an exact $\mathbb{U}$-category. We regard it as an extriangulated $\mathbb{U}$-category. Then our $\mathsf{M}(\mathcal{E})$ coincides with the Grothendieck monoid of an exact category which is studied in [BG16, Eno22] by the universal property.

The Grothendieck monoid of an exact category has the following properties.

Proposition 2.4.1. *Let $\mathcal{C}$ be an extriangulated $\mathbb{U}$-category. Consider the following conditions.*

(1) *$\mathcal{C}$ is an exact category (with the usual extriangulated structure).*
(2) *If $X \to 0 \to Y \dashrightarrow$ is a conflation in $\mathcal{C}$, then $X \cong 0$ holds in $\mathcal{C}$ (or equivalently, $Y \cong 0$ holds).*
(3) *If $[A] = 0$ in $\mathsf{M}(\mathcal{C})$, then $A \cong 0$ holds in $\mathcal{C}$.*
(4) *$\mathsf{M}(\mathcal{C})$ is sharp (see Definition 2.1.3).*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) holds.

Proof. The conditions in (2) are easily seen to be equivalent by considering the long exact sequence associated to the conflation.

(1) $\Rightarrow$ (2): This is clear since an inflation $X \to 0$ in an exact category must be monic.

(2) $\Rightarrow$ (3): It suffices to show that $A \sim_c 0$ implies $A \cong 0$. Then there exists a conflation $X \to Y \to Z \dashrightarrow$ such that either (a) $Y \cong A$ and $X \oplus Z \cong 0$ or (b) $Y \cong 0$ and $X \oplus Z \cong A$. In the case (a), both X and Z are isomorphic to 0 since $X \oplus Z \cong 0$. Thus $A \cong Y \cong 0$ by the conflation $0 \to Y \to 0 \dashrightarrow$. In the case (b), both X and Z are isomorphic to 0 by the assumption. Thus $A \cong X \oplus Z \cong 0$.

(3) $\Rightarrow$ (2): Suppose that we have a conflation $X \to 0 \to Y \dashrightarrow$. Then we have $[X \oplus Y] = [X] + [Y] = [0] = 0$ in $\mathsf{M}(\mathcal{C})$. Thus (4) implies $X \oplus Y \cong 0$, so $X \cong Y \cong 0$ holds.

(3) $\Rightarrow$ (4): Suppose that $x + y = 0$ holds in $\mathsf{M}(\mathcal{C})$. There are X and Y in $\mathcal{C}$ satisfying $[X] = x$ and $[Y] = y$, and $[X \oplus Y] = [X] + [Y] = 0$ holds. Thus (4) implies $X \oplus Y \cong 0$, which shows $X \cong Y \cong 0$. Therefore, $x = y = 0$ holds. □

It is natural to ask whether the conditions in Proposition 2.4.1 are equivalent. However, there is a counterexample for (4) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) (see [BHST, Example 5.2, Remark 5.3]). Thus, we pose the following question.

Question 2.4.2. *Let $\mathcal{C}$ be an extriangulated $\mathbb{U}$-category. Are the following conditions equivalent?*
1. *If $X \to 0 \to Y \dashrightarrow$ is a conflation in $\mathcal{C}$, then $X \cong 0$ holds in $\mathcal{C}$ (or equivalently, $Y \cong 0$ holds).*
2. *$\mathsf{M}(\mathcal{C})$ is sharp.*

Next, we consider the Grothendieck monoid of a triangulated category.

Proposition 2.4.3. *Let $\mathcal{T}$ be a triangulated $\mathbb{U}$-category.*
1. *$\mathsf{M}(\mathcal{T})$ is a group.*
2. *There is a natural isomorphism $\mathsf{M}(\mathcal{T}) \cong \mathsf{K}_0(\mathcal{T})$ of groups.*

Proof. (1) For every element $[X] \in \mathsf{M}(\mathcal{T})$, there is a triangle

$$X \longrightarrow 0 \longrightarrow \Sigma X = \Sigma X.$$

This means that we have a conflation $X \to 0 \to \Sigma X \dashrightarrow$, which implies $[X] + [\Sigma X] = 0$ in $\mathsf{M}(\mathcal{T})$. Therefore, every element in $\mathsf{M}(\mathcal{T})$ is invertible, that is, $\mathsf{M}(\mathcal{T})$ is a group.

(2) It follows from Remark 2.3.6 (1). □

In general, the converse does not hold: there is an extriangulated $\mathbb{U}$-category which is not triangulated but whose Grothendieck monoid is a group (see Corollary 5.3.2 for example).

The following example says the natural inclusion from an abelian category to its bounded derived category *categorifies* the group completion of the Grothendieck monoid.

Example 2.4.4. Let $\mathcal{A}$ be an abelian $\mathbb{U}$-category. Then $\mathcal{A}$ can be regarded as an extension-closed subcategory of its bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. The natural inclusion functor $\mathcal{A} \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ induces a monoid homomorphism

$$\mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathsf{D}^{\mathrm{b}}(\mathcal{A})) = \mathsf{K}_0(\mathsf{D}^{\mathrm{b}}(\mathcal{A})) \cong \mathsf{K}_0(\mathcal{A})$$

by Proposition 2.4.3. In fact, the monoid homomorphism $\mathsf{M}(\mathcal{A}) \to \mathsf{K}_0(\mathcal{A})$ coincides with the group completion of $\mathsf{M}(\mathcal{A})$ (see Remark 2.3.6).

Chapter 3

Classifying subcategories via Grothendieck monoids

Throughout this chapter, we fix Grothendieck universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$ and use Convention 1.1.12 and 2.2.5. Moreover, we assume that every subcategory is a strictly full subcategory. Hereafter, $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a skeletally small extriangulated category.

In this chapter, we give classifications of several subcategories of $\mathcal{C}$ via its Grothendieck monoid $\mathsf{M}(\mathcal{C})$. More precisely, we consider the following assignments:

- For a subcategory $\mathcal{D}$ of $\mathcal{C}$, we define a subset $\mathsf{M}_{\mathcal{D}}$ of $\mathsf{M}(\mathcal{C})$ by
$$\mathsf{M}_{\mathcal{D}} := \{[D] \in \mathsf{M}(\mathcal{C}) \mid D \in \mathcal{D}\}.$$
- For a subset N of $\mathsf{M}(\mathcal{C})$, we define a subcategory $\mathcal{D}_N$ of $\mathcal{C}$ by
$$\mathcal{D}_N := \{X \in \mathcal{C} \mid [X] \in N\}.$$

In what follows, we will show that these maps give bijections between certain subcategories of $\mathcal{C}$ and certain subsets of $\mathsf{M}(\mathcal{C})$. We freely identify $\mathsf{M}(\mathcal{C})$ with $|\mathcal{C}|/{\approx_c}$ by Proposition 2.3.3. Note that the collection of subcategories of $\mathcal{C}$ forms a set since $\mathcal{C}$ is skeletally small.

3.1 Classifying c-closed subcategories via subsets

The key notion of this chapter is *c-closed subcategories* defined below. These are the largest class of subcategories which can be classified via the Grothendieck monoid (see Proposition 3.1.3). In §3.3 and 3.6, we will show that Serre subcategories and dense 2-out-of-3 subcategories are c-closed and classify them via the Grothendieck monoid.

Definition 3.1.1. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be *closed under c-equivalences* if for any conflation $A \to B \to C \dashrightarrow$, we have that $B \in \mathcal{D}$ if and only if $A \oplus C \in \mathcal{D}$. We also say that $\mathcal{D}$ is a *c-closed subcategory* for short.

The following lemma follows immediately from Proposition 2.3.3. We freely use this characterization in what follows.

Lemma 3.1.2. *The following are equivalent for a subcategory $\mathcal{D}$ of $\mathcal{C}$.*
1. *$\mathcal{D}$ is closed under c-equivalences.*
2. *For every $X, Y \in \mathcal{C}$, if $X \sim_c Y$ (cf. Definition 2.3.2) holds and X belongs to $\mathcal{D}$, then Y also belongs to $\mathcal{D}$.*
3. *For every $X, Y \in \mathcal{C}$, if $[X] = [Y]$ holds in $\mathsf{M}(\mathcal{C})$ and X belongs to $\mathcal{D}$, then Y also belongs to $\mathcal{D}$.*

The relation between the subcategories $\mathcal{D}_N$ defined above and c-closed subcategories is as follows. Note that a subcategory is not assumed to be additive.

Proposition 3.1.3. *The following hold.*

(1) $\mathcal{D}_N$ *is closed under c-equivalences for any subset* N *of* $\mathsf{M}(\mathcal{C})$.
(2) *The assignments* $\mathcal{D} \mapsto \mathsf{M}_{\mathcal{D}}$ *and* $N \mapsto \mathcal{D}_N$ *give mutually inverse bijections between the set of c-closed subcategories of* $\mathcal{C}$ *and the power set of* $\mathsf{M}(\mathcal{C})$.

Proof. (1) Let $X \in \mathcal{D}_N$ and $Y \in \mathcal{C}$, and suppose that $[X] = [Y]$ in $\mathsf{M}(\mathcal{C})$. Then we have $[Y] = [X] \in N$, and hence Y also belongs to $\mathcal{D}_N$. This proves that $\mathcal{D}_N$ is closed under c-equivalences.

(2) First, we prove that $\mathsf{M}_{\mathcal{D}_N} = N$ holds for each subset N of $\mathsf{M}(\mathcal{C})$. Clearly, we have $\mathsf{M}_{\mathcal{D}_N} \supseteq N$. Take $[X] \in \mathsf{M}_{\mathcal{D}_N}$. Then there is an object $Y \in \mathcal{D}_N$ such that $[X] = [Y]$ holds in $\mathsf{M}(\mathcal{C})$. Since $\mathcal{D}_N$ is closed under c-equivalences by (1), we have that X is also in $\mathcal{D}_N$, that is, $[X] \in N$. Hence, we obtain $\mathsf{M}_{\mathcal{D}_N} = N$.

Next, we prove that $\mathcal{D}_{\mathsf{M}_{\mathcal{D}}} = \mathcal{D}$ holds for a c-closed subcategory $\mathcal{D}$ of $\mathcal{C}$. Clearly, we have $\mathcal{D}_{\mathsf{M}_{\mathcal{D}}} \supseteq \mathcal{D}$. Take $X \in \mathcal{D}_{\mathsf{M}_{\mathcal{D}}}$, then we have $[X] \in \mathsf{M}_{\mathcal{D}}$, so there exists an object $Y \in \mathcal{D}$ satisfying $[X] = [Y]$ in $\mathsf{M}(\mathcal{C})$. Since $\mathcal{D}$ is closed under c-equivalences, we obtain $X \in \mathcal{D}$. Therefore, we obtain $\mathcal{D}_{\mathsf{M}_{\mathcal{D}}} = \mathcal{D}$, which completes the proof. $\square$

In the rest of this section, we will discuss the conditions under which all subcategories are classified via $\mathsf{M}(\mathcal{C})$, that is, all subcategories are c-closed. We recommend that the reader skips the remaining part of this section in the first reading. Let us begin with a simple observation.

Proposition 3.1.4. *The following conditions are equivalent.*
(1) *All subcategories of* $\mathcal{C}$ *are c-closed.*
(2) *For any conflation* $A \to B \to C \dashrightarrow$ *in* $\mathcal{C}$, *we have* $B \cong A \oplus C$.

Proof. (1) $\Rightarrow$ (2): Let $A \to B \to C \dashrightarrow$ be a conflation in $\mathcal{C}$. Consider the subcategory $\mathcal{X}$ consisting of objects isomorphic to B. Since $\mathcal{X}$ is c-closed, it contains $A \oplus C$. This means $A \oplus C \cong B$.

(2) $\Rightarrow$ (1): Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$. For any conflation $A \to B \to C \dashrightarrow$, we have $B \cong A \oplus C$ by the assumption (2). Thus $B \in \mathcal{X}$ if and only if $A \oplus C \in \mathcal{X}$. This proves that $\mathcal{X}$ is c-closed. $\square$

Let us call a conflation $A \to B \to C \dashrightarrow$ *quasi-split* if $B \cong A \oplus C$ holds. There are subtle differences between split and quasi-split conflations.

Example 3.1.5. There are two short exact sequences of abelian groups:

$$\delta\colon 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \quad \text{and} \quad \epsilon\colon 0 \to 0 \to (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{N}} \xrightarrow{=} (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{N}} \to 0.$$

Then the short exact sequence obtained by their direct sum

$$\delta \oplus \epsilon\colon 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{N}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{N}} \longrightarrow 0$$

is quasi-split but not split. Indeed, the element $\delta \oplus \epsilon$ corresponds to $(\delta, 0)$ by the natural isomorphism $\mathrm{Ext}^1_{\mathbb{Z}}\left(\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{N}}, \mathbb{Z}\right) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}\left((\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{N}}, \mathbb{Z}\right)$, and it is nonzero since δ is not a split exact sequence.

From this example, a quasi-split conflation is not a split conflation in general. However, there are the following examples where this claim holds.

Example 3.1.6. In the following classes of extriangulated categories $\mathcal{C}$, every quasi-split conflation is a split conflation.

- $\mathcal{C}$ is an exact R-category over a commutative ring R such that $\mathcal{C}(A, B)$ is an R-module of finite length for any $A, B \in \mathcal{C}$. Indeed, for any quasi-split conflation $0 \to A \to B \xrightarrow{f} C \to 0$, we have an exact sequence

  $$0 \longrightarrow \mathcal{C}(C, A) \longrightarrow \mathcal{C}(C, B) \xrightarrow{\mathcal{C}(C,f)} \mathcal{C}(C, C)$$

  of R-modules. Now we have $\mathcal{C}(C, B) \cong \mathcal{C}(C, A) \oplus \mathcal{C}(C, C)$ by the assumption. Considering the lengths of R-modules in the above exact sequence, we conclude that $\mathcal{C}(C, f)$ is surjective, and this implies that the conflation $0 \to A \to B \xrightarrow{f} C \to 0$ splits.
- $\mathcal{C}$ is an extension-closed subcategory of $\mathsf{mod}\,\Lambda$, where Λ is an algebra over a commutative noetherian ring R with $\Lambda \in \mathsf{mod}\,R$. This follows from the result in [Miy67, Theorem 1], which states that any quasi-split short exact sequence in $\mathsf{mod}\,\Lambda$ actually splits.

The above examples lead us to the next question.

Question 3.1.7. *When does a quasi-split conflation become a split conflation? More concretely, in the following classes of exact categories $\mathcal{C}$, does this question have a positive answer?*

- *$\mathcal{C}$ is an exact R-category over a commutative noetherian ring R such that $\mathcal{C}(A,B) \in \operatorname{mod} R$ holds for any $A, B \in \mathcal{C}$.*
- *$\mathcal{C}$ is a noetherian exact category, that is for any $X \in \mathcal{C}$, the poset of admissible subobjects of X satisfies the ascending chain condition (see [Eno22, Section 2] for this poset).*

We now return to considering when all subcategories are classified via $\mathsf{M}(\mathcal{C})$. An extriangulated category $\mathcal{C}$ is said to be *quasi-split* if every conflation in $\mathcal{C}$ is quasi-split. Recall that this condition is equivalent to the condition under which all subcategories of $\mathcal{C}$ can be classified via the Grothendieck monoid by Proposition 3.1.4. On the other hand, if every conflation of an extriangulated category $\mathcal{C}$ is a split conflation, then every inflation is a monomorphism, and every deflation is an epimorphism. Thus $\mathcal{C}$ becomes an exact category (see [NP19, Corollary 3.18]). Such an exact category $\mathcal{C}$ is called a *split exact category*. It is clear that a split exact category is quasi-split. Thus, we have the following natural question.

Question 3.1.8. *When does a quasi-split extriangulated category $\mathcal{C}$ become a split exact category?*

Example 3.1.9. Question 3.1.8 has a positive answer for the following classes of extriangulated categories $\mathcal{C}$.

- $\mathcal{C}$ is one of the exact categories in Example 3.1.6.
- More generally, $\mathcal{C}$ is an exact category in which Question 3.1.7 has a positive answer.
- $\mathcal{C}$ is an extriangulated category with enough projectives. See Proposition 3.1.10 below.

We recall some terminology to prove Proposition 3.1.10. An object P of $\mathcal{C}$ is *projective* if $\mathbb{E}(P, X) = 0$ for any $X \in \mathcal{C}$. We denote by $\operatorname{Proj} \mathcal{C}$ the category of projective objects of $\mathcal{C}$. Clearly $\operatorname{Proj} \mathcal{C}$ is closed under extensions and direct summands. An extriangulated category $\mathcal{C}$ *has enough projectives* if for any $A \in \mathcal{C}$, there exists a deflation $P \twoheadrightarrow A$ from a projective object P.

Proposition 3.1.10. *Suppose that an extriangulated category $\mathcal{C}$ has enough projectives. Then the following conditions are equivalent.*

1. *Every subcategory of $\mathcal{C}$ is c-closed.*
2. *Every additive subcategory of $\mathcal{C}$ is c-closed.*
3. *Every extension-closed subcategory of $\mathcal{C}$ is c-closed.*
4. *$\mathcal{C}$ is a split exact category.*
5. *$\mathcal{C}$ is a quasi-split extriangulated category.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are clear, so we only prove (3) $\Rightarrow$ (4). Let $A \rightarrowtail B \twoheadrightarrow C \dashrightarrow$ be a conflation in $\mathcal{C}$. Since $\mathcal{C}$ has enough projectives, there is a conflation $X \rightarrowtail P \twoheadrightarrow C \dashrightarrow$ in $\mathcal{C}$ with $P \in \operatorname{Proj} \mathcal{C}$. The assumption (3) implies that $\operatorname{Proj} \mathcal{C}$ is c-closed, and hence $C \oplus X \in \operatorname{Proj} \mathcal{C}$. Then C is also projective since $\operatorname{Proj} \mathcal{C}$ is closed under direct summands. Therefore the conflation $A \rightarrowtail B \twoheadrightarrow C \dashrightarrow$ splits. This proves that $\mathcal{C}$ is a split exact category. ☐

3.2 Preliminaries: faces

In this section, we study faces a class of submonoids which corresponds to Serre subcategories in §3.3. Hereafter, M is a monoid.

Definition 3.2.1.

1. A submonoid F of M is called a *face* if for all $x, y \in M$, we have that $x + y \in F$ if and only if both $x \in F$ and $y \in F$.
2. $\operatorname{Face} M$ denotes the set of faces of M.

Remark 3.2.2. The set $M^{\times}$ of units (see Remark 2.1.4) is the smallest face, and M itself is the largest face. In particular, $\operatorname{Face} M$ is a singleton if and only if M is a group.

Example 3.2.3. Let M be a free monoid of rank 2 with basis e_1 and e_2.

(1) $\mathbb{N}(e_1+e_2)$ is a submonoid of M but not a face.
(2) $\operatorname{Face} M = \{M, \mathbb{N}e_1, \mathbb{N}e_2, 0\}$ holds.

Let us give an explicit description of the face generated by a subset, which is useful to study faces.

Fact 3.2.4 (cf. [Ogu18, Proposition I.1.4.2])**.** *Let S be a subset of a monoid M.*

(1) *The* submonoid $\langle S\rangle_{\mathbb{N}}$ of M generated by S, *i.e. the submonoid*

$$\langle S\rangle_{\mathbb{N}} := \left\{ \sum_{i=1}^{m} n_i x_i \,\middle|\, m, n_i \in \mathbb{N}, x_i \in S \right\}$$

is the smallest submonoid of M containing S. Note that $\langle\emptyset\rangle_{\mathbb{N}} = \{0\}$.

(2) *The* face $\langle S\rangle_{\mathrm{face}}$ of M generated by S, *i.e. the submonoid*

$$\langle S\rangle_{\mathrm{face}} := \{x \in M \mid \textit{there exists } y \in M \textit{ such that } x+y \in \langle S\rangle_{\mathbb{N}}\}$$

is the smallest face of M containing S.

Let $f\colon M \to N$ be a monoid homomorphism. For any face F of N, the inverse image $f^{-1}(F)$ is also a face of M. Thus, we have an inclusion-preserving map $\operatorname{Face}(f)\colon \operatorname{Face}(N) \to \operatorname{Face}(M)$. The following lemma is obvious but useful.

Lemma 3.2.5. *The map $\operatorname{Face}(f)\colon \operatorname{Face}(N) \to \operatorname{Face}(M)$ is injective for a surjective monoid homomorphism $f\colon M \to N$.*

Proof. It is straightforward. □

Let us consider a finiteness condition on a monoid and classify faces of a monoid satisfying it.

Definition 3.2.6.

(1) M is *finitely generated* if $M = \langle S\rangle_{\mathbb{N}}$ for a finite subset S of M.
(2) A face F of M is *finitely generated* if $F = \langle S\rangle_{\mathrm{face}}$ for a finite subset S of F.

Remark 3.2.7.

(1) If M is finitely generated, then it is finitely generated as a face.
(2) A face F of M is finitely generated if and only if $F = \langle x\rangle_{\mathrm{face}}$ for some element $x \in M$. Indeed, if $F = \langle S\rangle_{\mathrm{face}}$ for a finite subset S of M, then we can easily see that $F = \left\langle \sum_{s\in S} s\right\rangle_{\mathrm{face}}$.

Lemma 3.2.8. *If M is generated by a (not necessarily finite) subset $S \subseteq M$, then the map*

$$\langle - \rangle_{\mathrm{face}} : \mathsf{P}(S) \to \operatorname{Face}(M), \quad A \mapsto \langle A\rangle_{\mathrm{face}}$$

is an inclusion-preserving surjection, where $\mathsf{P}(S)$ is the power set of S.

Proof. Let F be a face of M and set $S_F := \{x \in S \mid x \in F\}$. We want to show that $F = \langle S_F\rangle_{\mathrm{face}}$. We may assume that $F \neq 0$. It is clear that $F \supseteq \langle S_F\rangle_{\mathrm{face}}$. Take $0 \neq x \in F$. Then $x = \sum_{i=1}^{m} n_i s_i$ for some $s_i \in S$ and $0 \neq n_i \in \mathbb{N}$ by Fact 3.2.4. Since F is a face, we obtain that $s_i \in F$ for all i, which shows $x \in \langle S_F\rangle_{\mathrm{face}}$. Thus, we conclude that $F = \langle S_F\rangle_{\mathrm{face}}$ and $\langle - \rangle_{\mathrm{face}} : \mathsf{P}(S) \to \operatorname{Face}(M)$ is surjective. Note that we actually proved $F = \langle S_F\rangle_{\mathbb{N}}$. □

Corollary 3.2.9. *If M is finitely generated, then* $\operatorname{Face}(M)$ *is a finite set.*

Proof. There is a finite subset S of M such that $M = \langle S\rangle_{\mathbb{N}}$ because M is finitely generated. Then $\mathsf{P}(S)$ is also a finite set, and we conclude that $\operatorname{Face}(M)$ is a finite set by Lemma 3.2.8. □

Example 3.2.10. Let M be a free monoid with basis $\{e_i \mid i \in I\}$. Then it is clear that the map $\langle - \rangle_{\mathrm{face}} : \mathsf{P}(\{e_i \mid i \in I\}) \to \operatorname{Face}(M)$ is bijective.

3.3 Classifying Serre subcategories via faces

In this section, we establish a bijection between the set of Serre subcategories of $\mathcal{C}$ and the set of faces of $\mathsf{M}(\mathcal{C})$, which generalizes [Bro97].

Definition 3.3.1. An additive subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a *Serre subcategory* if for any conflation

$$X \longrightarrow Y \longrightarrow Z \dashrightarrow$$

in $\mathcal{C}$, we have that both X and Z are in $\mathcal{D}$ if and only if $Y \in \mathcal{D}$.

In particular, a Serre subcategory is extension-closed, so it can be regarded as an extriangulated category in itself.

The relation between Serre subcategories and c-closed subcategories is the following.

Proposition 3.3.2. *A subcategory of $\mathcal{C}$ is Serre if and only if it is closed under finite direct sums, direct summands, and c-equivalences.*

Proof. It is clear that a Serre subcategory is closed under finite direct sums and direct summands. Let $A \to B \to C \dashrightarrow$ be a conflation in $\mathcal{C}$. For any additive subcategory $\mathcal{D}$ closed under direct summands, $A \oplus C$ belongs to $\mathcal{D}$ if and only if both A and C belong to $\mathcal{D}$. Thus, we can conclude that $\mathcal{D}$ is Serre if and only if it is closed under c-equivalences by comparing Definition 3.1.1 and 3.3.1. □

Now we can establish a bijection between Serre subcategories and faces.

Proposition 3.3.3. *The bijection in Proposition 3.1.3 (2) restricts to the bijection between the set* $\mathsf{Serre}\,\mathcal{C}$ *of Serre subcategories of $\mathcal{C}$ and the set* $\mathsf{Face}\,\mathsf{M}(\mathcal{C})$ *of faces of* $\mathsf{M}(\mathcal{C})$ *(see Definition 3.2.1).*

Proof. Let $\mathcal{S}$ be a Serre subcategory of $\mathcal{C}$ and F a face of $\mathsf{M}(\mathcal{C})$. We already know $\mathcal{D}_{\mathsf{M}_{\mathcal{S}}} = \mathcal{S}$ and $\mathsf{M}_{\mathcal{D}_F} = F$ by Propositions 3.1.3 and 3.3.2. Hence, we only need to show that $\mathsf{M}_{\mathcal{S}}$ is a face and $\mathcal{D}_F$ is a Serre subcategory.

We first prove that $\mathsf{M}_{\mathcal{S}}$ is a face. Note that $\mathcal{S}$ is closed under finite direct sums, direct summands, and c-equivalences by Proposition 3.3.2. It is clear that $\mathsf{M}_{\mathcal{S}}$ is a submonoid of $\mathsf{M}(\mathcal{C})$ since $\mathcal{S}$ is closed under direct sums. Suppose that $[X] + [Y] \in \mathsf{M}_{\mathcal{S}}$ for some objects $X, Y \in \mathcal{C}$. By the definition of $\mathsf{M}_{\mathcal{S}}$, there exists an object $Z \in \mathcal{S}$ such that $[Z] = [X] + [Y] = [X \oplus Y]$. Then we have $X \oplus Y \in \mathcal{S}$ because $\mathcal{S}$ is closed under c-equivalences. Since $\mathcal{S}$ is closed under direct summands, both X and Y belong to $\mathcal{S}$, and hence $[X], [Y] \in \mathsf{M}_{\mathcal{S}}$. This proves that $\mathsf{M}_{\mathcal{S}}$ is a face of $M(\mathcal{C})$.

Next, we prove that $\mathcal{D}_F$ is a Serre subcategory. It is obvious that $\mathcal{D}_F$ is closed under finite direct sums since F is a submonoid of $\mathsf{M}(\mathcal{C})$. We already know that $\mathcal{D}_F$ is c-closed by Proposition 3.1.3 (1). Thus, it is enough to show that $\mathcal{D}_F$ is closed under direct summands by Proposition 3.3.2. Let X and Y be objects in $\mathcal{C}$ with $X \oplus Y \in \mathcal{D}_F$. Then $[X] + [Y] = [X \oplus Y] \in F$. Because F is a face of $\mathsf{M}(\mathcal{C})$, both $[X]$ and $[Y]$ belong to F, which implies both X and Y belong to $\mathcal{D}_F$. Therefore, $\mathcal{D}_F$ is a Serre subcategory of $\mathcal{C}$. □

Finally, we compare $\mathsf{M}_{\mathcal{D}}$ with $\mathsf{M}(\mathcal{D})$ for an extension-closed subcategory $\mathcal{D} \subseteq \mathcal{C}$. The natural inclusion functor $\iota\colon \mathcal{D} \hookrightarrow \mathcal{C}$ induces the monoid homomorphism $\mathsf{M}(\iota)\colon \mathsf{M}(\mathcal{D}) \to \mathsf{M}(\mathcal{C})$ by Proposition 2.3.4. Clearly, the image of $\mathsf{M}(\iota)$ coincides with $\mathsf{M}_{\mathcal{D}}$. Thus, we have a surjective monoid homomorphism $\mathsf{M}(\mathcal{D}) \twoheadrightarrow \mathsf{M}_{\mathcal{D}}$. This monoid homomorphism is not injective in general, as the following example shows.

Example 3.3.4. Consider the polynomial ring $\Bbbk[T]$ over a field $\Bbbk$. Then the natural inclusion functor $\mathsf{mod}\,\Bbbk[T] \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Bbbk[T])$ induces the group completion

$$\mathsf{M}(\mathsf{mod}\,\Bbbk[T]) \to \mathsf{M}\left(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Bbbk[T])\right) \cong K_0(\mathsf{mod}\,\Bbbk[T])$$

by Example 2.4.4. It is not injective. Indeed, $[\Bbbk[T]/(T)]$ is non-zero in $\mathsf{M}(\mathsf{mod}\,\Bbbk[T])$ by Proposition 2.4.1, but it is zero in $\mathsf{K}_0(\mathsf{mod}\,\Bbbk[T])$ because there is a short exact sequence

$$0 \longrightarrow \Bbbk[T] \xrightarrow{\ T\ } \Bbbk[T] \longrightarrow \Bbbk[T]/(T) \longrightarrow 0.$$

In spite of this example, if we consider Serre subcategories, then the natural monoid homomorphism is injective:

Proposition 3.3.5. *Let $\mathcal{S}$ be a Serre subcategory of $\mathcal{C}$ and $\iota\colon \mathcal{S} \to \mathcal{C}$ the inclusion functor. Then the monoid homomorphism*

$$\mathsf{M}(\iota)\colon \mathsf{M}(\mathcal{S}) \to \mathsf{M}(\mathcal{C})$$

is injective. In particular, it induces an isomorphism $\mathsf{M}(\mathcal{S}) \xrightarrow{\sim} \mathsf{M}_{\mathcal{S}} \subseteq \mathsf{M}(\mathcal{C})$ of monoids.

Proof. Suppose that $A, B \in \mathcal{S}$ satisfies $\mathsf{M}(\iota)(A) = \mathsf{M}(\iota)(B)$ in $\mathsf{M}(\mathcal{C})$, that is, $A \approx_c B$ in $\mathcal{C}$. There is a sequence of objects $A_0 = A, A_1, \dots, A_n = B$ such that $A_i \sim_c A_{i+1}$ in $\mathcal{C}$ for all i. Then $A_i \in \mathcal{S}$ for all i since $\mathcal{S}$ is c-closed. Thus, it is enough to show that $A \sim_c B$ in $\mathcal{C}$ implies $A \sim_c B$ in $\mathcal{S}$. Since $A \sim_c B$ in $\mathcal{C}$, there is a conflation

$$X \longrightarrow Y \longrightarrow Z \dashrightarrow \tag{3.3.1}$$

in $\mathcal{C}$ satisfying either (a) $Y \cong A$ and $X \oplus Z \cong B$ or (b) $Y \cong B$ and $X \oplus Z \cong A$. Since $A, B \in \mathcal{S}$ and $\mathcal{S}$ is closed under direct summands, we have $X, Y, Z \in \mathcal{S}$ in both cases. Then the sequence (3.3.1) is a conflation in $\mathcal{S}$, which implies $A \sim_c B$ in $\mathcal{S}$. □

This injectivity is remarkable since it is false for Grothendieck groups and one has to consider the higher K-group K_1 to deal with its failure.

Example 3.3.6. Consider the subcategory $\mathcal{S}$ of $\mathsf{mod}\, \Bbbk[T]$ consisting of finitely generated torsion $\Bbbk[T]$-modules. It is clearly a Serre subcategory. The natural inclusion functor $\mathcal{S} \hookrightarrow \mathsf{mod}\, \Bbbk[T]$ induces an injective monoid morphism $\mathsf{M}(\mathcal{S}) \hookrightarrow \mathsf{M}(\mathsf{mod}\, \Bbbk[T])$ on the Grothendieck monoids by Proposition 3.3.5. On the other hand, it induces a zero morphism $\mathsf{K}_0(\mathcal{S}) \xrightarrow{0} \mathsf{K}_0(\mathsf{mod}\, \Bbbk[T])$ on the Grothendieck groups. Indeed, every object in $\mathcal{S}$ is a finite direct sum of finitely generated indecomposable torsion $\Bbbk[T]$-modules, and such a module M has a free resolution

$$0 \longrightarrow \Bbbk[T] \xrightarrow{f} \Bbbk[T] \longrightarrow M \longrightarrow 0$$

for some polynomial $f \in \Bbbk[T]$ by the structure theorem for finitely generated modules over a principal ideal domain. This implies $[M] = 0$ in $\mathsf{K}_0(\mathsf{mod}\, \Bbbk[T])$.

Remark 3.3.7. Propositions 3.3.3 and 3.3.5 are not entirely new. Those are originally mentioned in [Bro97, Proposition 16.8] for the category $\mathsf{Mod}\,\Lambda$ of *all* modules over a (*not necessarily noetherian*) ring Λ. Brookfield defined the Grothendieck monoids $\mathsf{M}(\mathcal{S})$ for Serre subcategories $\mathcal{S}$ of $\mathsf{Mod}\,\Lambda$. He studied mainly the case $\mathcal{S} = \mathsf{noeth}\,\Lambda$, the category of noetherian Λ-modules, and used the bijection to identify $\mathsf{M}(\mathsf{noeth}\,\Lambda)$ with the face $\mathsf{M}_{\mathsf{noeth}\,\Lambda} \subseteq \mathsf{M}(\mathsf{Mod}\,\Lambda)$ in our terminologies. Our approach using the notion of c-closed subcategories is quite different from Brookfield's one and has a broader application for classifying certain subcategories. For example, see §3.6.

3.4 Classifying Serre subcategories of a length exact category

In this section, we concentrate on length exact categories (see Definition 3.4.1 below). We give concrete examples of classifying Serre subcategories of a length exact category $\mathcal{E}$ via its Grothendieck monoid $\mathsf{M}(\mathcal{E})$. Our strategy is the following:

(1) Relate the Grothendieck monoid $\mathsf{M}(\mathcal{E})$ with an abstract monoid M.
(2) Classify faces of the abstract monoid M.
(3) Classify Serre subcategories of $\mathcal{E}$ by using (1) and (2).

Although we think that some results in this section are well-known to experts, we give the proofs from the viewpoint of Grothendieck monoids. In what follows, $\mathcal{E}$ is a skeletally small exact category.

We quickly review terminologies related to composition series to introduce finiteness conditions of exact categories. Let X be an object of $\mathcal{E}$. Two inflations $Y \rightarrowtail X$ and $Z \rightarrowtail X$ are *equivalent* if there is an isomorphism $Y \xrightarrow{\cong} Z$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & & \\ {\scriptstyle\cong}\downarrow & \searrow & \\ & & X \\ Z & \nearrow & \end{array}.$$

An *admissible subobject* of X is the equivalence class of an inflation $Y \rightarrowtail X$. We often say that Y is an admissible subobject of X and denote the cone of $Y \rightarrowtail X$ by X/Y. We omit the adjective *admissible* if $\mathcal{E}$ is an abelian category. The collection of admissible subobjects of X forms a set since $\mathcal{E}$ is a skeletally small. For two admissible subobjects Y and Z of X, we write $Y \leq Z$ if there exists an inflation $Y \rightarrowtail Z$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \searrow & \\ \downarrow & & X \\ Z & \nearrow & \end{array} .$$

This binary relation $\leq$ yields a partial order on the set of admissible subobjects of X. See [Eno22, Section 2] for a detailed study of the poset of admissible subobjects.

An *admissible subobject series* of X is a finite sequence $0 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ of admissible subobjects of X. It is *proper* if $X_{i+1}/X_i \neq 0$ for all i. In this case, we say that this proper admissible subobject series has *length* n. Two admissible subobject series $0 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ and $0 = Y_0 \leq Y_1 \leq \cdots \leq Y_m = Y$ are *isomorphic* if $n = m$ and there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1}$ for all $1 \leq i \leq n$.

A nonzero object $X \in \mathcal{E}$ is said to be *simple* if it has no admissible subobject except 0 and X itself. We denote by $\mathsf{sim}\,\mathcal{E}$ the set of isomorphism classes of simple objects of $\mathcal{E}$. An admissible subobject series $0 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ of $X \in \mathcal{E}$ is a *composition series* if X_{i+1}/X_i is simple for all i.

Definition 3.4.1.
(1) An object X of $\mathcal{E}$ is *of finite length* if the lengths of proper admissible subobject series of X have an upper bound.
(2) $\mathcal{E}$ is said to be *length* if every object in $\mathcal{E}$ is of finite length.
(3) A length exact category $\mathcal{E}$ *satisfies the Jordan-Hölder property* if, for every $X \in \mathcal{E}$, all composition series of X are isomorphic to each other.

Note that any object of finite length has a composition series since proper admissible subobject series of a maximal length are composition series (see [Eno22, Proposition 2.5]).

Example 3.4.2.
(1) A *length-like function* is an additive function $\ell\colon |\mathcal{E}| \to \mathbb{N}$ such that $\ell(X) = 0$ implies $X \cong 0$. If $\mathcal{E}$ has a length-like function, then $\mathcal{E}$ is a length exact category (see [Eno22, Lemma 4.3]).
(2) Let Λ be a finite dimensional algebra over a field $\Bbbk$. Then $\mathsf{mod}\,\Lambda$ is a length abelian category since the dimension as vector spaces gives rise to a length-like function $\dim_{\Bbbk}\colon |\mathsf{mod}\,\Lambda| \to \mathbb{N}$. An extension-closed subcategory of $\mathsf{mod}\,\Lambda$ is also a length exact category.
(3) A length abelian category satisfies the Jordan-Hölder property (see [Ste75, p.92, Examples 2]).

The following facts are basics to study the Grothendieck monoid of a length exact category.

Fact 3.4.3 ([Eno22, Proposition 4.8])**.** *If $\mathcal{E}$ is a skeletally small length exact category, then $\mathsf{M}(\mathcal{E})$ is generated by the set $\{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$. Moreover, $\mathsf{M}(\mathcal{E})$ is finitely generated if and only if $\mathsf{sim}\,\mathcal{E}$ is a finite set.*

Fact 3.4.4 ([Eno22, Theorem 4.12])**.** *The following are equivalent.*
(1) *$\mathcal{E}$ satisfies the Jordan-Hölder property.*
(2) *$\mathsf{M}(\mathcal{E})$ is a free monoid with basis $\{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$.*
In particular, if $\mathcal{A}$ is a skeletally small length abelian category, then $\mathsf{M}(\mathcal{A})$ is a free monoid with a basis $\{[S] \mid S \in \mathsf{sim}\,\mathcal{A}\}$.

Example 3.4.5. Let Λ be a finite dimensional algebra over a field $\Bbbk$. Then $\mathsf{M}(\mathsf{mod}\,\Lambda) \cong \mathbb{N}^{\oplus n}$, where n is the number of isomorphism classes of simple Λ-modules. The number of maximal right ideals of Λ is also n. Thus, if Λ is local, we have $\mathsf{M}(\mathsf{mod}\,\Lambda) \cong \mathbb{N}$.

We will now begin to classify Serre subcategories of a length exact category. For a subcategory $\mathcal{X}$ of an exact category $\mathcal{E}$, the *Serre subcategory generated by* $\mathcal{X}$ is the smallest Serre subcategory $\langle \mathcal{X} \rangle_{\mathrm{Serre}}$ containing $\mathcal{X}$. A Serre subcategory of the form $\langle X \rangle_{\mathrm{Serre}}$ for some $X \in \mathcal{E}$ is said to be *finitely generated.*

Proposition 3.4.6. *Let $\mathcal{E}$ be a skeletally small length exact category. Then we have an inclusion-preserving surjection*

$$\langle - \rangle_{\mathrm{Serre}} : \mathsf{P}(\mathsf{sim}\,\mathcal{E}) \to \mathrm{Serre}(\mathcal{E}), \quad \mathcal{X} \mapsto \langle \mathcal{X} \rangle_{\mathrm{Serre}}.$$

Proof. It follows from Proposition 3.3.3, Lemma 3.2.8 and Fact 3.4.3. □

As a corollary, we obtain a classification of Serre subcategories of an exact category satisfying the Jordan-Hölder property.

Corollary 3.4.7. *Let $\mathcal{E}$ be a skeletally small exact category satisfying the Jordan-Hölder property. Then we have an inclusion-preserving bijection*

$$\langle - \rangle_{\mathrm{Serre}} : \mathsf{P}(\mathsf{sim}\,\mathcal{E}) \to \mathrm{Serre}(\mathcal{E}), \quad \mathcal{X} \mapsto \langle \mathcal{X} \rangle_{\mathrm{Serre}}.$$

Proof. It follows from Example 3.2.10, Fact 3.4.4 and Proposition 3.4.6. □

We give a nontrivial example of classifying Serre subcategories of a length exact category which does not satisfy the Jordan-Hölder property. We first introduce the Cayley quiver, which is a monoid version of the Cayley graph of a group.

Definition 3.4.8 ([Eno22, Definition 7.5])**.** Let M be a monoid generated by $A \subseteq M$. Then the *Cayley quiver* of M with respect to A is a quiver defined as follows:

- The vertex set is M.
- For each $a \in A$ and $m \in M$, we draw a (labeled) arrow $m \xrightarrow{a} m + a$.

For a length exact category $\mathcal{E}$, the natural choice of A above is $\{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$.

Example 3.4.9 (cf. [Eno22, Section 7.2])**.** Let Λ be the path algebra of the quiver $1 \leftarrow 2$ over a field $\Bbbk$. Then $\mathsf{mod}\,\Lambda$ is a length abelian category whose indecomposable objects are exactly two simple modules S_1, S_2 and one projective injective module P. Thus $\mathsf{M}(\mathsf{mod}\,\Lambda) = \mathbb{N}[S_1] \oplus \mathbb{N}[S_2] \cong \mathbb{N}^{\oplus 2}$ by Fact 3.4.4. We identify $\mathsf{M}(\mathsf{mod}\,\Lambda)$ with $\mathbb{N}^{\oplus 2}$ via this isomorphism. Set $N := \mathbb{N}(m,n) \subseteq \mathsf{M}(\mathsf{mod}\,\Lambda)$ for $(0,0) \neq (m,n) \in \mathbb{N}^{\oplus 2}$. Consider the extension-closed subcategory $\mathcal{D}_N$ of $\mathsf{mod}\,\Lambda$ corresponding to N. Then $\mathcal{D}_N$ is a length exact category by Example 3.4.2. The structure of $\mathsf{M}(\mathcal{D}_N)$ is determined by Enomoto [Eno22, Proposition 7.6] as follows:

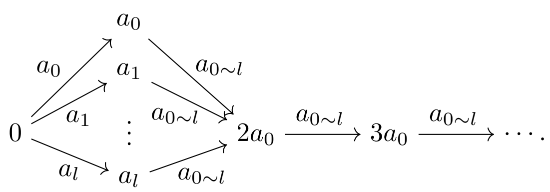
(1) $\mathcal{D}_N$ has exactly $l+1$ distinct simple objects $A_0, \ldots, A_l$, where $l := \min\{m,n\}$ and

$$A_i := P^{\oplus i} \oplus S_1^{\oplus(m-i)} \oplus S_2^{\oplus(n-i)}.$$

Thus $\mathsf{M}(\mathcal{D}_N)$ is generated by $[A_0], \ldots, [A_l]$.

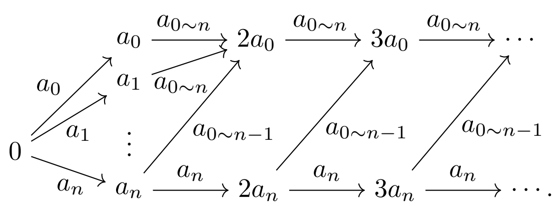
(2) Set $a_i := [A_i]$ for $0 \leq i \leq l$. Then the Cayley quiver of $\mathsf{M}(\mathcal{D}_N)$ with respect to $\{a_i \mid 0 \leq i \leq l\}$ is determined as follows, where $\xrightarrow{a_{0\sim k}}$ denotes $k+1$ arrows $a_0, \ldots, a_k$ for $0 \leq k \leq l$.

(Case 1) The case $m \neq n$:



In particular, $\mathsf{M}(\mathcal{E})$ is free if and only if either $m = 0$ or $n = 0$.

(Case 2) The case $m = n$:



Now we determine the faces of $\mathsf{M}(\mathcal{D}_N)$ to classify the Serre subcategories of $\mathcal{D}_N$:

(Case 1) Any face F of $\mathsf{M}(\mathcal{D}_N)$ is of the form $\langle a_i \mid i \in I\rangle_{\text{face}}$ for some $I \subseteq \{0, \ldots, l\}$ by Lemma 3.2.8. If I is not empty, then F contains $2a_0$. Thus all a_i belong to F since it is a face, and then $F = \mathsf{M}(\mathcal{D}_N)$. Therefore $\mathcal{D}_N$ has no nontrivial Serre subcategories.

(Case 2) Let $F = \langle a_i \mid i \in I\rangle_{\text{face}}$ be a face of $\mathsf{M}(\mathcal{D}_N)$ for some $I \subseteq \{0, \ldots, n\}$. If $i \in I$ for $0 \leq i \leq n-1$, then $2a_0 \in F$, and thus $F = \mathsf{M}(\mathcal{D}_N)$. Unlike the case $m \neq n$, $\mathsf{M}(\mathcal{D}_N)$ has a nontrivial face $F = \langle a_n\rangle_{\text{face}}$. Hence $\mathcal{D}_N$ has exactly three Serre subcategories 0, $\mathcal{D}_N$ and $\langle P^{\oplus n}\rangle_{\text{Serre}}$.

3.5 Preliminaries: cofinal subgroups

In this section, we study subtractive submonoids and cofinal submonoids. A subtractive submonoid can be thought as a submonoid which comes from a subgroup of the group completion. In §3.6, we will see that cofinal subtractive monoids classify dense 2-out-of-3 subcategories. Hereafter, M is a monoid.

Definition 3.5.1. Let S be a subset of M.

(1) S is *subtractive* if $x + y \in S$ and $x \in S$ imply $y \in S$ for any $x, y \in M$.

(2) S is *cofinal* if, for any $x \in M$, there exists $y \in M$ satisfying $x + y \in S$.

Remark 3.5.2.

(1) If M is a group, then a subtractive submonoid is nothing but a subgroup.

(2) If M is a group, then any submonoid N of M is cofinal since $x + (-x) = 0 \in N$ for all $x \in M$.

(3) We can define a pre-order $\leq$ on any monoid M by

$$x \leq y :\Leftrightarrow \text{there exists some } a \in M \text{ such that } y = x + a.$$

A cofinal subset of M defined as above is nothing but a cofinal subset of M with respect to this preorder $\leq$.

Since it is easier to deal with subgroups of a group than with submonoids of a monoid, we study the relation between submonoids of a monoid and subgroups of its group completion. As a consequence of this, we can classify certain subcategories via the Grothendieck groups.

Let $\rho\colon M \to \mathsf{gp}M$ be the group completion. We define a preorder on $\mathsf{gp}M$ by

$$x \leq y :\Leftrightarrow \text{there exists } a \in M \text{ such that } x + \rho(a) = y.$$

We set $S^+ := \{x \in S \mid x \geq 0\}$ for a subset S of $\mathsf{gp}M$. A subgroup H of $\mathsf{gp}M$ is *directed* if $H = \langle H^+\rangle_{\mathbb{Z}}$. Here $\langle S\rangle_{\mathbb{Z}}$ is the subgroup of $\mathsf{gp}M$ generated by a subset S. A subset S of $\mathsf{gp}M$ is *cofinal* if, for any $x \in \mathsf{gp}M$, there exists $y \in S$ such that $x \leq y$.

Remark 3.5.3.

(1) A subset S of $\mathsf{gp}M$ is cofinal if and only if, for any $x \in \mathsf{gp}M$, there exists $a \in M$ such that $x + \rho(a) \in S$.

(2) $\mathsf{gp}M^+ = \rho(M)$ is a cofinal submonoid of $\mathsf{gp}M$.

Example 3.5.4. Consider $M_1 := \mathbb{N}^{\oplus n}$. Then the preorder on $\mathsf{gp}M_1 = \mathbb{Z}^{\oplus n}$ induced by M_1 is the following:

$$(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n) \quad \Leftrightarrow \quad x_i \leq y_i \text{ for all } 1 \leq i \leq n. \tag{3.5.1}$$

A subgroup H of $\mathbb{Z}^{\oplus n}$ is cofinal if and only if it contains $(x_1, \cdots, x_n) \in \mathbb{Z}^{\oplus n}$ such that $x_i > 0$ for all i.

On the other hand, if we consider $M_2 := \mathbb{Z} \oplus \mathbb{N}^{\oplus(n-1)}$, then $\mathsf{gp}M_2 = \mathbb{Z}^{\oplus n}$ but the preorder on $\mathbb{Z}^{\oplus n}$ is different from (3.5.1). In this case, we have

$$(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n) \quad \Leftrightarrow \quad x_i \leq y_i \text{ for all } 2 \leq i \leq n.$$

Consider the following two maps:

$$\{\text{submonoids of } M\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{\text{subgroups of } \mathsf{gp}M\},$$

where $\Phi(N) = \langle \rho(N)\rangle_{\mathbb{Z}}$ and $\Psi(H) = \rho^{-1}(H)$.

Proposition 3.5.5. *The following hold.*

(1) $\Psi\Phi(N) \supseteq N$ *holds for any submonoid* N *of* M*.*
(2) $\Phi\Psi(H) \subseteq H$ *holds for any subgroup* H *of* $\mathsf{gp}M$*.*
(3) Φ *and* Ψ *restrict to inclusion-preserving bijections between* $\operatorname{Im}\Phi$ *and* $\operatorname{Im}\Psi$*.*
(4) $\operatorname{Im}\Phi = \{$*directed subgroups of* $\mathsf{gp}M\}$*.*
(5) $\operatorname{Im}\Psi \subseteq \{$*subtractive submonoids of* $M\}$*.*

Proof. (1),(2) and (5) are straightforward, hence we leave them to the reader. (3) is a formal consequence of (1) and (2). Indeed, we have that $\Phi(N) \supseteq \Phi\Psi\Phi(N) \supseteq \Phi(N)$ and $\Psi(H) \supseteq \Psi\Phi\Psi(H) \supseteq \Psi(H)$ for any submonoid N of M and any subgroup H of $\mathsf{gp}M$. We now prove (4). It is clear that $\Phi(N) = \langle\rho(N)\rangle_{\mathbb{Z}}$ is directed. We show that a directed subgroup H of $\mathsf{gp}M$ belongs to the image of Φ. Since $H = \langle H^+\rangle_{\mathbb{Z}}$, we have that

$$\Phi(\rho^{-1}(H^+)) = \big\langle\rho(\rho^{-1}(H^+))\big\rangle_{\mathbb{Z}} = \big\langle H^+\big\rangle_{\mathbb{Z}} = H.$$

Here the second equality holds since $H^+ \subseteq \mathsf{gp}M^+ = \operatorname{Im}\rho$. □

We will restrict this bijection to cofinal subtractive submonoids in Proposition 3.5.7. We need the following lemma for this.

Lemma 3.5.6. *Let* S *be a subset of* M*, and let* T *be a subset of* $\mathsf{gp}M$*.*

(1) T *is cofinal in* $\mathsf{gp}M$ *if and only if* $\rho^{-1}(T)$ *is cofinal in* M*.*
(2) *If* S *is a cofinal in* M*, then* $\rho(S)$ *is cofinal in* $\mathsf{gp}M$*.*
(3) *A cofinal subgroup of* $\mathsf{gp}M$ *is directed.*

Proof. (1) Suppose that T is cofinal in $\mathsf{gp}M$. Take any $x \in M$. Then there exists $a \in M$ such that $\rho(x+a) = \rho(x) + \rho(a) \in T$. Thus $x + a \in \rho^{-1}(T)$, which proves $\rho^{-1}(T)$ is cofinal in M.

Conversely, suppose that $\rho^{-1}(T)$ is cofinal in M. Take any $x \in \mathsf{gp}M$. Since $\rho(M)$ is cofinal in $\mathsf{gp}M$, there exists $a \in M$ such that $x \leq \rho(a)$. Because $\rho^{-1}(T)$ is cofinal in M, there is $b \in M$ such that $a + b \in \rho^{-1}(T)$. Hence, we have $x \leq \rho(a) \leq \rho(a) + \rho(b) = \rho(a+b) \in T$. This shows that T is cofinal in $\mathsf{gp}M$.

(2) It easily follows from $\rho^{-1}(\rho(S)) \supseteq S$ and (1).

(3) Let H be a cofinal subgroup of $\mathsf{gp}M$. Take any $h \in H$. Then $h = \rho(x) - \rho(y)$ holds for some $x, y \in M$. Since $\rho^{-1}(H)$ is cofinal in M by (1), there is some $y' \in M$ such that $y + y' \in \rho^{-1}(H)$. Then we have $h = \rho(x+y') - \rho(y+y')$. We also have $\rho(x+y') = h + \rho(y+y') \in H$, which implies $\rho(x+y') \in H \cap \mathsf{gp}M^+ = H^+$. Thus $h = \rho(x+y') - \rho(y+y') \in \langle H^+\rangle_{\mathbb{Z}}$. This proves that H is directed. □

Proposition 3.5.7. *The following hold.*

(1) *For a subgroup* H *of* $\mathsf{gp}M$*, it is cofinal in* $\mathsf{gp}M$ *if and only if* $\Psi(N)$ *is a cofinal submonoid of* M*.*
(2) *If* N *is a cofinal submonoid of* M*, then* $\Phi(N)$ *is a cofinal subgroup of* $\mathsf{gp}M$*.*
(3) *If* N *is a cofinal subtractive submonoid of* M*, then* $N = \Psi\Phi(N)$ *holds.*

Therefore we have the following commutative diagram:

$$\begin{array}{ccc}
\{\textit{submonoids of } M\} & \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} & \{\textit{subgroups of } \mathsf{gp}M\} \\
\cup\!\!| & & \cup\!\!| \\
\{\textit{subtractive submonoids of } M\} & & \{\textit{directed subgroups of } \mathsf{gp}M\} \\
\cup\!\!| & & \| \\
\operatorname{Im}\Psi & \xleftrightarrow{\ \cong\ } & \operatorname{Im}\Phi \\
\cup\!\!| & & \cup\!\!| \\
\{\textit{cofinal subtractive submonoids of } M\} & \xleftrightarrow{\cong} & \{\textit{cofinal subgroups of } \mathsf{gp}M\}.
\end{array}$$

Proof. (1) and (2) follow from Lemma 3.5.6. We only prove (3). Let N be a cofinal subtractive submonoid of M. Clearly $N \subseteq \Psi\Phi(N)$ holds. Suppose $x \in \Psi\Phi(N) = \rho^{-1}(\langle\rho(N)\rangle_{\mathbb{Z}})$. Then since $\rho(x) \in \langle\rho(N)\rangle_{\mathbb{Z}}$ and N is a submonoid of M, there are elements n_1 and n_2 in N such that $\rho(x) = \rho(n_1) - \rho(n_2)$, hence $\rho(x + n_2) = \rho(n_1)$. Therefore, there is an element $m \in M$ such that $x + n_2 + m = n_1 + m$ in M (see the argument below Definition 5.1.1). Then, since N is cofinal in M, there is $m' \in M$ satisfying $m + m' \in N$. Then we have $x + (n_2 + m + m') = n_1 + m + m'$. Now $x \in N$ follows since N is subtractive and $x + (n_2 + m + m')$ and $n_2 + m + m'$ belong to N. □

Corollary 3.5.8. *Let S be a cofinal subset of M. Then Φ and Ψ restrict to the following bijections:*

$$\{\textit{subtractive submonoids of } M \textit{ containing } S\} \underset{\Psi_S}{\overset{\Phi_S}{\rightleftarrows}} \{\textit{subgroups of } \mathsf{gp}M \textit{ containing } \rho(S)\}.$$

Proof. It is easily checked that Φ_S and Ψ_S are well-defined using Proposition 3.5.5 (5). Clearly any submonoid N of M containing S is cofinal, hence $\Psi\Phi(N) = N$ holds by Proposition 3.5.7 (3). Therefore, it suffices to show that a subgroup H of $\mathsf{gp}M$ containing $\rho(S)$ is directed. But this follows from Lemma 3.5.6 (3). □

3.6 Classifying dense 2-out-of-3 subcategories via cofinal subgroups

In this section, we give classifications of dense 2-out-of-3 subcategories of $\mathcal{C}$ in terms of $\mathsf{M}(\mathcal{C})$ and $\mathsf{K}_0(\mathcal{C})$, which generalize Thomason's classification of dense triangulated subcategories of a triangulated category [Tho97] in terms of $\mathsf{K}_0(\mathcal{C})$ and remove the unnecessary assumption on [Mat18, ZZ21].

Definition 3.6.1. Let $\mathcal{D}$ be an additive subcategory of $\mathcal{C}$.

(1) $\mathcal{D}$ is a *dense* subcategory if $\mathsf{add}\,\mathcal{D} = \mathcal{C}$ holds, that is, for every $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}$ satisfying $C \oplus C' \in \mathcal{D}$.

(2) $\mathcal{D}$ is a *2-out-of-3* subcategory if it satisfies 2-out-of-3 for conflations, that is, if two of three objects X, Y, Z in a conflation $X \to Y \to Z \dashrightarrow$ belong to $\mathcal{D}$, then so does the third.

We note that 2-out-of-3 subcategories closed under direct summands are called *thick subcategories*, see Definition 4.2.2.

Example 3.6.2. Let $\mathcal{T}$ be a triangulated subcategory. Then a subcategory of $\mathcal{T}$ is a 2-out-of-3 subcategory if and only if it is a triangulated subcategory (see e.g. [Tho97, 1.1]).

Remark 3.6.3. Let $\mathcal{D}$ be a 2-out-of-3 subcategory of $\mathcal{C}$. If $X \oplus Y \in \mathcal{D}$ and $X \in \mathcal{D}$, then $Y \in \mathcal{D}$ by a split conflation $X \to X \oplus Y \to Y \dashrightarrow$. We will freely use this property in what follows.

We can relax the 2-out-of-3 condition of dense 2-out-of-3 subcategories by the following observation.

Proposition 3.6.4 ([ZZ21, Lemma 5.5])**.** *Let $\mathcal{D}$ be a dense additive subcategory of $\mathcal{C}$. Then the following are equivalent.*

(1) *For every conflation $X \to Y \to Z \dashrightarrow$ in $\mathcal{C}$, if X and Y belong to $\mathcal{D}$, then so does Z.*

(2) *For every conflation $X \to Y \to Z \dashrightarrow$ in $\mathcal{C}$, if Y and Z belong to $\mathcal{D}$, then so does X.*

A key observation in this section is as follows.

Proposition 3.6.5. *Let $\mathcal{D}$ be a dense 2-out-of-3 subcategory of $\mathcal{C}$. Then $\mathcal{D}$ is closed under c-equivalences.*

Proof. Take any conflation

$$X \xrightarrow{f} Y \xrightarrow{g} Z \dashrightarrow \tag{3.6.1}$$

in $\mathcal{C}$. It suffices to show that Y belongs to $\mathcal{D}$ if and only if so does $X \oplus Z$.

First, suppose that Y belongs to $\mathcal{D}$. Since $\mathcal{D}$ is dense, there is some $W \in \mathcal{C}$ satisfying $Y \oplus Z \oplus W \in \mathcal{D}$. By taking the direct sum of (3.6.1) and a split conflation $Z \to Z \oplus W \to W \dashrightarrow$, we obtain the following conflation.

$$X \oplus Z \xrightarrow{\begin{bmatrix} f & 0 \\ 0 & \mathsf{id}_Z \\ 0 & 0 \end{bmatrix}} Y \oplus Z \oplus W \xrightarrow{\begin{bmatrix} g & 0 & 0 \\ 0 & 0 & \mathsf{id}_W \end{bmatrix}} Z \oplus W \dashrightarrow \tag{3.6.2}$$

Since $\mathcal{D}$ is 2-out-of-3, $Y \oplus (Z \oplus W) \in \mathcal{D}$ and $Y \in \mathcal{D}$ implies $Z \oplus W \in \mathcal{D}$. Therefore, (3.6.2) implies that $X \oplus Z$ belongs to $\mathcal{D}$.

Conversely, suppose that $X \oplus Z$ belongs to $\mathcal{D}$. By taking the direct sum of (3.6.1) and a split conflation $Z \to Z \oplus X \to X \dashrightarrow$, we obtain the following conflation:

$$X \oplus Z \xrightarrow{\begin{bmatrix} f & 0 \\ 0 & 0 \\ 0 & \mathsf{id}_Z \end{bmatrix}} Y \oplus X \oplus Z \xrightarrow{\begin{bmatrix} 0 & \mathsf{id}_X & 0 \\ g & 0 & 0 \end{bmatrix}} X \oplus Z \dashrightarrow \tag{3.6.3}$$

Since $\mathcal{D}$ is 2-out-of-3 and $X \oplus Z \in \mathcal{D}$, we have $Y \oplus (X \oplus Z) \in \mathcal{D}$, which implies $Y \in \mathcal{D}$ by $X \oplus Z \in \mathcal{D}$. □

Now we can state the following classification of dense 2-out-of-3 subcategories.

Theorem 3.6.6. *Let $\mathcal{C}$ be a skeletally small extriangulated category. There are bijections between the following sets.*
(1) *The set of dense 2-out-of-3 subcategories of $\mathcal{C}$.*
(2) *The set of cofinal subtractive submonoids of $\mathsf{M}(\mathcal{C})$.*
(3) *The set of cofinal subgroups of $\mathsf{K}_0(\mathcal{C})$.*

Proof. There is a bijection between (2) and (3) by Proposition 3.5.7. Thus, we only construct a bijection between (1) and (2). Due to Proposition 3.1.3 (2) and Proposition 3.6.5, we only have to check the following well-definedness of maps:
(i) $\mathsf{M}_{\mathcal{D}}$ is a cofinal subtractive submonoid for a dense 2-out-of-3 subcategory $\mathcal{D}$.
(ii) $\mathcal{D}_N$ is a dense 2-out-of-3 subcategory of $\mathcal{C}$ for a cofinal subtractive submonoid N.

(i) Let $\mathcal{D}$ be a dense 2-out-of-3 subcategory of $\mathcal{C}$. Since $\mathcal{D}$ is closed under direct sums, $\mathsf{M}_{\mathcal{D}}$ is a submonoid of $\mathsf{M}(\mathcal{C})$. To show that $\mathsf{M}_{\mathcal{D}}$ is cofinal in $\mathsf{M}(\mathcal{C})$, take any $[C] \in \mathsf{M}(\mathcal{C})$. Since $\mathcal{C}$ is dense, there is some C' satisfying $C \oplus C' \in \mathcal{D}$. This implies $[C] + [C'] = [C \oplus C'] \in \mathsf{M}_{\mathcal{D}}$. Thus $\mathsf{M}_{\mathcal{D}}$ is cofinal in $\mathsf{M}(\mathcal{C})$.

Next, to show that $\mathsf{M}_{\mathcal{D}}$ is subtractive, suppose that $x + y$ and x belong to $\mathsf{M}_{\mathcal{D}}$. Take $X, Y \in \mathcal{C}$ satisfying $[X] = x$ and $[Y] = y$. Then $[X \oplus Y]$ and $[X]$ belong to $\mathsf{M}_{\mathcal{D}}$. Since $\mathcal{D}$ is c-closed by Proposition 3.6.5, we have $\mathcal{D} = \mathcal{D}_{\mathsf{M}_{\mathcal{D}}}$ by Proposition 3.1.3. Therefore, $X \oplus Y$ and X belong to $\mathcal{D}$. Since $\mathcal{D}$ is 2-out-of-3, we obtain $Y \in \mathcal{D}$. Thus $y = [Y] \in \mathsf{M}_{\mathcal{D}}$ holds.

(ii) Let N be a cofinal subtractive submonoid of $\mathsf{M}(\mathcal{D})$. To show that $\mathcal{D}_N$ is dense, take any $C \in \mathcal{C}$. Since N is cofinal, there is some $C' \in \mathcal{C}$ satisfying $[C \oplus C'] = [C] + [C'] \in N$. Thus $C \oplus C' \in \mathcal{D}_N$ holds.

Next, we will check that $\mathcal{D}_N$ is 2-out-of-3. Take any conflation $X \to Y \to Z \dashrightarrow$ in $\mathcal{C}$. Then we have $[Y] = [X] + [Z]$ in $\mathsf{M}(\mathcal{C})$. If X and Z belong to $\mathcal{D}_N$, then $[X]$ and $[Z]$ belong to N, and hence so does $[Y] = [X] + [Z]$ since N is a submonoid. Thus Y belongs to $\mathcal{D}_N$. Similarly, if X and Y belong to $\mathcal{D}_N$, then $[X]$ and $[Y] = [X] + [Z]$ belong to N, and hence so does $[Z]$ since N is subtractive. Thus Z belongs to $\mathcal{D}_N$. The same argument works if Y and Z belong to $\mathcal{D}_N$. ☐

As a corollary, we can immediately deduce the following classification of dense triangulated subcategories.

Corollary 3.6.7 ([Tho97, Theorem 2.1])**.** *Let $\mathcal{T}$ be a skeletally small triangulated category. Then there exists a bijection between the following two sets:*
- *The set of dense triangulated subcategories of $\mathcal{T}$.*
- *The set of subgroups of $\mathsf{K}_0(\mathcal{T})$.*

Proof. Since $\mathsf{M}(\mathcal{T}) \cong \mathsf{K}_0(\mathcal{T})$ holds by Proposition 2.4.3 and dense triangulated subcategories of $\mathcal{T}$ are precisely dense 2-out-of-3 subcategories, we only have to check that a subset of $\mathsf{K}_0(\mathcal{T})$ is a cofinal subtractive submonoid if and only if it is a subgroup. This follows from Remark 3.5.2. ☐

Using this observation, we can obtain all dense 2-out-of-3 subcategories in an abelian length category with finitely many simples. First, recall the following description of the Grothendieck monoid. Let $\mathcal{A}$ be an abelian length category, that is, an abelian category such that every object has a composition series. Suppose that $\{S_1, \dots S_n\}$ is the set of all non-isomorphic simple objects in $\mathcal{A}$. Then for $C \in \mathcal{A}$, define $\underline{\dim}\, C := (x_1, \dots, x_n) \in \mathbb{N}^n$, where x_i is the multiplicity of S_i in the composition series of C (this is well-defined due to the Jordan-Hölder theorem). Then $\underline{\dim}$ respects conflations, and moreover, it induces the following isomorphisms of monoids and groups:

$$\begin{array}{ccc} \mathsf{M}(\mathcal{A}) & \xrightarrow[\sim]{\underline{\dim}} & \mathbb{N}^n \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\iota} \\ \mathsf{K}_0(\mathcal{A}) & \xrightarrow[\sim]{\underline{\dim}} & \mathbb{Z}^n, \end{array} \tag{3.6.4}$$

where ρ is the group completion and ι is the natural inclusion.

Corollary 3.6.8. *Let $\mathcal{A}$ be an abelian length category with n simple objects up to isomorphism. Then there are bijections between the following two sets:*
- *The set of dense 2-out-of-3 subcategories $\mathcal{C}$ of $\mathcal{A}$.*

- *The set of subgroups H of $\mathbb{Z}^n$ containing a strictly positive element.*

Here an element $x = (x_1, \ldots, x_n)$ of $\mathbb{Z}^n$ is strictly positive *if $x_i > 0$ for all i. The maps are given by $\mathcal{C} \mapsto \langle \{\underline{\dim}\, C \mid C \in \mathcal{C}\}\rangle_{\mathbb{Z}}$ and $H \mapsto \{C \in \mathcal{C} \mid \underline{\dim}\, C \in H\}$.*

Proof. It follows from Example 3.5.4 and Theorem 3.6.6. □

Certain classes of dense 2-out-of-3 subcategories were classified via the Grothendieck group in [Mat18] (for the exact case) and [ZZ21] (for the extriangulated case). We explain that their results can be immediately deduced from ours. Let us explain some terminology to state them.

Definition 3.6.9. Let $\mathcal{C}$ be an extriangulated category. Then a set $\mathcal{G}$ of objects in $\mathcal{C}$ is called a *generator* if for every $C \in \mathcal{C}$ there is a conflation $X \to G \to C \dashrightarrow$ in $\mathcal{C}$ with $G \in \mathcal{G}$.

Now we can deduce their results as follows.

Corollary 3.6.10 ([Mat18, Theorem 2.7], [ZZ21, Theorem 5.7])**.** *Let $\mathcal{C}$ be a skeletally small extriangulated category and $\mathcal{G}$ a generator of $\mathcal{C}$. Then there is a bijection between the following two sets:*

(1) *The set of dense 2-out-of-3 subcategories of $\mathcal{C}$ containing $\mathcal{G}$.*

(2) *The set of subgroups of $\mathsf{K}_0(\mathcal{C})$ containing the image of $\mathcal{G}$.*

Proof. We will show that these sets are in bijection with the following one:

(3) The set of subtractive submonoids of $\mathsf{M}(\mathcal{C})$ containing the image of $\mathcal{G}$.

Denote by $[\mathcal{G}] \subseteq \mathsf{M}(\mathcal{C})$ the image of $\mathcal{G}$ in $\mathsf{M}(\mathcal{C})$, then $[\mathcal{G}]$ is cofinal in $\mathsf{M}(\mathcal{C})$. Indeed, for every $[C] \in \mathsf{M}(\mathcal{C})$, there is a conflation $X \to G \to C \dashrightarrow$ in $\mathcal{C}$, and thus $[C] + [X] = [G] \in [\mathcal{G}]$ holds in $\mathsf{M}(\mathcal{C})$. Therefore, by Corollary 3.5.8, we have a bijection between (2) and (3), and every submonoid in (3) is cofinal. Therefore, (1) and (3) are subsets of the two sets in Theorem 3.6.6. Hence, it suffices to observe the following well-definedness, which are immediate from definitions: If $\mathcal{D}$ is a dense 2-out-of-3 subcategory containing $\mathcal{G}$, then $\mathsf{M}_{\mathcal{D}}$ contains $[\mathcal{G}]$, and if N is a submonoid of $\mathsf{M}(\mathcal{C})$ containing $[\mathcal{G}]$, then $\mathcal{D}_N$ contains $\mathcal{G}$. □

Chapter 4

Quotient of monoids and localization of extriangulated categories

The purpose of this chapter is to prove Theorem 4.3.1, which describes the Grothendieck monoid of the localization of extriangulated categories as a monoid quotient of the Grothendieck monoid.

Throughout this chapter, we fix Grothendieck universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$ and use Convention 1.1.12 and 2.2.5. Moreover, we assume that every category, functor, and subcategory is additive. In particular, every subcategory is strictly full and nonempty.

4.1 Preliminaries: quotient of monoids

We introduce the notion of quotient of monoids, which is a natural analogue of that of abelian groups. Hereafter, M is a monoid.

To each submonoid of a monoid, we can associate the congruence as follows.

Definition 4.1.1. Let N be a submonoid of M. Define a congruence on M as follows:

$$x \sim y :\Leftrightarrow \text{there exist } n, n' \in N \text{ such that } x + n = y + n'.$$

Then the monoid $M/N := M/\sim$ is called the *quotient monoid* of M by N. We write $x \equiv y \bmod N$ if $x \sim y$ holds. The equivalence class of $x \in M$ is denoted by $x \bmod N$.

It is easily seen that the quotient monoids have the following universal property.

Proposition 4.1.2. *Let N be a submonoid of a monoid M, and let $\pi\colon M \to M/N$ be the quotient homomorphism. Then $\pi(N) = 0$ holds, and for any monoid homomorphism $f\colon M \to X$ such that $f(N) = 0$, there exists a unique monoid homomorphism $\overline{f}\colon M/N \to X$ satisfying $\overline{f}\pi = f$. This means that the diagram*

$$N \overset{\iota}{\underset{0}{\rightrightarrows}} M \xrightarrow{\pi} M/N$$

is a coequalizer diagram in Mon*, where ι is the inclusion map.*

Unlike the case of abelian groups, submonoids of M/N do not correspond to those of M containing N:

Example 4.1.3. Let $M := \mathbb{N}^{\oplus 2}$ and $N := \mathbb{N}(1,0) + \mathbb{N}(1,1) \subseteq M$. Then we have $M/N = 0$ but M and N are distinct submonoids of M containing N.

However, we have a bijection for faces.

Proposition 4.1.4. *Let N be a submonoid of M, and let $\pi\colon M \to M/N$ be the quotient homomorphism.*

(1) *If F is a face of M containing N, then $F/N := \pi(F)$ is also a face of M/N.*

(2) *If F' is a face of M/N, then $\pi^{-1}(F')$ is also a face of M containing N.*

(3) *The assignments given in (1) and (2) give inclusion-preserving bijections between the set of faces of M containing N and that of M/N.*

Proof. We only prove (1) and (3) since the proof of (2) is straightforward.

(1) It is clear that F/N is a submonoid of M/N. Let $a, b \in M$ such that $\pi(a) + \pi(b) \in F/N$. Then there exist $x \in F$ and $n, n' \in N$ such that $x + n = (a + b) + n'$ in M. Since $x + n \in F$ and F is a face of M, we have that $a, b \in F$. Thus both $\pi(a)$ and $\pi(b)$ belong to F/N, which proves F/N is a face.

(3) We have that $\pi^{-1}(F')/N = \pi(\pi^{-1}(F')) = F'$ since π is surjective. It is easy to check that $\pi^{-1}(F/N) \supseteq F$. It remains to show that $\pi^{-1}(F/N) \subseteq F$. Let $a \in \pi^{-1}(F/N)$. Then we have $\pi(a) \in F/N$. There exist $x \in F$ and $n, n' \in N$ such that $x + n = a + n'$. Since $x + n \in F$ and F is a face, we have $a \in F$, which proves $\pi^{-1}(F/N) \subseteq F$. □

Corollary 4.1.5. *Let N be a submonoid of a monoid M. There is an inclusion-preserving bijection between* $\operatorname{Face}(M/N)$ *and* $\operatorname{Face}(M/\langle N\rangle_{\mathrm{face}})$. *In particular, we have an inclusion-preserving bijection* $\operatorname{Face}(M) \cong \operatorname{Face}(M/M^{\times})$, *where $M^{\times}$ is the set of units of M.*

Proof. Proposition 4.1.4 shows that both $\operatorname{Face}(M/N)$ and $\operatorname{Face}(M/\langle N\rangle_{\mathrm{face}})$ are in bijection with $\{X \in \operatorname{Face} M \mid X \supseteq N\}$ by the definition of $\langle N\rangle_{\mathrm{face}}$. Thus, the former assertion holds. The latter assertion follows from Remark 3.2.2 and the former one by putting $N := 0$. □

4.2 Preliminaries: localization of extriangulated categories

We first recall the localization of an extriangulated category following [NOS22].

Definition 4.2.1. Let $\mathcal{C}$ be an extriangulated category, and let $S \subseteq \operatorname{Mor}\mathcal{C}$ be a collection of morphisms. A pair $(\mathcal{C}_S, Q)$ of an extriangulated category $\mathcal{C}_S$ and an exact functor $Q\colon \mathcal{C} \to \mathcal{C}_S$ is the *exact localization of $\mathcal{C}$ with respect to S* if it satisfies the following conditions:

(i) $F(s)$ is an isomorphism in $\mathcal{C}_S$ for any $s \in S$.

(ii) For any extriangulated category $\mathcal{D}$ and any exact functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for any $s \in S$, there exists a unique exact functor $F_S\colon \mathcal{C}_S \to \mathcal{D}$ satisfying $F = F_S \circ Q$.

If the exact localization exists, it is unique up to exact isomorphisms. Note that the exact localization is closed under exact *isomorphisms*, but not closed under exact *equivalences*, see Remark 4.2.7 below.

Nakaoka–Ogawa–Sakai [NOS22] constructs the exact localization of an extriangulated category by a collection of morphisms under some assumptions. We only recall the construction of the localization of an extriangulated category by the collection of morphisms determined by a thick subcategory, as we shall explain.

From now on, $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category.

Definition 4.2.2. A subcategory $\mathcal{N}$ of $\mathcal{C}$ is a *thick* subcategory if it satisfies the following conditions:

(i) $\mathcal{N}$ is closed under direct summands.

(ii) $\mathcal{N}$ satisfies 2-out-of-3 for conflations in $\mathcal{C}$, that is, if two of three objects A, B, C in a conflation $A \to B \to C \dashrightarrow$ belong to $\mathcal{N}$, then so does the third.

For a thick subcategory $\mathcal{N} \subseteq \mathcal{C}$, we set the following collections of morphisms:

$$\mathcal{L} := \{\ell \in \operatorname{Mor}\mathcal{C} \mid \text{there is a conflation } A \xrightarrow{\ell} B \to N \dashrightarrow \text{ with } N \in \mathcal{N}\},$$
$$\mathcal{R} := \{r \in \operatorname{Mor}\mathcal{C} \mid \text{there is a conflation } N \to A \xrightarrow{r} B \dashrightarrow \text{ with } N \in \mathcal{N}\}.$$

We define $S_{\mathcal{N}}$ to be the collection of all finite compositions of morphisms in $\mathcal{L}$ and $\mathcal{R}$. We can easily check $\mathcal{L} \circ \mathcal{L} \subseteq \mathcal{L}$ and $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$. Thus, a morphism s in $S_{\mathcal{N}}$ is of the form $s = \cdots \ell_{n-1} r_n \ell_{n+1} r_{n+2} \cdots$ for some $\ell_i \in \mathcal{L}$ and $r_j \in \mathcal{R}$. The thick subcategory $\mathcal{N}$ can be recovered from $S_{\mathcal{N}}$ since we have

$$\mathcal{N} = \{A \in \mathcal{C} \mid \text{both } A \to 0 \text{ and } 0 \to A \text{ belong to } S_{\mathcal{N}}\} \tag{4.2.1}$$

by [NOS22, Lemma 4.5]. In the following, we consider the exact localization $\mathcal{C}/\mathcal{N} := \mathcal{C}_{S_{\mathcal{N}}}$ of $\mathcal{C}$ with respect to $S_{\mathcal{N}}$. This localization satisfies the following natural universal property.

Proposition 4.2.3. *Let $\mathcal{C}$ be an extriangulated category and $\mathcal{N}$ a thick subcategory of $\mathcal{C}$. Suppose that the exact localization $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N} := \mathcal{C}_{S_{\mathcal{N}}}$ exists. Then it satisfies the following conditions.*
- (i) *$Q(N) \cong 0$ holds for every $N \in \mathcal{N}$.*
- (ii) *For any extriangulated category $\mathcal{D}$ and an any exact functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(N) \cong 0$ holds for every $N \in \mathcal{N}$, there exists a unique exact functor $F_{\mathcal{N}}\colon \mathcal{C}/\mathcal{N} \to \mathcal{D}$ satisfying $F = F_{\mathcal{N}} \circ Q$.*

Proof. By comparing the claimed properties with Definition 4.2.1, it suffices to show the following claim: for an extriangulated category $\mathcal{D}$ and an exact functor $F\colon \mathcal{C} \to \mathcal{D}$, we have that $F(N) \cong 0$ holds for every $N \in \mathcal{N}$ if and only if $F(s)$ is an isomorphism for every $s \in S_{\mathcal{N}}$.

To see the "only if" part, suppose that $F(N) \cong 0$ holds for every $N \in \mathcal{N}$. It suffices to check that $F(\ell)$ and $F(r)$ are isomorphisms for $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ respectively. By the definition of $\mathcal{L}$, there is a conflation $A \xrightarrow{\ell} B \to N \dashrightarrow$ with $N \in \mathcal{N}$. Since F is an exact functor, we obtain a conflation $F(A) \xrightarrow{F(\ell)} F(B) \to F(N) \dashrightarrow$ in $\mathcal{D}$. Then $F(N) \cong 0$ implies that $F(\ell)$ is an isomorphism in $\mathcal{D}$ by considering the associated long exact sequence (cf. [NP19, Corollary 3.12]). The same proof applies to $r \in \mathcal{R}$.

To see the "if" part, suppose that $F(s)$ is an isomorphism for every $s \in S_{\mathcal{N}}$, and let $N \in \mathcal{N}$. Then $0 \to N$ clearly belongs to $\mathcal{L} \subseteq S_{\mathcal{N}}$, so $0 \to F(N)$ is an isomorphism in $\mathcal{D}$. Thus the assertion holds. $\square$

Let $\overline{\mathcal{C}} := \mathcal{C}/[\mathcal{N}]$ be the quotient by the ideal $[\mathcal{N}]$ consisting of morphisms which factor through objects in $\mathcal{N}$, and let $p\colon \mathcal{C} \to \overline{\mathcal{C}}$ be the canonical functor. In what follows, we write $\overline{f} := p(f)$ for any morphism f in $\mathcal{C}$. Set $\overline{S_{\mathcal{N}}} := p(S_{\mathcal{N}})$[1]. We recall a condition in [NOS22] under which the exact localization $\mathcal{C}/\mathcal{N} = (\mathcal{C}/\mathcal{N}, Q)$ exists.

Condition 4.2.4. Let $\mathcal{N}$ be a thick subcategory of $\mathcal{C}$.
- (i) $f \in S_{\mathcal{N}}$ holds for any split monomorphism $f\colon A \to B$ in $\mathcal{C}$ such that $\overline{f}$ is an isomorphism in $\overline{\mathcal{C}}$. (This is equivalent to the dual condition by [NOS22, Lemma 3.2]: $f \in S_{\mathcal{N}}$ holds for any split epimorphism $f\colon A \to B$ in $\mathcal{C}$ such that $\overline{f}$ is an isomorphism in $\overline{\mathcal{C}}$.)
- (ii) $\overline{S_{\mathcal{N}}}$ satisfies 2-out-of-3 with respect to compositions in $\overline{\mathcal{C}}$.
- (iii) $\overline{S_{\mathcal{N}}}$ is a multiplicative system in $\overline{\mathcal{C}}$.
- (iv) The set $\{\overline{txs} \mid x \text{ is an inflation in } \mathcal{C} \text{ and } s, t \in S_{\mathcal{N}}\}$ is closed under compositions. Dually, the set $\{\overline{tys} \mid y \text{ is a deflation in } \mathcal{C} \text{ and } s, t \in S_{\mathcal{N}}\}$ is closed under compositions.

Fact 4.2.5 ([NOS22, Theorem 3.5, Lemma 3.32])**.** *Let $\mathcal{N}$ be a thick subcategory of $\mathcal{C}$. If it satisfies Condition 4.2.4, then there exists the exact localization $\mathcal{C}/\mathcal{N}$ satisfying the following properties.*
- (1) *$\mathcal{C}/\mathcal{N}$ is constructed as the category $\overline{S_{\mathcal{N}}}^{-1}\overline{\mathcal{C}}$ of fractions. In particular, every morphism in $\mathcal{C}/\mathcal{N}$ can be described as a right or left roof of morphisms in $\overline{\mathcal{C}}$.*
- (2) *For any inflation α in $\mathcal{C}/\mathcal{N}$, there exist an inflation f in $\mathcal{C}$ and isomorphisms β, γ in $\mathcal{C}/\mathcal{N}$ satisfying $\alpha = \beta \circ Q(f) \circ \gamma$. Dually, for any deflation α in $\mathcal{C}/\mathcal{N}$, there exist a deflation f in $\mathcal{C}$ and isomorphisms β, γ in $\mathcal{C}/\mathcal{N}$ satisfying $\alpha = \beta \circ Q(f) \circ \gamma$.*

Remark 4.2.6. Let us confirm that Condition 4.2.4 implies the conditions in [NOS22, Theorem 3.5]. Suppose that $S_{\mathcal{N}}$ satisfies Condition 4.2.4. It is clear that $S_{\mathcal{N}}$ satisfies (M0) in [NOS22, Section 3]. The condition (MR1), (MR2), and (MR4) in [NOS22, Theorem 3.5] are nothing but (i), (iii), and (iv) of Condition 4.2.4, respectively. By [NOS22, Lemma 4.6], $S_{\mathcal{N}}$ satisfies (M3) in [NOS22, Corollary 3.4]. Thus, it also satisfies (MR3) by the condition (i) and [NOS22, Lemma 3.2, Claim 3.6]. Therefore $S_{\mathcal{N}}$ satisfies all the conditions in [NOS22, Theorem 3.5].

Remark 4.2.7. Some readers may find the definition of exact localizations unsatisfactory since it is not preserved by exact equivalences. In fact, there is a notion of *exact 2-localizations*, which is preserved by exact equivalences. For two extriangulated categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ the category of exact functors $\mathcal{C} \to \mathcal{D}$ and natural transformations of them. Any exact functor $F\colon \mathcal{C} \to \mathcal{C}'$ induces a functor $F^* : \mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}', \mathcal{D}) \to \mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ defined by $F^*(G) := G \circ F$.

For a collection $S \subseteq \mathsf{Mor}\,\mathcal{C}$ of morphisms in an extriangulated category $\mathcal{C}$, the *exact 2-localization* of $\mathcal{C}$ with respect to S is a pair $(\mathcal{C}_S, Q)$ of an extriangulated category $\mathcal{C}_S$ and an exact functor $Q\colon \mathcal{C} \to \mathcal{C}_S$ which satisfies the following conditions:

[1] Our notation $\overline{S_{\mathcal{N}}}$ is different from the one in [NOS22]. However, they coincide if (i) of Condition 4.2.4 is satisfied. See [NOS22, Lemma 3.2].

(i) $F(s)$ is an isomorphism in $\mathcal{C}_S$ for any $s \in S$.
(ii) For any extriangulated category $\mathcal{D}$ and an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for any $s \in S$, there exist an exact functor $\widetilde{F}\colon \mathcal{C}_S \to \mathcal{D}$ and a natural isomorphism $F \cong \widetilde{F} \circ Q$ of exact functors.
(iii) The functor

$$Q^*\colon \mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}_S, \mathcal{D}) \to \mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$$

is fully faithful for every extriangulated category $\mathcal{D}$.

In fact, the exact localization $\mathcal{C}/\mathcal{N}$ obtained by Fact 4.2.5 is also the exact 2-localization by [NOS22, Theorem 3.5].

Although Condition 4.2.4 seems a little technical, there is a sufficient condition which we can easily verify, see Fact 4.2.12 and 4.2.13. In this paper, we focus on the situation where Condition 4.2.4 is satisfied so that we can use the additional properties (1)–(2) of Fact 4.2.5 freely.

Remark 4.2.8. We do not assume in Condition 4.2.4 that $S_{\mathcal{N}}$ is a multiplicative system. However, any morphism $f\colon X \to Y$ in $\mathcal{C}/\mathcal{N}$ is of the form $f = Q(s)^{-1}Q(g)$ for some morphisms $g\colon X \to A$ in $\mathcal{C}$ and $s\colon Y \to A$ in $S_{\mathcal{N}}$. Indeed, let $\overline{Q}\colon \overline{\mathcal{C}} \to \overline{S_{\mathcal{N}}}^{-1}\overline{\mathcal{C}} = \mathcal{C}/\mathcal{N}$ be the canonical functor. For any morphism $f\colon X \to Y$ in $\mathcal{C}/\mathcal{N}$, there are morphisms $g\colon X \to A$ in $\overline{\mathcal{C}}$ and $\overline{s}\colon Y \to A$ in $\overline{S_{\mathcal{N}}}$ satisfying $f = \overline{Q}(\overline{s})^{-1}\overline{Q}(\overline{g})$. Since $\overline{Q}(\overline{\phi}) = Q(\phi)$ holds for any morphism ϕ in $\mathcal{C}$, the claim follows.

We introduce some convenient classes of thick subcategories.

Definition 4.2.9. Let $\mathcal{N}$ be a thick subcategory of $\mathcal{C}$.
(1) $\mathcal{N}$ is called *biresolving* if for any $C \in \mathcal{C}$, there exist an inflation $C \to N$ and a deflation $N' \to C$ in $\mathcal{C}$ with $N, N' \in \mathcal{N}$.
(2) $\mathcal{N}$ is called *percolating* if for any morphism $f\colon X \to Y$ in $\mathcal{C}$ factoring through some object in $\mathcal{N}$, there exist a deflation $g\colon X \to N$ and an inflation $h\colon N \to Y$ satisfying $N \in \mathcal{N}$ and $f = hg$[2].

For the triangulated case, we have the following observation.

Example 4.2.10. Let $\mathcal{T}$ be a triangulated category with shift functor Σ.
(1) A thick subcategory of $\mathcal{T}$ in the sense of Definition 4.2.2 coincides with the usual one, that is, a subcategory of $\mathcal{T}$ closed under cones, shifts, and direct summands. We can easily check it by considering the following conflations:

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \dashrightarrow, \quad X \to 0 \to \Sigma X \dashrightarrow, \quad \text{and} \quad \Sigma^{-1}X \to 0 \to X \dashrightarrow .$$

(2) Any thick subcategory of $\mathcal{T}$ is biresolving because there exist conflations $C \xrightarrow{0} N \to N \oplus \Sigma C \dashrightarrow$ and $\Sigma^{-1}C \oplus N \to N \xrightarrow{0} C \dashrightarrow$.
(3) Similarly, any thick subcategory of $\mathcal{T}$ is percolating because every morphism is both an inflation and a deflation in $\mathcal{T}$.

Typical examples of percolating subcategories are Serre subcategories of *admissible* extriangulated categories, as we shall explain. A morphism $f\colon A \to B$ in $\mathcal{C}$ is called *admissible* if it has a factorization $f = i \circ d$ such that i is an inflation and d is a deflation. We call this factorization a *deflation-inflation factorization*. We also say that $\mathcal{C}$ *is admissible* if every morphism in $\mathcal{C}$ is admissible. For examples, abelian categories and triangulated categories are admissible.

Example 4.2.11. Let $\mathcal{C}$ be an admissible extriangulated category. Then every Serre subcategory $\mathcal{N}$ of $\mathcal{C}$ is percolating. Indeed, let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ having a factorization $X \xrightarrow{x} N \xrightarrow{y} Y$ with $N \in \mathcal{N}$. Consider a deflation-inflation factorization $X \xrightarrow{d_1} M_1 \xrightarrow{i_1} N$ of x. Since $\mathcal{N}$ is a Serre subcategory, an inflation i_1 implies $M_1 \in \mathcal{N}$. Then consider a deflation-inflation factorization $M_1 \xrightarrow{d_2} M_2 \xrightarrow{i_2} Y$ of $M_1 \xrightarrow{i_1} N \xrightarrow{y} Y$. We have $M_2 \in \mathcal{N}$ by a deflation d_2, and $X \xrightarrow{d_2 d_1} M_2 \xrightarrow{i_2} Y$ is a desired decomposition. It is obvious that Serre subcategories are thick, and thus $\mathcal{N}$ is percolating.

The following two facts are useful conditions where Condition 4.2.4 is satisfied for biresolving and percolating subcategories.

[2] This definition is different from [NOS22, Definition 4.28], but they are equivalent by [NOS22, Lemma 4.29].

Fact 4.2.12 ([NOS22, Propostion 4.26])**.** *If $\mathcal{N}$ is biresolving, then Condition 4.2.4 is satisfied. In this case, $\mathcal{C}/\mathcal{N}$ is a triangulated category.*

Fact 4.2.13 ([NOS22, Corollary 4.42])**.** *Let $\mathcal{N}$ be a thick subcategory of $\mathcal{C}$. Consider the following conditions:*

(EL1) *$\mathcal{N}$ is percolating.*

(EL2) *For any split monomorphism $f\colon A \to B$ in $\mathcal{C}$ such that $\overline{f}$ is an isomorphism in $\overline{\mathcal{C}}$, there exist $N \in \mathcal{N}$ and $j\colon N \to B$ in $\mathcal{C}$ such that $[f\ j]\colon A \oplus N \to B$ is an isomorphism in $\mathcal{C}$.*

(EL3) *For every conflation $A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$ in $\mathcal{C}$, both of the following hold:*

$$\operatorname{Ker}\left(\mathcal{C}(-,A) \xrightarrow{f\circ(-)} \mathcal{C}(-,B)\right) \subseteq [\mathcal{N}](-,A),$$
$$\operatorname{Ker}\left(\mathcal{C}(C,-) \xrightarrow{(-)\circ g} \mathcal{C}(B,-)\right) \subseteq [\mathcal{N}](C,-).$$

(EL4) *$\mathcal{C}$ is admissible, namely, every morphism $f\colon A \to B$ in $\mathcal{C}$ has a factorization $f = i \circ d$ such that i is an inflation and d is a deflation.*

Then the following statements hold.

(1) *If* (EL1)–(EL3) *are satisfied, then Condition 4.2.4 is satisfied, and $\mathcal{C}/\mathcal{N}$ is an exact category.*

(2) *If* (EL1)–(EL4) *are satisfied, then $\mathcal{C}/\mathcal{N}$ is an abelian category (endowed with the natural extriangulated structure).*

The situation can be summarized as follows.

$$\mathcal{N} \text{ is biresolving} \xrightarrow{\text{Fact 4.2.12}} \text{Condition 4.2.4} + \mathcal{C}/\mathcal{N} \text{ is triangulated}$$
$$\text{(EL1)–(EL3) (+ (EL4))} \xrightarrow{\text{Fact 4.2.13}} \text{Condition 4.2.4} + \mathcal{C}/\mathcal{N} \text{ is exact (+ abelian)}$$

Our main interests are the above two cases, namely, the case where $\mathcal{N}$ is biresolving, and the case where (EL1)–(EL3) are satisfied (so $\mathcal{N}$ is percolating).

Remark 4.2.14.

(1) Consider the following condition:

(WIC) Let $h = gf$ be a morphism in $\mathcal{C}$. If h is an inflation, then so is f. Dually, if h is a deflation, then so is g.

Then (WIC) implies (EL2) by [NOS22, Remark 4.31 (2)]. A triangulated category satisfies (WIC) since every morphism is both an inflation and a deflation. More generally, an extension-closed subcategory of a triangulated category which is closed under direct summands satisfies (WIC), see Lemma 4.2.16 below.

(2) (EL2) implies Condition 4.2.4 (i) by [NOS22, Lemma 3.2, 4.34].

(3) Under the condition (EL1), the condition (EL3) is equivalent to that $\overline{f}$ (resp. $\overline{g}$) is a monomorphism (resp. an epimorphism) in $\overline{\mathcal{C}}$ for every conflation $A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$ in $\mathcal{C}$, see Lemma 4.3.5.

Example 4.2.15. The following are typical examples of the case where $\mathcal{N}$ is biresolving or $\mathcal{N}$ satisfies (EL1)–(EL4).

(1) Let $\mathcal{T}$ be a triangulated category. A thick subcategory $\mathcal{N}$ of $\mathcal{T}$ is biresolving as mentioned in Example 4.2.10, so the exact localization $\mathcal{T}/\mathcal{N}$ exists and becomes a triangulated category by Fact 4.2.12. This coincides with the Verdier quotient of $\mathcal{T}$ by $\mathcal{N}$.

(2) Let $\mathcal{F}$ be a Frobenius exact category, and let $\mathcal{N}$ be the subcategory of projective objects in $\mathcal{F}$. Then $\mathcal{N}$ is a biresolving subcategory of $\mathcal{F}$. In this case, the exact localization $\mathcal{F}/\mathcal{N}$ coincides with the usual stable (triangulated) category $\underline{\mathcal{F}} = \mathcal{F}/[\mathcal{N}]$.

(3) Let $\mathcal{A}$ be an abelian category. A Serre subcategory $\mathcal{S}$ of $\mathcal{A}$ is percolating as mentioned in Example 4.2.11. Moreover, it satisfies (EL1)–(EL4) in Fact 4.2.13. Indeed, (EL2) is satisfied by Remark 4.2.14 (1) since $\mathcal{A}$ satisfies (WIC), and (EL3) is satisfied since every inflation and deflation in $\mathcal{A}$ is a monomorphism and an epimorphism respectively. Therefore, the exact localization $\mathcal{A}/\mathcal{S}$ exists and becomes an abelian category by Fact 4.2.13. This coincides with the Serre quotient of $\mathcal{A}$ by $\mathcal{S}$.

We have the following convenient criterion for the condition (WIC).

Lemma 4.2.16. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated category $\mathcal{T}$ with shift functor Σ. If $\mathcal{C}$ is closed under direct summands in $\mathcal{T}$, then it satisfies* (WIC)*.*

Proof. Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms in $\mathcal{C}$. Suppose that $h := gf$ is an inflation, and thus there is a conflation $A \xrightarrow{h} C \xrightarrow{c} D \overset{\delta}{\dashrightarrow}$ in $\mathcal{C}$. Then there exists a conflation $A \xrightarrow{f} B \xrightarrow{b} X \overset{\tau}{\dashrightarrow}$ in $\mathcal{T}$ by taking cone of f. Applying [LN19, Proposition 1.20] (see also [Nee01, Proposition 1.4.3]) to $\mathcal{T}$, we obtain a morphism $x : X \to D$ in $\mathcal{T}$ which gives a morphism of conflations

$$\begin{array}{ccccccc} B & \xrightarrow{b} & X & \xrightarrow{\tau} & \Sigma A & \overset{-\Sigma f}{\dashrightarrow} & \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle x} & & \| & & \\ C & \xrightarrow[c]{} & D & \xrightarrow[\delta]{} & \Sigma A & \underset{-\Sigma h}{\dashrightarrow} & \end{array}$$

in $\mathcal{T}$ and makes $B \xrightarrow{\left[\begin{smallmatrix} b \\ g \end{smallmatrix}\right]} X \oplus C \xrightarrow{[x\ -c]} D \dashrightarrow$ a conflation in $\mathcal{T}$. Since $B, D \in \mathcal{C}$ and $\mathcal{C}$ is extension-closed, we have $X \oplus C \in \mathcal{C}$, which implies $X \in \mathcal{C}$ because $\mathcal{C}$ is closed under direct summands. Hence f is an inflation in $\mathcal{C}$. We can dually prove that, if h is a deflation, then so is g, so we omit it. $\square$

4.3 The Grothendieck monoid of the localization of an extriangulated category

In this section, we investigate the relations between Grothendieck monoids and the exact localization. Throughout this section, $\mathcal{C}$ is a skeletally small extriangulated category and $\mathcal{N}$ is a thick subcategory of $\mathcal{C}$. Note that, if Nakaoka-Ogawa-Sakai exact localization $\mathcal{C}/\mathcal{N}$ exists, it is also skeletally small by the construction (cf. Example 1.1.10 and Fact 4.2.5).

The following is the main theorem of this section. We refer the reader to Definition 4.1.1 for the quotient monoid by a submonoid.

Theorem 4.3.1. *Suppose that the following two conditions are satisfied:*

(i) *Condition 4.2.4 is satisfied. Thus, the exact localization $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ exists, and we can freely use Fact 4.2.5.*

(ii) *$S_{\mathcal{N}}$ is saturated, that is, for every morphism f in $\mathcal{C}$, we have $f \in S_{\mathcal{N}}$ if $Q(f)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$.*

Then $\mathsf{M}(Q)\colon \mathsf{M}(\mathcal{C}) \to \mathsf{M}(\mathcal{C}/\mathcal{N})$ induces an isomorphism of monoids:

$$\mathsf{M}(\mathcal{C})/\mathsf{M}_{\mathcal{N}} \xrightarrow{\sim} \mathsf{M}(\mathcal{C}/\mathcal{N}).$$

We first prove this theorem, and then discuss the condition (ii) of this theorem.

Lemma 4.3.2. *Assume the same conditions as in Theorem 4.3.1. Let X and Y be objects in $\mathcal{C}$. If $X \cong Y$ in $\mathcal{C}/\mathcal{N}$, then $[X] \equiv [Y] \bmod \mathsf{M}_{\mathcal{N}}$.*

Proof. Let $f\colon X \xrightarrow{\sim} Y$ be an isomorphism in $\mathcal{C}/\mathcal{N}$. We have morphisms $g\colon X \to A$ in $\mathcal{C}$ and $s\colon Y \to A$ in $S_{\mathcal{N}}$ such that $f = Q(s)^{-1}Q(g)$ by Remark 4.2.8. Then $Q(g)$ is also an isomorphism in $\mathcal{C}/\mathcal{N}$, and thus $g \in S_{\mathcal{N}}$ because $S_{\mathcal{N}}$ is saturated. Therefore, it is enough to show that $[X] \equiv [Y] \bmod \mathsf{M}_{\mathcal{N}}$ if there is a morphism $s\colon X \to Y$ in $S_{\mathcal{N}}$. Since a morphism in $S_{\mathcal{N}}$ is a finite composition of morphisms in $\mathcal{L}$ and $\mathcal{R}$, we may assume that $s \in \mathcal{L}$ or $s \in \mathcal{R}$. If $s \in \mathcal{L}$, then there is a conflation $X \xrightarrow{s} Y \to N \dashrightarrow$ with $N \in \mathcal{N}$. Then $[X] + [N] = [Y]$ holds in $\mathsf{M}(\mathcal{C})$, which implies $[X] \equiv [Y] \bmod \mathsf{M}_{\mathcal{N}}$. The case $s \in \mathcal{R}$ is similar. $\square$

Proof of Theorem 4.3.1. The homomorphism $\mathsf{M}(Q)\colon \mathsf{M}(\mathcal{C}) \to \mathsf{M}(\mathcal{C}/\mathcal{N})$ induces a homomorphism $\phi\colon \mathsf{M}(\mathcal{C})/\mathsf{M}_{\mathcal{N}} \to \mathsf{M}(\mathcal{C}/\mathcal{N})$ satisfying $\phi([A] \bmod \mathsf{M}_{\mathcal{N}}) = [Q(A)]$ by Proposition 4.1.2 since $Q(\mathcal{N}) = 0$ holds by Proposition 4.2.3. Clearly ϕ is surjective since Q is the identity on objects. We have to show that ϕ is injective. Suppose that $[Q(A)] = [Q(B)]$ in $\mathsf{M}(\mathcal{C}/\mathcal{N})$ for $A, B \in \mathcal{C}$, or equivalently, $A \approx_c B$ in $\mathcal{C}/\mathcal{N}$. We want to show that $[A] \equiv [B] \bmod \mathsf{M}_{\mathcal{N}}$. It suffices to prove it for the case $A \sim_c B$ in $\mathcal{C}/\mathcal{N}$.

Since $A \sim_c B$ in $\mathcal{C}/\mathcal{N}$, there exists a conflation $X \to Y \to Z \dashrightarrow$ in $\mathcal{C}/\mathcal{N}$ such that either (a) $X \oplus Z \cong A$ and $Y \cong B$ in $\mathcal{C}/\mathcal{N}$ or (b) $X \oplus Z \cong B$ and $Y \cong A$ in $\mathcal{C}/\mathcal{N}$. Clearly we only have to deal with the case (a). By Fact 4.2.5 (2), we can find a conflation $X' \xrightarrow{f} Y' \xrightarrow{g} Z' \dashrightarrow$ in $\mathcal{C}$ such that we have an isomorphism of conflations

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \dashrightarrow & \\ \wr\| & & \wr\| & & \wr\| & & \\ X' & \xrightarrow{Q(f)} & Y' & \xrightarrow{Q(g)} & Z' & \dashrightarrow & \end{array}$$

in $\mathcal{C}/\mathcal{N}$. Thus, by Lemma 4.3.2, we have

$$[A] = [X \oplus Z] \equiv [X' \oplus Z'] = [Y'] \equiv [Y] = [B] \bmod \mathsf{M}_{\mathcal{N}}.$$

This proves the injectivity of ϕ. □

In the following, we discuss the kernel of the localization functor $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ and study when $S_{\mathcal{N}}$ is saturated. We would like to thank Hiroyuki Nakaoka for sharing the results on the kernel of the localization functor. We prepare some lemmas.

Lemma 4.3.3. *Suppose* (i) *of Condition 4.2.4 and let $A \in \mathcal{C}$. If there exists a split monomorphism $u\colon A \to N$ with $N \in \mathcal{N}$, then $A \in \mathcal{N}$.*

Proof. The morphism $\overline{u}$ is a split monomorphism in $\overline{\mathcal{C}}$ since so is u in $\mathcal{C}$. Then the split monomorphism $A \xrightarrow{\overline{u}} N \cong 0$ should be an isomorphism in $\overline{\mathcal{C}}$. Thus, (i) of Condition 4.2.4 implies that $u\colon A \to N$ belongs to $S_{\mathcal{N}}$. By composing this with $N \to 0$, which is in $S_{\mathcal{N}}$, we obtain that $A \to 0$ is in $S_{\mathcal{N}}$. Dually, $0 \to A$ is also in $S_{\mathcal{N}}$ by considering a retraction $q\colon N \to A$ of u. By (4.2.1), we conclude that $A \in \mathcal{N}$. □

Lemma 4.3.4. *Suppose* (i) *of Condition 4.2.4. Then the following hold.*

(1) *Let $N \xrightarrow{k} Y \xrightarrow{r} Z \overset{\delta}{\dashrightarrow}$ be a conflation with $N \in \mathcal{N}$ and $\overline{r} = 0$ in $\overline{\mathcal{C}}$. Then $Y \in \mathcal{N}$.*

(2) *Dually, let $X \xrightarrow{\ell} Y \xrightarrow{c} N \overset{\delta}{\dashrightarrow}$ be a conflation with $N \in \mathcal{N}$ and $\overline{\ell} = 0$ in $\overline{\mathcal{C}}$. Then $Y \in \mathcal{N}$.*

In each case, all objects in the above conflations belong to $\mathcal{N}$ since $\mathcal{N}$ is a thick subcategory.

Proof. We only prove (1) since (2) follows dually. The assumption $\overline{r} = 0$ implies that it has a factorization $Y \xrightarrow{f} N' \xrightarrow{g} Z$ in $\mathcal{C}$ with $N' \in \mathcal{N}$. Then there is a morphism of conflations

$$\begin{array}{ccccccc} N & \xrightarrow{a} & M & \xrightarrow{c} & N' & \overset{g^*\delta}{\dashrightarrow} & \\ \| & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle g} & & \\ N & \xrightarrow[k]{} & Y & \xrightarrow[r]{} & Z & \underset{\delta}{\dashrightarrow} & . \end{array}$$

The top row implies $M \in \mathcal{N}$ because $\mathcal{N}$ is extension-closed. We can choose $b\colon M \to Y$ so that $M \xrightarrow{\left[\begin{smallmatrix} c \\ b \end{smallmatrix}\right]} Y \oplus N' \xrightarrow{[g\ -r]} Z \dashrightarrow$ is a conflation by [LN19, Proposition 1.20]. This implies that the following diagram is a weak pullback diagram, and hence a section $\phi\colon Y \to M$ of b is induced:

$$\begin{array}{ccccc} Y & & \overset{f}{\frown\!\!\!\longrightarrow} & & N' \\ & \overset{\phi}{\searrow} & & & \\ & & M & \xrightarrow{c} & N' \\ & & {\scriptstyle b}\downarrow & \text{wPB} & \downarrow{\scriptstyle g} \\ \| & & Y & \xrightarrow[r]{} & Z. \end{array}$$

Since $M \in \mathcal{N}$, we obtain $Y \in \mathcal{N}$ by Lemma 4.3.3. □

Set $\overline{\mathcal{L}} := p(\mathcal{L})$ and $\overline{\mathcal{R}} := p(\mathcal{R})$. We have the following interpretation of (EL3) for the case of percolating subcategories.

Lemma 4.3.5. *Let $\mathcal{N}$ be a percolating subcategory, and let $f\colon A \to B$ be a morphism in $\mathcal{C}$.*

(1) *$\overline{f}$ is a monomorphism in $\overline{\mathcal{C}}$ if and only if* $\operatorname{Ker}\left(\mathcal{C}(-, A) \xrightarrow{f \circ (-)} \mathcal{C}(-, B)\right) \subseteq [\mathcal{N}](-, A)$ *holds.*

(2) $\overline{f}$ *is an epimorphism in* $\overline{\mathcal{C}}$ *if and only if* $\operatorname{Ker}\left(\mathcal{C}(B,-) \xrightarrow{(-)\circ f} \mathcal{C}(A,-)\right) \subseteq [\mathcal{N}](B,-)$ *holds.*

Proof. Since (2) is the dual of (1), we only prove (1). To show the "only if" part, suppose that $\overline{f}$ is a monomorphism in $\overline{\mathcal{C}}$. Let $x \in \operatorname{Ker}\left(\mathcal{C}(X,A) \xrightarrow{f\circ(-)} \mathcal{C}(X,B)\right)$, that is, $x\colon X \to A$ satisfies $fx = 0$ in $\mathcal{C}$. Because $\overline{f}$ is a monomorphism, we have $\overline{x} = 0$ in $\overline{\mathcal{C}}$. This means $x \in [\mathcal{N}](X,A)$.

To show the "if" part, suppose that $\operatorname{Ker}\left(\mathcal{C}(-,A) \xrightarrow{f\circ(-)} \mathcal{C}(-,B)\right) \subseteq [\mathcal{N}](-,A)$ holds. Let $x\colon X \to A$ be a morphism in $\mathcal{C}$ with $\overline{fx} = 0$ in $\overline{\mathcal{C}}$, that is, fx factors through an object in $\mathcal{N}$. Then fx has a deflation-inflation factorization $X \xrightarrow{d} N \xrightarrow{i} B$ with $N \in \mathcal{N}$ since $\mathcal{N}$ is percolating. In particular, we have a conflation $K \xrightarrow{\ell} X \xrightarrow{d} N \dashrightarrow$. Note that $\ell \in \mathcal{L}$. Then we have $fx\ell = id\ell = 0$, and by the assumption, we conclude that $\overline{x\ell} = 0$ in $\overline{\mathcal{C}}$. Since $\overline{\ell} \in \overline{\mathcal{L}}$ is an epimorphism in $\overline{\mathcal{C}}$ by [NOS22, Lemma 4.7 (2)], we have $\overline{x} = 0$. This proves that $\overline{f}$ is a monomorphism. □

Corollary 4.3.6. *Suppose that either of the following conditions holds.*
(i) $\mathcal{N}$ *is a biresolving subcategory.*
(ii) $\mathcal{N}$ *satisfies* (EL1)–(EL3).
Then any morphism in $\overline{S_{\mathcal{N}}}$ *is both a monomorphism and an epimorphism in* $\overline{\mathcal{C}}$.

Proof. By [NOS22, Lemma 4.7], every morphism in $\overline{\mathcal{L}}$ (resp. $\overline{\mathcal{R}}$) is an epimorphism (resp. a monomorphism) in $\overline{\mathcal{C}}$. Thus, it suffices to show that every morphism in $\overline{\mathcal{L}}$ is a monomorphism in each case, since the proof for $\overline{\mathcal{R}}$ follows dually.

(i) This case is precisely [NOS22, Lemma 4.24].

(ii) In this case, $\mathcal{N}$ is a percolating subcategory by (EL1). Thus, the assertion follows from (EL3) and Lemma 4.3.5, since every morphism in $\mathcal{L}$ is an inflation in $\mathcal{C}$ by definition. □

Now we can show that the kernel of $\mathcal{C} \to \mathcal{C}/\mathcal{N}$ coincides with $\mathcal{N}$ under some conditions. Recall that a kernel $\ker F$ of an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of objects $C \in \mathcal{C}$ satisfying $F(C) \cong 0$ in $\mathcal{D}$.

Proposition 4.3.7. *Suppose that Condition 4.2.4 is satisfied, and that every morphism in* $\overline{S_{\mathcal{N}}}$ *is a monomorphism in* $\overline{\mathcal{C}}$. *Then* $\operatorname{Ker}(Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}) = \mathcal{N}$ *holds.*

Proof. We have $\operatorname{Ker}(Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}) \supseteq \mathcal{N}$ by Proposition 4.2.3. Conversely, suppose that $X \in \mathcal{C}$ satisfies $Q(X) \cong 0$ in $\mathcal{C}/\mathcal{N}$. Then there exists $Y \in \mathcal{C}$ satisfying $0 \in \overline{S_{\mathcal{N}}}(X,Y)$ since $\mathcal{C}/\mathcal{N}$ is constructed as the category of fractions $\overline{S_{\mathcal{N}}}^{-1}\overline{\mathcal{C}}$ by Fact 4.2.5. Hence, we obtain $s \in S_{\mathcal{N}}(X,Y)$ satisfying $\overline{s} = 0$ in $\overline{\mathcal{C}}$. By the construction of $S_{\mathcal{N}}$, we can write either $s = t\ell$ or $s = ur$ for some $t,u \in S_{\mathcal{N}}$, $\ell \in \mathcal{L}$, and $r \in \mathcal{R}$. We consider the case $s = t\ell$. Since $\overline{t}$ is a monomorphism in $\overline{\mathcal{C}}$ by the assumption, $\overline{s} = \overline{t\ell} = 0$ implies $\overline{\ell} = 0$. Thus, Lemma 4.3.4 (2) implies $X \in \mathcal{N}$. The case $s = ur$ can be proved similarly using Lemma 4.3.4 (1). □

Consequently, we can describe the kernel of the localization for the cases we are interested in.

Corollary 4.3.8. *Suppose that either of the following conditions holds.*
(i) $\mathcal{N}$ *is a biresolving subcategory.*
(ii) $\mathcal{N}$ *satisfies* (EL1)–(EL3).
Then $\operatorname{Ker}(Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}) = \mathcal{N}$ *holds.*

Proof. For each case, Condition 4.2.4 is satisfied by Fact 4.2.12 and 4.2.13, and every morphism in $\overline{S_{\mathcal{N}}}$ is a monomorphism by Corollary 4.3.6. Thus, the assertion follows from Proposition 4.3.7. □

Using this result, we next consider whether $S_{\mathcal{N}}$ is saturated. Assume Condition 4.2.4, and let $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ be the localization functor. Recall that $S_{\mathcal{N}}$ is *saturated* if $f \in S_{\mathcal{N}}$ holds for any morphism f in $\mathcal{C}$ such that $Q(f)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$.

Proposition 4.3.9. *If* $\mathcal{N}$ *is a biresolving subcategory of* $\mathcal{C}$, *then* $S_{\mathcal{N}}$ *is saturated.*

Proof. Let $f\colon X\to Y$ be a morphism in $\mathcal{C}$ with $Q(f)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$. Since $\mathcal{N}$ is biresolving, there is a conflation $X \xrightarrow{x} N \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ with $N\in\mathcal{N}$. Then there is a morphism of conflations

$$\begin{array}{ccccccc} X & \xrightarrow{x} & N & \xrightarrow{y} & C & \dashrightarrow & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \| & & \\ Y & \xrightarrow[a]{} & M & \xrightarrow[b]{} & C & \dashrightarrow & . \end{array} \tag{4.3.1}$$

We can choose a morphism $g\colon N\to M$ so that

$$X \xrightarrow{\left[\begin{smallmatrix} x \\ f \end{smallmatrix}\right]} N\oplus Y \xrightarrow{[\,g\ -a\,]} M \overset{b^*\delta}{\dashrightarrow} \tag{4.3.2}$$

is a conflation by [LN19, Proposition 1.20]. We have a factorization $f=[\,0\ \mathsf{id}_Y\,]\circ\left[\begin{smallmatrix} x \\ f \end{smallmatrix}\right]$ for $[\,0\ \mathsf{id}_Y\,]: N\oplus Y\to Y$ and $\left[\begin{smallmatrix} x \\ f \end{smallmatrix}\right]: X\to N\oplus Y$. Clearly, $[\,0\ \mathsf{id}_Y\,]$ belongs to $\mathcal{R}$. Thus, it is enough to show that $\left[\begin{smallmatrix} x \\ f \end{smallmatrix}\right]$ belongs to $\mathcal{L}$, that is, $M\in\mathcal{N}$. Applying Q to (4.3.1), we have a morphism of conflation

$$\begin{array}{ccccccc} X & \xrightarrow{Q(x)} & N & \xrightarrow{Q(y)} & C & \dashrightarrow & \\ {\scriptstyle Q(f)}\downarrow & & \downarrow{\scriptstyle Q(g)} & & \| & & \\ Y & \xrightarrow[Q(a)]{} & M & \xrightarrow[Q(b)]{} & C & \dashrightarrow & . \end{array}$$

in $\mathcal{C}/\mathcal{N}$ by Remark 2.2.3. Since $Q(f)$ is an isomorphism by the assumption, $Q(g)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$ by [NP19, Corollary 3.6]. Thus, we obtain $M\cong N\cong 0$ in $\mathcal{C}/\mathcal{N}$, which shows $M\in\operatorname{Ker}Q$. Since $\operatorname{Ker}Q=\mathcal{N}$ holds by Corollary 4.3.8, we obtain $M\in\mathcal{N}$. This proves $\left[\begin{smallmatrix} x \\ f \end{smallmatrix}\right]\in\mathcal{L}$ by (4.3.2). □

Unfortunately, for the percolating case, we do not know whether the condition (EL1)–(EL3) implies that $S_{\mathcal{N}}$ is saturated, although we have the following criterion.

Proposition 4.3.10. *Suppose that $\mathcal{N}$ satisfies* (EL1)–(EL3)*. Then the following conditions are equivalent.*

(1) *$S_{\mathcal{N}}$ is saturated.*
(2) *For a morphism $f\colon X\to Y$ in $\mathcal{C}$, if $Q(f)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$, then f is admissible in $\mathcal{C}$, that is, there exists a deflation-inflation factorization of f.*

Proof. (1) ⇒ (2): We have $S_{\mathcal{N}}=\mathcal{L}\circ\mathcal{R}$ by [NOS22, Lemma 4.37]. Hence, if $Q(f)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$, then $f\in S_{\mathcal{N}}=\mathcal{L}\circ\mathcal{R}$ since $S_{\mathcal{N}}$ is saturated, which proves (2) since a morphism in $\mathcal{L}$ (resp. $\mathcal{R}$) is an inflation (resp. a deflation).

(2) ⇒ (1): Let $f\colon X\to Y$ be a morphism in $\mathcal{C}$ such that $Q(f)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$. There is a deflation-inflation factorization $X\xrightarrow{d}A\xrightarrow{i}Y$ of f by (2). In particular, we have conflations $K\xrightarrow{k}X\xrightarrow{d}A\dashrightarrow$ and $A\xrightarrow{i}Y\xrightarrow{c}C\dashrightarrow$. Since Q is an exact functor, we obtain conflations $Q(K)\xrightarrow{Q(k)}Q(X)\xrightarrow{Q(d)}Q(A)\dashrightarrow$ and $Q(A)\xrightarrow{Q(i)}Q(Y)\xrightarrow{Q(c)}Q(C)\dashrightarrow$ in $\mathcal{C}/\mathcal{N}$. In particular, $Q(d)$ is a deflation in $\mathcal{C}/\mathcal{N}$. Thus $Q(d)$ is an epimorphism because $\mathcal{C}/\mathcal{N}$ is an exact category by Fact 4.2.13. On the other hand, $Q(d)$ is a split monomorphism since $Q(f)=Q(i)\circ Q(d)$ is an isomorphism. Thus $Q(d)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$. Dually, $Q(i)$ is also an isomorphism in $\mathcal{C}/\mathcal{N}$. Hence, both $Q(K)$ and $Q(C)$ are isomorphic to 0 in $\mathcal{C}/\mathcal{N}$, which implies $K,C\in\operatorname{Ker}Q=\mathcal{N}$ by Corollary 4.3.8. We conclude that $d\in\mathcal{R}$ and $i\in\mathcal{L}$, and hence $f=i\circ d\in S_{\mathcal{N}}$. □

Therefore, if one assumes in addition that (EL4) holds, that is, $\mathcal{C}$ is admissible, then we can show the saturatedness of $S_{\mathcal{N}}$. We can summarize our results of the saturatedness as follows.

Corollary 4.3.11. *Suppose that either of the following conditions holds.*

(i) *$\mathcal{N}$ is a biresolving subcategory.*
(ii) *$\mathcal{N}$ satisfies* (EL1)–(EL4).

Then $S_{\mathcal{N}}$ is saturated.

Proof. (i) This is Proposition 4.3.9.

(ii) Since $\mathcal{N}$ satisfies (EL1)–(EL3), we can apply Proposition 4.3.10. In addition, every morphism in $\mathcal{C}$ is admissible by (EL4), and hence the condition in Proposition 4.3.10 (2) automatically holds. Therefore, $S_{\mathcal{N}}$ is saturated. □

As a consequence, we can deduce the following result for the cases we are interested in.

Corollary 4.3.12. *Suppose that either of the following conditions holds.*
(i) *$\mathcal{N}$ is a biresolving subcategory.*
(ii) *$\mathcal{N}$ satisfies* (EL1)–(EL4).
Then the monoid homomorphism $\mathsf{M}(Q)\colon \mathsf{M}(\mathcal{C}) \to \mathsf{M}(\mathcal{C}/\mathcal{N})$ for the exact localization $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ induces an isomorphism of monoids:

$$\mathsf{M}(\mathcal{C})/\mathsf{M}_{\mathcal{N}} \xrightarrow{\sim} \mathsf{M}(\mathcal{C}/\mathcal{N}).$$

Proof. This follows from Theorem 4.3.1 and Corollary 4.3.11. □

Remark 4.3.13. Ogawa [Oga] developed localization theory of triangulated categories with respect to an *extension-closed subcategory*. We briefly introduce his results and explain the relation to ours. Let $\mathcal{T}$ be a triangulated category and $\mathcal{N}$ an extension-closed subcategory of $\mathcal{T}$. We consider $\mathcal{T}$ as the extriangulated category $(\mathcal{T}, \mathbb{E}, \mathfrak{s})$. Then there is a subfunctor $\mathbb{E}_{\mathcal{N}} \subseteq \mathbb{E}$ satisfying the following:

- $\mathcal{T}^{\mathcal{N}} := (\mathcal{T}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{|\mathbb{E}_{\mathcal{N}}})$ is an extriangulated category.
- $\mathcal{N}$ is a thick subcategory of $\mathcal{T}^{\mathcal{N}}$ and satisfies Condition 4.2.4. In particular, the exact localization $\mathcal{T}^{\mathcal{N}}/\mathcal{N}$ exists by Fact 4.2.5.
- $S_{\mathcal{N}}$ is saturated in $\mathcal{T}^{\mathcal{N}}$.

Thus, we can apply Theorem 4.3.1 and obtain a monoid isomorphism $\mathsf{M}\left(\mathcal{T}^{\mathcal{N}}\right)/\mathsf{M}_{\mathcal{N}} \xrightarrow{\sim} \mathsf{M}\left(\mathcal{T}^{\mathcal{N}}/\mathcal{N}\right)$ for any extension-closed subcategory $\mathcal{N}$ of $\mathcal{T}$. However, we do not know a relation between $\mathsf{M}(\mathcal{T})$ and $\mathsf{M}\left(\mathcal{T}^{\mathcal{N}}\right)$ at this moment.

Finally, we give applications of our description Theorem 4.3.1 of the Grothendieck monoid of the exact localization.

First, by applying this to the abelian case, we obtain the following consequence on the Serre quotient of an abelian category.

Corollary 4.3.14. *Let $\mathcal{A}$ be a skeletally small abelian category, $\mathcal{S}$ a Serre subcategory of $\mathcal{A}$, and $\iota\colon \mathcal{S} \hookrightarrow \mathcal{A}$ and $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ the inclusion and the localization functor respectively. Then the following holds.*
(1) *$\mathsf{M}(\iota)\colon \mathsf{M}(\mathcal{S}) \to \mathsf{M}(\mathcal{A})$ is injective, so we have an isomorphism $\mathsf{M}(\mathcal{S}) \cong \mathsf{M}_{\mathcal{S}}$.*
(2) *$\mathsf{M}(Q)\colon \mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A}/\mathcal{S})$ induces an isomorphism of monoids $\mathsf{M}(\mathcal{A})/\mathsf{M}_{\mathcal{S}} \cong \mathsf{M}(\mathcal{A}/\mathcal{S})$.*

Proof. (1) This is Proposition 3.3.5.

(2) Example 4.2.15 (3) shows that $\mathcal{S}$ satisfies (EL1)–(EL4). Therefore, we can apply Corollary 4.3.12 to this setting. □

This may be seen as a "short exact sequence" of monoids:

$$0 \longrightarrow \mathsf{M}(\mathcal{S}) \longrightarrow \mathsf{M}(\mathcal{A}) \longrightarrow \mathsf{M}(\mathcal{A}/\mathcal{S}) \longrightarrow 0. \tag{4.3.3}$$

This description of $\mathsf{M}(\mathcal{A}/\mathcal{S})$ gives the following description of Serre subcategories of $\mathcal{A}/\mathcal{S}$, which seems to be a folklore.

Corollary 4.3.15. *Let $\mathcal{A}$ be a skeletally small abelian category and $\mathcal{S}$ a Serre subcategory of $\mathcal{A}$. Then there is a bijection between the following two sets:*
(1) $\operatorname{Serre}(\mathcal{A}/\mathcal{S})$.
(2) $\{\mathcal{S}' \in \operatorname{Serre}\mathcal{A} \mid \mathcal{S} \subseteq \mathcal{S}'\}$.

Proof. There is a bijection between $\operatorname{Serre}(\mathcal{A}/\mathcal{S})$ and $\operatorname{Face}\mathsf{M}(\mathcal{A}/\mathcal{S})$ by Proposition 3.3.3. On the other hand, $\mathsf{M}(\mathcal{A}/\mathcal{S})$ is isomorphic to the quotient monoid $\mathsf{M}(\mathcal{A})/\mathsf{M}_{\mathcal{S}}$ by Corollary 4.3.14 (2). Thus, Proposition 4.1.4 (3) shows that $\operatorname{Face}(\mathsf{M}(\mathcal{A})/\mathsf{M}_{\mathcal{S}})$ is in bijection with the set of faces of $\mathsf{M}(\mathcal{A})$ containing $\mathsf{M}_{\mathcal{S}}$. Since $\operatorname{Face}\mathsf{M}(\mathcal{A})$ are in bijection with $\operatorname{Serre}\mathcal{A}$ by Proposition 3.3.3 again and since $\mathcal{S}$ corresponds to $\mathsf{M}_{\mathcal{S}}$ in this bijection, we conclude that $\operatorname{Face}(\mathsf{M}(\mathcal{A})/\mathsf{M}_{\mathcal{S}})$ are in bijection with (2). The situation can be summarized as the following figure.

$$\begin{array}{ccccccc}
\operatorname{Serre}(\mathcal{A}/\mathcal{S}) & \longleftarrow\cdots\cdots\cdots\longrightarrow & & & \{\mathcal{S}' \in \operatorname{Serre}\mathcal{A} \mid \mathcal{S} \subseteq \mathcal{S}'\} & \subseteq & \operatorname{Serre}\mathcal{A} \\
\updownarrow & & & & \updownarrow & & \updownarrow \\
\operatorname{Face}\mathsf{M}(\mathcal{A}/\mathcal{S}) & \longleftrightarrow & \operatorname{Face}(\mathsf{M}(\mathcal{A})/\mathsf{M}_{\mathcal{S}}) & \longleftrightarrow & \{F \in \operatorname{Face}\mathsf{M}(\mathcal{A}) \mid \mathsf{M}_{\mathcal{S}} \subseteq F\} & \subseteq & \operatorname{Face}\mathsf{M}(\mathcal{A})
\end{array}$$

□

It is well-known that the similar sequence to (4.3.3) for the Grothendieck group is only right exact. Actually, we can deduce it using our result as follows.

Corollary 4.3.16. *Suppose that all the items in Condition 4.2.4 are satisfied and $S_{\mathcal{N}}$ is saturated (e.g. the assumption in Corollary 4.3.11 holds). Then the following sequence is exact:*

$$\mathsf{K}_0(\mathcal{N}) \xrightarrow{\mathsf{K}_0(\iota)} \mathsf{K}_0(\mathcal{C}) \xrightarrow{\mathsf{K}_0(Q)} \mathsf{K}_0(\mathcal{C}/\mathcal{N}) \longrightarrow 0,$$

where $\iota\colon \mathcal{N} \hookrightarrow \mathcal{C}$ is the inclusion functor and $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ is the exact localization functor.

Proof. The diagram

$$\mathsf{M}(\mathcal{N}) \underset{0}{\overset{\mathsf{M}(\iota)}{\rightrightarrows}} \mathsf{M}(\mathcal{C}) \xrightarrow{\mathsf{M}(Q)} \mathsf{M}(\mathcal{C}/\mathcal{N})$$

is a coequalizer diagram in Mon by Theorem 4.3.1 (see also Proposition 4.1.2). Since the group completion functor $\mathsf{gp}\colon \mathsf{Mon} \to \mathsf{Ab}$ is a left adjoint of the forgetful functor $\mathsf{Ab} \hookrightarrow \mathsf{Mon}$ by Proposition 2.1.8, it preserves colimits. Hence, we have a coequalizer diagram

$$\mathsf{K}_0(\mathcal{N}) \underset{0}{\overset{\mathsf{K}_0(\iota)}{\rightrightarrows}} \mathsf{K}_0(\mathcal{C}) \xrightarrow{\mathsf{K}_0(Q)} \mathsf{K}_0(\mathcal{C}/\mathcal{N})$$

in Ab because $\mathsf{gp} \circ \mathsf{M} = \mathsf{K}_0$ by Remark 2.3.6. This is nothing but the claimed exact sequence. □

Chapter 5

Localization of monoids and intermediate subcategories

In this chapter, we address a categorification of a *monoid localization*, which makes certain elements of a monoid invertible (see §5.1). For this purpose, we study *intermediate subcategories* of the derived category in detail, which also gives a concrete example of the theory developed in Chapter 3 and 4 for an extriangulated category which is neither abelian nor triangulated.

Throughout this chapter, we fix Grothendieck universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$ and use Convention 1.1.12 and 2.2.5. Moreover, we assume that every category, functor, and subcategory is additive. In particular, every subcategory is strictly full and nonempty.

5.1 Preliminaries: localization of monoids

In this section, we introduce the localization of a monoid, which is a monoid analogue of the localization of a commutative ring.

Definition 5.1.1. Let M be a monoid and S a subset of M. The *localization of M with respect to S* is a monoid M_S together with a monoid homomorphism $\rho\colon M \to M_S$ which satisfies the following universal property:

(i) $\rho(s)$ is invertible in M_S for each $s \in S$.

(ii) For any monoid homomorphism $\phi\colon M \to X$ such that $\phi(s)$ is invertible for each $s \in S$, there is a unique monoid homomorphism $\overline{\phi}\colon M_S \to X$ satisfying $\phi = \overline{\phi}\rho$.

The localization of a monoid M with respect to a subset $S \subseteq M$ does exist, which is constructed as follows: Define a binary relation on $M \times \langle S \rangle_{\mathbb{N}}$ by

$$(x, s) \sim (y, t) :\Leftrightarrow \text{there exist } u \in \langle S \rangle_{\mathbb{N}} \text{ such that } x + t + u = y + s + u \text{ in } M.$$

It is a congruence on the monoid $M \times \langle S \rangle_{\mathbb{N}}$, and hence the quotient set $M_S := M \times \langle S \rangle_{\mathbb{N}}/{\sim}$ becomes a monoid. We denote by $[x, s]$ the equivalence class of $(x, s) \in M \times \langle S \rangle_{\mathbb{N}}$. We can think of $[x, s]$ as "$x - s$." Then it is straightforward to check that the monoid M_S together with a monoid morphism $\rho\colon M \to M_S$ defined by $\rho(m) = [m, 0]$ is the localization of M with respect to S. We call $\rho\colon M \to M_S$ the *localization homomorphism* of M with respect to S.

We reveal the relationship between faces of M_S and those of M in Proposition 5.1.4 below. Let us prove two lemmas for this purpose.

Lemma 5.1.2. *Let S be a subset of M. Then the natural monoid homomorphism*

$$M_S \to M_{\langle S \rangle_{\mathrm{face}}}, \quad [x, s] \mapsto [x, s]$$

is an isomorphism.

Proof. Let $\rho\colon M \to M_S$ be the localization homomorphism. We have that $\rho^{-1}(M_S^{\times}) = \langle S \rangle_{\mathrm{face}}$ by [Ogu18, The text following Proposition 1.4.4]. Then the conclusion follows immediately from the universal property. □

Lemma 5.1.3. *Let F be a face of M.*
(1) *M/F is sharp (see Definition 2.1.3 (1)).*
(2) *The monoid homomorphism*

$$\phi\colon M_F/M_F^\times \to M/F, \quad [x,s] \bmod M_F^\times \mapsto x \bmod F$$

is an isomorphism.

Proof. (1) Let $x, y \in M$ such that $x+y \equiv 0 \bmod F$. There are elements $s, t \in F$ such that $x+y+s = t$ in M. Since F is a face, both x and y belong to F, which implies both $x \equiv 0 \bmod F$ and $y \equiv 0 \bmod F$. Therefore M/F is sharp.

(2) The quotient homomorphism $M \to M/F$ induces a monoid homomorphism $\phi'\colon M_F \to M/F$ by the universal property of M_F. Then $\phi'(M_F^\times) = 0$ since M/F is sharp. Thus ϕ' induces a monoid homomorphism $\phi\colon M_F/M_F^\times \to M/F$ by the universal property of $M_F/M_F^\times$. The homomorphism ϕ is clearly surjective. We prove that ϕ is injective. Let $[x,s], [y,t] \in M_F$ such that $x \equiv y \bmod F$. Then there are $n, n' \in F$ such that $x+n = y+n'$ in M. Hence, we have the following equalities in M_F:

$$[x,s] + [s+n, 0] = [x+s+n, s] = [x+n, 0] = [y+n', 0] = [y,t] + [t+n', 0].$$

We conclude that $[x,s] \equiv [y,t] \bmod M_F^\times$ because $[s+n,0], [t+n',0] \in M_F^\times$. Therefore ϕ is injective. □

Proposition 5.1.4. *Let M be a monoid and S a subset of M. Then there is an inclusion-preserving bijection between the following sets:*
(1) $\mathrm{Face}(M_S)$.
(2) $\{F \in \mathrm{Face}\, M \mid F \supseteq S\}$.

Proof. We have the following inclusion-preserving bijections:

$$\mathrm{Face}(M_S) \cong \mathrm{Face}\left(M_{\langle S\rangle_{\mathrm{face}}}\right) \cong \mathrm{Face}\left(M_{\langle S\rangle_{\mathrm{face}}}/M^\times_{\langle S\rangle_{\mathrm{face}}}\right) \cong \mathrm{Face}(M/\langle S\rangle_{\mathrm{face}})$$

by Lemma 5.1.2, 5.1.3, and Corollary 4.1.5. The set $\mathrm{Face}(M/\langle S\rangle_{\mathrm{face}})$ corresponds bijectively to the set (2) by Proposition 4.1.4. □

5.2 Intermediate subcategories of the derived category

In the remaining of this chapter, $\mathcal{A}$ denotes a skeletally small abelian category. We denote by $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$, which we regard as an extriangulated category. Note that $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is skeletally small since so is $\mathcal{A}$. We also denote by $H^i\colon \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{A}$ the i-th cohomology functor. We often identify $\mathcal{A}$ with the essential image of the canonical embedding $\mathcal{A} \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathcal{A})$, that is, the subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ consisting of X such that $H^i(X) = 0$ for $i \neq 0$.

The following observation is useful throughout this section, and can be proved easily by using the truncation functor.

Lemma 5.2.1. *Let $\mathcal{A}$ be a skeletally small abelian category and $\mathcal{B}$ and $\mathcal{B}'$ subcategories of $\mathcal{A}$.*
(1) *We have the following equality in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$:*

$$\mathcal{B}[1] * \mathcal{B}' = \{X \in \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \mid H^i(X) = 0 \text{ for } i \neq 0, -1,\ H^{-1}(X) \in \mathcal{B}, \text{ and } H^0(X) \in \mathcal{B}'\}.$$

(2) *For every $X \in \mathcal{A}[1] * \mathcal{A}$, we have the following conflation in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, which is natural on X:*

$$H^{-1}(X)[1] \longrightarrow X \longrightarrow H^0(X) \dashrightarrow .$$

Moreover, every conflation $A[1] \to X \to B \dashrightarrow$ with $A, B \in \mathcal{A}$ is isomorphic to the above conflation.
(3) *For every $X \in \mathcal{A}[1] * \mathcal{A}$, there is some $Y \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ with $X \cong Y$ in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ such that Y is a complex concentrated in degree 0 and -1.*

Let us introduce the main topic of this section.

Definition 5.2.2. Let $\mathcal{A}$ be a skeletally small abelian category. An *intermediate subcategory* of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is a subcategory $\mathcal{C}$ satisfying the following conditions:
(1) $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{A}[1] * \mathcal{A}$ holds.
(2) $\mathcal{C}$ is closed under extensions in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.
(3) $\mathcal{C}$ is closed under direct summands in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.
By (2), we regard an intermediate subcategory as an extriangulated category.

See Figure 0.2.1 for the intuition of this notion. The simplest example of an intermediate subcategory is $\mathcal{A}$ itself, and generally intermediate subcategories are larger than $\mathcal{A}$ but not too large so that they are contained in $\mathcal{A}[1] * \mathcal{A}$.

First, we will see that intermediate subcategories can be described using torsionfree classes in $\mathcal{A}$. Here a subcategory $\mathcal{F}$ of $\mathcal{A}$ is called a *torsionfree class* if $\mathcal{F}$ is closed under subobjects and extensions. Note that torsionfree classes do not necessarily correspond to torsion classes and torsion pairs in general.

Theorem 5.2.3. *Let $\mathcal{A}$ be a skeletally small abelian category. Then the following hold.*
(1) *If $\mathcal{F}$ is a torsionfree class in $\mathcal{A}$, then $\mathcal{F}[1] * \mathcal{A}$ is an intermediate subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.*
(2) *If $\mathcal{C}$ is an intermediate subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, then $H^{-1}(\mathcal{C})$ is a torsionfree class in $\mathcal{A}$, and we have $\mathcal{C} = H^{-1}(\mathcal{C})[1] * \mathcal{A}$.*
(3) *The assignments given in* (1) *and* (2) *give bijections between the set of torsionfree classes in $\mathcal{A}$ and that of intermediate subcategories of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.*

Proof. (1) Let $\mathcal{F}$ be a torsionfree class in $\mathcal{A}$. Recall from Lemma 5.2.1 (1) that $\mathcal{F}[1] * \mathcal{A}$ consists of $X \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ satisfying $H^i(X) = 0$ for $i \neq 0, -1$ and $H^{-1}(X) \in \mathcal{F}$. We clearly have $\mathcal{A} \subseteq \mathcal{F}[1] * \mathcal{A} \subseteq \mathcal{A}[1] * \mathcal{A}$. Next, we show that $\mathcal{F}[1] * \mathcal{A}$ is closed under extensions in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. Suppose that there is a conflation (triangle)

$$X \longrightarrow Y \longrightarrow Z \dashrightarrow$$

in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ with $X, Z \in \mathcal{F}[1] * \mathcal{A}$. It clearly suffices to show that $H^{-1}(Y) \in \mathcal{F}$. Since H is a cohomological functor, we have the following exact sequence in $\mathcal{A}$:

$$0 = H^{-2}(Z) \longrightarrow H^{-1}(X) \longrightarrow H^{-1}(Y) \longrightarrow H^{-1}(Z).$$

Since we have $H^{-1}(X), H^{-1}(Z) \in \mathcal{F}$ and $\mathcal{F}$ is closed under subobjects and extensions, we obtain $H^{-1}(Y) \in \mathcal{F}$. Since clearly $H^i(Y) = 0$ for $i \neq 0, -1$, we obtain $Y \in \mathcal{F}[1] * \mathcal{A}$, so $\mathcal{C}$ is closed under extensions in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.

Since $H^i(X \oplus Y) \cong H^i(X) \oplus H^i(Y)$ holds and $\mathcal{F}$ is closed under direct summands in $\mathcal{A}$, we can easily check that $\mathcal{F}[1] * \mathcal{A}$ is closed under direct summands in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.

(2) Suppose that $\mathcal{C}$ is an intermediate subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, that is, $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{A}[1] * \mathcal{A}$ holds and $\mathcal{C}$ is closed under extensions and direct summands in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. We first show the equality $\mathcal{C} = H^{-1}(\mathcal{C})[1] * \mathcal{A}$. Lemma 5.2.1 (2) implies $\mathcal{C} \subseteq H^{-1}(\mathcal{C})[1] * \mathcal{A}$. To show the converse, it suffices to show $H^{-1}(\mathcal{C})[1] \subseteq \mathcal{C}$ since $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{C}$ is closed under extensions.

Take any $C \in \mathcal{C}$ and we will see $H^{-1}(C)[1] \in \mathcal{C}$. By Lemma 5.2.1 (3), we may assume that $C = C^\bullet$ is a complex concentrated in degree 0 and -1, that is, C is of the form $C = [\cdots \to 0 \to C^{-1} \to C^0 \to 0 \to \cdots]$. Lemma 5.2.1 (2) gives the following triangle:

$$H^{-1}(C)[1] \xrightarrow{\ f\ } C \longrightarrow H^0(C) \dashrightarrow \ .$$

On the other hand, define the cochain map $g \colon C^0 \to C$ by the following:

$$\begin{array}{ccccccccccccc}
C^0 & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
{\scriptstyle g}\big\downarrow & & & & \big\downarrow & & \big\downarrow & & \big\| & & \big\downarrow & & \\
C & & \cdots & \longrightarrow & 0 & \longrightarrow & C^{-1} & \longrightarrow & C^0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Then by consider the mapping cocone of $[f,\ g] \colon H^{-1}(C)[1] \oplus C^0 \to C$, we obtain the following triangle in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$:

$$A \longrightarrow H^{-1}(C)[1] \oplus C^0 \xrightarrow{[f,\ g]} C \longrightarrow A[1].$$

We claim that A belongs to $\mathcal{A}$. To show that, consider the cohomology long exact sequence induced from the above triangle:

$$\begin{aligned}0 = H^{-2}(C) \longrightarrow H^{-1}(A) \longrightarrow H^{-1}(H^{-1}(C)[1] \oplus C^0) \xrightarrow{\overline{f}} H^{-1}(C)\\ \longrightarrow H^{0}(A) \longrightarrow H^{0}(H^{-1}(C)[1] \oplus C^0) \xrightarrow{\overline{g}} H^{0}(C)\\ \longrightarrow H^{1}(A) \longrightarrow H^{1}(H^{-1}(C)[1] \oplus C^0) = 0\end{aligned}$$

Clearly we have $H^i(A) = 0$ for $i \neq -1, 0, 1$. On the other hand, $\overline{f}$ is isomorphic to $H^{-1}(f)$, which is an isomorphism. Hence $H^{-1}(A) = 0$ holds. Moreover, $\overline{g}$ is isomorphic to $H^0(g)\colon C^0 \to H^0(C)$, which is surjective. Hence $H^1(A) = 0$ holds. Therefore, we have shown that $H^i(A) = 0$ for $i \neq 0$. Thus $A \in \mathcal{A}$ holds.

It follows that $H^{-1}(C)[1] \oplus C^0$ belongs to $\mathcal{C}$, since $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{C}$ is closed under extensions. Therefore, we obtain $H^{-1}(C)[1] \in \mathcal{C}$ because $\mathcal{C}$ is closed under direct summands. Thus, we have shown that $\mathcal{C} = H^{-1}(\mathcal{C})[1] * \mathcal{A}$ holds.

We claim that $H^{-1}(\mathcal{C})$ is a torsionfree class in $\mathcal{A}$. Put $\mathcal{F} := H^{-1}(\mathcal{C})$ for simplicity, and then we have $\mathcal{C} = \mathcal{F}[1] * \mathcal{A}$ by the above argument. Suppose that we have a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{A}$. Note that this gives a triangle $X \to Y \to Z \to X[1]$ in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. If X and Z belong to $\mathcal{F}$, then $X[1]$ and $Z[1]$ belongs to $\mathcal{F}[1] \subseteq \mathcal{C}$. Therefore, the conflation

$$X[1] \longrightarrow Y[1] \longrightarrow Z[1] \dashrightarrow$$

in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ implies that $Y[1] \in \mathcal{C}$ since $\mathcal{C}$ is closed under extensions. Then we obtain $Y = H^{-1}(Y[1]) \in H^{-1}(\mathcal{C}) = \mathcal{F}$.

Similarly, suppose that Y belongs to $\mathcal{F}$. Then we have a conflation

$$Z \longrightarrow X[1] \longrightarrow Y[1] \dashrightarrow$$

in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, and we have $Z \in \mathcal{A} \subseteq \mathcal{C}$ and $Y[1] \in \mathcal{F}[1] \subseteq \mathcal{C}$. Therefore, $X[1] \in \mathcal{C}$ holds since $\mathcal{C}$ is closed under extensions. Hence, we obtain $X = H^{-1}(X[1]) \in H^{-1}(\mathcal{C}) = \mathcal{F}$. Therefore, $\mathcal{F}$ is a torsionfree class in $\mathcal{A}$.

(3) Let $\mathcal{F}$ be a torsionfree class in $\mathcal{A}$. Then Lemma 5.2.1 (1) shows $H^{-1}(\mathcal{F}[1] * \mathcal{A}) \subseteq \mathcal{F}$. Since we clearly have $\mathcal{F} \subseteq H^{-1}(\mathcal{F}[1] * \mathcal{A})$, we obtain $H^{-1}(\mathcal{F}[1] * \mathcal{A}) = \mathcal{F}$. Conversely, if $\mathcal{C}$ is an intermediate subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, then we have $\mathcal{C} = H^{-1}(\mathcal{C})[1] * \mathcal{A}$ by the proof of (2). Thus, the assignments in (1) and (2) are mutually inverse. □

5.3 The Grothendieck monoid of an intermediate subcategory

By the previous section, an intermediate subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is of the form $\mathcal{F}[1] * \mathcal{A}$ for a torsionfree class $\mathcal{F}$ in $\mathcal{A}$. Next, we will calculate the Grothendieck monoid of this extriangulated category. Note that the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{F}[1] * \mathcal{A}$ induces a monoid homomorphism $\mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{F}[1] * \mathcal{A})$.

Theorem 5.3.1. *Let $\mathcal{A}$ be a skeletally small abelian category and $\mathcal{F}$ a torsionfree class in $\mathcal{A}$, and put $\mathcal{C} := \mathcal{F}[1] * \mathcal{A}$. Then the natural monoid homomorphism $\mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{C})$ induces the following isomorphism of monoids*

$$\mathsf{M}(\mathcal{A})_{\mathsf{M}_{\mathcal{F}}} \cong \mathsf{M}(\mathcal{C}),$$

where the left-hand side is the localization of $\mathsf{M}(\mathcal{A})$ with respect to $\mathsf{M}_{\mathcal{F}}$ (see Definition 5.1.1).

Proof. Let $\iota\colon \mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{C})$ be the natural monoid homomorphism. We check that ι satisfies the universal property of the localization of $\mathsf{M}(\mathcal{A})$ with respect to $\mathsf{M}_{\mathcal{F}}$.

First, we show that every element in $\iota(\mathsf{M}_{\mathcal{F}})$ is invertible. Take any $[F] \in \mathsf{M}_{\mathcal{F}}$ with $F \in \mathcal{F}$, then $\iota[F] = [F]$ in $\mathsf{M}(\mathcal{C})$. Now we have a conflation

$$F \longrightarrow 0 \longrightarrow F[1] \dashrightarrow$$

in $\mathcal{C} = \mathcal{F}[1] * \mathcal{A}$. Therefore, $[F] + [F[1]] = 0$ holds in $\mathsf{M}(\mathcal{C})$, that is, $\iota[F] \in \mathsf{M}(\mathcal{C})$ is invertible.

Next, we will check the universal property. Let $\varphi\colon \mathsf{M}(\mathcal{A}) \to M$ be a monoid homomorphism such that $\varphi[F]$ is invertible in M for every $F \in \mathcal{F}$. We have to show that there is a unique monoid homomorphism $\overline{\varphi}\colon \mathsf{M}(\mathcal{C}) \to M$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathsf{M}(\mathcal{A}) & \xrightarrow{\iota} & \mathsf{M}(\mathcal{C}) \\ {\scriptstyle\varphi}\downarrow & \swarrow {\scriptstyle\overline{\varphi}} & \\ M & & \end{array}$$

We first check the uniqueness. Suppose that there is such a map $\overline{\varphi}$. Let $C \in \mathcal{C}$ be any object. Then since $\mathcal{C} = \mathcal{F}[1] * \mathcal{A}$, there is a conflation in $\mathcal{C}$

$$H^{-1}(C)[1] \longrightarrow C \longrightarrow H^0(C) \dashrightarrow$$

with $H^{-1}(C) \in \mathcal{F}$ by Lemma 5.2.1. Thus $[C] = [H^0(C)] + [H^{-1}(C)[1]] = [H^0(C)] - [H^1(C)]$ in $\mathsf{M}(\mathcal{C})$. Since $\overline{\varphi}$ is a monoid homomorphism, it preserves the inverse. Therefore, we must have

$$\overline{\varphi}[C] = \overline{\varphi}[H^0(C)] - \overline{\varphi}[H^1(C)] = \overline{\varphi}\iota[H^0(C)] - \overline{\varphi}\iota[H^1(C)] = \varphi[H^0(C)] - \varphi[H^1(C)].$$

Therefore, the uniqueness of $\overline{\varphi}$ holds.

Next, we will construct $\overline{\varphi}$. Consider the following map $\psi\colon |\mathcal{C}| \to M$:

$$\psi[X] := \varphi[H^0(X)] - \varphi[H^{-1}(X)].$$

Note that $\varphi[H^{-1}(X)]$ has an inverse in M since $H^{-1}(X) \in \mathcal{F}$. We will show that ψ respects conflations in $\mathcal{C}$. Clearly $\psi[0] = 0$ holds. Take any conflation

$$X \longrightarrow Y \longrightarrow Z \dashrightarrow$$

in $\mathcal{C}$. Since this is a triangle in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, we obtain the following long exact sequence in $\mathcal{A}$:

$$\begin{aligned} H^{-2}(Z) = 0 \longrightarrow H^{-1}(X) \longrightarrow H^{-1}(Y) \longrightarrow H^{-1}(Z) \\ \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^2(X) = 0 \end{aligned}$$

By decomposing this long exact sequence into short exact sequences in $\mathcal{A}$, we can easily obtain the equality

$$[H^{-1}(X)] + [H^{-1}(Z)] + [H^0(Y)] = [H^{-1}(Y)] + [H^0(X)] + [H^0(Z)]$$

in $\mathsf{M}(\mathcal{A})$. Therefore, by applying φ, we obtain

$$\varphi[H^{-1}(X)] + \varphi[H^{-1}(Z)] + \varphi[H^0(Y)] = \varphi[H^{-1}(Y)] + \varphi[H^0(X)] + \varphi[H^0(Z)],$$

which clearly implies $\psi[Y] = \psi[X] + \psi[Z]$ in M. Thus ψ respects conflations, so it induces a monoid homomorphism $\overline{\varphi}\colon \mathsf{M}(\mathcal{C}) \to M$. Moreover, for any $A \in \mathcal{A}$, we have $\overline{\varphi}\iota[A] = \varphi[H^0(A)] - \varphi[H^{-1}(A)] = \varphi[A] - 0 = \varphi[A]$, and hence $\overline{\varphi}\iota = \varphi$ holds. □

As an immediate consequence, we obtain the following example of an extriangulated category whose Grothendieck monoid is a group:

Corollary 5.3.2. *Let $\mathcal{A}$ be a skeletally small abelian category. Then the natural homomorphism $\mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A}[1] * \mathcal{A})$ induces an isomorphism $\mathsf{K}_0(\mathcal{A}) \cong \mathsf{M}(\mathcal{A}[1] * \mathcal{A})$.*

Proof. By Theorem 5.3.1, we have that $\mathsf{M}(\mathcal{A}[1] * \mathcal{A})$ is isomorphic to the localization of $\mathsf{M}(\mathcal{A})$ with respect to the whole set $\mathsf{M}(\mathcal{A})$. This is nothing but the group completion of $\mathsf{M}(\mathcal{A})$, which is $\mathsf{K}_0(\mathcal{A})$ (see Remark 2.3.6). □

The above corollary says that the inclusion $\mathcal{A} \hookrightarrow \mathcal{A}[1] * \mathcal{A}$ *categorifies* the group completion $\mathsf{M}(\mathcal{A}) \to \mathsf{K}_0(\mathcal{A})$. Furthermore, we can realize all the localization of $\mathsf{M}(\mathcal{A})$ in this way, as follows.

Remark 5.3.3. Consider the localization $\mathsf{M}(\mathcal{A})_S$ of $\mathsf{M}(\mathcal{A})$ with respect to any subset $S \subseteq \mathsf{M}(\mathcal{A})$. We have a monoid isomorphism $\mathsf{M}(\mathcal{A})_S \cong \mathsf{M}(\mathcal{A})_{\langle S \rangle_{\mathrm{face}}}$ by Lemma 5.1.2. Then $\mathcal{F} := \mathcal{D}_{\langle S \rangle_{\mathrm{face}}}$ is a Serre subcategory of $\mathcal{A}$ and $\mathsf{M}_{\mathcal{F}} = \langle S \rangle_{\mathrm{face}}$ by Proposition 3.3.3. In particular, it is a torsionfree class of $\mathcal{A}$. By Theorem 5.3.1, we have a monoid isomorphism

$$\mathsf{M}(\mathcal{A})_S \cong \mathsf{M}(\mathcal{A})_{\langle S \rangle_{\mathrm{face}}} = \mathsf{M}(\mathcal{A})_{\mathsf{M}_{\mathcal{F}}} \cong \mathsf{M}(\mathcal{F}[1] * \mathcal{A}).$$

Thus, we can realize all localizations of $\mathsf{M}(\mathcal{A})$ as the Grothendieck monoids of intermediate subcategories of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. Therefore, the natural inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ to an intermediate subcategory yields a categorification of a localization of the monoid $\mathsf{M}(\mathcal{A})$.

Next, we describe the Grothendieck monoid of an intermediate subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ when $\mathcal{A}$ is an abelian length category with finitely many simples. Recall that an *abelian length category* is a skeletally small abelian category such that every object has a finite length. For an abelian length category $\mathcal{A}$, we denote by $\operatorname{sim} \mathcal{A}$ the set of isomorphism classes of simple objects in $\mathcal{A}$. The following observation on the localization of a monoid can be easily checked.

Lemma 5.3.4. *Let M and N be monoids. Then the localization of $M \oplus N$ with respect to M is isomorphic to* $(\mathsf{gp} M) \oplus N$.

Corollary 5.3.5. *Let $\mathcal{A}$ be an abelian length category and $\mathcal{F}$ a torsionfree class in $\mathcal{A}$. Define $\operatorname{sim}_{\mathcal{F}} \mathcal{A}$ as follows:*

$$\operatorname{sim}_{\mathcal{F}} \mathcal{A} := \{[S] \in \operatorname{sim} \mathcal{A} \mid S \textit{ appears as a composition factor of a some object in } \mathcal{F}\}.$$

Then we have an isomorphism of monoids

$$\mathsf{M}(\mathcal{F}[1] * \mathcal{A}) \cong \mathbb{Z}^{\oplus \operatorname{sim}_{\mathcal{F}} \mathcal{A}} \oplus \mathbb{N}^{\oplus (\operatorname{sim} \mathcal{A} \setminus \operatorname{sim}_{\mathcal{F}} \mathcal{A})}.$$

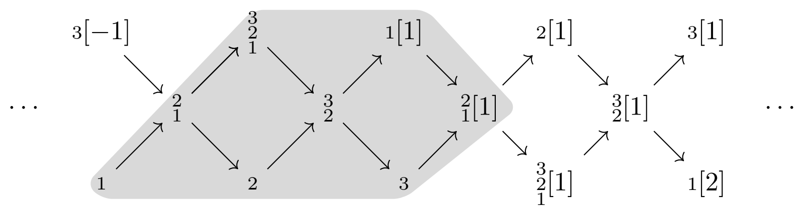
Proof. Since $\mathcal{A}$ is an abelian length category and the Jordan-Hölder theorem holds in $\mathcal{A}$, we have that $\mathsf{M}(\mathcal{A})$ is a free commutative monoid with the basis $\{[S] \in \mathsf{M}(\mathcal{A}) \mid [S] \in \operatorname{sim} \mathcal{A}\}$, see [Eno22, Corollary 4.10]. Thus, we have an isomorphism $\mathsf{M}(\mathcal{A}) \cong \mathbb{N}^{\oplus \operatorname{sim} \mathcal{A}}$, which sends $[A] \in \mathsf{M}(\mathcal{A})$ to $\sum n_i [S_i]$, where n_i is the multiplicity of S_i in the composition series of M. In the rest of this proof, we identify $\mathsf{M}(\mathcal{A})$ with $\mathbb{N}^{\oplus \operatorname{sim} \mathcal{A}}$.

Theorem 5.3.1 implies that $\mathsf{M}(\mathcal{F}[1] * \mathcal{A})$ is the localization of $\mathsf{M}(\mathcal{A})$ with respect to $\mathsf{M}_{\mathcal{F}}$. By Lemma 5.1.2, this localization coincides with that with respect to the smallest face $\langle \mathsf{M}_{\mathcal{F}} \rangle_{\mathrm{face}}$ of $\mathsf{M}(\mathcal{A})$ containing $\mathsf{M}_{\mathcal{F}}$. It is easily checked that the following holds:

$$\langle \mathsf{M}_{\mathcal{F}} \rangle_{\mathrm{face}} = \mathbb{N}^{\oplus \operatorname{sim}_{\mathcal{F}} \mathcal{A}} \subseteq \mathbb{N}^{\oplus \operatorname{sim}_{\mathcal{F}} \mathcal{A}} \oplus \mathbb{N}^{\oplus (\operatorname{sim} \mathcal{A} \setminus \operatorname{sim}_{\mathcal{F}} \mathcal{A})} = \mathsf{M}(\mathcal{A}).$$

Then Lemma 5.3.4 immediately deduces the assertion. □

Example 5.3.6. Let $\Bbbk$ be a field and Q a quiver $1 \leftarrow 2 \leftarrow 3$, and let $\mathcal{A} := \operatorname{mod} \Bbbk Q$. Then the Auslander-Reiten quiver of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is as follows:



Consider the torsionfree class $\mathcal{F} := \mathsf{add}\{1, \begin{smallmatrix}2\\1\end{smallmatrix}\}$ in $\mathcal{A}$. Then an intermediate subcategory $\mathcal{C} := \mathcal{F}[1] * \mathcal{A}$ is the additive closure of the gray region. Since $\mathcal{A}$ is length, $\mathsf{M}(\mathcal{A}) = \mathbb{N}[S_1] \oplus \mathbb{N}[S_2] \oplus \mathbb{N}[S_3]$, where $S_i = i$ is the simple module corresponding to each vertex i. Then $\mathsf{sim}_{\mathcal{F}} \mathcal{A} = \{S_1, S_2\}$, so Corollary 5.3.5 implies $\mathsf{M}(\mathcal{C}) = \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2] \oplus \mathbb{N}[S_3]$. For example, $[S_2]$ is invertible in $\mathsf{M}(\mathcal{C})$ although $[S_2[1]] \notin \mathcal{C}$, which can be checked alternatively as follows. We have $[S_2] = [\begin{smallmatrix}2\\1\end{smallmatrix}] + [1[1]]$ in $\mathsf{M}(\mathcal{C})$ by a conflation

$$\begin{smallmatrix}2\\1\end{smallmatrix} \longrightarrow 2 \longrightarrow 1[1] \dashrightarrow \quad .$$

On the other hand, $[\begin{smallmatrix}2\\1\end{smallmatrix}]$ and $[1[1]]$ have inverses $[\begin{smallmatrix}2\\1\end{smallmatrix}[1]]$ and $[1]$ in $\mathsf{M}(\mathcal{C})$ respectively. Thus $[S_2]$ has an inverse $[\begin{smallmatrix}2\\1\end{smallmatrix}[1]] + [1]$.

5.4 Serre subcategories of an intermediate subcategory

In this section, we classify Serre subcategories of an intermediate subcategory and study its localization.

First, we classify Serre subcategories of an intermediate subcategory via Serre subcategories of the original abelian category.

Proposition 5.4.1. *Let $\mathcal{A}$ be a skeletally small abelian category and $\mathcal{F}$ a torsionfree class in $\mathcal{A}$. Then there is a bijection between the following two sets.*

(1) $\mathrm{Serre}(\mathcal{F}[1] * \mathcal{A})$.

(2) $\{\mathcal{S} \in \mathrm{Serre}\,\mathcal{A} \mid \mathcal{F} \subseteq \mathcal{S}\}$.

The maps are given as follows: for $\mathcal{D}$ in (1), we consider $\mathcal{A} \cap \mathcal{D}$ in (2), and for $\mathcal{S}$ in (2) we consider $\mathcal{F}[1] * \mathcal{S}$ in (1).

Proof. Let $\mathcal{D}$ be a Serre subcategory of $\mathcal{F}[1] * \mathcal{A}$. Then clearly $\mathcal{A} \cap \mathcal{D}$ is a Serre subcategory of $\mathcal{A}$ since $\mathcal{A}$ is an extension-closed subcategory of $\mathcal{F}[1] * \mathcal{A}$. Moreover, for any $F \in \mathcal{F}$, we have a conflation

$$F \longrightarrow 0 \longrightarrow F[1] \dashrightarrow \tag{5.4.1}$$

in $\mathcal{F}[1] * \mathcal{A}$, which implies that $F \in \mathcal{D}$ and $F[1] \in \mathcal{D}$ since $\mathcal{D}$ is a Serre subcategory of $\mathcal{F}[1] * \mathcal{A}$ and $0 \in \mathcal{D}$. Thus $F \in \mathcal{A} \cap \mathcal{D}$, and hence $\mathcal{F} \subseteq \mathcal{A} \cap \mathcal{D}$.

Conversely, let $\mathcal{S}$ be a Serre subcategory of $\mathcal{A}$ with $\mathcal{F} \subseteq \mathcal{S}$, and put $\mathcal{D} := \mathcal{F}[1] * \mathcal{S}$. By Lemma 5.2.1 (1), we can describe $\mathcal{D}$ as follows:

$$\mathcal{F}[1] * \mathcal{S} = \{X \in \mathcal{F}[1] * \mathcal{A} \mid H^0(X) \in \mathcal{S}\}.$$

We claim that $\mathcal{D}$ is a Serre subcategory of $\mathcal{F}[1] * \mathcal{A}$. Suppose that there is a conflation

$$X \xrightarrow{f} Y \xrightarrow{g} Z \dashrightarrow$$

in $\mathcal{F}[1] * \mathcal{A}$. Then we obtain the following long exact sequence in $\mathcal{A}$:

$$H^{-1}(Z) \xrightarrow{\delta} H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z) \longrightarrow H^1(X) = 0$$

Thus, we obtain the following short exact sequence:

$$0 \longrightarrow \operatorname{Cok}\delta \longrightarrow H^0(Y) \xrightarrow{H^0(g)} H^0(Z) \longrightarrow 0.$$

If X and Z belong to $\mathcal{F}[1] * \mathcal{S}$, then we have $H^0(X), H^0(Z) \in \mathcal{S}$ by $X, Z \in \mathcal{F}[1] * \mathcal{A}$, and $\operatorname{Cok}\delta \in \mathcal{S}$ since $\mathcal{S}$ is closed under quotients in $\mathcal{A}$. Thus $H^0(Y) \in \mathcal{S}$ holds since $\mathcal{S}$ is closed under extensions. Similarly, if Y belongs to $\mathcal{F}[1] * \mathcal{S}$, then $H^0(Y) \in \mathcal{S}$, and hence $\operatorname{Cok}\delta, H^0(Z) \in \mathcal{S}$ since $\mathcal{S}$ is closed under subobjects and quotients. Moreover, we have another short exact sequence:

$$0 \longrightarrow \operatorname{Im}\delta \longrightarrow H^0(X) \longrightarrow \operatorname{Cok}\delta \longrightarrow 0.$$

Since $\operatorname{Im}\delta$ is a quotient of $H^{-1}(Z) \in \mathcal{F} \subseteq \mathcal{S}$, we have $\operatorname{Im}\delta \in \mathcal{S}$. Thus $H^0(X) \in \mathcal{S}$ holds since $\mathcal{S}$ is closed under extensions. Therefore, we obtain $X, Z \in \mathcal{F}[1] * \mathcal{S}$, so $\mathcal{F}[1] * \mathcal{S}$ is a Serre subcategory of $\mathcal{F}[1] * \mathcal{A}$.

Finally, we check that these maps are inverse to each other. Let $\mathcal{D}$ be a Serre subcategory of $\mathcal{F}[1] * \mathcal{A}$, and we will see $\mathcal{D} = \mathcal{F}[1] * (\mathcal{A} \cap \mathcal{D})$. By the conflation (5.4.1), we have $\mathcal{F}[1] \subseteq \mathcal{D}$. Thus, we obtain $\mathcal{F}[1] * (\mathcal{A} \cap \mathcal{D}) \subseteq \mathcal{D}$ since $\mathcal{D}$ is closed under extensions. Conversely, let $X \in \mathcal{D}$. Then since $X \in \mathcal{D} \subseteq \mathcal{F}[1] * \mathcal{A}$, we have the conflation

$$F[1] \longrightarrow X \longrightarrow A \dashrightarrow$$

in $\mathcal{F}[1] * \mathcal{A}$ with $F \in \mathcal{F}$ and $A \in \mathcal{A}$. Since $\mathcal{D}$ is a Serre subcategory of $\mathcal{F}[1] * \mathcal{A}$, we have in addition $A \in \mathcal{D}$, and hence $A \in \mathcal{A} \cap \mathcal{D}$. Thus $X \in \mathcal{F}[1] * (\mathcal{A} \cap \mathcal{D})$. Next, let $\mathcal{S}$ be a Serre subcategory of $\mathcal{A}$ containing $\mathcal{F}$. Clearly we have $\mathcal{S} \subseteq \mathcal{F}[1] * \mathcal{S}$, so $\mathcal{S} \subseteq \mathcal{A} \cap (\mathcal{F}[1] * \mathcal{S})$. Conversely, let $A \in \mathcal{A} \cap (\mathcal{F}[1] * \mathcal{S})$. Then we have an isomorphism $A \cong H^0(A)$, and $H^0(A) \in \mathcal{S}$ holds by Lemma 5.2.1 (1). Therefore, we obtain $A \in \mathcal{S}$. □

Remark 5.4.2. The bijection in Proposition 5.4.1 can also be constructed purely combinatorially as follows: Proposition 3.3.3, Theorem 5.3.1, and Proposition 5.1.4 give the following diagram consisting of bijections.

$$\begin{array}{ccc} \mathrm{Serre}(\mathcal{F}[1] * \mathcal{A}) & & \{\mathcal{S} \in \mathrm{Serre}(\mathcal{A}) \mid \mathcal{F} \subseteq \mathcal{S}\} \\ \mathsf{M}_{(-)} \downarrow & & \downarrow \mathsf{M}_{(-)} \\ \mathrm{Face}\,\mathsf{M}(\mathcal{F}[1] * \mathcal{A}) = \mathrm{Face}\,\mathsf{M}(\mathcal{A})_{\mathsf{M}_{\mathcal{F}}} & \overset{\sim}{\longleftrightarrow} & \{F \in \mathrm{Face}\,\mathsf{M}(\mathcal{A}) \mid \mathsf{M}_{\mathcal{F}} \subseteq F\} \end{array}$$

Therefore, we can think of Proposition 5.4.1 as a *categorification* of Proposition 5.1.4 on faces.

Next, we discuss the localization of an intermediate subcategory by its Serre subcategory. Such a localization is always possible and yields an abelian category:

Proposition 5.4.3. *Let $\mathcal{A}$ be a skeletally small abelian category, $\mathcal{F}$ a torsionfree class in $\mathcal{A}$, and $\mathcal{D}$ a Serre subcategory of $\mathcal{F}[1] * \mathcal{A}$. Then $\mathcal{D}$ satisfies conditions* (EL1)–(EL4) *in Fact 4.2.13. Therefore, the exact localization $(\mathcal{F}[1] * \mathcal{A})/\mathcal{D}$ exists and is an abelian category.*

Proof. (EL1) Since $\mathcal{D}$ is a Serre subcategory of $\mathcal{F}[1] * \mathcal{A}$ and $\mathcal{F}[1] * \mathcal{A}$ is admissible as shown in (EL4) below, it is a percolating subcategory by Example 4.2.11.

(EL2) Lemma 5.2.1 (1) implies that $\mathcal{F}[1] * \mathcal{A}$ is closed under direct summands in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. Therefore, $\mathcal{F}[1] * \mathcal{A}$ satisfies the condition (WIC) by Lemma 4.2.16, and thus (EL2) is satisfied by [NOS22, Remark 4.31 (2)].

(EL3) First, we will check the first condition in (EL3). Let $f\colon X \to Y$ be an inflation in $\mathcal{F}[1] * \mathcal{A}$ and $\varphi\colon W \to X$ in $\mathcal{F}[1] * \mathcal{A}$ a morphism satisfying $f\varphi = 0$. We have the following triangle in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ with $Z \in \mathcal{F}[1] * \mathcal{A}$:

$$\begin{array}{ccccccc} & & W & & & & \\ & \overline{\varphi} \swarrow & \downarrow \varphi & \searrow 0 & & & \\ Z[-1] & \xrightarrow{h} & X & \xrightarrow{f} & Y & \longrightarrow & Z. \end{array}$$

Since $f\varphi = 0$ and h is a weak kernel of f in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, it follows that φ factors through h, so we obtain the above $\overline{\varphi}\colon W \to Z[-1]$ satisfying $\varphi = h\overline{\varphi}$. Then, since $Z \in \mathcal{F}[1] * \mathcal{A}$, we have the following triangle in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ with $F \in \mathcal{F}$ and $A \in \mathcal{A}$.

$$\begin{array}{ccccccc} & & W & & & & \\ & \psi \swarrow & \downarrow \overline{\varphi} & \searrow 0 & & & \\ F & \xrightarrow{a} & Z[-1] & \xrightarrow{b} & A[-1] & \longrightarrow & F[1]. \end{array}$$

Here $b\overline{\varphi} = 0$ holds since both $\mathsf{D}^{\mathrm{b}}(\mathcal{A})(\mathcal{F}[1], \mathcal{A}[-1])$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A})(\mathcal{A}, \mathcal{A}[-1])$ vanish. Therefore, we obtain the above ψ satisfying $\overline{\varphi} = a\psi$. Thus $\varphi = h\overline{\varphi} = ha\psi$ holds, so φ factors through the object $F \in \mathcal{F}$. On the other hand, by the proof of Proposition 5.4.1, we have $\mathcal{F} \subseteq \mathcal{D}$. Hence φ factors through an object in $\mathcal{D}$.

Next, we will prove the second condition. Let $g\colon Y \to Z$ be a deflation in $\mathcal{F}[1] * \mathcal{A}$ and $\varphi\colon Z \to W$ in $\mathcal{F}[1] * \mathcal{A}$ a morphism satisfying $\varphi g = 0$. We have the following triangle in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ with $X \in \mathcal{F}[1] * \mathcal{A}$:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ & & & \searrow^{0} & \downarrow{\scriptstyle\varphi} & \swarrow_{\overline{\varphi}} & \\ & & & & W & & \end{array}$$

Since $\varphi g = 0$ and h is a weak cokernel of g in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, it follows that φ factors through h, so we obtain the above $\overline{\varphi}\colon X[1] \to W$ satisfying $\varphi = \overline{\varphi} h$. Then, since $W \in \mathcal{F}[1] * \mathcal{A}$, we have the following triangle in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ with $F \in \mathcal{F}$ and $A \in \mathcal{A}$.

$$\begin{array}{ccccccc} & & X[1] & & & & \\ & \swarrow^{\psi} & \downarrow{\scriptstyle\overline{\varphi}} & \searrow^{0} & & & \\ F[1] & \xrightarrow[a]{} & W & \xrightarrow[b]{} & A & \longrightarrow & F[2]. \end{array}$$

Here $b\overline{\varphi} = 0$ holds since $X[1] \in \mathcal{F}[2] * \mathcal{A}[1]$ and both $\mathsf{D}^{\mathrm{b}}(\mathcal{A})(\mathcal{F}[2], \mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A})(\mathcal{A}[1], \mathcal{A})$ vanish. Therefore, we obtain the above ψ satisfying $\overline{\varphi} = a\psi$. Thus $\varphi = \overline{\varphi} h = a\psi h$ holds, so φ factors through the object $F[1] \in \mathcal{F}[1]$. On the other hand, by the proof of Proposition 5.4.1, we have $\mathcal{F}[1] \subseteq \mathcal{D}$. Hence φ factors through an object in $\mathcal{D}$, and thus (EL3) is satisfied.

(EL4) The proof of this part is essentially the same as the proof of the fact that the heart of a t-structure is an abelian category. We denote by $(-)^{\leq 0}, (-)^{\geq 1}\colon \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{A}$ the truncation functors with respect to the standard t-structure of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.

Let $f\colon X \to Y$ be any morphism in $\mathcal{F}[1] * \mathcal{A}$. By taking the mapping cocone K of f and the truncation of K and using the octahedral axiom, we obtain the following commutative diagram consisting of triangles in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$:

$$\begin{array}{ccccccc} K^{\leq 0} & = & K^{\leq 0} & & & & \\ \downarrow & & \downarrow & & & & \\ K & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & K[1] \\ \downarrow & & \downarrow{\scriptstyle p} & & \| & & \downarrow \\ K^{\geq 1} & \longrightarrow & W & \xrightarrow[i]{} & Y & \longrightarrow & K^{\geq 1}[1] \\ \downarrow & & \downarrow & & & & \\ K^{\leq 0}[1] & = & K^{\leq 0}[1] & & & & \end{array} \tag{5.4.2}$$

We claim that $f = ip$ is a deflation-inflation factorization. It suffices to show the following three assertions.

(1) $K^{\leq 0} \in \mathcal{F}[1] * \mathcal{A}$

(2) $K^{\geq 1}[1] \in \mathcal{A} \subseteq \mathcal{F}[1] * \mathcal{A}$.

(3) $W \in \mathcal{F}[1] * \mathcal{A}$.

(1) By the second row of (5.4.2), we have the following long exact sequence in $\mathcal{A}$:

$$\begin{array}{l} 0 = H^{-2}(Y) \longrightarrow H^{-1}(K) \longrightarrow H^{-1}(X) \\ \longrightarrow H^{-1}(Y) \longrightarrow H^{0}(K) \longrightarrow H^{0}(X) \\ \longrightarrow H^{0}(Y) \longrightarrow H^{1}(K) \longrightarrow H^{1}(X) = 0. \end{array}$$

It easily follows that $H^i(K) = 0$ for $i \notin \{-1, 0, 1\}$. Moreover, since $H^{-1}(X) \in \mathcal{F}$, we have $H^{-1}(K) \in \mathcal{F}$ because $\mathcal{F}$ is closed under subobjects. Since the i-th cohomology of $K^{\leq 0}$ is zero for $i > 0$ and the same as K for $i \leq 0$, we obtain that $K^{\leq 0}$ belongs to $\mathcal{F}[1] * \mathcal{A}$ by Lemma 5.2.1 (1).

(2) Because the i-th cohomology of K vanishes for $i \geq 1$ except for $i = 1$ by the proof of (1), we have $K^{\geq 1} \in \mathcal{A}[-1]$. Thus $K^{\geq 1}[1] \in \mathcal{A}$ holds.

(3) Since $H^i(X) = 0$ and $H^i(K^{\le 0}[1]) = H^{i+1}(K^{\le 0}) = 0$ for $i \ge 1$, we obtain that $H^i(W) = 0$ for $i \ge 1$ by the second column of (5.4.2). On the other hand, the third row of (5.4.2) shows the following exact sequence in $\mathcal{A}$:

$$0 = H^{-1}(K^{\ge 1}) \longrightarrow H^{-1}(W) \longrightarrow H^{-1}(Y) \longrightarrow H^0(K^{\ge 1}) = 0$$

Thus $H^i(W) = 0$ holds for $i \le -2$ and $H^{-1}(W) \cong H^{-1}(Y) \in \mathcal{F}$ holds. Therefore, we obtain $W \in \mathcal{F}[1]*\mathcal{A}$ by Lemma 5.2.1 (1). $\square$

Actually, we can describe the localization of an intermediate subcategory as a usual Serre quotient of an abelian category as follows.

Theorem 5.4.4. *Let $\mathcal{A}$ be a skeletally small abelian category, $\mathcal{F}$ a torsionfree class in $\mathcal{F}$, and $\mathcal{S}$ a Serre subcategory of $\mathcal{A}$ with $\mathcal{F} \subseteq \mathcal{S}$. We have the following commutative diagram consisting of exact functors of extriangulated categories, where Q_1 and Q_2 are localization functors:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota} & \mathcal{F}[1] * \mathcal{A} \\ {\scriptstyle Q_1}\downarrow & & \downarrow{\scriptstyle Q_2} \\ \mathcal{A}/\mathcal{S} & \xrightarrow{\Phi} & (\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S}) \end{array}$$

Then Φ gives an equivalence of extriangulated categories (or equivalently, abelian categories).

Proof. First, recall that an exact functor from an extriangulated category to an exact category (e.g. an abelian category) is precisely an additive functor preserving conflations by Lemma 2.2.4. It is easily checked from the universality of the localization that an exact functor Φ exists (see Proposition 4.2.3). Furthermore, since both $\mathcal{A}/\mathcal{S}$ and $(\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S})$ are abelian categories by Proposition 5.4.3, we only have to check that Φ is just an equivalence of an additive category.

Actually, we will construct a quasi-inverse $\Psi\colon (\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S}) \to \mathcal{A}/\mathcal{S}$ as follows. Consider the following diagram.

$$\begin{array}{ccc} \mathcal{F}[1] * \mathcal{A} & \xrightarrow{H^0} & \mathcal{A} \\ {\scriptstyle Q_2}\downarrow & & \downarrow{\scriptstyle Q_1} \\ (\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S}) & \overset{\Psi}{\dashrightarrow} & \mathcal{A}/\mathcal{S} \end{array} \tag{5.4.3}$$

We claim that the composition $Q_1 H^0$ is an exact functor. To show this, take any conflation

$$X \xrightarrow{f} Y \xrightarrow{g} Z \dashrightarrow$$

in $\mathcal{F}[1] * \mathcal{A}$. Then since it is a triangle in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, we obtain the following long exact sequence in $\mathcal{A}$:

$$H^{-1}(Z) \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) = 0$$

Then Lemma 5.2.1 (1) implies $H^{-1}(Z) \in \mathcal{F}$. Now since $\mathcal{F}$ is contained in $\mathcal{S}$, we obtain the following short exact sequence in $\mathcal{A}/\mathcal{S}$ by applying an exact functor Q_1 to the above exact sequence:

$$0 \longrightarrow Q_1 H^0(X) \longrightarrow Q_1 H^0(Y) \longrightarrow Q_1 H^0(Z) \longrightarrow 0.$$

Therefore, Lemma 2.2.4 shows that $Q_1 H^0$ is an exact functor between extriangulated categories. Hence, an exact functor Ψ in (5.4.3) is induced. Now we obtain the following strict commutative diagram consisting of additive functors between additive categories.

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\iota} & \mathcal{F}[1] * \mathcal{A} & \xrightarrow{H^0} & \mathcal{A} \\ {\scriptstyle Q_1}\downarrow & & \downarrow{\scriptstyle Q_2} & & \downarrow{\scriptstyle Q_1} \\ \mathcal{A}/\mathcal{S} & \xrightarrow[\Phi]{} & (\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S}) & \xrightarrow[\Psi]{} & \mathcal{A}/\mathcal{S} \end{array}$$

Since $H^0\iota$ is naturally isomorphic to $\mathsf{id}_{\mathcal{A}}$, we have $Q_1 H^0 \iota \simeq Q_1$. Thus, the universal property of Q_1 as an exact 2-localization (see Remark 4.2.7) implies that $\Psi\Phi$ is naturally isomorphic to the identity functor.

Similarly, we have the following strict commutative diagram.

$$\begin{array}{ccccc} \mathcal{F}[1] * \mathcal{A} & \xrightarrow{H^0} & \mathcal{A} & \xrightarrow{\iota} & \mathcal{F}[1] * \mathcal{A} \\ {\scriptstyle Q_2}\downarrow & & \downarrow{\scriptstyle Q_1} & & \downarrow{\scriptstyle Q_2} \\ (\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S}) & \xrightarrow[\Psi]{} & \mathcal{A}/\mathcal{S} & \xrightarrow[\Phi]{} & (\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S}) \end{array}$$

We claim that $Q_2 \iota H^0$ is naturally isomorphic to Q_2. In fact, for any morphism $f\colon X \to Y$ in $\mathcal{F}[1] * \mathcal{A}$, we obtain the following morphism of conflations in $\mathcal{F}[1] * \mathcal{A}$:

$$\begin{array}{ccccccc} H^{-1}(X)[1] & \longrightarrow & X & \longrightarrow & \iota H^0(X) & \dashrightarrow & \\ {\scriptstyle H^{-1}(f)[1]}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \iota H^0(f)} & & \\ H^{-1}(Y)[1] & \longrightarrow & Y & \longrightarrow & \iota H^0(Y) & \dashrightarrow & \end{array}$$

Since Q_2 is an exact functor which sends every object in $\mathcal{F}[1]$ to 0, we obtain the following exact commutative diagram in an abelian category $(\mathcal{F}[1] * \mathcal{A})/(\mathcal{F}[1] * \mathcal{S})$:

$$\begin{array}{ccccccc} 0 = Q_2 H^{-1}(X)[1] & \longrightarrow & Q_2(X) & \longrightarrow & Q_2 \iota H^0(X) & \longrightarrow & 0 \\ {\scriptstyle Q_2 H^{-1}(f)[1]}\downarrow & & \downarrow{\scriptstyle Q_2(f)} & & \downarrow{\scriptstyle Q_2 \iota H^0(f)} & & \\ 0 = Q_2 H^{-1}(Y)[1] & \longrightarrow & Q_2(Y) & \longrightarrow & Q_2 \iota H^0(Y) & \longrightarrow & 0 \end{array}$$

Therefore, $Q_2 \iota H^0$ is naturally isomorphic to Q_2. Thus, the universal property of an exact 2-localization Q_1 (Remark 4.2.7) implies that $\Phi\Psi$ is naturally isomorphic to the identity functor. Therefore, Φ and Ψ are mutually quasi-inverse. $\square$

Chapter 6

The spectra of Grothendieck monoids

In this chapter, we study the monoid spectrum of the Grothendieck monoid and introduce a topology on the set of Serre subcategories. As a consequence, we classify finitely generated Serre subcategories by using this topology.

Throughout this chapter, we fix Grothendieck universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$ and use Convention 1.1.12 and 2.2.5. Moreover, we assume that every subcategory is a strictly full subcategory. Hereafter, $\mathcal{C}$ is a skeletally small extriangulated category.

6.1 Preliminaries: the spectrum of a commutative monoid

In this section, we review the spectrum of a monoid. The main reference is [Ogu18]. We often refer to [GW20], a textbook of scheme theory since many constructions are analogies of the spectrum of a commutative ring. Throughout this section, M is a monoid.

Definition 6.1.1.

(1) A subset I of M is called an *ideal* if for all $x \in I$ and $a \in M$, we have $x + a \in I$.

(2) An ideal $\mathfrak{p}$ of M is said to be *prime* if it satisfies (i) $\mathfrak{p} \neq M$ and (ii) $x + y \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ for all $x, y \in M$.

(3) The *monoid spectrum* of M is the set $\operatorname{MSpec} M$ of prime ideals of M.

For a subset S of M, the *ideal* $\langle S \rangle_{\text{ideal}}$ *generated by* S is the smallest ideal of M containing S. We can describe it as follows:

$$\langle S \rangle_{\text{ideal}} := \{x + a \mid x \in S, a \in M\}.$$

Remark 6.1.2.

(1) The set $M^+ := M \setminus M^\times$ of non-units is the unique maximal ideal of M. It is also a prime ideal of M.

(2) The empty set $\emptyset$ is the unique minimal ideal of M. It is also a prime ideal of M.

(3) The monoid spectrum $\operatorname{MSpec} M$ is never empty and has the maximum and minimum element with respect to inclusion by (1) and (2). $\operatorname{MSpec} M$ consists of exactly one point if and only if M is a group.

The relation between faces and prime ideals is the following.

Fact 6.1.3 ([Ogu18, Section I.1.4])**.**

(1) $\mathfrak{p}^c := M \setminus \mathfrak{p}$ *is a face of* M *for any prime ideal* $\mathfrak{p}$ *of* M*.*

(2) $F^c := M \setminus F$ *is a prime ideal of* M *for any face* F *of* M*.*

(3) *The assignments given in (1) and (2) give inclusion-reversing bijections between* $\operatorname{Face}(M)$ *and* $\operatorname{MSpec} M$*.*

We will now endow $\operatorname{MSpec} M$ with the structure of a topological space. For a subset S of M, we set

$$V(S) := \{\mathfrak{p} \in \operatorname{MSpec} M \mid \mathfrak{p} \supseteq S\}.$$

Note that $V(S) = V(\langle S \rangle_{\text{ideal}})$ holds. They satisfy the following equalities (cf. [GW20, Lemma 2.1]):

- $V(M) = \emptyset$ and $V(\emptyset) = \operatorname{MSpec} M$.
- $\bigcap_{\alpha \in A} V(S_\alpha) = V\left(\bigcup_{\alpha \in A} S_\alpha\right)$ for a family $\{S_\alpha\}_{\alpha \in A}$ of subsets of M.
- $V(I) \cup V(J) = V(I \cap J)$ for ideals I, J of M.

These equalities show that we can define a topology on $\operatorname{MSpec} M$ by taking the subsets of the form $V(S)$ to be the closed subsets. We call it the *Zariski topology* on $\operatorname{MSpec} M$. Note that $\operatorname{MSpec} M$ has a unique closed point M^+ and a unique generic point $\emptyset$ by Remark 6.1.2. In particular, $\operatorname{MSpec} M$ is an irreducible topological space.

Let

$$D(f) := \{\mathfrak{p} \in \operatorname{MSpec} M \mid f \notin \mathfrak{p}\}$$

for each element $f \in M$. They are open in $\operatorname{MSpec} M$ since $D(f) = \operatorname{MSpec} M \setminus V(f)$. They satisfy

$$D(f) \cap D(g) = D(f + g)$$

for any $f, g \in M$. Open subsets of $\operatorname{MSpec} M$ of this form are called *principal open subsets* of $\operatorname{MSpec} M$. The set of principal open subsets $D(f)$ forms a basis of the Zariski topology on $\operatorname{MSpec} M$ (cf. [GW20, Proposition 2.5]).

We define a preorder on a topological space.

Definition 6.1.4. Let X be a topological space.

(1) For two points $x, y \in X$, we say that x is a *specialization* of y or that y is a *generalization* of x if x belongs to the topological closure $\overline{\{y\}}$ of $\{y\}$ in X. Define a preorder $\preceq$ on X by

$$x \preceq y :\Leftrightarrow x \text{ is a specialization of } y.$$

We call it the *specialization order* on X. When we regard X as a poset by the specialization order, it is denoted by $X_{\text{spcl}} := (X, \preceq)$.

(2) A subset A of X is *specialization-closed* (resp. *generalization-closed*) if for any $x \in A$ and every its specialization (resp. generalization) $x' \in X$, we have that $x' \in A$.

Remark 6.1.5. Let X be a topological space.

(1) A subset A of X is specialization-closed if and only if its complement $A^c = X \setminus A$ is generalization-closed.
(2) Any closed subset is specialization-closed. Also, any open subset is generalization-closed.
(3) Recall that X is called a T_0*-space* if for any distinct points, there exists an open subset containing exactly one of them. In this case, the specialization order on X is a partial order. That is, $x \preceq y$ and $y \preceq x$ imply $x = y$ for any $x, y \in X_{\text{spcl}}$.

The specialization order on $\operatorname{MSpec} M$ recovers the inclusion-order on prime ideals.

Proposition 6.1.6. *The following hold.*

(1) $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ *for any prime ideal* $\mathfrak{p} \subseteq M$.
(2) $(\operatorname{MSpec} M)_{\text{spcl}}$ *is isomorphic to* $\operatorname{MSpec} M$ *ordered by reverse inclusion as posets.*

Proof. We omit the proof since it is straightforward. □

We give a topological characterization of principal open subsets $D(f)$. We will use it to classify finitely generated Serre subcategories in Proposition 6.2.6.

Definition 6.1.7. A topological space X is *strongly quasi-compact* if for every open covering $\{U_i\}_{i \in I}$ of X, there exists $i \in I$ such that $X = U_i$.

Lemma 6.1.8. *Let M be a monoid. A nonempty open subset U of* $\operatorname{MSpec} M$ *is strongly quasi-compact if and only if $U = D(f)$ for some $f \in M$.*

Proof. We first show that $D(f)$ is nonempty and strongly quasi-compact. The subset $D(f)$ has the maximum element $\langle f\rangle_{\text{face}}^c$ with respect to inclusions. Indeed, for any $\mathfrak{p} \in D(f)$, we have $f \notin \mathfrak{p}$. Since $\mathfrak{p}^c$ is a face and $f \in \mathfrak{p}^c$, we obtain $\langle f\rangle_{\text{face}} \subseteq \mathfrak{p}^c$. Thus, we conclude that $\langle f\rangle_{\text{face}}^c \supseteq \mathfrak{p}$. Let $\{U_i\}_{i\in I}$ be an open covering of $D(f)$. Then there exists $i \in I$ such that $\langle f\rangle_{\text{face}}^c \in U_i$, which implies $D(f) = U_i$ because U_i is generalization-closed. This proves $D(f)$ is strongly quasi-compact.

Conversely, suppose that U is a nonempty strongly quasi-compact open subset of $\operatorname{MSpec} M$. Since the principal open subsets are a basis of Zariski topology, the open subset U is covered by them. Thus $U = D(f)$ for some $f \in M$ because U is strongly quasi-compact. □

The monoid spectrum $\operatorname{MSpec} M$ is equipped with a natural sheaf of monoids.

Fact 6.1.9 ([Ogu18, Section II.1.2]). *There is a sheaf $\mathscr{O}_M$ of monoids on* $\operatorname{MSpec} M$*, which is called the* structure sheaf*, satisfying the following.*

(1) *For any element $f \in M$, we have that $\mathscr{O}_M(D(f)) = M_f$.*
(2) *In particular, we have that $\mathscr{O}_M(\operatorname{MSpec} \mathsf{M}(\mathcal{A})) = M$.*
(3) *For any point $\mathfrak{p} \in \operatorname{MSpec} M$, the stalk $\mathscr{O}_{M,\mathfrak{p}}$ of $\mathscr{O}_M$ is isomorphic to the localization $M_{\mathfrak{p}^c}$ of M with respect to $\mathfrak{p}^c := M \setminus \mathfrak{p}$.*

A monoidal space is a pair $(X, \mathscr{M})$ of a topological space X and a sheaf $\mathscr{M}$ of monoids on X. A morphism $(f, f^\flat)\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ of monoidal spaces is a pair of continuous map $f\colon X \to Y$ and a morphism $f^b\colon f^{-1}\mathscr{N} \to \mathscr{M}$ of sheaves of monoids such that the map on the stalks $\mathscr{N}_{f(x)} \to \mathscr{M}_x$ are local monoid homomorphism for all $x \in X$. Here a monoid morphism $\phi\colon M \to N$ is *local* if $\phi^{-1}(N^\times) = M^\times$. A morphism $(f, f^\flat)\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ is an isomorphism if and only if f is a homeomorphism and $f^\flat$ is an isomorphism of sheaves. An *affine monoid scheme* is a monoidal space isomorphic to $(\operatorname{MSpec} M, \mathscr{O}_M)$ for some monoid M.

Remark 6.1.10. An affine monoid scheme $(\operatorname{MSpec} M, \mathscr{O}_M)$ was first introduced by Kato [Kat94] to study toric singularities. Deitmar [Dei05] used it to construct a theory of "schemes over the field $\mathbb{F}_1$ with one element". See [LP11] for more information.

6.2 The spectrum of the Grothendieck monoid

In this section, we first introduce a topology on the set $\operatorname{Serre}(\mathcal{C})$ of Serre subcategories and study the relationship between the topologies on $\operatorname{Serre}(\mathcal{C})$ and $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$. Next, we classify finitely generated Serre subcategories by using this topology. Finally, we introduce a sheaf $\mathscr{M}$ of monoids on $\operatorname{Serre}(\mathcal{A})$ for an abelian category $\mathcal{A}$, which is related to the quotient abelian category $\mathcal{A}/\mathcal{S}$, and compare it with the structure sheaf $\mathscr{O}_{\mathsf{M}(\mathcal{A})}$ of $\operatorname{MSpec} \mathsf{M}(\mathcal{A})$.

Let us begin with the bijections which follow from Proposition 3.3.3 and Fact 6.1.3.

Proposition 6.2.1. *There are bijections between the following sets:*

(1) *The set* $\operatorname{Serre}(\mathcal{C})$ *of Serre subcategories of* $\mathcal{C}$*.*
(2) *The set* $\operatorname{Face} \mathsf{M}(\mathcal{C})$ *of faces of* $\mathsf{M}(\mathcal{C})$*.*
(3) *The set* $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$ *of prime ideals of* $\mathsf{M}(\mathcal{C})$*.*

Moreover, the bijection between (1) and (2) is inclusion-preserving while the one between (2) and (3) is inclusion-reversing.

The bijection between (1) and (3) induces a topology on $\operatorname{Serre}(\mathcal{C})$ from $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$. In the following, we describe this topology explicitly. For a subcategory $\mathcal{X}$ of $\mathcal{C}$, we set

$$V(\mathcal{X}) := \{\mathcal{S} \in \operatorname{Serre}(\mathcal{C}) \mid \mathcal{S} \cap \mathcal{X} = \emptyset\}.$$

We can easily check that the following equalities hold:

- $V(\mathcal{C}) = \emptyset$ and $V(\emptyset) = \operatorname{Serre}(\mathcal{C})$.
- $\bigcap_{\alpha\in A} V(\mathcal{X}_\alpha) = V(\bigcup_{\alpha\in A} \mathcal{X}_\alpha)$ for a family $\{\mathcal{X}_\alpha\}_{\alpha\in A}$ of subcategories of $\mathcal{C}$.
- $V(\mathcal{X}) \cup V(\mathcal{Y}) = V(\mathcal{X} \oplus \mathcal{Y})$ for subcategories $\mathcal{X}$, $\mathcal{Y}$ of $\mathcal{C}$, where $\mathcal{X} \oplus \mathcal{Y} := \{X \oplus Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$.

Thus, we can define a topology on $\operatorname{Serre}(\mathcal{C})$, which is called the *Zariski topology*, by taking the subsets of the form $V(\mathcal{X})$ to be the closed subsets.

For an object $X \in \mathcal{C}$, we put

$$U_X := \{\mathcal{S} \in \mathrm{Serre}(\mathcal{C}) \mid X \in \mathcal{S}\}.$$

We can easily check that $U_X \cap U_Y = U_{X\oplus Y}$ for any $X, Y \in \mathcal{C}$.

We now compare the Zariski topology on $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$ with the one on $\mathrm{Serre}(\mathcal{C})$.

Proposition 6.2.2. *The following hold.*

(1) *The bijection* $\Phi\colon \mathrm{Serre}(\mathcal{C}) \xrightarrow{\cong} \operatorname{MSpec} \mathsf{M}(\mathcal{C})$ *in Proposition 6.2.1 is a homeomorphism.*
(2) *The set of subsets of the form* U_X *forms an open basis of* $\mathrm{Serre}(\mathcal{C})$*.*
(3) $\mathrm{Serre}(\mathcal{C})_{\mathrm{spcl}} \cong (\mathrm{Serre}(\mathcal{C}), \subseteq)$ *as posets (see Definition 6.1.4).*

Proof. We first note that $\Phi(\mathcal{S}) = \mathsf{M}_{\mathcal{S}}^c := \mathsf{M}(\mathcal{C}) \setminus \mathsf{M}_{\mathcal{S}}$ for any $\mathcal{S} \in \mathrm{Serre}(\mathcal{C})$.

(1) Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$, and let $\mathcal{S}$ be a Serre subcategory of $\mathcal{C}$. Then $\mathcal{S} \cap \mathcal{X} = \emptyset$ if and only if $\mathsf{M}_{\mathcal{S}} \cap \mathsf{M}_{\mathcal{X}} = \emptyset$ since $\mathcal{S}$ is c-closed by Proposition 3.3.2. It is equivalent to $\Phi(\mathcal{S}) = \mathsf{M}_{\mathcal{S}}^c \supseteq \mathsf{M}_{\mathcal{X}}$. Thus, we obtain $\Phi(V(\mathcal{X})) = V(\mathsf{M}_{\mathcal{X}})$, which implies Φ is a homeomorphism since $\mathsf{M}_{\mathcal{X}}$ runs through all subsets of $\mathsf{M}(\mathcal{C})$ by Proposition 3.1.3.

(2) It is clear since $\Phi(U_X) = D([X])$ for any $X \in \mathcal{C}$.

(3) It follows from Proposition 6.1.6 and the fact that Φ is inclusion-reversing. □

Remark 6.2.3. The topology on $\mathrm{Serre}(\mathcal{C})$ is a natural analogue of the topology on the set of thick subcategories of a triangulated category, which is introduced by Balmer [Bal05] (see also [MT20, Mat21]).

Next, we will classify finitely generated Serre subcategories of $\mathcal{C}$ by using the Zariski topology on $\mathrm{Serre}(\mathcal{C})$. Recall that a Serre subcategory $\mathcal{S}$ of $\mathcal{C}$ is finitely generated if $\mathcal{S} = \langle X \rangle_{\mathrm{Serre}}$ for some object $X \in \mathcal{C}$. We need two lemmas for the open subsets U_X of $\mathrm{Serre}(\mathcal{C})$.

Lemma 6.2.4. *Let* U *be a nonempty open subset of* $\mathrm{Serre}(\mathcal{C})$*. Then* U *is strongly quasi-compact if and only if* $U = U_X$ *for some* $X \in \mathcal{C}$*.*

Proof. Let $\Phi\colon \mathrm{Serre}(\mathcal{C}) \xrightarrow{\cong} \operatorname{MSpec} \mathsf{M}(\mathcal{C})$ be the homeomorphism in Proposition 6.2.2. This lemma immediately follows from Lemma 6.1.8 and $\Phi(U_X) = D([X])$. □

Lemma 6.2.5. *Let* X *and* Y *be objects of* $\mathcal{C}$*. Then* $U_X \subseteq U_Y$ *if and only if* $\langle X \rangle_{\mathrm{Serre}} \supseteq \langle Y \rangle_{\mathrm{Serre}}$*. In particular,* $U_X = U_Y$ *if and only if* $\langle X \rangle_{\mathrm{Serre}} = \langle Y \rangle_{\mathrm{Serre}}$*.*

Proof. Suppose that $U_X \subseteq U_Y$. Then $\langle X \rangle_{\mathrm{Serre}} \in U_X \subseteq U_Y$, which implies $Y \in \langle X \rangle_{\mathrm{Serre}}$. Thus, we have that $\langle Y \rangle_{\mathrm{Serre}} \subseteq \langle X \rangle_{\mathrm{Serre}}$. Conversely, suppose that $\langle X \rangle_{\mathrm{Serre}} \supseteq \langle Y \rangle_{\mathrm{Serre}}$. Take $\mathcal{S} \in U_X$. Then $X \in \mathcal{S}$, which implies $Y \in \langle Y \rangle_{\mathrm{Serre}} \subseteq \langle X \rangle_{\mathrm{Serre}} \subseteq \mathcal{S}$, and hence $\mathcal{S} \in U_Y$. Thus, we have that $U_X \subseteq U_Y$. □

Proposition 6.2.6. *There are bijections between the following sets:*

(1) *The set of finitely generated Serre subcategories of* $\mathcal{C}$*.*
(2) *The set of nonempty strongly quasi-compact open subsets of* $\mathrm{Serre}(\mathcal{C})$*.*
(3) *The set of nonempty strongly quasi-compact open subsets of* $\operatorname{MSpec} \mathsf{M}(\mathcal{C})$*.*

The bijection from (1) to (2) is given by $\mathcal{X} = \langle X \rangle_{\mathrm{Serre}} \mapsto U_X$*.*

Proof. Let $\Phi\colon \mathrm{Serre}(\mathcal{C}) \xrightarrow{\cong} \operatorname{MSpec} \mathsf{M}(\mathcal{C})$ be the homeomorphism in Proposition 6.2.2. It is clear that there is a bijection between (2) and (3) induced by Φ. Let us construct a bijection between (1) and (2). For any $X, Y \in \mathcal{C}$, $U_X = U_Y$ if and only if $\langle X \rangle_{\mathrm{Serre}} = \langle Y \rangle_{\mathrm{Serre}}$ by Lemma 6.2.5. Thus, the assignment $\mathcal{X} = \langle X \rangle_{\mathrm{Serre}} \mapsto U_X$ is well-defined and injective. On the other hand, it is surjective by Lemma 6.2.4. Therefore, the assignment $\mathcal{X} = \langle X \rangle_{\mathrm{Serre}} \mapsto U_X$ gives a bijection from (1) to (2). □

Finally, we construct a sheaf of monoids on $\mathrm{Serre}(\mathcal{A})$ for a skeletally small abelian category $\mathcal{A}$, which is related to the quotient abelian category $\mathcal{A}/\mathcal{S}$. There is no application of this sheaf at this moment. However, it may be interesting from the viewpoints of geometry over the field $\mathbb{F}_1$ with one element and noncommutative algebraic geometry. Even if the reader skips the rest of this section, there is no harm to read the other sections.

We begin with a review of the notion of abelian quotient categories. See [Pop73, Section 4.3] for details. For a Serre subcategory $\mathcal{S}$ of $\mathcal{A}$, there are an abelian category $\mathcal{A}/\mathcal{S}$ and an exact functor $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ which satisfy the following universal property:

- For any exact functor $F\colon \mathcal{A} \to \mathcal{C}$ of abelian categories such that $F(\mathcal{S}) = 0$, there exists a unique exact functor $\overline{F}\colon \mathcal{A}/\mathcal{S} \to \mathcal{C}$ satisfying $F = \overline{F}Q$.

We call $\mathcal{A}/\mathcal{S}$ the *abelian quotient category* of $\mathcal{A}$ with respect to $\mathcal{S}$, and $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ the *quotient functor*. The following facts are useful to study the abelian quotient category $\mathcal{A}/\mathcal{S}$.

Fact 6.2.7 ([Pop73, Lemma 4.3.4, 4.3.7, 4.3.9])**.** *Let $\mathcal{S}$ be a Serre subcategory of an abelian category $\mathcal{A}$, and let $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ be the quotient functor.*

(1) *$\mathcal{S} = \{X \in \mathcal{A} \mid Q(X) = 0\}$ holds.*
(2) *Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$.*
   - *$Q(f)$ is a monomorphism in $\mathcal{A}/\mathcal{S}$ if and only if $\mathrm{Ker}(f) \in \mathcal{S}$.*
   - *$Q(f)$ is an epimorphism in $\mathcal{A}/\mathcal{S}$ if and only if $\mathrm{Cok}(f) \in \mathcal{S}$.*
   - *$Q(f)$ is an isomorphism in $\mathcal{A}/\mathcal{S}$ if and only if $\mathrm{Ker}(f), \mathrm{Cok}(f) \in \mathcal{S}$.*
(3) *Any morphism of $\mathcal{A}/\mathcal{S}$ can be written by $Q(s)^{-1}Q(f)Q(t)^{-1}$ for some morphisms s, t, f in $\mathcal{A}$.*

Let $\mathcal{A}$ be a skeletally small abelian category. Let us construct a sheaf of monoids on $\mathrm{Serre}(\mathcal{A})$. Let $\mathcal{B}$ be the set of strongly quasi-compact open subsets of $\mathrm{Serre}(\mathcal{A})$. Explicitly, we have that $\mathcal{B} = \{U_X \mid X \in \mathcal{A}\}$ by Lemma 6.2.4. Then $\mathcal{B}$ is an open basis of $\mathrm{Serre}(\mathcal{A})$ by Proposition 6.2.2. Note that $U_X \supseteq U_Y$ if and only if $\langle X \rangle_{\mathrm{Serre}} \subseteq \langle Y \rangle_{\mathrm{Serre}}$ for any $X, Y \in \mathcal{A}$ by Lemma 6.2.5. In this case, there is an exact functor $F_{X,Y}\colon \mathcal{A}/\langle X \rangle_{\mathrm{Serre}} \to \mathcal{A}/\langle Y \rangle_{\mathrm{Serre}}$ induced by the universal property of the abelian quotient category $\mathcal{A}/\langle X \rangle_{\mathrm{Serre}}$. In particular, we obtain a monoid homomorphism $r_{X,Y} := \mathsf{M}(F_{X,Y})\colon \mathsf{M}(\mathcal{A}/\langle X \rangle_{\mathrm{Serre}}) \to \mathsf{M}(\mathcal{A}/\langle Y \rangle_{\mathrm{Serre}})$. Thus, the assignment

$$U_X \mapsto \mathscr{M}_{\mathcal{A}}(U_X) := \mathsf{M}(\mathcal{A}/\langle X \rangle_{\mathrm{Serre}})$$

defines a presheaf of monoids on $\mathcal{B}$. Define a presheaf $\mathscr{M}_{\mathcal{A}}$ on $\mathrm{Serre}(\mathcal{A})$ by

$$V \mapsto \mathscr{M}_{\mathcal{A}}(V) := \varprojlim_{U} \mathscr{M}_{\mathcal{A}}(U),$$

where U runs through the set of $U \in \mathcal{B}$ with $U \subseteq V$. Then it satisfies the condition of [GW20, Proposition 2.20] since $U \in \mathcal{B}$ is strongly quasi-compact. Thus $\mathscr{M}_{\mathcal{A}}$ is a sheaf on $\mathrm{Serre}(\mathcal{A})$. We need the following lemma to study this sheaf $\mathscr{M}_{\mathcal{A}}$.

Lemma 6.2.8. *Let $\mathcal{S}$ be a Serre subcategory of a skeletally small abelian category $\mathcal{A}$, and let $X, Y \in \mathcal{A}$.*

(1) *If X is a subobject of Y in $\mathcal{A}/\mathcal{S}$, then there is $M \in \mathcal{S}$ such that X remains a subobject of Y in $\mathcal{A}/\langle M \rangle_{\mathrm{Serre}}$.*
(2) *If Y is a quotient of X in $\mathcal{A}/\mathcal{S}$, then there is $M \in \mathcal{S}$ such that Y remains a quotient of X in $\mathcal{A}/\langle M \rangle_{\mathrm{Serre}}$.*
(3) *If $X \cong Y$ in $\mathcal{A}/\mathcal{S}$, then there is $M \in \mathcal{S}$ such that $X \cong Y$ in $\mathcal{A}/\langle M \rangle_{\mathrm{Serre}}$.*

Proof. The proof of (2) is similar to that of (1), and (3) is a consequence of (1) and (2). Thus we only prove (1). Any monomorphism $X \hookrightarrow Y$ in $\mathcal{A}/\mathcal{S}$ can be written as $Q(s)^{-1}Q(f)Q(t)^{-1}$ for some morphisms s, t, f in $\mathcal{A}$ by Fact 6.2.7 (3). We set

$$M := \mathrm{Ker}(s) \oplus \mathrm{Ker}(t) \oplus \mathrm{Cok}(s) \oplus \mathrm{Cok}(t) \oplus \mathrm{Ker}(f).$$

Then $M \in \mathcal{S}$ by Fact 6.2.7 (2). Since $\mathrm{Ker}(s)$, $\mathrm{Ker}(t)$, $\mathrm{Cok}(s)$, $\mathrm{Cok}(t)$ and $\mathrm{Ker}(f)$ belong to $\langle M \rangle_{\mathrm{Serre}}$, the morphisms s and t are isomorphisms in $\mathcal{A}/\langle M \rangle_{\mathrm{Serre}}$, and f is a monomorphism in $\mathcal{A}/\langle M \rangle_{\mathrm{Serre}}$. Thus, there is a monomorphism $X \hookrightarrow Y$ in $\mathcal{A}/\langle M \rangle_{\mathrm{Serre}}$, and hence X remains a subobject of Y in $\mathcal{A}/\langle M \rangle_{\mathrm{Serre}}$. ☐

The following fact is useful to study the Grothendieck monoid of an abelian category.

Fact 6.2.9 ([Bro98, Proposition 3.3])**.** *Let $\mathcal{A}$ be a skeletally small abelian category. For any two objects $X, Y \in \mathcal{A}$, the equality $[X] = [Y]$ holds in $\mathsf{M}(\mathcal{A})$ if and only if X and Y have isomorphic subobject series (see §3.4 for the terminologies).*

Proposition 6.2.10. *Let $\mathcal{A}$ be a skeletally small abelian category and $\mathscr{M}_{\mathcal{A}}$ a sheaf on $\mathrm{Serre}(\mathcal{A})$ constructed as above.*

(1) *For any $X \in \mathcal{A}$, we have $\mathscr{M}_{\mathcal{A}}(U_X) = \mathsf{M}(\mathcal{A}/\langle X \rangle_{\mathrm{Serre}})$.*

(2) *In particular, we have* $\mathscr{M}_{\mathcal{A}}(\mathrm{Serre}(\mathcal{A})) = \mathsf{M}(\mathcal{A})$.
(3) *For any point* $\mathcal{S} \in \mathrm{Serre}(\mathcal{A})$*, the stalk* $\mathscr{M}_{\mathcal{A},\mathcal{S}}$ *of* $\mathscr{M}_{\mathcal{A}}$ *is isomorphic to* $\mathsf{M}(\mathcal{A}/\mathcal{S})$.

Proof. We only prove (3) because (1) and (2) are obvious by the definition of $\mathscr{M}_{\mathcal{A}}$. Let $\mathcal{S}$ be a Serre subcategory of $\mathcal{A}$. For any $X \in \mathcal{A}$ with $\mathcal{S} \in U_X$, we have the natural exact functor $\mathcal{A}/\langle X \rangle_{\mathrm{Serre}} \to \mathcal{A}/\mathcal{S}$. They induce a monoid homomorphism

$$\phi\colon \mathscr{M}_{\mathcal{A},\mathcal{S}} = \varinjlim_{U_X \ni \mathcal{S}} \mathscr{M}_{\mathcal{A}}(U_X) = \varinjlim_{U_X \ni \mathcal{S}} \mathsf{M}(\mathcal{A}/\langle X \rangle_{\mathrm{Serre}}) \to \mathsf{M}(\mathcal{A}/\mathcal{S}).$$

It is clear that ϕ is surjective. We now prove ϕ is injective. We first note that the natural map $\mathsf{M}(\mathcal{A}) \to \mathscr{M}_{\mathcal{A},\mathcal{S}}$ is surjective since the natural map $\mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A}/\langle M \rangle_{\mathrm{Serre}})$ is surjective. We denote by $[X]_{\mathcal{S}}$ the element of $\mathscr{M}_{\mathcal{A},\mathcal{S}}$ represented by $X \in \mathcal{A}$. Suppose that $\phi([X]_{\mathcal{S}}) = \phi([Y]_{\mathcal{S}})$ for some $X, Y \in \mathcal{A}$. Then there are admissible subobject series $0 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ and $0 = Y_0 \leq Y_1 \leq \cdots \leq Y_n = Y$ such that $X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1}$ in $\mathcal{A}/\mathcal{S}$ for some permutation $\sigma \in \mathfrak{S}_n$ by Fact 6.2.9. Applying Lemma 6.2.8 to $X_{i-1} \leq X_i$, $Y_{i-1} \leq Y_i$ and $X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1}$, and taking their direct sum, we get $M \in \mathcal{S}$ such that X and Y remain having isomorphic subobject series in $\mathcal{A}/\langle M \rangle_{\mathrm{Serre}}$. Thus $[X] = [Y]$ in $\mathsf{M}(\mathcal{A}/\langle M \rangle_{\mathrm{Serre}})$ and $\mathcal{S} \in U_M$. This proves $[X]_{\mathcal{S}} = [Y]_{\mathcal{S}}$ in $\mathscr{M}_{\mathcal{A},\mathcal{S}}$. $\square$

Finally, we compare $(\mathrm{Serre}(\mathcal{A}), \mathscr{M}_{\mathcal{A}})$ with $(\mathrm{MSpec}\,\mathsf{M}(\mathcal{A}), \mathscr{O}_{\mathsf{M}(\mathcal{A})})$ as monoidal spaces. Define a sheaf $\overline{\mathscr{O}}_{\mathsf{M}(\mathcal{A})}$ on $\mathrm{MSpec}\,\mathsf{M}(\mathcal{A})$ by the sheafification of presheaf

$$U \mapsto \mathscr{O}_{\mathsf{M}(\mathcal{A})}(U)/\mathscr{O}_{\mathsf{M}(\mathcal{A})}(U)^{\times}.$$

For any object $X \in \mathcal{A}$, we have an isomorphism

$$\mathscr{O}_{\mathsf{M}(\mathcal{A})}(D([X]))/\mathscr{O}_{\mathsf{M}(\mathcal{A})}(D([X]))^{\times} = \mathsf{M}(\mathcal{A})_{[X]}/\mathsf{M}(\mathcal{A})_{[X]}^{\times} \xrightarrow{\cong} \mathsf{M}(\mathcal{A})/\langle [X] \rangle_{\mathrm{face}} \xrightarrow{\cong} \mathsf{M}(\mathcal{A}/\langle X \rangle_{\mathrm{Serre}}).$$

by Theorem 4.3.1, Lemmas 5.1.2 and 5.1.3. Thus, we have a natural isomorphism

$$\overline{\mathscr{O}}_{\mathsf{M}(\mathcal{A})}(D([X])) \xrightarrow{\cong} \mathscr{M}_{\mathcal{A}}(U_X). \tag{6.2.1}$$

Let $\Phi\colon \mathrm{Serre}(\mathcal{C}) \xrightarrow{\cong} \mathrm{MSpec}\,\mathsf{M}(\mathcal{A})$ be the homeomorphism in Proposition 6.2.2. Then (6.2.1) gives rise to an isomorphism $\Phi^{-1}\overline{\mathscr{O}}_{\mathsf{M}(\mathcal{A})} \to \mathscr{M}_{\mathcal{A}}$ of sheaves of monoids. Thus we have the following proposition.

Proposition 6.2.11. *Let* $\mathcal{A}$ *be a skeletally small abelian category. The bijection in Proposition 6.2.1 induces an isomorphism of monoidal spaces*

$$(\mathrm{Serre}(\mathcal{A}), \mathscr{M}_{\mathcal{A}}) \cong (\mathrm{MSpec}\,\mathsf{M}(\mathcal{A}), \overline{\mathscr{O}}_{\mathsf{M}(\mathcal{A})}).$$

Part II

Applications

Chapter 7

The Grothendieck monoids of smooth projective curves

Throughout this chapter, we fix Grothendieck universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$ and use Convention 1.1.12 and 2.2.5. Moreover, we assume that every category, functor, and subcategory is additive. In particular, every subcategory is strictly full and nonempty. Hereafter C is a smooth projective curve over an algebraically closed field $\mathbb{k}$.

There are three exact categories related to C:

- The category $\mathsf{coh}(C)$ of coherent sheaves on C.
- The category $\mathsf{vect}(C)$ of vector bundles on C.
- The category $\mathsf{tor}(C)$ of coherent torsion sheaves on C.

We determine the Grothendieck monoids of them and classify Serre subcategories of them. For the basics of algebraic geometry, we refer to [Har77, GW20]. We will review the categorical properties of $\mathsf{coh}\, C$ in each of the following sections. Note that $\mathsf{coh}(C)$ is skeletally small, and hence so are $\mathsf{vect}(C)$ and $\mathsf{tor}(C)$.

7.1 The case of coherent torsion sheaves

We first review a categorical characterization of coherent torsion sheaves on a curve. Let $i : Z \hookrightarrow X$ be a closed immersion into a noetherian scheme X and let $\mathscr{I}$ be the quasi-coherent ideal sheaf corresponding to Z. Then the functor $i_* : \mathsf{coh}\, Z \to \mathsf{coh}\, X$ is a fully faithful exact functor whose essential image $\operatorname{Im} i_*$ is the subcategory consisting of coherent sheaves $\mathscr{F}$ such that $\mathscr{I}\mathscr{F} = 0$ (cf. [SP, Tag 01QX]). It follows immediately that $\operatorname{Im} i_*$ is closed under subobjects in $\mathsf{coh}\, X$. This means that there is no difference between subobjects of $\mathscr{F} \in \mathsf{coh}\, Z$ and subobjects of $i_*\mathscr{F} \in \mathsf{coh}\, X$. For a closed point $x \in X$, consider the natural closed immersion $i\colon \operatorname{Spec} \kappa(x) \hookrightarrow X$. Then $\mathscr{O}_x := i_*\mathscr{O}_{\operatorname{Spec}\kappa(x)}$ is a simple object of $\mathsf{coh}\, X$ by the above discussion.

Lemma 7.1.1. *The following are equivalent for a coherent sheaf $\mathscr{F}$ on a noetherian scheme X:*

(1) *$\mathscr{F}$ is a simple object in $\mathsf{coh}\, X$.*

(2) *$\mathscr{F} \cong \mathscr{O}_x$ for some closed point $x \in X$.*

Proof. We have already proved that (2) implies (1). Hence, we only prove that (1) implies (2). Suppose that $\mathscr{F}$ is a simple object in $\mathsf{coh}\, X$. Recall that a simple object is nonzero, so we have $\operatorname{Supp} \mathscr{F} \neq \emptyset$. There is a closed point x of $\operatorname{Supp} \mathscr{F}$ because X is noetherian (cf. [GW20, Lemma1.25, Exercise 3.13]). Let $i\colon \operatorname{Spec} \kappa(x) \hookrightarrow X$ be the natural closed immersion. Then $\mathscr{F}(x) := i^*\mathscr{F} = \mathscr{F}_x/\mathscr{F}_x\mathfrak{m}_x \in \mathsf{mod}\, \kappa(x)$ is nonzero by Nakayama's lemma. Because the unit morphism $\mathscr{F} \to i_*i^*\mathscr{F} = i_*\mathscr{F}(x)$ is surjective and $\mathscr{F}$ is simple, we have that $\mathscr{F} \xrightarrow{\cong} i_*\mathscr{F}(x)$. Then $\mathscr{F}(x)$ is also a simple object in $\mathsf{mod}\, \kappa(x)$. This means $\mathscr{F}(x) \cong \kappa(x)$, and we obtain the desired result. $\square$

Lemma 7.1.2. *The following are equivalent for a coherent sheaf $\mathscr{F}$ on a noetherian scheme X:*

(1) *$\mathscr{F}$ is of finite length in $\mathsf{coh}\, X$ (see Definition 3.4.1).*

(2) *$\operatorname{Supp}(\mathscr{F})$ consists of only finitely many closed points.*

In this case, the following hold:

(i) $\mathscr{F}_x$ *is an* $\mathscr{O}_{X,x}$*-module of finite length for any* $x \in X$.
(ii) *The natural morphism* $\mathscr{F} \to \bigoplus_{x \in \operatorname{Supp} \mathscr{F}} i_{x*}\mathscr{F}_x$ *is an isomorphism, where* i_x *is the natural morphism* $\operatorname{Spec} \mathscr{O}_{X,x} \to X$.

Proof. It is clear when $\mathscr{F} = 0$. We assume that $\mathscr{F} \neq 0$, and hence $\operatorname{Supp} \mathscr{F} \neq \emptyset$.

(1) $\Rightarrow$ (2): There is a composition series $0 = \mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}_n = \mathscr{F}$ in $\operatorname{coh} X$ since $\mathscr{F}$ is of finite length. Then $\mathscr{F}_i/\mathscr{F}_{i-1} \cong \mathscr{O}_{x_i}$ for some closed point $x_i \in X$ by Lemma 7.1.1. Thus, we have $\operatorname{Supp} \mathscr{F} = \bigcup_{i=1}^{n} \operatorname{Supp} \mathscr{O}_{x_i} = \{x_i \mid 1 \leq i \leq n\}$.

(2) $\Rightarrow$ (1): We regard $Z := \operatorname{Supp} \mathscr{F}$ as a closed subscheme of X which corresponds to the annihilator $\operatorname{Ann}(\mathscr{F})$ of $\mathscr{F}$ (cf. [GW20, Subsection 7.17]). Note that $\mathscr{O}_{Z,x} \cong \mathscr{O}_{X,x}/\operatorname{Ann}_{\mathscr{O}_{X,x}}(\mathscr{F}_x)$ as rings. Then the natural morphism $\coprod_{x \in \operatorname{Supp} \mathscr{F}} \operatorname{Spec} \mathscr{O}_{Z,x} \to Z$ is an isomorphism and $\mathscr{O}_{Z,x}$ is an artinian local ring by (2) and [GW20, Proposition 5.11]. Since $\mathscr{F}_x$ is finitely generated over the artinian ring $\mathscr{O}_{Z,x}$, it is of finite length as an $\mathscr{O}_{Z,x}$-module, and hence (i) also holds. Let $j \colon Z \hookrightarrow X$ and $j_x \colon \operatorname{Spec} \mathscr{O}_{Z,x} \to Z$ be the natural closed immersions. Then we have

$$j^*\mathscr{F} \cong \bigoplus_{x \in \operatorname{Supp} \mathscr{F}} j_{x*}(j^*\mathscr{F})_x \cong \bigoplus_{x \in \operatorname{Supp} \mathscr{F}} j_{x*}\left(\mathscr{F}_x/\mathscr{F}_x \cdot \operatorname{Ann}_{\mathscr{O}_{X,x}}(\mathscr{F}_x)\right) = \bigoplus_{x \in \operatorname{Supp} \mathscr{F}} j_{x*}\mathscr{F}_x.$$

Thus, we obtain isomorphisms

$$\mathscr{F} \xrightarrow{\cong} j_*j^*\mathscr{F} \cong j_*\left(\bigoplus_{x \in \operatorname{Supp} \mathscr{F}} j_{x*}\mathscr{F}_x\right) \cong \bigoplus_{x \in \operatorname{Supp} \mathscr{F}} (jj_x)_*\mathscr{F}_x. \tag{7.1.1}$$

See [GW20, Remark 7.36] for the first isomorphism. Since jj_x is a closed immersion and $\mathscr{F}_x$ is of finite length, we conclude that $\mathscr{F}$ is also of finite length in $\operatorname{coh} X$. Then (ii) holds by (7.1.1) and the following commutative diagram:

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & X \\ {\scriptstyle j_x}\uparrow & & \uparrow{\scriptstyle i_x} \\ \operatorname{Spec} \mathscr{O}_{Z,x} & \hookrightarrow & \operatorname{Spec} \mathscr{O}_{X,x}. \end{array}$$

□

Let us characterize coherent sheaves of finite length on a smooth projective curve C. For any closed point $x \in C$, we set $\mathscr{O}_{nx} := i_*\left(\mathscr{O}_{C,x}/\mathfrak{m}_x^n\right)$, where i is the natural morphism $\operatorname{Spec} \mathscr{O}_{C,x} \to C$.

Lemma 7.1.3. *The following are equivalent for a coherent sheaf* $\mathscr{F}$ *on* C*:*
(1) $\mathscr{F}$ *is of finite length in* $\operatorname{coh} C$.
(2) $\operatorname{Supp}(\mathscr{F})$ *has only finitely many points.*
(3) $\mathscr{F}_\eta = 0$ *holds, where* η *is the generic point of* C.
In this case, the following hold:
(i) $\mathscr{F}_x$ *is a torsion* $\mathscr{O}_{C,x}$*-module for any* $x \in C$.
(ii) $\mathscr{F} \cong \bigoplus_{x \in \operatorname{Supp} \mathscr{F}} \mathscr{O}_{n_x x}$ *for some positive integers* $n_x > 0$.

Proof. It is clear when $\mathscr{F} = 0$. We assume that $\mathscr{F} \neq 0$, and hence $\operatorname{Supp} \mathscr{F} \neq \emptyset$. Since C is a 1-dimensional integral scheme of finite type over $\Bbbk$, the following are equivalent for a non-empty closed subset Z of C (cf. [GW20, Proposition 5.20]):
- $\dim Z = 0$.
- Z has only finitely many points.
- Z consists of finitely many closed points.
- $\eta \notin Z$.

The equivalence of (1), (2) and (3) follows from Lemma 7.1.2 and the above. For a finitely generated module over the discrete valuation ring $\mathscr{O}_{C,x}$, it is of finite length if and only if it is torsion. Moreover, it is of the form $\mathscr{O}_{C,x}/\mathfrak{m}_x^{n_x}$ for some integer $n_x \geq 0$. Thus (i) and (ii) follow from Lemma 7.1.2. □

A coherent sheaf $\mathscr{F}$ on C is said to be *torsion* if it satisfies the equivalent conditions of Lemma 7.1.3. We denote by $\operatorname{tor} C$ the category of coherent torsion sheaves. It is immediate that $\operatorname{tor} C$ is a Serre subcategory of the abelian category $\operatorname{coh} C$. It is also clear that $\operatorname{tor} C$ is a length abelian category.

We will calculate the Grothendieck monoid $\mathsf{M}(\operatorname{tor} C)$ and classify Serre subcategories of $\operatorname{tor} C$. For this, we recall divisors on C. Let $C(\Bbbk)$ be the set of closed points of C. We denote by $\operatorname{Div}(C)$ the free abelian group generated by $C(\Bbbk)$. An element $D = \sum_{i=1}^n m_i x_i$ of $\operatorname{Div}(C)$ is called a *divisor* on C. The integer $\deg D := \sum_{i=1}^n m_i$ is called the *degree* of D. A divisor $D = \sum_{i=1}^n m_i x_i$ is said to be *effective* if $m_i \geq 0$ for all i. $\operatorname{Div}^+(C)$ denotes the set of effective divisors on C, which is a submonoid of $\operatorname{Div}(C)$.

Proposition 7.1.4. *The following hold.*

(1) $\operatorname{sim}(\operatorname{tor} C) = \{\mathscr{O}_x \mid x \in C(\Bbbk)\}$ *holds (see §3.4 for the notation).*
(2) *There is a monoid isomorphism*

$$\operatorname{Div}^+(C) \xrightarrow{\cong} \mathsf{M}(\operatorname{tor} C), \quad \sum_{i=1}^n m_i x_i \mapsto \sum_{i=1}^n m_i [\mathscr{O}_{x_i}].$$

Proof. It follows from Lemma 7.1.1 and Fact 3.4.4. □

Corollary 7.1.5. *There is an inclusion-preserving bijection*

$$\mathsf{P}(C(\Bbbk)) \xrightarrow{\cong} \operatorname{Serre}(\operatorname{tor} C), \quad A \mapsto \langle \mathscr{O}_x \mid x \in A \rangle_{\text{Serre}}.$$

Proof. It follows from Example 3.4.2 (3), Corollary 3.4.7 and Proposition 7.1.4. □

Note that $\mathsf{P}(C(\Bbbk))$ is exactly the set of specialization-closed subsets except C itself (see Definition 6.1.4).

7.2 The case of vector bundles

We begin with a review of vector bundles on a noetherian scheme X. A *locally free sheaf* of rank n on X is a coherent sheaf $\mathscr{F}$ such that $\mathscr{F}_x \cong \mathscr{O}_{X,x}^{\oplus n}$ for all $x \in X$ (cf. [GW20, Proposition 7.41]). We call a locally free sheaf of finite rank on X a *vector bundle* over X. We denote by $\operatorname{vect} X$ the category of vector bundled over X. Then $\operatorname{vect} X$ is an extension-closed subcategory of $\operatorname{coh} X$. Indeed, for any exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ in $\operatorname{coh} X$ with $\mathscr{F}, \mathscr{H} \in \operatorname{vect} X$ and any $x \in X$, the exact sequence $0 \to \mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x \to 0$ splits since $\mathscr{H}_x$ is a free $\mathscr{O}_{X,x}$-module. Thus $\mathscr{G}_x \cong \mathscr{F}_x \oplus \mathscr{H}_x$ is also a free $\mathscr{O}_{X,x}$-module for any $x \in X$. This implies $\mathscr{G} \in \operatorname{vect} X$, and hence $\operatorname{vect} X$ is extension-closed. Then $\operatorname{vect} X$ is a length exact category because the ranks of vector bundles give rise to a length-like function $\operatorname{rk}\colon |\operatorname{vect} X| \to \mathbb{N}$. An admissible subobject in $\operatorname{vect} X$ is called a *subbundle*.

Before studying the Grothendieck monoid $\mathsf{M}(\operatorname{vect} C)$, we recall the structure of the Grothendieck group $\mathsf{K}_0(\operatorname{vect} C)$. For this, we will introduce the Picard group of a noetherian scheme X. A *line bundle* $\mathscr{L}$ is a vector bundle of rank 1. It gives rise to an exact equivalence $- \otimes \mathscr{L}\colon \operatorname{coh} X \xrightarrow{\sim} \operatorname{coh} X$, which restricts to an exact equivalence $\operatorname{vect} X \xrightarrow{\sim} \operatorname{vect} X$. It is clear that $\operatorname{rk}(\mathscr{U} \otimes \mathscr{V}) = \operatorname{rk}(\mathscr{U})\operatorname{rk}(\mathscr{V})$ for any vector bundles $\mathscr{U}$ and $\mathscr{V}$. In particular, we have that $\operatorname{rk}(\mathscr{L} \otimes \mathscr{V}) = \operatorname{rk}(\mathscr{V})$ if $\mathscr{L}$ is a line bundle. The set $\operatorname{Pic} X$ of isomorphism classes of line bundles over X becomes a group whose operation is the tensor product $\otimes$ and unit is $\mathscr{O}_X$. The inverse of $\mathscr{L}$ in $\operatorname{Pic}(X)$ is given by the dual $\mathscr{L}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$ of $\mathscr{L}$. The group $\operatorname{Pic} X$ is called the *Picard group* of X. We can assign a vector bundle $\mathscr{V}$ of rank $r \geq 1$ with a line bundle $\det \mathscr{V} := \bigwedge^r \mathscr{V}$, which is called the *determinant bundle* of $\mathscr{V}$. We define the determinant bundle of the zero sheaf 0 by $\det(0) := \mathscr{O}_X$. It gives rise to an additive function $\det\colon |\operatorname{vect} X| \to \operatorname{Pic} X$.

Fact 7.2.1 ([LeP97, Section 2.6])**.** *The following holds for a smooth projective curve C.*

(1) *The inclusion functor* $\operatorname{vect} C \hookrightarrow \operatorname{coh} C$ *induces a group isomorphism* $K_0(\operatorname{vect} C) \xrightarrow{\cong} \mathsf{K}_0(\operatorname{coh} C)$.
(2) *There is a group isomorphism*

$$K_0(\operatorname{vect} C) \xrightarrow{\cong} \operatorname{Pic}(C) \times \mathbb{Z}, \quad [\mathscr{V}] \mapsto (\det \mathscr{V}, \operatorname{rk} \mathscr{V}).$$

We will determine the Grothendieck monoid $\mathsf{M}(\operatorname{vect} C)$ in Proposition 7.2.4 below. Let us give a few preliminaries for Proposition 7.2.4. A coherent sheaf $\mathscr{F}$ on a noetherian scheme X is *globally generated* if there exists a surjective morphism $\mathscr{O}_X^{\oplus n} \twoheadrightarrow \mathscr{F}$. We do not define *very ample* line bundles which appear in the following fact. See [Har77, Section II.5, page 120] for the definition. We only note that any projective variety has a very ample line bundle.

Fact 7.2.2 (Serre [Ser55, Theorem 66.2], cf. [Har77, Theorem II.5.17])**.** *Let X be a projective variety over $\Bbbk$, and let $\mathscr{O}(1)$ be a very ample line bundle on X. Then for any coherent sheaf $\mathscr{F}$ on X, there is an integer n_0 such that $\mathscr{F} \otimes \mathscr{O}(1)^{\otimes n}$ is globally generated for all $n \geq n_0$.*

Fact 7.2.3 (Atiyah [Ati57, Theorem 2])**.** *Let X be a smooth projective variety of dimension d over $\Bbbk$, and let $\mathscr{V}$ be a globally generated vector bundle of rank r over X. If $r > d$, then $\mathscr{V}$ contains a trivial subbundle of rank $r - d$, that is, there is an inflation $\mathscr{O}_X^{\oplus(r-d)} \rightarrowtail \mathscr{V}$ in $\mathsf{vect}\, X$.*

We prepare notations to use the following proof. Let $\mathscr{O}(1)$ be a very ample line bundle on a smooth projective curve C. We set $\mathscr{O}(n) := \mathscr{O}(1)^{\otimes n}$ when $n \geq 0$ and $\mathscr{O}(n) := (\mathscr{O}(1)^{\vee})^{\otimes |n|}$ when $n < 0$. For a coherent sheaf $\mathscr{F}$ on C, we set $\mathscr{F}(n) := \mathscr{F} \otimes \mathscr{O}(n)$. Then $\mathscr{F}(n) \otimes \mathscr{O}(m) \cong \mathscr{F}(n+m)$ holds for any integers n and m.

Proposition 7.2.4. *The following hold.*
(1) *A vector bundle is simple in $\mathsf{vect}\, C$ if and only if it is a line bundle.*
(2) *$\mathsf{M}(\mathsf{vect}\, C)$ is a cancellative monoid, that is, the natural monoid homomorphism $\mathsf{M}(\mathsf{vect}\, C) \to \mathsf{K}_0(\mathsf{vect}\, C)$ is injective (see Definition 2.1.3 and Remark 2.3.6).*
(3) *There is a monoid isomorphism*

$$\mathsf{M}(\mathsf{vect}\, C) \xrightarrow{\cong} (\operatorname{Pic} C \times \mathbb{N}^+) \cup \{(\mathscr{O}_C, 0)\} \subseteq \operatorname{Pic} C \times \mathbb{Z}, \quad [\mathscr{V}] \mapsto (\det \mathscr{V}, \operatorname{rk} \mathscr{V}),$$

where $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ is the semigroup of strictly positive integers.

Proof. (1) Let $\mathscr{V}$ be a vector bundle of rank r. Then there is some integer n such that $\mathscr{V}(n)$ is globally generated by Fact 7.2.2. If $r > 1$, then there is an inflation $\mathscr{O}_C^{\oplus(r-1)} \rightarrowtail \mathscr{V}(n)$ in $\mathsf{vect}(C)$ by Fact 7.2.3. Since the functor $- \otimes \mathscr{O}(-n) \colon \mathsf{vect}\, C \xrightarrow{\sim} \mathsf{vect}\, C$ is exact, we have an inflation $\mathscr{O}(-n)^{\oplus(r-1)} \rightarrowtail \mathscr{V}$. Thus, a simple object in $\mathsf{vect}\, C$ has to be a line bundle. Conversely, a line bundle is a simple object in $\mathsf{vect}\, C$ because $\operatorname{rk} \colon |\mathsf{vect}\, C| \mapsto \mathbb{N}$ is a length-like function.

(2) Define a monoid homomorphism by $\Phi := (\det, \operatorname{rk}) \colon \mathsf{M}(\mathsf{vect}\, C) \to \operatorname{Pic} C \times \mathbb{N}$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathsf{M}(\mathsf{vect}\, C) & \longrightarrow & \mathsf{K}_0(\mathsf{vect}\, C) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\cong} \\ \operatorname{Pic} C \times \mathbb{N} & \hookrightarrow & \operatorname{Pic} C \times \mathbb{Z}. \end{array}$$

It is enough to show that Φ is injective. Take vector bundles $\mathscr{U}$ and $\mathscr{V}$ such that $\Phi(\mathscr{U}) = \Phi(\mathscr{V})$. That is, they satisfy $\det \mathscr{U} \cong \det \mathscr{V}$ and $r := \operatorname{rk} \mathscr{U} = \operatorname{rk} \mathscr{V}$. It follows from Fact 7.2.2 that $\mathscr{U}(n)$ and $\mathscr{V}(n)$ are globally generated for some same integer n. Then there are conflations

$$0 \to \mathscr{O}(-n)^{\oplus(r-1)} \to \mathscr{U} \to \mathscr{L} \to 0 \quad \text{and} \quad 0 \to \mathscr{O}(-n)^{\oplus(r-1)} \to \mathscr{V} \to \mathscr{M} \to 0$$

in $\mathsf{vect}\, C$ by Fact 7.2.3. Here $\mathscr{L}$ and $\mathscr{M}$ are line bundles. Then we have

$$\mathscr{L} = \det \mathscr{L} \cong \det \mathscr{U} \otimes \det \left(\mathscr{O}(-n)^{\oplus(r-1)}\right)^{\vee} \cong \det \mathscr{V} \otimes \det \left(\mathscr{O}(-n)^{\oplus(r-1)}\right)^{\vee} \cong \det \mathscr{M} = \mathscr{M}.$$

Hence, we obtain $[\mathscr{U}] = [\mathscr{L}] + (r-1)[\mathscr{O}(-n)] = [\mathscr{M}] + (r-1)[\mathscr{O}(-n)] = [\mathscr{V}]$ in $\mathsf{M}(\mathsf{vect}\, C)$. This proves Φ is injective.

(3) It follows from $\operatorname{Im} \Phi = (\operatorname{Pic} C \times \mathbb{N}^+) \cup \{(\mathscr{O}_C, 0)\}$. □

Corollary 7.2.5. *The exact category $\mathsf{vect}\, C$ has no nontrivial Serre subcategories.*

Proof. It is enough to show that the monoid $M := (\operatorname{Pic} C \times \mathbb{N}^+) \cup \{(\mathscr{O}_C, 0)\} \subseteq \operatorname{Pic} C \times \mathbb{Z}$ has no nontrivial faces by Proposition 3.3.3 and Proposition 7.2.4 (3). Let F be a nonzero face of M. There is $(\mathscr{L}, r) \in F$ such that $(\mathscr{L}, r) \neq (\mathscr{O}_C, 0)$. Then we have $(\mathscr{O}_C, 1) \in F$ since $2(\mathscr{L}, r) = (\mathscr{L}^{\otimes 2}, 2r-1) + (\mathscr{O}_C, 1)$ in M and F is a face. For any non-zero element $(\mathscr{M}, s) \in M$, we obtain $(\mathscr{M}, s) + (\mathscr{M}^{\vee}, s) = (\mathscr{O}_C, 2s) = 2s(\mathscr{O}_C, 1) \in F$, and thus $(\mathscr{M}, s) \in F$. This means $F = M$, and hence M has no nontrivial faces. □

7.3 The case of coherent sheaves

We finally deal with the case of the category $\operatorname{\mathsf{coh}} C$ of coherent sheaves. We begin with the relationship between $\operatorname{\mathsf{tor}} C$, $\operatorname{\mathsf{vect}} C$ and $\operatorname{\mathsf{coh}} C$.

Lemma 7.3.1. *The following hold.*
(1) $\operatorname{Hom}_{\mathscr{O}_C}(\mathscr{T}, \mathscr{V}) = 0$ *holds for all* $\mathscr{T} \in \operatorname{\mathsf{tor}} C$ *and* $\mathscr{V} \in \operatorname{\mathsf{vect}} C$.
(2) *For every coherent sheaf* $\mathscr{F}$ *on* C*, there exists an exact sequence*

$$0 \to \mathscr{F}_{\mathrm{tor}} \to \mathscr{F} \to \mathscr{F}_{\mathrm{vect}} \to 0$$

in $\operatorname{\mathsf{coh}} C$ *such that* $\mathscr{F}_{\mathrm{tor}} \in \operatorname{\mathsf{tor}} C$ *and* $\mathscr{F}_{\mathrm{vect}} \in \operatorname{\mathsf{vect}} C$.
In particular, $(\operatorname{\mathsf{tor}} C, \operatorname{\mathsf{vect}} C)$ *is a torsion pair in* $\operatorname{\mathsf{coh}} C$ *(see [Ste75, Section VI.2] for the definition).*

Proof. (1) Let $f\colon \mathscr{T} \to \mathscr{V}$ be a morphism from a coherent torsion sheaf to a vector bundle in $\operatorname{\mathsf{coh}} C$. Then $\mathscr{T}_x$ is a torsion module and $\mathscr{V}_x$ is a free module for any $x \in C$ by Lemma 7.1.3 and the definition of vector bundles. Hence $f_x\colon \mathscr{T}_x \to \mathscr{V}_x$ is equal to zero for all $x \in C$. This implies $f = 0$.

(2) Let η be the generic point of C and $K(C) := \mathscr{O}_{C,\eta}$ the function field of C. Consider the natural morphism $j\colon \operatorname{Spec} K(C) \to C$. Define a coherent sheaf $\mathscr{F}_{\mathrm{tor}}$ by the kernel of the unit morphism $\mathscr{F} \to j_*j^*\mathscr{F} = j_*\mathscr{F}_\eta$. Note that $j_*\mathscr{F}_\eta$ is a constant sheaf on C with value $\mathscr{F}_\eta$. Thus, we have $\mathscr{F}_{\mathrm{tor}}(U) = \{s \in \mathscr{F}(U) \mid s_\eta = 0\}$ for every open subset U of C. Then it is clear that $\mathscr{F}_{\mathrm{tor},\eta} = 0$, and thus $\mathscr{F}_{\mathrm{tor}}$ is a coherent torsion sheaf. Set $\mathscr{F}_{\mathrm{vect}} := \mathscr{F}/\mathscr{F}_{\mathrm{tor}} \in \operatorname{\mathsf{coh}} C$. Then $\mathscr{F}_{\mathrm{vect}}$ is a subsheaf of the constant sheaf $j_*\mathscr{F}_\eta$. Hence $(\mathscr{F}_{\mathrm{vect}})_x$ is an $\mathscr{O}_{C,x}$-submodule of $\mathscr{F}_\eta$ for every point $x \in C$. This implies $(\mathscr{F}_{\mathrm{vect}})_x$ is a torsionfree $\mathscr{O}_{C,x}$-module, and thus it is a free $\mathscr{O}_{C,x}$-module since $\mathscr{O}_{C,x}$ is a discrete valuation ring. For this reason, $\mathscr{F}_{\mathrm{vect}}$ is a vector bundle. □

We will determine the structure of the Grothendieck monoid $\mathsf{M}(\operatorname{\mathsf{coh}} C)$ in Proposition 7.3.2 below. For this, we recall the relation between divisors and line bundles. We can attach to a divisor D a line bundle $\mathscr{O}_C(D)$. It gives rise to a group homomorphism

$$\operatorname{Div} C \to \operatorname{Pic} C, \quad D \mapsto \mathscr{O}_C(D).$$

For any effective divisor $D = \sum_{i=1}^n n_i x_i$ on C, we set $\mathscr{O}_D := \bigoplus_{i=1}^n \mathscr{O}_{n_i x_i}$ (see the sentence before Lemma 7.1.3 for the definition of $\mathscr{O}_{nx}$). Then there is the following exact sequence in $\operatorname{\mathsf{coh}} C$:

$$0 \to \mathscr{O}_C(-D) \to \mathscr{O}_C \to \mathscr{O}_D \to 0. \tag{7.3.1}$$

Note that the abelian category $\operatorname{\mathsf{coh}} C$ is not length since there is an infinite subobject series of $\mathscr{O}_C$:

$$\cdots \subsetneq \mathscr{O}_C(-3x) \subsetneq \mathscr{O}_C(-2x) \subsetneq \mathscr{O}_C(-x) \subsetneq \mathscr{O}_C,$$

where x is a closed point of C. Thus we cannot use the results in §3.4.

Proposition 7.3.2. *The following hold.*
(1) *The inclusion functors* $\operatorname{\mathsf{tor}} C \hookrightarrow \operatorname{\mathsf{coh}} C$ *and* $\operatorname{\mathsf{vect}} C \hookrightarrow \operatorname{\mathsf{coh}} C$ *induce injective monoid homomorphisms* $\mathsf{M}(\operatorname{\mathsf{tor}} C) \hookrightarrow \mathsf{M}(\operatorname{\mathsf{coh}} C)$ *and* $\mathsf{M}(\operatorname{\mathsf{vect}} C) \hookrightarrow \mathsf{M}(\operatorname{\mathsf{coh}} C)$*, respectively.*
(2) *For any line bundle* $\mathscr{L}$ *and any effective divisor* D*, we have* $[\mathscr{L}] + [\mathscr{O}_D] = [\mathscr{L} \otimes \mathscr{O}_C(D)]$.
(3) $\mathsf{M}(\operatorname{\mathsf{coh}} C)$ *is the disjoint union of* $\mathsf{M}_{\operatorname{\mathsf{tor}} C}$ *and* $\mathsf{M}^+_{\operatorname{\mathsf{vect}} C} := \mathsf{M}_{\operatorname{\mathsf{vect}} C} \setminus \{0\}$ *as a set.*

Proof. (1) The natural monoid homomorphism $\mathsf{M}(\operatorname{\mathsf{tor}} C) \to \mathsf{M}(\operatorname{\mathsf{coh}} C)$ is injective by Proposition 3.3.5. We prove that the natural monoid homomorphism $\iota\colon \mathsf{M}(\operatorname{\mathsf{vect}} C) \to \mathsf{M}(\operatorname{\mathsf{coh}} C)$ is injective. Recall that $\mathsf{M}(\operatorname{\mathsf{vect}} C)$ is cancellative and the natural homomorphism $\mathsf{K}_0(\operatorname{\mathsf{vect}} C) \to \mathsf{K}_0(\operatorname{\mathsf{coh}} C)$ is an isomorphism by Proposition 7.2.4 and Fact 7.2.1. It follows that ι is injective by the following commutative diagram:

$$\begin{array}{ccc} \mathsf{M}(\operatorname{\mathsf{vect}} C) & \xrightarrow{\iota} & \mathsf{M}(\operatorname{\mathsf{coh}} C) \\ \downarrow & & \downarrow \\ \mathsf{K}_0(\operatorname{\mathsf{vect}} C) & \xrightarrow{\cong} & \mathsf{K}_0(\operatorname{\mathsf{coh}} C). \end{array}$$

(2) We first note that $\mathscr{T}\otimes\mathscr{L}\cong\mathscr{T}$ for any coherent torsion sheaf $\mathscr{T}$. Applying the exact functor $-\otimes(\mathscr{L}\otimes\mathscr{O}_C(D))\colon \operatorname{coh} C\cong\operatorname{coh} C$ to the exact sequence (7.3.1), we get an exact sequence

$$0\to\mathscr{L}\to\mathscr{L}\otimes\mathscr{O}_C(D)\to\mathscr{O}_D\to 0.$$

Hence, we have the equality $[\mathscr{L}]+[\mathscr{O}_D]=[\mathscr{L}\otimes\mathscr{O}_C(D)]$.

(3) For any coherent sheaf $\mathscr{F}$, there exists a coherent torsion sheaf $\mathscr{T}$ and a vector bundle $\mathscr{V}$ such that $[\mathscr{F}]=[\mathscr{T}]+[\mathscr{V}]$ by Lemma 7.3.1. Then there is an effective divisor D such that $\mathscr{T}\cong\mathscr{O}_D$. We can write $[\mathscr{V}]=\sum_{i=1}^{r}[\mathscr{L}_i]$ for some line bundles $\mathscr{L}_i$ by Proposition 7.2.4. If $\mathscr{V}$ is a nonzero vector bundle, we have

$$[\mathscr{F}]=[\mathscr{O}_D]+\sum_{i=1}^{r}[\mathscr{L}_i]=[\mathscr{L}_1\otimes\mathscr{O}_C(D)]+\sum_{i=2}^{r}[\mathscr{L}_i]=\left[(\mathscr{L}_1\otimes\mathscr{O}_C(D))\oplus\bigoplus_{i=2}^{r}\mathscr{L}_i\right]\in\mathsf{M}_{\mathsf{vect}\,C}.$$

This proves the desired conclusion. ◻

As a corollary of Proposition 7.3.2, we recover Fact 0.2.1 for smooth projective curves. See Definition 6.1.4 for the definition of specialization-closed subsets.

Corollary 7.3.3 (cf. [Gab62, Proposition VI.2.4])**.** *There is an inclusion-preserving bijection between the following sets:*

- *The set of Serre subcategories of* $\operatorname{coh} C$.
- *The set of specialization-closed subsets of* C.

Proof. It is enough to classify faces of $\mathsf{M}(\operatorname{coh} C)$ by Proposition 3.3.3. Let F be a face of $\mathsf{M}(\operatorname{coh} C)$. If $[\mathscr{V}]\in F$ for some nonzero vector bundle $\mathscr{V}$, it contains $\mathsf{M}_{\mathsf{vect}\,C}$ by Corollary 7.2.5. Then F must coincide with $\mathsf{M}(\operatorname{coh} C)$ by the exact sequence (7.3.1). Thus, if $F\neq\mathsf{M}(\operatorname{coh} C)$, it is contained in $\mathsf{M}_{\mathsf{tor}\,C}$. The faces of $\mathsf{M}(\operatorname{tor} C)$ bijectively correspond to the subsets of the set $C(\Bbbk)$ of closed points by Corollary 7.1.5. Extending this bijection by assigning $\mathsf{M}(\operatorname{coh} C)$ with C, we obtain the desired bijection. ◻

We also have the following corollary of Proposition 7.3.2. This corollary is used in Example 8.0.5 later.

Corollary 7.3.4. *Let C_1 and C_2 be smooth projective curves over an algebraically closed field $\Bbbk$. If $\mathsf{M}(\operatorname{coh} C_1)\cong\mathsf{M}(\operatorname{coh} C_2)$ as monoids, then $\operatorname{Pic} C_1\cong\operatorname{Pic} C_2$ as groups.*

Proof. We first recall some terminologies and a result. A nonzero element x of a monoid M is called an *atom* if $x=y+z$ for $y,z\in M$ implies either $y=0$ or $z=0$. We denote by $\mathsf{Atom}\,M$ the set of atoms in M. For a skeletally small exact category $\mathcal{E}$, we have the following bijection by [Eno22, Proposition 3.6] (see §3.4 for the notation):

$$\mathsf{sim}\,\mathcal{E}\xrightarrow{\cong}\mathsf{Atom}(\mathsf{M}(\mathcal{E})),\quad S\mapsto[S].$$

Thus $\mathsf{Atom}(\mathsf{M}(\operatorname{coh} C_i))=\{[\mathscr{O}_x]\mid x \text{ is a closed point of } C_i\}$ by Lemma 7.1.1.

Let $\phi\colon\mathsf{M}(\operatorname{coh} C_1)\xrightarrow{\cong}\mathsf{M}(\operatorname{coh} C_2)$ be a monoid isomorphism. Then ϕ preserves atoms, and hence it restricts to a monoid isomorphism $\mathsf{M}_{\mathsf{tor}\,C_1}\xrightarrow{\cong}\mathsf{M}_{\mathsf{tor}\,C_2}$. Since $\mathsf{M}(\operatorname{coh} C)=\mathsf{M}_{\mathsf{tor}\,C}\sqcup\mathsf{M}^{+}_{\mathsf{vect}\,C}$ by Proposition 7.3.2, the monoid isomorphism ϕ also restricts to $\mathsf{M}_{\mathsf{vect}\,C_1}\xrightarrow{\cong}\mathsf{M}_{\mathsf{vect}\,C_2}$. By Proposition 7.2.4, we have $\mathsf{Atom}(\mathsf{M}(\operatorname{vect} C_i))=\{[\mathscr{L}]\mid\mathscr{L}\text{ is a line bundle on } C_i\}$. Thus, there is a some line bundle $\mathscr{L}$ on C_2 such that $\phi([\mathscr{O}_{C_1}])=[\mathscr{L}]$. Twisting this isomorphism by $\mathsf{M}(-\otimes\mathscr{L}^{\vee})\colon\mathsf{M}(\operatorname{vect} C_2)\xrightarrow{\cong}\mathsf{M}(\operatorname{vect} C_2)$, we have a monoid isomorphism $\psi\colon\mathsf{M}(\operatorname{vect} C_1)\xrightarrow{\cong}\mathsf{M}(\operatorname{vect} C_2)$ such that $\psi([\mathscr{O}_{C_1}])=[\mathscr{O}_{C_2}]$.

It is enough to show that $\mathsf{M}(\operatorname{vect} C_i)/\langle[\mathscr{O}_{C_i}]\rangle_{\mathbb{N}}\cong\operatorname{Pic} C_i$ as monoids. Consider the monoid homomorphism $\det\colon\mathsf{M}(\operatorname{vect} C_i)\to\operatorname{Pic} C_i$. It induces a monoid homomorphism $\mathsf{M}(\operatorname{vect} C_i)/\langle[\mathscr{O}_{C_i}]\rangle_{\mathbb{N}}\to\operatorname{Pic} C$. It is clear that this homomorphism is surjective. We now prove that it is injective. Suppose that $\det\mathscr{U}\cong\det\mathscr{V}$ for some vector bundles $\mathscr{U}$ and $\mathscr{V}$. We may assume that $\operatorname{rk}\mathscr{U}\geq\operatorname{rk}\mathscr{V}$. Set $d:=\operatorname{rk}\mathscr{U}-\operatorname{rk}\mathscr{V}$. Then we have $\det(\mathscr{U})\cong\det\left(\mathscr{V}\oplus\mathscr{O}_{C_i}^{\oplus d}\right)$ and $\operatorname{rk}(\mathscr{U})=\operatorname{rk}\left(\mathscr{V}\oplus\mathscr{O}_{C_i}^{\oplus d}\right)$. Since $(\det,\operatorname{rk})\colon\mathsf{M}(\operatorname{vect} C_i)\to\operatorname{Pic} C_i\times\mathbb{N}$ is injective as proved in Proposition 7.2.4 (2), we have the following equality in $\mathsf{M}(\operatorname{vect} C_i)$:

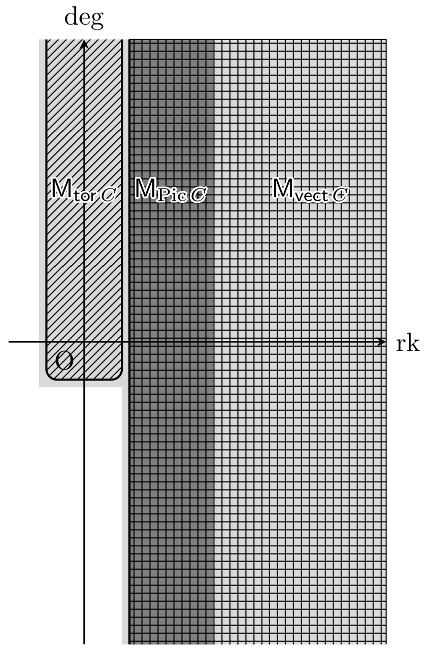
$$[\mathscr{U}]=\left[\mathscr{V}\oplus\mathscr{O}_{C_i}^{\oplus d}\right]=[\mathscr{V}]+d[\mathscr{O}_{C_i}].$$

This means that $[\mathscr{U}]\equiv[\mathscr{V}]\mod\langle[\mathscr{O}_{C_i}]\rangle_{\mathbb{N}}$, and hence the monoid homomorphism $\mathsf{M}(\operatorname{vect} C_i)/\langle[\mathscr{O}_{C_i}]\rangle_{\mathbb{N}}\to\operatorname{Pic} C$ is injective. ◻

Now we compare the Grothendieck monoid $\mathsf{M}(\mathsf{coh}\, C)$ with the Grothendieck group $\mathsf{K}_0(\mathsf{coh}\, C)$. There are unique group homomorphisms $\deg, \mathrm{rk}\colon \mathsf{K}_0(\mathsf{coh}\, C) \to \mathbb{Z}$ satisfying the following conditions (see [LeP97, Section 2.6]):

- $\mathrm{rk}(\mathscr{F}) = \mathrm{rk}(\mathscr{F}_{\mathrm{vect}})$ for any coherent sheaf $\mathscr{F}$ on C.
- $\deg(\mathscr{O}_C(D)) = \deg D$ for any divisor D on C.
- $\deg(\mathscr{O}_D) = \deg D$ for any effective divisor D on C.

The image of the map $(\mathrm{rk}, \deg)\colon \mathsf{K}_0(\mathsf{coh}\, C) \to \mathbb{Z}^{\oplus 2}$ is illustrated as follows:



Here the gray region corresponds to the Grothendieck monoid $\mathsf{M}(\mathsf{coh}\, C)$. Let $\rho\colon \mathsf{M}(\mathsf{coh}\, C) \to \mathsf{K}_0(\mathsf{coh}\, C)$ be the natural map. The map ρ is injective on $\mathsf{M}_{\mathsf{vect}\, C}$ by Proposition 7.2.4 and 7.3.2. Whereas, the map ρ loses a lot of information on $\mathsf{M}_{\mathsf{tor}\, C}$. Indeed, for two effective divisors D and E, the equality $[\mathscr{O}_D] = [\mathscr{O}_E]$ holds in $\mathsf{K}_0(\mathsf{coh}\, C)$ if and only if $\mathscr{O}_C(D) = \mathscr{O}_C(E)$ in $\mathrm{Pic}\, C$.

Example 7.3.5. Let $\mathbb{P}^1$ be the projective line. Then $\deg\colon \mathrm{Pic}\, C \to \mathbb{Z}$ is a group isomorphism (cf. [GW20, Example 11.45]). In particular, the map $(\mathrm{rk}, \deg)\colon \mathsf{K}_0(\mathsf{coh}\, \mathbb{P}^1) \to \mathbb{Z}^{\oplus 2}$ is a group isomorphism. For two effective divisors D and E on $\mathbb{P}^1$, the equality $[\mathscr{O}_D] = [\mathscr{O}_E]$ holds in $\mathsf{K}_0(\mathsf{coh}\, C)$ if and only if $\deg D = \deg E$. Thus, the map ρ loses all information except the degrees for torsion sheaves. In particular, the equality $[\mathscr{O}_x] = [\mathscr{O}_y]$ holds in $\mathsf{K}_0(\mathsf{coh}\, C)$ for any closed points $x, y \in \mathbb{P}^1(\Bbbk)$. Thus, the Grothendieck group $\mathsf{K}_0(\mathsf{coh}\, \mathbb{P}^1)$ has no information about closed points of $\mathbb{P}^1$. In contrast, the Grothendieck monoid $\mathsf{M}(\mathsf{coh}\, \mathbb{P}^1)$ remembers all closed points of $\mathbb{P}^1$ because $\mathsf{M}(\mathsf{coh}\, C) \supseteq \mathsf{M}_{\mathsf{tor}\, C} = \bigoplus_{x \in \mathbb{P}^1(\Bbbk)} \mathbb{N}[\mathscr{O}_x]$.

This example has another consequence. Let $\Bbbk Q$ be the path algebra of Kronecker quiver. It is well-known that the bounded derived categories $D^b(\mathsf{coh}\, \mathbb{P}^1)$ and $D^b(\mathsf{mod}\, \Bbbk Q)$ are triangulated equivalent. However, we have a monoid isomorphism $\mathsf{M}(\mathsf{mod}\, \Bbbk Q) \cong \mathbb{N}^{\oplus 2}$ by Fact 3.4.4. Thus $\mathsf{M}(\mathsf{coh}\, \mathbb{P}^1)$ and $\mathsf{M}(\mathsf{mod}\, \Bbbk Q)$ are not isomorphic as monoids. This implies the Grothendieck monoids are not derived invariants.

Finally, we will introduce the notion of the *twisted disjoint union* to describe the structure of $\mathsf{M}(\mathsf{coh}\, C)$ in terms of purely monoid-theoretic language. The rest of this section does not affect the other sections and can be skipped. We first recall the notion of a monoid action. Let M be a monoid. An *M-action* on a set X is a monoid homomorphism $\sigma\colon M \to \mathrm{End}_{\mathsf{Set}}(X) := \mathrm{Hom}_{\mathsf{Set}}(X, X)$. The pair $X = (X, \sigma)$ is called an *M-set*. Set $\sigma_m := \sigma(m)$ and $m \cdot x := \sigma_m(x)$ for all $m \in M$ and $x \in X$. A map $f\colon X \to Y$ between M-sets is *M-equivariant* if $f(m \cdot x) = m \cdot f(x)$ holds for all $m \in M$ and $x \in X$.

Let X, Y and Z be M-sets. A map $\alpha\colon X \times Y \to Z$ is an *M-bimorphism* if it satisfies $m \cdot \alpha(x, y) = \alpha(m \cdot x, y) = \alpha(x, m \cdot y)$ for all $m \in M$, $x \in X$ and $y \in Y$. An *M-semigroup* is an M-set S with an M-bimorphism $\alpha\colon S \times S \to S$ satisfying associativity and commutativity. In other words, it is a (commutative) semigroup S with an M-action satisfying $m \cdot (x+y) = m \cdot x + y = x + m \cdot y$ for all $m \in M$ and $x, y \in S$. An *M-semigroup homomorphism* is an M-equivariant map $f\colon S \to T$ satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in S$. We denote by $\mathsf{SemiGrp}_M$ the category of M-semigroups and M-semigroups homomorphisms.

Example 7.3.6. Let $\phi\colon M \to X$ be a monoid homomorphism. Then ϕ defines an action of M on X by $m \cdot x := \phi(m) + x$ for $m \in M$ and $x \in X$. We can easily check that X is an M-semigroup by this action.

Let M be a monoid, and let S be an M-semigroup whose action is given by $\sigma\colon M \to \operatorname{End}_{\mathsf{Set}}(S)$. The *twisted disjoint union* $M \sqcup_\sigma S$ of M and S is the set-theoretic disjoint union $M \sqcup S$ with a binary operation given by

$$x + y := \begin{cases} x +_M y & \text{if both } x \in M \text{ and } y \in M, \\ x +_S y & \text{if both } x \in S \text{ and } y \in S, \\ \sigma_x(y) & \text{if } x \in M \text{ and } y \in S, \\ \sigma_y(x) & \text{if } x \in S \text{ and } y \in M, \end{cases}$$

where $+_M$ (resp. $+_S$) denotes the binary operation on M (resp. S). We can check easily that $M \sqcup_\sigma S$ is a (commutative) monoid. The natural inclusion $i\colon M \hookrightarrow M \sqcup_\sigma S$ is a monoid homomorphism. Hence, we can think of $M \sqcup_\sigma S$ as an M-semigroup by Example 7.3.6. Then the natural inclusion $j\colon S \hookrightarrow M \sqcup_\sigma S$ is an M-semigroup homomorphism.

We describe a universal property of the twisted disjoint union. We denote by $\mathsf{Mon}_{M/}$ the slice category of Mon under a monoid M. That is, its objects are monoid homomorphisms $M \to X$, and morphisms between $\phi\colon M \to X$ and $\psi\colon M \to Y$ are monoid homomorphisms $f\colon X \to Y$ satisfying $f\phi = \psi$.

Proposition 7.3.7. *Let $\phi\colon M \to X$ be a monoid homomorphism. We regard X as an M-semigroup. Let S be an M-semigroup, and let $i\colon M \to M \sqcup_\sigma S$ and $j\colon S \to M \sqcup_\sigma S$ be the natural inclusions. Then there is a natural isomorphism*

$$\operatorname{Hom}_{\mathsf{Mon}_{M/}}(M \sqcup_\sigma S, X) \xrightarrow{\cong} \operatorname{Hom}_{\mathsf{SemiGrp}_M}(S, X), \quad h \mapsto hj.$$

Proof. We omit the proof since it is straightforward. □

Consider the Grothendieck monoid $\mathsf{M}(\mathsf{coh}\, C)$. Then $\mathsf{M}(\mathsf{coh}\, C)$ is an $\mathsf{M}_{\mathsf{tor}\, C}$-semigroup by the inclusion homomorphism $\mathsf{M}_{\mathsf{tor}\, C} \hookrightarrow \mathsf{M}(\mathsf{coh}\, C)$. The subsemigroup $\mathsf{M}^+_{\mathsf{vect}\, C} := \mathsf{M}_{\mathsf{vect}\, C} \setminus \{0\}$ is also an $\mathsf{M}_{\mathsf{tor}\, C}$-semigroup whose $\mathsf{M}_{\mathsf{tor}\, C}$-action is given by $\sigma_{[\mathscr{O}_D]}([\mathscr{V}]) := [\mathscr{O}_D] + [\mathscr{V}]$. Then the natural inclusion map $\mathsf{M}^+_{\mathsf{vect}\, C} \hookrightarrow \mathsf{M}(\mathsf{coh}\, C)$ is an $\mathsf{M}_{\mathsf{tor}\, C}$-semigroup homomorphism. It induces a monoid homomorphism $h\colon \mathsf{M}_{\mathsf{tor}\, C} \sqcup_\sigma \mathsf{M}^+_{\mathsf{vect}\, C} \to \mathsf{M}(\mathsf{coh}\, C)$ by Proposition 7.3.7. It is clear that h is an isomorphism. Thus the following statement follows.

Corollary 7.3.8. *There is a monoid isomorphism*

$$\operatorname{Div}^+(C) \sqcup_\sigma (\operatorname{Pic} C \times \mathbb{N}^+) \xrightarrow{\cong} \mathsf{M}(\mathsf{coh}\, C),$$

where the $\operatorname{Div}^+(C)$-action on $\operatorname{Pic} C \times \mathbb{N}^+$ is defined by $\sigma_D(\mathscr{L}, r) := (\mathscr{L} \otimes \mathscr{O}_C(D), r)$.

Chapter 8

Reconstruction of the topology of a noetherian scheme

Throughout this chapter, we fix Grothendieck universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$ and use Convention 1.1.12 and 2.2.5. We assume that all subcategories are strictly full subcategories. Hereafter X is a noetherian scheme.

In this chapter, we recover the topology of X from the Grothendieck monoid $\mathsf{M}(\operatorname{coh} X)$. We first construct an immersion from X to $\operatorname{Serre}(\operatorname{coh} X)$ as topological spaces. For any point $x \in X$, define a subcategory of $\operatorname{coh} X$ by

$$\operatorname{coh}^x X := \{\mathscr{F} \in \operatorname{coh} X \mid \mathscr{F}_x = 0\}.$$

It is clear that $\operatorname{coh}^x X$ is a Serre subcategory of $\operatorname{coh} X$. Let $j\colon X \to \operatorname{Serre}(\operatorname{coh} X)$ be a map defined by $j(x) := \operatorname{coh}^x X$.

Lemma 8.0.1. *The map $j\colon X \to \operatorname{Serre}(\operatorname{coh} X)$ is an immersion of topological spaces. That is, it is a homeomorphism onto a subspace of $\operatorname{Serre}(\operatorname{coh} X)$.*

Proof. We first prove that j is injective. Let $x, y \in X$ be distinct points. Since any scheme is T_0-space (cf. [GW20, Proposition 3.25]), the specialization order on X is a partial order by Remark 6.1.5 (2). Thus $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$ hold. We may assume that $x \notin \overline{\{y\}}$. Then $\mathscr{O}_{\overline{\{x\}}} \notin \operatorname{coh}^x X$ but $\mathscr{O}_{\overline{\{y\}}} \in \operatorname{coh}^x X$, where we consider $\overline{\{x\}}$ and $\overline{\{y\}}$ as reduced subschemes of X. Hence $\operatorname{coh}^x X \neq \operatorname{coh}^y X$, which proves j is injective. For a coherent sheaf $\mathscr{F}$ on X, we have

$$j^{-1}(U_{\mathscr{F}}) = \{x \in X \mid \mathscr{F} \in \operatorname{coh}^x X\} = \{x \in X \mid \mathscr{F}_x = 0\} = X \setminus \operatorname{Supp} \mathscr{F}.$$

Thus j is continuous. Let Z be a closed subset of X. We consider Z as a reduced subscheme of X. Then it is straightforward that $\operatorname{coh}^x X \in V(\{\mathscr{O}_Z\})$ if and only if $x \in \operatorname{Supp}(\mathscr{O}_Z) = Z$ for any $x \in X$. Thus, we have $j(Z) = j(X) \cap V(\{\mathscr{O}_Z\})$, and hence $j(Z)$ is a closed subset of $j(X)$. Therefore j is a homeomorphism onto the subspace $j(X)$ of $\operatorname{Serre}(\operatorname{coh} X)$. □

Next, we determine the image of the immersion $j\colon X \hookrightarrow \operatorname{Serre}(\operatorname{coh} X)$. A Serre subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ is *meet-irreducible* if $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{S}$ implies $\mathcal{X} \subseteq \mathcal{S}$ or $\mathcal{Y} \subseteq \mathcal{S}$ for any $\mathcal{X}, \mathcal{Y} \in \operatorname{Serre}(\mathcal{A})$.

Proposition 8.0.2. *For a Serre subcategory $\mathcal{S}$ of $\operatorname{coh} X$, it is meet-irreducible if and only if $\mathcal{S} = \operatorname{coh}^x X$ for some point $x \in X$. In particular, we have*

$$j(X) = \{\mathcal{S} \in \operatorname{Serre}(\operatorname{coh} X) \mid \mathcal{S} \text{ is meet-irreducible}\}.$$

Proof. By Gabriel's classification of Serre subcategories (Fact 0.2.1), there is a poset isomorphism

$$\operatorname{Spcl}(X) \xrightarrow{\cong} \operatorname{Serre}(\operatorname{coh} X), \quad Z \mapsto \operatorname{coh}_Z X := \{\mathscr{F} \in \operatorname{coh} X \mid \operatorname{Supp} \mathscr{F} \subseteq Z\},$$

where $\operatorname{Spcl}(X)$ is the set of specialization-closed subsets of X ordered by inclusion. Then for any $Z \in \operatorname{Spcl}(X)$, the Serre subcategory $\operatorname{coh}_Z X$ is meet-irreducible if and only if so is Z in the following sense:

- A specialization-closed subset Z is *meet-irreducible* if $A \cap B \subseteq Z$ implies $A \subseteq Z$ or $B \subseteq Z$ for any $A, B \in \operatorname{Spcl}(X)$.

We also introduce the dual notion for generalization-closed subsets. We denote by $\operatorname{Genl}(X)$ the set of generalization-closed subsets of X.

- A generalization-closed subset U is *join-irreducible* if $A \cup B \supseteq U$ implies $A \supseteq U$ or $B \supseteq U$ for any $A, B \in \operatorname{Genl}(X)$.

Then $Z \in \operatorname{Spcl}(X)$ is meet-irreducible if and only if $Z^c (:= X \setminus Z) \in \operatorname{Genl}(X)$ is join-irreducible. Therefore, it is equivalent that determining meet-irreducible Serre subcategories of $\mathsf{coh}\, X$ and determining join-irreducible generalization-closed subsets of X.

We now determine join-irreducible generalization-closed subsets of X. For any point $x \in X$, we denote by $\langle x \rangle_{\mathrm{genl}}$ the set of generalizations of x. It is clear that $\langle x \rangle_{\mathrm{genl}}$ is a join-irreducible generalization-closed subset for any $x \in X$. We prove that any join-irreducible generalization-closed subset is of the form $\langle x \rangle_{\mathrm{genl}}$ for some $x \in X$. Let $A \in \operatorname{Genl}(X)$ be join-irreducible. For any $x \in A$, there is a minimal element $y \in A$ with respect to the specialization-order such that $y \preceq x$. Indeed, if $\overline{\{x\}} \cap A$ has exactly one point x, then x itself is a minimal element of A. If $\overline{\{x\}} \cap A$ contains a point x_1 such that $x \neq x_1$, then we have a sequence $x \succeq x_1$ of points of A. Repeating this operation, we have a sequence $x \succeq x_1 \succeq x_2 \succeq \cdots$ of points of A. This sequence terminates since X is noetherian. Thus, we get a minimal element $y \in A$ such that $y \preceq x$. Let I be the set of minimal elements of A. Then $A = \bigcup_{a \in I} \langle a \rangle_{\mathrm{genl}}$ by the discussion above. Fix $x \in I$. Since

$$A = \langle x \rangle_{\mathrm{genl}} \cup \bigcup_{a \in I, a \neq x} \langle a \rangle_{\mathrm{genl}}$$

and A is join-irreducible, we have that $A = \langle x \rangle_{\mathrm{genl}}$ or $A = \bigcup_{a \in I, a \neq x} \langle a \rangle_{\mathrm{genl}}$. If $A = \bigcup_{a \in I, a \neq x} \langle a \rangle_{\mathrm{genl}}$, then $x \in A = \bigcup_{a \in I, a \neq x} \langle a \rangle_{\mathrm{genl}}$. Hence, there is $a \in I$ such that $a \neq x$ and $x \in \langle a \rangle_{\mathrm{genl}}$, which contradicts the minimality of x. Thus we have $A = \langle x \rangle_{\mathrm{genl}}$.

We have proved that a subset A of X is join-irreducible generalization-closed subsets if and only if $A = \langle x \rangle_{\mathrm{genl}}$ for some $x \in X$. We can easily see that $\mathscr{F}_x = 0$ if and only if $\operatorname{Supp} \mathscr{F} \subseteq \langle x \rangle_{\mathrm{genl}}^c$ for any $\mathscr{F} \in \mathsf{coh}\, X$ because $\operatorname{Supp} \mathscr{F}$ is specialization-closed. Thus $\mathsf{coh}_{\langle x \rangle_{\mathrm{genl}}^c} X = \mathsf{coh}^x X$, which proves the proposition. □

Let X be a noetherian scheme. Define $\operatorname{Serre}(\mathsf{coh}\, X)_{\mathrm{irred}}$ by the set of meet-irreducible Serre subcategories of $\mathsf{coh}\, X$. We consider it as a subspace of $\operatorname{Serre}(\mathsf{coh}\, X)$. Then the immersion $j \colon X \xrightarrow{\cong} \operatorname{Serre}(\mathsf{coh}\, X)$ induces a homeomorphism $X \xrightarrow{\cong} \operatorname{Serre}(\mathsf{coh}\, X)_{\mathrm{irred}}$ by Lemma 8.0.1. Thus, we can recover the topological space X from the topological space $\operatorname{Serre}(\mathsf{coh}\, X)$. In particular, the Grothendieck monoid $\mathsf{M}(\mathsf{coh}\, X)$ recovers the topology of X. Moreover, $\operatorname{Serre}(\mathsf{coh}\, X)_{\mathrm{irred}}$ has the following property.

Lemma 8.0.3. *Let X and Y be noetherian schemes. Any homeomorphism $\operatorname{Serre}(\mathsf{coh}\, X) \xrightarrow{\cong} \operatorname{Serre}(\mathsf{coh}\, Y)$ restricts to a homeomorphism $\operatorname{Serre}(\mathsf{coh}\, X)_{\mathrm{irred}} \xrightarrow{\cong} \operatorname{Serre}(\mathsf{coh}\, Y)_{\mathrm{irred}}$.*

Proof. Let $\Psi \colon \operatorname{Serre}(\mathsf{coh}\, X) \xrightarrow{\cong} \operatorname{Serre}(\mathsf{coh}\, Y)$ be a homeomorphism. Then it is clear that Ψ is also a poset isomorphism $\operatorname{Serre}(\mathsf{coh}\, X)_{\mathrm{spcl}} \xrightarrow{\cong} \operatorname{Serre}(\mathsf{coh}\, Y)_{\mathrm{spcl}}$. Thus Ψ is an isomorphism of the poset $\operatorname{Serre}(\mathsf{coh}\, X)$ and $\operatorname{Serre}(\mathsf{coh}\, Y)$ ordered by inclusion by Proposition 6.2.2 (3). Therefore Ψ preserves meet-irreducible Serre subcategories, and hence we have $\Psi(\operatorname{Serre}(\mathsf{coh}\, X)_{\mathrm{irred}}) = \operatorname{Serre}(\mathsf{coh}\, Y)_{\mathrm{irred}}$. □

Based on the above considerations, we obtain the following.

Theorem 8.0.4. *Consider the following conditions for noetherian schemes X and Y.*

(1) *$X \cong Y$ as schemes.*
(2) *$\mathsf{M}(\mathsf{coh}\, X) \cong \mathsf{M}(\mathsf{coh}\, Y)$ as monoids.*
(3) *$\operatorname{MSpec} \mathsf{M}(\mathsf{coh}\, X) \cong \operatorname{MSpec} \mathsf{M}(\mathsf{coh}\, Y)$ as topological spaces.*
(4) *$\operatorname{Serre}(\mathsf{coh}\, X) \cong \operatorname{Serre}(\mathsf{coh}\, Y)$ as topological spaces.*
(5) *$X \cong Y$ as topological spaces.*

Then "(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5)" hold.

Proof. The implications "(1) $\Rightarrow$ (2) $\Rightarrow$ (3)" are obvious. The equivalence "(3) $\Leftrightarrow$ (4)" follows from Proposition 6.2.2. The implication "(4) $\Rightarrow$ (5)" follows from Lemma 8.0.1 and 8.0.3. □

The implications "(2) ⇒ (1)" and "(5) ⇒ (2)" of Theorem 8.0.4 do not hold. Example 0.2.2 (1) is a counter example of the former. We now give a counter example of the latter. The following example is suggested by Professor Kazuhiro Fujiwara and Professor Sho Tanimoto.

Example 8.0.5. Let C be a smooth projective curve over a field $\Bbbk$. Then C has 1 generic point and $\max\{|\mathbb{N}|, |\Bbbk|\}$ closed points. Hence, the underlying topological space of C depends only on the cardinality of $\Bbbk$. Consider the projective line $\mathbb{P}^1$ and an elliptic curve E over the complex number field $\mathbb{C}$. Then $\mathbb{P}^1 \cong E$ as topological spaces by the discussion above. On the other hand, we can deduce $\operatorname{Pic}\mathbb{P}^1 \ncong \operatorname{Pic} E$ since $\operatorname{Pic}\mathbb{P}^1 \cong \mathbb{Z}$ is countable but $\operatorname{Pic} E \cong E(\mathbb{C}) \times \mathbb{Z}$ is uncountable. Here $E(\mathbb{C})$ is the group of $\mathbb{C}$-rational points of the elliptic curve E. Thus, we can conclude that $\mathsf{M}(\mathsf{coh}\,\mathbb{P}^1) \ncong \mathsf{M}(\mathsf{coh}\,E)$ by Corollary 7.3.4.

We finally comment on [BKS07] and our approach.

Remark 8.0.6. Let X be a noetherian scheme. Buan, Krause and Solberg reconstructed the topological space X from the poset $\operatorname{Serre}(\mathsf{coh}\,X)$ of Serre subcategories in [BKS07]. We review their approach and compare it with ours.

We first recall the spectrum of a frame. See [BKS07] and [PP12] for detailed explanations. A *frame* is a poset $L = (L, \leq)$ satisfying the following conditions:

- L is a complete lattice, that is, any subset A of L admits a supremum $\sup A = \bigvee_{a\in A} a$ and an infimum $\inf A = \bigwedge_{a\in A} a$. We denote by $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$.
- L satisfies the distributed law:
$$(\bigvee_{a\in A} a) \wedge b = \bigvee_{a\in A} (a \wedge b)$$
for any subset $A \subseteq L$ and any element $b \in L$.

An element p of a frame L is *meet-irreducible* if $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$ for any $x, y \in L$. The set $\operatorname{LSpec} L$ of meet-irreducible elements of L is called the *lattice spectrum* of L. The set $\operatorname{LSpec} L$ has a topology whose closed subsets are of the form

$$V(a) := \{p \in \operatorname{LSpec} L \mid a \leq p\}, \quad a \in L.$$

We can endow the underlying set $\operatorname{LSpec} L$ with a new topology by taking subsets of the following form to be the open subsets:

$$Y = \bigcup_{i\in I} Y_i \quad \text{such that } X \setminus Y_i \text{ is a quasi-compact open in } \operatorname{LSpec} L \text{ for all } i \in I. \tag{8.0.1}$$

We denote this new space by $\operatorname{LSpec}^* L$ and call this topology the *dual topology* of $\operatorname{LSpec} L$, which was introduced by Hochster in [Hoc69].

We now come back to the case $L = \operatorname{Serre}(\mathsf{coh}\,X)$. Buan, Krause and Solberg proved that X and $\operatorname{LSpec}^*(\operatorname{Serre}(\mathsf{coh}\,X))$ are homeomorphic for any noetherian scheme X in [BKS07, Section 9] by using Hochster duality [Hoc69, Proposition 8]. This gives another proof of "(4) ⇒ (5)" in Theorem 8.0.4. Indeed, consider the following condition:

(4.5) $\operatorname{Serre}(\mathsf{coh}\,X) \cong \operatorname{Serre}(\mathsf{coh}\,Y)$ as posets.

Then "(4) ⇒ (4.5)" holds by Proposition 6.2.2 (3). If $\operatorname{Serre}(\mathsf{coh}\,X) \cong \operatorname{Serre}(\mathsf{coh}\,Y)$ as posets, then we have homeomorphisms

$$X \cong \operatorname{LSpec}^*(\operatorname{Serre}(\mathsf{coh}\,X)) \cong \operatorname{LSpec}^*(\operatorname{Serre}(\mathsf{coh}\,Y)) \cong Y.$$

Thus "(4.5) ⇒ (5)" holds.

We now compare our approach with that of [BKS07]. An advantage of our approach is that we can avoid the dual topology and Hochster duality. Although $\operatorname{Serre}(\mathsf{coh}\,X)_{\mathrm{irred}} = \operatorname{LSpec}(\operatorname{Serre}(\mathsf{coh}\,X))$ as subsets of $\operatorname{Serre}(\mathsf{coh}\,X)$, they have different topologies. $\operatorname{Serre}(\mathsf{coh}\,X)$ has the correct topology in the sense that X can be embedded in $\operatorname{Serre}(\mathsf{coh}\,X)$ as a topological space.

On the other hand, an advantage of the approach of [BKS07] is that it is more general than ours. Indeed, we can recover the poset structure of $\operatorname{Serre}(\mathsf{coh}\,X)$ from the topology of $\operatorname{Serre}(\mathsf{coh}\,X)$ by considering the specialization-order (Proposition 6.2.2 (3)). However, the author does not know whether the topology on $\operatorname{Serre}(\mathsf{coh}\,X)$ which we introduced in this paper can be recovered from the poset structure of $\operatorname{Serre}(\mathsf{coh}\,X)$. In particular, we cannot prove "(4.5) ⇒ (5)" by our approach.

In summary, our approach is simpler than [BKS07] and avoids heavy facts such as Hochster duality, but the approach of [BKS07] is more general than ours.

Chapter 9

The Grothendieck groups of periodic derived categories

The m-periodic derived category of an abelian category is a natural $\mathbb{Z}/m\mathbb{Z}$-periodic analogue of the usual derived category. In this chapter, we determine the Grothendieck group (= Grothendieck monoid by Proposition 2.4.3) of the periodic derived category of a skeletally small abelian category with enough projectives. In particular, we prove that the Grothendieck group of the m-periodic derived category of finitely generated modules over an Artin algebra is a free $\mathbb{Z}$-module if m is even, and is an $\mathbb{F}_2$-vector space if m is odd. Moreover, in both cases of parity of m, the rank of the Grothendieck group is equal to the number of isomorphism classes of simple modules in both cases. As an application, we prove that the number of non-isomorphic summands of a strict periodic tilting object T, which was introduced by the author in [Sai1] as a periodic analogue of tilting objects, is independent of the choice of T.

Throughout this chapter, we fix Grothendieck universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$ and use Convention 1.1.12 and 2.2.5. Moreover, we assume that every category, functor, and subcategory is additive. In particular, every subcategory is strictly full and nonempty. We denote by Σ the suspension functor of a triangulated category. For a positive integer m, we denote by $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ the cyclic group of order m.

9.1 Grothendieck groups of periodic triangulated categories

In this section, we investigate properties of the Grothendieck group of a periodic triangulated category.

Definition 9.1.1. Let $\mathcal{T}$ be a triangulated category.

(1) For a positive integer $m > 0$, $\mathcal{T}$ is *m-periodic* if $\Sigma^m \simeq \mathrm{Id}_{\mathcal{T}}$ as additive functors.
(2) The *period* of $\mathcal{T}$ is the smallest positive integer m such that $\mathcal{T}$ is m-periodic.

9.1.1 Even periodic case

Let m be an even integer, and $\mathcal{T}$ be a skeletally small m-periodic triangulated category.

Lemma 9.1.2. *A cohomological functor $F\colon \mathcal{T} \to \mathcal{A}$ induces a homomorphism*

$$\phi\colon \mathsf{K}_0(\mathcal{T}) \longrightarrow \mathsf{K}_0(\mathcal{A}), \quad [X] \longmapsto \sum_{i=0}^{m-1} (-1)^i \left[F^i(X)\right].$$

Proof. We need to show that for any exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}$, the equality $\phi(X) - \phi(Y) + \phi(Z) = 0$ holds in $\mathsf{K}_0(\mathcal{A})$. We have two exact sequences in $\mathcal{A}$:

$$F^{m-1}(Z) \xrightarrow{f} F^m(X) \simeq F^0(X) \xrightarrow{g} F^0(Y), \tag{9.1.1}$$

$$0 \to \operatorname{Ker} g \to F^0(X) \xrightarrow{g} F^0(Y) \to \cdots \to F^{m-1}(Y) \to F^{m-1}(Z) \xrightarrow{f} \operatorname{Im} f \to 0. \tag{9.1.2}$$

By (9.1.2), we have $[\operatorname{Ker} g] = \phi(X) - \phi(Y) + \phi(Z) + (-1)^m[\operatorname{Im} f]$. Then (9.1.1) and the assumption that m is even imply the equality $\phi(X) - \phi(Y) + \phi(Z) = 0$. □

9.1.2 Odd periodic case

Let m be an odd integer, and $\mathcal{T}$ be a skeletally small m-periodic triangulated category.

Lemma 9.1.3. *$\mathsf{K}_0(\mathcal{T})$ is an $\mathbb{F}_2$-vector space. That is, for any element $\alpha \in \mathsf{K}_0(\mathcal{T})$, we have $2\alpha = 0$.*

Proof. By the axiom of triangulated categories, a triangle $X \to 0 \to \Sigma X \to \Sigma X$ is exact for any $X \in \mathcal{T}$. It implies $[\Sigma X] = -[X]$ in $\mathsf{K}_0(\mathcal{T})$. Hence, we have $[X] = [\Sigma^m X] = (-1)^m[X] = -[X]$, which yields $2[X] = 0$ for any $X \in \mathcal{T}$. □

Lemma 9.1.4. *A cohomological functor $F \colon \mathcal{T} \to \mathcal{A}$ induces a homomorphism*

$$\phi_{\mathbb{F}_2} \colon \mathsf{K}_0(\mathcal{T}) \longrightarrow \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2}, \quad [X] \longmapsto \sum_{i=0}^{m-1} (-1)^i \left[F^i(X)\right] \bmod 2\mathsf{K}_0(\mathcal{A}),$$

where $\mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2} := \mathsf{K}_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \simeq \mathsf{K}_0(\mathcal{A})/2\mathsf{K}_0(\mathcal{A})$.

Proof. Let $X \to Y \to Z \to \Sigma X$ be an exact triangle in $\mathcal{T}$. Then we have, for some object $K \in \mathcal{A}$,

$$[K] = \phi_{\mathbb{F}_2}(X) - \phi_{\mathbb{F}_2}(Y) + \phi_{\mathbb{F}_2}(Z) + (-1)^m[K]$$

in $\mathsf{K}_0(\mathcal{A})$ by the same calculation as in Lemma 9.1.2. Since m is odd, we get $\phi_{\mathbb{F}_2}(X) - \phi_{\mathbb{F}_2}(Y) + \phi_{\mathbb{F}_2}(Z) = 2[K] \equiv 0 \bmod 2\mathsf{K}_0(\mathcal{A})$. Hence, the map $[\mathcal{T}] \to \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2}$, $[X] \mapsto \phi_{\mathbb{F}_2}(X) \bmod 2\mathsf{K}_0(\mathcal{A})$ induces a homomorphism $\phi_{\mathbb{F}_2} \colon \mathsf{K}_0(\mathcal{T}) \to \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2}$. □

9.2 Preliminaries: periodic derived categories

In this section, we give a review of periodic derived categories. See [Sai1, §3] for a detailed account. Let us fix a positive integer m.

Definition 9.2.1. Let $\mathcal{C}$ be an additive category.

(1) An *m-periodic complex* V is a family $(V, d_V) = (V^i, d_V^i)_{i \in \mathbb{Z}_m}$ of objects $V_i \in \mathcal{C}$ and morphisms $d_V^i \colon V^i \to V^{i+1}$ in $\mathcal{C}$ satisfying $d_V^{i+1} d_V^i = 0$ for all $i \in \mathbb{Z}_m$.

(2) A *periodic chain map* $f \colon V \to W$ between m-periodic complexes V and W is a family $(f_i)_{i \in \mathbb{Z}_m}$ of morphisms in $\mathcal{C}$ satisfying $f^{i+1} d_V^i = d_W^i f^i$ for all $i \in \mathbb{Z}_m$.

(3) $\mathsf{C}_m(\mathcal{C})$ denotes the category of m-periodic complexes and periodic chain maps.

We can say roughly that an m-periodic complex is a $\mathbb{Z}_m$-graded complex. Replacing $\mathbb{Z}_m$ by $\mathbb{Z}$ in the above Definition 9.2.1, we recover the usual notion of complexes, chain maps and their category.

Example 9.2.2. Let $\mathcal{C}$ be an additive category.

(1) A 1-periodic complex is a morphism $d \colon V \to V$ in $\mathcal{C}$ with $d^2 = 0$.

(2) A 2-periodic complex is a diagram $V^0 \underset{d^1}{\overset{d^0}{\rightleftarrows}} V^1$ in $\mathcal{C}$ with $d^1 d^0 = d^0 d^1 = 0$.

Two periodic chain maps $f, g \colon V \to W$ of m-periodic complexes are called *homotopic* if there exist $s^i \colon V^i \to W^{i-1}$ $(i \in \mathbb{Z}_m)$ such that $f^i - g^i = d_W^{i-1} s^i + s^{i+1} d_V^i$ for all $i \in \mathbb{Z}_m$. This homotopy condition defines an equivalence relation $\sim_h$ on the set $\mathrm{Hom}_{\mathsf{C}_m(\mathcal{C})}(V, W)$. The *homotopy category* $\mathsf{H}_{\mathrm{m}}(\mathcal{C})$ *of m-periodic complexes* is the category whose objects are the same as those of $\mathsf{C}_m(\mathcal{C})$, and whose morphism set is defined by $\mathrm{Hom}_{\mathsf{H}_{\mathrm{m}}(\mathcal{C})}(V, W) = \mathrm{Hom}_{\mathsf{C}_m(\mathcal{C})}(V, W)/{\sim_h}$. The shift functor $[1] : \mathsf{H}_{\mathrm{m}}(\mathcal{C}) \to \mathsf{H}_{\mathrm{m}}(\mathcal{C})$ is defined by

$$V \mapsto V[1] := (V^{i+1}, -d_V^{i+1})_{i \in \mathbb{Z}_m}.$$

The homotopy category $\mathsf{H}_{\mathrm{m}}(\mathcal{C})$ is a triangulated category with suspension functor $[1] : \mathsf{H}_{\mathrm{m}}(\mathcal{C}) \to \mathsf{H}_{\mathrm{m}}(\mathcal{C})$.

For an abelian category $\mathcal{A}$, the category $\mathsf{C}_m(\mathcal{A})$ is also an abelian category, where a sequence $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ in $\mathsf{C}_m(\mathcal{A})$ is exact if and only if $0 \to U^i \xrightarrow{f^i} V^i \xrightarrow{g^i} W^i \to 0$ is exact in $\mathcal{A}$ for all $i \in \mathbb{Z}_m$. Define the *ith cohomology* of $V \in \mathsf{C}_m(\mathcal{A})$ by $H^i(V) := \operatorname{Ker} d_V^i / \operatorname{Im} d_V^{i-1}$ for $i \in \mathbb{Z}_m$. It gives rise to a functor $H^i \colon \mathsf{H}_{\mathrm{m}}(\mathcal{A}) \to \mathcal{A}$ for all $i \in \mathbb{Z}_m$. A periodic chain map $f \colon V \to W$ of m-periodic complexes is called a *quasi-isomorphism* if $H^i(f) \colon H^i(V) \to H^i(W)$ is an isomorphism for all $i \in \mathbb{Z}_m$.

Definition 9.2.3. For an abelian category $\mathcal{A}$, the *m-periodic derived category* $\mathsf{D}_m(\mathcal{A})$ is the localization of $\mathsf{H}_{\mathrm{m}}(\mathcal{A})$ with respect to quasi-isomorphisms.

By the standard argument, one can find that the category $\mathsf{D}_m(\mathcal{A})$ is a triangulated category and the localization functor $\mathsf{H}_{\mathrm{m}}(\mathcal{A}) \to \mathsf{D}_m(\mathcal{A})$ is a triangulated functor.

Remark 9.2.4. The periodic derived category is a typical example of periodic triangulated categories. A caution is that *the period of* $\mathsf{D}_m(\mathcal{A})$ *is not necessarily equal to* m, and it actually depends on the parity of m. By [Sai1, Proposition 5.1], we have the following three cases for the period p of $\mathsf{D}_m(\mathcal{A})$.
(i) m is even and $p = m$.
(ii) m is odd and $p = m$.
(iii) m is odd and $p = 2m$.
This phenomenon, which might look strange at first glance, is caused by the change of signs of differential by the shift functor [1]. For example, the shift of a 1-periodic complex is $(M, d)[1] = (M, -d)$. Hence (M, d) and $(M, d)[1]$ are not isomorphic in general.

The ith cohomology functor $H^i\colon \mathsf{H}_{\mathrm{m}}(\mathcal{A}) \to \mathcal{A}$ induces a functor $\mathsf{D}_m(\mathcal{A}) \to \mathcal{A}$. We also denote the latter by $H^i\colon \mathsf{D}_m(\mathcal{A}) \to \mathcal{A}$. One of the advantage of considering the localized category $\mathsf{D}_m(\mathcal{A})$ is that an exact sequence in $\mathcal{A}$ yields an exact triangle in $\mathsf{D}_m(\mathcal{A})$.

Fact 9.2.5 ([Sai1, Propostions 3.12, 3.19])**.** *Let $\mathcal{A}$ be an abelian category.*
(1) *The natural functor $\mathsf{C}_m(\mathcal{A}) \to \mathsf{D}_m(\mathcal{A})$ is an exact functor of extriangulated categories.*
(2) *The ith cohomology functor $H^i\colon \mathsf{D}_m(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor.*

The purpose of this paper is to study the Grothendieck group of a periodic derived category. First, we should find out the condition of a periodic derived category under which we can define the Grothendieck group. It turns out to be sufficient to find the condition under which the periodic derived category considered is skeletally small. We will give a solution of this problem in Proposition 9.2.8 below.

Let us give a few preliminary for Proposition 9.2.8. For an abelian category $\mathcal{A}$, we denote by $\mathsf{Proj}\,\mathcal{A}$ the full subcategory of $\mathcal{A}$ consisting of projective objects.

Fact 9.2.6 ([Gor, Lemma 9.5], c.f. [Sai1, Corollary 3.28])**.** *Let $\mathcal{A}$ be an abelian category of finite global dimension with enough projectives. Then the natural functor $\mathsf{H}_{\mathrm{m}}(\mathsf{Proj}\,\mathcal{A}) \to \mathsf{D}_m(\mathcal{A})$ is a triangulated equivalence.*

Fact 9.2.7 ([Gor, Proposition 9.7], c.f. [Sai1, Lemma 3.26])**.** *Let $\mathcal{A}$ be an abelian category of finite global dimension with enough projectives. Then the smallest triangulated subcategory of $\mathsf{D}_m(\mathcal{A})$ containing $\mathcal{A}$ coincides with $\mathsf{D}_m(\mathcal{A})$.*

These facts yield the following statements on the Grothendieck group of a periodic derived category.

Proposition 9.2.8. *Let $\mathcal{A}$ be a skeletally small abelian category of finite global dimension with enough projectives.*
(1) *The periodic derived category $\mathsf{D}_m(\mathcal{A})$ is skeletally small. In particular, we can define the Grothendieck group of $\mathsf{D}_m(\mathcal{A})$.*
(2) *The natural functor $\mathcal{A} \to \mathsf{D}_m(\mathcal{A})$ is an exact functor of extriangulated categories. In particular, we have an induced homomorphism $\psi\colon \mathsf{K}_0(\mathcal{A}) \to \mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$.*
(3) *The smallest triangulated subcategory of $\mathsf{D}_m(\mathcal{A})$ containing $\mathcal{A}$ coincides with $\mathsf{D}_m(\mathcal{A})$. In particular, the homomorphism $\psi\colon \mathsf{K}_0(\mathcal{A}) \to \mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$ is surjective.*

Proof. (1) It is not obvious in general that $\mathrm{Hom}_{\mathsf{D}_m(\mathcal{A})}(V, W)$ forms a set since $\mathsf{D}_m(\mathcal{A})$ is a localization of $\mathsf{H}_{\mathrm{m}}(\mathcal{A})$. However, it is enough to show that $\mathsf{H}_{\mathrm{m}}(\mathsf{Proj}\,\mathcal{A})$ is skeletally small since the natural functor $\mathsf{H}_{\mathrm{m}}(\mathsf{Proj}\,\mathcal{A}) \xrightarrow{\sim} \mathsf{D}_m(\mathcal{A})$ is an equivalence by Fact 9.2.6. Note that $\mathsf{Mor}\,\mathcal{A} = \bigcup_{M,N \in |\mathcal{A}|} \mathrm{Hom}_{\mathcal{A}}(M, N)$ forms a set since $\mathcal{A}$ is skeletally small. Then $\mathrm{Hom}_{\mathsf{H}_{\mathrm{m}}(\mathcal{A})}(V, W)$ is a quotient of the set $\mathrm{Hom}_{\mathsf{C}_m(\mathcal{A})}(V, W) \subseteq \prod_{i=0}^{m-1} \mathrm{Hom}_{\mathcal{A}}(V^i, W^i)$ and $|\mathsf{H}_{\mathrm{m}}(\mathsf{Proj}\,\mathcal{A})|$ is a subset of the set $\prod_{i=0}^{m-1} \mathsf{Mor}\,\mathcal{A}$. Thus $\mathsf{H}_{\mathrm{m}}(\mathsf{Proj}\,\mathcal{A})$ is skeletally small.

(2) Since the natural inclusion $\mathcal{A} \hookrightarrow \mathsf{C}_m(\mathcal{A})$ and the natural functor $\mathsf{C}_m(\mathcal{A}) \to \mathsf{D}_m(\mathcal{A})$ are exact, their composition $\mathcal{A} \to \mathsf{D}_m(\mathcal{A})$ is also exact.

(3) For a collection S of objects in a triangulated category, it is well-known that every object of the smallest triangulated category containing S is a (finite) iterated extension of shifts of objects belonging

to S. Hence, Fact 9.2.7 implies that each object of $\mathsf{D}_m(\mathcal{A})$ is an iterated extension of shifts of objects of $\mathcal{A}$, and thus ψ is surjective. $\square$

Finally, we explain the relation between periodic complexes and usual complexes. We will not use the following results and explanations in the rest of this paper but they give a good picture of periodic derived categories.

Let $\mathcal{A}$ be an abelian category. The symbol $\mathsf{C}(\mathcal{A})$ (resp. $\mathsf{C}^b(\mathcal{A})$) denotes the category of complexes (resp. bounded complexes) over $\mathcal{A}$. We introduce the functors ι and π as follows.

$$\iota\colon \mathsf{C}_m(\mathcal{A}) \longrightarrow \mathsf{C}(\mathcal{A}), \quad \iota(V) := \left(V^{(i \bmod m)}, d_V^{(i \bmod m)}\right)_{i\in\mathbb{Z}},$$
$$\pi\colon \mathsf{C}^b(\mathcal{A}) \longrightarrow \mathsf{C}_m(\mathcal{A}), \quad \pi(V) := \left(\bigoplus_{j\equiv i \bmod m} V^j, \bigoplus_{j\equiv i \bmod m} d_V^j\right)_{i\in\mathbb{Z}_m}.$$

The functors ι and π preserve quasi-isomorphisms, and induce triangulated functors $\iota\colon \mathsf{D}_m(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ and $\pi\colon \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathsf{D}_m(\mathcal{A})$, respectively. The functor $\pi\colon \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathsf{D}_m(\mathcal{A})$ is called the *covering functor*. This name comes from the following fact.

Fact 9.2.9 ([Sai1, Corollary 3.29])**.** *Let $\mathcal{A}$ be a skeletally small abelian category of finite global dimension with enough projectives. Then, for any $V, W \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$, we have*

$$\mathrm{Hom}_{\mathsf{D}_m(\mathcal{A})}(\pi V, \pi W) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathcal{A})}(V, W[mi]).$$

In general, for an additive category $\mathcal{C}$ and an auto-equivalence $F\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$, the *orbit category* $\mathcal{C}/F$ of $\mathcal{C}$ by F is defined by

$$\mathsf{Ob}(\mathcal{C}/F) := \mathsf{Ob}\,\mathcal{C}, \quad \mathrm{Hom}_{\mathcal{C}/F}(X,Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, F^iY).$$

The composition of two morphisms $f\colon X \to F^pY$ and $g\colon Y \to F^qZ$ is defined by $(F^pg)\circ f\colon X \to F^{p+q}Z$. The natural functor $\pi\colon \mathcal{C} \to \mathcal{C}/F$ is called the *covering functor* in general. The identity id_{FX} gives rise to a natural isomorphism $\pi(FX) \xrightarrow{\sim} \pi(X)$ in $\mathcal{C}/F$ for all $X \in \mathcal{C}$. Roughly speaking, the orbit category $\mathcal{C}/F$ is obtained by identifying the $\{F^n\}_{n\in\mathbb{Z}}$-orbits of objects of $\mathcal{C}$.

Fact 9.2.9 means that $\mathrm{Im}\,\pi \subseteq \mathsf{D}_m(\mathcal{A})$ can be identified with the orbit category $\mathsf{D}^{\mathrm{b}}(\mathcal{A})/[m]$ of the bounded derived category by the m-shift functor. In fact, $\mathsf{D}_m(\mathcal{A})$ is the smallest triangulated category containing the orbit category $\mathsf{D}^{\mathrm{b}}(\mathcal{A})/[m]$. See [Kel05, Zha14] for the detailed accounts.

An abelian category is called *hereditary* if it is of global dimension 1 and has enough projectives. The periodic derived categories of hereditary abelian categories are rather simple.

Fact 9.2.10 ([Bri13, Lemma 4.2], [Sta18, Lemma 5.1], c.f. [Sai1, Proposition 3.32])**.** *Let $\mathcal{A}$ be a hereditary abelian category. Then for any m-periodic complex $V \in \mathsf{D}_m(\mathcal{A})$, there exists an isomorphism $V \simeq \bigoplus_{i\in\mathbb{Z}_m} H^i(V)[-i]$ in $\mathsf{D}_m(\mathcal{A})$.*

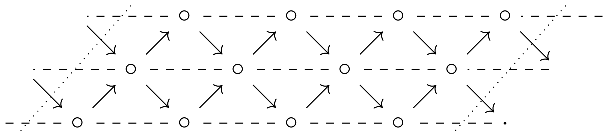
In particular, the covering functor $\pi\colon \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathsf{D}_m(\mathcal{A})$ is essentially surjective, and thus the m-periodic derived category $\mathsf{D}_m(\mathcal{A})$ can be identified with the orbit category $\mathsf{D}^{\mathrm{b}}(\mathcal{A})/[m]$.

Example 9.2.11. Let $\Bbbk Q$ be a path algebra over a field $\Bbbk$. By Fact 9.2.10, an indecomposable object of $\mathsf{D}_m(\mathsf{mod}\,\Bbbk Q)$ is of the form $M[i]$ for some $M \in \mathsf{mod}\,\Bbbk Q$ and some $i \in \mathbb{Z}_m$. If $m \geq 2$, then we have

$$\mathrm{Hom}_{\mathsf{D}_m(\mathsf{mod}\,\Bbbk Q)}(M, N[i]) = \begin{cases} \mathrm{Hom}_{\Bbbk Q}(M,N) & \text{if } i \equiv 0 \bmod m \\ \mathrm{Ext}^1_{\Bbbk Q}(M,N) & \text{if } i \equiv 1 \bmod m \\ 0 & \text{if } i \not\equiv 0, 1 \bmod m \end{cases}$$

for any $M, N \in \mathsf{mod}\,\Bbbk Q$ by Fact 9.2.9.

The category $\mathsf{D}_m(\mathsf{mod}\,\Bbbk Q)$ admits Auslander-Reiten sequences [Fu12, Theorem 2.10], and the covering functor $\pi\colon \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Bbbk Q) \to \mathsf{D}_m(\mathsf{mod}\,\Bbbk Q)$ preserves Auslander-Reiten sequences [Fu12, Theorem 3.1]. For example, let us consider the quiver $Q = 1 \leftarrow 2 \leftarrow 3$. Then the Auslander-Reiten quiver of $\mathsf{D}_2(\mathsf{mod}\, \Bbbk Q)$ is the following:



9.3 The Grothendieck groups of periodic derived categories

The following theorem is the main result of this chapter. It indicates that the m-periodic derived categories behave similarly as the usual derived categories if m is even. In contrast, they behave strangely if m is odd. The even periodic case is a direct generalization of [Fu12, Proposition 2.11], but the odd periodic case is a new one and includes a new insight of odd periodic triangulated categories.

Theorem 9.3.1. *Let $\mathcal{A}$ be a skeletally small abelian category with enough projectives. Suppose that the global dimension of $\mathcal{A}$ is finite. Then we have an isomorphism*

$$\mathsf{K}_0(\mathsf{D}_m(\mathcal{A})) \simeq \begin{cases} \mathsf{K}_0(\mathcal{A}) & \text{if } m \text{ is even,} \\ \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2} := \mathsf{K}_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \text{if } m \text{ is odd,} \end{cases}$$

which is induced by the natural functor $\mathcal{A} \to \mathsf{D}_m(\mathcal{A})$. Here $\mathbb{F}_2$ is the finite field of two elements.

Proof. Let m be a positive integer, and p be the period of $\mathsf{D}_m(\mathcal{A})$. As explained in Remark 9.2.4, we have the following three cases.

(i) m is even and $p = m$.
(ii) m is odd and $p = m$.
(iii) m is odd and $p = 2m$.

We prove Theorem 9.3.1 separately in these cases. The proof of the case (iii) will also work in the case (ii), but we give separate proofs since the proof of the case (ii) is simple and motivates the proof of (iii).

In all the cases, we have a surjective homomorphism $\psi\colon \mathsf{K}_0(\mathcal{A}) \to \mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$ induced by the natural exact functor $\mathcal{A} \to \mathsf{D}_m(\mathcal{A})$ by Proposition 9.2.8. Now we separate the argument into each case.

(i) If m is even, then the 0th cohomology functor $H^0 : \mathsf{D}_m(\mathcal{A}) \to \mathcal{A}$ induces a homomorphism

$$\phi\colon \mathsf{K}_0(\mathsf{D}_m(\mathcal{A})) \longrightarrow \mathsf{K}_0(\mathcal{A}), \quad [V] \longmapsto \sum_{i=1}^{m} (-1)^i [H^i(V)]$$

by Lemma 9.1.2. The homomorphism ϕ is a retraction of ψ, and hence ψ is injective. Thus ψ is an isomorphism.

(ii) If m is odd and $p = m$, then $\mathsf{D}_m(\mathcal{A})$ is an odd periodic triangulated category. Thus $\mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$ is an $\mathbb{F}_2$-vector space by Lemma 9.1.3, and ψ induces a surjective homomorphism $\psi_{\mathbb{F}_2}\colon \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2} \to \mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$. The 0th cohomology functor $H^0 : \mathsf{D}_m(\mathcal{A}) \to \mathcal{A}$ also induces a homomorphism

$$\phi_{\mathbb{F}_2}\colon \mathsf{K}_0(\mathsf{D}_m(\mathcal{A})) \longrightarrow \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2}, \quad [V] \longmapsto \sum_{i=1}^{m} [H^i(V)] \bmod 2\mathsf{K}_0(\mathcal{A})$$

by Lemma 9.1.4. The homomorphism $\phi_{\mathbb{F}_2}$ is clearly a retraction of $\psi_{\mathbb{F}_2}$, and hence $\psi_{\mathbb{F}_2}$ is an isomorphism.

(iii) If m is odd and $p = 2m$, then $\mathsf{D}_m(\mathcal{A})$ is an even periodic triangulated category. Before starting the correct argument, let us give a try to do an analogous discussion as the case (i). Applying Lemma 9.1.2 to the 0th cohomology functor $H^0 : \mathsf{D}_m(\mathcal{A}) \to \mathcal{A}$, we have an induced homomorphism $\phi\colon \mathsf{K}_0(\mathsf{D}_m(\mathcal{A})) \to \mathsf{K}_0(\mathcal{A})$, which turns out to be the zero map. Indeed, we have

$$\psi(V) = \sum_{i=1}^{2m} (-1)^i [H^i(V)] = \sum_{i=1}^{m} (-1)^i [H^i(V)] + \sum_{i=m+1}^{2m} (-1)^i [H^i(V)]$$

$$= \sum_{i=1}^{m}(-1)^i[H^i(V)] - \sum_{i=1}^{m}(-1)^i[H^i(V)] = 0.$$

Thus, the proof of the case (i) does not work in the case (iii).
Although $\mathsf{D}_m(\mathcal{A})$ is even periodic, we can prove the similar results as Lemmas 9.1.3 and 9.1.4, that is, $\mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$ is an $\mathbb{F}_2$-vector space and the assignment $\phi_{\mathbb{F}_2}(V) := \sum_{i=1}^{m}[H^i(V)] \bmod 2\mathsf{K}_0(\mathcal{A})$ defines a homomorphism $\phi_{\mathbb{F}_2} \colon \mathsf{K}_0(\mathsf{D}_m(\mathcal{A})) \to \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2}$. We first prove that $\mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$ is an $\mathbb{F}_2$-vector space. An m-periodic complex $V = (V^i, d^i)_{i\in\mathbb{Z}_m}$ and its m-shift $\Sigma^m V = (V^i, -d^i)_{i\in\mathbb{Z}_m}$ are not necessary isomorphic in $\mathsf{D}_m(\mathcal{A})$ in general. However, we have $[V] = [\Sigma^m V]$ in $\mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$. We prove this by induction on the number n_V of $i \in \mathbb{Z}_m$ with $V^i \neq 0$. It is clear if $n_V = 0$ or 1. Suppose $n_V \geq 2$. Then there exists $i \in \mathbb{Z}_m$ such that $V^i \neq 0$. We may assume that $i = 0$. There exists the following exact sequences in $C_m(\mathcal{A})$.

$$\begin{array}{cccccccc}
0 & & & 0 & & 0 & & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow \\
U & :(\cdots \longrightarrow & V^{m-1} & \longrightarrow & Z^0(V) & \xrightarrow{0} & V^1 & \longrightarrow \cdots) \\
\downarrow & & \| & & \downarrow & & \| & \\
V & :(\cdots \longrightarrow & V^{m-1} & \longrightarrow & V^0 & \longrightarrow & V^1 & \longrightarrow \cdots) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
V^0/Z^0(V) & :(\cdots \longrightarrow & 0 & \longrightarrow & V^0/Z^0(V) & \longrightarrow & 0 & \longrightarrow \cdots), \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 &
\end{array}$$

and

$$\begin{array}{cccccccc}
0 & & 0 & & 0 & & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
Z^0(V) & :(\cdots \longrightarrow & 0 & \longrightarrow & Z^0(V) & \longrightarrow & 0 & \longrightarrow \cdots) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
U & :(\cdots \longrightarrow & V^{m-1} & \longrightarrow & Z^0(V) & \xrightarrow{0} & V^1 & \longrightarrow \cdots) \\
\downarrow & & \| & & \downarrow & & \| & \\
W & :(\cdots \longrightarrow & V^{m-1} & \longrightarrow & 0 & \longrightarrow & V^1 & \longrightarrow \cdots). \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 &
\end{array}$$

Noting that $V^i = (\Sigma^m V)^i$ and $Z^i(V) = Z^i(\Sigma^m V)$, we also have exact sequences $0 \to \Sigma^m U \to \Sigma^m V \to V^0/Z^0(V) \to 0$ and $0 \to Z^0(V) \to \Sigma^m U \to \Sigma^m W \to 0$. Since $n_W < n_V$, we have $[W] = [\Sigma^m W]$ in $\mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$ by the induction hypothesis. The canonical exact functor $C_m(\mathcal{A}) \to \mathsf{D}_m(\mathcal{A})$ carries the exact sequences above to exact triangles in $\mathsf{D}_m(\mathcal{A})$, and thus we have

$$[V] = [W] + [Z^0(V)] + [V^0/Z^0(V)] = [\Sigma^m W] + [Z^0(V)] + [V^0/Z^0(V)] = [\Sigma^m V] = -[V].$$

Hence $\mathsf{K}_0(\mathsf{D}_m(\mathcal{A}))$ is an $\mathbb{F}_2$-vector space. The method above is Gorsky's induction technique for periodic complexes, which appears in the proof of Fact 9.2.7. See [Gor, Proposition 9.7].
Next, we prove that the assignment $\phi_{\mathbb{F}_2}(V) := \sum_{i=1}^{m}[H^i(V)] \bmod 2\mathsf{K}_0(\mathcal{A})$ defines a homomorphism $\phi_{\mathbb{F}_2} \colon \mathsf{K}_0(\mathsf{D}_m(\mathcal{A})) \to \mathsf{K}_0(\mathcal{A})_{\mathbb{F}_2}$. Let $U \to V \to W \to \Sigma U$ be an exact triangle in $\mathsf{D}_m(\mathcal{A})$. Note that $H^0(U) = H^m(U)$ holds. Then we have two exact sequences in $\mathcal{A}$:

$$H^{m-1}(W) \xrightarrow{f} H^m(U) \simeq H^0(U) \xrightarrow{g} H^0(V),$$

$$0 \to \operatorname{Ker} g \to H^0(U) \xrightarrow{g} H^0(V) \to \cdots \to H^{m-1}(V) \to H^{m-1}(W) \xrightarrow{f} \operatorname{Im} f \to 0.$$

A similar discussion as in Lemma 9.1.4 implies $\phi_{\mathbb{F}_2}(U) - \phi_{\mathbb{F}_2}(V) + \phi_{\mathbb{F}_2}(W) \equiv 0 \bmod 2\mathsf{K}_0(\mathcal{A})$. The rest of the proof is similar to (ii).

The proof is now finished. □

As an immediate corollary, we have the following.

Corollary 9.3.2. *Let Λ be an Artin algebra of finite global dimension, and let n be the number of isomorphism classes of simple modules. Then we have*

$$\mathsf{K}_0(\mathsf{D}_m(\mathsf{mod}\,\Lambda)) \simeq \begin{cases} \mathbb{Z}^{\oplus n} & \textit{if } m \textit{ is even,} \\ \mathbb{F}_2^{\oplus n} & \textit{if } m \textit{ is odd.} \end{cases}$$

Let us explain the motivation of Theorem 9.3.1 and Corollary 9.3.2. In [Sai1], the author proves the *periodic* tilting theorem, which gives a sufficient condition for a given triangulated category to be equivalent to the periodic derived category of an algebra. We review some definitions and results in [Sai1].

Definition 9.3.3. Let $\mathcal{T}$ be an m-periodic triangulated category.

(1) An object $T \in \mathcal{T}$ is called *m-periodic tilting* if it satisfies $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = 0$ for any $i \in \mathbb{Z} \setminus m\mathbb{Z}$ and the smallest thick triangulated category containing T coincides with $\mathcal{T}$.

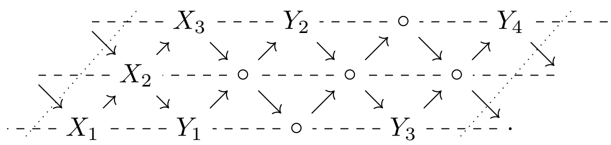
(2) An m-periodic tilting object $T \in \mathcal{T}$ is called *strict* if the global dimension of the endomorphism algebra $\mathrm{End}_{\mathcal{T}}(T)$ is less than m.

We will not give the definitions of the conditions *algebraic* and *idempotent complete* in the following theorem. See [Sai1] for the precise description. We can say that these conditions are mild, and in fact they are satisfied by almost all concrete triangulated categories appearing in the study of representations of algebras.

Fact 9.3.4 (The periodic tilting theorem [Sai1, Corollary 5.4])**.** *Let $\mathcal{T}$ be an idempotent complete algebraic m-periodic triangulated category over a perfect field $\Bbbk$. Suppose that $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is finite dimensional over $\Bbbk$ for all objects $X, Y \in \mathcal{T}$. If $\mathcal{T}$ has a strict m-periodic tilting object T, then there exists a triangulated equivalence $\mathcal{T} \to \mathsf{D}_m(\mathsf{mod}\,\Lambda)$, where $\Lambda := \mathrm{End}_{\mathcal{T}}(T)$.*

The periodic tilting theorem and periodic tilting objects are periodic analogue of the usual tilting theorem and tilting objects (c.f. [Tilt07]). Hence, we expect that periodic tilting objects have properties similar to the usual tilting objects. However, we have the following example which was taught by Professor Osamu Iyama in the conference *Algebraic Lie Theory and Representation Theory, 2021*.

Example 9.3.5. Let $\Bbbk A_3$ be the path algebra of the quiver $1 \leftarrow 2 \leftarrow 3$ of type A_3 over a perfect field $\Bbbk$. The Auslander-Reiten quiver of $\mathsf{D}_2(\mathsf{mod}\,\Bbbk A_3)$ is the following (see also Example 9.2.11):



Then $X := \bigoplus_{i=1}^{3} X_i$ and $Y := \bigoplus_{i=1}^{4} Y_i$ are both 2-periodic tilting objects in $D_2(\mathsf{mod}\,\Bbbk A_3)$. Thus, the number of non-isomorphic summands of a periodic tilting object is not constant, while it is known that the number of non-isomorphic summands of a tilting object is constant.

In the above Example 9.3.5, we observe that $\mathrm{End}(X) \simeq \Bbbk A_3$ and $\mathrm{End}(Y)$ is isomorphic to a self-injective Nakayama algebra, and hence X is strict but Y is not. Thus, we can expect that the number of non-isomorphic summands of a *strict* periodic tilting object is constant. It is true as the following Corollary 9.3.6 shows.

Corollary 9.3.6. *Let m be a positive integer, and $\mathcal{T}$ be an idempotent complete algebraic m-periodic triangulated category over a perfect field $\Bbbk$. Suppose that $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is finite dimensional over $\Bbbk$ for all objects $X, Y \in \mathcal{T}$. Then the number of non-isomorphic summands of a strict periodic tilting object is constant.*

Proof. Let $T_1, T_2 \in \mathcal{T}$ be strict m-periodic tilting objects, and set $\Lambda_i := \mathrm{End}_{\mathcal{T}}(T_i)$ $(i = 1, 2)$. Then we have two triangulated equivalences $\mathcal{T} \xrightarrow{\sim} \mathsf{D}_m(\mathsf{mod}\,\Lambda_i)$ by Fact 9.3.4. Their composition induces an isomorphism $\mathsf{K}_0(\mathsf{D}_m(\mathsf{mod}\,\Lambda_1)) \simeq \mathsf{K}_0(\mathsf{D}_m(\mathsf{mod}\,\Lambda_2))$ of the Grothendieck groups. Hence Λ_1 and Λ_2 have the same number of isomorphism classes of simple modules by Corollary 9.3.2. Since the number of non-isomorphic summands of T_i is equal to the number of isomorphism classes of simple modules over Λ_i, we have the statement. $\square$

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