Corrigendum

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1. p.3, l.10, does → is
2. p.4, l.-16, Cuoco-Monsky → Cuoco–Monsky
3. p.5, l.3, an unnecessary period after the equation
4. p.5, l.5, $M_p^n \to M_{p^n}$
5. p.16, l.10, an unnecessary period after the equation
6. p.24, l.-10, an unnecessary period after the equation
7. p.30, l.-1, $W(2^n) \to W(n)$

The Iwasawa invariants of $\mathbb{Z}_p^d$-covers of links

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Abstract

Let p be a prime number. In this thesis, we define the Iwasawa invariants of links and prove two asymptotic formulae for the first homology groups of $\mathbb{Z}_p^d$-covers of links in rational homology 3-spheres, which are generalizations of the Iwasawa type formulae proven by Hillman–Matei–Morishita and Kadokami–Mizusawa under a mild assumption. We also provide examples of these formulae. Moreover, when $d \leq 2$, considering the twisted Whitehead links, we prove that Iwasawa μ-invariants can be arbitrary non-negative integers. This thesis also includes an example of p-adic torsions for $d = 2$. This thesis is based on a paper [28] that is a joint work with Jun Ueki.

Contents

1 Introduction

Let p be a prime number. For an abelian group G whose p-torsion subgroup is a finite group, let $e(G)$ denote the p-exponent of the order of the p-torsion subgroup. For a number field k, let $Cl(k)$ denote the ideal class group of k. The size of $Cl(k)$ is known to be finite and is called the class number of k.

In [10], Iwasawa proved the following result, which is so-called Iwasawa's class number formula. Let $\mathbb{Z}_p$ denote the ring of p-adic integers.

Theorem ([10, Theorem 4]). *Let k_∞/k be a $\mathbb{Z}_p$-extension and k_{p^n} be the subfields corresponding to the subgroups $p^n\mathbb{Z}_p$ of $\mathbb{Z}_p$. Then there exist invariants $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$, depending only on k_∞/k, such that*

$$e(Cl(k_{p^n})) = \mu p^n + \lambda n + \nu$$

for every sufficiently large n.

This result is known to be the first asymptotic formula that describes the regularity of the variation of the class numbers in certain towers of number fields. The values μ, λ, ν are called the Iwasawa invariants of k_∞/k.

On the other hand, let M be a closed connected orientable 3-manifold. M is called a *rational homology* 3*-sphere* ($\mathbb{Q}$HS^3) if $H_i(M;\mathbb{Q}) \cong H_i(S^3;\mathbb{Q})$ for all $i \geq 0$. M is called an *integral homology* 3*-sphere* ($\mathbb{Z}$HS^3) if $H_i(M;\mathbb{Z}) \cong H_i(S^3;\mathbb{Z})$ for all $i \geq 0$. By the Poincare duality and the universal coefficient theorem, we have

- M is a $\mathbb{Q}$HS^3 if and only if $H_1(M;\mathbb{Z})$ is a finite group.
- M is a $\mathbb{Z}$HS^3 if and only if $H_1(M;\mathbb{Z}) = 0$.

In this sense with a deeper background as we explain later, $\mathbb{Q}$HS^3's in topology correspond to number fields in number theory, and $\mathbb{Z}$HS^3's correspond to number fields with the class number 1. In what follows, we write $H_1(M) = H_1(M;\mathbb{Z})$.

After a work of Hillman–Matei–Morishita [7, Theorem 5.1.7], Kadokami and Mizusawa proved the following topological analogue of Iwasawa's formula. This is so called the Iwasawa type formula.

Theorem ([13, Theorem 2.1]). *Let L be a link in a $\mathbb{Q}HS^3$ M, let $X = M - N(L)$ denote the exterior of an open tubular neighbourhood of L, let $\tau : \pi_1(X) \twoheadrightarrow \mathbb{Z}$ be a surjective homomorphism, and let $X_\infty \to X$ denote the corresponding $\mathbb{Z}$-cover. Let $(M_{p^n} \to M)_n$ denote the system of the branched $\mathbb{Z}/p^n\mathbb{Z}$-covers obtained by the Fox completion and suppose that every M_{p^n} is a $\mathbb{Q}HS^3$. Then there exist invariants $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$, depending only on $X_\infty \to X$ and p, such that*

$$e(H_1(M_{p^n})) = \mu p^n + \lambda n + \nu$$

for every sufficiently large n.

These μ, λ, ν are called the Iwasawa invariants of $(M_{p^n} \to M)_n$. As a remark, Ueki proved that this formula also holds for a $\mathbb{Z}_p$-cover that does not necessarily derived from a $\mathbb{Z}$-cover in [31].

Let $d \geq 2$. In [4], Cuoco and Monsky generalized the result of Iwasawa to $\mathbb{Z}_p^d$-extensions.

Theorem ([4, Theorem I]). *Let k_∞/k be a $\mathbb{Z}_p^d$-extension of number fields and k_{p^n} be the subfields corresponding to the subgroups $(p^n\mathbb{Z}_p)^d$ of $\mathbb{Z}_p^d$. Then there exist invariants $\mu, \lambda \in \mathbb{Z}_{\geq 0}$, depending only on k_∞/k, such that*

$$e(Cl(k_{p^n})) = (\mu p^n + \lambda n + O(1))p^{(d-1)n},$$

where O is the Bachmann–Landau notation with respect to n.

These μ, λ are called the Iwasawa invariants of k_∞/k. We remark that Monsky showed in [18] that the $O(p^{(d-1)n})$ part can be refined to $\alpha^* + O(np^{(d-2)n})$ for some $\alpha^* \in \mathbb{R}$ ($\alpha^* \in \mathbb{Q}$ if $d = 2$).

To show this result, the power series ring $\Lambda := \mathbb{Z}_p[[T_1, \ldots, T_d]]$ and modules over Λ play important roles. Let k be a number field and k_∞/k a $\mathbb{Z}_p^d$-extension. Let k_{p^n} denote the fixed field of $p^n\mathbb{Z}_p^d$ and let l_∞/k_∞ be the maximal abelian unramified pro-p extension of k_∞. Put $\Gamma := \mathrm{Gal}(k_\infty/k)$ and $\mathcal{X} := \mathrm{Gal}(l_\infty/k_\infty)$. Then it can be shown that l_∞/k is Galois. Put $\mathcal{G} := \mathrm{Gal}(l_\infty/k)$. Then there is a well-defined action

$$\Gamma \times \mathcal{X} \ni (\sigma, x) \mapsto \tilde{\sigma} x \tilde{\sigma}^{-1} \in \mathcal{X},$$

where $\tilde{\sigma} \in \mathcal{G}$ is a lifting of σ from Γ to $\mathcal{G}$. Let $\mathbb{Z}_p[[\Gamma]]$ denote the complete group ring of Γ over $\mathbb{Z}_p$. Then $\mathcal{X}$ becomes a $\mathbb{Z}_p[[\Gamma]]$-module via this action. It is known that the Iwasawa-Serre homomorphism

$$\mathbb{Z}_p[[\Gamma]] \ni \gamma_i \to 1 + T_i \in \mathbb{Z}_p[[T_1, \ldots, T_d]]$$

is an isomorphism of rings (cf. [27, Theorem 3.3.9]) and $\mathcal{X}$ is a finitely generated torsion Λ-module.

On the other hand, let M be a $\mathbb{Q}\mathrm{HS}^3$ and L a link in M, that is, the image of a tame embedding of $S^1 \sqcup \cdots \sqcup S^1$ into M. Let $N(L)$ be an open tubular neighbourhood of L and put $X = M - N(L)$. Let $\tau : \pi_1(X) \twoheadrightarrow \mathbb{Z}^d$ be a surjective homomorphism and let $X_\infty \to X$ denote the $\mathbb{Z}^d$-cover corresponding to $\ker(\tau)$. Here, d does not need to coincide with the number of components of L. In addition, for each $n \geq 1$, let $X_n \to X$ denote the $(\mathbb{Z}^d/(n\mathbb{Z})^d =)(\mathbb{Z}/n\mathbb{Z})^d$-subcover of $X_\infty \to X$ and let $M_n \to M$ denote the branched $(\mathbb{Z}/n\mathbb{Z})^d$-covers obtained by the Fox completion. An inverse system $\widetilde{M} = (h_{p^n} : M_{p^n} \to M)_n$ is called *the branched $\mathbb{Z}_p^d$-cover* over (M, L) derived from $X_\infty \to X$.

Let $\Lambda_\mathbb{Z} = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. The Alexander polynomial $\Delta(t_1, \ldots, t_d) \in \Lambda_\mathbb{Z}$ of $X_\infty \to X$ is defined to be a generator of the divisorial hull of the finitely generated $\Lambda_\mathbb{Z}$-module $H_1(X_\infty)$. Details of the theory of Alexander polynomials are written in [6, Chapter 3 and 4]. Note that the ring $\Lambda_\mathbb{Z}$ can be regarded as a subring of Λ by the homomorphism of rings

$$\Lambda_\mathbb{Z} \ni t_i \to 1 + T_i \in \Lambda.$$

Hence $\Delta(t_1, \ldots, t_d)$ in $\Lambda_\mathbb{Z}$ can be seen as an element $\Delta(1 + T_1, \ldots, 1 + T_d)$ of Λ.

In this thesis, we prove the following topological analogue of the result of Cuoco and Monsky. This is a generalization of Hillman–Matei–Morishita and Kadokami–Mizusawa under a mild additional assumption. Put $W := \{\xi \in \overline{\mathbb{Q}}_p \mid \xi^{p^n} = 1 \text{ for some } n \geq 0\}$.

Main result 1 (Theorem 7.4)**.** *Let $\widetilde{M}$ be the branched $\mathbb{Z}_p^d$-cover over (M, L) derived from a $\mathbb{Z}^d$-cover $X_\infty \to X$, that is, the inverse system consisting of $(\mathbb{Z}/p^n\mathbb{Z})^d$-branched covers $M_{p^n} \to M$ derived from $X_\infty \to X$, defined as above, and suppose that every M_{p^n} is a $\mathbb{Q}HS^3$. Let $\Delta(t_1, \ldots, t_d)$ denote the Alexander polynomial of $X_\infty \to X$ and suppose that $\Delta(t_1, \ldots, t_d)$ does not vanish on W^d. Then there exist invariants $\mu, \lambda \in \mathbb{Z}_{\geq 0}$, depending only on $X_\infty \to X$ and p, such that*

$$e(H_1(M_{p^n})) = (\mu p^n + \lambda n + O(1))p^{(d-1)n}.$$

We remark that, when $d = 1$, it is known that every M_{p^n} is a $\mathbb{Q}\mathrm{HS}^3$ if and only if $\Delta(t)$ does not vanish on $W \setminus \{1\}$ [31, Theorem 4.17]. Hence the $\mathbb{Q}\mathrm{HS}^3$ assumption in the result of Kadokami–Mizusawa is slightly weaker than the assumption on Alexander polynomials for our result. Also, as was the case with the result of Cuoco-Monsky, the $O(p^{(d-1)n})$ part can be refined to $\alpha^* + O(np^{(d-2)n})$ for some $\alpha^* \in \mathbb{R}$ ($\alpha^* \in \mathbb{Q}$ if $d = 2$).

In order to obtain this result, we first establish a similar formula for a general $\mathbb{Z}_p$-cover of compact connected orientable 3-manifolds derived from a $\mathbb{Z}^d$-cover (Theorem 7.3).

Greenberg conjectured that the p-exponent of the class numbers in $\mathbb{Z}_p^d$-towers of number fields ramified at finitely many primes is given by a polynomial in p^n and n of total degree at most d for every sufficiently large n (cf. [4, Section 7]). This means that the $O(p^{(d-1)n})$ part of the result of Cuoco–Monsky would be precisely described by such a polynomial. By restricting the $\mathbb{Q}\mathrm{HS}^3$'s we consider to $\mathbb{Z}\mathrm{HS}^3$'s, we obtained an evidence of Greenberg's conjecture in the link side.

If L is a d-component link in a $\mathbb{Q}\mathrm{HS}^3$ M and $X_\infty \to X$ is its unique $\mathbb{Z}^d$-cover, then we will denote the Alexander polynomial by $\Delta_L(t_1, \ldots, t_d)$. In this situation, Iwasawa invariants μ and λ are determined only by L and p. Hence we denote them by μ_L and λ_L.

Main result 2 (Theorem 7.6)**.** *Let $\widetilde{M}$ be the branched $\mathbb{Z}_p^d$-cover over (M, L) derived from a $\mathbb{Z}^d$-cover $X_\infty \to X$. Suppose in addition that M is a $\mathbb{Z}HS^3$, L consists of d components, and the Alexander polynomial $\Delta_L(t_1, \ldots, t_d)$ of L does not vanish on $(W \setminus \{1\})^d$. Then, every* *branched* $(\mathbb{Z}/p^n\mathbb{Z})^d$*-cover* M_{p^n} *is a* $\mathbb{Q}HS^3$*, and there exists a unique* $f(U,V) \in \mathbb{Q}[U,V]$ *with* $\deg_V f \leq 1$ *and total degree* $\deg f \leq d$ *such that*

$$e(H_1(M_{p^n})) = f(p^n, n).$$

for every sufficiently large n*.*

This means, we can express $e(H_1(M_p^n))$ by

$$e(H_1(M_{p^n})) = \mu_L p^{dn} + \lambda_L n p^{(d-1)n} + \mu_{d-1} p^{(d-1)n} + \lambda_{d-1} n p^{(d-2)n} + \ldots + \mu_1 p^n + \lambda_1 n + \nu$$

for every sufficiently large n, where $\mu_{d-1}, \ldots, \mu_1, \lambda_{d-1}, \ldots, \lambda_1, \nu \in \mathbb{Q}$. Hence the $O(p^{(d-1)n})$ part of Cuoco–Monsky type formula can be described by a polynomial with the form Greenberg conjectured in the case of a pair of a $\mathbb{Z}\mathrm{HS}^3$ and a d-component link.

In order to prove Main result 1, we investigate some basic properties of $\mathbb{Z}^d$-covers of 3-manifolds. Especially, the following result plays a key role. We define ideals I_n of $\Lambda_{\mathbb{Z}}$ by $I_n = (t_1^n - 1, t_2^n - 1, \ldots, t_d^n - 1)$.

Main result 3 (Theorem 6.1)**.** *Let* $X_\infty \to X$ *be a* $\mathbb{Z}^d$*-cover of a compact connected orientable* 3*-manifold. Then we have*

$$e(H_1(X_{p^n})) = e(H_1(X_\infty)/I_{p^n} H_1(X_\infty)) + O(n) \ (n \to \infty).$$

We also exhibit several examples. For the twisted Whitehead link W_{2p^k}, we have

$$|H_1(M_{p^n}, \mathbb{Z}_p)| = p^{(kp^n + 2n - 2k)p^n - 2n + k}.$$

We will calculate this example in Section 9. For this purpose, we calculate the Alexander polynomial of the twisted Whitehead links W_{2m}, which is already known, in Section 8 and obtain

$$\Delta_{W_{2m}}(x, y) = m(x-1)(y-1).$$

i.e.,

$$\Delta_{W_{2m}}(1+X, 1+Y) = mXY.$$

In particular, we obtain $\mu_{W_{2p^k}} = k$. Therefore, we have

Main result 4 (Theorem 9.7)**.** *Suppose* $d = 2$*. Then, for arbitrary* $m \geq 0$*, there exists a link* L *in* S^3 *such that* $\mu_L = m$*.*

This can be viewed as a topological analogue of the number theoretical results of Iwasawa [11, Theorem 1] and Ozaki [22, Theorem 2]. We remark that a result of Kadokami–Mizusawa [13] yields a similar result for $d = 1$ (Theorem 9.7), which is a refinement of a result of Ueki [30, Theorem 5.2].

Our calculation yields a new example of Kionke's p-adic torsions for $d = 2$ as well. In [32], Ueki and Yoshizaki proved the following result.

Theorem ([32, Theorem B]). *Let $(X_{p^n} \to X)_n$ be a $\mathbb{Z}_p$-cover of compact 3-manifold X. Then, the sizes of the torsion subgroups $H_1(X_{p^n})_{\mathrm{tor}}$, those of the non-p torsion subgroups $H_1(X_{p^n})_{\text{non-}p}$, and those of the l-torsion subgroups $H_1(X_{p^n})_{(l)}$ for each prime number l, of the 1st homology groups converge in $\mathbb{Z}_p$.*

For $L = 6_1^2$ (see Example 9.3) and $p \neq 3$, we will show that $|H_1(M_{p^n})| = 3^{p^n-1}$ and this sequence converges in $\mathbb{Z}_p$. This can be regarded as an example of the above result for $d = 2$.

The structure of this thesis is as follows. In Section 2, we briefly explain a historical context of our motivation and introduce some basic analogies between number theory and knot(link) theory. In Section 3, we introduce the notion of Iwasawa invariants μ and λ for links and explain what corresponds to the Weierstrass preparation theorem of one variable Iwasawa theory for our case. In Section 4, we study some basic properties on Alexander polynomials. In Section 5, we review results of Cuoco and Monsky on Λ-modules that we use in Section 7. In Section 6, we prove a fundamental result for 3-manifolds that is crucial to attain our main results. In Section 7, we prove our main results. In Section 8, we calculate the Alexander polynomial of the twisted Whitehead links. In Section 9, as stated above, we provide several examples to reinforce our results. By using Sage Math, we also place a table of the Iwasawa invariants μ and λ for links that appear in tables of the Rolfsen's book [24]. In Section 10, we give some remarks on our results.

2 Backgrounds

In this section, we briefly explain a historical context of our motivation and review basic analogies between number theory and topology.

Let k be a number field, i.e., a finite extension of the field $\mathbb{Q}$ of rational numbers. The class number of k is the size of the ideal class group $Cl(k)$ of k. The notion of class numbers in number theory has been crucial as a research object since the era of Kummer. Kummer invented the notion of class numbers and successfully proved that the Fermat last theorem holds for a prime number p if p does not divide the class number of p-th cyclotomic field. He also proved that the class numbers of cyclotomic fields are related with the particular values of the Riemann zeta function, which is closely related to the distribution of prime numbers.

That is being said, the regularity of class numbers has been mysterious and it is basically difficult to control. The Gauss conjecture, which states that there are infinitely many quadratic real fields whose class numbers are one, is still an open problem. We do not even know whether there are infinitely many number fields whose class numbers are one or not.

Under such a background, as stated in the introduction, Iwasawa found a formula that controls the p-exponent of the class numbers in any $\mathbb{Z}_p$-tower of number fields.

To prove this formula, class field theory plays a key role. Let k be a number field and let $k_{\mathrm{ab}}^{\mathrm{ur}}$ denote the Hilbert class field, i.e., the maximal unramified abelian extension of k. Then class field theory states that there is an isomorphism

$$\mathrm{Gal}(k_{\mathrm{ab}}^{\mathrm{ur}}/k) \cong Cl(k).$$

This isomorphism is one of the major corollaries of the so-called Artin reciprocity law, and this allows one to regard class numbers as sizes of Galois groups. Given a $\mathbb{Z}_p$-extension, Iwasawa constructed the corresponding tower of Hilbert p-class fields and obtained the aforementioned formula.

Because this isomorphism of class field theory can be seen as an analogue of the Hurewicz isomorphism, we see that class numbers in the number theoretical side correspond to the sizes of first homology groups in the topological side.

Hillman, Matei, and Morishita proved the formula for the first p-homology groups of p^n-fold cyclic covers of links in the 3-sphere S^3 corresponding to Iwasawa's class number formula. Kadokami and Mizusawa generalized this result to any rational homology 3-sphere.

On the other hand, in 1981, Cuoco and Monsky proved that the corresponding formula holds for $\mathbb{Z}_p^d$-extension over number fields. Since the abelianizations of link groups are free abelian groups, we commenced believing that the Iwasawa type formula for $d \geq 2$ should hold, and it is supposed to correspond to the result of Cuoco–Monsky.

Number theory	Knot theory
number field k (the ring of integers $\operatorname{Spec} \mathcal{O}_k$)	closed connected orientable 3-manifold
prime ideal $\mathfrak{p} : \operatorname{Spec} \mathcal{O}_k/\mathfrak{p} \hookrightarrow \operatorname{Spec} \mathcal{O}_k$ family of primes $= \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$	knot $K : S^1 \hookrightarrow M$ link $L : \sqcup S^1 \hookrightarrow M$
$\mathbb{Q}$	S^3
$Cl(k)$	$H_1(M;\mathbb{Z})$ or $\operatorname{tor}_{\mathbb{Z}} H_1(M;\mathbb{Z})$
Fact: $h(k)$ is finite	(Assumption: M is a $\mathbb{Q}\mathrm{HS}^3$)
$\mathbb{Z}_p$-extension k_∞/k	branched $\mathbb{Z}_p$-cover $\widetilde{M} \to M$
subfield k_{p^n} corresponding to $p^n\mathbb{Z}_p$	subcover M_{p^n} corresponding to $p^n\mathbb{Z}$
Artin reciprocity law	Hurewicz isomorphism
$Cl(K_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p$	$H_1(M_{p^n}, \mathbb{Z}_p)$
$\mathbb{Z}_p^d$-extension k_∞/k	branched $\mathbb{Z}_p^d$-cover $\widetilde{M} \to M$
subfield k_{p^n} corresponding to $(p^n\mathbb{Z}_p)^d$	cover M_{p^n} corresponding to $(p^n\mathbb{Z})^d$

3 On some estimates

In this section, we collect several results on some estimates that are related with p^n-th roots of unity.

3.1 Weierstrass preparation theorem

In this subsection, we introduce the notion of the Iwasawa invariants for Iwasawa modules over Iwasawa algebras with multiple variables. We basically review the Section 1 of the paper of Cuoco–Monsky [4] and check what corresponds to the Weierstrass preparation theorem [34, Theorem 7.3] for the Iwasawa algebra with one variable in our situation.

Let Γ be a free $\mathbb{Z}_p$-module of rank d written multiplicatively and fix a basis $\{\gamma_1, \ldots, \gamma_d\}$ of Γ. Let $\mathbb{Z}_p[[\Gamma]]$ denote the complete group ring of Γ over $\mathbb{Z}_p$ and let $\Lambda := \mathbb{Z}_p[[T_1, \ldots, T_d]]$ be the power series ring over $\mathbb{Z}_p$. It is known that the map

$$\mathbb{Z}_p[[\Gamma]] \ni \gamma_i \to 1 + T_i \in \mathbb{Z}_p[[T_1, \ldots, T_d]]$$

is an isomorphism of rings (cf. [27, Theorem 3.3.9]). Hence Γ is a closed multiplicative subgroup of Λ generated by $1+T_j$ via this isomorphism. Each element of $\mathrm{GL}_d(\mathbb{Z}_p)$ induces an automorphism of Γ that prolongs to a ring automorphism of Λ. We call such automorphisms linear automorphisms of Λ. Let $\sigma \in \Gamma \backslash \Gamma^p$, where $\Gamma^p := \{\gamma \in \Gamma \mid \gamma = \gamma'^p \text{ for some } \gamma' \in \Gamma\}$. Then there is a linear automorphism mapping σ to $1 + T_j$. Let $\Omega := \mathbb{F}_p[[T_1, \ldots, T_d]]$. For $F \in \Lambda$, let $\overline{F} \in \Omega$ denote the mod p reduction of F. Since $(\overline{T}_j)$ is a height one prime ideal of Ω, so is $(\overline{\sigma - 1})$. If $\mathfrak{p}$ is a prime ideal of height 1, then let $v_{\mathfrak{p}}$ be the associated discrete valuation. From now on, let F be a nonzero element of Λ. Then there exists some nonzero $F_0 \in \Lambda$ and $\mu = \mu(F) \in \mathbb{Z}_{\geq 0}$ such that $F = p^{\mu(F)} F_0$ and $p \nmid F_0$. Define

$$\lambda = \lambda(F) := \sum v_{\mathfrak{p}}(\overline{F_0}),$$

where the sum runs over all $\mathfrak{p}$ of the form $(\overline{\sigma - 1}), \sigma \in \Gamma \backslash \Gamma^p$. Let

$$W := \{\xi \in \overline{\mathbb{Q}}_p \mid \xi^{p^n} = 1 \text{ for some } n \geq 0\}.$$

Let $v : \overline{\mathbb{Q}}_p \to \mathbb{Q}$ be the order function normalized so that $v(p) = 1$. We make the unusual convention $v(0) = 0$. Let $\zeta = (\zeta_1, \ldots, \zeta_d) \in W^d$. Put $F(\zeta - 1) := F(\zeta_1 - 1, \ldots, \zeta_d - 1)$. Since $v(\zeta_j - 1) > 0$, one has $v(F(\zeta - 1)) \geq 0$. For $n \geq 0$, define

$$\Sigma_n(F) := \sum_{\zeta \in W(n)^d} v(F(\zeta - 1)),$$

where $W(n) := \{\xi \in W \mid \xi^{p^n} = 1\}$.

Example 3.1. (1) Let

$$\sigma = \gamma_1 \gamma_2 = (1 + T_1)(1 + T_2).$$

Then

$$\overline{\sigma - 1} = \overline{T_1 T_2} + \overline{T}_1 + \overline{T}_2.$$

(2) Suppose $p \neq 2$. Let

$$\sigma = \gamma_1^2 = (1 + T_1)^2.$$

Then

$$\overline{\sigma - 1} = \overline{T}_1^2 + 2\overline{T}_1 = \overline{T_1(T_1 + 2)}.$$

Since $T_1 + 2 \in \Lambda^*$, one has $(\overline{\sigma - 1}) = (\overline{T}_1)$ as ideals. In general, if $\sigma = \gamma_1^k$ and $p \nmid k$, then $(\overline{\sigma - 1}) = (\overline{T}_1)$.

Example 3.2. (1) For arbitrary $m \in \mathbb{Z}_{\geq 0}$, we have

$$\Sigma_n(p^m) = \sum_{\zeta \in W(n)^d} v(p^m) = \sum_{\zeta} mv(p) = m \sum_{\zeta} 1 = mp^{dn}.$$

(2) If we fix a primitive p^2-th root of unity ξ, then

$$\begin{aligned}\Sigma_2(T_1) &= \sum_{\zeta \in W(2)^d} v(\zeta_1 - 1) \\ &= (v(\xi - 1) + v(\xi^2 - 1) + \ldots + v(\xi^{p^2} - 1))p^{2(d-1)} \\ &= \left(\frac{p-1}{p-1} + \frac{p(p-1)}{p(p-1)}\right) p^{2(d-1)} = 2p^{2(d-1)}.\end{aligned}$$

In general, we have

$$\Sigma_n(T_1) = np^{n(d-1)}.$$

(3) If $\gamma_1^{e_1} \cdots \gamma_d^{e_d} \in \Gamma \backslash \Gamma^p$, then we find that

$$\Sigma_n((1+T_1)^{e_1} \cdots (1+T_d)^{e_d} - 1) = \sum_{\zeta \in W(n)^d} v(\zeta_1^{e_1} \cdots \zeta_d^{e_d} - 1) = \sum_{\zeta \in W(n)^d} v(\zeta_1 - 1) = \Sigma_n(T_1).$$

Lemma 3.3 ([4, Lemma 1.4, 1.5, 1.6])**.** (1) *Let $G \in \Lambda$ with $\overline{F} = \overline{G} \neq 0$. Then*

$$\Sigma_n(F) - \Sigma_n(G) = O(p^{(d-1)n}).$$

(2) *If there exist $F_1, F_2 \in \Lambda$ such that $F = F_1 F_2$, then*

$$\Sigma_n(F) = \Sigma_n(F_1) + \Sigma_n(F_2) + O(p^{(d-1)n}).$$

(3) *If $\mu(F) = \lambda(F) = 0$, then $\Sigma_n(F) = O(p^{(d-1)n})$.*

One can show Lemma 3.3 by using results of Monsky [17] on Λ.

Proposition 3.4 ([4, Theorem 1.7])**.** *Let F be a nonzero element of Λ. Then we have*

$$\Sigma_n(F) = (\mu(F)p^n + \lambda(F)n + O(1))p^{(d-1)n}.$$

Proof. Write $F = p^{\mu(F)} \cdot F_0$ with $\overline{F}_0 \neq 0$. Since Ω is a unique factorization domain, there exist $F_1, \ldots, F_k \in \Lambda$ with $\overline{F}_j$ irreducible in Ω such that $\overline{F}_0 = \overline{F}_1 \cdots \overline{F}_k$. By Lemma 3.3 (1), (2), we have

$$\Sigma_n(F) = \Sigma_n(p^{\mu(F)}) + \Sigma_n(F_1) + \cdots + \Sigma_n(F_k) + O(p^{(d-1)n}).$$

By Lemma 3.3 (3), if $\lambda(F_j) = 0$, then $\Sigma_n(F_j) = O(p^{(d-1)n})$. By Example 3.2 (1), (2), (3), we have

$$\begin{aligned}\Sigma_n(F) &= \mu(F)p^{dn} + \lambda(F)np^{(d-1)n} + O(p^{(d-1)n}) \\ &= (\mu(F)p^n + \lambda(F)n + O(1))p^{(d-1)n}.\end{aligned}$$

□

Remark 3.5. The proof of Proposition 3.4 is corresponding to the Weierstrass prepartion theorem of one variable Iwasawa theory.

3.2 Results of Monsky

In this subsection, we briefly review a result of Monsky, which is vital for our proof of Main result 2.

Let E_d denote the $\mathbb{Z}_p$-module $\mathrm{Hom}(W^d, W)$.

Definition 3.6. $S \subset W^d$ is said to be *semi-algebraic* if it is a finite union of subsets each of which is defined by finitely many conditions of the following three types

(a) $\tau(\zeta) = \varepsilon$,

(b) $\tau(\zeta) \neq \varepsilon$,

(c) $\log_p |\langle \tau(\zeta) \rangle| \geq \log_p |\langle \tau'(\zeta) \rangle| + r$,

where $\tau, \tau' \in E_d$, $\varepsilon \in W$, and $r \in \mathbb{Z}$.

Lemma 3.7. *$(W \setminus \{1\})^d$ is semi-algebraic.*

Proof. For each $1 \leq i \leq d$, let π_i denote the projection $W^d \to W$. Then we have

$$(W \setminus \{1\})^d = \bigcap_{1 \leq i \leq d} \{\zeta \in W^d \mid \pi_i(\zeta) \neq 1\}.$$

□

Proposition 3.8 ([17, Theorem 5.6])**.** *Let $S \subset W^d$ be a semi-algebraic set and let $F \in \Lambda$. Then there exists a unique $f(U, V) \in \mathbb{Q}[U, V]$ with $\deg_V f \leq 1$ and total degree $\deg f \leq d$ such that*

$$\sum_{\zeta \in S \cap W(n)^d} v(F(\zeta - 1)) = f(p^n, n)$$

for every sufficiently large n.

Proposition 3.9. *Let $F \in \Lambda$. Then there exist $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ such that*

$$\sum_{\zeta \in (W(n) \setminus \{1\})^d} v(F(\zeta - 1)) = (\mu p^n + \lambda n + O(1)) p^{(d-1)n}.$$

Proof. We have

$$\sum_{\zeta \in W(n)^d} v(F(\zeta - 1)) = \sum_{\zeta \in (W(n) \setminus \{1\})^d} v(F(\zeta - 1)) + \sum_{\text{others}} v(F(\zeta - 1)).$$

By Proposition 3.8, we have

$$\sum_{\text{others}} v(F(\zeta - 1)) = O(p^{(d-1)n}).$$

By Proposition 3.4, we complete the proof. □

4 On Alexander polynomials

In this section, we study some basic properties on Alexander polynomials. Details of the theory of Alexander polynomials are written in [6, Chapter 3 and 4].

Let R be a Noetherian unique factorization domain and let $\mathcal{M}, \mathcal{N}$ be finitely generated R-modules. $\mathcal{M}$ is said to be *pseudo-isomorphic* to $\mathcal{N}$ if and only if, for each prime ideal $\mathfrak{p}$ of height one in R, the induced $R_{\mathfrak{p}}$-homomorphism $\mathcal{M}_{\mathfrak{p}} \to \mathcal{N}_{\mathfrak{p}}$ is an isomorphism, where $\mathcal{M}_{\mathfrak{p}}, \mathcal{N}_{\mathfrak{p}}$ denote the localizations of $\mathcal{M}, \mathcal{N}$ at $\mathfrak{p}$ respectively. Also, since R is Noetherian, we may choose an exact sequence

$$R^r \to R^s \to \mathcal{M} \to 0.$$

The ideal of R that is generated by the s-subdeterminants of the presentation matrix of $R^r \to R^s$ is called the Fitting ideal of $\mathcal{M}$, and we denote it by $\mathrm{Fitt}(\mathcal{M})$. If $r < s$, then we define $\mathrm{Fitt}(\mathcal{M}) = 0$. It is known that the definition of Fitting ideals is independent of the choices of exact sequences. Details of the theory of Fitting ideals are written in [20, Chapter 3]. The *divisorial hull* $\mathrm{d.h.}(\mathfrak{a})$ of $\mathfrak{a}$ is the intersection of the principal ideals that contain $\mathfrak{a}$.

Proposition 4.1. *Let $\mathcal{M}$ be a finitely generated R-module. If* $\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M})) \neq 0$, *then $\mathcal{M}$ is a torsion R-module.*

Proof. By the definition of the divisorial hull, if $\mathrm{Fitt}(\mathcal{M}) = 0$, then $\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M})) = 0$. By the theory of Fitting ideals, we have $\mathrm{Fitt}(\mathcal{M}) \subset \mathrm{Ann}_R(\mathcal{M})$, where $\mathrm{Ann}_R(\mathcal{M})$ denotes the annihilator of $\mathcal{M}$ over R. Thus, the assumption $\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M})) \neq 0$ implies $\mathrm{Fitt}(\mathcal{M}) \neq 0$, and hence $\mathrm{Ann}_R(\mathcal{M}) \neq 0$. Therefore, $\mathcal{M}$ is a torsion R-module. □

Remark 4.2. When $d = 1$, for link modules, it is known that $\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M})) \neq 0$ holds if and only if $\mathcal{M}$ is a torsion R-module (cf. [13, Lemma 3.1]).

An R-module of the form $\bigoplus_{i=1}^{s} R/\mathfrak{p}_i^{m_i}$, where $\mathfrak{p}_i$ are height one prime ideals in R, is called an *elementary R-module*.

Let $\mathcal{M}$ be a finitely generated torsion R-module. Then, by [27, Proposition 3.1.6] or [21, Theorem 2.3.6(Japanese)], there is an elementary R-module $\mathcal{E} := \bigoplus_{i=1}^{s} R/\mathfrak{p}_i^{m_i}$ such that $\mathcal{M}$ is pseudo-isomorphic to $\mathcal{E}$. Since R is a Noetherian unique factorization domain, every height one prime ideal is principal. Hence $\prod \mathfrak{p}_i^{m_i}$ is generated by an element. The element up to multiplication by units is called the *characteristic element* of $\mathcal{M}$, and we denote it by $\mathrm{Char}\mathcal{M}$. For a finitely generated non-torsion R-module, we make a convention $\mathrm{Char}\mathcal{M} = 0$.

Proposition 4.3. *Let $\mathcal{M}$ be a finitely generated torsion R-module. Then we have*

$$(\mathrm{Char}\mathcal{M}) = \mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M})).$$

Proof. Since $\mathcal{M}$ is pseudo-isomorphic to an elementary R-module $\mathcal{E} := \bigoplus_{i=1}^{s} R/\mathfrak{p}_i^{m_i}$, for each height one prime ideal $\mathfrak{p}$ in R, we have

$$\mathcal{M}_{\mathfrak{p}} \cong (\bigoplus R/\mathfrak{p}_i^{m_i})_{\mathfrak{p}}.$$

This implies

$$\mathrm{Fitt}(\mathcal{M})_{\mathfrak{p}} = \mathrm{Fitt}(\mathcal{M}_{\mathfrak{p}}) = \mathrm{Fitt}((\bigoplus R/\mathfrak{p}_i^{m_i})_{\mathfrak{p}}) = \mathrm{Fitt}(\bigoplus R/\mathfrak{p}_i^{m_i})_{\mathfrak{p}}.$$

Since $\mathrm{Fitt}(\bigoplus R/\mathfrak{p}_i^{m_i}) = \prod_i \mathfrak{p}_i^{m_i}$, we have

$$\mathrm{Fitt}(\mathcal{M})_{\mathfrak{p}} = \prod_{i, \mathfrak{p}_i = \mathfrak{p}} \mathfrak{p}_i^{m_i}.$$

By [6, Lemma 3.2], we have

$$\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M})) = \bigcap_{\mathfrak{p}} \mathrm{Fitt}(\mathcal{M})_{\mathfrak{p}}.$$

Therefore, we obtain

$$\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M})) = \prod \mathfrak{p}_i^{m_i} = (\mathrm{Char}\mathcal{M}).$$

□

Now, we have an embedding

$$\Lambda_{\mathbb{Z}} \ni t_i \to 1 + T_i \in \Lambda.$$

Let $\mathcal{M}$ be a finitely generated $\Lambda_{\mathbb{Z}}$-module. Then, for each $n \geq 0$, $\mathcal{M}_n := \mathcal{M}/I_n\mathcal{M}$ is a $\mathbb{Z}[t_1^{\mathbb{Z}/n\mathbb{Z}}, \ldots, t_d^{\mathbb{Z}/n\mathbb{Z}}]$-module. Hence $\mathcal{M}_n \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p[t_1^{\mathbb{Z}/n\mathbb{Z}}, \ldots, t_d^{\mathbb{Z}/n\mathbb{Z}}]$-module. Therefore, for any $m \geq n$, $\mathcal{M}_{p^n} \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p[t_1^{\mathbb{Z}/p^m\mathbb{Z}}, \ldots, t_d^{\mathbb{Z}/p^m\mathbb{Z}}]$-module. Accordingly, $\mathcal{M}_{p^n} \otimes \mathbb{Z}_p$ are Λ-modules. Thus $\varprojlim_n \mathcal{M}_{p^n} \otimes \mathbb{Z}_p$ is also a Λ-module. Let $\widehat{\mathcal{M}}$ denote the Λ-module $\varprojlim_n \mathcal{M}_{p^n} \otimes \mathbb{Z}_p$.

Lemma 4.4. *Let $a_1, \ldots, a_n$ be elements in $\Lambda_{\mathbb{Z}}$ and let g be their greatest common divisor in $\Lambda_{\mathbb{Z}}$. Then g is their greatest common divisor in Λ as well.*

Proof. By the definition of g, for each $1 \leq i \leq n$, there exists $a_i' \in \Lambda_{\mathbb{Z}}$ such that $a_i = g a_i'$. Suppose that there exists a prime element P of Λ and there exists a_i'' for each $1 \leq i \leq d$ such that $a_i' = P a_i''$. Since completions satisfy the going down property, $P\Lambda \cap \Lambda_{\mathbb{Z}}$ is a height one prime ideal of $\Lambda_{\mathbb{Z}}$. Since $\Lambda_{\mathbb{Z}}$ is a unique factorization domain, there exists a prime element $P_{\mathbb{Z}}$ of $\Lambda_{\mathbb{Z}}$ such that $P_{\mathbb{Z}}\Lambda_{\mathbb{Z}} = P\Lambda \cap \Lambda_{\mathbb{Z}}$. Therefore, for each $1 \leq i \leq n$, a_i can be divided by $P_{\mathbb{Z}}$ in $\Lambda_{\mathbb{Z}}$. This contradicts g is a greatest common divisor in $\Lambda_{\mathbb{Z}}$. Therefore, there exists no such a prime element, and so g is a greatest common divisor in Λ as well. □

Lemma 4.5. *Let $\mathfrak{a}$ be an ideal of $\Lambda_{\mathbb{Z}}$. Then we have*

$$(\mathrm{d.h.}(\mathfrak{a}))\Lambda = \mathrm{d.h.}(\mathfrak{a}\Lambda)$$

in Λ.

Proof. Let $\mathcal{S}$ be a system of generators of $\mathfrak{a}$. Since $\Lambda_{\mathbb{Z}}$ is a unique factorization domain, $\mathrm{d.h.}(\mathfrak{a})$ is the principal ideal generated by the greatest common divisor of the elements in $\mathcal{S}$. Since $\mathcal{S}$ generates $\mathfrak{a}\Lambda$ in Λ and Λ is also a unique factorization domain, $\mathrm{d.h.}(\mathfrak{a}\Lambda)$ is also the principal ideal generated by the greatest common divisor of the elements in $\mathcal{S}$. Therefore, by Lemma 4.4, we have

$$(\mathrm{d.h.}(\mathfrak{a}))\Lambda = \mathrm{d.h.}(\mathfrak{a}\Lambda)$$

in Λ. □

Proposition 4.6. *Let $\mathcal{M}$ be a finitely generated torsion $\Lambda_{\mathbb{Z}}$-module. Then we have*

$$(\mathrm{Char}\mathcal{M}) = (\mathrm{Char}\widehat{\mathcal{M}})$$

in Λ.

Proof. Consider an exact sequence

$$\Lambda_{\mathbb{Z}}^r \to \Lambda_{\mathbb{Z}}^s \to \mathcal{M} \to 0.$$

Since taking tensor products, quotients, and projective limits are right exact, we have an exact sequence

$$\Lambda^r \to \Lambda^s \to \widehat{\mathcal{M}} \to 0.$$

Since the presentation matrices of these exact sequences coincide, we have

$$\mathrm{Fitt}(\mathcal{M})\Lambda = \mathrm{Fitt}(\widehat{\mathcal{M}}).$$

Hence

$$\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M})\Lambda) = \mathrm{d.h.}(\mathrm{Fitt}(\widehat{\mathcal{M}})).$$

Therefore, by Lemma 4.5 and Proposition 4.3, we obtain

$$\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M}))\Lambda = \mathrm{d.h.}(\mathrm{Fitt}(\widehat{\mathcal{M}})) = (\mathrm{Char}\widehat{\mathcal{M}}).$$

□

Definition 4.7. Let L be a link in a $\mathbb{Q}\mathrm{HS}^3$ M and $X := M - N(L)$. Let $X_\infty \to X$ be a $\mathbb{Z}^d$-cover. Then, $H_1(X_\infty)$ is a finitely generated $\Lambda_{\mathbb{Z}}$-module, and a generator of $\mathrm{d.h.}(\mathrm{Fitt}(\mathcal{M}))$ up to multiplication by units is called *the Alexander polynomial* of $X_\infty \to X$. We will denote it by $\Delta(t_1, \dots, t_d)$.

If L is a d-component link in a $\mathbb{Q}\mathrm{HS}^3$ M and $X_\infty \to X$ is its unique $\mathbb{Z}^d$-cover, then the Alexander polynomial of $X_\infty \to X$ is called *the Alexander polynomial* of L. We will denote it by $\Delta_L(t_1, \dots, t_d)$.

Remark 4.8. By Proposition 4.3, we have $\mathrm{Char}H_1(X_\infty) = \Delta(t_1, \dots, t_d)$ up to multiplication by units. Moreover, by Proposition 4.1, if $\Delta \neq 0$ in $\Lambda_{\mathbb{Z}}$, then $H_1(X_\infty)$ is a finitely generated torsion $\Lambda_{\mathbb{Z}}$-module.

5 Lemmas on Λ-modules

In this section, we briefly review some results of Cuoco–Monsky [4, Theorem I] on Λ-modules, which will be used in Section 7.

For each $n \geq 1$, let $\mathcal{I}_{p^n}$ be the ideal of Λ generated by $\{(1+T_j)^{p^n} - 1 \mid 1 \leq j \leq d\}$.

Proposition 5.1 ([4, Theorem 3.4]). *Let $\mathcal{M}$ be a torsion Λ-module with characteristic element F. Suppose* $\mathrm{rank}_{\mathbb{Z}_p}(\mathcal{M}/\mathcal{I}_{p^n}\mathcal{M}) = O(p^{(d-2)n})$. *Then we have*

$$e(\mathcal{M}/I_{p^n}\mathcal{M}) = (\mu(F)p^n + \lambda(F)n + O(1))p^{(d-1)n},$$

where $\mu(F), \lambda(F)$ are non-negative integers defined in Section 3.1.

Remark 5.2. The assumption $\mathrm{rank}_{\mathbb{Z}_p}(\mathcal{M}/\mathcal{I}_{p^n}\mathcal{M}) = O(p^{(d-2)n})$ is equivalent to the condition that $\Delta(1+T_1,\ldots,1+T_d)$ has no special prime factors in Λ in the sense of Cuoco–Monsky [4, Theorem 3.13]. Also, Monsky [18, Theorem 3.12 with $S=\emptyset$] showed that the $O(p^{(d-1)n})$ part can be refined to $\alpha^* + O(np^{(d-2)n})$ for some $\alpha^* \in \mathbb{R}$ ($\alpha^* \in \mathbb{Q}$ if $d=2$).

For $\zeta = (\zeta_1, \cdots, \zeta_d) \in W^d$, define $\mathbb{Z}_p[\zeta] = \mathbb{Z}_p[\zeta_1,\ldots,\zeta_d]$. Let $\mathcal{M}$ be a finitely generated torsion Λ-module. For each $\zeta \in W^d$, put $\mathcal{M}_\zeta = \mathcal{M} \otimes_\Lambda \mathbb{Z}_p[\zeta]$. Then $\mathcal{M}_\zeta$ is a finitely generated $\mathbb{Z}_p[\zeta]$-module via the ring homomorphism $\Lambda \ni F \mapsto F(\zeta - 1) \in \mathbb{Z}_p[\zeta]$. Let $r_\zeta(\mathcal{M}_\zeta) := \mathrm{rank}_{\mathbb{Z}_p[\zeta]} \mathcal{M}_\zeta$. Define

$$Z(\mathcal{M}) := \{\zeta \in W^d \mid r_\zeta(\mathcal{M}_\zeta) \geq 1\}$$

and

$$Z_n(\mathcal{M}) := Z(\mathcal{M}) \cap W(n)^d.$$

Lemma 5.3 ([4, Lemma 3.7]). *There exists $s \geq 1$ such that*

$$|Z_n(\mathcal{M})| \leq \mathrm{rank}_{\mathbb{Z}_p}(\mathcal{M}/\mathcal{I}_{p^n}\mathcal{M}) \leq s|Z_n(\mathcal{M})|$$

for all $n \geq 1$.

Lemma 5.4 ([4, Lemma 3.3]). *Let $\mathcal{M}, \mathcal{N}$ be finitely generated torsion Λ-modules. Suppose $\mathcal{M}$ is pseudo-isomorphic to $\mathcal{N}$. Then we have*

$$|\mathrm{rank}_{\mathbb{Z}_p}(\mathcal{M}/\mathcal{I}_{p^n}\mathcal{M}) - \mathrm{rank}_{\mathbb{Z}_p}(\mathcal{N}/\mathcal{I}_{p^n}\mathcal{N})| = O(p^{(d-2)n}).$$

Lemma 5.5. *Let $\mathcal{M}$ be a finitely generated torsion Λ-module and F the characteristic element of $\mathcal{M}$. Suppose that $F(T_1,\ldots,T_d)$ does not vanish on $\{\zeta - 1 \in \overline{\mathbb{Q}}_p^d \mid \zeta \in W^d\}$. Then we have*

$$\mathrm{rank}_{\mathbb{Z}_p}(\mathcal{M}/\mathcal{I}_{p^n}\mathcal{M}) = O(p^{(d-2)n}).$$

Proof. Since $\mathcal{M}$ is torsion over Λ, $\mathcal{M}$ is pseudo-isomorphic to an elementary Λ-module $\mathcal{E}$. By the definition of characteristic element, we have $F(T_1,\ldots,T_d)\mathcal{E} = 0$. By the definition of the action of $\mathbb{Z}_p[\zeta]$ on $\mathcal{E}_\zeta$, we have $F(\zeta-1)\mathcal{E}_\zeta = 0$. By assumption, we have $F(\zeta - 1) \neq 0$ for any $\zeta \in W^d$. Therefore, $\mathcal{E}_\zeta$ is a torsion $\mathbb{Z}_p[\zeta]$-module. Hence $r_\zeta(\mathcal{E}_\zeta) = 0$. Therefore, we have $Z(\mathcal{E}) = \emptyset$, and so is $Z_n(\mathcal{E})$. By Lemma 5.3, we have

$$\mathrm{rank}_{\mathbb{Z}_p}(\mathcal{E}/\mathcal{I}_{p^n}\mathcal{E}) = 0.$$

Thus, by Lemma 5.4, we have the assertion. □

Remark 5.6. If there exists a polynomial $f(t_1, \ldots, t_d) \in \mathbb{Z}[t_1, \ldots, t_d]$ such that $f(1+T_1, \ldots, 1+T_d) = F(T_1, \ldots, T_d)$, then the assumption can be replaced by "$f(t_1, \ldots, t_d)$ does not vanish on W^d."

6 Fundamental result for 3-manifolds

In this section, we prove that the sizes of the torsion parts of the first homology groups of certain 3-manifolds are sufficiently close to the sizes of the torsion parts of quotients of homology groups of certain infinite coverings. The result in this section allows one to apply multi-variable Iwasawa theory to link cases.

Let X be a compact connected orientable 3-manifold with a surjective homomorphism $\pi_1(X) \twoheadrightarrow \mathbb{Z}^d$, where d is a positive integer, and let $X_\infty \to X$ denote the corresponding $\mathbb{Z}^d$-cover. Since $\pi_1(X)/\pi_1(X_\infty) \cong \mathbb{Z}^d$, we may choose a basis $\{t_1, \ldots, t_d\}$ of $\pi_1(X)/\pi_1(X_\infty)$. Let $\Lambda_\mathbb{Z} := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. Then $H_1(X_\infty)$ is a finitely generated $\Lambda_\mathbb{Z}$-module. Let $I_n := (t_1^n - 1, t_2^n - 1, \ldots, t_d^n - 1)$ be the ideal of $\Lambda_\mathbb{Z}$ and let X_n be the $(\mathbb{Z}/n\mathbb{Z})^d$-cover corresponding to $\ker(\pi_1(X) \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^d)$.

Theorem 6.1. *Under the setting as above, we have*

$$e(H_1(X_{p^n})) = e(H_1(X_\infty)/I_{p^n}H_1(X_\infty)) + O(n) \ (n \to \infty).$$

Proof. Put $N := \pi_1(X_\infty)^{\mathrm{ab}} = H_1(X_\infty)$. We have an exact sequence of groups

$$1 \to N \to \pi_1(X)/\pi_1(X_\infty)^c \xrightarrow{\mathcal{P}} \pi_1(X)/\pi_1(X_\infty) \to 1,$$

where $\pi_1(X_\infty)^c$ denotes the commutator subgroup of $\pi_1(X_\infty)$. Let $x_1, \ldots, x_d$ be elements in the inverse images $\mathcal{P}^{-1}(t_1), \ldots, \mathcal{P}^{-1}(t_d)$ respectively. Then we have

$$H_1(X_n) = \langle x_1^n, \ldots, x_d^n, N\rangle / \langle [x_i^n, x_j^n], I_nN (1 \le i < j \le d)\rangle.$$

Indeed, we have an exact commutative diagram of groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & \pi_1(X)/\pi_1(X_\infty)^c & \xrightarrow{\mathcal{P}} & \pi_1(X)/\pi_1(X_\infty) & \longrightarrow & 1 \\
 & & \| & & \cup & & \cup & & \\
1 & \longrightarrow & N & \longrightarrow & \mathcal{P}^{-1}((n\mathbb{Z})^d) & \longrightarrow & (n\mathbb{Z})^d & \longrightarrow & 1.
\end{array}$$

Since

$$\pi_1(X_n)/\pi_1(X_\infty) = \mathcal{P}^{-1}((n\mathbb{Z})^d) = \langle x_1^n, \ldots, x_d^n, N\rangle$$

and

$$(t_i - 1)y = y^{x_i}y^{-1} = x_i y x_i^{-1} y^{-1} \text{ for each } y \in N,$$

we obtain

$$\begin{aligned} H_1(X_n) &= \langle x_1^n, \ldots, x_d^n, N\rangle^{\mathrm{ab}} \\ &= \langle x_1^n, \ldots, x_d^n, N\rangle / \langle [x_i^n, x_j^n], I_n N (1 \le i < j \le d)\rangle. \end{aligned}$$

Since $\mathcal{P}^{-1}((n\mathbb{Z})^d)$ is a normal subgroup of $\pi_1(X)/\pi_1(X_\infty)^c$, so is $(\mathcal{P}^{-1}((n\mathbb{Z})^d))^c = \langle [x_i^n, x_j^n], I_n N(1 \le i < j \le d)\rangle$. This implies that $\langle [x_i^n, x_j^n], I_n N(1 \le i < j \le d)\rangle$ is a $\Lambda_{\mathbb{Z}}$-module. Therefore,

$$0 \to N/\langle [x_i^n, x_j^n], I_n N(1 \le i < j \le d)\rangle \to H_1(X_n) \to (n\mathbb{Z})^d \to 0$$

is an exact sequence of $\Lambda_{\mathbb{Z}}$-modules. Since this exact sequence splits, we have

$$\mathrm{tor}_{\mathbb{Z}}(H_1(X_n)) \cong \mathrm{tor}_{\mathbb{Z}}(N/\langle [x_i^n, x_j^n], I_n N(1 \le i < j \le d)\rangle).$$

as $\mathbb{Z}$-modules. We compare this to $\mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_n H_1(X_\infty)) = \mathrm{tor}_{\mathbb{Z}}(N/I_n N)$. If we put $c_{ij} := [x_i, x_j] \in N$ and $\mathcal{T}_n(t) := \sum_{k=0}^{n-1} t^k$, then we have $[x_i^n, x_j^n] = \mathcal{T}_n(t_i)\mathcal{T}_n(t_j)c_{ij}$ since

$$\begin{aligned} [x_i^n, x_j^n] &= x_i(x_i^{n-1}x_j^n x_i^{-(n-1)} x_j^{-n})x_i^{-1}(x_i x_j^n x_i^{-1} x_j^{-n}) \\ &= [x_i^{n-1}, x_j^n]^{x_i}[x_i, x_j^n] \\ &= ([x_i^{n-2}, x_j^n]^{x_i}[x_i, x_j^n])^{x_i}[x_i, x_j^n] \\ &= \ldots \\ &= (t_i^{n-1} + \cdots + t_i + 1)[x_i, x_j^n] \\ &= (t_i^{n-1} + \cdots + t_i + 1)[x_i, x_j][x_i, x_j^{n-1}]^{x_j} \\ &= \ldots \\ &= (t_i^{n-1} + \cdots + t_i + 1)(t_j^{n-1} + \cdots + t_j + 1)[x_i, x_j] \\ &= \mathcal{T}_n(t_i)\mathcal{T}_n(t_j)c_{ij}. \end{aligned}$$

Put $c_{ij}^{(n)} := [x_i^n, x_j^n]$. Then we find that $c_{ii}^{(n)} = 0$ and $c_{ji}^{(n)} = -c_{ij}^{(n)}$. Let C_n be the $\Lambda_{\mathbb{Z}}$-submodule of N that is generated by $\{c_{ij}^{(n)}(1 \le i < j \le d)\}$. Then we have an exact sequence of $\Lambda_{\mathbb{Z}}$-modules

$$0 \to (C_n + I_n N)/I_n N \to N/I_n N \to N/(C_n + I_n N) \to 0.$$

Since $\langle [x_i^n, x_j^n], I_n N(1 \le i < j \le d)\rangle$ is a $\Lambda_{\mathbb{Z}}$-module, we have

$$\langle [x_i^n, x_j^n], I_n N(1 \le i < j \le d)\rangle = C_n + I_n N. \tag{6.1}$$

Therefore, we have $\mathrm{tor}_{\mathbb{Z}}(N/(C_n + I_n N)) \cong \mathrm{tor}_{\mathbb{Z}}(H_1(X_n))$.

We shall show that the action of t_i on $(C_n + I_n N)/I_n N$ is trivial. Indeed, for i, j, we have

$$(t_i - 1)c_{ij}^{(n)} = (t_i^n - 1)\mathcal{T}_n(t_j)c_{ij} \in I_n N$$

and

$$(t_j-1)c_{ij}^{(n)} = (t_j^n-1)\mathcal{T}_n(t_i)c_{ij} \in I_nN.$$

On the other hand, for $k \neq i,j$, by the Hall-Witt identity ($[x,[y^{-1},z]]^y \cdot [y,[z^{-1},x]]^z \cdot [z,[x^{-1},y]]^x = 1$), we have

$$t_j(t_i^{-1}-1)[x_j^{-1},x_k] + t_k(t_j-1)[x_k^{-1},x_i^{-1}] + t_i^{-1}(t_k-1)c_{ij} = 0.$$

(We remark that, by a further calculation, we have $(t_i-1)c_{jk}+(t_j-1)c_{ki}+(t_k-1)c_{ij}=0$.) Mutiplying this equation by $\mathcal{T}_n(t_j)\mathcal{T}_n(t_i)$, we obtain

$$-t_jt_i^{-1}(t_i^n-1)\mathcal{T}_n(t_j)[x_j^{-1},x_k] + t_k(t_j^n-1)\mathcal{T}_n(t_i)[x_k^{-1},x_i^{-1}] + t_i^{-1}(t_k-1)c_{ij}^{(n)} = 0.$$

Multiplying by t_i, we conclude that

$$(t_k-1)c_{ij}^{(n)} = t_j(t_i^n-1)\mathcal{T}_n(t_j)[x_j^{-1},x_k] - t_it_k(t_j^n-1)\mathcal{T}_n(t_i)[x_k^{-1},x_i^{-1}] \in I_nN.$$

Therefore, t_i acts trivially on $(C_n+I_nN)/I_nN$. Moreover, by (6.1), we see that $(C_n+I_nN)/I_nN$ is a $\mathbb{Z}$-module generated by at most $\frac{d(d-1)}{2}$ elements. Let P_n be the inverse image of $\mathrm{tor}_{\mathbb{Z}}(N/(C_n+I_nN))$ under $N/I_nN \to N/(C_n+I_nN)$. Then, we have $\mathrm{tor}_{\mathbb{Z}}(P_n) = \mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_nH_1(X_\infty))$, and

$$0 \to (C_n+I_nN)/I_nN \to P_n \to \mathrm{tor}_{\mathbb{Z}}(N/(C_n+I_nN)) \to 0$$

is an exact sequence of $\Lambda_{\mathbb{Z}}$-modules. For an arbitrary pair of positive integers $n \mid m$, we have an exact commutative diagram of $\Lambda_{\mathbb{Z}}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (C_m+I_mN)/I_mN & \longrightarrow & P_m & \longrightarrow & \mathrm{tor}_{\mathbb{Z}}(N/(C_m+I_mN)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'_{m,n}} & & \downarrow{\scriptstyle \varphi_{m,n}} & & \downarrow{\scriptstyle \varphi''_{m,n}} & & \\ 0 & \longrightarrow & (C_n+I_nN)/I_nN & \longrightarrow & P_n & \longrightarrow & \mathrm{tor}_{\mathbb{Z}}(N/(C_n+I_nN)) & \longrightarrow & 0, \end{array} \qquad (6.2)$$

where $\varphi'_{m,n}, \varphi_{m,n}, \varphi''_{m,n}$ are the maps induced by the identity map of N respectively. Define a $\Lambda_{\mathbb{Z}}$-homomorphism $\psi_{n,m} : (C_n+I_nN)/I_nN \to (C_m+I_mN)/I_mN$ by $\psi_{n,m} = \mathcal{T}_{m/n}(t_1^n)\cdots\mathcal{T}_{m/n}(t_d^n)$. This is well-defined since

$$\begin{aligned} & \mathcal{T}_{m/n}(t_i^n)\mathcal{T}_n(t_i) \\ = \; & ((t_i^n)^{m/n-1}+\cdots+t_i^n+1)(t_i^{n-1}+\cdots+t_i+1) \\ = \; & t_i^{m-1}+\cdots+t_i+1 \\ = \; & \mathcal{T}_m(t_i) \end{aligned}$$

implies that t_i^n-1 maps to t_i^m-1 via $\times\mathcal{T}_{m/n}(t_i^n)$. Since t_i acts trivially on $(C_n+I_nN)/I_nN$, the map multiplying by $\prod_{1\leq i\leq d}((t_i^n)^{m/n-1}+(t_i^n)^{m/n-2}+\cdots+t_i^n+1))$ becomes the map multiplying by $\prod_{1\le i\le d}(1+1+\cdots+1+1)$ via $\varphi'_{m,n}\circ\psi_{n,m}$ and $\psi_{n,m}\circ\varphi'_{m,n}$. This implies $\varphi'_{pn,n}\circ\psi_{n,pn}=p^d$ and $\psi_{n,pn}\circ\varphi'_{pn,n}=p^d$. Since we have

$$\begin{aligned} c_{ij}^{(m)} &= \mathcal{T}_m(t_i)\mathcal{T}_m(t_j)c_{ij} \\ &= \mathcal{T}_{m/n}(t_i)\mathcal{T}_{m/n}(t_j)\mathcal{T}_n(t_i)\mathcal{T}_n(t_j)c_{ij} \\ &= \mathcal{T}_{m/n}(t_i)\mathcal{T}_{m/n}(t_j)c_{ij}^{(n)} \end{aligned}$$

and t_i acts trivially on $(C_n+I_nN)/I_nN$, one finds that

$$\operatorname{im}\varphi'_{pn,n}=p^2((C_n+I_nN)/I_nN).$$

Likewise, since

$$\mathcal{T}_{m/n}(t_1)\cdots\mathcal{T}_{m/n}(t_d)c_{d-1,d}^{(n)}=\mathcal{T}_{m/n}(t_1)\cdots\mathcal{T}_{m/n}(t_{d-2})c_{d-1,d}^{(m)},$$

we have

$$\operatorname{im}\psi_{n,pn}=p^{d-2}((C_{pn}+I_{pn}N)/I_{pn}N).$$

Since there are $\mathbb{Z}$-homomorphisms onto subgroups of finite indices each other, the finitely generated $\mathbb{Z}$-modules $(C_n+I_nN)/I_nN$ and $(C_{pn}+I_{pn}N)/I_{pn}N$ have the same $\mathbb{Z}$-rank, and so $\ker\varphi'_{pn,n}$ and $\ker\psi_{n,pn}$ are finite. We shall show that

$$\psi_{n,pn}(\operatorname{tor}_{\mathbb{Z}}((C_n+I_nN)/I_nN))=p^{d-2}\operatorname{tor}_{\mathbb{Z}}((C_{pn}+I_{pn}N)/I_{pn}N).$$

Indeed, we have

$$\begin{aligned} \operatorname{im}\psi_{n,pn} &= p^{d-2}((C_{pn}+I_{pn}N)/I_{pn}N) \\ &= p^{d-2}(\mathbb{Z}^r\oplus(\operatorname{tor}_{\mathbb{Z}}((C_{pn}+I_{pn}N)/I_{pn}N)) \\ &= p^{d-2}\mathbb{Z}^r\oplus p^{d-2}\operatorname{tor}_{\mathbb{Z}}((C_{pn}+I_{pn}N)/I_{pn}N). \end{aligned}$$

for some $r\ge 0$. Now, $\psi_{n,pn}$ induces a map

$$\psi_{n,pn}^{-1}(p^{d-2}\operatorname{tor}_{\mathbb{Z}}((C_{pn}+I_{pn}N)/I_{pn}N))\to p^{d-2}\operatorname{tor}_{\mathbb{Z}}((C_{pn}+I_{pn}N)/I_{pn}N)),$$

and we have

$$\operatorname{tor}_{\mathbb{Z}}((C_n+I_nN)/I_nN)\subset\psi_{n,pn}^{-1}(p^{d-2}\operatorname{tor}_{\mathbb{Z}}((C_{pn}+I_{pn}N)/I_{pn}N)).$$

Since the kernel of this map is finite, we have that both the image and the kernel of this map are $\mathbb{Z}$-torsion. Therefore, we must have

$$\operatorname{tor}_{\mathbb{Z}}((C_n+I_nN)/I_nN)=\psi_{n,pn}^{-1}(p^{d-2}\operatorname{tor}_{\mathbb{Z}}((C_{pn}+I_{pn}N)/I_{pn}N)),$$

i.e.,

$$\psi_{n,pn}(\operatorname{tor}_{\mathbb{Z}}((C_n+I_nN)/I_nN))=p^{d-2}\operatorname{tor}_{\mathbb{Z}}((C_{pn}+I_{pn}N)/I_{pn}N).$$

Since

$$0 \to \mathrm{tor}_{\mathbb{Z}}((C_n + I_nN)/I_nN) \to \mathrm{tor}_{\mathbb{Z}}(P_n) \to \mathrm{tor}_{\mathbb{Z}}(N/(C_n + I_nN))$$

is an exact sequence of $\Lambda_{\mathbb{Z}}$-modules, we have

$$\frac{|\mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_nH_1(X_\infty)|}{|\mathrm{tor}_{\mathbb{Z}}((C_n + I_nN)/I_nN)|} \text{ divides } |\,\mathrm{tor}_{\mathbb{Z}}(H_1(X_n))|.$$

Since

$$\psi_{n,pn}(\mathrm{tor}_{\mathbb{Z}}((C_n + I_nN)/I_nN)) = p^{d-2}\mathrm{tor}_{\mathbb{Z}}((C_{pn} + I_{pn}N)/I_{pn}N),$$

we have

$$\begin{aligned} & |\mathrm{tor}_{\mathbb{Z}}((C_n + I_nN)/I_nN)| \\ = \ & |\ker \psi_{n,pn}| \cdot |p^{d-2}\mathrm{tor}_{\mathbb{Z}}((C_{pn} + I_{pn}N)/I_{pn}N)| \\ = \ & |\ker \psi_{n,pn}| \cdot \frac{|\mathrm{tor}_{\mathbb{Z}}((C_{pn} + I_{pn}N)/I_{pn}N)|}{|\mathrm{tor}_{\mathbb{Z}}((C_{pn} + I_{pn}N)/I_{pn}N)/p^{d-2}\mathrm{tor}_{\mathbb{Z}}((C_{pn} + I_{pn}N)/I_{pn}N)|}. \end{aligned}$$

Since $(C_{pn} + I_{pn}N)/I_{pn}N$ is generated by at most $\frac{d(d-1)}{2}$ elements over $\mathbb{Z}$, we must have

$$|\mathrm{tor}_{\mathbb{Z}}((C_{pn} + I_{pn}N)/I_{pn}N)/p^{d-2}\mathrm{tor}_{\mathbb{Z}}((C_{pn} + I_{pn}N)/I_{pn}N)| \text{ divides } p^{\frac{d(d-1)(d-2)}{2}}.$$

Therefore,

$$|\mathrm{tor}_{\mathbb{Z}}((C_{pn} + I_{pn}N)/I_{pn}N)| \text{ divides } p^{\frac{d(d-1)(d-2)}{2}}|\mathrm{tor}_{\mathbb{Z}}((C_n + I_nN)/I_nN)|.$$

By iterating this and putting $A_1 := |\mathrm{tor}_{\mathbb{Z}}((C_1 + I_1N)/I_1N)|$, we obtain that

$$|\mathrm{tor}_{\mathbb{Z}}((C_{p^m} + I_{p^m}N)/I_{p^m}N)| \text{ divides } A_1p^{m\frac{d(d-1)(d-2)}{2}},$$

and so

$$|\mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_{p^m}H_1(X_\infty))| \text{ divides } A_1p^{m\frac{d(d-1)(d-2)}{2}}|\mathrm{tor}_{\mathbb{Z}}(H_1(X_{p^m}))|.$$

This implies

$$e(H_1(X_\infty)/I_{p^n}H_1(X_\infty)) \le e(H_1(X_{p^n})) + O(n) \ (n \to \infty). \tag{6.3}$$

Applying the snake lemma to the commutative diagram (6.2), we have an exact sequence of $\Lambda_{\mathbb{Z}}$-modules

$$0 \to \ker \varphi'_{pn,n} \to \ker \varphi_{pn,n} \to \ker \varphi''_{pn,n} \to \mathrm{coker}\,\varphi'_{pn,n} \to \mathrm{coker}\,\varphi_{pn,n} \to \mathrm{coker}\,\varphi''_{pn,n} \to 0.$$

Since $\ker \varphi'_{pn,n}, \ker \varphi''_{pn,n}, \mathrm{coker}\,\varphi'_{pn,n}, \mathrm{coker}\,\varphi''_{pn,n}$ are finite, so are $\ker \varphi_{pn,n}, \mathrm{coker}\,\varphi_{pn,n}$. Hence we have

$$\frac{|\ker \varphi''_{pn,n}|}{|\mathrm{coker}\,\varphi''_{pn,n}|} = \frac{|\ker \varphi_{pn,n}|}{|\mathrm{coker}\,\varphi_{pn,n}|}\frac{|\mathrm{coker}\,\varphi'_{pn,n}|}{|\ker \varphi'_{pn,n}|}.$$

By the homomorphism theorem, we have

$$\frac{|\ker \varphi''_{pn,n}|}{|\operatorname{coker} \varphi''_{pn,n}|} = \frac{|\mathrm{tor}_{\mathbb{Z}} H_1(X_{pn})|}{|\mathrm{tor}_{\mathbb{Z}} H_1(X_n)|}.$$

Since $\ker \varphi_{pn,n}$ is finite, we have

$$\varphi_{pn,n}^{-1}(\mathrm{tor}_{\mathbb{Z}}(P_n)) = \mathrm{tor}_{\mathbb{Z}}(P_{pn}).$$

Let $\varphi''' : \mathrm{tor}_{\mathbb{Z}}(P_{pn}) \to \mathrm{tor}_{\mathbb{Z}}(P_n)$ be the restriction map of $\varphi_{pn,n}$. Then we have $\ker \varphi''' = \ker \varphi_{pn,n}$ and $\operatorname{coker} \varphi''' \subset \operatorname{coker} \varphi_{pn,n}$. Therefore, we have

$$\frac{|\ker \varphi'''|}{|\operatorname{coker} \varphi'''|} = \frac{|\mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_{pn} H_1(X_\infty))|}{|\mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_n H_1(X_\infty))|}.$$

Accordingly, we obtain

$$\frac{|\mathrm{tor}_{\mathbb{Z}} H_1(X_{pn})|}{|\mathrm{tor}_{\mathbb{Z}} H_1(X_n)|} = \frac{|\mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_{pn} H_1(X_\infty))|}{|\mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_n H_1(X_\infty))|} \frac{|\operatorname{coker} \varphi'_{pn,n}|}{|\operatorname{coker} \varphi_{pn,n}/\operatorname{coker} \varphi'''| \cdot |\ker \varphi'_{pn,n}|}.$$

Put $A_2 := |\mathrm{tor}_{\mathbb{Z}}(H_1(X_1))|$. Since

$$|\operatorname{coker} \varphi'_{pn,n}| = |(C_n + I_n N)/(p^2 C_n + I_n N)| \text{ divides } p^{d(d-1)} (= p^{2\frac{d(d-1)}{2}}),$$

we obtain

$$|\mathrm{tor}_{\mathbb{Z}} H_1(X_{p^m})| \text{ divides } A_2 p^{d(d-1)m} |\mathrm{tor}_{\mathbb{Z}}(H_1(X_\infty)/I_{p^m} H_1(X_\infty))|.$$

This implies

$$e(H_1(X_{p^n})) \leq e(H_1(X_\infty)/I_{p^n} H_1(X_\infty)) + O(n) \ (n \to \infty). \tag{6.4}$$

Therefore, by (6.3) and (6.4), we obtain

$$e(H_1(X_{p^n})) = e(H_1(X_\infty)/I_{p^n} H_1(X_\infty)) + O(n) \ (n \to \infty).$$

□

7 The main results

In this section, we prove our three main results.

7.1 Iwasawa type formula for links

In this subsection, we prove results that form the cornerstone of our proof of Main result 1.

Let $\{A_n\}_n$ be an inverse system of abelian groups. We say $\{A_n\}_n$ satisfies the Mittag-Leffler condition(ML-condition) if and only if, for arbitrary $n \geq 0$, there exists $N_0 \geq n$ such that

$$N > N_0 \Longrightarrow \mathrm{im}(A_N \to A_n) = \mathrm{im}(A_{N_0} \to A_n).$$

If the all morphisms of $\{A_n\}_n$ are surjective, then it satisfies the ML-condition.

Lemma 7.1 ([12, §1])**.** *Let $\{A_n\}_n, \{B_n\}_n, \{C_n\}_n$ be inverse systems of profinite Λ-modules such that, for each $n \geq 0$, an exact sequence of Λ-modules*

$$0 \to A_n \to B_n \to C_n \to 0$$

is given. If $\{A_n\}_n$ satisfies the ML-condition, then the sequence

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

is exact.

Let X be a compact connected orientable 3-manifold with a surjective homomorphism $\pi_1(X) \twoheadrightarrow \mathbb{Z}^d$ with d a positive integer and let $X_\infty \to X$ denote the corresponding $\mathbb{Z}^d$-cover. Then $H_1(X_\infty)$ is a finitely generated $\Lambda_\mathbb{Z}$-module. Put $H_1(X_{p^n})' := H_1(X_\infty)/I_{p^n}H_1(X_\infty)$, $H_1(X_{p^n}, \mathbb{Z}_p)' := H_1(X_{p^n})' \otimes \mathbb{Z}_p$, and $\mathcal{H} := \varprojlim_n H_1(X_{p^n}, \mathbb{Z}_p)' = \widehat{H_1(X_\infty)}$. Since all of the morphisms of the inverse system $\{H_1(X_{p^n})'\}_n$ are surjective, the same statement holds for $\{H_1(X_{p^n}, \mathbb{Z}_p)'\}_n$ as well. Therefore, $\{H_1(X_{p^n}, \mathbb{Z}_p)'\}_n$ satisfies the ML-condition.

Lemma 7.2. *We have*

$$\mathcal{H}/\mathcal{I}_{p^n}\mathcal{H} \cong H_1(X_{p^n}, \mathbb{Z}_p)'.$$

as Λ-modules.

In particular, by Theorem 6.1, we have

$$e(\mathcal{H}/\mathcal{I}_{p^n}\mathcal{H}) = e(H_1(X_\infty)/I_{p^n}H_1(X_\infty)) = e(H_1(X_{p^n})) + O(n) \ (n \to \infty).$$

Proof. Let n and N be non-negative integers. Consider the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{p^{n+N}}H_1(X_\infty) & \longrightarrow & H_1(X_\infty) & \longrightarrow & H_1(X_{p^{n+N}})' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{p^n}H_1(X_\infty) & \longrightarrow & H_1(X_\infty) & \longrightarrow & H_1(X_{p^n})' & \longrightarrow & 0. \end{array}$$

By the snake lemma, we have

$$\ker(H_1(X_{p^{n+N}})' \to H_1(X_{p^n})') = \operatorname{coker}(I_{p^{n+N}}H_1(X_\infty) \to I_{p^n}H_1(X_\infty)).$$

Hence we have an exact sequence

$$0 \to I_{p^n}H_1(X_\infty)/I_{p^{n+N}}H_1(X_\infty) \to H_1(X_{p^{n+N}})' \to H_1(X_{p^n})' \to 0.$$

Since $I_{p^n}H_1(X_\infty)/I_{p^{n+N}}H_1(X_\infty) = I_{p^n}(H_1(X_\infty)/I_{p^{n+N}}H_1(X_\infty))$,

$$0 \to I_{p^n}H_1(X_{p^{n+N}})' \to H_1(X_{p^{n+N}})' \to H_1(X_{p^n})' \to 0$$

is exact. This induces the exact sequence of $\mathbb{Z}_p[t_1^{\mathbb{Z}/p^{n+N}\mathbb{Z}}, \ldots, t_d^{\mathbb{Z}/p^{n+N}\mathbb{Z}}]$-modules

$$0 \to I_{p^n} H_1(X_{p^{n+N}}, \mathbb{Z}_p)' \to H_1(X_{p^{n+N}}, \mathbb{Z}_p)' \to H_1(X_{p^n}, \mathbb{Z}_p)' \to 0.$$

Taking $N \to \infty$, by Lemma 7.1, we obtain the exact sequence of Λ-modules

$$0 \to \mathcal{I}_{p^n} \mathcal{H} \to \mathcal{H} \to H_1(X_{p^n}, \mathbb{Z}_p)' \to 0.$$

This completes the proof.

□

By Proposition 4.6, we have

$$(\mathrm{Char}\mathcal{H}) = (\mathrm{Char} H_1(X_\infty)).$$

in Λ.

Theorem 7.3. *Let X be a compact connected orientable 3-manifold with a surjective homomorphism $\pi_1(X) \twoheadrightarrow \mathbb{Z}^d$ with d a positive integer and let $X_\infty \to X$ be the corresponding $\mathbb{Z}^d$-cover. Let $F(t_1, \ldots, t_d)$ denote the characteristic element of $H_1(X_\infty)$ and suppose that $F(t_1, \ldots, t_d)$ does not vanish on W^d. Then there exist invariants $\mu, \lambda \in \mathbb{Z}_{\geq 0}$, depending only on $X_\infty \to X$ and p, such that*

$$e(H_1(X_{p^n})) = (\mu p^n + \lambda n + O(1)) p^{(d-1)n}.$$

Proof. Since $F(t_1, \ldots, t_d)$ does not vanish on W^d, by Lemma 5.5, we have

$$\mathrm{rank}_{\mathbb{Z}_p}(\mathcal{H}/\mathcal{I}_{p^n}\mathcal{H}) = O(p^{(d-2)n}).$$

By Lemma 7.2 and Proposition 5.1, we obtain

$$e(H_1(X_{p^n})) = (\mu(F) p^n + \lambda(F) n + O(1)) p^{(d-1)n}.$$

□

7.2 On branched $\mathbb{Z}_p^d$-covers

In this subsection, we prove our main result, which is a generalization of the Iwasawa type formula proven by Kadokami and Mizusawa to branched $\mathbb{Z}_p^d$-covers of links in $\mathbb{Q}\mathrm{HS}^3$'s with $d \geq 1$ under a mild assumption.

Theorem 7.4. *Let L be a link in a $\mathbb{Q}HS^3$ M and put $X = M - N(L)$. Let $(M_{p^n})_n$ be the branched $\mathbb{Z}_p^d$-cover consisting of $\mathbb{Q}HS^3$'s over (M, L) derived from a $\mathbb{Z}^d$-cover $X_\infty \to X$. Let $\Delta(t_1, \ldots, t_d)$ denote the Alexander polynomial of $X_\infty \to X$ and and suppose that $\Delta(t_1, \ldots, t_d)$ does not vanish on W^d. Then there exist $\mu, \lambda \in \mathbb{Z}_{\geq 0}$, depending only on $X_\infty \to X$ and p, such that*

$$e(H_1(M_{p^n})) = (\mu p^n + \lambda n + O(1)) p^{(d-1)n}.$$

Proof. We may assume that every component of L is truly branched in some $M_{p^n} \to M$. In every $\mathbb{Z}_p$-cover corresponding to the direct product component of the Galois group $\mathbb{Z}_p^d$, by the Hilbert ramification theory for knots (cf. [29, Section 2], [19, Chapter 5]), the inertia subgroup of $\mathbb{Z}_p$ is of finite index. Hence, for every sufficiently large n, each of the components of the inverse image of L in M_{p^n} is branched in some $\mathbb{Z}/p\mathbb{Z}$-subcover of $M_{p^{n+1}} \to M_{p^n}$ This means that the meridians form a part of a $\mathbb{Z}$-basis of the free quotient of $H_1(X_{p^n})$. Let α denote the meridians of the inverse image of L in M_{p^n} and let

$$H_1(M_{p^n}) \cong H_1(X_{p^n})/\langle\{\alpha\text{'s}\}\rangle$$

denote the natural isomorphism induced by the Fox completion. Since the meridians $[\alpha]$'s form a part of a $\mathbb{Z}$-basis of the free quotient of $H_1(X_{p^n})$, this map induces

$$\mathrm{tor}_{\mathbb{Z}}(H_1(M_{p^n})) \cong \mathrm{tor}_{\mathbb{Z}}(H_1(X_{p^n})).$$

In particular, we have

$$e(H_1(M_{p^n})) = e(H_1(X_{p^n})).$$

By Theorem 7.3, we obtain

$$e(H_1(M_{p^n})) = (\mu(\Delta)p^n + \lambda(\Delta)n + O(1))p^{(d-1)n}.$$

□

7.3 More precise estimate

In this subsection, after reviewing a result of Porti, we prove that the Bachmann-Landau O-notation can be removed in the case where M is a $\mathbb{Z}\mathrm{HS}^3$.

Let M be a $\mathbb{Z}\mathrm{HS}^3$ and L a link in M with d components $K_1, \ldots, K_d$. In this case, the variables $t_1, \ldots, t_d$ of Alexander polynomial $\Delta_L(t_1, \ldots, t_d)$ correspond to the components $K_1, \ldots, K_d$. Put $X := M - N(L)$. Let G be a finite abelian group and $\pi : \pi_1(X) \twoheadrightarrow G$ a surjective group homomorphism. Let M_π denote the covering of M branched along L corresponding to $\ker \pi$. Let

$$\widehat{G} := \{\widehat{\xi} : G \to \mathbb{C}^* \mid \widehat{\xi} \text{ is a group homomorphism}\}$$

be the Pontryagin dual of G. Fix meridians $\alpha_1, \ldots, \alpha_d \in H_1(X)$. Here, α_i are regarded as elements of $H_1(X)$ via π. For arbitrary $\widehat{\xi} \in \widehat{G}$, let $L_{\widehat{\xi}} := \bigcup_{\widehat{\xi}(\alpha_i) \neq 1} K_i$ be a sublink of L and $\Delta_{L_{\widehat{\xi}}}(t_{i_1}, \ldots, t_{i_k})$ denote the Alexander polynomial of $L_{\widehat{\xi}}$. For the trivial representation $G \to \mathbb{C}^*$, one has $L_1 = \emptyset$. We put $\Delta_{L_1} := 1$. Let

$$\widehat{G}^{(1)} := \{\widehat{\xi}' \in \widehat{G} \mid L_{\widehat{\xi}'} = K_i \text{ for some } 1 \leq i \leq d\}.$$

For arbitrary $\widehat{\xi}' \in \widehat{G}^{(1)}$, let $i(\widehat{\xi}')$ denote the corresponding i. Put

$$|H_1(M_\pi)| := \begin{cases} \# H_1(M_\pi) & \text{if finite} \\ 0 & \text{if infinite.} \end{cases}$$

Proposition 7.5 ([23, Theorem 1.1]). *We have*

$$|H_1(M_\pi)| = \pm \prod_{\widehat{\xi}\in\widehat{G}} \Delta_{L_{\widehat{\xi}}}(\widehat{\xi}(\alpha_{i_1}),\dots,\widehat{\xi}(\alpha_{i_k})) \frac{|G|}{\prod_{\widehat{\xi}'\in\widehat{G}^{(1)}}(1-\widehat{\xi}'(\alpha_{i(\widehat{\xi}')}))}.$$

We consider the case when $G = (\mathbb{Z}/p^n\mathbb{Z})^d$. Note that the equation of polynomials

$$\prod_{\xi\in W(n)\setminus\{1\}} (x-\xi) = x^{p^n-1} + x^{p^n-2} + \cdots + x^2 + x + 1$$

implies

$$\prod_{\xi\in W(n)\setminus\{1\}} (1-\xi) = p^n.$$

Therefore, we have

$$\frac{|G|}{(\prod_{\xi\in W(n)\setminus\{1\}}(1-\xi))^d} = 1.$$

Hence, by Proposition 7.5, we obtain

$$|H_1(M_\pi)| = \pm \prod_{L'} \prod_{\zeta\in(W(n)\setminus\{1\})^{c(L')}} \Delta_{L'}(\zeta),$$

where L' runs over the sublinks of L and $c(L')$ is the number of components of L'.

Theorem 7.6. *Let M be a $\mathbb{Z}HS^3$. Suppose $\Delta_L(t_1,\dots,t_d)$ does not vanish on $(W\setminus\{1\})^d$. Then, every branched $(\mathbb{Z}/p^n\mathbb{Z})^d$-cover M_{p^n} is a $\mathbb{Q}HS^3$, and there exists a unique $f(U,V)\in\mathbb{Q}[U,V]$ with $\deg_V f \le 1$ and total degree $\deg f \le d$ such that*

$$e(H_1(M_{p^n})) = f(p^n, n).$$

for every sufficiently large n.

Proof. Since Δ_L does not vanish on $(W\setminus\{1\})^d$, neither does $\Delta_{L'}$ on $(W\setminus\{1\})^{c(L')}$ for any $L'\subset L$ by the Torres condition (cf. [3]). By Proposition 7.5, we have

$$\begin{aligned} e(H_1(M_{p^n})) &= v(\pm\prod_{L'}\prod_{\zeta\in(W(n)\setminus\{1\})^{c(L')}} \Delta_{L'}(\zeta)) \\ &= \sum_{L'}\sum_{\zeta\in(W(n)\setminus\{1\})^{c(L')}} v(\Delta_{L'}(\zeta)). \end{aligned}$$

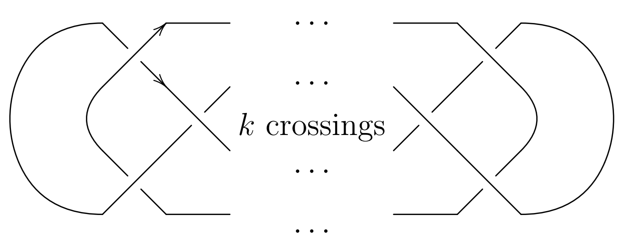
By Proposition 3.8, we have the assertion. □

Remark 7.7. By Proposition 3.9, the Iwasawa μ-invariants and the λ-invariants in this asymptotic formula for $\mathbb{Z}\mathrm{HS}^3$'s can also be obtained from Alexander polynomials in the way we introduced in Section 3.

8 Twisted Whitehead links

In this section, as a preparation to investigate examples, we recall the definition of twisted Whitehead links W_k ($k \in \mathbb{Z}$) and calculate the Alexander polynomial.

Definition 8.1. For each $k \in \mathbb{Z}_{\geq 0}$, the twisted Whitehead link W_k in S^3 is defined by the following diagram.



Proposition 8.2. (1) *If $m \geq 0$, then*

$$\Delta_{W_{2m+1}}(x, y) = 1 + m - mx - my + (1 + m)xy.$$

(2) *If $m \geq 1$, then*

$$\Delta_{W_{2m}}(x, y) = m(1 + xy - x - y).$$

The case (1) can be found in Kidwell's article [14, Section 3]. The case (2) is an exercise in Rolfsen's book [24, Chapter 7, I, Exercise 10]. In what follows, we present a proof.

8.1 Conway Potential function

We make use of the notion of the Conway potential function of a link defined in Hartley's article [8, Section 2]. Let $\nabla \in \Lambda_{\mathbb{Z}}$ denote the Conway potential function of a link L.

Lemma 8.3 ([8, (5.5)])**.** *We have*

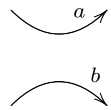
$$\nabla(t_1, \ldots, t_d) = (-1)^d \nabla(t_1^{-1}, \ldots, t_d^{-1}).$$

Lemma 8.4 ([8, (1.1)])**.** *We have*

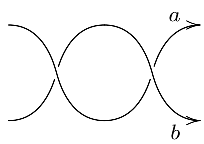
$$\nabla(t_1, \ldots, t_d) = \Delta(t_1^2, \ldots, t_d^2) t_1^{m_1} \cdots t_d^{m_d},$$

where Δ is the Alexander polynomial properly chosen with correct sign and m_i are integers that are uniquely determined by the requirement that ∇ satisfies the Lemma 8.3.

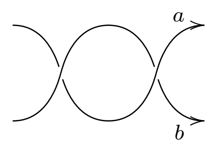
Lemma 8.5 (Replacement relations, [8, (5.1),(5.2)])**.** (1) *Let L_{00} be a link that contains a configuration*



where a, b are segments from distinct knots. Let L_{++}, L_{--} be the links obtained by replacing the configuration by



and



.

Let ∇_{++}, ∇_{--}, and ∇_{00} denote the Conway potential function of L_{++}, L_{--}, and L_{00} respectively. Then we have

$$\nabla_{++} + \nabla_{--} = (t_a t_b + t_a^{-1} t_b^{-1})\nabla_{00}.$$

(2) *Consider the case where one of the arcs of* (1) *is oppositely oriented. Then we have*

$$\nabla_{++} + \nabla_{--} = (t_a t_b^{-1} + t_a^{-1} t_b)\nabla_{00}.$$

8.2 The Alexander polynomials of the twisted Whitehead links

Let us prove Proposition 8.2.

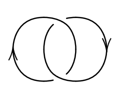
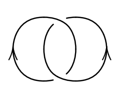
Proof. We prove

$$\nabla_{W_{2m+1}} = -(m+1)(t_a t_b + t_a^{-1} t_b^{-1}) + m(t_a t_b^{-1} + t_a^{-1} t_b)$$

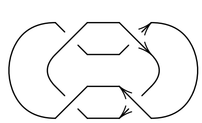
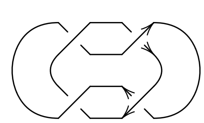
and

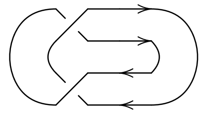
$$\nabla_{W_{2m}} = m(t_a t_b - t_a t_b^{-1} - t_a^{-1} t_b + t_a^{-1} t_b^{-1})$$

by induction on m. It is known that

1. If L is a split link, then $\nabla_L = 0$.
2. If $L :=$  , then $\nabla_L = 1$.
3. If $L :=$  , then $\nabla_L = -1$.

Let

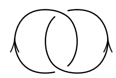
$L_{++} :=$  $L_{--} :=$ 



$L_{00} :=$

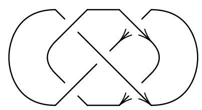
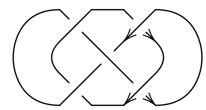
Then, by the replacement relation (1), we have

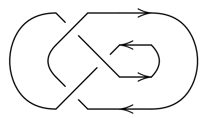
$$\nabla_{++} + \nabla_{--} = (t_a t_b + t_a^{-1} t_b^{-1})\nabla_{00}.$$

Since $L_{--} = W_1$ and $L_{00} =$  , we obtain

$$\nabla_{W_1} = -(t_a t_b + t_a^{-1} t_b^{-1}).$$

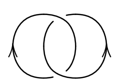
Let

$L_{++} :=$  $L_{--} :=$ 

$L_{00} :=$ 

Then, by the replacement relation (2), we have

$$\nabla_{++} + \nabla_{--} = (t_a t_b^{-1} + t_a^{-1} t_b)\nabla_{00}.$$

Since $L_{++} = W_1, L_{--} = W_2, L_{00} =$  , we have

$$\nabla_{W_1} + \nabla_{W_2} = -(t_a t_b^{-1} + t_a{}^{-1} t_b),$$

i.e.,

$$\nabla_{W_2} = t_a t_b - t_a t_b^{-1} - t_a^{-1} t_b + t_a^{-1} t_b^{-1}.$$

Iterating these arguments, we obtain

$$\nabla_{W_{2m+1}} = -\nabla_{W_{2m}} - (t_a t_b + t_a^{-1} t_b^{-1})$$

and

$$\nabla_{W_{2m}} = -\nabla_{W_{2m-1}} - (t_a t_b^{-1} + t_a^{-1} t_b).$$

By the induction hypotheses, we have

$$\begin{aligned} \nabla_{W_{2m+1}} &= -m(t_a t_b - t_a t_b^{-1} - t_a^{-1} t_b + t_a^{-1} t_b^{-1}) - (t_a t_b + t_a^{-1} t_b^{-1}) \\ &= -(m+1)(t_a t_b + t_a^{-1} t_b^{-1}) + m(t_a t_b^{-1} + t_a^{-1} t_b)) \end{aligned}$$

and

$$\begin{aligned}\nabla_{W_{2m}} &= m(t_at_b + t_a^{-1}t_b^{-1}) - (m-1)(t_at_b^{-1} + t_a^{-1}t_b)) - (t_at_b^{-1} + t_a^{-1}t_b) \\ &= m(t_at_b - t_at_b^{-1} - t_a^{-1}t_b + t_a^{-1}t_b^{-1}).\end{aligned}$$

By Lemma 8.4, we must have

$$\begin{aligned}\Delta_{W_{2m+1}}(x,y) &= \Delta_{W_{2m+1}}(t_a^2, t_b^2) \\ &= (m+1)(t_a^2t_b^2 + 1) - m(t_a^2 + t_b^2)\end{aligned}$$

and

$$\begin{aligned}\Delta_{W_{2m}}(x,y) &= \Delta_{W_{2m}}(t_a^2, t_b^2) \\ &= m(t_a^2t_b^2 - t_a^2 - t_b^2 + 1).\end{aligned}$$

□

9 Examples

In this section, we provide examples of Theorem 7.6 by considering twisted Whitehead links. Moreover, we show that these examples assure us that Iwasawa μ-invariants can be arbitrary non-negative integers when $d \leq 2$. We also introduce an example of so-called "p-adic limits" for $d = 2$ that we have succeeded in calculating. At the end of this section, we provide a table of the μ and λ invariants for the links that appear in tables of the Rolfsen's book [24]. We used Sage Math to compute the invariants.

9.1 $\mathbb{Z}_p^2$-covers

Here, based on Porti's result (Proposition 7.5), we explicitly calculate the sizes of the p-torsions in the $\mathbb{Z}_p^2$-covers of W_{2p^n} and $L = 6_1^2$, as examples of Theorem 7.6.

Example 9.1 (Twisted Whitehead links W_{2p^k})**.** Let p be any prime number and $k \in \mathbb{Z}_{\geq 0}$. Then branched $\mathbb{Z}_p^2$-cover $(M_{p^n} \to S^3)_n$ over (S^3, W_{2p^k}) satisfies the following:

$$e(H_1(M_{p^n})) = (kp^n + 2n - 2k)p^n - 2n + k.$$

Hence we have $\mu_{W_{2p^k}} = k$ and $\lambda_{W_{2p^k}} = 2$.

Proof. Since $\Delta_{W_{2p^k}}(x,y) = p^k(1+xy-x-y)$, we have

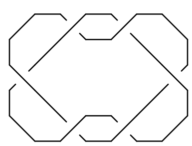
$$\begin{aligned}
|H_1(M_{p^n})| &= \prod_{1\neq\zeta_2\in W(n)} \prod_{1\neq\zeta_1\in W(n)} \Delta(\zeta_1,\zeta_2) \\
&= \prod_{\zeta_2}\prod_{\zeta_1} p^k(1-\zeta_1)(1-\zeta_2) \\
&= \prod_{\zeta_2} (p^k)^{p^n-1}(1-\zeta_2)^{p^n-1}(1-\zeta_1)(1-\zeta_1^2)\cdots(1-\zeta_1^{p^n-1}) \\
&= \prod_{\zeta_2} (p^k)^{p^n-1}(1-\zeta_2)^{p^n-1}p^n \\
&= (p^k)^{(p^n-1)(p^n-1)}(p^n)^{p^n-1}(p^n)^{p^n-1} \\
&= p^{k(p^{2n}-2p^n+1)+2np^n-2n} \\
&= p^{(kp^n+2n-2k)p^n-2n+k}.
\end{aligned}$$

□

In particular, we obtain

Theorem 9.2. *Suppose $d=2$. Then, for arbitrary $k \in \mathbb{Z}_{\geq 0}$, there exists a 2-component link L such $\mu_L = k$.*

Example 9.3 ($L = 6_1^2$)**.** The link $L = 6_1^2$ is defined by



and its Alexander polynomial is $\Delta_L(x,y) = x^2y^2+xy+1$, which we have calculated by using Rolfsen's table [24]. In the branched $\mathbb{Z}_p^2$-cover over (S^3, L), we have $|H_1(M_{p^n})| = 3^{p^n-1}$. If $p \neq 3$, then all the Iwasawa invariants are zero.

For $n \geq 1$, let Φ_n denote the n-th cyclotomic polynomial, i.e.,

$$\Phi_n = \prod_{1\leq k\leq n,\ \gcd(k,n)=1} (x - e^{2\pi i \frac{k}{n}}).$$

We utilize the following result.

Lemma 9.4 (Apostol, [2, Theorem 1])**.** *If $m > n > 1$ and $(m,n) > 1$, then we have*

$$\mathrm{Res}(\Phi_m, \Phi_n) = \begin{cases} l^{\varphi(n)} & \frac{m}{n} = l^e \\ 1 & \textit{otherwise,} \end{cases}$$

where l is a prime number, Res(f,g) *denotes the resultant of two polynomials, and $\varphi(n)$ is Euler's totient function.*

Proof of Example 9.3. Put

$$r_{p^n}(x) := \mathrm{Res}(y^{p^n} - 1, \Delta_L(x,y)).$$

Then we have

$$\begin{aligned} r_{p^n}(x) &= \prod_{\xi \in W(n)} (x^2\xi^2 + x\xi + 1) \\ &= \prod (x\xi - \omega)(x\xi - \omega^2) \\ &= \prod (x - \frac{\omega}{\xi})(x - \frac{\omega}{\xi}) \\ &= \prod (x - \xi\omega)(x - \xi\omega^2) \\ &= \prod_{k=0}^{n} \Phi_{3p^k}(x), \end{aligned}$$

where ω is a primitive cube root of unity. By Lemma 9.4, we have

$$\begin{aligned} & \mathrm{Res}(x^{p^n} - 1, r_{p^n}(x)) \\ = & \prod_{l,k} \mathrm{Res}(\Phi_{p^l}(x), \Phi_{3\cdot p^k}(x)) \\ = & \prod_{l=k} \mathrm{Res}(\Phi_{p^l}(x), \Phi_{3\cdot p^l}(x)) \\ = & \prod_{l=0}^{n} 3^{\varphi(p^l)} = 3^{1+\sum_{l=1}^{n} p^{l-1}(p-1)} \\ = & 3^{p^n}. \end{aligned}$$

On the other hand, we have

$$\Delta_L(1,1) = 3.$$

By Lemma 9.4, we also have

$$\mathrm{Res}(y^{p^n} - 1, \Delta_L(1,y)) = \mathrm{Res}(y^{p^n} - 1, \Phi_3) = \mathrm{Res}(\Phi_1, \Phi_3) = 3$$

and

$$\mathrm{Res}(x^{p^n} - 1, \Delta_L(x,1)) = 3.$$

Therefore, we obtain

$$|H_1(M_{p^n})| = \prod_{1 \neq \zeta_1, \zeta_2 \in W(2^n)} \Delta_L(\zeta_1, \zeta_2) = 3^{p^n - 1}.$$

□

9.2 TLN-cover and ν-invariant

Here, we briefly observe an example of the case with $(c, d) = (2, 1)$ and give a remark. The main reference for this section is [13]. Let L be a link in a $\mathbb{Q}\mathrm{HS}^3$ M consisting of null-homologous components. The total linking number cover (TLN-cover) is the $\mathbb{Z}$-cover corresponding to the surjective homomorphism $\tau : \pi_1(M - N(L)) \to \mathbb{Z}$ sending all positive meridians to 1. In this situation, the reduced Alexander polynomial $\widetilde{\Delta}_L(t) = \Delta_L(t, \ldots, t)$ and a polynomial $A_{X_\infty}(t) = (t-1)\widetilde{\Delta}_L(t)$ are defined, and we have

Proposition 9.5 (Kadokami–Mizusawa, [13, Theorem 3.3])**.** *Let L be a link in a $\mathbb{Q}HS^3$ M consisting of null-homologous components and let $(M_{p^n} \to M)_n$ denote the system of branched $\mathbb{Z}/p^n\mathbb{Z}$-covers obtained from the subcovers of the TLN-cover over (M, L) by the Fox completions. Then*

$$\frac{|H_1(M_{p^n})|}{\#H_1(M)} = \left| \prod_{1 \neq \xi \in W(n)} A_{X_\infty}(\xi) \right|.$$

Example 9.6. Let $L = W_{2p^k}$ in S^3 and let $(M_{p^n} \to S^3)_n$ denote the sequence of branched $\mathbb{Z}/p^n\mathbb{Z}$-covers obtained from the TLN-cover. Then we have

$$\widetilde{\Delta}_L(t) = \Delta_L(t, t) \in \Lambda_{\mathbb{Z}} = p^k(t-1)^2,$$

$$A_{X_\infty}(t) = (t-1)\widetilde{\Delta}_L(t) = p^k(t-1)^3,$$

and

$$e(H_1(M_{p^n})) = kp^n + 3n - k.$$

Proof. By Proposition 9.5, we have

$$\begin{aligned} |H_1(M_{p^n})| = \frac{|H_1(M_{p^n})|}{\#H_1(M)} &= |\prod_{1 \neq \xi \in W(n)} A_{X_\infty}(\xi)| \\ &= \prod_{\xi \neq 1} p^k(1-\xi)^3 \\ &= (p^k)^{p^n - 1}(p^n)^3 \\ &= p^{kp^n + 3n - k}. \end{aligned}$$

□

Therefore, Theorem 9.2 can be improved to

Theorem 9.7. *Both when $d = 1, 2$, for arbitrary $k \in \mathbb{Z}_{\geq 0}$, there exists a 2-component link L in S^3 such $\mu_L = k$.*

Remark 9.8. By Ueki–Yoshizaki [32], special interests of Iwasawa ν-invariants in $\mathbb{Z}_p$-covers are known. Example 9.6 indicates that Iwasawa ν-invariants can be arbitrarily small as well. Also, Example 9.1 indicates that Iwasawa ν-invariants for $d = 2$ can be arbitrarily large.

9.3 p-adic torsions

Kisilevsky proved that in a $\mathbb{Z}_p$-extension of a global field the class numbers p-adically converges [16, Corollary 1]. Yoshizaki and Ueki proved an analogous result for $\mathbb{Z}_p$-covers of 3-manifolds:

Proposition 9.9 (A part of Ueki–Yoshizaki, [32, Theorem 3.1])**.** *Let $(X_{p^n} \to X)_n$ be a compatible system of $\mathbb{Z}/p^n\mathbb{Z}$-covers of a compact 3-manifold X. Then the sizes of the non-p torsion subgroups $H_1(X_{p^n})_{non\text{-}p}$ converges in $\mathbb{Z}_p$.*

We remark that the p-adic limit of $H_1(M_{p^n})_{\text{non-}p}$ coincides with Kionke's p-adic torsion by [15, Theorem 1.1]. Kionke's framework is for arbitrary pro-p covers and the p-adic limits of $|H_1(M_{p^n})_{\text{non-}p}|$ in $\mathbb{Z}_p^d$-covers of links also give examples of the p-adic torsions. Here we attach an example for the case $c = d = 2$. Let $\mathbb{C}_p$ denote the p-adic completion of an algebraic closure of the p-adic numbers $\mathbb{Q}_p$ and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ of algebraic closure of $\mathbb{Q}$.

Example 9.10. Let $L = 6_1^2$ in S^3 and let $p \neq 3$. Then, by Example 9.3, we have

$$\lim_{n\to\infty} |H_1(M_{p^n})_{\text{non-}p}| = \lim_{n\to\infty} 3^{p^n-1} = \xi/3,$$

where ξ denotes the unique root of unity of order prime to p satisfying $|\xi - 3|_p < 1$, that is, $\xi \equiv 3 \bmod p$ holds. We have $\xi/3 \in \mathbb{Q}$ if and only if $p = 2$ and in this case we have $\xi/3 = 1/3$.

Proof. (1) By Fermat's small theorem, we have $3^{p^n} \equiv 3 \bmod p$, and hence $\xi^{p^n} = \xi$. By [32, Lemma 5.6 (1)], $\lim_{n\to\infty}(3^{p^n} - \xi^{p^n}) = 0$ in $\mathbb{Z}_p$. Thus we obtain the assertion.

(2) Let us apply [32, Theorem 5.7] to verify the consistency. If $p \neq 2, 3$, then $3^{p^n-1} = (\mathrm{sgn}(1-3)\cdot \mathrm{Res}(t^{p^n}-1, t-3)+1)/3 = ((-1)(-1)^p(\xi-1)+1)/3 = \xi/3$. If $p = 2$, then $3^{p^n-1} = (\mathrm{sgn}((1-3)(-1-3))\cdot \mathrm{Res}(t^{p^n}-1, t-3)+1)/3 = ((-1)^2(-1)^2(\xi-1)+1)/3 = \xi/3$. □

9.4 μ and λ invariants for Rolfsen's table

Table 1 is the table for μ and λ invariants of links that we have succeeded in calculating by using Sage Math. We cite the data of Alexander polynomial from a table in Rolfsen's book [24]. We shall observe several interesting examples.

- $\lambda(6_1^2) = 2$ if $p = 3$ since

$$\begin{aligned} & X^2Y^2 + 2X^2Y + 2XY^2 + X^2 + 5XY + Y^2 + 3X + 3Y + 3 \\ \equiv\ & ((1+X)(1+Y)-1)^2 \mod 3. \end{aligned}$$

- $\lambda(6_3^3) = 1$ since

$$\begin{aligned} & -XYZ - XY - XZ - YZ - X - Y - Z \\ =\ & -((1+X)(1+Y)(1+Z)-1). \end{aligned}$$

- $\lambda(8_3^4) = 2$ since

$$\begin{aligned} & WXYZ + WXY + WXZ + WYZ + XYZ + WY + XY + WZ + XZ \\ =\ & ((1+X)(1+W)-1)((1+Y)(1+Z)-1). \end{aligned}$$

Table 1: μ and λ invariants

link	Alexander polynomial of $\Delta_L(1+X, 1+Y)$	μ	λ
4_1^2	$XY + X + Y + 2$	0	1 if $p = 2$
5_1^2	XY	0	2
6_1^2	$X^2Y^2 + 2X^2Y + 2XY^2 + X^2 + 5XY + Y^2 + 3X + 3Y + 3$	0	2 if $p = 3$
6_2^2	$X^2Y + XY^2 + X^2 + 3XY + Y^2 + 3X + 3Y + 3$	0	0
6_3^2	$2XY + X + Y + 2$	0	0
7_1^2	$X^2Y^2 + X^2Y + XY^2 + XY - X - Y - 1$	0	0
7_2^2	$X^2Y^2 + X^2Y + XY^2 + 3XY + X + Y + 1$	0	0
7_3^2	$2XY$	1 if $p = 2$	2
7_4^2	$X^3Y + 2X^2Y + 2XY$	0	4 if $p = 2$ 2 if not
7_5^2	$X^3Y + X^3 + X^2Y + 3X^2 + XY + 3X + Y + 2$	0	1 if $p = 2$
7_6^2	$X^3Y + X^2Y + XY$	0	2
7_7^2	$X^3Y + X^3 + 3X^2Y + 3X^2 + 3XY + 3X + Y + 2$	0	1 if $p = 2$
7_8^2	XY	0	2
8_1^2	$X^3Y^3 + 3X^3Y^2 + 3X^2Y^3 + 3X^3Y + 9X^2Y^2 + 3XY^3 + 2X^3 + 10X^2Y + 10XY^2$ $+Y^3 + 7X^2 + 13XY + 4Y^2 + 9X + 6Y + 4$	0	0
8_2^2	$X^3Y + X^2Y^2 + XY^3 + X^3 + 4X^2Y + 4XY^2 + Y^3 + 4X^2 + 7XY$ $+4Y^2 + 6X + 6Y + 4$	0	0
8_3^2	$2X^2Y^2 + 3X^2Y + 3XY^2 + X^2 + 7XY + Y^2 + 3X + 3Y + 3$	0	0
8_4^2	$X^3Y^2 + X^2Y^3 + 2X^3Y + 4X^2Y^2 + 2XY^3 + X^3 + 7X^2Y + 7XY^2 + Y^3$ $+4X^2 + 10XY + 4Y^2 + 6X + 6Y + 4$	0	3 if $p = 2$
8_5^2	$X^2Y^2 - X^2 - XY - Y^2 - 3X - 3Y - 3$	0	0
8_6^2	$3XY + X + Y + 2$	0	1 if $p = 2$
8_7^2	$X^2Y^2 - XY - X - Y - 1$	0	0
8_8^2	$X^2Y^2 + XY + X + Y + 1$	0	0
8_9^2	$-X^3 - 2X^2Y - X^2 + 3X + Y + 2$	0	1 if $p = 2$
8_{10}^2	X^3Y	0	4
8_{11}^2	$-X^3Y + X^3 - X^2Y + 3X^2 - XY + 3X + Y + 2$	0	1 if $p = 2$
8_{12}^2	X^3Y	0	4
8_{13}^2	$X^3Y - X^2Y - XY$	0	2
8_{14}^2	$X^3Y + X^3 - X^2Y + 3X^2 - XY + 3X + Y + 2$	0	1 if $p = 2$
8_{15}^2	XY	0	2
8_{16}^2	$-X^3 - X^2 + 2XY + 3X + Y + 2$	0	1 if $p = 2$
9_1^2	$X^3Y^3 + 2X^3Y^2 + 2X^2Y^3 + X^3Y + 4X^2Y^2 + XY^3 + X^2Y + XY^2 - X^2$ $-2XY - Y^2 - 3X - 3Y - 2$	0	0
9_2^2	$X^3Y + X^2Y^2 + XY^3 + 2X^2Y + 2XY^2 - X^2 - XY - Y^2 - 3X - 3Y$ -2	0	0
9_3^2	$2X^2Y^2 + 2X^2Y + 2XY^2 + 3XY - X - Y - 1$	0	0
9_4^2	$X^3Y^2 + X^2Y^3 + X^3Y + 5X^2Y^2 + XY^3 + 5X^2Y + 5XY^2 + 5XY$	0	2
9_5^2	$X^3Y + 2X^2Y^2 + XY^3 + 4X^2Y + 4XY^2 + 4XY$	0	4 if $p = 2$ 2 if not
9_6^2	$-X^3Y^2 - X^2Y^3 - X^3Y - 3X^2Y^2 - XY^3 - 2X^2Y - 2XY^2 + X^2 + XY$ $+Y^2 + 3X + 3Y + 2$	0	1 if $p = 2$
9_7^2	$-X^3Y^2 - X^2Y^3 - X^3Y - 2X^2Y^2 - XY^3 - X^2Y - XY^2 + X^2 + 2XY$ $+Y^2 + 3X + 3Y + 2$	0	1 if $p = 2$
9_8^2	$2X^2Y + 2XY^2 + 3XY - X - Y - 1$	0	0
9_9^2	$X^3Y^2 - X^2Y^3 + X^3Y - 3X^2Y^2 - XY^3 - 3X^2Y - 3XY^2 - 2X^2 - 3XY$ $-2X$	0	1
9_{10}^2	$3XY$	1 if $p = 3$	2
9_{11}^2	$2X^2Y^2 + X^2Y + XY^2 + XY - X - Y - 1$	0	0
9_{12}^2	$X^2Y^2 - X^2Y - XY^2 - XY + X + Y + 1$	0	0
9_{13}^2	$X^5Y + 4X^4Y + 7X^3Y + 6X^2Y + 3XY$	0	4 if $p = 3$ 2 if not
9_{14}^2	$X^5 + 2X^4Y + 5X^4 + 6X^3Y + 10X^3 + 8X^2Y + 10X^2 + 4XY + 5X$ $+Y + 2$	0	1 if $p = 2$
9_{15}^2	$2X^3Y + 3X^2Y + 3XY$	0	4 if $p = 3$ 2 if not
9_{16}^2	$2X^3 + 4X^2Y + 5X^2 + 3XY + 3X + Y + 2$	0	1 if $p = 2$
9_{17}^2	$2X^3 + 3X^2Y + 5X^2 + 2XY + 3X + Y + 2$	0	1 if $p = 2$
9_{18}^2	$2X^3Y + 2X^2Y + 2XY$	1 if $p = 2$	2
9_{19}^2	$X^4Y + X^3Y^2 + 5X^3Y + 2X^2Y^2 + X^3 + 8X^2Y + 2XY^2 + 2X^2 + 6XY$ $+2X + Y + 1$	0	0
9_{20}^2	$X^4 + 2X^3Y + 2X^2Y^2 + 4X^3 + 7X^2Y + 2XY^2 + 7X^2 + 6XY + Y^2$ $+6X + 3Y + 3$	0	0
9_{21}^2	$X^4Y^2 + X^4Y + 3X^3Y^2 + 5X^3Y + 4X^2Y^2 + X^3 + 8X^2Y + 2XY^2 + 2X^2$ $+6XY + 2X + Y + 1$	0	0
9_{22}^2	$X^4Y + 2X^3Y^2 + X^4 + 5X^3Y + 4X^2Y^2 + 4X^3 + 10X^2Y + 3XY^2 + 7X^2$ $+8XY + Y^2 + 6X + 3Y + 3$	0	0
9_{23}^2	$X^3Y + 2X^2Y^2 + XY^3 + 3X^2Y + 3XY^2 - X^2 - Y^2 - 3X - 3Y - 2$	0	1 if $p = 2$
9_{24}^2	$3X^2Y^2 + 3X^2Y + 3XY^2 + X^2 + 7XY + Y^2 + 3X + 3Y + 3$	0	2 if $p = 3$

link	Alexander polynomial of $\Delta_L(1+X,1+Y,\ldots)$	μ	λ
9^2_{25}	X^3Y-2X^2Y-2XY	0	4 if $p=2$ 2 if not
9^2_{26}	$X^3Y-X^3-X^2Y-X^2+XY+3X+Y+2$	0	1 if $p=2$
9^2_{27}	$2X^3Y+3X^2Y+3XY$	0	4 if $p=3$ 2 if not
9^2_{28}	$X^3Y-X^3-X^2Y-3X^2-XY-3X-Y-2$	0	1 if $p=2$
9^2_{29}	$-2X^4Y-5X^3Y-6X^2Y-3XY+1$	0	0
9^2_{30}	$-X^3Y+2X^3+X^2Y+5X^2+3X+Y+2$	0	0
9^2_{31}	$X^5Y+3X^4Y+4X^3Y+2X^2Y+XY$	0	2
9^2_{32}	$2X^3Y+X^2Y+XY$	0	2
9^2_{33}	$2X^3Y+X^2Y+XY$	0	2
9^2_{34}	$X^4Y^2+X^4Y+2X^3Y^2+3X^3Y+2X^2Y^2+2X^2Y+XY^2-X^2-2X$ $-Y-1$	0	0
9^2_{35}	$X^4Y^2+X^4Y+2X^3Y^2+3X^3Y+2X^2Y^2+4X^2Y+XY^2+X^2+4XY$ $+2X+Y+1$	0	0
9^2_{36}	$2X^3Y+2X^2Y+2XY$	1 if $p=2$	2
9^2_{37}	$X^5Y+3X^4Y+5X^3Y+4X^2Y+2XY$	0	4 if $p=2$ 2 if not
9^2_{38}	$2X^3Y+X^2Y+X^2+2XY+3X+Y+2$	0	1 if $p=2$
9^2_{39}	$-X^4Y-2X^3Y^2-X^4-4X^3Y-4X^2Y^2-3X^3-7X^2Y-3XY^2-4X^2$ $-4XY-Y^2-2X-Y$	0	0
9^2_{40}	$X^4Y^2+2X^4Y+X^3Y^2+X^4+5X^3Y+4X^3+4X^2Y+XY^2+7X^2$ $+4XY+Y^2+6X+3Y+3$	0	2 if $p=3$
9^2_{41}	$X^3Y^3+2X^3Y^2+X^2Y^3+X^3Y+5X^2Y^2+3X^2Y+3XY^2+3XY$	0	4 if $p=3$ 2 if not
9^2_{42}	$X^4Y^2+X^4Y+X^3Y^2+2X^3Y-X^2Y-X^2-2XY-2X-Y$ -1	0	0
9^2_{43}	$X^5+5X^4+10X^3+10X^2+5X+Y+2$	0	1 if $p=2$
9^2_{44}	X^3Y+2X^2Y+2XY	0	4 if $p=2$ 2 if not
9^2_{45}	$2X^3+5X^2-XY+3X+Y+2$	0	1 if $p=2$
9^2_{46}	$2XY$	1 if $p=2$	2
9^2_{47}	XY	0	2
9^2_{48}	$2X^3+X^2Y+5X^2+3X+Y+2$	0	1 if $p=2$
9^2_{49}	$X^4+4X^3+X^2Y+7X^2+2XY+Y^2+6X+3Y+3$	0	2 if $p=3$
9^2_{50}	$-X^2Y-XY+Y^2+X+Y+1$	0	0
9^2_{51}	$X^4Y+X^4+3X^3Y+4X^3+4X^2Y+XY^2+7X^2+4XY+Y^2$ $+6X+3Y+3$	0	0
9^2_{52}	$X^2Y^2+X^2Y+XY-Y^2-X-Y-1$	0	0
9^2_{53}	$X^2Y^2+X^3+2X^2Y+2XY^2+Y^3+4X^2+5XY+4Y^2+6X$ $+6Y+4$	0	2 if $p=2$
9^2_{54}	$X^2Y+XY^2+XY-X-Y-1$	0	0
9^2_{55}	X^3Y+X^2Y+XY	0	2
9^2_{56}	X^3Y+X^2Y+XY	0	2
9^2_{57}	$X^3Y+2X^2Y+X^2+3XY+3X+Y+2$	0	0
9^2_{58}	$X^3Y+X^2Y+X^2+2XY+3X+Y+2$	0	0
9^2_{59}	$X^5Y+X^5+5X^4Y+5X^4+9X^3Y+10X^3+8X^2Y+10X^2+4XY$ $+5X+Y+2$	0	0
9^2_{60}	$X^3Y+2X^3+2X^2Y+5X^2+XY+3X+Y+2$	0	0
9^2_{61}	$X^3Y^2+2X^3Y+3X^2Y^2+XY^3+X^3+6X^2Y+6XY^2+Y^3+4X^2$ $+9XY+4Y^2+6X+6Y+4$	0	2 if $p=2$
6^3_1	$XY+XZ+YZ+X+Y+Z$	0	0
6^3_2	$-XYZ$	0	3
6^3_3	$-XYZ-XY-XZ-YZ-X-Y-Z$	0	1
7^3_1	$-XYZ+YZ-X+Y+Z$	0	0
8^3_1	$-X^2Y^2Z-2X^2YZ-XY^2Z-X^2Y+XY^2-X^2Z-3XYZ-X^2+Y^2$ $-2XZ-2X+2Y-Z$	0	0
8^3_2	$X^2Y^2+X^2YZ+XY^2Z+2X^2Y+2XY^2+X^2Z+2XYZ+Y^2Z+X^2$ $+4XY+Y^2+2XZ+2YZ+2X+2Y+Z$	0	0
8^3_3	$XYZ-XY-XZ-YZ-X-Y-Z$	0	1 if $p=2$
8^3_4	$X^3Y+2X^2YZ+X^3+3X^2Y+X^2Z+2XYZ+4X^2+2XY+2XZ$ $+4X$	0	3 if $p=2$ 1 if not
8^3_5	$X^2Y^2Z+X^2YZ+XY^2Z+2XYZ$	0	4 if $p=2$ 3 if not
8^3_6	$X^2Y^2Z+X^2YZ+XY^2Z+3XYZ+XZ+YZ+Z$	0	1
8^3_7	$-X^2Y^2Z-X^2Y^2-2X^2YZ-2XY^2Z-2X^2Y-2XY^2-X^2Z-4XYZ-Y^2Z$ $-X^2-4XY-Y^2-2XZ-2YZ-2X-2Y-Z$	0	1
8^3_8	$XY^2Z-X^2Y+XY^2+XYZ+Y^2Z-X^2+Y^2+2YZ-2X$ $+2Y+Z$	0	0
8^3_9	XYZ	0	3
8^3_{10}	$X^3YZ+X^3Y+X^3Z+3X^2YZ+X^3+3X^2Y+3X^2Z+2XYZ+4X^2$ $+2XY+2XZ+4X$	0	3 if $p=2$ 1 if not

link	Alexander polynomial of $\Delta_L(1+X, 1+Y, \ldots)$	μ	λ
9^3_1	$X^2Y^2Z + X^2YZ + X^2Y - XY^2 - XZ - YZ - Z$	0	0
9^3_2	$X^2Y^2Z + X^2YZ + X^2Y - XY^2 + 2XYZ + XZ + YZ + Z$	0	0
9^3_3	$-X^3YZ - X^2YZ - X^3 + X^2Y + X^2Z - XYZ - X^2 + XY + XZ$ $+YZ - X + Y + Z$	0	0
9^3_4	$X^3Y + X^3Z + 2X^2YZ + X^3 + 2X^2Y + 2X^2Z + 2XYZ + X^2 + 2XY$ $+2XZ + YZ + X + Y + Z$	0	0
9^3_5	$X^2YZ + XY^2Z + 2XYZ - Y^2Z + X^2 - Y^2 - 2YZ + 2X - 2Y$ $-Z$	0	0
9^3_6	$X^2Y^2Z + X^2YZ + X^2Y - XY^2 + XYZ - Y^2Z + X^2 - Y^2 - 2YZ$ $+2X - 2Y - Z$	0	0
9^3_7	$2XYZ - YZ + X - Y - Z$	0	1 if $p = 2$
9^3_8	$X^3Y + 2X^2YZ + 2X^2Y + XYZ$	0 0	3 if $p = 2, 3$ 2 if not
9^3_9	$X^3YZ + X^2YZ + XYZ$	0	3
9^3_{10}	X^2Y^2Z	0	5
9^3_{11}	$X^2Y^2Z - 8XYZ - Y^2Z + X^2 - Y^2 - 8XZ - 10YZ + 2X - 2Y$ $-9Z$	0	0
9^3_{12}	X^3YZ	0	5
9^3_{13}	$X^2Y^2Z + X^2Y^2 + X^2YZ + XY^2Z + X^2Y + XY^2 - XZ - YZ - Z$	0	0
9^3_{14}	$X^2Y^2Z + X^2Y^2 + X^2YZ + XY^2Z + X^2Y + XY^2 + 2XYZ + XZ + YZ$ $+Z$	0	0
9^3_{15}	$X^3Y + X^3Z + X^3 + 2X^2Y + 2X^2Z + X^2 + 2XY + 2XZ + YZ$ $+X + Y + Z$	0	0
9^3_{16}	$-2X^2YZ + X^3 - X^2Y - X^2Z - 2XYZ + X^2 - XY - XZ - YZ$ $+X - Y - Z$	0	0
9^3_{17}	$-X^3 + X^2Y + X^2Z - X^2 + XY + XZ + YZ - X + Y$ $+Z$	0	0
9^3_{18}	$-XYZ$	0	3
9^3_{19}	$-X^3YZ - X^3Y - 2X^2YZ - 2X^2Y - XYZ$	0	3
9^3_{20}	$-X^3Z - X^2YZ - 2X^2Z$	0	4 if $p = 2$ 3 if not
8^4_1	$-WXY - WXZ - WYZ - XYZ - WY - XY - WZ - XZ$	0	0
8^4_2	$-WXZ - WYZ - WY + XY - WZ - XZ$	0	0
8^4_3	$WXYZ + WXY + WXZ + WYZ + XYZ + WY + XY + WZ + XZ$	0	2
8^4_4	$WXYZ - WXZ - WYZ - WY + XY - WZ - XZ$	0	0

10 Remarks

- In [33], Wan proved that Greenberg's conjecture holds in the function field case. On the other hand, in [5], DuBose and Vallières proved that Greenberg's conjecture holds in the graph case. It is mysterious whether Greenberg's conjecture holds for our $\mathbb{Q}\mathrm{HS}^3$ case or not. It would be interesting if we could construct a non-Greenberg example. We should continue comparing these four fields: number fields, function fields, links, and graphs.

- As is well known, the relationship between the sizes of the torsion subgroups and the Betti numbers is like siblings as research objects. The polynomial periodicity of Betti numbers for $\mathbb{Z}^d$-coverings has been researched by Adams–Sarnak [1], Hironaka [9], and Sakuma [26]. While Betti numbers are always trivial in number theoretical side, the topological side deserves to be researched.

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