Doctoral Dissertation

Higher Courant-Dorfman algebras and associated higher Poisson vertex algebras

Courant-Dorfman代数と対応するPoisson頂点代数の高次化

Ryo Hayami

Nagoya University

Supervisor: Professor Hidetoshi Awata

March 6, 2023

Abstract

In this thesis, we consider a notion of a higher version of the relation between Courant-Dorfman algebras and Poisson vertex algebras. We define a higher Courant-Dorfman algebra, and study the relationship with graded symplectic geometry. In particular, we give graded Poisson algebras of degree $-n$ in the non-degenerate case. For higher Courant-Dorfman algebras coming from finite-dimensional vector bundles, they coincide with the algebras of functions of the associated differential-graded(dg) symplectic manifolds of degree n.

We define a higher Lie conformal algebra and Poisson vertex algebra, and give a higher (weak) Courant-Dorfman algebraic structure arising from them. Moreover, we prove that the higher Lie conformal algebras and higher Poisson vertex algebras have properties like Lie conformal algebras and Poisson vertex algebras. As an example, we obtain an algebraic description of Batalin-Fradkin-Vilkovisky(BFV) current algebras.

Acknowledgments

I want to thank my supervisor Prof. Hidetoshi Awata for his advices and supports. Also I want to thank Prof. Noriaki Ikeda for his comments and suggestions. Finally, I want to thank my family for its understanding and support.

Contents

Chapter 1

Introduction

A Courant algebroid is a 4-tuple $(E, \rho, \langle,\rangle, [,])$ where E is a vector bundle over a smooth manifold M, ρ is an anchor map to tangent bundle, $\langle,\rangle$ is a non-degenerate metric, and $[,]$ is a Courant bracket on the sections of the bundle, satisfying a set of compatibility conditions. It first appeared in [1] as the generalized tangent bundle $TM \oplus T^*M$ with a natural projection $\rho : TM \oplus T^*M \to TM$, a natural pairing $\langle . \rangle$, and a Dorfman bracket $[,]$, and a general definition was given in [10] to generalize the double of Lie bialgebroids (Lie algebroid analogue of Lie bialgebras[3]). We can obtain a map $d : C^\infty(M) \to \Gamma(E)$ by defining $\langle df, e\rangle = \rho(e)f$ for $f \in C^\infty(M), e \in \Gamma(E)$. Courant algebroids play an important role in some areas of mathematics and physics, for example, generalized geomtries[2], T-dualities[4], topological sigma models[5],supergravity[6], and double field theories[7]. Moreover, there is a one-to-one correspondence between the isomorphism class of differential-graded (dg for short) symplectic manifolds of degree 2 and isomorphism class of Courant algebroids[9].

A Courant-Dorfman algebra is a 5-tuple $(R, E, \partial, \langle,\rangle, [,])$, where R is a commutative algebra, E is an R-module, $\langle,\rangle : E \otimes E \to R$ is a symmetric bilinear form, $\partial : R \to E$ is a derivation, and $[,] : E \otimes E \to E$ is a Dorfman bracket, satisfying a set of compatibility conditions. A Courant algebroid gives a Courant-Dorfman algebra via $(C^\infty(M), \Gamma(E), d, \langle,\rangle, [,])$. Courant-Dorfman algebras generalize Courant algebroids in two directions: first allowing for more general commutative algebras R and modules E than algebras of smooth functions and modules of smooth sections, and second allowing for degenerate $\langle,\rangle$. The relation between Courant-Dorfman algebras and Poisson vertex algebras was found in the context of current algebras.

Current algebras are Poisson algebras consisting of functions on mapping spaces. In classical field theories, a Poisson algebraic structure of currents plays important roles when we consider symmetries of fields. The most basic example is Kac-Moody algebra, which is the Lie algebraic structure on $\mathrm{Map}(S^1, G)$, where G is a Lie group. Let $\mathfrak{g}$ be the Lie algebra of G and e_a be generators of $\mathfrak{g}$ such that $[e_a, e_b] = f_{ab}^c e_c$. The bracket is of the form

$$\{J_a(\sigma), J_b(\sigma')\} = f_{ab}^c J_c(\sigma)\delta(\sigma - \sigma') + k\delta_{ab}\delta'(\sigma - \sigma'), \tag{1.1}$$

where k is a constant. The algebra plays important roles as the symmetry of the Wess-Zumino-Witten model, 2-dimensional conformal invariant sigma model whose target space is a Lie group[11].

Alekseev and Strobl observed there was more general current algebra whose source manifold was S^1 but a target manifold was a general smooth manifold[25]. Let M be a smooth manifold and choose a vector field $v = v^i(x)\partial_i$ and a 1-form $\alpha = \alpha_i(x)dx^i$ on M. We associate to them a current,

$$J_{(v,\alpha)}(\sigma) = v^i(x(\sigma))p_i(\sigma) + \alpha_i(x(\sigma))\partial_\sigma x^i(\sigma). \tag{1.2}$$

The Poisson bracket of these currents is of the form,

$$\{J_{(v,\alpha)}(\sigma), J_{(u,\beta)}(\sigma')\} = J_{[(v,\alpha),(u,\beta)]}(\sigma)\delta(\sigma-\sigma') + \langle (v,\alpha),(u,\beta)\rangle(\sigma)\delta'(\sigma-\sigma'), \tag{1.3}$$

where u, v is a vector field on M, α, β is a 1-form on M, $[(v,\alpha),(u,\beta)] = ([v,u], L_v\beta - \iota_u d\alpha)$ is the Dorfman bracket on the generalized tangent bundle $TM \oplus T^*M$ and $\langle (v,\alpha),(u,\beta)\rangle = \iota_u\alpha + \iota_v\beta$. Let $M = G$ be a Lie group, and consider an Alekseev-Strobl current of the form

$$J = p(\sigma) - \frac{k}{4\pi} g^{-1}(\sigma)\partial_\sigma g(\sigma), \tag{1.4}$$

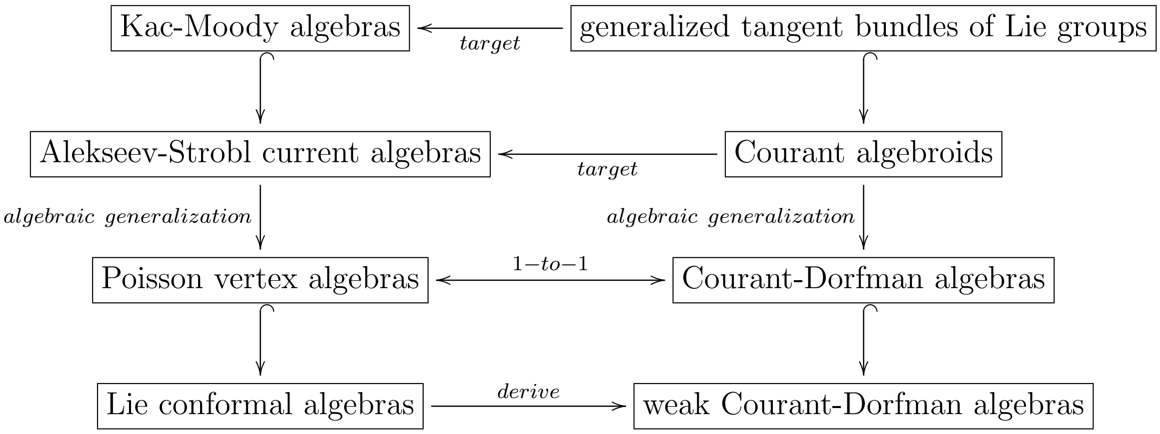
where $p \in TG$ is a left invariant momentum and $g^{-1}\partial g \in T^*G$ is the Maurer-Cartan form. (The current of this type was first given in [36].) We can identify $TG \oplus T^*G$ with $\mathfrak{g} \oplus \mathfrak{g}^*$ and, with the Killing form, we can identify $\mathfrak{g} \oplus \mathfrak{g}^*$ with $\mathfrak{g}$. Thus we can decompose J on a basis e_a of $\mathfrak{g}$, and the Poisson bracket of J_a's is a Kac-Moody algebra(1.1). Alekseev-Strobl currents appear in the description of symmetries of 2-dimensional σ-models.

Inspired by [25], Ekstrand and Zabzine studied the algebraic structure underlying more general current algebras on loop spaces[26]. They found that a weak notion of Courant-Dorfman algebras (weak Courant-Dorfman algebras) appears when we consider the Poisson bracket of currents. As an example, we can obtain the currents whose target are general Courant algebroids. In [30], (weak) Courant-Dorfman algebras were derived using the language of Lie conformal algebras (LCA for short) and Poisson vertex algebras (PVA for short).

A Lie conformal algebra is a module with a λ-bracket satisfying some conditions like a Lie algebra, and a Poisson vertex algebra is defined as an algebra which has a structure of a Lie conformal algebra and satisfies the Leibniz rule. They first appeared in the context of vertex algebras, and the relation with the Poisson bracket of currents was investigated in [15]. We can obtain a Lie conformal algebra from the Poisson bracket of currents, and we can obtain a Poisson vertex algebra by taking into account the multiplication of currents. A Poisson vertex algebra can be seen as an algebraic generalization of a Poisson algebraic structure on loop spaces, while a Lie conformal algebra can be seen as an algebraic generalization of a Lie bracket on loop spaces. In [30], Ekstrand gave weak Couarnt-Dorfman algebras from Lie conformal algebras and showed that the graded Poisson vertex algebras generated by elements of degree 0 and 1 are in one-to one correspondence with the Courant-Dorfman algebras.

The above discussions are summarized as follows.

(1.5)



(1.6)

Courant algebroids are in one-to-one correspondence with degree 2 dg symplectic manifolds, and Alekseev-Strobl current algebras can be described in the language of dg symplectic geometry[31]. Moreover, Poisson algebras on the mapping space whose source manifold was in higher dimensions were constructed (for example, [18], [19]) and a general framework explaining these current algebras were given using dg symplectic geometry.[32], [33], [34] These currents are called BFV(Batalin-Fradkin-Vilkovisky) current algebras. There Courant algebroids(degree 2 dg symplectic manifolds) are generalized to degree n dg symplectic manifolds. BFV current algebras and degree n dg symplectic manifolds can be seen as a higher analog of the second line of (1.6).

The aim of this paper is to give a higher analog of the third line and fourth line of (1.6). In other words, we consider how to make higher Poisson vertex algebras, higher Courant-Dorfman algebras, higher Lie conformal algebras and higher weak Courant-Dorfman algebras which are generalizations of BFV current algebras and algebras of functions of degree n dg symplectic manifolds. In particular, with higher Courant-Dorfman algebras and higher Poisson vertex algebras, we may be able to find and unify more general current algebras including the BFV current algebras, and use the techniques of Poisson vertex algebras in the higher setting.

In this paper, we give a higher analog of the relation between Poisson vertex algebras and Courant-Dorfman algebras. First, we define higher Courant-Dorfman algebras by taking an algebraic structure of functions of degree n dg symplectic manifolds. We give some examples, including ordinary Courant-Dorfman algebras and higher Dorfman bracket on $TM \oplus \wedge^{n-1}T^*M$. We also give an extended version of higher Courant-Dorfman algebras, whose definition is more natural when we consider the relation with higher PVAs.

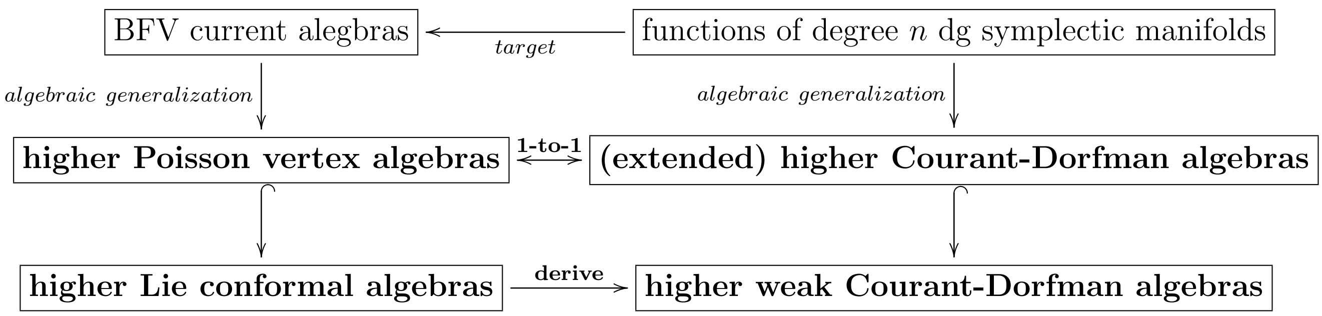
Second, we check that non-degenerate higher Courant-Dorfman algebras have a similar property to the non-degenerate Courant-Dorfman algebras. In particular, we make a graded Poisson algebra of degree $-n$ from a non-degenerate higher Courant-Dorfman algebra. This graded Poisson algebra is a generalization of the graded Poisson algebra of degree -2 introduced in [27], [8]. For a non-degenerate higher Courant-Dorfman algebra from a finite-dimensional graded vector bundle, this graded Poisson algebra is isomorphic to the algebra of functions of degree n dg symplectic manifolds.

Third, we define a higher analog of Lie conformal algebras and Poisson vertex algebras, which are to higher Courant-Dorfman algebras what Poisson vertex algebras are to Courant-Dorfman algebras. We derive a weak notion of higher Courant-Dorfman algebras from higher Lie conformal algebras, and give the correspondence between higher Poisson vertex algebras and higher Courant-Dorfman algebras. This correspondence is a higher generalization of the correspondence between Courant-Dorfman algebras and Poisson vertex algebras, and the main result of this thesis.

Theorem. *There is a bijection between higher Poisson vertex algebras generated by elements of degree* $0 \leq i \leq n-1$ *and extended higher Courant-Dorfman algebras.*

Moreover, we check higher Lie conformal algebras and higher Poisson vertex algebras have LCA-like and PVA-like properties. In particular, we show we can construct a graded Lie algebra out of the tensor product of a higher LCA and an arbitrary differential graded-commutative algebra (dgca for short) and a graded Poisson algebra out of the tensor product of a higher PVA and an arbitrary dgca. Taking a tensor product of the higher Courant-Dorfman algebra arising from a dg symplectic manifold of degree n and de-Rham complex of a $n-1$ dimensional manifold, we see the associated Poisson algebras can be seen as an algebraic description of BFV current algebras. This is the higher generalization of Alekseev-Strobl Poisson vertex algebras.

The higher generalization of (1.6) are summarized as follows.



(1.7)

In the case of $n = 2$, this coincides with (1.6). The bold parts (second line and third line) are defined and studied in this paper.

The organization of this thesis is as follows. In chapter 2, we recall some basics about dg symplectic geometry. In chapter 3, we review the relation between Poisson vertex algebras and Courant-Dorfman algebras, focusing on the Poisson structure of loop spaces. In chapter 4, we define the higher Courant-Dorfman algebra and give some examples. In chapter 5, we construct graded Poisson algebras of degree $-n$, generalizing Keller-Waldman Poisson algebras. In chapter 6, we define higher Lie conformal algebras and Poisson vertex algebras and see the relation with higher Courant-Dorfman algebras. Moreover, we show how we can see these algebras as higher generalization of ordinary LCAs and PVAs.

Chapter 2

Dg symplectic manifolds

In this chapter, we review some basics of dg symplectic manifolds. We refer to [20], [21], [37].

A graded vector space is a collection of vector spaces $V = \oplus_{i\in\mathbb{Z}} V_i$, where V_i is the vector space of degree i. Denote the dual of V by $V^* = \oplus_{i\in\mathbb{Z}}(V_i^*)^{-i}$. Define the tensor algebra of V^* by

$$\mathrm{Tens}(V^*) = \oplus_{i\geq 0}(V^*)^{\otimes i}, \tag{2.1}$$

and the symmetric algebra of V^* by

$$\mathrm{Sym}(V^*) = \mathrm{Tens}(V^*)/(v\otimes w - (-1)^{|v||w|}w\otimes v), \tag{2.2}$$

where $|v|, |w|$ is the degree of the homogeneous elements $v, w \in V^*$. The algebra of functions on V is identified with $\mathrm{Sym}(V^*)$.

A graded manifold $\mathcal{M}$ is a locally ringed space $(M, C^\infty(\mathcal{M}))$ which is locally isomorphic to $(U, C^\infty(U)\otimes \mathrm{Sym}V^*)$, where $U \subset \mathbb{R}^n$ is open, and V is a finite-dimensional graded vector space. A morphism of graded manifolds is a morphism of graded-commutative algebras of functions.

Let V be a graded vector space with homogeneous coordinates $(z^i)_{i=1}^n$ corresponding to a basis of V^*. A vector field X on V is an $\mathbb{R}$-linear derivation on V satisfying the Leibniz rule

$$X(fg) = X(f)g + (-1)^{k|f|}fX(g) \tag{2.3}$$

for $f, g \in V$. It is of the form

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial z^i} \tag{2.4}$$

where $X^i \in \mathrm{Sym}(V^*)$, and $\frac{\partial}{\partial z^i}$ is the dual basis of V. A vector field X acts on V^* according to the following rules:

$$\frac{\partial}{\partial z^i}(z^j) = \delta_i^j, \tag{2.5}$$

$$\frac{\partial}{\partial z^i}(fg) = \left(\frac{\partial}{\partial z^i}(f)\right)g + (-1)^{|z^i||f|}f\frac{\partial}{\partial z^i}(g). \tag{2.6}$$

A vector field X is graded if $|Xf| = |f| + k$ for homogeneous f and fixed $k \in \mathbb{Z}$. k is called the degree of X.

A graded vector field on a graded manifold $\mathcal{M}$ of degree k is a graded linear map

$$X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})[k], \tag{2.7}$$

where $W[k]^i = W^{k+i}$, which satisfies the graded Leibniz rule, i.e.

$$X(fg) = X(f)g + (-1)^{k|f|} f X(g) \tag{2.8}$$

holds for all homogeneous smooth functions $f, g \in : C^\infty(\mathcal{M})$.

Example 2.1. The Euler vector field E on $\mathcal{M}$ is a vector field of degree 0 which satisfies

$$Ef = |f|f, \tag{2.9}$$

for a homogeneous element $f \in C^\infty(\mathcal{M})$. Locally, it is of the form,

$$E = \sum_i |z^i| z^i \frac{\partial}{\partial z^i}. \tag{2.10}$$

Definition 2.1 ($[20, Definition 3.3.]$)**.** A cohomological vector field Q is a graded vector field of degree 1 which satisfies $Q^2 = 0$.

Every cohomological vector field on $\mathcal{M}$ corresponds to a differential on $C^\infty(\mathcal{M})$. A morphism of dg manifolds is a morphism of dg algebras of functions.

The space of graded differential forms consists of homomorphisms from the graded vector fields on $\mathcal{M}$ to the functions on $\mathcal{M}$,

$$\Omega^1(\mathcal{M}) := \mathrm{Hom}_{C^\infty(\mathcal{M})}(\mathfrak{X}(\mathcal{M}), C^\infty(\mathcal{M})). \tag{2.11}$$

Locally, the algebra of differential forms on a graded manifold $\mathcal{M}$ is constructed by adding new coordinates dz^i to $z^i(|dz^i| = |z^i| + 1)$. We denote a space of k-th differential forms by $\Omega^k(\mathcal{M})$. Define the de-Rham differential and the Lie derivative L_V (V:a vector field) by

$$\begin{aligned} d\omega(V_1, ..., V_{n+1}) = &\sum_{i=1}^{n} (-1)^{|V^i|(|\omega| - n + |V^1| + \cdots + |V^{i-1}|)} V^i \omega(V_1, ..., V_{i-1}, \hat{V}_i, V_{i+1}, ..., V_n), \\ &+ \sum_{1 \le i < j \le n} (-1)^{(|V^i| + \cdots + |V^{j-1}|)|V^j|} \omega(V_1, ..., V_{i-1}, [V_i, V_j], V_{i+1}, ..., V_n) \end{aligned} \tag{2.12}$$

$$L_V = \iota_V d + (-1)^{|V|} d \iota_V, \tag{2.13}$$

where $\omega \in \Omega^n(\mathcal{M}), V_i \in \mathfrak{X}$ and ι_V is the contraction.

Definition 2.2 ($[20, Definition 4.3.]$)**.** A graded symplectic form of degree k on a graded manifold $\mathcal{M}$ is a two-form ω which has the following properties;

- ω is homogeneous of degree k,
- ω is closed with respect to the de-Rham differential,
- ω is non-degenerate, i.e. the induced morphism,

$$\omega : T\mathcal{M} \to T^*[k]\mathcal{M}, \tag{2.14}$$

is an isomorphism. There $[k]$ means degree shifting the fibres of the vector bundle.

A graded symplectic manifold of degree k is a pair $(\mathcal{M}, \omega)$ of a graded manifold $\mathcal{M}$ and a graded symplectic form ω of degree k on $\mathcal{M}$.

Lemma 2.1 ([20, *Lemma*4.5.]). *Let ω be a graded symplectic form of degree $k \neq 0$. Then ω is exact.*

Proof. Let E be the Euler vector field. Then,

$$k\omega = L_E\omega = (d\iota_E + \iota_E d)\omega = d(\iota_E\omega), \tag{2.15}$$

which implies $\omega = \frac{d\iota_E\omega}{k}$($E$:Euler vector field). □

Definition 2.3 ([20, *Definition*4.6.]). Let ω be a graded symplectic form on a graded manifold $\mathcal{M}$. A vector field X is called symplectic if $L_X\omega = 0$, and Hamiltonian if there is a smooth function H such that $\iota_X\omega = dH$.

Lemma 2.2 ([20, *Lemma*4.7.]). *Let ω be a graded symplectic form of degree $k \neq 0$ and X be a symplectic vector field of degree l. If $k+l \neq 0$, then X is Hamiltonian.*

Proof. For the Euler vector field E,

$$\begin{aligned} -l\iota_X\omega = \iota_{[E,X]}\omega &= \iota_X d(\iota_E\omega) - d(\iota_X\iota_E\omega) \\ &= k\iota_X\omega + d(\iota_E\iota_X\omega) \end{aligned} \tag{2.16}$$

Let $H := \iota_E\iota_X\omega$, Then

$$dH = (k+l)\iota_X\omega. \tag{2.17}$$

Hence $\iota_X\omega = \frac{dH}{k+l}$.

□

For a degree k graded symplectic manifold $(\mathcal{M}, \omega)$, we can define a Poisson bracket $\{-,-\}$ on $C^\infty(\mathcal{M})$ via

$$\{f, g\} := (-1)^{|f|+1}X_f(g) \tag{2.18}$$

where X_f is the unique graded vector field that satisfies $\iota_{X_f}\omega = df$. X_f is called a Hamiltonian vector field of f. If the vector field Q is Hamiltonian, one can find a Hamiltonian function S such that

$$Q = \{S, -\}. \tag{2.19}$$

Since

$$Q^2(f) = \{\{S, S\}, f\} \tag{2.20}$$

$Q^2 = 0$(i.e. Q is cohomological) is equivalent to $\{S, S\}$ being a constant.

Assume that Q is a cohomological vector field. Then, $|S| = k + 1$, while $|\{-, -\}| = -k$. Consequently, $|\{S, S\}| = k + 2$. If $k \neq -2$, then

$$\{S, S\} = 0. \tag{2.21}$$

This equation is known as the classical master equation. A cohomological vector field with a Hamiltonian function S such that $Q = \{S, -\}$ is called a symplectic cohomological vector field.

Definition 2.4 ([20, $Definition 4.10.$]). A graded manifold endowed with a graded symplectic form and a symplectic cohomological vector field is called a differential graded symplectic manifold, or dg symplectic manifold for short.

A morphism between two dg symplectic manifolds is a morphism of the Poisson algebras of functions respecting the differential induced by the symplectic cohomological vector field.

We consider some special cases of dg symplectic manifolds (M, ω, S), where S is the Hamiltonian function associated to a cohomological vector field. When $k = -1$, these manifolds correspond to BV theories. A BV theory is a formulation of a Lagrangian formalism of a gauge theory based on dg manifold([22]). In this case, the Poisson bracket induced by ω corresponds to the BV antibracket and the Hamiltonian function corresponds to the BV action. When $k = 0$, they emerge in the BFV theories. A BFV theory is a formulation of a constrained Hamiltonian system based on a dg manifold, which is a Hamiltonian counterpart of the BV theory([23], [24]). In this case, the Poisson bracket induced by ω corresponds to the BFV Poisson bracket and the Hamiltonian function corresponds to the BRST charge. Note that the physical Hamiltonian cannot be decided from the dg symplectic manifold.

Suppose $k > 0$ and that all the coordinates are of non-negative degree. Then $\mathcal{M}$ is called an N-manifold. N-manifolds of degree 1 and 2 are analyzed in [9].

$k = 1$. Every graded symplectic manifold of degree 1 is canonically isomorphic to the graded cotangent bundle $T^*[1]M$ of the base manifold M. We denote the coordinates of degree 0 by x^i, and the coordinates in degree 1 by p_i.

The Hamiltonian S has degree 2, thus locally it must be of the form,

$$S = \frac{1}{2} \sum_{i,j=1}^{n} \pi^{ij}(x) p_i p_j. \tag{2.22}$$

Hence, locally S corresponds to a bivector field $\Pi = \pi^{ij}(x)\partial_i \wedge \partial_j$ and $\{S, S\} = 0$ implies that S corresponds to a Poisson bivector field.

Let $C^i(C^\infty(\mathcal{M}))$ be the subspace of $C^\infty(\mathcal{M})$ generated by degree i coordinates. For $f,g\in C^0(C^\infty(\mathcal{M}))$

$$\begin{aligned}\{\{f,S\},g\} &= \{\sum_{i,j=1}^{n}\frac{\partial f}{\partial x^i}\pi^{ij}p_j,g\}\\ &= \sum_{i,j=1}^{n}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}\pi^{ij},\end{aligned} \tag{2.23}$$

which is a Poisson manifold structure.

Hence, there is a one-to-one correspondence between isomorphism class of dg symplectic manifolds of degree 1 and isomorphism class of Poisson manifolds.

$k=2$. The graded symplectic structure induces an isomorphism between the coordinates of degree 0, which we denote by x^i, and the coordinates in degree 2, which we denote by p_i. We denote the coordinates in degree 1 by η^α. The graded symplectic form can be written as

$$\omega=\sum_{i=1}^{n}dp_i dx^i+\frac{1}{2}\sum_{\alpha,\beta=1}^{n}d(g_{\alpha\beta}(x)\eta^\alpha)d\eta^\beta \tag{2.24}$$

where $g_{\alpha\beta}$ is a symmetric non-degenerate form.

Globally, the dg symplectic manifold corresponds to the symplectic realization of $E[1]$ for a vector bundle E over M, equipped with a non-degenerate fibre pairing g.

The Hamiltonian S has degree 3, thus locally it must be of the form

$$S=\sum_{i,\alpha}\rho^i_\alpha(x)p_i\eta^\alpha+\frac{1}{6}\sum_{\alpha,\beta,\gamma}f_{\alpha\beta\gamma}(x)\eta^\alpha\eta^\beta\eta^\gamma. \tag{2.25}$$

For $e\in\Gamma(E)$, the first term corresponds to a bundle map $\rho:E\to TM$ defined by $\rho(e_\alpha)=\rho^i_\alpha(e)\partial_i$, while the second one gives a bracket $[,]$ on $\Gamma(E)$ defined by $[e_\alpha,e_\beta]=f^\gamma_{\alpha\beta}e_\gamma$. $\{S,S\}=0$ implies that (E,g) is a Courant algebroid[9].

Definition 2.5 ($[9, Definition 4.2.]$)**.** A Courant algebroid is a vector bundle E over a smooth manifold M, with a non-degenerate symmetric bilinear form $\langle,\rangle$, and a bilinear bracket $*$ on $\Gamma(E)$. The form and the bracket must be compatiable, in the meaning defined below, with the vector fields on M. We must have a smooth bundle map, the anchor

$$\pi:E\to TM. \tag{2.26}$$

These structure satisfy the following five axioms, for all $A,B,C\in\Gamma(E)$ and $f\in C^\infty(M)$.

Axiom.1 : $\pi(A*B)=[\pi(A),\pi(B)]$ (The bracket of the right hand side is the Lie bracket of vector fields).

Axiom.2 : $A*(B*C)=(A*B)*C+B*(A*C)$.

Axiom.3 : $A * (fB) = (\pi(A)f)B + f(A * B)$.

Axiom.4 : $\langle A, B * C + C * B\rangle = \pi(A)\langle B, C\rangle$.

Axiom.5 : $\pi(A)\langle B, C\rangle = \langle A * B, C\rangle + \langle B, A * C\rangle$.

From the above data, we can define a map, $\partial : C^\infty(M) \to \Gamma(E)$ by

$$\langle \partial f, A\rangle = \pi(A)f \tag{2.27}$$

for all $A \in \Gamma(E)$. A morphism of Courant algebroids is a bundle map respecting all the operations.

We give a correspondence between Courant algebroids and dg symplectic manifolds of degree 2. Denote $C^i(C^\infty(\mathcal{M})) = \{f \in C^\infty(\mathcal{M} : |f| \leq i\}$. Then

$$C^0(C^\infty(\mathcal{M})) \simeq C^\infty(M), C^1(C^\infty(\mathcal{M})) \simeq \Gamma(E). \tag{2.28}$$

For $f \in C^0(C^\infty(\mathcal{M}))$ and $A, B \in C^1(C^\infty(\mathcal{M}))$, we define the anchor π and the bilinear bracket $*$ as the derived brackets,

$$\{\{A, S\}, B\} = A * B, \tag{2.29}$$

$$\{\{A, S\}, f\} = \pi(A)f = \partial(f)A = \{\{S, f\}, A\}. \tag{2.30}$$

We can check this definition satisfies the conditions of a Courant algebroid.

Conversely, given a Courant algebroid $(E, M, \langle,\rangle, *, \pi)$, we can associate a degree 2 dg symplectic manifold $(\mathcal{M}, \omega, S)$. Locally,

$$S = \sum_{i,\alpha} \pi(e_\alpha)x^i p_i \eta^\alpha + \frac{1}{6}\sum_{\alpha,\beta,\gamma} \langle [e_\alpha, e_\beta], e_\gamma\rangle(x)\eta^\alpha\eta^\beta\eta^\gamma, \tag{2.31}$$

where $e_\alpha, e_\beta, e_\gamma \in \Gamma(E)$. Hence, there is a one-to-one correspondence between the isomorphism class of dg symplectic manifolds of degree 2 and isomorphism class of Courant algebroids.

Theorem 2.1 ($[9, Theorem 4.5.]$)**.** *Dg symplectic manifolds of degree 2 are in 1-1 correspondence with Courant algebroids.*

Chapter 3

Courant-Dorfman algebras and Poisson vertex algebras

In this chapter we review the definitions of Courant-Dorfman algebras and Poisson vertex algebras and the relation between these algebras.

Courant-Dorfman algebras are defined by Roytenberg in [8] as an algebraic generalization of Courant algebroids[10]. These are to Courant algebroids what Lie-Rinehart algebras are to Lie algebroids.

Definition 3.1 ($[8, Definition 2.1.]$)**.** A Courant-Dorfman algebra consists of the following data:

- a commutative algebra R,
- an R-module E,
- a symmetric bilinear form $\langle , \rangle : E \otimes E \to R$,
- a derivation $\partial : R \to E$,
- a Dorfman bracket $[,] : E \otimes E \to E$,

which satisfies the following conditions;

$$[e_1, fe_2] = f[e_1, e_2] + \langle e_1, \partial f\rangle e_2, \tag{3.1}$$

$$\langle e_1, \partial\langle e_2, e_3\rangle\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle, \tag{3.2}$$

$$[e_1, e_2] + [e_2, e_1] = \partial\langle e_1, e_2\rangle, \tag{3.3}$$

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]], \tag{3.4}$$

$$[\partial f, e] = 0, \tag{3.5}$$

$$\langle \partial f, \partial g\rangle = 0, \tag{3.6}$$

where $f, g \in R$ and $e_1, e_2, e_3 \in E$.

For a Courant-Dorfman algebra, when $\langle , \rangle$ is non-degenerate, we can make a graded Poisson algebra of degree -2, and when $R = C^\infty(M)$ and $E = \Gamma(F)$ for a vector bundle $F \to M$ (i.e. E is a Courant algebroid), the graded Poisson algebra is isomorphic to the Poisson algebra of functions of the associated degree n dg symplectic manifolds([27], [8]).

An important property of Courant-Dorfman algebras is a relation with Poisson vertex algebras.

Definition 3.2 ([12, *Definition* 2.7])**.** A Lie conformal algebra is a $\mathbb{C}[\partial]$-module W (i.e.∂ acts on elements of W) with a λ-bracket $\{_\lambda\} : W \otimes W \to W[\lambda]$, $\{a_\lambda b\} = \sum_{j\in\mathbb{Z}_+} \lambda^j a_{(j)}b$ (a product $a_{(j)}b \in W$ is called j-th bracket) which satisfies the following conditions. (Here λ is an indeterminate.)

Sesquilinearity :

$$\{\partial a_\lambda b\} = \lambda\{a_\lambda b\},\ \{a_\lambda \partial b\} = (\partial + \lambda)\{a_\lambda b\}, \tag{3.7}$$

(∂ is a derivation of the λ-bracket.)

Skew-symmetry :

$$\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}, \tag{3.8}$$

Jacobi-identity :

$$\{a_\lambda\{b_\mu c\}\} = \{\{a_\lambda b\}_{\mu+\lambda} c\} + \{b_\mu\{a_\lambda c\}\}. \tag{3.9}$$

Definition 3.3 ([15, *Definition* 1.14.])**.** A Poisson vertex algebra is a commutative algebra W with a derivation ∂(i.e.$\partial(ab) = (\partial a)b + a(\partial b)$) and λ-bracket $\{_\lambda\}$: $W \otimes W \to W[\lambda]$ such that W is a Lie conformal algebra and satisfies the Leibniz rule.

Leibniz rule :

$$\{a_\lambda b \cdot c\} = \{a_\lambda b\} \cdot c + b \cdot \{a_\lambda c\}. \tag{3.10}$$

Poisson vertex algebras appear when we consider functions on phase spaces T^*LM of loop spaces $LM = Map(S^1, M)$. We denote local coordinates on T^*LM by $X^i(\sigma), P_i(\sigma)$ with a coordinate σ on S^1, and define a Poisson bracket by

$$\{X^i(\sigma), P_i(\sigma')\} = \delta^i_j \delta(\sigma - \sigma'). \tag{3.11}$$

We can construct local functions on T^*LM out of the coordinates X, P and $\partial = \partial_\sigma$. We consider local functions of the form

$$A(X, \partial X, ..., \partial^k X, P, ..., \partial^l P) \tag{3.12}$$

where k, l are finite. We can create a functional out of A by

$$\epsilon(\sigma) \in C^\infty(S^1) \mapsto J_\epsilon(A) = \int_{S^1} \epsilon(\sigma) A(X, \partial X, ..., \partial^k X, P, ..., \partial^l P) d\sigma. \tag{3.13}$$

Considering the Poisson brackets between them, we can find geometric and algebraic structures on M. In [25], the Poisson brackets between currents parametrised by sections of a generalized tangent bundle $TM \oplus T^*M$ is written in terms of the Dorfman bracket. In [26], considering more general currents, weak Coutant-Dorfman algebras are derived.

Definition 3.4 ([30, $Definition 4.1$])**.** A weak Courant-Dorfman algebra $(E, R, \partial, \langle,\rangle, [,])$ is defined by the following data:

- a vector space R,
- a vector space E,
- a symmetric bilinear form $\langle,\rangle : E \otimes E \to R$,
- a map $\partial : R \to E$,
- a Dorfman bracket $[,] : E \otimes E \to E$,

which satisfy the following conditions:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]], \tag{3.14}$$

$$[A, B] + [B, A] = \partial\langle A, B\rangle, \tag{3.15}$$

$$[\partial f, A] = 0. \tag{3.16}$$

The differences with the definition of a Courant-Dorfman algebra are the properties related to the algebraic structure of $\mathcal{R}$ and $\mathcal{E}$. The relation between Poisson brackets on the local functionals and Lie conformal and Poisson vertex algebras is discussed in [15]. Denote the coordinates on T^*LM by $u^\alpha(\sigma) = \{X^i(\sigma), P_{i-d}(\sigma)\}^\alpha$, where $\alpha = 1, ..., 2d$ and let $u^{\alpha(n)} = \partial^n u^\alpha$. The local functions can be written as polynomials

$$a(u^\alpha, ..., u^{\alpha(N)}). \tag{3.17}$$

We have a total derivative operator by

$$\partial = u^{\alpha(1)}\frac{\partial}{\partial u^\alpha} + \cdots + u^{\alpha(N+1)}\frac{\partial}{\partial u^{\alpha(N)}}. \tag{3.18}$$

The algebra of these polynomials with the total derivative is called an algebra of differential equation $\mathcal{V}$. When we integrate functions over S^1, the function of the form $\partial_\sigma(\cdots)$ doesn't contribute. We can take the quotient $\mathcal{V}/\partial\mathcal{V}$. We denote the image of $a \in \mathcal{V}$ by $\int a \in \mathcal{V}/\partial\mathcal{V}$.

A local Poisson bracket on the phase space can be described by

$$\{u^\alpha(\sigma), u^\beta(\sigma')\} = H_0^{\alpha\beta}(\sigma')\delta(\sigma-\sigma') + H_1^{\alpha\beta}(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + \cdots + H_N^{\alpha\beta}(\sigma')\partial_{\sigma'}^N\delta(\sigma-\sigma'). \tag{3.19}$$

For $a, b \in \mathcal{V}$, we have

$$\{a(\sigma), b(\sigma')\} = \sum_{m,n}\frac{\partial a(\sigma)}{\partial u^{\alpha(m)}}\frac{\partial b(\sigma')}{\partial u^{\beta(n)}}\partial_\sigma^m\partial_{\sigma'}^n\{u^\alpha(\sigma), u^\beta(\sigma')\}. \tag{3.20}$$

Using the Fourier transformation of this Poisson bracket, we obtain a Poisson vertex algebra. Define the Fourier transformed bracket by

$$\{a_\lambda b\} = \int_{S^1} e^{\lambda(\sigma-\sigma')}\{a(\sigma), b(\sigma')\}d\sigma. \tag{3.21}$$

This bracket (called a λ-bracket) with $\mathcal{V}$ and ∂ satisfies the axioms of a Lie conformal algebra[15]. The algebra of differential functions $\mathcal{V}$ with ∂, the λ-bracket and the multiplication of polynomials on $\mathcal{V}$ is a Poisson vertex algebra. Therefore, we can translate the relation between (weak) Courant-Dorfman algebras and currents on the phase space into that between (weak) Courant-Dorfman algebras and Poisson vertex algebras (Lie conformal algebras).

From Lie conformal algebras and Poisson vertex algebras, we can make Lie algebras and Poisson algebras using formal power series. For a Lie conformal algebra W, $W \otimes \mathbb{C}[[t, t^{-1}]]/Im(\partial + \partial_t)$ is a Lie algebra with the Lie bracket

$$[a \otimes t^m, b \otimes t^n] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)}b) t^{m+n-j}. \tag{3.22}$$

Moreover, for a Poisson vertex algebra W, $W \otimes \mathbb{C}[[t, t^{-1}]]/Im(\partial + \partial_t) \cdot W \otimes \mathbb{C}[[t, t^{-1}]]$ is a Poisson algebra with the same Lie bracket.

If we define a formal distribution $a(z)(a \in W)$ by

$$a(z) := \sum_{m \in \mathbb{Z}} z^{-1-m} a t^m \tag{3.23}$$

and the formal δ-function

$$\delta(z - w) := \sum_{m \in \mathbb{Z}} z^{-m-1} w^m, \tag{3.24}$$

then we can check that

$$[a(z), b(w)] = \sum_{j \geq 0} (a_{(j)}b)(w) \partial_w^j \delta(z - w). \tag{3.25}$$

This Lie bracket has a similar form to the bracket of local functions.

We can derive the properties of a weak Courant-Dorfman algebra from a Lie conformal algebra by comparing the independent terms of λ on both sides of the axioms.

Let

$$[a_\lambda b] = \sum_{j \geq 0} a_{(j)} b \lambda^j, \ a_{(0)}b = [a, b], \ [a_\lambda b] - [a, b] = \langle a_\lambda b \rangle, \tag{3.26}$$

$$\langle a, b \rangle = \frac{1}{2}(\langle a_{-\partial} b \rangle + \langle b_{-\partial} a \rangle). \tag{3.27}$$

Then the sesquilinearity says that

$$[\partial a, b] + o(\lambda) = \{\partial a_\lambda b\} = \lambda \{a_\lambda b\} \Rightarrow [\partial a, b] = 0, \tag{3.28}$$

the skew-symmetry says that

$$[a, b] + o(\lambda) = \{a_\lambda b\} = -\{b_{-\lambda-\partial} a\} = -[b, a] + \partial \langle b_{-\partial} a \rangle + o(\lambda) \Rightarrow [a, b] + [b, a] = \partial \langle a, b \rangle, \tag{3.29}$$

and the Jacobi-identity says that

$$[a,[b,c]]+o(\lambda) = [[a,b],c]+[b,[a,c]]+o(\lambda) \Rightarrow [a,[b,c]] = [[a,b],c]+[b,[a,c]]. \quad (3.30)$$

The right formulas are the conditions of a weak Courant-Dorfman algebra.

Moreover, in [30], a one-to-one correspondence between graded Poisson vertex algebras generated by elements of degree 0 and 1 and Courant-Dorfman algebras is established as Theorem 1. In this case, the λ-bracket is of the form

$$\{a_\lambda b\} = [a,b] + \lambda\langle a,b\rangle. \quad (3.31)$$

Substituting this for the axioms of Poisson vertex algebras, we can obtain the axioms of Courant-Dorfman algebraic structure.

Theorem 3.1 ([30, $Theorem 4.1$])**.** *The Poisson vertex algebras that are graded and generated by elements of degree 0 and 1 are in a one-to-one correspondence with the Courant-Dorfman algebras via*

$$W^0 = R,\ W^1 = E,\ \partial = \partial \quad (3.32)$$

$$[e_\lambda e'] = [e,e'] + \lambda\langle e,e'\rangle,\ [e_\lambda f] = \langle e, \partial f\rangle \quad (3.33)$$

In the case of $E = TM \oplus T^*M$, the associated Poisson vertex algebra can be seen as the algebraic description of Alekseev-Strobl currents[25]. This correspondence is used to study the duality of currents[16], and non-commutative analog is considered [17].

Chapter 4

Definitions and examples of higher Courant-Dorfman algebras

In this chapter, we define higher Courant-Dorfman algebras of degree n and give examples. The definition of these algebras of degree 2 coincides with that of Courant-Dorfman algebras.

Let $R = E^0$ be a commutative algebra over a ring $K \supset \mathbb{Q}$, and $E = \oplus_{1\le i\le n-1} E^i$ be a graded R-module, where E^i has degree i. Define a pairing $\langle , \rangle : E \otimes E \to R$ such that $\langle a, b\rangle = 0$ unless $|a| + |b| = n$. Consider the graded-commutative algebra freely generated by E and denote it by $\tilde{\mathcal{E}} = (\mathcal{E}^k)_{k\in\mathbb{Z}}$. We restrict this graded-commutative algebra to the elements of degree $n - 1 \ge k \ge 0$ and denote it by $\mathcal{E} = (\mathcal{E}^k)_{n-1\ge k\ge 0}$. The pairing $\langle , \rangle$ can be extended to $\mathcal{E}$ by the Leibniz rule

$$\langle a, b \cdot c\rangle = \langle a, b\rangle \cdot c + (-1)^{(|a|-n)|b|} b \cdot \langle a, c\rangle. \tag{4.1}$$

Definition 4.1. $\mathcal{E} = (\mathcal{E}^k)_{n-1\ge k\ge 0}$ is *a higher Courant-Dorfman algebra* of degree n if $\mathcal{E}$ has a differential $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$ which satisfies $d^2 = 0$ and $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$ and a bracket $[,] : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ of degree $1 - n$ which satisfies the following condition:

sesquilinearity :

$$\langle da, b\rangle = -(-1)^{|a|-n}[a, b], [da, b] = 0. \tag{4.2}$$

skew-symmetry :

$$[a, b] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, a] = -(-1)^{|a|} d\langle a, b\rangle, \tag{4.3}$$

$$\langle a, b\rangle = -(-1)^{(|a|-n)(|b|-n)}\langle b, a\rangle. \tag{4.4}$$

Jacobi identity :

$$[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, [a, c]], \tag{4.5}$$

$$[a, \langle b, c\rangle] = \langle [a, b], c\rangle + (-1)^{(|a|+1-n)(|b|+1-n)}\langle b, [a, c]\rangle, \tag{4.6}$$

$$\langle a, \langle b, c\rangle\rangle = \langle\langle a, b\rangle, c\rangle + (-1)^{(|a|-n)(|b|-n)}\langle b, \langle a, c\rangle\rangle. \tag{4.7}$$

Leibniz rule :

$$[a \cdot b, c] = [a, b] \cdot c + (-1)^{(|a|+1-n)|b|} b \cdot [a, c]. \tag{4.8}$$

Restricting the bracket to $\mathcal{E}^{n-1} \otimes \mathcal{E}^{n-1} \to \mathcal{E}^{n-1}$, it follows that $\mathcal{E}^{n-1}$ is a Leibniz algebra by the Jacobi identity.

Definition 4.2. A Leibniz algebra is an R-module E with a bracket $[,] : E \otimes E \to E$ satisfying the Leibniz identity;

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]. \tag{4.9}$$

Next, we define the non-degeneracy condition, and fullness condition, like Courant-Dorfman algebras.

Definition 4.3. The bilinear form $\langle , \rangle$ gives rise to a map

$$(-)^\flat : E^i \to (E^{n-i})^\vee = Hom_R(E^{n-i}, R) \tag{4.10}$$

defined by

$$e^\flat(e') = \langle e, e' \rangle. \tag{4.11}$$

$\langle , \rangle$ is non-degenerate if $(-)^\flat$ is an isomorphism, and a higher Courant-Dorfman algebra is non-degenerate if $\langle , \rangle$ is strongly non-degenerate.

When a higher Courant-Dorfman algebra is non-degenerate, the inverse map is denoted by

$$(-)^\sharp : (E^i)^\vee \to E^{n-i} \tag{4.12}$$

and there is a graded-symmetric bilinear form

$$\{-, -\} : E^\vee \otimes_R E^\vee \to R \tag{4.13}$$

defined by

$$\{\lambda, \mu\} = \langle \lambda^\sharp, \mu^\sharp \rangle. \tag{4.14}$$

Definition 4.4. $\langle , \rangle$ is full if ,for every $1 \leq i \leq n-1$, every $a \in R$ can be written as a finite sum $a = \sum_j \langle x_j, y_j \rangle$ with $x_j \in E^i, y \in E^{m-i}$.

Define the anchor map

$$\rho : E^{n-1} \to \mathfrak{X} = Der(R, R) \tag{4.15}$$

by setting

$$\rho(e) \cdot f = \langle e, df \rangle. \tag{4.16}$$

We can define a Dirac submodule, like an ordinary Couarnt-Dorfman algebra.

Definition 4.5. Suppose $\mathcal{E}$ is a higher Courant-Dorfman algebra. An R-submodule $\mathcal{D} \subset \mathcal{E}$ is said to be a Dirac submodule if $\mathcal{D}$ is isotropic with respect to $\langle , \rangle$ and closed under $[-, -]$.

We give some examples.

Example 4.1. Consider the case $n = 2$. In this case, there is an R-module E^1, a pairing $\langle , \rangle : E^1 \otimes E^1 \to R$, a derivation $d : R \to E^1$, and three brackets $[,] : R \otimes E^1 \to R$,$[,] : E^1 \otimes R \to R$,and $[,] : E^1 \otimes E^1 \to E^1$.

From the sesquilinearity, we can check that $[e, f] = \langle e, df \rangle, [f, e] = -\langle df, e \rangle$. For other operations, one can see that the above definition reduces to the definition of a Courant-Dorfman algebra.

Example 4.2. Given a commutative algebra R, let $E^{n-1} = \mathfrak{X} = Der(R, R), E^1 = \Omega^1(K\ddot{a}hler\ differential)$. In this case, $\mathcal{E}^{n-1} = \mathfrak{X} \oplus \Omega^{n-1}$. It becomes a higher Courant-Dorfman algebra with respect to

$$\langle v, \alpha \rangle = \iota_v \alpha, \tag{4.17}$$

$$[v, \alpha] = L_v \alpha, [\alpha, v] = d(\iota_v \alpha) - L_v \alpha, \tag{4.18}$$

$$[v_1, v_2] = \iota_{v_1} \iota_{v_2} \omega \ (\omega \in \Omega^{n+1,cl}). \tag{4.19}$$

and d is the de-Rham differential on Ω^i. In the case of $R = C^\infty(M)$, $\mathcal{E}^{n-1} = TM \oplus \wedge^{n-1} T^* M$, and the bracket $[,]$ is called a higher Dorfman bracket.

Example 4.3. Let $(\mathcal{M}, \omega, \Theta)$ be a degree n dg symplectic manifold and $C = C^{n-1}(C^\infty(\mathcal{M})) = \{f \in C^\infty(\mathcal{M} : |f| \leq n - 1\}$. This is a higher Courant-Dorfman algebra with

$$[a, b] = \{\{a, \Theta\}, b\},\ \langle a, b \rangle = \{a, b\},\ da = \{\Theta, a\}. \tag{4.20}$$

In the previous example, the higher Courant-Dorfman algebra on $\mathcal{E}^{n-1} = TM \oplus \wedge^{n-1} T^* M$ coincides the algebra on $C = C^{n-1}(C^\infty(T^*[n]T[1]M))$.

Example 4.4. As a variant of Example 2, we can replace $\mathfrak{X}$ by a Lie-Rinehart algebra (R, L) and let $E^{n-1} = L, E^1 = \Omega^1$. In this case, $\mathcal{E}^{n-1} = L \oplus \Omega^{n-1}$. It becomes a higher Courant-Dorfman algebra with respect to

$$\langle a, \alpha \rangle = \iota_{\rho(a)} \alpha, \tag{4.21}$$

$$[v, \alpha] = L_{\rho(a)} \alpha, [\alpha, v] = d(\iota_{\rho(a)} \alpha) - L_{\rho(a)} \alpha, \tag{4.22}$$

$$[v_1, v_2] = \iota_{(\rho(v_1))} \iota_{(\rho(v_2))} \omega \ (\omega \in \Omega^{n+1,cl}), \tag{4.23}$$

and d is the de-Rham differential on Ω^i.

In order to focus on the relation with higher Poisson vertex algebras, we should define extended higher Courant-Dorfman algebras, relaxing the condition on $\langle , \rangle$.

Definition 4.6. Let $R = E^0$ be a commutative algebra, and $E = E^i (1 \leq i \leq n-1)$ be a graded R-module. Consider the graded-commutative algebra freely generated by E and denote it by $\tilde{\mathcal{E}} = (\mathcal{E}^k)_{k \in \mathbb{Z}}$. We restrict this graded-commutative algebra to the elements of degree $n - 1 \geq k \geq 0$ and denote it by $\mathcal{E} = (\mathcal{E}^k)_{n-1 \geq k \geq 0}$.

$\mathcal{E} = (\mathcal{E}^k)_{n-1 \geq k \geq 0}$ is *an extended higher Courant-Dorfman algebra* of degree n if $\mathcal{E}$ has a differential $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$ which satisfies $d^2 = 0$ and $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$, a pairing $\langle , \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ of degree $-n$ and a bracket $[,] : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ of degree $1 - n$ which satisfies the sesquilinearity, skew-symmetry, Jacobi identity, and Leibniz rule.

The difference from a higher Courant-Dorfman algebra is that an extended Courant-Dorfman algebra allows the pairing $\langle , \rangle : E^i \otimes E^j \to E^{i+j-n}$ with $i + j \geq n + 1$. From the viewpoint of graded geometry, these algebras include the case that the base manifold is a graded manifold.

Chapter 5

Non-degenerate higher Courant-Dorfman algebras and degree n dg symplectic manifolds

In this chapter, we consider the case that $\langle , \rangle$ is non-degenerate, and study the relationship between the algebras and functions of degree n dg symplectic manifolds. We construct a graded Poisson algebra of degree $-n$, generalizing the Keller-Waldman Poisson algebras[27]. We assume that each E^i is a projective, finitely generated module over R, and that $\langle , \rangle$ is non-degenerate and full.

Definition 5.1. We assume $r \geq n$. $\mathcal{C}^r(\mathcal{E}) \subset \oplus_{1\leq j\leq n-1}\oplus_{1\leq k\leq r-j}\oplus_{\sum_{t=1}^k i_t=r-j}\mathrm{Hom}_K(E^{n-i_1}\otimes \cdots \otimes E^{n-i_k}, E^j)$ consists of elements C for which there exists a K-multilinear map

$$\sigma_C \in \oplus_{1\leq k\leq r-j} \oplus_{\sum_{t'=1}^{k-1} i_{t'}=r-n} \mathrm{Hom}_K(E^{n-i_1}\otimes \cdots \otimes E^{n-i_{k-1}}, \mathfrak{X}), \tag{5.1}$$

satisfying the following conditions:

(1)For all $x_1, ..., x_{k-1}, u, w \in E$, we have

$$\sigma_C(x_1, ..., x_{k-1})\langle u, w\rangle = \langle C(x_1, ..., x_{k-1}, u), w\rangle + \langle u, C(x_1, ..., x_{k-1}, w)\rangle. \tag{5.2}$$

(2)For all $x_1, ..., x_k, u \in E$, we have

$$\begin{aligned}&\langle C(x_1, ...x_i, x_{i+1}, ..., x_k) - (-1)^{(|x_i|-n)(|x_{i+1}|-n)}C(x_1, ..., x_{i+1}, x_i, ..., x_k), u\rangle \\ &= \sigma_C(x_1, ..., x_{i-1}, x_{i+2}, ..., x_k, u)\langle x_i, x_{i+1}\rangle. \end{aligned} \tag{5.3}$$

Furthermore, $\mathcal{C}^0(\mathcal{E}) = R, \mathcal{C}^i(\mathcal{E}) = \mathcal{E}^i$ for $1 \leq i \leq n-1$ and define

$$\mathcal{C}^\bullet(\mathcal{E}) = \oplus_{r\geq 0}\mathcal{C}^r(\mathcal{E}). \tag{5.4}$$

We call σ_C the symbol of C.

Define $d_C \in \oplus_{1\leq l\leq r-n-k} \oplus_{\sum_{t'=1}^{l} i_{t'}=r-n-k} \mathrm{Hom}_K(E^{n-i_1}\otimes \cdots \otimes E^{n-i_l}, \mathfrak{X}\otimes E^k)$ by

$$\langle d_C(x_1, ..., x_l)a, y\rangle := \sigma_C(x_1, ..., x_l, y)a. \tag{5.5}$$

We can use instead of elements $C \in \mathcal{C}^r(\mathcal{E})$ K-multilinear forms $\omega \in \oplus_{1\leq k\leq r}\oplus_{\sum_{t=1}^k i_t=r}$ $\mathrm{Hom}_K(E^{n-i_1}\otimes \cdots \otimes E^{n-i_k}, R)$ defined by $\omega(x_1, ..., x_t) = \langle C(x_1, ..., x_{t-1}), x_t\rangle$.

Definition 5.2. For $r \geq 1$ the subspace $\Omega^r_{\mathcal{C}}(\mathcal{E}) \subset \oplus_{1\leq k\leq r} \oplus_{\sum_{t=1}^k i_t = r} \mathrm{Hom}_K(E^{n-i_1} \otimes \cdots \otimes E^{n-i_k}, R)$ consists of elements ω satisfying the following conditions;

(1)

$$\omega(x_1, ..., ax_k) = a\omega(x_1, ..., x_k), \tag{5.6}$$

for all $a \in R$.

(2)For $r \geq 2$, there exists a multilinear map,

$$\sigma_\omega \in \oplus_{1\leq k\leq r} \oplus_{\sum_{t'=1}^{k-2} i_{t'} = r-n} \mathrm{Hom}_K(E^{n-i_1} \otimes \cdots \otimes E^{n-i_{k-2}}, \mathfrak{X}), \tag{5.7}$$

such that

$$\begin{aligned} &\omega(x_1, ...x_i, x_{i+1}, ..., x_k) - (-1)^{(|x_i|-n)(|x_{i+1}|-n)}\omega(x_1, ..., x_{i+1}, x_i, ..., x_k) \\ &= \sigma_\omega(x_1, ...\overset{i}{\wedge}...\overset{i+1}{\wedge}, x_k)\langle x_i, x_{i+1}\rangle. \end{aligned} \tag{5.8}$$

By the non-degeneracy of $\langle , \rangle$, we obtain the following Lemma:

Lemma 5.1. *There is an isomorphism of graded R-modules*

$$\mathcal{C}^\bullet(\mathcal{E}) \to \Omega^\bullet_{\mathcal{C}}(\mathcal{E}), \tag{5.9}$$

given by

$$\omega(x_1, ..., x_t) = \langle C(x_1, ..., x_{t-1}), x_t\rangle. \tag{5.10}$$

Proposition 5.1. *The map*

$$[,] : \mathcal{C}^r(\mathcal{E}) \otimes \mathcal{C}^s(\mathcal{E}) \to \mathcal{C}^{r+s-n}(\mathcal{E}), \tag{5.11}$$

defined by

$$[a, b] = 0, [a, x] = 0 = [x, a], [x, y] = \langle x, y\rangle, [D, a] = \sigma_D a = -[a, D], \tag{5.12}$$

$$[C, x] = \iota_x C = -(-1)^{(r+n)(|x|+n)}[x, C], \tag{5.13}$$

for elements $a, b \in R, x, y \in \mathcal{C}^s(\mathcal{E})$ for $s \leq n$, $D \in \mathcal{C}^n(\mathcal{E}), C \in \mathcal{C}^r(\mathcal{E})$ for $r \geq n$, and by the recursion,

$$\iota_x[C_1, C_2] = [[C_1, C_2], x] = [C_1, [C_2, x]] - (-1)^{(|C_1|+n)(|C_2|+n)}[C_2, [C_1, x]], \tag{5.14}$$

is well-defined and makes $\mathcal{C}^\bullet(\mathcal{E})$ a graded Lie algebra.

Proof. It suffices to show that the recursion (5.14) is consistent with (5.12) and (5.13), that $[C_1, C_2] \in \mathcal{C}^{r+s-n}(\mathcal{E})$, and that the bracket satisfies the conditions for a graded Lie algebra.

The consistency can be checked as follows:

$$\begin{aligned} [[D, x], y] &= \langle D(x), y\rangle = (-1)^{(|x|-n)(|y|-n)}\langle D(y), x\rangle + \sigma_C\langle x, y\rangle \\ &= (-1)^{(|x|-n)(|y|-n)}[[D, y], x] + [D, [x, y]]. \end{aligned} \tag{5.15}$$

$$[[C,x],y] = \iota_y \iota_x C = (-1)^{(|x|-n)(|y|-n)} \iota_x \iota_y C + d_C\langle x, y\rangle$$
$$= (-1)^{(|x|-n)(|y|-n)}[[C,y],x] + [[C,[x,y]]. \tag{5.16}$$

Next, we check that $[C_1, C_2]$ is an element in $\mathcal{C}^{r+s-n}(\mathcal{E})$. For $N \leq 2n-1$, the claim is clear. For $N = 2n$, we consider three cases. If $a \in R$ and $C \in \mathcal{C}^{2n}(\mathcal{E})$, then $[C,a] = d_C a \in \mathcal{C}^n(\mathcal{E})$ and

$$[[C,a],b] = [C,[a,b]] - (-1)^n [a,[C,b]]. \tag{5.17}$$

If $x \in E^i$ and $C \in \mathcal{C}^{2n-i}(\mathcal{E})$, then $[C,a] = d_C a \in \mathcal{C}^n(\mathcal{E})$ and

$$[[C,x],a] = [C_1,[x,a]] - (-1)^{(n-i)(i+n)}[x,[C,a]]. \tag{5.18}$$

If $D_1, D_2 \in \mathcal{C}^n(\mathcal{E})$, then $[D_1, D_2] \in \mathcal{C}^r(\mathcal{E})$ with

$$\sigma_{[D_1,D_2]} a = \sigma_{D_1}\sigma_{D_2} a - \sigma_{D_2}\sigma_{D_1} a, \tag{5.19}$$

$$[[D_1,D_2],a] = [D_1,[D_2,a]] - [D_2,[D_1,a]]. \tag{5.20}$$

Let $C_1 \in \mathcal{C}^r(\mathcal{E}), C_2 \in \mathcal{C}^s(\mathcal{E})$ with $r+s \geq 2n+1$. Consider a map $h : R \to \mathcal{C}^{r+s-2n}(\mathcal{E})$ defined by

$$h(a) = [C_1,[C_2,a]] - (-1)^{(r+n)(s+n)}[C_2,[C_1,a]]. \tag{5.21}$$

Then $[C_1, C_2] \in \mathcal{C}^{r+s-n}(\mathcal{E})$ and the symbol is

$$\sigma_{[C_1,C_2]}(x_1,...,x_t)a = \langle h(a)(x_1,...,x_{t-1}), x_t\rangle. \tag{5.22}$$

The skew symmetry is clear by the construction. We check the Jacobi identity. It suffices to show

$$J(C_1,C_2,C_3) := [[C_1,C_2],C_3] - [C_1,[C_2,C_3]] - (-1)^{(|C_1|+n)(|C_2+n|)}[C_2,[C_1,C_3]] = 0. \tag{5.23}$$

We prove the claim by induction for $N = \sum |C^i|$. For $1 \leq N \leq 2n$, it is clear. By the recursion,

$$\begin{aligned}&[J(C_1,C_2,C_3),x]\\ =&(-1)^{(|C_2-n|)(|x|-n)+(|C_3|-n)(|x|-n)} J([C_1,x],C_2,C_3)\\ &+(-1)^{(|C_3|-n)(|x|-n)} J(C_1,[C_2,x],C_3) + J(C_1,C_2,[C_3,x])\end{aligned} \tag{5.24}$$

and by induction, we obtain $[J(C_1,C_2,C_3),x] = 0$. For $N \geq 2n+1$, $|J(C_1,C_2,C_3)| \geq 1$, therefore we conclude $J(C_1,C_2,C_3) = 0$. □

Proposition 5.2. *There exists an associative, graded-commutative K-bilinear product $\wedge$ of degree 0 on $\mathcal{C}^\bullet(\mathcal{E})$ uniquely defined by*

$$a \wedge b = ab = b \wedge a, a \wedge x = ax = x \wedge a, \tag{5.25}$$

for $a, b \in R$ and $x \in E$ and by the recursion rule

$$[C_1 \wedge C_2, x] = (-1)^{(r-n)s} C_2 \wedge [C_1,x] + C_1 \wedge [C_2,x]. \tag{5.26}$$

Proof. We prove that

$$(x_1, ..., x_t) \to [C_1 \wedge C_2, x_1](x_2, ..., x_t), \tag{5.27}$$

is an element in $\mathcal{C}^{r+s}(\mathcal{E})$, and that

$$[C_1 \wedge C_2, a] = (-1)^{(r-n)s} C_2 \wedge [C_1, a] + C_1 \wedge [C_2, a]. \tag{5.28}$$

If $N \leq n$, the claim is clear. If $N = r + s \geq n + 1$, the map

$$h(a) = (-1)^{(r-n)s} C_2 \wedge [C_1, a] + C_1 \wedge [C_2, a], \tag{5.29}$$

is $d_{C_1 \wedge C_2}$. □

Theorem 5.1. $(\mathcal{C}^\bullet(\mathcal{E}), [,], \wedge)$ *is a graded Poisson algebra of degree* $-n$.

Proof. It suffices to show the Leibniz rule

$$[C_1 \wedge C_2, C_3] = (-1)^{(r-n)s} C_2 \wedge [C_1, C_3] + C_1 \wedge [C_2, C_3]. \tag{5.30}$$

We can check by direct calculations and the recursions.

□

Since $\mathcal{C}^\bullet(\mathcal{E}) \simeq \Omega^\bullet_{\mathcal{C}}(\mathcal{E})$, we can define a graded Poisson algebraic structure on $\Omega^\bullet_{\mathcal{C}}(\mathcal{E})$. This bracket is an extension of $\{-, -\} : E^\vee \otimes_R E^\vee \to R$.

We can construct $m \in \Omega^\bullet_{\mathcal{C}}(\mathcal{E}) \simeq \mathcal{C}^r(\mathcal{E})$ from the map $\phi : \mathcal{E}^{i_1} \otimes \mathcal{E}^{i_2} \otimes \cdots \otimes \mathcal{E}^{i_m} \to \mathcal{E}^{i_1 + \cdots + i_m - mn + r}$ by

$$\omega_\phi(e_1, e_2, ..., e_k) = \langle \cdots \langle \phi(e_1, ..., e_m), e_{m+1} \rangle \cdots \rangle, e_k \rangle. \tag{5.31}$$

Let ϕ be the bracket of the higher Courant-Dorfman algebra. Then, ω_ϕ satisfies $|\omega_\phi| = n+1$ and $[\omega_\phi, \omega_\phi] = 0$ and the map $[\omega_\phi, -]$ is degree 1 and squares to 0, thus it defines a differential on $\mathcal{C}^\bullet(\mathcal{E})$. This is a higher derived bracket of this algebra [29].

Next, we define another Poisson algebra $\mathcal{R}^\bullet(\mathcal{E})$ generalizing the Rothstein algebra.

Definition 5.3. A connection ∇ for the graded module $E = (E^i)$ is a map $\nabla : \mathfrak{X} \times E \to E$ of degree 0 such that

$$\nabla_{aD} x = a \nabla_D x, \tag{5.32}$$

$$\nabla_D(ax) = a \nabla_D x + D(a) x, \tag{5.33}$$

for all $a \in R$ $x, y \in E$ and $D \in \mathfrak{X}$. If $\langle , \rangle : E \otimes E \to R$ is a K-bilinear form, then ∇ is called metric if in addition

$$D\langle x.y \rangle = \langle \nabla_D x, y \rangle + \langle x, \nabla_D y \rangle, \tag{5.34}$$

for all $x, y \in E$ and $D \in \mathfrak{X}$.

If each E^i is finitely generated and projective then it allows for a connection ∇. If $\langle,\rangle : E \otimes E \to R$ is non-degenerate, then ∇ can be chosen to be a metric connection. Indeed, if $\tilde{\nabla}$ is any connection and $\langle,\rangle$ is strongly non-degenerate then ∇ defined by

$$\langle \nabla_D x, y\rangle = \frac{1}{2}(\langle \tilde{\nabla}_D x, y\rangle - \langle x, \tilde{\nabla}_D y\rangle + D\langle x, y\rangle) \tag{5.35}$$

is a metric connection.

Next we introduce the curvature of ∇. A given connection for E extends to $\mathrm{Sym}(E)$ by imposing the Leibniz rule. Thus we can consider

$$R(D_1, D_2)\xi := \nabla_{D_1}\nabla_{D_2}\xi - \nabla_{D_2}\nabla_{D_1}\xi - \nabla_{[D_1,D_2]}\xi, \tag{5.36}$$

for $D_i \in \mathfrak{X}$ and $\xi \in \mathrm{Sym}(E)$. It defines an element

$$R(D_1, D_2) \in \mathrm{End}(\mathrm{Sym}(E)). \tag{5.37}$$

Restricting $R(D_1, D_2)$ to E gives a map $R(D_1, D_2) : E \to E$. For $x \in E^i$ and $y \in E^{n-i}$,

$$\langle R(D_1, D_2)x, y\rangle = (-1)^{i(n-i)}\langle R(D_1, D_2)y, x\rangle. \tag{5.38}$$

E^i is projective and finitely generated, thus using the strongly non-degenerate inner product $\langle,\rangle$ on E we can define $r(D_1, D_2) \in Sym^2E|_{deg=n}$ by

$$R(D_1, D_2)x = \langle r(D_1, D_2), x\rangle. \tag{5.39}$$

With this preparation, the higher Rothstein algebra can now be defined. (The ordinary Rothstein algebra is defined in [27].)

Definition 5.4. The higher Rothstein algebra is defined as a graded symmetric algebra by

$$\mathcal{R}^\bullet(\mathcal{E}) = \mathrm{Sym}(\oplus_{1\le i\le n-1}E^i[-i] \oplus \mathfrak{X}[-n]). \tag{5.40}$$

Theorem 5.2. *Let ∇ be a metric connection on E. Then there exists a unique graded Poisson structure $\{-,-\}_R$ on $\mathcal{R}^\bullet(\mathcal{E})$ of degree $-n$ such that*

$$\{a, b\}_R = 0 = \{a, x\}_R, \tag{5.41}$$
$$\{x, y\}_R = \langle x, y\rangle = -(-1)^{(|x|-n)(|y|-n)}\{y, x\}_R, \tag{5.42}$$
$$\{D, a\}_R = -D(a) = -\{a, D\}_R, \tag{5.43}$$
$$\{D, x\}_R = -\nabla_D x = -\{x, D\}_R, \tag{5.44}$$
$$\{D_1, D_2\}_R = -[D_1, D_2] - r(D_1, D_2) = -\{D_2, D_1\}_R, \tag{5.45}$$

for $a, b \in R$, $x, y \in E$ and $D_1, D_2 \in \mathfrak{X}$.

Proof. We can extend the bracket $\{\}_R$ to $\mathcal{R}(\mathcal{E})$ by the Leibniz rule from the above definition. The skew symmetry is clear by construction.

Jacobi identity follows from the following Bianchi identity.

$$\begin{aligned}&\nabla_{D_1}r(D_2, D_3) + \nabla_{D_2}r(D_3, D_1) + \nabla_{D_3}r(D_1, D_2)\\ &+r(D_1, [D_2, D_3]) + r(D_2, [D_3, D_1]) + r(D_3, [D_1, D_2]) = 0.\end{aligned} \tag{5.46}$$

□

Next, we find the relation between $\mathcal{R}^\bullet(\mathcal{E})$ and $\mathcal{C}^\bullet(\mathcal{E})$.

Definition 5.5. Let the R-linear map $\mathcal{J} : \mathcal{R}^\bullet(\mathcal{E}) \to \mathcal{C}^\bullet(\mathcal{E})$ be defined by

$$\mathcal{J}(a) = a, \mathcal{J}(x) = x, \mathcal{J}(D) = -\nabla_D \tag{5.47}$$

for $a \in R, x \in E$ and $D \in \mathfrak{X}$ and extend by the Leibniz rule.

Proposition 5.3. *(1) The map $\mathcal{J}$ is a homomorphism of Poisson algebras.*
(2) Let $\phi \in \mathcal{R}^\bullet(\mathcal{E})$ with $r \geq n$, then

$$\mathcal{J}(\phi)(x_1, ..., x_k) = \{\{...\{\phi, x_1\}_R, ...\}_R, x_k\}_R, \tag{5.48}$$

and

$$\sigma_{\mathcal{J}(\phi)}(x_1, ..., x_{k-1})a = \{\{...\{\phi, x_1\}_R, ...\}_R, x_{k-1}\}_R, a\}_R, \tag{5.49}$$

for all $x_i \in E$ and $a \in R$.

Proof. (1):From the definition this is obvious for generators and it is true for all $\mathcal{R}^\bullet(\mathcal{E})$ by the Leibniz rule.
(2)$[\mathcal{J}(\phi), x] = [\mathcal{J}(\phi), \mathcal{J}(x)] = \mathcal{J}(\{\phi, x\}_R)$ and induction for k. □

Lemma 5.2. *Let $\phi \in \mathcal{R}^r(\mathcal{E})$ with $r \geq 1$, then*

$$\mathcal{J}(\phi)(x_1, ..., x_k) = \{\{...\{\phi, x_1\}_R, ...\}_R, x_k\}_R = 0, \tag{5.50}$$

if and only if $\phi = 0$.

Proof. It is true for $r = 1, ..., n-1$ due to the non-degeneracy of $\langle, \rangle$. Suppose for it is true for $1, 2, ..., r-1$. For $\phi \in \mathcal{R}^r(\mathcal{E})$, we have $|\{\phi, x\}_R| < r$, thus it satisfies the condition if and only if $\{\phi, x\}_R = 0$ Then

$$\{\phi, \langle x, y\rangle\} = \{\phi, \{x, y\}\} = \{\{\phi, x\}, y\} + (-1)^{(\phi+n)(|x|+n)}\{x, \{\phi, y\}\} = 0, \tag{5.51}$$

and due to fullness $\{\phi, a\}_R = 0$. Then, $\phi = 0$. □

Corollary 5.1. *Let $\hat{\mathcal{C}}^\bullet(\mathcal{E})$ be the subalgebra of $\mathcal{C}^\bullet(\mathcal{E})$ generated by R, E and $\mathcal{C}^n(\mathcal{E})$. Then $\hat{\mathcal{C}}^\bullet(\mathcal{E})$ is closed under the bracket $[,]$ and $\mathcal{J}$ is an isomorphism of Poisson algebras*

$$\mathcal{J} : \mathcal{R}^\bullet(\mathcal{E}) \to \hat{\mathcal{C}}^\bullet(\mathcal{E}). \tag{5.52}$$

Proof. $\mathcal{J}$ is injective due to the above lemma. If $D \in \mathcal{C}^n(\mathcal{E})$ we can define an element $\xi \in Sym(E)|_{deg.=n}$ by $\langle \xi, x\rangle = D(x) - \Delta_{\sigma_D} x$, hence $D \in \mathcal{J}(\mathcal{R}^n(\mathcal{E}))$, therefore $\mathcal{C}^n(\mathcal{E}) \simeq \mathcal{R}^n(\mathcal{E})$. □

Lemma 5.3. *We have $\hat{\mathcal{C}}^{n+1}(\mathcal{E}) = \mathcal{C}^{n+1}(\mathcal{E})$.*

Proof. Let $C \in \mathcal{C}^{n+1}(\mathcal{E})$ and let $d_C \in Der(R, E^1)$ be given $\langle d_C r, x\rangle = \sigma_C(x)r$. We can find $D^1, ..., D^n \in \mathfrak{X}$ and $e_1, ..., e_n \in \mathcal{E}$ such that $d_C(r) = D^i(r)e_i$. Let $T = C - \nabla_{D^i} \wedge e_i$. Then, $T \in \mathcal{E}^{n+1}$. □

Let $m \in \mathcal{C}^{n+1}(\mathcal{E})$ with $[m,m]=0$. Then $\delta_m = [m,-]$ squares to 0, and we obtain a subcomplex $\hat{\mathcal{C}}^\bullet(\mathcal{E})$. This complex is isomorphic to $\mathcal{R}^\bullet(\mathcal{E})$ with differential $\delta_{\mathcal{J}(m)} = [\mathcal{J}(m),-]$.

When $R = C^\infty(M)$ and $E^i = \Gamma(M, F^i)$ for a graded vector bundle $F^i \to M$, this Poisson algebra is isomorphic to the associated dg symplectic manifold$(\mathcal{M}, \omega, \Theta)$.

Lemma 5.4. *Let $(F^i(1 \le i \le n-1))$ be a graded bundle over a smooth manifold M, and $\langle , \rangle : F^i \otimes F^{n-i} \to C^\infty(M)$ a fiberwise non-degenerate graded-symmetric bilinear form. Degree n graded symplectic manifolds (with a choice of splitting in the sense of Remark*5.1*) are in one-to-one correspondence with such graded vector bundles with $\langle , \rangle$.*

Proof. Any graded manifold is noncanonically diffeomorphic to a graded manifold associated to a graded vector bundle([28],Theorem 1). Let $(\mathcal{M}, \omega)$ be a degree n symplectic manifold and let F^i the associated graded vector bundle. Then, $E^n = \Gamma(TM)$ and the Poisson bracket of degree $-n$ induced by ω is an extension of $\langle , \rangle$ as a derivation. (In this case $C^\infty(\mathcal{M}) \simeq \mathcal{R}^\bullet(\mathcal{E})$.) □

Remark 5.1. The diffeomorphism between a graded manifold and a graded manifold associated to a graded vector bundle is noncanonical. Denote the algebra of degree i functions of a graded manifold $\mathcal{M}$ by $\mathcal{A}^i$. There exists a short exact sequence

$$0 \longrightarrow (\mathcal{A}^1)^2 \longrightarrow \mathcal{A}^2 \longrightarrow \Gamma(F^2) \longrightarrow 0, \tag{5.53}$$

where F^2 is a vector bundle over the base manifold M of $\mathcal{M}$. Fixing a splitting, we can identify $\mathcal{A}^2$ with $(\mathcal{A}^1)^2 \oplus \Gamma(F^2)$. For $\mathcal{A}^i (i \ge 2)$, we can choose such a splitting. Thus graded manifolds with a choice of splittings are in one-to-one correspondence with graded vector bundles.

Theorem 5.3. *Let $(R, E^i(1 \le i \le n-1), \langle , \rangle, d, [-,-])$ be a higher Courant-Dorfman algebra. Suppose $R = C^\infty(M)$ for a smooth manifold M, and each $E^i = \Gamma(F^i)$ for a graded vector bundle F^i over M. Degree n dg symplectic manifolds are in one-to-one correspondence with higher Courant-Dorfman algebras of these types.*

Proof. Let $(\mathcal{M}, \omega)$ be a degree n symplectic manifold corresponding to $(E^i, \langle , \rangle)$, with $\mathcal{A}$ its graded Poisson algebra of polynomial functions. Then $\mathcal{A}^0 = C^\infty(M)$ and $\mathcal{A}^i = \mathcal{E}^i$ for $1 \le i \le n-1$, and $\{-,-\}$ restricted to $\mathcal{A}^i$ is an extension of $\langle , \rangle$. Let $\Theta \in \mathcal{A}^{n+1}$ satisfy $\{\Theta, \Theta\} = 0$. Given arbitrary $e, e_1, e_2 \in \mathcal{A}^i$, define a differential d and bracket $[,]$ by

$$d(e) = \{\Theta, e\}, [e_1, e_2] = \{\{e_1, \Theta\}, e_2\}. \tag{5.54}$$

This construction gives a higher Courant-Dorfman algebra.

Conversely, given a higher Courant-Dorfman algebraic structure on $(E^i, \{,\})$, we can define $\Theta = \mathcal{J}(\omega_\phi)$. Locally, Θ can be written as follows. In a Darboux chart $(\xi^{a(k)}) = (q^{a(l)}, p^{a(n-l)})(1 \le k \le n, 1 \le l \le \lfloor \frac{n}{2} \rfloor)$, corresponding to a chart (x_i) on M and a local basis $e^{a(k)}$ of sections of E^k such that $\langle e^{a(k)}, e^{b(n-k)} \rangle = \delta^{ab}$

$$\Theta = \sum_{\sum i_t = n+1} \phi(q) \xi^{a_1(i_1)} \cdots \xi^{a_m i_m} \tag{5.55}$$

$$\phi(q) = \langle \cdots \langle [e^{a_1(n-i_1)}, e^{a_2(n-i_2)}], e^{a_3(n-i_3)} \rangle, \cdots, e^{a_m(n-i_m)} \rangle. \tag{5.56}$$

This satisfies $\{\Theta, \Theta\} = 0$ due to the properties of a higher Coutant-Dorfman algebra. □

Chapter 6

Higher PVAs from higher Courant-Dorfman algebras

In this chapter, we define higher PVAs corresponding to higher Courant-Dorfman algebras and check these algebras have a PVA-like property. In particular, a tensor product of a higher PVA and an arbitrary dgca has a structure of degree 0 graded Poisson algebra.

First, we define higher Lie conformal algebras and derive properties of higher weak Courant-Dorfman algebras, in a similar way that we derive the properties of Courant-Dorfman algebras from Lie conformal algebras.

Definition 6.1. *A higher Lie conformal algebra* of degree n is a graded $\mathbb{C}[d]$-module $W = \oplus_{m \in \mathbb{Z}_{\geq 0}} W^m$(i.e. d acts on elements of W) with $|d| = 1$, which has a degree $1-n$ map which we call Λ-bracket $[_\Lambda] : W \otimes W \to W[\Lambda]$ with $|\Lambda| = 1$ which satisfy the conditions. (Here, Λ is an indeterminate.)

Sesquilinearity

$$[d a_\Lambda b] = -(-1)^{-n} \Lambda [a_\Lambda b], [a_\Lambda db] = -(-1)^{|a|-n}(d + \Lambda)[a_\Lambda b] \tag{6.1}$$

Skewsymmetry

$$[a_\Lambda b] = -(-1)^{(|a|+1-n)(|b|+1-n)}[b_{-\Lambda-d} a] \tag{6.2}$$

Jacobi identity

$$[a_\Lambda [b_\Gamma c]] = [[a_\Lambda b]_{\Lambda+\Gamma} c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b_\Gamma [a_\Lambda c]]. \tag{6.3}$$

We derive the properties of higher weak Courant-Dorfman algebras from higher Lie conformal algebras.

The Λ-bracket is of the form

$$[a_\Lambda b] = \sum_{j \geq 0} \Lambda^j a_{(j)} b \ (a_{(j)} b \in W^{|a|+|b|+1-n-j}). \tag{6.4}$$

Let

$$[a, b] = a_{(0)} b, \ \langle a_\Lambda b \rangle = \sum_{j \geq 1} \Lambda^j a_{(j)} b. \tag{6.5}$$

$$\langle a, b\rangle = \langle a_{-d} b\rangle. \tag{6.6}$$

Then we derive the properties of a higher Courant-Dorfman algebra by comparing the independent terms of Λ on the both sides of the axioms.

From the sesquilinearity, we can see that

$$[da, b] + o(\Lambda) = \{da_\Lambda b\} = (-1)^{-n}\Lambda\{a_\Lambda b\} \Rightarrow [da, b] = 0, \tag{6.7}$$

from the skew-symmetry, we can see that

$$\begin{aligned} [a, b] + o(\Lambda) = \{a_\Lambda b\} &= -(-1)^{(|a|+1-n)(|b|+1-n)}\{b_{-\Lambda-d} a\} \\ &= -(-1)^{(|a|+1-n)(|b|+1-n)}([b, a] + d\langle b_{-d} a\rangle) + o(\Lambda) \\ \Rightarrow & [a, b] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, a] = (-1)^{(|a|+1-n)(|b|+1-n)} d\langle b, a\rangle, \end{aligned} \tag{6.8}$$

and from the Jacobi-identity, we can see that

$$\begin{aligned} [a, [b, c]] + o(\Lambda) &= [[a, b], c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, [a, c]] + o(\Lambda) \\ \Rightarrow [a, [b, c]] &= [[a, b], c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, [a, c]]. \end{aligned} \tag{6.9}$$

These are properties of a higher weak Courant-Dorfman algebras.

Definition 6.2. *A higher weak Courant-Dorfman algebra* of degree n consists of the following data:

- a graded vector space $\mathcal{E} = (\mathcal{E}^i)$,
- a graded symmetric bilinear form of degree $-n$ $\langle , \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$,
- a map of degree 1 $d : \mathcal{E} \to \mathcal{E}$,
- a Dorfman bracket of degree $1 - n$ $[,] : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$,

which satisfies the following conditions.

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + (-1)^{(|e_1|+1-n)(|e_2|+1-n)}[e_2, [e_1, e_3]], \tag{6.10}$$

$$[e_1, e_2] + (-1)^{(|e_1|+1-n)(e_2+1-n)}[e_2, e_1] = (-1)^{(|e_1|+1-n)(|e_2|+1-n)} d\langle e_2, e_1\rangle, \tag{6.11}$$

$$[de_1, e_2] = 0. \tag{6.12}$$

Next, we define higher Poisson vertex algebras. We did not assume that d is a differential so far. From now on, we assume $d^2 = 0$. Then, $C = (C^k, d)$ is a cochain complex.

Definition 6.3. Let $C = (C^k, d)$ a cochain complex. C is a higher Lie conformal algebra of degree n if it endows with a Λ-bracket $[_\Lambda] : C \otimes C \to C[\Lambda]$ defined by

$$a \otimes b \mapsto [a_\Lambda b] = a_{(0)} b + \Lambda a_{(1)} b \tag{6.13}$$

satisfying the axioms of higher Lie conformal algebras. C is a higher Poisson vertex algebra of degree n if it is a higher LCA and a differential graded-commutative algebra which satisfies

the Leibniz rule

$$[a_\Lambda bc] = [a_\Lambda b]c + (-1)^{(|a|+1-n)|b|} b[a_\Lambda c]. \tag{6.14}$$

From extended higher Courant-Dorfman algebras, we obtain the following theorem.

Theorem 6.1. *The above higher Poisson vertex algebras generated by elements of degree $0 \le i \le n-1$ are in one-to-one correspondence with the extended higher Courant-Dorfman algebras*

Proof. Assume we have a higher PVA $(C = (C^k, d), \{_\Lambda\})$. We denote $R = C^0, \mathcal{E}^i = C^i (1 \le i \le n-1)$. $C = (C^k, d)$ is a dgca, thus R is a commutative algebra and each $\mathcal{E}^i$ is an R-module. We denote the Λ-bracket by

$$a_{(0)}b = [a, b] \ a_{(1)}b = (-1)^{|a|}\langle a, b\rangle. \tag{6.15}$$

Sesquilinearity says that

$$(da)_{(0)}b + \Lambda(da)_{(1)}b = -(-1)^{-n}\Lambda(a_{(0)}b + \Lambda a_{(1)}b). \tag{6.16}$$

Comparing the 0th-order terms and the first-order terms of Λ, we have

$$[da, b] = 0, \ \langle a, b\rangle = -(-1)^{|a|-n}[a, b]. \tag{6.17}$$

In a similar way, from the skewsymmetry,

$$a_{(0)}b + \Lambda a_{(1)}b = -(-1)^{(|a|+1-n)(|b|+1-n)}(b_{(0)}a - (\Lambda + d)b_{(1)}a), \tag{6.18}$$

we can see that

$$[a, b] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, a] = -(-1)^{|a|}d\langle b, a\rangle, \tag{6.19}$$

$$\langle a, b\rangle = -(-1)^{(|a|-n)(|b|-n)}\langle b, a\rangle. \tag{6.20}$$

From the Jacobi-identity

$$\begin{aligned} &a_{(0)}(b_{(0)}c) + a_{(0)}(\Gamma b_{(1)}c) + \Lambda a_{(1)}(b_{(0)}c) + \Lambda a_{(1)}(\Gamma b_{(1)}c) \\ &= (a_{(0)}b)_{(0)}c + (\Lambda + \Gamma)(a_{(0)}b)_{(1)}c + (\Lambda a_{(1)}b)_{(0)}c + (\Lambda + \Gamma)(\Lambda a_{(1)}b)_{(1)}c \\ &+ (-1)^{(|a|+1-n)(|b|+1-n)}\{b_{(0)}(a_{(0)}c) + b_{(0)}(\Lambda a_{(1)}c) + \Gamma b_{(1)}(a_{(0)}c) + \Gamma b_{(1)}(\Lambda a_{(1)}c)\}, \end{aligned} \tag{6.21}$$

we can see that

$$[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, [a, c]], \tag{6.22}$$

$$[a, \langle b, c\rangle] = \langle [a, b], c\rangle + (-1)^{(|a|+1-n)(|b|+1-n)}\langle b, [a, c]\rangle, \tag{6.23}$$

$$\langle a, \langle b, c\rangle\rangle = \langle\langle a, b\rangle, c\rangle + (-1)^{(|a|-n)(|b|-n)}\langle b, \langle a, c\rangle\rangle. \tag{6.24}$$

From the Leibniz rule

$$a_{(0)}(bc) + \Lambda a_{(1)}(bc) = (a_{(0)}b)c + \Lambda(a_{(1)}b)c + (-1)^{(|a|+1-n)|b|}b(a_{(0)}c + \Lambda a_{(1)}c), \tag{6.25}$$

we can see that

$$[a \cdot b, c] = [a, b] \cdot c + (-1)^{(|a|+1-n)|b|} b \cdot [a, c], \tag{6.26}$$

$$\langle a \cdot b, c\rangle = \langle a, b\rangle \cdot c + (-1)^{(|a|-n)|b|} b \cdot \langle a, c\rangle. \tag{6.27}$$

The conditions coincide the definition of extended higher Courant-Dorfman algebras.

Conversely, assuming that we have an extended higher Courant-Dorfman algebra $(\mathcal{E} = (\mathcal{E}^k, d), \langle,\rangle, [,])$, define a Λ-bracket $\{a_\Lambda b\} = [a, b] + (-1)^{|a|}\Lambda\langle a, b\rangle$. Then, this bracket satisfies the conditions of a Λ-bracket.

□

Next, we check this algebra has a PVA-like property. In particular, we show we can construct a graded Lie algebra from a tensor product of a higher LCA and an arbitrary differential graded-commutative algebra (dgca for short) and a graded Poisson algebra out of that of a higher PVA and an arbitrary dgca.

Lemma 6.1. *Let $C = (C^k, d_1)$ be a higher LCA and (E, d_2) be a dgca. Then, the tensor product $C \otimes E$ of cochain complexes is also a higher LCA by defining a bracket as $[a \otimes f_\Lambda b \otimes g] = (-1)^{(|b|+1-n)|f|}[a_{\Lambda+d_2} b] \otimes fg, d(a \otimes f) = d_1 a \otimes f + (-1)^{|a|} a \otimes d_2 f$.*

Proof. sesquilinearity:

$$\begin{aligned}
[d(a \otimes f)_\Lambda b \otimes g] &= [d_1 a \otimes f_\Lambda b \otimes g] + (-1)^{|a|}[a \otimes d_2 f_\Lambda b \otimes g] \\
&= (-1)^{(|b|+1-n)|f|}\{[d_1 a_{\Lambda+d_2} b] \otimes fg + (-1)^{|a|+|b|+1-n}[a_{\Lambda+d_2} b] \otimes (d_2 f)g\} \\
&= (-1)^{(|b|+1-n)|f|}\{-(-1)^{-n}(\Lambda + d_2)[a_{\Lambda+d_2} b] \otimes fg + [a_{\Lambda+d_2} b] \otimes (d_2 f)g\} \\
&= (-1)^{(|b|+1-n)|f|}\{-(-1)^{-n}\Lambda[a_{\Lambda+d_2} b] \otimes fg - (-1)^{|a|+|b|+1-n}[a_{\Lambda+d_2} b] \otimes (d_2 f)g \\
&+ (-1)^{|a|+|b|+1-n}[a_{\Lambda+d_2} b] \otimes (d_2 f)g\} \\
&= -(-1)^{-n}\Lambda[a \otimes f_\Lambda b \otimes g].
\end{aligned} \tag{6.28}$$

skew-symmetry:

$$\begin{aligned}
[a \otimes f_\Lambda b \otimes g] &= [a_{\Lambda+d_2} b] \otimes fg] \\
&= -(-1)^{(|a|+1-n)(|b|+1-n)+(|b|+1-n)|f|}[b_{-\Lambda-d_2-d_1} a] \otimes fg \\
&= -(-1)^{(|a|+|f|+1-n)(|b|+|g|+1-n)-(|a|+1-n)|g|}[b_{-\Lambda-d} a] \otimes gf \\
&= -(-1)^{(|a|+|f|+1-n)(|b|+|g|+1-n)}[b \otimes g_{-\Lambda-d} a \otimes f].
\end{aligned} \tag{6.29}$$

Using the Jacobi identity of the original higher LCAs, we can check the Jacobi identity in a similar way.

□

Definition 6.4. A graded Lie algebra $\mathcal{C}$ of degree $n \in \mathbb{Z}$ is a cochain complex of vector spaces with a bilinear operation $[,] : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ of degree n satisfying:

(1)skew-symmetry:$[a, b] = -(-1)^{(|a|+n)(|b|+n)}[b, a]$,

(2)Jacobi identity:$[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+n)(|b|+n)}[b, [a, c]]$.

Lemma 6.2. *Let $C = (C^k, d)$ be a higher Lie conformal algebra of degree n. Then $C/\mathrm{Im}d$ is naturally a graded Lie algebra of degree $(1-n)$ with bracket*

$$[a + dC, b + dC] = [a_\Lambda b]_{\Lambda=0} + dC \tag{6.30}$$

Proof. The well-definedness follows from the sesquilinearity.

$$[d\alpha, b] = -(-1)^{-n}(\Lambda[\alpha_\Lambda b])_{\Lambda=0} = 0, \tag{6.31}$$

$$[a, d\beta] = -(-1)^{|a|-n}((\Lambda + d)[a_\Lambda \beta])_{\Lambda=0} = d[a_\Lambda \beta] \simeq 0. \tag{6.32}$$

The skew-symmetry follows from the skew-symmetry of the complex.

$$\begin{aligned}[a, b] &= [a_\Lambda b]_{\Lambda=0} \\ &= -(-1)^{(|a|+1-n)(|b|+1-n)}[b_{-\Lambda-d}a]_{\Lambda=0} \\ &\simeq -(-1)^{(|a|+1-n)(|b|+1-n)}[b_\Lambda a]_{\Lambda=0} \\ &= -(-1)^{(|a|+1-n)(|b|+1-n)}[b, a] \end{aligned}\tag{6.33}$$

In the similar way, we can check the Jacobi-identity follows from the Jacobi-identity of the complex.

□

Lemma 6.3. *Let L be a graded Lie algebra of degree n. Then, $L[-n]$ is a graded Lie algebra with the same bracket.*

Proof. It satisfies the skewsymmetry and the Jacobi identity due to the grade-shifting.

□

For any higher LCA of degree n C and dgca E, we put $L(C, E) = C \otimes E/\mathrm{Im}d$ and $\mathrm{Lie}(C, E) = L(C, E)[n-1]$. By the above lemmas, $\mathrm{Lie}(C, E)$ is a graded-Lie algebra via

$$\{a \otimes f, b \otimes g\} = (-1)^{(|b|+1-n)|f|}(a_{(0)}b \otimes fg + (-1)^{|a|}a_{(1)}b \otimes (df)g). \tag{6.34}$$

Next, we discuss the Poisson algebraic structure. Let $C = (C^n, d)$ be a higher PVA of degree n. Then, $C \otimes E[n-1]$ is a dgca with products $a \otimes f \cdot b \otimes g = (-1)^{|b||f|}a \cdot b \otimes f \cdot g$, and $\mathrm{Lie}(C, E)$ is a graded Lie algebra. We put $P(C, E) = C \otimes E[n-1]/(Imd) \cdot (C \otimes E[n-1])$.

Theorem 6.2. *$P(C, E)$ is a graded Poisson algebra with*

$$[a \otimes f] \cdot [b \otimes g] = (-1)^{|b||f|}[a \cdot b \otimes fg], \tag{6.35}$$

$$\{[a \otimes f], [b \otimes g]\} = (-1)^{(|b|+1-n)|f|}(a_{(0)}b \otimes fg + (-1)^{|a|}a_{(1)}b \otimes (df)g). \tag{6.36}$$

Proof. Let $I_d = (Imd) \cdot (C \otimes E[n-1])$. If $a, b \in I_d$, then $a \cdot b, da \in I_d$. Therefore, I_d is a dg ideal of $C \otimes E[n-1]$ and $P(C, E)$ is a dgca. If $a, b \in I_d/(Imd)$, then $[a, b] \in I_d/(Imd)$ by the Leibiniz identity of C, thus $I_d/(Imd)$ is a graded Lie ideal of $Lie(C, E)$ and $P(C, E)$ is a Lie algebra with the Lie bracket. The Leibiniz identity follows from the Leibniz identity of C. Therefore, $P(C, E)$ is a Poisson algebra.

□

By the above theorem, we obtain a graded Poisson algebra from a higher PVA and a dgca.

Example 6.1. We define the BFV analog of formal distribution Lie algebras. Define the algebra of power series

$$\mathbb{C}[[t_1, t_1^{-1}, ...t_n, t_n^{-1}]][\theta_1, ..., \theta_n] \tag{6.37}$$

where t_i are even coordinates of degree 0, θ_i are odd coordinates of degree 1.

Define the "de-Rham differential" as

$$df := \sum_i \frac{\partial f}{\partial t^i}\theta_i. \tag{6.38}$$

Let $C = (C^n, Q)$ be a higher LCA of degree $n+1$. For

$$V := C \otimes \mathbb{C}\left[[t_1, t_1^{-1}, ...t_n, t_n^{-1}]][\theta_1, ..., \theta_n]\right][n]/((Q\alpha)\otimes f + \alpha \otimes df), \tag{6.39}$$

the bracket

$$[a \otimes t_1^{p_1}\cdots t_n^{p_n}\theta^J, b \otimes t_1^{q_1}\cdots t_n^{q_n}\theta^K] \tag{6.40}$$

$$= (a_{(0)}b)t_1^{p_1+q_1}\cdots t_n^{p_n+q_n}\theta^{J\cdot K} + \sum_{k=1}^{n}(a_{(1)}b)p_k t_1^{p_1+q_1}\cdots t_k^{p_k+q_k-1}t_n^{p_n+q_n}\theta^{J\cdot\{k\}\cdot K}, \tag{6.41}$$

$$J, K \subset \{1, ..., n\}, J\cdot K = \begin{cases} \phi & (J\cap K \neq \phi), \\ J\cup K & (J\cap K = \phi), \end{cases} \tag{6.42}$$

makes the graded Lie algeraic structure.

We define a formal distribution,

$$\begin{aligned} a(Z_1, ..., Z_n) &= a(z_1, ..., z_n, \zeta_1, ..., \zeta_n) \\ &= \sum_{m_i\in\mathbb{Z}, J\subset\{1,...,n\}} z_1^{-1-m_1}\cdots z_n^{-1-m_n}\zeta^{\{1,...,n\}\setminus J}\alpha t_1^{m_1}\cdots t_n^{m_n}\theta^J, \end{aligned} \tag{6.43}$$

and the formal δ-function,

$$\begin{aligned} \delta(Z-W) &= \delta(z_1-w_1)\cdots\delta(z_n-w_n)\delta(\zeta_1-\xi_1)\cdots\delta(\zeta_n-\xi_n) \\ &= \sum_{m_i\in\mathbb{Z}} z_1^{-m_1-1}w_1^{m_1}\cdots z_n^{-m_n-1}w_n^{m_n}(\zeta_1-\xi_1)\cdots(\zeta_n-\xi_n), \end{aligned} \tag{6.44}$$

Then, we obtain

$$[a(Z), b(W)] = [a,b](W)\delta(Z-W) + \langle a,b\rangle(W)d(\delta(Z-W)). \tag{6.45}$$

(For another example of formal distribution Lie algebra using superfields, see [14].)

Consider the case $n = 2$. Let $C = (C^n, Q)$ be a higher PVA of degree 2. Then $P(C, \mathbb{C}[[t, t^{-1}]][\theta])$ is a graded Poisson algebra via

$$\{at^m, bt^n\} = (a_{(0)}b)t^{m+n} + (a_{(1)}b)mt^{m+n-1}\theta,\ \{at^m\theta, bt^n\} = (a_{(0)}b)t^{m+n}\theta. \tag{6.46}$$

An extended higher Courant-Dorfman algebra of degree 2 is the same as a Courant-Dorfman algebra, and there is a PVA corresponding to a given higher PVA. We denote the PVA by $\tilde{C}$. We restrict $P(C, \mathbb{C}[[t, t^{-1}]][\theta])$ to the degree 0 part. Explicitly,

$$P(C, \mathbb{C}[[t, t^{-1}]][\theta])|_{degree0} = \{at^{m_1}, bt^{m_2}\theta | a \in C^1, b \in C^0, m_1, m_2 \in \mathbb{Z}\}. \quad (6.47)$$

We can define an isomorphism of Poisson algebras between $P(C, \mathbb{C}[[t, t^{-1}]][\theta])|_{degree0}$ and the Poisson algebra arising from the associated Poisson vertex algebra $\tilde{C} \otimes \mathbb{C}[[t, t^{-1}]]/Im(d + \partial_t) \cdot \tilde{C} \otimes \mathbb{C}[[t, t^{-1}]]$ by sending $at^{m_1}(a \in C^1)$ to at^{m_1}, and $bt^{m_2}\theta(b \in C^0)$ to bt^{m_2}. This subalgebra corresponds to the physical current algebra of a BFV current algebra.

Example 6.2. Let $(\mathcal{M}, \omega, Q = \{\Theta, -\})$ be a degree n dg symplectic manifold and $C = C^{n-1}(C^\infty(\mathcal{M})) = \{a \in C^\infty(\mathcal{M}) : |a| \leq n-1\}$ and consider a higher Courant-Dorfman algebra on C. Let Σ_{n-1} be a $n-1$ dimensional manifold and $E = (\Omega^\bullet(\Sigma_{n-1}), D)$ be their de-Rham complex. Then, $P(C, E)$ is equipped with degree 0 Poisson bracket with

$$[a \otimes \epsilon_1, b \otimes \epsilon_2] = \{\{a, \Theta\}, b\} \otimes \epsilon_1\epsilon_2 + \{a, b\} \otimes (D\epsilon_1)\epsilon_2, \quad (6.48)$$

where $a, b \in C$ and $\epsilon_1, \epsilon_2 \in E$. This is an algebraic description of BFV current algebras from dg symplectic manifolds[32], [34].

BFV current algebras are Poisson brackets on $C^\infty(Map(T[1]\Sigma_{n-1}, \mathcal{M}))$, where $T[1]\Sigma_{n-1}$ is the shifted tangent space of Σ_{n-1}. In order to obtain the currents, we have to take a proper Lagrangian submanifold of $Map(T[1]\Sigma_{n-1}, \mathcal{M})$. One way is the zero-locus reduction[35].

Proposition 6.1 ([34, *Proposition*1])**.** *We take a degree $-n$ graded Poisson algebra P with a differential Z, and take a quotient P/I_Z, where I_Z is the ideal of P generated by Z-exact terms.*

Then, P/I_Z is a degree $-n+1$ Poisson algebra with the derived bracket

$$\{[a], [b]\} = [\{a, Z(b)\}]. \quad (6.49)$$

Applying to the BFV current algebras we obtain the Poisson bracket on $C^\infty(Map(T[1]\Sigma_{n-1}, \mathcal{M}))/I_{\tilde{D}+\tilde{Q}}$, where $\tilde{D}$ and $\tilde{Q}$ is a differential on $Map(T[1]\Sigma_{n-1}, \mathcal{M})$ induced by D and Q.

For $a \in C^\infty(\mathcal{M})$ and $\epsilon \in C^\infty(T[1]\Sigma_{n-1})$, We define $J_\epsilon(a) \in C^\infty(Map(T[1]\Sigma_{n-1}, \mathcal{M}))$ by

$$J_\epsilon(a)(\phi) = \int_{T[1]\Sigma_{n-1}} \epsilon \cdot \phi^*(a)(\sigma, \theta) d^{n-1}\sigma d^{n-1}\theta, \quad (6.50)$$

where $\epsilon \in C^\infty(\Sigma_{n-1})$ are test functions on $T[1]\Sigma_{n-1}$, σ, θ are coordinates on $T[1]\Sigma_{n-1}$ of degree 0 and 1, $\phi \in Map(T[1]\Sigma_{n-1}, \mathcal{M})$ and $\phi^*(a)$ is the pullback of a. Then the Poisson bracket is

$$\begin{aligned}
&\{J_{\epsilon_1}(a), J_{\epsilon_2}(b)\}(\phi) \\
&= \int_{T[1]\Sigma_{n-1}} \epsilon_1\epsilon_2 \cdot \phi^*(\{\{a, \Theta\}, b\})(\sigma, \theta) d^{n-1}\sigma d^{n-1}\theta \\
&+ \int_{T[1]\Sigma_{n-1}} (D\epsilon_1)\epsilon_2 \cdot \phi^*(\{a, b\})(\sigma, \theta) d^{n-1}\sigma d^{n-1}\theta, \quad (6.51)
\end{aligned}$$

where $\epsilon_1, \epsilon_2 \in C^\infty(T[1]\Sigma_{n-1})$ are test functions on $T[1]\Sigma_{n-1}$, σ, θ are coordinates on $T[1]\Sigma_{n-1}$ of degree 0 and 1,

Comparing to (6.48) and (6.51), we see that taking the quotient corresponds to the zero-locus reduction.

Chapter 7

Outlooks

In this thesis, we gave higher analogs of Lie conformal algebras and Poisson vertex algebras. It is natural to ask whether they have the same applications as ordinary Lie conformal algebras and Poisson vertex algebras. For example, our higher PVAs may be used to analyze multi-variable Hamiltonian PDEs. Considering the algebraic description of more general currents would be important.

Another interesting problem is the non-commutative analog. In [17], non-commutative version of Courant-Dorfman algebras and Poisson vertex algebras, which are called double Courant-Dorfman algebras and double Poisson vertex algebras, are considered. Their higher generalization would be given using our algebras. Another way taking the non-commutative version is the quantization, in analogy with vertex algebras.

References

[1] Ted Courant.: *Dirac manifolds*, Transactions of the American Mathematical Society Vol. 319, No. 2 (Jun., 1990), pp. 631-661

[2] Marco Gualtieri.: *Generalized complex geometry and T-duality*,arxiv:0401221

[3] V. G. Drinfel'd.: *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations* Dokl. Akad. Nauk SSSR 268.2 (1983), pp. 285–287.

[4] Gil R. Cavalcanti, Marco Gualtieri.: *Generalized complex geometry*,A Celebration of the Mathematical Legacy of Raoul Bott (CRM Proceedings and Lecture Notes), American Mathematical Society, 2010, pp. 341-366.arxiv:1106.1747

[5] Dmitry Roytenberg.: *AKSZ-BV Formalism and Courant Algebroid-induced Topological Field Theories*,Lett.Math.Phys.79:143-159,2007,arxiv:0608150

[6] A. Coimbra, C. Strickland-Constable and D. Waldram,: *Supergravity as Generalised Geometry I: Type II Theories* JHEP 1111 (2011) 091,arxiv:1107.1733

[7] Izu Vaisman.: *Towards a double field theory on para-Hermitian manifolds*,J. Math. Phys. 54, 123507 (2013) arxiv:1209.0152

[8] Dmitry Roytenberg.: *Courant-Dorfman algebras and their cohomology*,Lett. Math. Phys. (2009) 90:311-351, arxiv:0902.4862

[9] Dmitry Roytenberg.: *On the structure of graded symplectic supermanifolds and Courant algebroids*, Contemp. Math., Vol. 315, Amer. Math. Soc., Providence, RI, 2002, arxiv:0203110

[10] Zhang-Ju Liu, Alan Weinstein, Ping Xu.: *Manin Triples for Lie Bialgebroids*,J. Diff. Geom., 45:547-574 arxiv:9508013

[11] V. G. Knizhnik and A. B. Zamolodchikov.: *Current algebra and Wess-Zumino model in two dimensions*, Nucl.Phys.B 247 (1984) 83-103

[12] Victor G. Kac; *Vertex algebras for beginners*, Lecture Ser., vol 10, AMS, 1996. Second edition, 1998

[13] Victor Kac.: *Introduction to vertex algebras, Poisson vertex algebras, and integrable Hamiltonian PDE*, arxiv:1512.00821

[14] Reimundo Heluani, Victor G. Kac: *Supersymmetric vertex algebras*,Commun.Math.Phys.271:103-178,2007 arxiv:0603633

[15] Aliaa Barakat, Alberto De Sole, Victor G. Kac.: *Poisson vertex algebras in the theory of Hamiltonian equations*,Jpn. J. Math. 4 (2009), no. 2, 141-252 arxiv:0907.1275

[16] Pedram Hekmati, Varghese Mathai.: *T-duality of current algebras and their quantization*,Contemporary Mathematics, 584 (2012) 17-38 arxiv:1203.1709

[17] Luis Álvarez-Cónsul, David Fernández, Reimundo Heluani.: *Noncommutative Poisson vertex algebras and Courant-Dorfman algebras*, arxiv:2106.00270

[18] Giulio Bonelli, Maxim Zabzine.: *From current algebras for p-branes to topological M-theory*,JHEP 0509 (2005) 015 arxiv:0507051

[19] Machiko Hatsuda, Tetsuji Kimura.: *Canonical approach to Courant brackets for D-branes*,Journal of High Energy Physics 34 (2012) arxiv:1203.5499

[20] Alberto S. Cattaneo, Florian Schaetz.: *Introduction to supergeometry*, arxiv:1011.3401

[21] Jian Qiu, Maxim Zabzine.: *Introduction to Graded Geometry, Batalin-Vilkovisky Formalism and their Applications*, arxiv:1105.2680

[22] I. A. Batalin, G. A. Vilkovisky,: *Gauge algebra and quantization*, Phys. Lett., 102B:27, 1981

[23] I. A. Batalin and G. A. Vilkovisky,: *Relativistic S-matrix of dynamical systems with boson and fermion constraints*, Phys. Lett. B 69 (1977), 309–312

[24] I. A. Batalin and E. S. Fradkin,: *A generalized canonical formalism and quantization of reducible gauge theories*, Phys. Lett. B 122, 157–164 (1983)

[25] Anton Alekseev, Thomas Strobl.: *Current Algebras and Differential Geometry*,JHEP 0503 (2005) 035 arxiv:0410183

[26] Joel Ekstrand, Maxim Zabzine.: *Courant-like brackets and loop spaces*,JHEP 1103:074,2011 arxiv:0903.3215

[27] Frank Keller, Stefan Waldmann.: *Deformation theory of Courant algebroids via the Rothstein algebra*,Journal of Pure and Applied Algebra 219(8) arxiv:0807.0584

[28] G. Bonavolonta, N. Poncin; *On the category of Lie n-algebroids*, J. Geom. Phys., 73 (2013), 70-90 arxiv:1207.3590

[29] Theodore Voronov.: *Higher derived brackets and homotopy algebras*,J. Pure and Appl. Algebra. 202 (2005), Issues 1-3, 1 November 2005, 133-153 arxiv:0304038

[30] Joel Ekstrand: *Going round in circles From sigma models to vertex algebras and back*, Doctoral thesis,2011.

[31] Noriaki Ikeda, Kozo Koizumi.: *Current Algebras and QP Manifolds*,International Journal of Geometric Methods in Modern Physics 10(6) arxiv:1108.0473

[32] Noriaki Ikeda, Xiaomeng Xu.: *Current Algebras from DG Symplectic Pairs in Supergeometry*, arxiv:1308.0100

[33] Taiki Bessho, Marc A. Heller, Noriaki Ikeda, Satoshi Watamura: *Topological Membranes, Current Algebras and H-flux - R-flux Duality based on Courant Algebroids*, J. High Energ. Phys. 2016, 170 (2016). arxiv:1511.03425

[34] Alex S. Arvanitakis.: *Brane current algebras and generalised geometry from QP manifolds*, J. High Energ. Phys. 2021, 114 (2021). arxiv:2103.08608

[35] M.A. Grigoriev, A.M. Semikhatov, I.Yu. Tipunin.: *BRST Formalism and Zero Locus Reduction*,J.Math.Phys. 42 (2001) 3315-3333 arxiv:0001081

[36] A.Yu.Alekseev, P.Schaller, T.Strobl.: *The Topological G/G WZW Model in the Generalized Momentum Representation*,Phys.Rev. D52 (1995) 7146-7160 arxiv:9505012

[37] Noriaki Ikeda.: *Lectures on AKSZ Sigma Models for Physicists*Noncommutative Geometry and Physics 4, Workshop on Strings, Membranes and Topological Field Theory: 79-169, 2017, World scientific, Singapore, arxiv:1204.3714