Special Mathematics Lecture

Groups and their representations

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Goals of these Lectures notes:

Provide the necessary background information for understanding the main ideas and concepts of group theory
These notes correspond to 14 lectures lasting 90 minutes each.

Website for this course:
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About these lecture notes:
These notes are mainly based on reference [1]. Other references will be cited when necessary.

Chapter 1

Groups

In this chapter, we discuss the definitions and basic concepts required to understand group theory.

1.1 Groups and subgroups

We start with the definition of a group.

Definition 1.1.1 (Group)**.** *A* group *is a set G together with a map $G \times G \to G$ (denoted by "$\cdot$", " ", or "$+$") that satisfies the following three conditions $\forall a, b, c \in G$:*

1) $(ab)c = a(bc)$ (associativity),

2) $\exists\, e \in G$ *such that* $ea = ae = a$ (existence of an identity element),

3) $\forall\, a \in G,\ \ \exists\, a^{-1} \in G$ *such that* $aa^{-1} = a^{-1}a = e$ (existence of an inverse for each element).

In this definition we have used the multiplicative notation. If the additive notation is chosen, then the map is denoted by "+", the identity element is denoted by 0, and the inverse a^{-1} of a is denoted by $-a$. We sometimes write $(G, +)$ of $(G, \cdot)$ if we want to emphasize the additive notation or the multiplicative notation, but the simpler notation G for a group is also commonly used. Most of the time in these notes, the multiplicative notation will be preferred. Let us immediately mention some easy consequences of this definition.

Remark 1.1.2. *1) Observe that for any group G, the identity element e is unique.*

2) Observe that $e^{-1} = e$, $\left(a^{-1}\right)^{-1} = a$, $(ab)^{-1} = b^{-1}a^{-1}$. It also follows from the definition that for any element a, its inverse a^{-1} is unique.

3) The equality $ab = ac$ implies the equality $b = c$. Similarly, $ba = ca$ implies $b = c$.

Exercise 1.1.3. *Prove the statement contained in the previous remark.*

Let us now mention two special instances of groups:

Definition 1.1.4 (Abelian and finite groups)**.** *1) The group G is* Abelian *or* commutative *if $ab = ba$ for all $a, b \in G$,*

2) The group G is finite *if it contains a finite number of elements.*

Usually, we write $|G|$ for the cardinality of a set. Thus, the group G is finite if $|G| < \infty$.

Examples 1.1.5 (Examples of Groups)**.** *1) The most common groups are* $(\mathbb{Z}^n, +)$, $(\mathbb{R}^n, +)$ *for* $n \in \mathbb{N}$, *and* $(\mathbb{R}_+, \cdot)$.

2) The cyclic group C_n: *For* $n \in \mathbb{N}$, *consider the set* $C_n = \{e, a^1, a^2, a^3, a^4, \dots, a^{n-1}\}$ *endowed with the following rules:* $e \equiv a^0 \equiv a^n$, $a^j a^k = a^{j+k \mod n}$, $(a^j)^{-1} = a^{n-j}$. *One can check that this set with these rules define an Abelian group.*

3) The symmetric group S_n: *For* $n \in \mathbb{N}$, *the group* S_n *corresponds to the group of permutations of n elements, or equivalently the group of all bijective maps from a set of n elements to itself. This group contains* $n!$ *elements, and is not Abelian if* $n \geq 3$. *A convenient way to represent it is obtained by using the two-line notation for each bijection. For example, for* $n = 3$ *its elements can be represented by*

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

The group operation of S_n *is the standard composition of permutations, or the composition of bijective maps.*

4) Groups of $n \times n$ *matrices are very useful and commonly used. We introduce the main ones. The group law is always given by the multiplication of matrices. We set* Det *for the determinant, and denote by* M^{-1} *the inverse of a matrix M, and by* M^T *its transpose.*
The general linear group $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, *the set of* $n \times n$ *invertible matrices,*
The special linear group $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$, *the set of* $n \times n$ *invertible matrices with determinant* 1,
The unitary group $\mathrm{U}(n)$, *the set of invertible matrices such that* $M^* := \overline{M}^T = M^{-1}$. *The matrix* M^* *is called the* adjoint *matrix of M, and it follows from the property* $M^* = M^{-1}$ *that* $|\mathrm{Det}(M)| = 1$,
The special unitary group $\mathrm{SU}(n)$, *the set* $\{M \in \mathrm{U}(n) \mid \mathrm{Det}(M) = 1\}$,
The orthogonal group $\mathrm{O}(n)$, *the subset of* $\mathrm{GL}(n, \mathbb{R})$ *that contains all A satisfying* $A^T = A^{-1}$. *It follows from this property that* $\mathrm{Det}(A) = \pm 1$,
The special orthogonal group $\mathrm{SO}(n)$, *the set* $\{A \in \mathrm{O}(n) \mid \mathrm{Det}(A) = 1\}$.

It is easily observed that all these groups of matrices consist of subsets of the set $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ of all real or complex invertible matrices. In fact, they are subgroups:

Definition 1.1.6 (Subgroup)**.** *A* subgroup G_0 *of a group G is a subset of the group G which is also a group itself. A subgroup* G_0 *is* proper *if* $G_0 \neq G$, *and* G_0 *is* not trivial *if* $G_0 \neq \{e\}$.

Exercise 1.1.7. *Determine some subgroups in each groups introduced in Examples 1.1.5. Prove that they are indeed subgroups.*

1.2 Conjugation and equivalence classes

We now introduce some relations between elements of a group G.

Definition 1.2.1 (Conjugation)**.** *For* $a, b \in G$, *we say that a is* conjugate *to b if* $\exists\, c \in G$ *such that* $a = cbc^{-1}$. *In this case, we write* $a \sim b$.

Exercise 1.2.2. *Prove that the conjugation defines an* equivalence relation, *namely the following three properties are satisfied:*

1) $a \sim a$ (reflexivity),

2) If $a \sim b$ *then* $b \sim a$ (symmetry), *3) If $a \sim b$ and $b \sim c$, then $a \sim c$* (transitivity).

Whenever an equivalence relation is available, it is natural to put together the elements which are conjugate.

Definition 1.2.3 (Equivalence class, conjugacy class)**.** *Let G be a set endowed with an equivalence relation $\sim$. For any $a \in G$, the* equivalence class *$[a]$ containing a is defined by $[a] := \{b \in G \mid b \sim a\}$. When the equivalence relation is given by a conjugation, $[a]$ is also called the* conjugacy class *containing a.*

This notion of conjugacy class for a group G is very important, and the following properties should be checked:

1) Each element $a \in G$ is in a single conjugacy class,

2) The identity element e generates a class on its own,

3) If G is Abelian, each element generates its own class,

4) If G_0 is a subgroup of G and for any $c \in G$, the set

$$cG_0c^{-1} := \{cac^{-1} \mid a \in G_0\}$$

defines another subgroup of G. One then says that cG_0c^{-1} is a subgroup *conjugated to* G_0.

The construction of a conjugated subgroup leads naturally to the following definition:

Definition 1.2.4 (Invariant or normal subgroup)**.** *Let G be a group, and G_0 be a subgroup. We say that G_0 is an* invariant *subgroup, or a* normal *subgroup of G if $cG_0c^{-1} = G_0$ for all $c \in G$. We write $G_0 \triangleleft G$ for a normal subgroup G_0 of G.*

Some examples of invariant subgroups are provided below. It is interesting to guess other examples among the groups introduced in Examples 1.1.5.

Examples 1.2.5 (Examples of invariant subgroups)**.** *1) For $G = (\mathbb{R}, +)$, $G_0 = (\mathbb{Z}, +)$ is a normal subgroup.*

2) For $G = \mathrm{GL}(n, \mathbb{C})$, $G_0 = \mathbb{C}\mathbb{1}_{n\times n}$ is a normal (and Abelian) subgroup.

The notion of normal subgroup leads then to the concept of simple or semi-simple groups:

Definition 1.2.6 (Simple and semi-simple group)**.** *A group G is* simple *if $\{e\}$ is the only proper and normal subgroup of G. The group G is* semi-simple *if $\{e\}$ is the only proper and normal Abelian subgroup of G.*

Clearly, a semi-simple group might or might not be simple, while any simple group is automatically semi-simple. For this reason, simplicity is a stronger requirement than semi-simplicity.

Exercise 1.2.7. *Show that* $\mathrm{SO}(3)$ *is a simple group.*

Let us now introduce another definition leading to equivalence classes.

Definition 1.2.8 (Left conjugation)**.** *Let G be a group, and let G_0 be a subgroup. For any $a, b \in G$ we set $a \overset{\ell}{\sim} b$ if $a^{-1}b \in G_0$, or equivalently if $b = ac$ for some $c \in G_0$.*

One easily checks the following properties:

1) $a \overset{\ell}{\sim} a$ for any $a \in G$,

2) If $a \overset{\ell}{\sim} b$, then $b \overset{\ell}{\sim} a$,

3) If $a \overset{\ell}{\sim} b$ and $b \overset{\ell}{\sim} c$, then $a \overset{\ell}{\sim} c$.

As in Exercise 1.2.2, one infers from these properties that $\overset{\ell}{\sim}$ defines an equivalence relation. It is then natural to define equivalence classes, as introduced in Definition 1.2.3: For any $a \in G$

$$_{G_0}[a] := \{b \mid a \overset{\ell}{\sim} b\} = aG_0. \tag{1.2.1}$$

These equivalence classes are also called *left coset*. Equivalently, we can define $a \overset{r}{\sim} b$ if $ba^{-1} \in G_0$ and check that $\overset{r}{\sim}$ also defines an equivalence relation. The corresponding equivalence classes are denoted by $[\ ]_{G_0}$ and are called *right coset*. Observe that $[a]_{G_0} = G_0 a$. Let us emphasize that in general, aG_0 and $G_0 a$ are not subgroups of G, and $\{_{G_0}[a] \mid a \in G\}$ and $\{[a]_{G_0} \mid a \in G\}$ are not groups. However, the following statement holds:

Proposition 1.2.9. *Let G be a group and G_0 a subgroup.*

1. *G_0 is a normal subgroup if and only if $_{G_0}[a] = [a]_{G_0}$ for any $a \in G$,*
2. *If G_0 is a normal subgroup, then the operation $[a]_{G_0}[b]_{G_0} := [ab]_{G_0}$ defines a product on the equivalence classes. By defining $[a]_{G_0}^{-1} := [a^{-1}]_{G_0}$ and the identity given by G_0 itself, these operations define a group, denoted by G/G_0 and called the* quotient group *or the* factor group.

As an example of the previous construction, recall that $(\mathbb{Z}, +)$ is a normal subgroup of $(\mathbb{R}, +)$. Then, the quotient group $(\mathbb{R}, +)/(\mathbb{Z}, +)$ corresponds to $([0, 1), +\text{mod } 1)$, often denoted by $\mathbb{S}^1$.

Exercise 1.2.10. *Show that the above proposition holds. In addition, if G if finite, and if G_0 is a normal subgroup of G, show that the following equality holds:*

$$\big|G/G_0\big| = \frac{|G|}{|G_0|}.$$

We also define another notion which is of central importance for the study of groups: the set of elements which commute with all the other ones:

Definition 1.2.11 (Center)**.** *The* center *$Z(G)$ of a group G is defined by $Z(G) := \{a \in G \mid ab = ba\ \forall\ b \in G\}$.*

Exercise 1.2.12. *Prove that $Z(G)$ is an Abelian and normal subgroup of G.*

So far, we have considered only one group G. We shall now consider two groups, and some relations or maps between them.

Definition 1.2.13 (Homomorphism, isomorphism, endomorphism, automorphism)**.** *Let G and G' be two groups. A (group)* homomorphism *is a map $\phi : G \to G'$ such that $\forall\ a, b \in G$, $\phi(ab) = \phi(a)\,\phi(b)$. An* isomorphism *is a bijective homomorphism, and if an isomorphism exists between the two groups G and G', we write $G \simeq G'$ and say that G and G' are isomorphic. A homomorphism from a group G to itself is called an* endomorphism*, and a bijective endomorphism is called an* automorphism.

Let us stress that the relation $\phi(ab) = \phi(a)\,\phi(b)$ involves the group law of G and of G': The product on the left-hand side is the product in G, while on the right-hand side it is the product in G'. Usually, we say that the map ϕ preserves the group laws of G and G'. The following statements then hold for ϕ a group homomorphism from G to G':

Proposition 1.2.14. *1) $\phi(e_G) = e_{G'}$ and $\phi(a^{-1}) = (\phi(a))^{-1}$,*

2) $\mathrm{Ker}(\phi) := \{a \in G \mid \phi(a) = e_{G'}\}$ is a normal subgroup of G,

3) $G/\mathrm{Ker}(\phi)$ is isomorphic to $\phi(G)$ through the isomorphism $\tilde{\phi}$ defined by $\tilde{\phi}([a]_{\mathrm{Ker}(\phi)}) := \phi(a)$ for any $a \in G$, *4) If G_0 is a subgroup of G, then $\phi(G_0)$ is a subgroup of G'.*

Exercise 1.2.15. *Prove the statements of the previous proposition.*

Let us provide an example of a homomorphism which plays an important role in various fields. For it, we firstly introduce the Pauli matrices

$$\sigma_1 \equiv \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2.2}$$

together with the 2×2 identity matrix $\sigma_0 := \mathbb{1}$. It is known that for any $A \in M_2(\mathbb{C})$ there exists $a_0, a_1, a_2, a_3 \in \mathbb{C}$ such that

$$A = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 =: a_0\mathbb{1} + a \cdot \sigma,$$

with $a = (a_1, a_2, a_3)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, and that $\mathrm{Det}(A) = (a_0)^2 - a^2$. In the next statement, Tr denotes the usual trace of a matrix.

Proposition 1.2.16. *The map $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ defined for any $U \in \mathrm{SU}(2)$ and for $j, k \in \{1, 2, 3\}$ by*

$$\phi(U)_{jk} = \frac{1}{2}\mathrm{Tr}(\sigma_j U \sigma_k U^{-1})$$

is a surjective homomorphism, with $\mathrm{Ker}(\phi) = \{\mathbb{1}, -\mathbb{1}\}$, where $\{\mathbb{1}, -\mathbb{1}\}$ corresponds to the group containing just these two elements.

Exercise 1.2.17. *Prove this proposition (it is a standard result available in many textbooks). An illustration of this map is provided in Figure 1.1.*

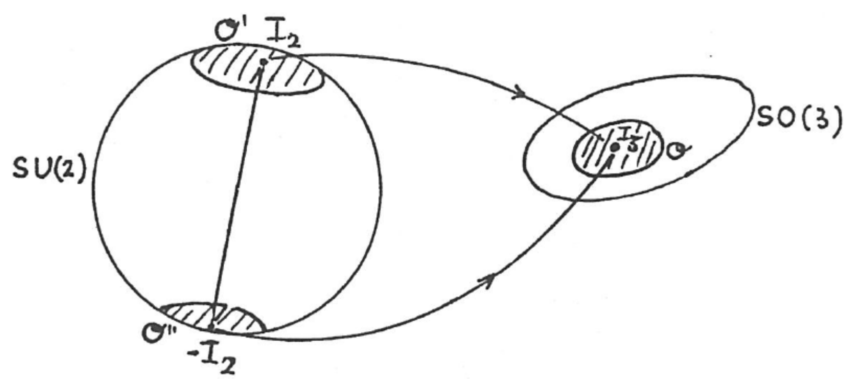

Figure 1.1: Schematic representation of the 2 – 1 map from SU(2) to SO(3).

1.3 Direct and semi-direct products

In this section, we briefly mention how two groups can generate a third one, and provide some converse constructions.

Definition 1.3.1 (Direct product)**.** *For $j \in \{1, 2\}$ let G_j be a group with identity element denoted by e_j. The* direct product *of G_1 and G_2 is defined by the set $G := \{(a_1, a_2) \mid a_j \in G_j\}$ together with the product $(a_1, a_2)(b_1, b_2) := (a_1b_1, a_2b_2)$, the inverse $(a_1, a_2)^{-1} := (a_1^{-1}, a_2^{-1})$, and the identity $e := (e_1, e_2)$. This group is usually denoted by $G_1 \times G_2$.*

The proof that $G_1 \times G_2$ is indeed a group is left as an easy exercise. One also observes that $\{(a_1, e_2) \in G_1 \times G_2 \mid a_1 \in G_1\}$ and $\{(e_1, a_2) \in G_1 \times G_2 \mid a_2 \in G_2\}$ define normal subgroups of $G_1 \times G_2$. These subgroups can naturally be identified with G_1 and G_2, respectively.

Conversely, suppose that a group G possesses two normal subgroups G_1 and G_2 satisfying the following properties:

1) $G_1 \cap G_2 = \{e\}$ with e the identity element of G,

2) Each element a of G admits a decomposition $a = a_1 a_2$ with $a_1 \in G_1$ and $a_2 \in G_2$.

Then G is isomorphic to $G_1 \times G_2$, and one has $G/G_1 \simeq G_2$ and $G/G_2 \simeq G_1$.

Examples 1.3.2. *Clearly for any $m, n \in \mathbb{N}$, $(\mathbb{R}^{m+n}, +)$ is isomorphic to $(\mathbb{R}^m, +) \times (\mathbb{R}^n, +)$. If m and n do not possess any common divisor except 1, then $C_m \times C_n$ is isomorphic to C_{mn}.*

Exercise 1.3.3. *For n odd, show that $\mathrm{O}(n)$ is isomorphic to $\mathrm{SO}(n) \times \{\mathbb{1}, -\mathbb{1}\}$. Is it still the case if n is even ?*

In the previous construction, the normality of the two subgroups is a very strong requirement. If only one subgroup is normal, then one ends up with the following concept:

Definition 1.3.4 (Inner semi-direct product)**.** *A group G is called an* inner semi-direct product *if there exist two subgroups N and H of G satisfying the following properties*

1) N is normal,

2) $N \cap H = \{e\}$ with e the identity element of G,

3) Each element a of G admits a decomposition $a = nh$ with $n \in N$ and $h \in H$.

In this case we write $G = N \rtimes H$, and often say that G is the inner semi-direct product of N and H.

It is then natural to wonder if G can be constructed from two groups N and H, as for the direct product. It is indeed possible, but the construction is slightly more involved, and is called the *outer semi-direct product*. For this, we firstly recall that an automorphism ϕ of a group N is a bijective map $\phi : N \to N$ satisfying $\phi(ab) = \phi(a)\phi(b)$ for any $a, b \in N$, and observe that the set $\mathrm{Aut}(N)$ of all automorphisms of N is itself a group, with the usual composition of maps. We now consider two groups N and H, with identity e_N and e_H, and consider a map $\psi : H \to \mathrm{Aut}(N)$. We then consider the set $\{(n, h) \mid n \in N, h \in H\}$ together with the product

$$(n_1, h_1)(n_2, h_2) := (n_1[\psi(h_1)](n_2), h_1 h_2),$$

the inverse $(n, h)^{-1} := ([\psi(h^{-1})](n^{-1}), h^{-1})$, and the identity $e := (e_N, e_H)$. It turns out that this set and these operations define a group, denoted by $N \rtimes_\psi H$ and called the outer-direct product of N and H. One can naturally identify N with $\{(n, e_H) \mid n \in N\}$ and H with $\{(e_N, h) \mid h \in H\}$. With these identifications, one observes that N is a normal subgroup of $N \rtimes_\psi H$, and that $N \rtimes_\psi H$ corresponds to the inner semi-direct product of N and H.

Exercise 1.3.5. *Check the assertions about $N \rtimes_\psi H$, in particular check that it is a group and that N is a normal subgroup.*

Exercise 1.3.6 (Dihedral groups)**.** *For any $n \in \mathbb{N}$, define the* dihedral group *D_n with $2n$ elements, and show that this group is an inner semi-direct product, or that this group is is isomorphic to the outer semi-direct product of the cyclic groups C_n and C_2.*

Exercise 1.3.7 (♥)**.** *Check that any inner semi-direct product is also an outer semi-direct product. In other words, check that these two concepts are equivalent. You can get some inspiration from* https://en.wikipedia.org/wiki/Semidirect_product

1.4 Transformation groups

Quite often, groups are related to a space and to some properties associated with this space. This leads to the following notion:

Definition 1.4.1 (Transformation group)**.** *Let X be a set and let x denote the element of X. A* transformation group of X *consists in a group G and of a map $\circ : G \times X \to X$ satisfying for any $x \in X$ the following two properties: $e \circ x = x$ for any $x \in X$ and $a \circ (b \circ x) = (ab) \circ x$ for any $a, b \in G$.*

Let us immediately stress that the notation $\circ$ is introduced for this abstract definition, but in the applications various notations are used, depending on the context. Also, it is often assumed that a certain property of X is preserved under the application of the transformation group G.

Examples 1.4.2. *1) Let X be usual Euclidean space $\mathbb{R}^n$, and let $d : X \times X \to [0, \infty)$ denote the distance function, namely for any $x, y \in X$,*

$$\mathrm{d}(x, y) := \|x - y\| = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}. \tag{1.4.1}$$

Then, the translation group *$T(n)$ corresponds to $(\mathbb{R}^n, +)$ acting as*

$$a \circ x := x + a \qquad \text{for any } a \in T(n) \text{ and } x \in \mathbb{R}^n.$$

Clearly, $T(n)$ preserves the distance, namely $\mathrm{d}(a \circ x, a \circ y) = \mathrm{d}(x, y)$ for any $a \in T(n)$ and $x, y \in \mathbb{R}^n$. In this situation, it is an "accident" that the space X and the transformation group $T(n)$ acting on X can both be identified with $\mathbb{R}^n$. Clearly, $T(n)$ contains several subgroups, as for example $(\mathbb{Z}^n, +)$.

2) Let X be usual Euclidean space $\mathbb{R}^n$, and let $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ denote the scalar product *on X defined for any $x, y \in X$ by*

$$\langle x, y \rangle := \sum_{j=1}^{n} x_j y_j. \tag{1.4.2}$$

Then, the rotation group *$R(n)$ consists in all linear transformations $\mathbb{R}^n \to \mathbb{R}^n$ preserving the scalar product, namely $\langle a \circ x, a \circ y \rangle = \langle x, y \rangle$ for any $a \in R(n)$ and $x, y \in X$. Since linear transformations of $\mathbb{R}^n$ are described by matrices, for any $a \in R(n)$ there exists $A \in \mathrm{GL}(n, \mathbb{R})$ such that $a \circ x = Ax$, and then the invariance relation reads:*

$$\langle x, \mathbb{1} y \rangle = \langle x, y \rangle = \langle a \circ x, a \circ b \rangle = \langle Ax, Ay \rangle = \langle x, A^T Ay \rangle$$

meaning that $A^T A = \mathbb{1}$, or equivalently $A^T = A^{-1}$. Thus, the rotation group can be identified with the group $\mathrm{O}(n)$[1]*, as introduced in Examples 1.1.5.*

Before going on with additional examples, let us provide a few more natural definitions.

Definition 1.4.3 (Orbit and stabilizer)**.** *Let G be a transformation group of a set X, and let $x \in X$. The set $O_x := \{a \circ x \mid a \in G\} \subset X$ is called* the orbit *of x. The set $G_x := \{a \in G \mid a \circ x = x\} \subset G$ is called* the stabilizer *of x.*

[1]Be aware that some authors would call rotations only the elements of $\mathrm{SO}(n)$, and that this different might lead to some confusions.

For the next statement, we recall that a *partition* of a set Ω consists in a family $\{\Omega_j\}_j$ of subsets of Ω satisfying $\cup_j \Omega_j = \Omega$ and $\Omega_j \cap \Omega_k = \emptyset$ whenever $j \neq k$.

Lemma 1.4.4. *For any transformation group G of a set X one has:*

1) The set of orbits defines a partition of X,

2) G_x is a subgroup of G, for any $x \in X$,

3) If $x' \in O_x$, then $G_{x'} \simeq G_x$.

The proof is not difficult and can be done as an exercise. Whenever G is a finite group, an additional relation holds:

Lemma 1.4.5. *Let G be a finite transformation group of a set X. Then, for any $x \in X$ one has*

$$|G_x|\,|O_x| = |G|.$$

1.5 Euclidean group and Poincaré group

In this section, we consider two famous transformation groups. The first one was studied much before the development of group theory.

For the Euclidean group, the set X corresponds the Euclidean space $\mathbb{R}^n$, and recall that $d : X \times X \to [0, \infty)$ is the distance function introduced in (1.4.1).

Definition 1.5.1 (Euclidean group)**.** *The* Euclidean group $E(n)$ *consists in the group of all transformations of $\mathbb{R}^n$ that preserve the Euclidean distance between any two points, namely* $\mathrm{d}(a \circ x, a \circ y) = \mathrm{d}(x, y)$ *for any $a \in E(n)$ and for any $x, y \in \mathbb{R}^n$.*

Clearly, the group $T(n)$ introduced in Examples 1.4.2 is a subgroup of $E(n)$, since it preserves the distance. The rotation group $R(n)$ introduced in Examples 1.4.2 is also a subgroup of $E(n)$. Indeed, since any element of $\in R(n)$ can be represented by $B \in \mathrm{O}(n)$, it is sufficient to observe that for $B \in \mathrm{O}(n)$ one has

$$\begin{aligned}\mathrm{d}(Bx, By)^2 = \|Bx - By\|^2 &= \langle B(x - y), B(x - y)\rangle = \langle (x - y), B^T B(x - y)\rangle \\ &= \langle (x - y), \mathbb{1}(x - y)\rangle = \langle (x - y), (x - y)\rangle = \|x - y\|^2 = \mathrm{d}(x, y)^2.\end{aligned}$$

More generally, any pair (b, B) with $b \in T(n)$ and $B \in \mathrm{O}(n)$ defines an element of $E(n)$ by acting on $x \in X$ as $(b, B) \circ x := Bx + b$. In fact, it turns out that all elements of $E(n)$ are of this form. It thus follows that the composition law on $E(n)$ is given by

$$(b, B)(b', B') = (b + Bb', BB') \qquad b, b' \in T(n) \text{ and } B, B' \in \mathrm{O}(n),$$

the inverse of (b, B) is given by $(b, B)^{-1} = (-B^{-1}b, B^{-1})$, and the identity is $e = (0, \mathbb{1})$.

Exercise 1.5.2. *Check that $(T(n), \mathbb{1})$ is a normal subgroup of $E(n)$, and that $E(N)$ is isomorphic to the semi-direct product $T(n) \rtimes R(n)$. Exhibit different types of subgroups of $E(n)$, see for example*

https://en.wikipedia.org/wiki/Euclidean_group

We now move to the study of the Poincaré group. For that purpose, we consider $X = \mathbb{R}^4$ and denote its element by $x = (x^0, x^1, x^2, x^3)$ with $x^j \in \mathbb{R}$ for $j \in \{0, 1, 2, 3\}$. We endow this space with the following bilinear form:

$$x \cdot y = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \qquad \text{for any } x, y \in \mathbb{R}^4. \tag{1.5.1}$$

It is indeed easily observed that the map $\mathbb{R}^4 \times \mathbb{R}^4 \ni (x, y) \mapsto x \cdot y \in \mathbb{R}$ is linear in both arguments. If we introduce the diagonal matrix

$$g := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and use the notation of the scalar product defined in (1.4.2), then the following equality holds:

$$x \cdot y = \langle gx, y \rangle. \tag{1.5.2}$$

The set $\mathbb{R}^4$ together with this bilinear form is called *the Minkowski space* and is denoted by $\mathbb{M}$.

Definition 1.5.3 (Lorentz group)**.** *The* Lorentz group $\mathcal{L}$ *consists in the set of all matrices* $\Lambda \in M_4(\mathbb{R})$ *that preserve the bilinear map, namely*

$$(\Lambda x) \cdot (\Lambda y) = x \cdot y \qquad \textit{for any } x, y \in \mathbb{M}. \tag{1.5.3}$$

Note that the relation introduced in this definition has also a purely matricial version. By taking (1.5.2) into account the above relation reads

$$\Lambda^T g \Lambda = g. \tag{1.5.4}$$

Thus, the Lorentz group $\mathcal{L}$ consists of all $\Lambda \in M_4(\mathbb{R})$ satisfying (1.5.4).

Exercise 1.5.4. *Define the restricted Lorentz group, and study the notions of orthochronous and proper Lorentz transformations.*

The Euclidean group is the group preserving the distance defined by the Euclidean norm. We can now define the Poincaré group with a similar approach. For that purpose, we define an analog of d^2 introduced in (1.4.1) but in the Minkowski setting:

$$t(x, y) := (x - y) \cdot (x - y).$$

Definition 1.5.5 (Poincaré group)**.** *The* Poincaré group $\mathcal{P}$ *(also called the* Lorentz inhomogeneous group*) consists in the group of all transformations of* $\mathbb{M}$ *that leave invariant the quantity* $t(x, y)$ *invariant, namely* $t(a \circ x, a \circ y) = t(x, y)$ *for any* $a \in \mathcal{P}$ *and* $x, y \in \mathbb{M}$ *.*

It turns out that the elements of $\mathcal{P}$ consist of pairs (b, Λ) with $b \in T(4)$ and $\Lambda \in \mathcal{L}$. Their action on $x \in \mathbb{M}$ is given by $(b, \Lambda) \circ x := \Lambda x + b$, which leads to the composition law of $\mathcal{P}$:

$$(b, \Lambda)(b', \Lambda') = (b + \Lambda b', \Lambda\Lambda'), \tag{1.5.5}$$

the inverse of (b, Λ) is given by $(b, \Lambda)^{-1} = (-\Lambda^{-1} b, \Lambda^{-1})$, and the identity is $e = (0, \mathbb{1})$.

Exercise 1.5.6. *Check that* $(T(4), \mathbb{1})$ *is a normal subgroup of* $\mathcal{P}$*, and that* $\mathcal{P}$ *is isomorphic to the semi-direct product* $T(4) \rtimes \mathcal{L}$*.*

Chapter 2

Linear representations

In this chapter, we introduce the notion of linear representations of a group, which play a very important role in many fields.

2.1 Vector spaces and Hilbert spaces

In this section we recall the notion of a vector space, and then concentrate on Hilbert spaces. They are special instances of vector spaces endowed with a scalar product. The underlying vector space can be of finite or infinite dimension. Usually, we consider complex vector spaces and complex Hilbert spaces, but real versions also exist.

Recall that a *complex vector space* $\mathcal{V}$ is a set endowed with two operations: An *addition*, which gives one new element from two elements of this set, usually written $+$, and a *scalar multiplication* of the elements by any complex number. Additional compatibility requirements are summarized here

https://en.wikipedia.org/wiki/Vector_space

A *linear map* on $\mathcal{V}$ is a function $T : \mathcal{V} \to \mathcal{V}$ satisfying $T(f + \lambda g) = Tf + \lambda Tg$ for any $f, g \in \mathcal{V}$ and $\lambda \in \mathbb{C}$. The set of all linear maps on $\mathcal{V}$ is denoted by $\mathcal{L}(\mathcal{V})$. Observe that $\mathbb{1}$ defined by $\mathbb{1}f = f$ is an element of $\mathcal{L}(\mathcal{V})$.

Definition 2.1.1 (Hilbert space)**.** *A (complex)* Hilbert space $\mathcal{H}$ *is a complex vector space, endowed with a scalar product* $\langle \cdot, \cdot \rangle$ *which is complete for the induced norm* $\|f\| := \langle f, f \rangle^{1/2}$. *We also assume* $\mathcal{H}$ *to be* separable[1].

Recall that a scalar product satisfies the following conditions, for any $f, g, h \in \mathcal{H}$ and $\lambda \in \mathbb{C}$:

1) $\langle g, f \rangle = \overline{\langle f, g \rangle}$,

2) $\langle f, \lambda g + h \rangle = \lambda \langle f, g \rangle + \langle f, h \rangle$,

3) $\langle f, f \rangle \geq 0$, with equality if and only if $f = 0$.

Note that we have chosen the linearity of the scalar product in the second argument, but we could also have

[1] Separable means that there exists a countable basis for $\mathcal{H}$. Not all Hilbert spaces are separable, but the non-separable ones are less frequently used.

chosen the linearity in the first argument. Completeness means that any *Cauchy sequence*[2] converges in $\mathcal{H}$. An analogy to this concept can be made with $\mathbb{Q}$ and $\mathbb{R}$: $\mathbb{Q}$ is not complete because many converging sequences have their limit in $\mathbb{R}$ and not in $\mathbb{Q}$, even if all the elements of the sequence are in $\mathbb{Q}$. Let us also emphasize that the scalar product satisfies the *Schwarz inequality*

$$|\langle f, g\rangle| \leq \|f\|\,\|g\|,$$

and that $\|f\| := \langle f, f\rangle^{1/2}$ is a norm, which means that it satisfies the *triangle inequality*

$$\|f + g\| \leq \|f\| + \|g\|.$$

Examples 2.1.2. *We present a few Hilbert spaces which appear very often:*

1) $\mathbb{C}^n$ *with scalar product* $\langle a, b\rangle := \sum_{j=1}^n \overline{a_j} b_j$ *for* $a, b \in \mathbb{C}^n$*,*

2) $L^2(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \mid \int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x < \infty\}$ *with scalar product*

$$\langle f, g\rangle := \int_{\mathbb{R}^n} \overline{f(x)} g(x)\,\mathrm{d}x \qquad \text{for } f, g \in L^2(\mathbb{R}^n),$$

3) $\ell^2(\mathbb{Z}^n) := \{(a_j)_{j\in\mathbb{Z}^n} \mid \sum_{j\in\mathbb{Z}^n} |a_j|^2 < \infty\}$ *with scalar product*

$$\big\langle (a_j)_{j\in\mathbb{Z}^n}, (b_j)_{j\in\mathbb{Z}^n}\big\rangle = \sum_{j\in\mathbb{Z}^n} \overline{a_j} b_j \qquad \text{for } (a_j)_{j\in\mathbb{Z}^n}, (b_j)_{j\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n).$$

The next definition contains a generalization of matrices acting on $\mathbb{C}^n$. Note that a matrix is always bounded, but this is not always the case in the more general framework of a Hilbert space.

Definition 2.1.3 (Bounded linear operator)**.** *A* bounded linear operator T *on a Hilbert space* $\mathcal{H}$ *consists in a linear map* $T : \mathcal{H} \to \mathcal{H}$ *satisfying* $\|Tf\| \leq c\|f\|$ *for some* $c > 0$ *and all* $f \in \mathcal{H}$*. The infimum over all* c *is denoted by* $\|T\|$ *and is called the* norm *of* T*. The set of all bounded linear operators is denoted by* $\mathcal{B}(\mathcal{H})$*.*

For example, the operator $\mathbb{1}$ acting as $\mathbb{1}f = f$ belongs to $\mathcal{B}(\mathcal{H})$, with $\|\mathbb{1}\| = 1$. Observe also that if $T, R \in \mathcal{B}(\mathcal{H})$, then TR is defined by $[TR](f) := T(Rf)$, it belongs to $\mathcal{B}(\mathcal{H})$ and its norm satisfies $\|TR\| \leq \|T\|\,\|R\|$.

Exercise 2.1.4. *In* $L^2(\mathbb{R})$ *or in* $\ell^2(\mathbb{Z})$*, exhibit a linear operator which is not bounded (and prove that it is not bounded). Can you also exhibit one in an abstract Hilbert space as introduced in Definition 2.1.1.*

Like for matrices, we define an *adjoint* for any $T \in \mathcal{B}(\mathcal{H})$, namely the adjoint T^* of T is the bounded linear operator satisfying for any $f, g \in \mathcal{H}$:

$$\langle f, Tg\rangle = \langle T^* f, g\rangle.$$

It can be shown that this adjoint always exists and is unique. Note that if $\mathcal{H} = \mathbb{C}^n$, then $\mathcal{B}(\mathbb{C}^n) = M_n(\mathbb{C})$ which means that any $T \in \mathcal{B}(\mathcal{H})$ is nothing but a matrix. In this situation, T^* corresponds to the adjoint matrix (transpose and complex conjugate).

Let us now look at special instances of bounded linear operators.

Definition 2.1.5 (Self-adjoint, positive, unitary, invertible operators)**.** *Let* $T \in \mathcal{B}(\mathcal{H})$*.*

[2] A Cauchy sequence in $\mathcal{H}$ is a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ satisfying the condition: for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| \leq \epsilon$ for all $n, m \geq N$. Note that any convergent sequence is Cauchy, but the converse is not true.

1) T is self-adjoint *or* Hermitian *if $T^* = T$,*

2) T is positive *if $\langle f, Tf\rangle \geq 0$, for any $f \in \mathcal{H}$,*

3) T is unitary *if $TT^* = T^*T = \mathbb{1}$,*

4) T is invertible *(in $\mathcal{B}(\mathcal{H})$) if T is bijective.*

Observe that the first three properties depend on the scalar product, and thus can not be defined for a vector space $\mathcal{V}$ without a scalar product. On the other hand, the invertibility property does not depend on the scalar product, and can also be defined on $\mathcal{V}$. We set $\mathrm{GL}(\mathcal{V})$ or $\mathrm{GL}(\mathcal{H})$ for the set of invertible elements of $\mathcal{V}$ or of $\mathcal{H}$, and observe that they are groups with identity $\mathbb{1}$.

Remark 2.1.6. *Observe that a finite dimensional complex vector space can always be identified with $\mathbb{C}^n$ for some $n \in \mathbb{N}$, and that $\mathbb{C}^n$ is endowed with a scalar product. It means that any finite dimensional complex vector space can always be identified with a finite dimensional Hilbert space. The same is not true for infinite dimensional vector spaces.*

2.2 Linear representations

We can now define the notion of a linear representation of a group. For the initial definition, observe that the scalar product is not really necessary, which means that the vector space structure is sufficient. However, the theory is much richer if the representation is taking place in a Hilbert space, but it is also less general!

Definition 2.2.1 (Linear representation)**.** *Let G be a group, and let $\mathcal{V}$ be a vector space. A* linear representation *of G in $\mathcal{V}$ corresponds to a map $U : G \to \mathcal{L}(\mathcal{V})$ satisfying $U(e) = \mathbb{1}$ and $U(ab) = U(a)U(b)$ for any $a, b \in G$. One writes $(\mathcal{V}, U)$ for this representation, or $(\mathcal{H}, U)$ if the vector space $\mathcal{V}$ is a Hilbert space $\mathcal{H}$.*

In particular, it follows from this definition that $U(a^{-1}) = U(a)^{-1}$, which means that all elements of the range of U are invertible, or equivalently $\mathrm{Ran}(U) \subset \mathrm{GL}(\mathcal{V})$. As a consequence, the map U corresponds to a homomorphism $G \to \mathrm{GL}(\mathcal{V})$. In the sequel, we shall simply say *a representation* instead of a *linear representation*, since these representations are the most common ones. Also, let us stress that all statements for $(\mathcal{V}, U)$ apply to $(\mathcal{H}, U)$ (since any Hilbert space is a special instance of a vector space) but the converse is not true: some statements for $(\mathcal{H}, U)$ are simply meaningless for $(\mathcal{V}, U)$.

Remark 2.2.2 (Unitary representation)**.** *If the representation is taking place in a Hilbert space $\mathcal{H}$, and if the operator $U(a)$ is unitary for any $a \in G$, then the map $G \to \mathcal{U}(\mathcal{H})$ is called* a unitary representation *of G. Here, we have used the notation $\mathcal{U}(\mathcal{H})$ for the set of all unitary operators on $\mathcal{H}$.*

Definition 2.2.3 (Trivial, faithful representation, dimension)**.** *A representation $(\mathcal{V}, U)$ of a group G is* trivial *if $U(a) = \mathbb{1}$ for any $a \in G$, while U is* faithful *if $U(a) \neq \mathbb{1}$ for any $a \in G \setminus \{e\}$. The* dimension *of a representation corresponds to the dimension of $\mathcal{V}$, denoted by* $\dim(\mathcal{V})$.

Clearly, the definition of dimension generates two types of representations: the finite dimensional representations, with $U(a)$ being a matrix for any $a \in G$, and the infinite dimensional ones, with $U(a)$ being a linear map on the infinite dimensional vector space $\mathcal{V}$.

Lemma 2.2.4. *Let G be a group, and $\mathcal{V}$ be a vector space.*

1. *If $U : G \to \mathcal{L}(\mathcal{V})$ is a representation, then the set $G_0 := \{a \in G \mid U(a) = \mathbb{1}\}$ is a normal subgroup of G,*
2. *If G is simple, then all non-trivial representations are faithful,*

3. *If G_0 is a normal subgroup of G and if $U : G/G_0 \to \mathcal{L}(\mathcal{V})$ is a representation, then the map* $\mathrm{U} : G \to \mathcal{L}(\mathcal{V})$ *given by* $\mathrm{U}(a) := U([a]_{G_0})$*, for any $a \in G$, defines a representation of G.*

The proof of this lemma is an easy exercise. Let us emphasize the meaning of the third statement: Starting from a representation of the quotient group, we can define a representation of the group itself. Quite often, we are trying to identify representations which are either equivalent, or inequivalent, in a precise sense.

Definition 2.2.5 (Equivalent or similar representations)**.** *Let $U : G \to \mathcal{L}(\mathcal{V})$ and $U' : G \to \mathcal{L}(\mathcal{V}')$ be two representations of a group G. These representations are* equivalent *or* similar *if there exists a bijective linear map $\mathcal{T} : \mathcal{V} \to \mathcal{V}'$ satisfying $U'(a) = \mathcal{T}U(a)\mathcal{T}^{-1}$ for any $a \in G$. In this case, we write $(\mathcal{V}, U) \simeq (\mathcal{V}', U')$. The bijective map $\mathcal{T}$ is called a* similarity transformation*. If $\mathcal{V}, \mathcal{V}'$ are Hilbert spaces, and if $\mathcal{T}$ is a unitary map between them, then the two representations are said to be* unitarily equivalent*.*

In this definition, we used the concept of a unitary map between two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. It means that $\mathcal{T}^*\mathcal{T} = \mathbb{1}$ and $\mathcal{T}\mathcal{T}^* = \mathbb{1}$, where $\mathcal{T}^*$ is defined by the equality

$$\langle f', \mathcal{T} f\rangle_{\mathcal{H}'} = \langle \mathcal{T}^* f', f\rangle_{\mathcal{H}} \qquad \text{for } f \in \mathcal{H} \text{ and } f' \in \mathcal{H}'.$$

The scalar product has been indexed for clarity.

Before moving on, observe that the notion of equivalent representations define an equivalence relation, as presented in Exercise 1.2.2. We now present a result which says that for finite groups, only unitary representations are really important. Keep in mind that this statement is not true for general groups.

Proposition 2.2.6. *Let G be a finite group, and let $(\mathcal{H}, U)$ be a representation of G in a Hilbert space $\mathcal{H}$. Then, U is equivalent to unitary representation $(\mathcal{H}', U')$.*

A proof for this result is provided in [1, Thm. 2.8] and is based on an average over the group. For finite groups, such an average is well defined, while it is generally not the case for infinite groups. Nevertheless, a similar statement exists for other groups, as for example compact Lie groups.

2.3 Reducible / irreducible representations

Let $\mathcal{V}_0$ be a *subspace* of a vector space $\mathcal{V}$, meaning that $\mathcal{V}_0$ is stable for the addition of its elements and for the multiplication by elements of $\mathbb{C}$. Another subspace $\mathcal{V}_1$ of $\mathcal{V}$ is called a *complementary subspace* if any $f \in \mathcal{V}$ admits a unique decomposition $f = f_0 + f_1$ with $f_0 \in \mathcal{V}_0$ and $f_1 \in \mathcal{V}_1$. In this case we write $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$. If $\mathcal{V}$ is a Hilbert space $\mathcal{H}$, we assume that the subspaces are *closed*, meaning that they are complete for the norm of $\mathcal{H}$ [3]. Then, if $\mathcal{H}_0$ is a subspace of $\mathcal{H}$, there exists a unique subspace $\mathcal{H}_0^\perp$ complementary to $\mathcal{H}_0$ satisfying $\langle f, g\rangle = 0$ for any $f \in \mathcal{H}_0$ and $g \in \mathcal{H}_0^\perp$. In this setting, we always choose this distinct subspace and call $\mathcal{H}_0^\perp$ the *orthogonal complement*. We still write $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$.

Let us now consider $T \in \mathcal{L}(\mathcal{V})$. In a decomposition $\mathcal{V}_0 \oplus \mathcal{V}_1$ of $\mathcal{V}$, this operator takes the form

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}$$

with clear meaning for each entry of the matrix. We are now ready for the main definition of this section.

Definition 2.3.1 (Invariance, minimality, and irreducibility)**.** *Let $(\mathcal{V}, U)$ be a representation of a group G.*

[3] The completeness means that any Cauchy sequence in $\mathcal{H}_0$ converges in $\mathcal{H}_0$. By analogy, one can think about $\{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ as a closed real subspace of $\mathbb{R}^2$.

1. *A subspace $\mathcal{V}_0 \subset \mathcal{V}$ is* invariant *if $U(a)\mathcal{V}_0 \subset \mathcal{V}_0$ for any $a \in G$. This subspace is* proper *if $\mathcal{V}_0 \neq \mathcal{V}$, and* non-trivial *if $\mathcal{V}_0 \neq \{0\}$. It is* minimal *if it does not contain any other non-trivial and proper invariant subspace.*

2. *The representation is* irreducible *if $\{0\}$ and $\mathcal{V}$ are the only invariant subspaces, and* reducible *otherwise.*

Based on this definition, one directly infers the following:

Exercise 2.3.2. *If G is finite and if the representation $(\mathcal{V}, U)$ is irreducible, show that $\mathcal{V}$ is finite dimensional and has dimension at most equal to $|G|$.*

Observe that if a subspace $\mathcal{V}_0 \subset \mathcal{V}$ is invariant, and if $\mathcal{V}_1$ is a complementary subspace, then in the decomposition $\mathcal{V}_0 \oplus \mathcal{V}_1$ of $\mathcal{V}$, any operator $U(a)$ takes the form $\begin{pmatrix} U_{00}(a) & U_{01}(a) \\ 0 & U_{11}(a) \end{pmatrix}$, for any $a \in G$. If $\mathcal{V}_1$ is also invariant, then $U(a)$ takes the simplest form $\begin{pmatrix} U_{00}(a) & 0 \\ 0 & U_{11}(a) \end{pmatrix}$, for any $a \in G$. In this case, one says that the representation is *decomposable* with respect to the direct sum $\mathcal{V}_0 \oplus \mathcal{V}_1$.

Definition 2.3.3 (Complete reducibility)**.** *A representation $(\mathcal{V}, U)$ of a group G is* completely reducible *if for any invariant subspace $\mathcal{V}_0 \subset \mathcal{V}$, there exists a complementary subspace which is also invariant.*

As already mentioned, if the vector space is a Hilbert space $\mathcal{H}$ and if $\mathcal{H}_0 \subset \mathcal{H}$ is an invariant subspace, then the distinct complementary subspace is the orthogonal complement $\mathcal{H}_0^\perp$. Clearly, if $(\mathcal{V}, U)$ is completely reducible and if $\dim(\mathcal{V}) < \infty$, then this representation can be decomposed into a direct sum of irreducible representations. If $\dim(\mathcal{V}) = \infty$, the statement is not true in general since the decomposition into invariant subspaces might never end. Nevertheless it can be shown that some representations are completely reducible.

Theorem 2.3.4. *1) If G is a finite group, any representation $(\mathcal{H}, U)$ in a Hilbert space is completely reducible,*

2) Any unitary representation $(\mathcal{H}, U)$ of a group G in a Hilbert space is completely reducible. In particular, if $\dim(\mathcal{H}) < \infty$, then the Hilbert space admits an orthogonal decomposition $\mathcal{H} = \oplus_k \mathcal{H}^k$ and each subspace $\mathcal{H}^k$ is a minimal invariant subspace.

By Remark 2.1.6, observe that the first statement also holds for an arbitrary representation $(\mathcal{V}, U)$ if $\mathcal{V}$ is finite dimensional.

Exercise 2.3.5. *Provide a proof for the previous statement. In particular, show that if a subspace $\mathcal{H}_0$ of $\mathcal{H}$ is invariant under a unitary representation of G, then $\mathcal{H}_0^\perp$ is also invariant under this representation.*

The next statement provides a criterion for the equivalence of two representations. We refer to [14, p. 55] for the proof, or leave it as an exercise.

Lemma 2.3.6 (Schur's Lemma)**.** *Let $(\mathcal{V}, U)$, $\mathcal{V}', U'$ be two irreducible representations of a group G. Assume that there exists a linear map $\mathcal{T} : \mathcal{V} \to \mathcal{V}'$ satisfying for all $a \in G$*

$$\mathcal{T}\, U(a) = U'(a)\mathcal{T}.$$

Then, either $\mathcal{T} = 0$, or $\mathcal{T}$ defines a similarity transformation, as introduced in Definition 2.2.5. In particular, if $(\mathcal{V}, U)$ and $(\mathcal{V}', U')$ are inequivalent, then $\mathcal{T} = 0$.

Let us state and prove two corollaries of Schur's Lemma.

Corollary 2.3.7. *Let $(\mathcal{V}, U)$ be a finite dimensional irreducible representation of a group G, and assume that there exists $T \in \mathcal{L}(\mathcal{V})$ satisfying $T\, U(a) = U(a)T$ for all $a \in G$. Then $T = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$.*

Proof. Since $\mathcal{V}$ is finite dimensional, T is a matrix which has at least one eigenvalue λ, which means that $\mathrm{Ker}(T - \lambda\mathbb{1}) \neq \emptyset$. It then follows from the assumption that $(T - \lambda\mathbb{1})\,U(a) = U(a)\,(T - \lambda\mathbb{1})$ for any $a \in G$. Since $T - \lambda\mathbb{1}$ can not be bijective, it follows from Schur's Lemma that $(T - \lambda\mathbb{1}) = 0$, meaning that $T = \lambda\mathbb{1}$. □

Corollary 2.3.8. *If G is Abelian, any finite dimensional irreducible representation of G is of dimension* 1.

Proof. Let $(\mathcal{V}, U)$ be a finite and irreducible representation of G. Since $U(a)\,U(b) = U(b)\,U(a)$ for any $a, b \in G$ it follows from Corollary 2.3.7 that $U(b) = \lambda(b)\mathbb{1}$ for some $\lambda(b) \in \mathbb{C}$ which depends on b. Then, for any $f \in \mathcal{V}$ with $f \neq 0$, for any $\alpha \in \mathbb{C}$, and for any $b \in G$ one has $U(b)\,\alpha f = \lambda(b)\,\alpha f$, which means that $\mathbb{C}f$ is an invariant subspace. Since the representation $(\mathcal{V}, U)$ is irreducible, it follows that $\mathcal{V} = \mathbb{C}f$. As a consequence, $\mathcal{V}$ is one dimensional. □

The last statement of this section is also about finite groups, and complements the content of Exercise 2.3.2. Its proof is left as an exercise, but it is not completely trivial, see for example [1, Prop. 2.19] or [12, Corollary p. 25].

Proposition 2.3.9. *Let G be a finite group and assume that G_0 is an Abelian subgroup of G. Then any irreducible representation of G is of dimension at most $|G|/|G_0|$.*

2.4 Representation of finite groups

In this section, we concentrate on finite groups and develop some of the special features of their representations. In particular, we are interested in inequivalent irreducible representations of such groups. Thanks to Exercise 2.3.2 or to Proposition 2.3.9, we know that all irreducible representations of a finite group are finite dimensional, which means that all vector spaces in this section are finite dimensional.

Proposition 2.4.1 (Orthogonality relation)**.** *Let $(\mathcal{V}, U)$ and $(\mathcal{V}', U')$ be two irreducible representations of a finite group G, and let $T : \mathcal{V} \to \mathcal{V}'$ be a linear map. Set*

$$Z_T := \frac{1}{|G|}\sum_{a\in G} U'(a)\,T\,U(a)^{-1}. \tag{2.4.1}$$

Then,

1. *If $(\mathcal{V}, U) \not\approx (\mathcal{V}', U')$, then $Z_T = 0$,*
2. *If $(\mathcal{V}, U) = (\mathcal{V}', U')$, then $Z_T = \frac{1}{n}\mathrm{Tr}(T)\,\mathbb{1}$ with $n = \dim(\mathcal{V})$.*

Proof. 1) One easily checks that $U'(b)\,Z_T = Z_T\,U(b)$ for any $b \in G$. Thus, it follows from Lemma 2.3.6 that $Z_T = 0$.

2) By Corollary 2.3.7 one infers that $Z_T = \lambda\mathbb{1}$ for some $\lambda \in \mathbb{C}$. Then, observe that

$$n\lambda = \mathrm{Tr}(\lambda\mathbb{1}) = \mathrm{Tr}\Big(\frac{1}{|G|}\sum_{a\in G} U'(a)\,T\,U(a)^{-1}\Big) = \frac{1}{|G|}\sum_{a\in G}\mathrm{Tr}\big(U'(a)\,T\,U(a)^{-1}\big) = \mathrm{Tr}(T),$$

leading to $\lambda = \frac{1}{n}\mathrm{Tr}(T)$. Note that the relation $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ has been used for the last equality. □

Recall that the notion of equivalent representation (see Definition 2.2.5) defines an equivalent relation, which allows us to define the set $\{\eta^k\}_k$ of all equivalent classes of inequivalent irreducible representations of a finite group G. For each equivalent class, we choose a representative which is a unitary representation $(\mathcal{H}^k, U^k)$. Thanks to Proposition 2.2.6, note that there is no restriction in requiring that this representation is unitary.

For an irreducible unitary representation $(\mathcal{H}^k, U^k)$, let us fix an orthonormal basis $\{\mathrm{e}_j^k\}_{j=1}^{n_k}$ with $n_k := \dim(\mathcal{H}^k)$. For any $a \in G$ and for $i, j \in \{1, \dots, n_k\}$ we also set

$$U_{ij}^k(a) \equiv U^k(a)_{ij} := \langle \mathrm{e}_i^k, U^k(a)\,\mathrm{e}_j^k \rangle_{\mathcal{H}^k} \tag{2.4.2}$$

for the element ij of the matrix $U^k(a)$. Clearly, the finite sequence $(U_{ij}^k(a))_{a\in G}$ belongs to $\mathbb{C}^{|G|}$. In the sequel we shall use the notation δ_{mn} for the Kronecker delta function, namely $\delta_{mn} = 0$ if $m \neq n$ while $\delta_{mn} = 1$ if $m = n$.

Let us now consider a special linear operator T in Proposition 2.4.1, namely the operator $T := |\mathrm{e}_s^\ell\rangle\langle \mathrm{e}_j^k| : \mathcal{H}^k \to \mathcal{H}^\ell$ with $s \in \{1, \dots, n_\ell\}$ and $j \in \{1, \dots, n_k\}$. This operator acts on any $f \in \mathcal{H}^k$ as

$$|\mathrm{e}_s^\ell\rangle\langle \mathrm{e}_j^k| f := \langle \mathrm{e}_j^k, f \rangle\, \mathrm{e}_s^\ell.$$

If then follows that

$$\langle \mathrm{e}_r^\ell, Z_T \mathrm{e}_i^k \rangle = \frac{1}{|G|} \sum_{a\in G} \langle \mathrm{e}_r^\ell, U^l(a)\,\mathrm{e}_s^\ell \rangle \langle \mathrm{e}_j^k, U^k(a)^{-1} \mathrm{e}_i^k \rangle = \frac{1}{|G|} \sum_{a\in G} U_{rs}^\ell(a) \overline{U_{ij}^k(a)}. \tag{2.4.3}$$

If we choose $\ell \neq k$, then it follows from Proposition 2.4.1 that $Z_T = 0$, which implies by (2.4.3) that $0 = \frac{1}{|G|} \sum_{a\in G} U_{rs}^\ell(a) \overline{U_{ij}^k(a)}$. In other words, one has $U_{rs}^\ell \perp U_{ij}^k$ in $\mathbb{C}^{|G|}$. On the other hand, if we choose $\ell = k$, it also follows from Proposition 2.4.1 that

$$Z_T = \frac{1}{n_k} \mathrm{Tr}(|\mathrm{e}_s^k\rangle\langle \mathrm{e}_j^k|)\, \mathbb{1} = \frac{1}{n_k} \delta_{sj}\, \mathbb{1},$$

which leads by (2.4.3) to

$$\frac{1}{|G|} \sum_{a\in G} U_{rs}^k(a) \overline{U_{ij}^k(a)} = \langle \mathrm{e}_r^k, Z_T \mathrm{e}_i^k \rangle = \frac{1}{n_k} \delta_{sj} \langle \mathrm{e}_r^k, \mathbb{1} \mathrm{e}_i^k \rangle = \frac{1}{n_k} \delta_{sj} \delta_{ri}.$$

Thus, if we summarize these relations one has shown that

$$\frac{1}{|G|} \sum_{a\in G} U_{rs}^\ell(a) \overline{U_{ij}^k(a)} = \frac{1}{n_k} \delta_{k\ell} \delta_{sj} \delta_{ri}. \tag{2.4.4}$$

The following statement is a direct consequence of the previous orthogonality relation. Note that a stronger statement will be proved later on.

Corollary 2.4.2. *Let G be a finite group, and let $\{\eta^k\}_k$ be the set of equivalence classes of inequivalent irreducible representations of G, each of dimension n_k. Then the following relation holds:*

$$\sum_k n_k^2 \leq |G|.$$

Proof. For each equivalent class of representations, one has n_k^2 elements $(U_{ij}^k)_{i,j=1}^{n_k}$ which are orthogonal in $\mathbb{C}^{|G|}$. Since $\dim(\mathbb{C}^{|G|}) = |G|$, it follows that $\sum_k n_k^2 \leq |G|$. □

Let us emphasize one important outcome of the previous statement: the inequivalent and irreducible representations of a finite group are only in finite number. We now introduce a very useful concept for representations of finite groups.

Definition 2.4.3 (Character)**.** *Let $(\mathcal{H}, U)$ be a finite dimensional representation of a finite group G. For any $a \in G$, the* character of a in U *is defined by*

$$\chi(a) := \mathrm{Tr}(U(a)).$$

Since for any invertible matrix B one has $\mathrm{Tr}(BAB^{-1}) = \mathrm{Tr}(A)$, it follows that elements in a conjugacy class have the same character, and that characters are identical for two equivalent representations. One also observes that $\chi(\cdot) \in \mathbb{C}^{|G|}$, but more can be said:

Corollary 2.4.4. *Let $(\mathcal{H}^k, U^k)$ and $(\mathcal{H}^\ell, U^\ell)$ be two unitary and irreducible representations of a finite group G, with respective characters denoted by χ^k and χ^ℓ. Then,*

$$\frac{1}{|G|}\sum_{a\in G}\overline{\chi^k(a)}\chi^\ell(a) = \begin{cases} 1 & \text{if } (\mathcal{H}^k, U^k) \simeq (\mathcal{H}^\ell, U^\ell) \\ 0 & \text{otherwise} \end{cases}.$$

Exercise 2.4.5. *Provide a proof of this statement, by using* (2.4.4).

Consider now $(\mathcal{H}, U)$ a finite dimensional representation of a finite group G. By Theorem 2.3.4 this representation is completely reducible, and therefore there exists a unique set $\{\nu_k\}_k \subset \mathbb{N}$ such that $\mathcal{H} \simeq \oplus_k \nu_k \mathcal{H}^k$ and $U \simeq \oplus_k \nu_k U^k$. Here, $(\mathcal{H}^k, U^k)$ represents a unitary irreducible representation of G in the equivalence class η^k, and the notation $2\mathcal{H}^k$ means $\mathcal{H}^k \oplus \mathcal{H}^k$ (and similarly for any natural number bigger than 2). In the sequel, we shall assume that $(\mathcal{H}, U)$ is unitary, and identify it with the direct sum of irreducible representations. The following statement provides a formula for computing ν_k.

Theorem 2.4.6. *Let $(\mathcal{H}, U)$ be a unitary and finite dimensional representation of a finite group G, and let $\{\nu_k\}_k \subset \mathbb{N}$ the set mentioned above.*

1) $\nu_k = \frac{1}{|G|}\sum_{a\in G}\overline{\chi(a)}\chi^k(a)$, where $\chi(a)$ and $\chi^k(a)$ denote the character of a in U and in U^k, respectively,

2) $(\mathcal{H}, U)$ is irreducible if and only if $\frac{1}{|G|}\sum_{a\in G}|\chi(a)|^2 = 1$,

3) If $(\mathcal{H}', U')$ is a second finite dimensional representation of G, then $(\mathcal{H}, U) \simeq (\mathcal{H}', U')$ if and only if their characters are equal.

We provide below the proof for the first two statements. For the third one, the necessity condition has already been mentioned after Definition 2.4.3. For the sufficiency, we refer to [1, Prop. 2.26] or to [13, Cor. III.2.6].

Proof. 1) By Theorem 2.3.4 this representation is completely reducible. Thus, writing $U = \oplus_{k'}\nu_{k'}U^{k'}$, one gets $\chi(a) = \sum_{k'}\nu_{k'}\chi^{k'}(a)$. Then, by Corollary 2.4.4 one infers that

$$\frac{1}{|G|}\sum_{a\in G}\overline{\chi(a)}\chi^k(a) = \frac{1}{|G|}\sum_{k'}\nu_{k'}\sum_{a\in G}\overline{\chi^{k'}(a)}\chi^k(a) = \sum_{k'}\nu_{k'}\,\delta_{kk'} = \nu_k.$$

2) As before, we write $U = \oplus_{k'}\nu_{k'}U^{k'}$, then one gets

$$\frac{1}{|G|}\sum_{a\in G}|\chi(a)|^2 = \frac{1}{|G|}\sum_{a\in G}\Big(\sum_k \nu_k\overline{\chi^k(a)}\Big)\Big(\sum_\ell \nu_\ell\chi^\ell(a)\Big) = \sum_k \nu_k\sum_\ell \nu_\ell\,\frac{1}{|G|}\sum_{a\in G}\overline{\chi^k(a)}\chi^\ell(a)$$

$$= \sum_k \nu_k \sum_\ell \nu_\ell \delta_{k\ell} = \sum_k \nu_k^2.$$

Clearly, the equality $\sum_k \nu_k^2 = 1$ holds if and only if there exists only one $\nu_k = 1$, and all the other ones are 0. This situation corresponds to an irreducible representation, leading directly to the statement. □

We now introduce a representation which plays a very important role: the group acting on itself. In the next statement, $\ell^2(G)$ means all functions from G to $\mathbb{C}$ (they are automatically square summable when G is finite), and observe that $\ell^2(G)$ can be identified with $\mathbb{C}^{|G|}$, scalar product included.

Definition 2.4.7 (Regular representation)**.** *Let G be a finite group, and consider the Hilbert space $\mathcal{H}^{\mathrm{reg}} := \ell^2(G)$. The* regular representation *of G is given by $(\mathcal{H}^{\mathrm{reg}}, U^{\mathrm{reg}})$ with $[U^{\mathrm{reg}}(a)f](b) = f(a^{-1}b)$ for any $f \in \mathcal{H}^{\mathrm{reg}}$.*

Exercise 2.4.8. *Check that the regular representation is indeed a representation. Is it a unitary representation ?*

As before, the regular representation is completely reducible, which means that it can be written as $\mathcal{H}^{\mathrm{reg}} = \oplus_k \nu_k \mathcal{H}^k$ and $U^{\mathrm{reg}} = \oplus_k \nu_k U^k$ with $\sum_k \nu_k n_k = \dim(\mathcal{H}^{\mathrm{reg}}) = |G|$. Let us now choose an orthonormal basis of $\mathcal{H}^{\mathrm{reg}}$ given by $\{\delta_a\}_{a\in G}$ with $\delta_a(b) = 1$ if $b = a$ and $\delta_a(b) = 0$ if $b \neq a$. Then one has

$$U^{\mathrm{reg}}_{bc}(a) := \langle \delta_b, U^{\mathrm{reg}}(a)\delta_c\rangle = \langle \delta_b, \delta_c(a^{-1}\cdot)\rangle = \sum_{d\in G} \overline{\delta_b(d)}\delta_c(a^{-1}d) = \delta_c(a^{-1}b) = \begin{cases} 1 & \text{if } a^{-1}b = c \\ 0 & \text{otherwise.} \end{cases}$$

In particular one has

$$U^{\mathrm{reg}}_{bb}(a) = \begin{cases} 1 & \text{if } a^{-1}b = b \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } a = e \\ 0 & \text{otherwise} \end{cases},$$

from which one infers that

$$\chi^{\mathrm{reg}}(a) = \begin{cases} |G| & \text{if } a = e \\ 0 & \text{otherwise} \end{cases}. \tag{2.4.5}$$

The following important result can now be deduced:

Theorem 2.4.9. *Consider a finite group G and let $\{\eta^k\}_k$ be the set of equivalence classes of its inequivalent irreducible representations, with $(\mathcal{H}^k, U^k)$ a unitary irreducible representation in the class η^k and with* $\dim(\mathcal{H}^k) = n_k$. *Then,*

1) $U^{\mathrm{reg}} = \oplus_k n_k U^k$,

2) $\sum_k n_k^2 = |G|$.

Let us comment on the first statement: it means that the regular representation contains each irreducible representations a number of times equal to their dimension, namely $\nu_k = n_k$. Then, the second statement can be directly inferred from

$$|G| = \dim(\mathcal{H}^{\mathrm{reg}}) = \sum_k \nu_k n_k = \sum_k n_k^2.$$

Proof. For the statement 1) and with the notation already introduced one gets from (2.4.5)

$$\nu_k = \frac{1}{|G|} \sum_a \overline{\chi^{\mathrm{reg}}(a)} \chi^k(a) = \frac{1}{|G|} |G| \chi^k(e) = n_k$$

since the trace of the $n_k \times n_k$ identity matrix is n_k. □

Let us add one more statement about the conjugacy classes of a finite group. These classes were introduced in Definition 1.2.3. The proof is left as an exercise, but is not completely trivial, see for example [1, Prop. 2.29 & Thm 2.30].

Theorem 2.4.10 (♥)**.** *For any finite group, the number of its conjugacy classes is equal to the number of inequivalent irreducible representations.*

Exercise 2.4.11. *Look at a few examples of finite groups and determine all their inequivalent irreducible representations.*

2.5 Tensor product of representations

In this section, we construct new representations, and decompose them... but let us start with the underlying Hilbert space. The first construction is abstract, but the subsequent lemma will help a lot.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and consider $f_j \in \mathcal{H}_j$ for $j \in \{1,2\}$. We set $f_1 \otimes f_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ acting on $(g_1, g_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ as

$$[f_1 \otimes f_2](g_1, g_2) = \langle g_1, f_1\rangle_{\mathcal{H}_1} \langle g_2, f_2\rangle_{\mathcal{H}_2}.$$

Clearly, $f_1 \otimes f_2$ is a bi-antilinear map on $\mathcal{H}_1 \times \mathcal{H}_2$. We denote by $\mathcal{E}$ the set of linear combinations of such $f_1 \otimes f_2$, and define a scalar product on $\mathcal{E}$ by

$$\langle f_1 \otimes f_2, f_1' \otimes f_2'\rangle := \langle f_1, f_1'\rangle_{\mathcal{H}_1} \langle f_2, f_2'\rangle_{\mathcal{H}_2}. \tag{2.5.1}$$

Based on this, we provide the main definition for a new Hilbert space:

Definition 2.5.1 (Tensor product of Hilbert spaces)**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. The completion of the set $\mathcal{E}$ with respect to the norm defined by the scalar product (2.5.1) is a new Hilbert space, denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$ and called* the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$.

The previous construction is abstract, but a basis of this Hilbert space is easy to exhibit. We refer to [10, Sec. II.4] for more details and a proof of the following statement.

Lemma 2.5.2. *Let $\{e_j^1\}_j$ and $\{e_k^2\}_k$ be orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then, the set $\{e_j^1 \otimes e_k^2\}_{j,k}$ defines a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

Let us now consider $A_1 \in \mathcal{B}(\mathcal{H}_1)$ and $A_2 \in \mathcal{B}(\mathcal{H}_2)$. We can define a new operator $A_1 \otimes A_2$ belonging to $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by $[A_1 \otimes A_2](f_1 \otimes f_2) = (A_1 f_1) \otimes (A_2 f_2)$, simply written $A_1 f_1 \otimes A_2 f_2$. We then get the multiplication rule $(A_1 \otimes A_2)(B_1 \otimes B_2) = A_1 B_1 \otimes A_2 B_2$. In addition, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, then we also get $\mathrm{Tr}(A_1 \otimes A_2) = \mathrm{Tr}(A_1)\,\mathrm{Tr}(A_2)$.

Exercise 2.5.3. *Check the above simple statements, and show that $\|A_1 \otimes A_2\| = \|A_1\|\,\|A_2\|$.*

We now come back to the representations of groups. Assume that $(\mathcal{H}_1, U_1)$ is a representation of a group G_1, and that $(\mathcal{H}_2, U_2)$ is a representation of a group G_2. Then we can define for $a_1 \in G_1$ and $a_2 \in G_2$ the element $(a_1, a_2) \in G_1 \times G_2$, and the operator $\mathrm{U}((a_1, a_2)) := U_1(a_1) \otimes U_2(a_2)$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$. We formulate in the first part of the next statement some easy outcomes of this construction, and leave the proof as an exercise. For the second part of the statement, we refer to [1, Prop. 2.37].

Lemma 2.5.4. *1) The map $\mathrm{U} : G_1 \times G_2 \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a linear representation of the direct product group $G_1 \times G_2$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is also finite dimensional, and the following* *equality holds for the characters:* $\chi_{\mathrm{U}}((a_1, a_2)) = \chi_1(a_1)\chi_2(a_2)$*, where* χ_j *is the character in the representation* $(\mathcal{H}_j, U_j)$ *for* $j \in \{1, 2\}$*.*

2) If $(\mathcal{H}_1, U_1)$ *is an irreducible representation of a finite group* G_1*, and* $(\mathcal{H}_2, U_2)$ *is an irreducible representation of a finite group* G_2*, then* $(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathrm{U})$ *is an irreducible representation of* $G_1 \times G_2$*, and all irreducible representations of* $G_1 \times G_2$ *are of this form.*

The above construction is called the *tensor product representation* of $G_1 \times G_2$. If we consider now representations of a single group G, the situation is very different. More precisely, we assume that $(\mathcal{H}_1, U_1)$ and $(\mathcal{H}_2, U_2)$ are representations of the same group G, and define the tensor product representation $\mathrm{U} : G \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by $\mathrm{U}(a) := U_1(a) \otimes U_2(a)$ for any $a \in G$. Even if $(\mathcal{H}_1, U_1)$ and $(\mathcal{H}_2, U_2)$ are irreducible representations, it is not clear if $(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathrm{U})$ is an irreducible representation of G, and in general it is not. However, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, observe that the representations $(\mathcal{H}_1 \otimes \mathcal{H}_2, U_1 \otimes U_2)$ and $(\mathcal{H}_2 \otimes \mathcal{H}_1, U_2 \otimes U_1)$ are equivalent. This property follows from the equalities valid for any $a \in G$:

$$\chi_{U_1 \otimes U_2}(a) = \chi_1(a)\chi_2(a) = \chi_2(a)\chi_1(a) = \chi_{U_2 \otimes U_1}(a)$$

and from the uniqueness of the characters, as mentioned in 3) of Theorem 2.4.6.

Let us come back to the representation $(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathrm{U})$ with $\mathrm{U} = U_1 \otimes U_2$ We shall assume that G is finite, and consider $(\mathcal{H}_1, U_1)$ and $(\mathcal{H}_2, U_2)$ two irreducible unitary representations of G. Recall that $(\mathcal{H}^\ell, U^\ell)$ denotes a unitary representation in the equivalence class η^ℓ of the set of all irreducible representations of G. Then there exist j, k such that $(\mathcal{H}_1, U_1) = (\mathcal{H}^j, U^j)$ and $(\mathcal{H}_2, U_2) = (\mathcal{H}^k, U^k)$. The decomposition of the tensor product representation into irreducible representations can be obtained with the formula introduced in 1) of Theorem 2.4.6, namely $\mathcal{H}^j \otimes \mathcal{H}^k = \oplus_\ell \nu_\ell \mathcal{H}^\ell$ and $\mathrm{U} = U^j \otimes U^k = \oplus_\ell \nu_\ell U^\ell$ with

$$\nu_\ell = \frac{1}{|G|} \sum_{a \in G} \overline{\chi_{\mathrm{U}}(a)} \chi^\ell(a) = \frac{1}{|G|} \sum_{a \in G} \overline{\chi^j(a)\chi^k(a)} \chi^\ell(a).$$

We also fix an orthonormal basis $\{e_i^\ell\}_{i=1}^{n_\ell}$ for each Hilbert space $\mathcal{H}^\ell$. As mentioned in Lemma 2.5.2, the set $\{e_r^j \otimes e_s^k\}_{r,s}$ defines an orthonormal basis of $\mathcal{H}^j \otimes \mathcal{H}^k$, sometimes called the *uncoupled basis*. On the other hand, the family $\{e_i^{\ell,m} \mid 1 \leq m \leq \nu_\ell, i \in \{1, \dots, n_\ell\}\}_\ell$ defines a basis of the direct sum $\oplus_\ell \nu_\ell \mathcal{H}^\ell$. Thus, having two natural bases for the same Hilbert space, one can express the elements of one basis in terms of the other basis. Such relations are known under the name of *Clebsch-Gordan coefficients* and have been extensively studied by physicists. The weakness of this notion is that these coefficients depend on the choice of a basis in each space $\mathcal{H}^\ell$. With the notations introduced above, one has

$$e_i^{\ell,m} = \sum_{r,s} C(\ell, m, i; j, k)_{r,s} e_r^j \otimes e_s^k$$

where the coefficients $C(\ell, m, i; j, k)_{r,s}$ are precisely the Clebsch-Gordan coefficients. They express one vector of the basis of the direct sum as a linear combination of the vectors of the basis of the tensor product of Hilbert spaces.

Let us briefly sketch further results in the same direction. These results appear in the framework of quantum mechanics but have their roots in representation theory.

Let $(\mathcal{H}, U)$ be a unitary representation of a group G, and observe that this representation induces a representation of G in $\mathcal{B}(\mathcal{H})$. Indeed, $\mathcal{B}(\mathcal{H})$ is a vector space, and for any $T \in \mathcal{B}(\mathcal{H})$ and any $a \in G$ one can set

$$\mathcal{U}(a)T := U(a)\,T\,U(a)^{-1}.$$

Exercise 2.5.5. *Check that $\mathcal{U} : G \to \mathcal{L}(\mathcal{B}(\mathcal{H}))$ defines a representation.*

Then, let us assume that this representation $(\mathcal{B}(\mathcal{H}), \mathcal{U})$ can be decomposed into a direct sum $\mathcal{B}(\mathcal{H}) = \oplus_\ell \mu_\ell \mathcal{L}^\ell$ and $\mathcal{U} = \oplus_\ell \mu_\ell \mathcal{U}^\ell$ of irreducible representations, with $\mu_\ell \in \mathbb{N}$. Note that $\mathcal{L}^\ell$ is made of linear operators acting on $\mathcal{H}$. If $\mathcal{H}$ is finite dimensional, then $\mathcal{B}(\mathcal{H})$ is also finite dimensional and such a decomposition holds, but if $\mathcal{H}$ is infinite dimensional, one may have to accept that $\mu_\ell = \infty$. The fact that $(\mathcal{L}^\ell, \mathcal{U}^\ell)$ is an irreducible representation means that there exist an irreducible representation $(\mathcal{H}^\ell, U^\ell)$ of the equivalent class η^ℓ of all irreducible representations of G, and a bijective map $\mathcal{T}_\ell : \mathcal{H}^\ell \to \mathcal{L}^\ell$ satisfying (see Definition 2.2.5) for any $f \in \mathcal{H}^\ell$ and any $a \in G$

$$\mathcal{T}_\ell U^\ell(a) f = \mathcal{U}(a)\mathcal{T}_\ell f \Leftrightarrow \mathcal{T}_\ell(U^\ell(a)f) = \mathcal{U}(a)\mathcal{T}_\ell(f) \Leftrightarrow \mathcal{T}_\ell(U^\ell(a)f) = U(a)\mathcal{T}_\ell(f)\,U(a)^{-1}.$$

Similarly, the initial unitary representation $(\mathcal{H}, U)$ can be decomposed into a direct sum of irreducible representations $\mathcal{H} = \sum_j \nu_j \mathcal{H}^j$ and $U = \oplus_j \nu_j U^j$. In this framework, it turns out that information on the quantity

$$\langle f_k, \mathcal{T}_\ell(f) f_j\rangle \tag{2.5.2}$$

can be obtained, for $f_k \in \mathcal{H}^{k,m}$, $f_j \in \mathcal{H}^{j,n}$ for $m \in \{1, \dots, \nu_k\}$, $n \in \{1, \dots, \nu_j\}$, and for $f \in \mathcal{H}^\ell$. More precisely:

Theorem 2.5.6 (Selection rule)**.** *In the framework introduced above, one has $\langle f_k, \mathcal{T}_\ell(f) f_j\rangle = 0$ except if there exists a representation of the class η^k in the decomposition of the tensor product representation $(\mathcal{H}^\ell \otimes \mathcal{H}^j, U^\ell \otimes U^j)$ into irreducible representations of G.*

When a representation of the class η^ℓ appears in the decomposition of the tensor product representation $(\mathcal{H}^j \otimes \mathcal{H}^k, U^j \otimes U^k)$ into irreducible representations of G, then the quantity (2.5.2) can be different from 0. It can be computed by using the Clebsch-Gordan coefficients already introduced, and such a result is known as Wigner-Eckart theorem.

Exercise 2.5.7. *Provide a proof of the selection rule, and study the Wigner-Eckart theorem.*

2.6 Symmetries and projective representations

It is very often useful to consider elements of a Hilbert space *modulo* $\mathbb{C}$, or more precisely $\hat{\mathcal{H}} := \mathcal{H}/\mathbb{C}$. It means that for any element $\hat{f} \in \hat{\mathcal{H}}$ there exists $f \in \mathcal{H}$ with $\|f\| = 1$ and $\hat{f} = \{\lambda f \mid \lambda \in \mathbb{C}\}$. In other words, each element of $\hat{\mathcal{H}}$ is a one dimensional vector space. The space $\hat{\mathcal{H}}$ is also called the *projective Hilbert space* and its elements are called *rays* or *projective rays*. The idea behind this construction is that the complex phase of a quantum system can never be recovered.

Alternatively, the elements of $\hat{\mathcal{H}}$ are also in bijection with the set of *pure states* which plays a very important role in quantum mechanics. The set of pure states can be described by the one dimensional projection $|f\rangle\langle f|$ for $f \in \mathcal{H}$ and $\|f\| = 1$, and acting as $|f\rangle\langle f|g = \langle f, g\rangle f$ for any $g \in \mathcal{H}$. One easily observes that $|f\rangle\langle f|$ is an orthogonal projection, namely an element $P \in \mathcal{B}(\mathcal{H})$ satisfying $P^2 = P = P^*$. For any ray $\hat{f} \in \hat{\mathcal{H}}$ we denote by $P_{\hat{f}}$ the pure state defined by $P_{\hat{f}} := \big|f\rangle\langle f\big|$.

Exercise 2.6.1. *Check that rays and pure states are in bijection.*

Definition 2.6.2 (Transition probability)**.** *For any rays $\hat{f}, \hat{g} \in \hat{\mathcal{H}}$, the* transition probability *from $\hat{f}$ to $\hat{g}$ is defined by*

$$\mathrm{Tr}(P_{\hat{f}} P_{\hat{g}}) = |\langle f, g\rangle|^2.$$

We are now looking for operations which do no change these transition probabilities.

Definition 2.6.3 (Symmetry)**.** *A* symmetry *is a map $S : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ satisfying*

$$\mathrm{Tr}(P_{S\hat{f}} P_{S\hat{g}}) = \mathrm{Tr}(P_{\hat{f}} P_{\hat{g}}).$$

Clearly, if $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator, then we can set $S_U \hat{f} := \widehat{Uf}$, and S_U is a symmetry. Indeed, observe firstly that

$$\|Uf\|^2 = \langle Uf, Uf\rangle = \langle U^* Uf, f\rangle = \langle f, f\rangle = \|f\|^2 = 1$$

and that the following equalities hold:

$$\mathrm{Tr}(P_{S_U\hat{f}} P_{S_U\hat{g}}) = |\langle Uf, Ug\rangle|^2 = |\langle U^* Uf, g\rangle|^2 = |\langle f, g\rangle|^2 = \mathrm{Tr}(P_{\hat{f}} P_{\hat{g}}).$$

Note that if $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, then λU is another unitary operator in $\mathcal{H}$ but S_U and $S_{\lambda U}$ define the same symmetry. Observe also that the same construction holds if $U : \mathcal{H} \to \mathcal{H}$ is an anti-unitary operator, namely if U satisfies $U(f + \lambda g) = Uf + \overline{\lambda} Ug$ and $\langle Uf, Ug\rangle = \overline{\langle f, g\rangle}$. As an example of an anti-unitary operator U on $\mathbb{C}^n$, one can consider the complex conjugation: $U(a_1, a_2, \ldots, a_n) = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n})$. Then, a rather deep theorem of Wigner states that all symmetries are implemented by unitary or anti-unitary operators. We state the result below, and refer to the following link for more information:

https://en.wikipedia.org/wiki/Wigner's_theorem

For shortness, we introduce the notation $\mathbb{T}$ for $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.

Theorem 2.6.4 (Wigner's theorem)**.** *Let $S : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ be a symmetry. Then there exists $U : \mathcal{H} \to \mathcal{H}$, either unitary or anti-unitary, such that $S = S_U$. The operator U is unique modulo $\lambda \in \mathbb{T}$.*

Let us now extend this notion of symmetries to the notion of *group of symmetries*. More precisely, we shall consider a map S from a group G to the set of symmetries satisfying $S(ab) = S(a)S(b)$ and $S(e) = \mathbb{1}$, where $S(a)$ and $S(b)$ are symmetries for any $a, b \in G$. By Wigner's theorem, for each $a \in G$ there exists a unitary or an anti-unitary operator $U(a)$ acting on $\mathcal{H}$ satisfying $S(a) = S_{U(a)}$ with the notation introduced above. For simplicity, suppose that all $U(a)$ are unitary. Then, a natural question is about the map $G \ni a \mapsto U(a) \in \mathcal{U}(\mathcal{H})$, is this map a unitary representation ? Unfortunately (or fortunately because it makes life more interesting) the answer is NO. Indeed, if for any $a \in G$ we fix a unitary operator $U(a)$ satisfying $S(a) = S_{U(a)}$, then we only get

$$U(a)\,U(b) = \omega(a, b)\,U(ab)$$

with $\omega(a, b) \in \mathbb{T}$. This additional factor is coming from the non-uniqueness of the unitary operator corresponding to any symmetry. Observe also that if we had chosen $U'(a) := \rho(a)U(a)$ for some $\rho(a) \in \mathbb{T}$ and any $a \in G$, then one would get

$$U'(a)\,U'(b) = \rho(a)\rho(b)\,U(a)\,U(b) = \rho(a)\rho(b)\,\omega(a, b)\,U(ab) = \frac{\rho(a)\rho(b)}{\rho(ab)}\omega(a, b)\,U'(ab) \equiv \omega'(a, b)\,U'(ab),$$

meaning that a different choice of unitary operators would provide a change of ω of the form

$$\omega'(a, b) = \frac{\rho(a)\rho(b)}{\rho(ab)}\omega(a, b) \tag{2.6.1}$$

for any $a, b \in G$.

Having this motivation in mind, one is naturally led to a more general definition for the representation of a group.

Definition 2.6.5 (Projective representation)**.** *Let G be a group, and let $\mathcal{V}$ be a vector space. A* projective representation *of G in $\mathcal{V}$ corresponds to a map $U : G \to \mathcal{L}(\mathcal{V})$ satisfying $U(e) = \mathbb{1}$ and $U(a)\,U(b) = \omega(a,b)\,U(ab)$ for any $a,b \in G$, with $\omega(a,b) \in \mathbb{C}^*$. The map $\omega : G \times G \to \mathbb{C}^*$ is called* a 2-cocycle*. We denote by $(\mathcal{V}, U, \omega)$ any projective representation.*

Let us immediately mention one important property of the 2-cocycles:

Exercise 2.6.6. *Check that any 2-cocycle satisfies the following property for any $a,b,c \in G$:*

$$\omega(a,b)\,\omega(ab,c) = \omega(a,bc)\,\omega(b,c).$$

This relation, called the 2-cocycle relation*, can be obtained by computing $U(a)\,U(b)\,U(c)$ by two different approaches, using the associativity of the product of operators acting on $\mathcal{H}$.*

In addition, there exists a natural notion of equivalence of 2-cocycles, as already exhibited in (2.6.1).

Definition 2.6.7 (Equivalence and triviality of 2-cocycles)**.** *Two 2-cocycles $\omega : G\times G \to \mathbb{C}^*$ and $\omega' : G\times G \to \mathbb{C}^*$ are called* equivalent *if there exists $\rho : G \to \mathbb{C}^*$ such that relation (2.6.1) holds, for any $a,b \in G$. We say that a 2-cocycle ω is* trivial *if there exists $\rho : G \to \mathbb{C}^*$ satisfying $\omega(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}$.*

Let us enumerate several remarks related to 2-cocycles:

1) Projective representations are very natural and appear quite often. They are more general than linear representations and contain them,

2) If ω is trivial, then the map $\rho(a)^{-1}U(a)$ define a linear representation,

3) If the 2-cocycles ω and ω' are equivalent, we say that the two projective representations $(\mathcal{V}, U, \omega)$ and $(\mathcal{V}, U', \omega')$ are *equivalent*,

4) In the above definition for ω and ρ we have not assumed any regularity (continuity, measurability, ...) of these maps. Depending on the context, and in particular if the group G has additional structures, then some regularity conditions have to be imposed on these functions,

5) Quite often, the maps ω and ρ are taking values in $\mathbb{T}$ and not in $\mathbb{C}^*$.

A new natural question occurs in this context: Can one always trivialize a 2-cocycle ? In general, the answer is NO, but it depends on the group. Investigations in this direction corresponds to the study of *group cohomology*.

Let us finally provide another example of how projective representations appear. Suppose that G is isomorphic to a quotient group, see Proposition 1.2.9. More precisely, we assume that there exist a group $\mathfrak{G}$ with a normal subgroup $\mathfrak{G}_0$ and a bijective homomorphism $\phi : \mathfrak{G}/\mathfrak{G}_0 \to G$. Let also $\mathcal{U} : \mathfrak{G} \to \mathcal{L}(\mathcal{V})$ be a linear representation of $\mathfrak{G}$, and assume that $\mathcal{U}(\mathfrak{a}) = \sigma(\mathfrak{a})\mathbb{1}$ for any $\mathfrak{a} \in \mathfrak{G}_0$ with $\sigma(\mathfrak{a}) \in \mathbb{C}^*$. For any $a \in G$, let us denote by $\mathfrak{a}$ an element of $\mathfrak{G}$ satisfying $\phi([\mathfrak{a}]_{\mathfrak{G}_0}) = a$. We then define a map $U : G \to \mathcal{L}(\mathcal{V})$ by $U(a) := \mathcal{U}(\mathfrak{a})$ and check that this map U defines a projective representation of G.

For the proof of this statement, let us consider $a,b \in G$, set $c := ab$, and let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ some chosen elements of $\mathfrak{G}$ which satisfy $\phi([\mathfrak{a}]_{\mathfrak{G}_0}) = a$, $\phi([\mathfrak{b}]_{\mathfrak{G}_0}) = b$, and $\phi([\mathfrak{c}]_{\mathfrak{G}_0}) = c$. Note that these elements are not unique. Then we have

$$\phi([\mathfrak{c}]_{\mathfrak{G}_0}) = c = ab = \phi([\mathfrak{a}]_{\mathfrak{G}_0})\phi([\mathfrak{b}]_{\mathfrak{G}_0}) = \phi([\mathfrak{a}]_{\mathfrak{G}_0}[\mathfrak{b}]_{\mathfrak{G}_0}) = \phi([\mathfrak{a}\mathfrak{b}]_{\mathfrak{G}_0})$$

which implies that $[\mathfrak{c}]_{\mathfrak{G}_0} = [\mathfrak{a}\mathfrak{b}]_{\mathfrak{G}_0}$. Thus, there exists $\mathfrak{d} \in \mathfrak{G}_0$ (which depends on the initial choice of $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$) such that $\mathfrak{a}\mathfrak{b} = \mathfrak{d}\mathfrak{c}$. As a consequence, it follows that

$$U(a)\,U(b) = \mathcal{U}(\mathfrak{a})\mathcal{U}(\mathfrak{b}) = \mathcal{U}(\mathfrak{a}\mathfrak{b}) = \mathcal{U}(\mathfrak{d}\mathfrak{c}) = \mathcal{U}(\mathfrak{d})\mathcal{U}(\mathfrak{c}) = \sigma(\mathfrak{d})\mathcal{U}(\mathfrak{c}) = \sigma(\mathfrak{d})\,U(c) = \sigma(\mathfrak{d})\,U(ab).$$

Thus, if we set $\omega(a,b) := \sigma(\mathfrak{d})$, which depends indeed on a and b, and on the choices made above, then we observe that we have obtained a projective representation of G.

If we summarize this construction, observe that a representation of $\mathfrak{G}$ having the property that it takes scalar values on a normal subgroup $\mathfrak{G}_0$ leads naturally to a projective representation of its quotient group $\mathfrak{G}/\mathfrak{G}_0$ (or any group isomorphic to its quotient group). A converse question then holds: Given a group G, can one find a larger group $\mathfrak{G}$ with a normal subgroup $\mathfrak{G}_0$ such that the quotient group is isomorphic to G, and such that any projective representations of G can be lift to a linear representation of $\mathfrak{G}$? If so, the group $\mathfrak{G}$ is called the *universal cover* of G, or the *universal covering group*. Sometimes such a cover group exists, sometimes not. For example, SU(2) is the universal cover of SO(3), with $\mathfrak{G}_0 = \{\mathbb{1}, -\mathbb{1}\}$, and any finite group possesses a universal cover which can be constructed explicitly. But such a construction is not possible for all groups.

Exercise 2.6.8 (♥). *Describe the universal cover of any finite group.*

Chapter 3

Lie groups and Lie algebras

Lie groups are special groups with an additional differential structure compatible with the group law. In this chapter, we introduce them and provide some examples. We also introduce Lie algebras and link them with Lie groups.

3.1 Topological notions and manifolds

We start by introducing a few notions from topology, since Lie groups are special instances of topological spaces. Topology is at the root of many subjects in mathematics, as for example calculus...

Definition 3.1.1 (Topological space)**.** *A* topological space $(\mathcal{M}, \mathcal{T})$ *consists of a set* $\mathcal{M}$ *together with a collection* $\mathcal{T}$ *of subsets of* $\mathcal{M}$ *satisfying*

1) $\varnothing, \mathcal{M} \in \mathcal{T}$,

2) If $V_\alpha \in \mathcal{T}$, *then* $\bigcup_\alpha V_\alpha \in \mathcal{T}$ *(stability of* $\mathcal{T}$ *under arbitrary union),*

3) If $V_1, \dots, V_n \in \mathcal{T}$, *then* $\bigcap_{i=1}^n V_i \in \mathcal{T}$ *(stability of* $\mathcal{T}$ *under finite intersection).*

The elements of $\mathcal{T}$ *car called* open sets *and their complements* $\mathcal{M} \setminus V$ *are called* closed sets, *for any* $V \in \mathcal{T}$.

An example of a topological space is provided by $\mathbb{R}$, together with the set of all open intervals, their arbitrary unions and their finite intersections. Clearly, it is not easy to describe all open sets, and a better notion will be introduced below. We continue with a few additional definitions related to topological spaces. The first one is related to all open sets containing a given point.

Definition 3.1.2 (Neighborhood)**.** *Let* $(\mathcal{M}, \mathcal{T})$ *be a topological space and let* $p \in \mathcal{M}$ *be one point in* $\mathcal{M}$. *A* neighborhood *of* p *is any open set containing* p. *We write* ν_p *for the set of all neighborhoods of the point* p, *or in other words for the set of all open sets containing* p.

The next definition is about the separability of points: can one always find neighborhoods of two distinct points with an empty intersection ? Yes, is the space is Hausdorff ! Fortunately, most of the usual spaces have the Hausdorff property.

Definition 3.1.3 (Hausdorff property)**.** *A topological space* $(\mathcal{M}, \mathcal{T})$ *is* Hausdorff *if for any* $p_1, p_2 \in \mathcal{M}$ *with* $p_1 \neq p_2$ *there exist* $V_1 \in \nu_{p_1}$ *and* $V_2 \in \nu_{p_2}$ *with* $V_1 \cap V_2 = \varnothing$.

Exercise 3.1.4. *Provide an example of a topological space which is not Hausdorff.*

As already mentioned, providing a list of all open sets is rather long and difficult. As a result, we introduce the notion of a basis of a topological space.

Definition 3.1.5 (Basis of a topological space)**.** *A subset* $\mathfrak{B} := \{V_\alpha\}_\alpha \subset \mathfrak{T}$ *is a* basis *of* $(\mathcal{M}, \mathfrak{T})$ *is for any* $p \in \mathcal{M}$ *and any* $V \in \nu_p$*, there exists* $U \in \mathfrak{B}$ *with* $p \in U \subset V$*.*

It is not difficult to observe that $\mathfrak{B} := \{(a, b) \mid a, b \in \mathbb{R}\}$ define a basis of $\mathbb{R}$ with the open sets mentioned above. The same holds for $\mathbb{R}^n$ with $\mathfrak{B} := \{\mathrm{B}(X, r) \mid X \in \mathbb{R}^n, r > 0\}$ the set of all open balls (here $\mathrm{B}(X, r) := \{Y \in \mathbb{R}^n \mid \|X - Y\| < r\}$). A natural question is then about the size of a basis, is it countable (meaning in bijection with $\mathbb{N}$) or not ?

Definition 3.1.6 (Second countable)**.** *A topological space* $(\mathcal{M}, \mathfrak{T})$ *is* second countable *if it admits a countable basis.*

Exercise 3.1.7. *Show that* $\mathbb{R}^n$ *with the usual topology provided by open sets is second countable.*

There is one more notion which is defined without any additional concept: the notion of continuous map.

Definition 3.1.8 (Continuous map)**.** *Let* $(\mathcal{M}, \mathfrak{T})$ *and* $(\mathcal{N}, \mathfrak{S})$ *be two topological spaces, and let* $f : \mathcal{M} \to \mathcal{N}$*. The map* f *is* continuous *if* $f^{-1}(U) \in \mathfrak{T}$ *for any* $U \in \mathfrak{S}$*, where*

$$f^{-1}(U) = \{p \in \mathcal{M} \mid f(p) \in U\}.$$

Since the set $\mathfrak{S}$ is usually difficult to describe, one conveniently observes that f is continuous if $f^{-1}(U) \in \mathfrak{T}$ for all U is a basis of $(\mathcal{N}, \mathfrak{S})$. Clearly, there is now one exercise which has to be done:

Exercise 3.1.9. *Let* $\mathcal{M} = \mathcal{N} = \mathbb{R}$ *with the usual topology defined by open sets. Check that the notion of continuity introduced above corresponds to the standard definition of a continuous function* $f : \mathbb{R} \to \mathbb{R}$ *in terms of* ϵ *and* δ*.*

In the framework of Definition 3.1.8, if f is continuous and bijective, and if f^{-1} is also continuous, we say that the two topological spaces $(\mathcal{M}, \mathfrak{T})$ and $(\mathcal{N}, \mathfrak{S})$ are *homeomorphic*, and that f is a *homeomorphism*. Having these topological notions in mind, we can now define the next central object.

Definition 3.1.10 (Topological manifold)**.** *A* topological manifold *of dimension n (and without boundary) is a Hausdorff and second countable topological space* $(\mathcal{M}, \mathfrak{T})$ *such that for any* $p \in \mathcal{M}$ *there exist an open set* $V \in \nu_p$ *and a continuous and injective function* $\varphi : V \to \mathbb{R}^n$ *with* $\varphi(V)$ *open and* $\varphi^{-1} : \varphi(V) \to \mathcal{M}$ *also continuous.*

In this definition, note that checking the continuity of φ means checking that for any open set W of $\mathbb{R}^n$, $\varphi^{-1}(W \cap \varphi(V))$ belongs to $\mathfrak{T}$ [1]. We usually say that φ is a homeomorphism from V to its image, or simply a *local homeomorphism*. Let us also mention that for any p, the map φ, defined on a neighborhood V of p, provides a *local coordinate system* or a *local chart* at p. Indeed, if we set $(x_1(q), x_2(q) \dots, x_n(q)) := \varphi(q)$ for any q in V, then the map

$$V \ni q \mapsto (x_1(q), x_2(q) \dots, x_n(q)) \in \mathbb{R}^n$$

defines a local description of $\mathcal{M}$ around the point p. If necessary, we can even fix φ such that $\varphi(p) = 0 \in \mathbb{R}^n$. In simpler terms, it means that the manifold $\mathcal{M}$ can be parameterized locally by n real parameters.

[1] More generally, if $\Omega \subset \mathcal{M}$ is a subset of a topological space $(\mathcal{M}, \mathfrak{T})$, the *subspace topology* on Ω is provided by the set $\mathfrak{T}_\Omega := \{V \cap \Omega \mid V \in \mathfrak{T}\}$, and $(\Omega, \mathfrak{T}_\Omega)$ is a topological space.

Let us mention one "famous" outcome of this definition: A cup and a doughnut can not be distinguished by a mathematician. Indeed, both manifolds are locally and globally identical, from a topological point of view.

For Lie groups, continuity conditions are not enough, we need smoothness, and therefore a stronger version of the previous definition. Note however that one can not really define the meaning of differentiability directly on the manifold, another trick is necessary. For this, let us denote generically by φ the local homeomorphic maps, namely the bijective and bi-continuous maps from an open set of $\mathcal{M}$ to an open set of $\mathbb{R}^n$.

Definition 3.1.11 (Smooth manifold)**.** *A* smooth manifold *of dimension n is a topological manifold of dimension n with the composition maps $\varphi_j \circ \varphi_k^{-1}$ and $\varphi_k \circ \varphi_j^{-1}$ of class C^∞, wherever and whenever they exist.*

In the previous definition, the function $\varphi_j \circ \varphi_k^{-1}$ exists if $\mathrm{Dom}(\varphi_j) \cap \mathrm{Dom}(\varphi_k) =: V_{jk} \neq \emptyset$ and then $\varphi_j \circ \varphi_k^{-1}$ is defined from $\varphi_k(V_{jk}) \subset \mathbb{R}^n$ to $\varphi_j(V_{jk}) \subset \mathbb{R}^n$. A similar definition holds for $\varphi_k \circ \varphi_j^{-1}$, and these functions are called *transition functions*.

3.2 Lie groups

The setting introduced in the previous definition has no relation with groups. However, some groups have the structure of a smooth manifold, and such groups correspond precisely to Lie groups.

Definition 3.2.1 (Lie group)**.** *A* Lie group *G is a group that is also a finite dimensional smooth manifold, for which the group law and the inversion are smooth maps.*

As already mentioned, the smoothness condition can not be directly read on the manifold, it appears through the local charts.

Exercise 3.2.2. *Write the smoothness condition for the product and for the inversion in terms of local charts, as precisely as possible.*

Let us mention a few examples of Lie groups: $(\mathbb{R}^n, +)$, $((0,\infty), \cdot)$, $(\mathbb{T}, \cdot)$ are very simple Lie groups. The rotation group mentioned in Section 1.4, the Euclidean group, the Lorentz group, and the Poincaré group introduced in Section 1.5 are also examples of Lie groups. The groups of $n \times n$ matrices introduced in Example 1.1.5 are Lie groups as well. We shall come back to most of these groups in the sequel.

Our first aim is to consider Lie groups which have properties quite similar to finite groups. Clearly, Lie groups contain an infinite number of elements, so what is the concept of smallness for infinite sets ?

Definition 3.2.3 (Compact space, compact subset)**.** *A topological space $(\mathcal{M}, \mathcal{T})$ is* compact *if any covering of $\mathcal{M}$ by open sets admits a finite subcover. A subset $\Omega \subset \mathcal{M}$ is compact is any covering of Ω by open sets of $\mathcal{M}$ admits a finite subcover.*

More explicitly, it means that if one covers $\mathcal{M}$ by a family of open sets V_α (meaning that any $p \in \mathcal{M}$ belongs to at least one set V_α), then one can select a finite family of these open sets which still covers entirely $\mathcal{M}$. For the subset Ω, observe that the definition corresponds to the compactness of the topological space $(\Omega, \mathcal{T}_\Omega)$, where $\mathcal{T}_\Omega$ is the subspace topology.

Exercise 3.2.4. *On $\mathbb{R}$ show that any closed interval is a compact set, while any open interval is not compact. More generally, any closed and bounded set in $\mathbb{R}^n$ is compact, can you prove it ?*

Examples of compact Lie groups are $\mathbb{T}$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. As a result of this definition, *compact Lie groups* can be considered as small Lie groups, and their properties are quite similar to finite groups. In particular, compact Lie groups possess a left and right invariant finite *Haar measure*. In the next statement, we do not provide the exact definition of a Haar measure, but we emphasize its effect for the definition of the integral on G. For it, we introduce the set $C_c(G)$ of *continuous and compactly supported functions* on G with values in $\mathbb{C}$. Compactly supported means that these functions are not 0 only on a compact set.

Proposition 3.2.5. *Let G be a Lie group. There exist a map $I : C_c(G) \to \mathbb{C}$ satisfying the following properties for any $f, g \in C_c(G)$, $a \in G$, and $\lambda \in \mathbb{C}$:*

1) $I(f + \lambda g) = I(f) + \lambda I(g)$ (linearity),

2) If $f \geq 0$, then $I(f) \geq 0$ (positivity),

3) $I(f(a\,\cdot)) = I(f)$ (invariance by left multiplication).

If in addition the group G is compact, then the following properties also hold:

4) For $f = 1$, one has $I(f) = 1$ (normalization),

5) $I(f(\cdot\, a)) = I(f)$ (invariance by right multiplication),

6) $I(f\,\cdot^{-1}) = I(f)$ (invariance under taking the inverse).

One usually writes $I(f) =: \int_G f(a)\mu(\mathrm{d}a)$, where μ denotes the Haar measure on G.

Let us stress that the properties 4), 5) and 6) do not hold in general if the group G is not compact. In this case, Haar measures have to be divided into left Haar measures and right Haar measures. The difference between these measures can be encoded into the so-called *modular function*. In fact, existence of a Haar measure holds for all *locally compact Hausdorff groups*, such groups are more general than Lie groups.

Exercise 3.2.6 (♥). *Study the definitions of locally compact Hausdorff groups, left Haar measures, right Haar measures, and modular functions.*

One of the main interests of the Haar measure for compact groups is that the averaging process mentioned for example in (2.4.1) can be replaced by the average

$$\int_G U(a)\,T\,U(a)^{-1}\mu(\mathrm{d}a) \tag{3.2.1}$$

once the different objects appearing in this expression are defined. For non compact groups, this averaging process is usually not well defined, because of the lack of property 4).

Let us now be precise about linear representations of Lie groups. Clearly, the setting is still the same: a map $U : G \to \mathcal{L}(\mathcal{V})$ or a map $U : G \to \mathcal{B}(\mathcal{H})$ satisfying $U(e) = \mathbb{1}$ and $U(ab) = U(a)\,U(b)$ for any $a, b \in G$. However, we shall impose some continuity properties to the map U, and these properties will depend on the context. In the Hilbert space setting, one usually considers *strongly continuous representations*, meaning that for any fixed $f \in \mathcal{H}$ one has $\|U(a)f - U(a_0)f\| \to 0$ whenever $a \to a_0$. Alternatively, once can also consider *weakly continuous representations*, meaning that for any fixed $f, g \in \mathcal{H}$, one has $\langle f, (U(a)g - U(a_0))g\rangle \to 0$ whenever $a \to a_0$, or *uniformly continuous representations*, meaning that $\|U(a) - U(a_0)\| \to 0$ whenever $a \to a_0$.

Exercise 3.2.7. *Show that a uniformly continuous representation is also a strongly continuous representation, and that a strongly continuous representation is also a weakly continuous representation. If $\mathcal{H}$ is finite dimensional, show that the three notions coincide. In this case, one just speak about a* continuous representation *of G.*

Based on (3.2.1), several results available for any finite group can be extended to any compact Lie group G. We list a few of them:

1) For any continuous representation of G in a finite dimensional Hilbert space $\mathcal{H}$, there exists a new scalar product on $\mathcal{H}$ making this representation unitary, see [13, Thm. VII.9.1]. This result is similar to Proposition 2.2.6 and allows us to consider only unitary representations G on finite dimensional Hilbert spaces,

2) All strongly continuous irreducible representations of G are finite dimensional. This statement is similar to the content of Exercise 2.3.2 for finite group and its proof can be found in [6, Thm. 5.2],

3) Any strongly continuous unitary representation of G is a direct sum of irreducible representations. This statement corresponds for compact Lie groups to Theorem 2.3.4 and to the infinite direct sum mentioned in the last part of Section 2.5. The proof is also provided in [6, Thm. 5.2] or in [13, Thm. VII.10.8],

4) The analogue of formula (2.4.4) holds for G, namely

$$\int_G U^\ell_{rs}(a)\overline{U^k_{ij}(a)}\mu(\mathrm{d}a) = \frac{1}{n_k}\delta_{k\ell}\delta_{sj}\delta_{ri},$$

where $U^\ell_{rs}(a)$ and $U^k_{ik}(a)$ are defined in (2.4.2), with the indices k and ℓ referring to irreducible representations and the other indices indicating the elements of matrices, see [13, Thm. VII.9.5],

5) For characters, Corollary 2.4.4 can be adapted to G and reads

$$\int_G \overline{\chi^k(a)}\chi^\ell(a)\mu(\mathrm{d}a) = \delta_{k\ell}.$$

If $(\mathcal{H}, U)$ is a finite dimensional unitary representation of G, the number of times the representation $(\mathcal{H}^k, U^k)$ appears in its decomposition into irreducible representations is given by $\nu_k = \int_G \overline{\chi(a)}\chi^k(a)\mu(\mathrm{d}a)$, as in Theorem 2.4.6. We refer to [13, Thm. VII.9.5 & Corol. VII.9.6] for these statements,

6) Definition 2.4.7 of the regular representation holds for G, and the first statement of Theorem 2.4.9 holds as well, namely $\mathcal{H}^{\mathrm{reg}} = L^2(G,\mu) = \oplus_k n_k \mathcal{H}^k$ and $U^{\mathrm{reg}} = \oplus_k n_k U^k$. This statement is part of the so-called Peter-Weyl Theorem, see for example [6, Thm. 5.12] or [13, Corol. VII.10.2].

The representations theory of general Lie groups is more involved, since infinite dimensional irreducible representations exist.

3.3 Lie algebras

Let us start this section by a few additional topological definitions.

Definition 3.3.1 (Connected, path-connected, simply connected). *Let $(\mathcal{M}, \mathcal{T})$ be a topological space.*

1) $\mathcal{M}$ is connected *if it is not the disjoint union of two non-empty open sets,*

2) $\mathcal{M}$ is path-connected *if for any $a, b \in \mathcal{M}$ there exists a continuous map $f : [0,1] \to \mathcal{M}$ with $f(0) = a$ and $f(1) = b$,*

3) $\mathcal{M}$ is simply connected *if it is path-connected and if any loop defined by a continuous map $f : [0,1] \to \mathcal{M}$ with $f(0) = f(1)$ can be continuously contracted to a point.*

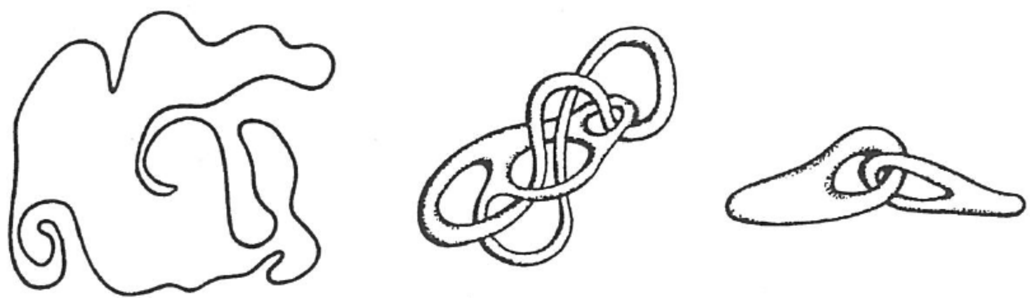

Figure 3.1: Two path connected sets (one simply connected, the other one not simply connected) and a disconnected set.

Some of these notions are represented in Figure 3.1. Be aware that connected but not path-connected sets exist, but they are not easy to exhibit.

In a Lie group G, the *identity component* G_0 is going to play an important role. It is defined as the set of all points which are path-connected to the identity e. One important property of G_0 is stated in the following exercise:

Exercise 3.3.2. *Show that the identity component G_0 of any Lie group is a normal subgroup.*

Before studying the structure of the identity component of G, let us provide the following abstract definition:

Definition 3.3.3 (Lie algebra)**.** *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A* Lie algebra *on $\mathbb{K}$ is a finite dimensional vector space L on $\mathbb{K}$ endowed with a composition rule $[\cdot,\cdot] : L \times L \to L$ satisfying for any $X, Y, Z \in L$ and $\alpha, \beta \in \mathbb{K}$:*

1) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ *(linearity),*

2) $[X, Y] = -[Y, X]$ *(anti-commutativity).*

3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jacobi identity).*

The composition rule $[\cdot,\cdot]$ is called the Lie bracket.

Clearly, it follows from this definition that $[X, X] = 0$, and that the Lie algebra is called *Abelian* or *commutative* if $[X, Y] = 0$ for all $X, Y \in L$. The following statement can also be easily checked:

Exercise 3.3.4. *Let L be a set of $n \times n$ matrices and assume that $[X, Y] := XY - YX \in L$ for any X, Y in L. Check that the above properties then hold.*

If we consider a basis $\{X_1, \ldots, X_d\}$ of L, then any element of L can be expressed as a linear combination of these d elements, and so does the expression $[X_j, X_k]$. Thus, let us set

$$[X_j, X_k] := \sum_{\ell=1}^{d} c_{jk}^{\ell} X_\ell \tag{3.3.1}$$

and call the coefficients $c_{jk}^{\ell} \in \mathbb{K}$ the *structure coefficients of L*. Observe that these coefficients are not independent, since for example $c_{jk}^{\ell} = -c_{kj}^{\ell}$, from the property 2). Additional relations follow from the Jacobi identity, namely

$$\sum_{r=1}^{d} \left(c_{ir}^{s} c_{jk}^{r} + c_{jr}^{s} c_{ki}^{r} + c_{kr}^{s} c_{ij}^{r} \right) = 0, \qquad \forall i, j, k \in \{1, \ldots, n\}. \tag{3.3.2}$$

Note however that these coefficients depend on the choice of the initial basis.

Exercise 3.3.5. *Check* (3.3.2).

For the link between Lie groups and Lie algebras, we shall mainly consider linear Lie groups. Not all Lie groups are linear, but quite many of them belong to this family, and their theory is simpler (less concepts of differential geometry are necessary). First of all, let us consider the groups $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$, and define the distance function d given by

$$\mathrm{d}(A,B) := \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}|a_{jk}-b_{jk}|^2\Big)^{1/2}$$

where $A=(a_{jk})$ and $B=(b_{jk})$ belong to $\mathrm{GL}(n,\mathbb{K})$ with $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Clearly, we use the same notation for the absolute value of a real number or for the modulus of a complex number. Based on this notion we can construct open balls $\mathrm{B}(A,r) := \{C\in\mathrm{GL}(n,\mathbb{K}) \mid \mathrm{d}(C,A)<r\}$ and then a basis for a topology of $\mathrm{GL}(n,\mathbb{K})$. With these topologies, these groups become topological groups, and the notions of open or closed sets are then available.

Definition 3.3.6 (Linear Lie groups)**.** *A* linear Lie group*, or* matrix Lie group *is a closed subgroup of* $\mathrm{GL}(n,\mathbb{R})$ *or of* $\mathrm{GL}(n,\mathbb{C})$*. For linear Lie groups, the identity element is denoted by* $\mathbb{1}$ *instead of* e*.*

As one of the simplest example, observe that the set $\Big\{\begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix} \mid \theta\in\mathbb{R}\Big\}$ defines the Lie group SO(2).

Exercise 3.3.7. *1) Check that the Euclidean group and the Poincaré group, introduced in Section 1.5, can be rewritten as linear Lie groups. The underlying concept is the notion of* affine group.

2) Provide additional example of linear Lie groups.

Observe that the definition of Lie groups provided in Definition 3.2.1 does not correspond to the above one, and the link is not so clear. However, it is a consequence of a rather deep theorem, namely *Cartan's theorem*, or *closed subgroup theorem*

https://en.wikipedia.org/wiki/Closed-subgroup_theorem

that every linear Lie group in the sense of Definition 3.3.6 is a Lie group in the sense of Definition 3.2.1.

The advantage of considering linear Lie groups is that they are very concrete objects, and that a local parametrization of its elements are usually easy to exhibit. Thus, let us consider a linear Lie group $G\subset\mathrm{GL}(n,\mathbb{R})$ of dimension d, and let (V,φ) be a local coordinate system at an element $B\in G$, namely a neighborhood V of B and a local homeomorphic map $\varphi: V\to\mathbb{R}^d$. Because of the underlying structure (and this is not trivial) it turns out that the map $\varphi^{-1}:\varphi(V)\to G\subset\mathrm{GL}(n,\mathbb{R})$ is smooth, which means that the n^2 entries $(\varphi^{-1})_{jk}:\varphi(V)\to\mathbb{R}$ are C^∞ functions. Similarly, if we consider a linear Lie group $G\subset\mathrm{GL}(n,\mathbb{C})$, then the $2n^2$ maps $\Re((\varphi^{-1})_{jk}):\varphi(V)\to\mathbb{R}$ and $\Im((\varphi^{-1})_{jk}):\varphi(V)\to\mathbb{R}$ are C^∞ functions, where we have used $\Re$ and $\Im$ for the real and the imaginary part of a complex number. Since $\varphi(V)\subset\mathbb{R}^d$, we can think about φ^{-1} as a d-dimensional parametrization of the linear Lie group G in a neighborhood of the element $B\in G$. Let us stress that even if G is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ or of $\mathrm{GL}(n,\mathbb{C})$, the dimension d of G is independent of n (but is smaller than or equal to n^2 or to $2n^2$).

Let us now fix the special element $B=\mathbb{1}$ and assume that $\varphi(\mathbb{1})=0\in\mathbb{R}^d$ (this is always possible, by a translation of φ, if necessary). Then for $\ell\in\{1,\dots,d\}$ we set

$$X_\ell := \lim_{t\to 0}\frac{\varphi^{-1}(tE_\ell)-\mathbb{1}}{t}, \tag{3.3.3}$$

where $\{E_\ell\}_{\ell=1}^d$ is the standard basis of $\mathbb{R}^d$ with the vector E_ℓ taking the value 1 at the entry ℓ, and 0 elsewhere. If we think about entries of a matrix, we have

$$(X_\ell)_{jk} := \lim_{t\to 0} \frac{(\varphi^{-1}(tE_\ell))_{jk} - \delta_{jk}}{t}.$$

Clearly, X_ℓ belong to $M_n(\mathbb{R})$ or to $M_n(\mathbb{C})$. Sometimes, the definition provided in (3.3.3) is simply written $X_\ell = [\partial_\ell \varphi^{-1}](0)$.

Let us now list two easy consequences of the previous construction:

1) The d matrices $X_1, \dots, X_d$ are linearly independent on $\mathbb{R}$. These d matrices generate a real vector space (it means that the coefficients for any linear combination are real) which is denoted by $L(G)$. This space is called the *tangent space* of G at $\mathbb{1}$, and has dimension d.

2) If $x : (-\epsilon, \epsilon) \ni t \mapsto x(t) \in \varphi(V) \subset \mathbb{R}^d$ is a smooth parametric curve in $\mathbb{R}^d$, with $x(0) = 0$, then the map $X : (-\epsilon, \epsilon) \ni t \mapsto X(t) := \varphi^{-1}(x(t)) \in G$ with $X(0) = \mathbb{1}$ defines a smooth curve in G, and one has

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \varphi^{-1}(x(t))|_{t=0} = \sum_{\ell=1}^d x'_\ell(0) X_\ell \ \in L(G).$$

   It means that the derivative of any smooth curve at $\mathbb{1}$ belongs to the tangent space $L(G)$, and the set of derivatives of all such curves generates this tangent space.

The next statement contains the link between linear Lie groups and Lie algebras, as introduced in Definition 3.3.3. Recall that the commutator of two matrices has been introduced in Exercise 3.3.4 and is defined by $[X, Y] := XY - YX$.

Proposition 3.3.8. *Let G be a linear Lie group, and let $L(G)$ be the vector space generated by the family $\{X_\ell\}_{\ell=1}^d$ introduced in (3.3.3). Endowed with the commutator $[\cdot, \cdot]$, $L(G)$ turns into a real Lie algebra of dimension d.*

The following proof holds if one assumes that the two maps (defined in the proof below) $t \mapsto A(t)$ and $t \mapsto B(t)$ are analytic. With this requirement, these functions admit a Taylor expansion. In general, smoothness and analyticity are two different concepts, but in the context of groups, any Lie group admits a unique real analytic structure (which is thus implicitly chosen). Note that some authors call *analytic group* any connected Lie group.

Proof. As emphasized in Exercise 3.3.4, one only has to check that $[X_j, X_k]$ belongs to $L(G)$. For that purpose, let us consider two analytic curves on G, namely $t \mapsto A(t)$ and $t \mapsto B(t)$ satisfying $A(0) = B(0) = \mathbb{1}$, $A'(0) = X_j$, and $B'(0) = X_k$, where $A'(t) := \frac{\mathrm{d}}{\mathrm{d}t} A(t)$ and $B'(t) = \frac{\mathrm{d}}{\mathrm{d}t} B(t)$. For example, one can choose $A(t) := \varphi^{-1}(tE_j)$ and $B(t) := \varphi^{-1}(tE_k)$ for $|t|$ small enough and for (V, φ) an analytic local coordinate system at $\mathbb{1}$, but other analytic curves are possible. By performing a Taylor expansion near $t = 0$, observe that $A(\sqrt{t}) = \mathbb{1} + \sqrt{t} X_j + tA_2 + o(t)$ and that $B(\sqrt{t}) = \mathbb{1} + \sqrt{t} X_k + tB_2 + o(t)$, with $A_2 := \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} A(t)|_{t=0}$ and $B_2 := \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} B(t)|_{t=0}$. Accordingly, observe that $A(\sqrt{t})^{-1} = \mathbb{1} - \sqrt{t} X_j + t(X_j^2 - A_2) + o(t)$ and that $B(\sqrt{t})^{-1} = \mathbb{1} - \sqrt{t} X_k + t(X_k^2 - B_2) + o(t)$. Then, by considering the map

$$t \mapsto C(t) := A(\sqrt{t}) B(\sqrt{t}) A(\sqrt{t})^{-1} B(\sqrt{t})^{-1} \ \in G$$

(the image is in G, since it is a group) one observes that $C(t) = \mathbb{1} + t[X_j, X_k] + o(t)$, which means that $\frac{\mathrm{d}}{\mathrm{d}t} C(t)|_{t=0} = [X_j, X_k]$ belongs to the tangent space at $\mathbb{1}$, or equivalently $[X_j, X_k] \in L(G)$. □

Since each linear Lie group defines a Lie algebra, one usually keeps a related name for the two objects. For example, the Lie algebra of $\mathrm{SU}(n)$ is denoted by $\mathrm{su}(n)$, while the Lie algebra of $\mathrm{SO}(n)$ is denoted by $\mathrm{so}(n)$. More generally, the name of the Lie group is written with uppercase letters, while the name of the corresponding Lie algebra is denoted with lowercase letters.

Remark 3.3.9. *So far, only linear Lie groups have been studied, mainly because the definition of* (3.3.3) *would not hold in the general framework of Lie groups. However, let us observe that the use of smooth parametric curves on G for defining the tangent space is available in the general framework of Lie groups, see the above observation 2). In fact, the tangent space at p of any smooth manifold is defined as the vector space generated by the derivative at 0 of smooth curves $(-\epsilon, \epsilon) \ni t \mapsto \gamma(t) \in G$ with $\gamma(0) = p$. Then, in the context of Lie groups, a composition rule can be defined on the tangent space at e, and this turns the tangent space into a real Lie algebra. As a consequence, Proposition 3.3.8 holds even in the general context of Lie groups, once the right notions of tangent space and of Lie bracket are introduced. We refer to [2, Sec. IV.7] for more information on this construction.*

3.4 More relations between Lie groups and Lie algebras

In this section, we further develop some relations between Lie groups and Lie algebras. For simplicity, we still concentrate on linear Lie groups, but most of the results are valid in the more general framework of arbitrary Lie groups. We recall that if $B \in M_n(\mathbb{C})$, one sets

$$\exp(B) := \sum_{j=0}^{\infty} \frac{1}{j!} B^j$$

with $\exp(B) \in M_n(\mathbb{C})$ satisfying the norm estimate $\|\exp(B)\| \leq \mathrm{e}^{\|B\|}$.

Proposition 3.4.1. *Let G be a linear Lie group, and let $L(G)$ be its Lie algebra. Fix $X \in L(G)$ and consider $s, t \in \mathbb{R}$.*

1) The element $\exp(tX)$ belongs to the identity component G_0 of G,

2) The set $\{A(t)\}_{t\in\mathbb{R}}$ with $A(t) := \exp(tX)$ is a 1-parameter family *of elements of G_0, namely the following equalities hold: $A(0) = \mathbb{1}$, $A(s)A(t) = A(s + t)$, and $A(t)^{-1} = A(-t)$,*

3) The previous 1-parameter family is the only one satisfying $\frac{\mathrm{d}}{\mathrm{d}t}A(t)|_{t=0} = X$.

We refer to [3, Thm. 8.3.1] for a proof of this statement. Note that the following equalities also hold, with the notation $A(t) := \exp(tX)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = XA(t) = A(t)X.$$

With the previous statement, one infers that the exponentiation of elements of the Lie algebra generates elements of the identity component of the corresponding linear Lie group. The next statement says that it is possible to generate all elements of G_0, by a suitable procedure. Note that since $L(G)$ is a real vector space, tX belongs to $L(G)$ whenever $X \in L(G)$ and $t \in \mathbb{R}$.

Proposition 3.4.2. *Let G be a linear Lie group, and let $L(G)$ be its Lie algebra.*

1) There exists an open set V in G_0 with $\mathbb{1} \in V$ such that for any $A \in V$, $A = \exp(X)$ for some $X \in L(G)$, *2) For any $A \in G_0$, there exist $X_1, X_2, \dots, X_N \in L(G)$ with $N < \infty$ such that*

$$A = \exp(X_1)\exp(X_2)\dots\exp(X_N), \tag{3.4.1}$$

3) If G is compact, then we can always choose $N = 1$, which means that for any $A \in G_0$, there exists $X \in L(G)$ with $A = \exp(X)$.

The proof of the above result is more involved. We refer for example to [3, Thm. 8.5.VII & 8.5.VIII] and to the references cited therein, or to [1, Sec. 4.2.2].

An easy consequence of 1) is that in a neighborhood V of $\mathbb{1}$ all elements A satisfy $A = B^2$ for some $B \in V$. Observe also that (3.4.1) triggers a natural question: do we have $\exp(X)\exp(Y) = \exp(X + Y)$ for arbitrary elements of $M_n(\mathbb{C})$? The answer is clearly NO, but the following formula holds:

$$\exp(X)\exp(Y) = \exp(Z)$$

with

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\{[X,[X,Y]] + [Y,[Y,X]]\} + \dots, \tag{3.4.2}$$

where the r.h.s. is a series containing commutators of increasingly higher orders. This formula is known as *Campbell-Baker-Hausdorff formula*. Note that an integral version of this formula also exists, which provides a closed formula (without the ...). A systematic presentation is provided in [8, Sec. 3.2–3.5], but the general formula involves the logarithm of a matrix, a concept that we have not introduced so far.

Let us now look at additional relations between linear Lie groups and Lie algebras through representations. Recall that the representation of a group was introduced in Definition 2.2.1. For a Lie algebra, as introduced in Definition 3.3.3 and with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, one sets:

Definition 3.4.3 (Representation of a Lie algebra)**.** *A* representation of a Lie algebra *L consists in a pair $(\mathcal{V}, h)$ with $\mathcal{V}$ a $\mathbb{K}$-vector space, and $h : L \to \mathcal{L}(\mathcal{V})$ a homomorphism, namely h satisfies $h(0) = 0$, $h(\alpha X + \beta Y) = \alpha h(X) + \beta h(Y)$, and*

$$h(X)h(Y) - h(Y)h(X) = h([X,Y])$$

for any $X, Y \in L$ and $\alpha, \beta \in \mathbb{K}$.

The following statement can be proved as an exercise, see also [8, Prop. 4.4].

Lemma 3.4.4. *Let $(\mathcal{V}, U)$ be a representation of a Lie group in a (real or complex) finite dimensional vector space $\mathcal{V}$. Then the map $\Gamma : L(G) \to \mathcal{L}(\mathcal{V})$ given for any $X \in L(G)$ by*

$$\Gamma(X) := \frac{\mathrm{d}}{\mathrm{d}t} U\big(\exp(tX)\big)\Big|_{t=0} \tag{3.4.3}$$

defines a representation of its Lie algebra $L(G)$. In addition, the following equality holds for any $t \in \mathbb{R}$:

$$\exp\big(t\Gamma(X)\big) = U\big(\exp(tX)\big).$$

It is then natural to wonder if a converse statement is true, but clearly it can not be. Indeed, the Lie algebra provides only information on the identity component G_0 of the corresponding Lie group G. More precisely, since two linear Lie groups which are isomorphic in a neighborhood of their respective identity, have isomorphic Lie algebras[2], any representation of these Lie algebras won't be able to provide distinct information on

[2]Two Lie algebras L_1, L_2 are isomorphic if there exists a bijective linear map $\phi : L_1 \to L_2$ satisfying $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for any $X, Y \in L_1$. We then write $L_1 \simeq L_2$.

the two corresponding groups. However, a partial converse is true for connected linear Lie groups, see [8, Prop. 4.5] for a precise statement.

Let us now mention a very important application of Lie algebras. Assume that the evolution of a quantum system is described by the unitary evolution group $\{e^{-itH}\}_{t\in\mathbb{R}}$, where H is a self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that there exists a Lie group G with a unitary representation $(\mathcal{H}, U)$ commuting with e^{-itH} for all $t \in \mathbb{R}$, namely $U(a)e^{-itH} = e^{-itH}U(a)$ for all $t \in \mathbb{R}$ and $a \in G$. Then any $X \in L(G)$ defines a *constant of motion*. More precisely, for any $X \in L(G)$ the operator $\Gamma(X)$ defined in (3.4.3) satisfies $e^{-itH}\Gamma(X)e^{itH} = \Gamma(X)$, meaning that this operator is constant under the evolution. Note that if $\mathcal{H}$ is infinite dimensional, the operator $\Gamma(X)$ can be unbounded, and some care is necessary for the previous equality. Observe also that if the Lie group G is d-dimensional, then there exist d independent constants of motion.

Let us gather below a few remarks about the linear Lie groups $O(3)$, $SO(3)$, $U(2)$ which appear in several contexts. Recall that a surjective homomorphism $\phi : SU(2) \to SO(3)$ has been introduced in Proposition 1.2.16, with kernel $\{\mathbb{1}, -\mathbb{1}\}$.

The following properties can be checked, see various sections and examples in [3] and in [8], or in [1, Sec. 4.3]:

1) The three groups $O(3)$, $SO(3)$, $U(2)$ are compact linear Lie groups of dimension 3,

2) $O(3)$ is not connected,

3) $SO(3)$ is connected but not simply connected,

4) $SU(2)$ is simply connected,

5) $SO(3)$ and $SU(2)$ are isomorphic near the identity, which means that $L(SO(3))$ is isomorphic to $L(SU(2))$,

6) With the Pauli matrices σ_j introduced in (1.2.2), the set $\{X_j\}_{j=1}^3$, with $X_j := -\frac{i}{2}\sigma_j$, defines the Lie algebra of $su(2)$ and verifies $[X_j, X_k] := \varepsilon_{jk\ell}X_\ell$, where

$$\varepsilon_{jk\ell} := \begin{cases} 1 & \text{if } (j,k,\ell) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (j,k,\ell) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{otherwise.} \end{cases}$$

It means that the structure coefficients for $su(2)$ are given by $c_{jk}^{\ell} = \varepsilon_{jk\ell}$. Similarly, the three matrices

$$Y_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Y_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4.4}$$

define the Lie algebra $so(3)$ and satisfy the same commutation relations, namely $[Y_j, Y_k] := \varepsilon_{jk\ell}Y_\ell$.

Exercise 3.4.5. *Prove part of or all the above statements.*

Exercise 3.4.6 (♥)**.** *Study the finite dimensional representations of* $SU(2)$*, or equivalently the finite dimensional representation of* $su(2)$*. There exists a unique irreducible representation of* $SU(2)$ *(up to equivalence) in each vector space* $\mathbb{C}^n$*.*

3.5 Complexification

In this short section, we describe an importance construction, the complexification of real Lie algebras.

Let us firstly stress that despite the appearance of the factor i in the statement 6) above for the Lie algebra of su(2), it does not mean that it is a complex Lie algebra. It is still a real Lie algebra, as it is the case for all Lie algebras obtained from Lie groups. On the other hand, it is often useful to *complexify* a real Lie algebra. We start with the simplest situation: Let $\{X_1, \dots, X_d\}$ be a basis for a real Lie algebra, and assume that these elements are also linearly independent over $\mathbb{C}$. In this case, one can directly consider complex linear combinations of these d elements, and one obtains a complex Lie algebra of the same dimension and with the same structure coefficients. This procedure applies for example to the Lie algebra su(2), with a basis given by the set

$$\left\{\frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right\}. \tag{3.5.1}$$

Let us however observe that this procedure does not always lead to a new object. Consider for example the real Lie algebra $\mathrm{sl}(n, \mathbb{C})$ given by the set of matrices

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}\right\}.$$

Clearly, these matrices are linearly independent over $\mathbb{R}$ (and generate the real Lie algebra) but they are not linearly independent over $\mathbb{C}$. In this case, the previous construction does not produce anything interesting. When the elements of a real Lie algebra L are not linearly independent over $\mathbb{C}$, the complexification of L is slightly more involved: It consists of pairs $(X, Y) \in L \times L$ with the addition defined componentwise, with the multiplication by complex numbers given by $(\alpha + i\beta)(X, Y) := (\alpha X - \beta Y, \alpha Y + \beta X)$ for any $\alpha, \beta \in \mathbb{R}$, and with the composition rule defined by

$$[(X_1, Y_1), (X_2, Y_2)] := ([X_1, X_2] - [Y_1, Y_2], [X_1, Y_2] + [Y_1, X_2]).$$

With these structures, the set $L \times L$ defines a complex Lie algebra, usually denoted by $L_{\mathbb{C}}$ and is called the complexification of L. Note that this construction is also available if the initial vectors are linearly independent over $\mathbb{C}$.

In the following exercise, it is asked to show that $L_{\mathbb{C}}$ is indeed a Lie algebra, but also to show the equivalence of the two constructions if the elements of L are linearly independent over $\mathbb{C}$.

Exercise 3.5.1. *1) Show that the above construction leads to a complex Lie algebra $L_{\mathbb{C}}$ of the same dimension as the initial real Lie algebra L. Show that a basis of $L_{\mathbb{C}}$ is given by $\{(X_\ell, 0)\}_{\ell=1}^d$, if $\{X_\ell\}_{\ell=1}^d$ is a basis of L.*

2) If the element of L are linearly independent over $\mathbb{C}$, set $\phi\big((X, Y)\big) := X + iY$ for any $(X, Y) \in L_{\mathbb{C}}$ and show that ϕ defines an isomorphism of complex Lie algebra between $L_{\mathbb{C}}$ and the initial construction provided, see also [5, Sec. 13.3].

Consider a complex Lie algebra L. We say that L' is a *real form* of L if the complexification of L' is isomorphic to L. Note that not all complex Lie algebras have a real form, and two real forms of a complex Lie algebra can be non isomorphic. For example, $\mathrm{u}(n)_{\mathbb{C}}$ and $\mathrm{gl}(n, \mathbb{R})_{\mathbb{C}}$ are both complex algebras isomorphic to $\mathrm{gl}(\mathrm{n}, \mathbb{C})$, but the algebras $\mathrm{u}(n)$ and $\mathrm{gl}(n, \mathbb{R})$ are not isomorphic for $n \geq 2$.

Exercise 3.5.2. *Prove the previous statement about* $\mathrm{u}(n)_{\mathbb{C}}$, $\mathrm{gl}(n, \mathbb{R})_{\mathbb{C}}$, *and* $\mathrm{gl}(\mathrm{n}, \mathbb{C})$.

Chapter 4

Semi-simple theory

In this chapter, we further develop the theory for semi-simple Lie groups and Lie algebras. Several important Lie groups are of this form, as already seen in Chapter 1.

4.1 Simple, semi-simple Lie groups and Lie algebras

We gather in this first section the main definitions related to simplicity and semi-simplicity.

Definition 4.1.1 (Lie subalgebra, invariant Lie subalgebra)**.** *Let L be a Lie algebra over $\mathbb{K}$.*

1) A Lie subalgebra *of L consists in a subset of L which is itself a Lie algebra over $\mathbb{K}$ with the same Lie bracket,*

2) A Lie subalgebra L' of L is invariant *if $[X, Y] \in L'$ whenever $X \in L'$ and $Y \in L$.*

In the context of Lie algebras, a Lie subalgebra will simply be called a subalgebra. Note that an invariant subalgebra of L is also referred to as an *ideal*. One example of an Abelian ideal is provided by the next definition:

Definition 4.1.2 (Center)**.** *The* center *of a Lie algebra L is defined by $\{Y \in L \mid [Y, X] = 0$ for all $X \in L\}$.*

In order to make a link between Lie subalgebras and subgroups of Lie groups, let us recall that the notion of subgroup has been introduced in Definition 1.1.6. For Lie groups, this notion has to be strengthened, namely a Lie subgroup G' of a Lie group G consists in a subgroup of G which is itself a Lie group and such that the inclusion map $G' \hookrightarrow G$ is an immersion[1]. By the already mentioned theorem of Cartan, any closed subgroup of a Lie group is a Lie group, but not all Lie subgroups are of this form. In fact, the latter situation corresponds to embedding instead of an immersion. For the link with Lie subalgebras, one has (see [9, Sec. 9.1] or [1, Lem. 5.14]) :

Proposition 4.1.3. *Let G be a connected Lie group. A Lie subgroup G_0 is normal if and only if the corresponding Lie algebra $L(G_0)$ is an invariant subalgebra of $L(G)$.*

Let us now move to the definition of simplicity and semi-simplicity for a Lie algebra.

[1]The notions of *immersion* and *embedding* are central concepts of differential geometry, and the difference is rather subtle. We refer to https://en.wikipedia.org/wiki/Immersion_(mathematics) for more information, or to any book on differential geometry.

Definition 4.1.4 (Simple, semi-simple Lie algebra)**.** *1) A Lie algebra is* simple *if it is not Abelian and does not possess a proper and non-trivial invariant subalgebra,*

2) A Lie algebra is semi-simple *if it does not possess a non-trivial Abelian invariant subalgebra.*

Observe that a semi-simple algebra can not be Abelian, because the any algebra is an invariant subalgebra of itself. The related notions of simple or semi-simple groups have already been introduced in Definition 1.2.6. Note that the following definitions are not universally accepted and differ slightly depending on the authors.

Definition 4.1.5 (Simple, semi-simple Lie group)**.** *1) A connected Lie group G is a* simple Lie group *if G is non-Abelian, and $\{e\}$ is the only proper and normal, connected and closed subgroup,*

2) A connected Lie group G is a semi-simple Lie group *if G is non-Abelian, and $\{e\}$ is the only proper and normal, connected and closed Abelian subgroup.*

Let us emphasize an important technical point: a simple Lie group may contain discrete normal subgroups, hence being a simple Lie group is different from being simple as an abstract group, in the sense of Definition 1.2.6. The same remark applies to semi-simple Lie groups.

In the next statement, we provide links between Lie groups and Lie algebras, and also mention a relation with complexification introduced in Section 3.5. We refer to [5, Sec. 13.3] for the proof.

Theorem 4.1.6. *1) A connected Lie group group is a simple Lie group if and only if its Lie algebra is simple,*

2) A connected Lie group group is a semi-simple Lie group if and only if its Lie algebra is semi-simple,

3) A real Lie algebra L is semi-simple if and only if its complex Lie algebra $L_{\mathbb{C}}$ is semi-simple,

4) For a real Lie algebra L, if its complexification $L_{\mathbb{C}}$ is simple, then L is also simple.

Additional information on the structure of Lie groups and Lie algebras will be provided in the subsequent sections

4.2 Adjoint representation and Killing form

Let L be a Lie algebra over $\mathbb{K}$, and let us consider the map ad defined by

$$\mathrm{ad} : L \ni X \mapsto \mathrm{ad}_X \in \mathcal{L}(L)$$

with $\mathrm{ad}_X(Y) := [X, Y]$. Clearly, the following properties hold, for any $X, X' \in L$ and $\beta \in \mathbb{K}$

1) $\mathrm{ad}_0 = 0$,

2) $\mathrm{ad}_{X+\beta X'} = \mathrm{ad}_X + \beta\,\mathrm{ad}_{X'}$,

3) $\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X = [\mathrm{ad}_X, \mathrm{ad}_Y]$.

Exercise 4.2.1. *Check that the above properties hold, and that* ad *defines a representation of L in $\mathcal{L}(L)$ in the sense provided in Definition 3.4.3. Check that 3) can also be rewritten as*

$$\mathrm{ad}_X([Y, Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]. \tag{4.2.1}$$

This representation of the Lie algebra L in $\mathcal{L}(L)$ is called the *adjoint representation*. In other words, the set $\{\mathrm{ad}_X \mid X \in L\}$ is a vector space over $\mathbb{K}$, and once endowed with the composition rule $[\mathrm{ad}_X, \mathrm{ad}_Y] := \mathrm{ad}_{[X,Y]}$, it becomes a Lie algebra over $\mathbb{K}$, called the *adjoint Lie algebra*.

Since L, and consequently $\mathcal{L}(L)$, are finite dimensional, the following definition is meaningful:

Definition 4.2.2 (Killing[2] form)**.** *The* Killing form *of L consists of the symmetric bilinear map $K : L\times L \to \mathbb{C}$ defined by $K(X,Y) := \mathrm{Tr}(\mathrm{ad}_X\,\mathrm{ad}_Y)$ for any $X,Y \in L$.*

Some properties of the Killing form are gathered in the following exercise:

Exercise 4.2.3. *Check the following properties of the Killing form:*

1) If $\{X_1,\dots,X_d\}$ is a basis of L with structure coefficients c^ℓ_{jk}, as defined in (3.3.1), then the following equalities hold:

$$g_{jk} := K(X_j,X_k) = \sum_{r,s=1}^{d} c^s_{jr}\,c^r_{ks}, \tag{4.2.2}$$

2) $K([X,Y],Z) = K(X,[Y,Z])$ for any $X,Y,Z \in L$,

3) The following statements are equivalent:

(a) $K(Y,X) = 0$ for all $Y \in L \Rightarrow X = 0$,

(b) $\mathrm{Det}((g_{jk})) \neq 0$,

4) The property $\mathrm{Det}((g_{jk})) \neq 0$ is independent of the initial choice for the basis chosen for defining the structure coefficients c^ℓ_{jk}.

If the property $\mathrm{Det}((g_{jk})) \neq 0$ holds, we say that the Killing form is *non-degenerate*. In fact, this property is very important, as seen in the following statement. The subsequent results are all borrowed from [3, Sec. 11.2] to which we refer for the proofs.

Theorem 4.2.4 (Cartan's criterion)**.** *A Lie algebra L is semi-simple if and only if its Killing form is non-degenerate.*

For the next statement (whose proof relies on the Killing form), we use the notion of irreducible representation of a Lie algebra by analogy to the representation of a group, see Definition 2.3.1: The representation is irreducible if $\{0\}$ or the full space are the only invariant subspaces.

Lemma 4.2.5. *If L is a semi-simple Lie algebra, its adjoint representation is faithful, namely $\mathrm{ad}_X \neq \mathrm{ad}_Y$ whenever $X \neq Y$. In addition, if L is simple, then its adjoint representation is irreducible.*

The proof of the following statement can be found in [5, Sec. 14.2] and in the corresponding appendix.

Lemma 4.2.6 (Weyl's lemma)**.** *A semi-simple connected Lie group G is compact if and only if the Killing form of its Lie algebra $L(G)$ is negative definite, namely if and only if $K(X,X) < 0$ for all $X \in L(G)$ with $X \neq 0$.*

The previous results can be used in particular for showing that $\mathrm{SU}(n)$ is a semi-simple compact Lie group for any $n \geq 2$.

Exercise 4.2.7 (♥)**.** *For $\mathrm{su}(n)$, show that $K(X,Y) = 2n\mathrm{Tr}(XY)$ for any $X,Y \in \mathrm{su}(n)$. Check also that $\mathrm{su}(n) = \{X \in M_n(\mathbb{C}) \mid X = -X^* \text{ and } \mathrm{Tr}(X) = 0\}$. Deduce from these results and from the above statements that $\mathrm{SU}(n)$ is a semi-simple compact Lie group.*

[2] Wilhelm Karl Joseph Killing (10 May 1847–11 February 1923) was a German mathematician who made important contributions to the theories of Lie algebras, Lie groups, and non-Euclidean geometry.

Let us close with section with one more result about the structure of semi-simple Lie algebras, showing that their study reduces to the study of simple Lie algebras. It corresponds to [5, Thm. VI p. 488] with a proof given in the appendix E of this reference.

Proposition 4.2.8. *Every semi-simple Lie algebra is either simple or is the direct sum of a finite set of simple Lie algebras, that is*

$$L = L_1 \oplus L_2 \oplus \ldots \oplus L_N$$

with L_j simple Lie algebras. Moreover, this decomposition is unique.

4.3 Roots of complex semi-simple Lie algebras

Recall that any real Lie algebra can be complexified, and that the adjoint representation of a Lie algebra L is a representation of L taking place in $\mathcal{L}(L)$. In particular, for any $X \in L$ with $X \neq 0$, the map $\mathrm{ad}_X : L \to L$ is a linear map, and since L is finite dimensional, we can look for eigenvalues and eigenvectors of ad_X. Namely, we look for $\lambda \in \mathbb{C}$ and $Y \in L$ such that the equality

$$\mathrm{ad}_X(Y) = \lambda Y \Longleftrightarrow [X, Y] = \lambda Y$$

holds. Note that 0 is always an eigenvalue, with corresponding eigenvectors X, since the equalities

$$\mathrm{ad}_X(X) = [X, X] = 0 = 0\, X$$

always hold.

Exercise 4.3.1. *Recall that for any element of $M_n(\mathbb{C})$, there exist d eigenvalues (multiplicity included), which correspond to the roots of the characteristic polynomial. However, be aware that it does not mean that there exist d eigenvectors. Study the notion of generalized eigenvectors for an arbitrary matrix in $M_n(\mathbb{C})$, and also the Jordan normal form of this matrix. Observe that not all elements of $M_n(\mathbb{C})$ are diagonalizable, and provide an example of a matrix which is not diagonalizable.*

Let us consider a special subalgebra of any complex semi-simple Lie algebra.

Definition 4.3.2 (Cartan subalgebra)**.** *Let L be a complex semi-simple Lie algebra. A* Cartan subalgebra *L_0 of L is a maximal Abelian subalgebra of L such that for all $X \in L_0$ the linear maps ad_X are simultaneously diagonalizable. In other words, L_0 is a complex subspace of L such that the following conditions hold:*

1) If $X_1, X_2 \in L_0$, then $[X_1, X_2] = 0$, (Abelian subalgebra)

2) If $[X, Y] = 0$ for all $X \in L_0$, then $Y \in L_0$, (maximality)

3) For any $X \in L_0$, the linear map ad_X is diagonalizable (diagonalization)

In fact, one easily check that $[X_1, X_2] = 0$ implies that $[\mathrm{ad}_{X_1}, \mathrm{ad}_{X_2}] = 0$. Then, this commutation relation and the fact that all ad_X can be diagonalized, imply that they can be diagonalized simultaneously.

It can be shown that any complex semi-simple Lie algebra possesses at least one Cartan subalgebra, and that if it possesses more than one, then all of them are isomorphic. In particular, their dimension is the same. This property leads to the following definition:

Definition 4.3.3 (Rank)**.** *The* rank *of a complex semi-simple Lie algebra is defined as the dimension of any of its Cartan subalgebra. If the rank of the complex semi-simple Lie algebra is denoted by d, then its rank is denoted by d_0.*

Let us now fix a Cartan subalgebra L_0 of a complex semi-simple Lie algebra, and let $\{Y_1, \dots, Y_d\}$ be a basis of L satisfying $\mathrm{ad}_X(Y_j) := \lambda_j(X)\, Y_j$ for any $X \in L_0$. This condition means that the basis of L is chosen according to the diagonalization of the linear maps ad_X for any $X \in L_0$. Then, observe that for any $X, X' \in L_0$ and any $\beta \in \mathbb{C}$ one has

$$\mathrm{ad}_{X+\beta X'}(Y_j) = [X + \beta X', Y_j] = [X, Y_j] + \beta[X', Y_j] = \lambda_j(X)Y_j + \beta\lambda_j(X')Y_j = (\lambda_j(X) + \beta\lambda_j(X'))Y_j. \tag{4.3.1}$$

However, since L_0 is also a complex vector space, then $X+\beta X'$ belongs to L_0 and $\mathrm{ad}_{X+\beta X'}(Y_j) = \lambda_j(X+\beta X')Y_j$. These two expressions imply that

$$\lambda_j(X + \beta X') = \lambda_j(X) + \beta\lambda_j(X'),$$

or in other words the map $\lambda_j : L_0 \to \mathbb{C}$ is linear, for any $j \in \{1, \dots, d\}$. Note that the set of all linear maps from L_0 to $\mathbb{C}$ is called the *dual space* of L_0, and is usually denoted by L_0^*. Thus, we have obtained that $\lambda_j \in L_0^*$ for any $j \in \{1, \dots, d\}$.

Remark 4.3.4. *Since $\mathrm{ad}_X(Y) = 0$ for any $X, Y \in L_0$, it is possible to choose the basis $\{Y_1, \dots, Y_d\}$ such that $Y_1, \dots, Y_{d_0}$ belong to L_0, and $Y_{d_0+1}, \dots, Y_d$ do not belong to L_0. Thus, for $j \in \{1, \dots, d_0\}$ one has $\lambda_j(X) = 0$ for all $X \in L_0$. On the other hand, for any $j \in \{d_0 + 1, \dots, d\}$, the map $L_0 \ni X \mapsto \lambda_j(X) \in \mathbb{C}$ can not be the 0-map, since otherwise the maximality of the Cartan subalgebra would be violated. As a consequence, there exist $d - d_0$ elements of L_0^* which are not the 0-maps.*

Let us try to make the above construction less dependent on the choice of a basis, and make it more abstract.

Definition 4.3.5 (Root)**.** *For a complex semi-simple Lie algebra L with Cartan subalgebra L_0, a* root *of L is an element $\alpha \in L_0^*$ with α not the 0-map, such that there exists $Y_\alpha \in L$, $Y_\alpha \neq 0$, with $\mathrm{ad}_X(Y_\alpha) = \alpha(X)Y_\alpha$ for all $X \in L_0$. The set of all roots is denoted by $\mathcal{R}$.*

Clearly, the notion of root corresponds to a generalization of an eigenvalue, when several linear maps are commuting. More precisely, Y_α is a common eigenvector for all ad_X with $X \in L_0$, and the corresponding eigenvalues are $\alpha(H)$. Note that if we fix a basis $X_1, \dots, X_{d_0}$ of L_0, then α is fully determined by the d_0 complex values $\alpha(X_1), \dots, \alpha(X_{d_0})$.

Now, for any $\alpha \in \mathcal{R}$ we set

$$L_\alpha := \{Y \in L \mid \mathrm{ad}_X(Y) = \alpha(X)Y,\ \forall X \in L_0\}, \tag{4.3.2}$$

and call it the *root subspace* associated with the root α. Since the linear maps ad_X commute, for all $X \in L_0$, and can be diagonalized simultaneously, one infers that

$$L = L_0 \oplus \bigoplus_{\alpha \in \mathcal{R}} L_\alpha.$$

Note that we can not say that these direct sums are orthogonal direct sums, since no scalar product has been introduced so far. In this representation the linear map ad_X takes the form

$$\mathrm{ad}_X = 0 \oplus \bigoplus_{\alpha \in \mathcal{R}} \alpha(X)\mathbb{1}.$$

Exercise 4.3.6. *Check the above statement.*

For the next statement, we write L_α as defined in (4.3.2) if $\alpha \in \mathcal{R}$, $L_0 := L_0 \dots$ ☺, and $L_\alpha = \{0\}$ if $\alpha \notin \mathcal{R}$ and $\alpha \neq 0$.

Lemma 4.3.7. *For any $\alpha, \beta \in L_0^*$ and for $Y_\alpha \in L_\alpha$ and $Y_\beta \in L_\beta$ one has $[Y_\alpha, Y_\beta] \in L_{\alpha+\beta}$.*

Let us be more explicit about the content of this lemma. Clearly, if α or β are not roots or 0, then $[Y_\alpha, Y_\beta] = 0$. This equality also holds if α and β are roots, but if $\alpha + \beta$ is not a root or 0. On the other hand, if α and $-\alpha$ are roots, then $[Y_\alpha, Y_{-\alpha}] \in L_0$.

Exercise 4.3.8. *Prove the previous lemma by using the Jacobi identity, see also [8, Prop. 6.18].*

We now add one more result about roots. The proof is given in [8, Prop. 6.19 & Thm. 6.20], but involves a few tools not introduced in these notes.

Proposition 4.3.9. *Let L be complex semi-simple Lie algebra of rank d_0.*

1) If $\alpha \in \mathcal{R}$, then $-\alpha \in \mathcal{R}$,

2) If $\alpha \in \mathcal{R}$, then the only multiples of α that are roots are α and $-\alpha$,

3) If $\alpha \in \mathcal{R}$, then L_α is one dimensional,

4) There exist $d - d_0$ different roots,

5) The roots span L_0^*,

Note that there are a lot of algebraic relations on the set of roots and on the choices of bases, and that this theory is very well developed. It would be too long to present these results, we simply mention one main outcome: It is possible to endow L with a basis having some specific properties. More precisely, there exists a *standard basis* $\{H_1, \dots, H_{d_0}, E_\alpha, E_{-\alpha}, \dots, E_\gamma, E_{-\gamma}\}$ of L having the following properties:

1) $[H_i, H_j] = 0$ for all $i, j \in \{1, \dots, d_0\}$,

2) $[H_j, E_\alpha] = \mathrm{ad}_{H_j}(E_\alpha) = \alpha(H_j)E_\alpha$ with $\alpha(H_j) \in \mathbb{R}$,

3) $[E_\alpha, E_\beta] = \begin{cases} \sum_{j=1}^{d_0} \alpha(H_j)H_j & \text{if } \alpha + \beta = 0, \\ \tau_{\alpha\beta}E_{\alpha+\beta} & \text{if } \alpha + \beta \in \mathcal{R}, \quad (\text{where } \tau_{\alpha\beta} \neq 0) \\ 0 & \text{otherwise.} \end{cases}$

In addition, if the roots are arranged in the order $\alpha, -\alpha, \beta, -\beta, \dots$, then the matrix (g_{jk}) given by $g_{jk} = \mathrm{Tr}(\mathrm{ad}_{X_j}\mathrm{ad}_{X_k})$ with $X_j, X_k \in \{H_1, \dots, H_{d_0}, E_\alpha, \dots, E_{-\gamma}\}$ takes the form

$$(g_{jk}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.3.3}$$

We refer to [1, Sec. 5.4.3] for this construction.

In the above basis, all $\alpha(H_j) \in \mathbb{R}$ for $j \in \{1, \dots, d_0\}$, which means that roots can be identified with elements of $\mathbb{R}^{d_0}$. We then say that a root α is *positive* if the first non-zero of the list $(\alpha(H_{d_0}), \alpha(H_{d_0-1}), \dots, \alpha(H_1))$ is positive, and the root is negative otherwise. Thus, the set of roots can be divided into two sets: the set of positive roots $\mathcal{R}_+$ and the set of negative roots $\mathcal{R}_-$ satisfying $\mathcal{R} = \mathcal{R}_- \cup \mathcal{R}_+$ and $\mathcal{R}_- \cap \mathcal{R}_+ = \emptyset$. We can also endow the set of roots with the *lexicographic order*, namely $\alpha > \beta$ if $\alpha - \beta \in \mathcal{R}_+$. Some positive roots are playing a special role:

Definition 4.3.10 (Simple root)**.** *With respect to the standard basis introduced above, a root is* simple *if it is positive and can not be expressed as a linear combination(with positive coefficients) of other positive roots.*

We now provide a final statement about roots, and refer to [1, Prop. 5.33] for its proof.

Theorem 4.3.11. *Let L be a complex semi-simple Lie algebra of dimension d and of rank d_0, and endowed with the standard basis introduced above.*

1) There are exactly d_0 simple roots $\alpha^1, \dots \alpha^{d_0}$,

2) These d_0 simple roots are linearly independent and general L_0^*,

3) For any $\beta \in \mathcal{R}$ there exist $a_1, \dots a_{d_0} \in \mathbb{Z}$ with $\beta = \sum_{j=1}^{d_0} a_j \alpha^j$, and either all $a_j > 0$ or all $a_j < 0$,

4) If α, β are simple roots, then $\alpha - \beta$ is not a root,

Let us now illustrate the above construction with two examples, and a few exercises.

Exercise 4.3.12. *Show that* $\mathrm{su}(n) = \{X \in M_n(\mathbb{C}) \mid X = -X^*$ *and* $\mathrm{Tr}(X) = 0\}$*, and that its dimension d is $n^2 - 1$. Show also that the elements of this Lie algebra are linearly independent over $\mathbb{C}$. Since a Cartan subalgebra of* $\mathrm{su}(n)_{\mathbb{C}}$ *consists of d_0 matrices simultaneously diagonalizable, and since all elements of* $\mathrm{su}(n)_{\mathbb{C}}$ *have a trace equal to* 0*, deduce that the rank d_0 of* $\mathrm{su}(n)_{\mathbb{C}}$ *is equal to $n - 1$.*

In the special case $\mathrm{su}(2)_{\mathbb{C}}$, one basis has been exhibited in (3.5.1), but it is clearly not a standard basis. In this case, observe that the rank of $\mathrm{su}(2)_{\mathbb{C}}$ is 1, and one can choose:

$$H := \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & -\frac{1}{2\sqrt{2}} \end{pmatrix}, \quad E_\alpha := \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} := \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

In this standard basis, one has $\alpha(H) = \frac{1}{\sqrt{2}}$, and $-\alpha(H) = -\frac{1}{\sqrt{2}}$, and

$$(g_{jk}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $\mathrm{su}(3)_{\mathbb{C}}$, its dimension is 8 and its rank is 2. Thus, there exist 6 different roots in $\mathcal{R}$. The standard basis is given by

$$H_1 := \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 := \frac{1}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E_\alpha := \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-\alpha} := \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$E_\beta := \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-\beta} := \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_\gamma := \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-\gamma} := \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.3.4}$$

In this basis, the matrix (g_{jk}) possesses the standard form of (4.3.3), and the roots are

$$\alpha = \Big(\frac{1}{2\sqrt{3}}, \frac{1}{2}\Big), \quad \beta = \Big(-\frac{1}{2\sqrt{3}}, \frac{1}{2}\Big), \quad \gamma = \Big(\frac{1}{\sqrt{3}}, 0\Big).$$

together with $-\alpha, -\beta, -\gamma$. Note that only β and γ are simple roots, since $\alpha = \beta + \gamma$.

4.4 Weights of complex semi-simple Lie algebras

Let L be a complex semi-simple Lie algebra, and let $(\mathcal{V}, h)$ be a finite dimensional representation of L, as introduced in Definition 3.4.3. As for the adjoint representation we look for elements $v \in \mathcal{V}$, with $v \neq 0$, such that the equality $h(H)v = \mu(H)v$ holds for all H in the Cartan subalgebra L_0 of L, and where $\mu(H) \in \mathbb{C}$. If such an element v exists, it is called a *weight vector* of the representation, and the map $\mu : L_0 \to \mathbb{C}$ is called a *weight* of the representation. By a computation similar to (4.3.1), one easily infers that $\mu \in L_0^*$, which means that μ is a linear map on L_0. More generally, for any $\mu \in L_0^*$ we can set

$$L_\mu := \{v \in \mathcal{V} \mid h(H)v = \mu(H)v, \ \forall H \in L_0\},$$

and the dimension of L_μ is called the *multiplicity of the weight* μ. Observe that this construction is in fact a generalization of the one provided for roots, which correspond to weights for the adjoint representation ($\mathcal{V} = L$ and $h = \mathrm{ad}.$).

Let us consider again the standard basis $\{H_1, \dots, H_{d_0}, E_\alpha, E_{-\alpha}, \dots, E_\gamma, E_{-\gamma}\}$ of L, and set $\mathfrak{H}_j := h(H_j)$, $\mathfrak{E}_\alpha := h(E_\alpha)$. Then the following relations hold:

1) $[\mathfrak{H}_i, \mathfrak{H}_j] = 0$ for all $i, j \in \{1, \dots, d_0\}$,

2) $[\mathfrak{H}_j, \mathfrak{E}_\alpha] = \alpha(H_j)\mathfrak{E}_\alpha$ with $\alpha(H_j) \in \mathbb{R}$,

3) $$[\mathfrak{E}_\alpha, \mathfrak{E}_\beta] = \begin{cases} \sum_{j=1}^{d_0} \alpha(H_j)\mathfrak{H}_j & \text{if } \alpha + \beta = 0, \\ \tau_{\alpha\beta}\mathfrak{E}_{\alpha+\beta} & \text{if } \alpha + \beta \in \mathcal{R}, \quad (\text{where } \tau_{\alpha\beta} \neq 0) \\ 0 & \text{otherwise.} \end{cases}$$

The main difference with the same relations for the elements of the basis $\{H_1, \dots, H_{d_0}, E_\alpha, E_{-\alpha}, \dots, E_\gamma, E_{-\gamma}\}$ of L is that the linear independence of the elements $\{\mathfrak{H}_1, \dots, \mathfrak{H}_{d_0}, \mathfrak{E}_\alpha, \mathfrak{E}_{-\alpha}, \dots, \mathfrak{E}_\gamma, \mathfrak{E}_{-\gamma}\}$ is no more ensured, it depends on the representation.

We now stress the role of roots for any representation.

Proposition 4.4.1. *Let L be a complex semi-simple Lie algebra endowed with the standard basis, and let $(\mathcal{V}, h)$ be a finite dimensional representation of L. Let μ be a weight with a weight vector $v \in L_\mu$.*

1. *$\mu(H_j) \in \mathbb{R}$ for any $j \in \{1, \dots, d_0\}$,*
2. *For any $\alpha \in \mathcal{R}$, if $\mathfrak{E}_\alpha v \neq 0$, then $\mathfrak{E}_\alpha v \in L_{\mu+\alpha}$ and $\mu + \alpha$ is a weight,*
3. *The weight vectors associated with different weights are linearly independent,*
4. *The number of different weights for $(\mathcal{V}, h)$ is at most equal to $\dim(\mathcal{V})$.*

The first statement corresponds in fact to a corollary of Proposition 4.4.2 but we prefer to present it immediately, for simplicity. Let us prove the second statement, the third one being slightly more involved, and the fourth one following directly from the third one. One has for any $H \in L_0$:

$$h(H)\mathfrak{E}_\alpha v = \mathfrak{E}_\alpha h(H)v + [h(H), \mathfrak{E}_\alpha]v = \mu(H)\mathfrak{E}_\alpha v + \alpha(H)\mathfrak{E}_\alpha v = (\mu(H) + \alpha(H))\mathfrak{E}_\alpha v,$$

which means precisely that $\mathfrak{E}_\alpha v \in L_{\mu+\alpha}$. Because of these relations, for α or for $-\alpha$, the operators $\mathfrak{E}_\alpha$ is often called a *raising operator* and $\mathfrak{E}_{-\alpha}$ a *lowering operator*.

We now consider a root α and a weight μ, and define

$$\alpha \cdot \mu := \sum_{j=1}^{d_0} \sum_{k=1}^{d_0} g^{jk} \alpha(H_k)\mu(H_j), \tag{4.4.1}$$

with (g^{jk}) the inverse matrix of the matrix (g_{jk}) introduced in (4.2.2). In fact, it turns out that this expression is independent of the choice of a basis $\{X_1, \dots, X_{d_0}\}$ of L_0. In particular, in the standard basis of L, one has $g^{jk} = \delta_{jk}$ for $j, k \in \{1, \dots, d_0\}$, and then $\alpha \cdot \mu = \sum_{j=1}^{d_0} \alpha(H_j)\mu(H_j)$.

If we consider for $(\mathcal{V}, h)$ the adjoint representation, then $\mu = \beta$ for some $\beta \in \mathcal{R}$, and we end up with expressions of the form $\alpha \cdot \beta$ between roots. In particular, for $\alpha \in \mathcal{R}$ we set

$$\|\alpha\|^2 := \alpha \cdot \alpha = \sum_{j=1}^{d_0} \alpha(H_j)\alpha(H_j) > 0$$

if the standard basis is chosen for L_0.

Let us now answer a natural question. It follows from Proposition 4.4.1 that if μ is a weight, then $\mu + k\alpha$ can also be a weight, for some $k \in \mathbb{Z}$. Clearly, this can not be the case for too many k since the number of weights is at most equal to the dimension of $\mathcal{V}$. Thus, for which k is $\mu + k\alpha$ still a weight ?

For answering this question, let us again endow L with the standard basis, and set $\alpha_j := \alpha(H_j) \in \mathbb{R}$, and also set $\mu_j := \mu(H_j) \in \mathbb{R}$ for a weight of a representation $(\mathcal{V}, h)$. We also define $\alpha^\perp := \{x \in \mathbb{R}^{d_0} \mid \alpha \cdot x = 0\}$ the hyperplane perpendicular to α. Finally, let $\bar{\mu}$ denote the point in $\mathbb{R}^{d_0}$ obtained from μ by a mirror symmetry with respect to the hyperplane $\alpha^\perp$, namely

$$\bar{\mu} := \mu - 2\frac{\mu \cdot \alpha}{\|\alpha\|}\frac{\alpha}{\|\alpha\|} = \mu - 2\frac{\mu \cdot \alpha}{\|\alpha\|^2}\alpha.$$

The following statement provides several properties of this geometric construction. Its proof is rather technical, we refer for example to [1, Prop. 5.37].

Proposition 4.4.2. *In the framework introduced above:*

1. $N := -2\frac{\mu \cdot \alpha}{\|\alpha\|^2} \in \mathbb{Z}$,
2. *For any* $k \in \mathbb{Z} \cap [0, N]$*, or any* $k \in \mathbb{Z} \cap [N, 0]$ *if* $N \leq 0$*, the vector* $\mu + k\alpha$ *is a weight,*
3. $\mu - 2\frac{\mu \cdot \alpha}{\|\alpha\|^2}\alpha$ *is also a weight.*

Let us illustrate this statement with Figure 4.1, with a Cartan sublagebra of dimension 2, and six roots. The three positive roots generate three planes which are represented, and the weights have to satisfy the relations mentioned in the above statement.

In order to illustrate more concretely the above construction, let us come back to the Lie algebra $\mathrm{su}(2)_{\mathbb{C}}$ already introduced at the end of Section 4.3. We always assume that the algebra is endowed with its standard basis. Let $(\mathcal{V}, h)$ be a finite dimensional representation of this Lie algebra. Since L_0 is one dimensional, all weights belong to $\mathbb{R}$, and therefore there exists a maximal weight. Let us denote by μ this maximal weight. Since $\mathcal{R}$ contains only the two roots $\alpha = \pm\frac{1}{\sqrt{2}}$, it follows that $\|\alpha\|^2 = \frac{1}{2}$, and then for $\alpha = \frac{1}{\sqrt{2}}$ one gets $N := -2\frac{\mu \cdot \alpha}{\|\alpha\|^2} = -\frac{4}{\sqrt{2}}\mu$. As a consequence of the previous proposition (and since $\mu > 0$) one infers that $N \in \mathbb{Z}$ with $N < 0$, and that $\mu = -\frac{\sqrt{2}}{4}N$. Therefore, the possible weights are

$$\mu, \mu - \frac{1}{\sqrt{2}}, \mu - 2\frac{1}{\sqrt{2}}, \dots, \mu - \frac{4}{\sqrt{2}}\mu\frac{1}{\sqrt{2}} = \mu - 2\mu = -\mu.$$

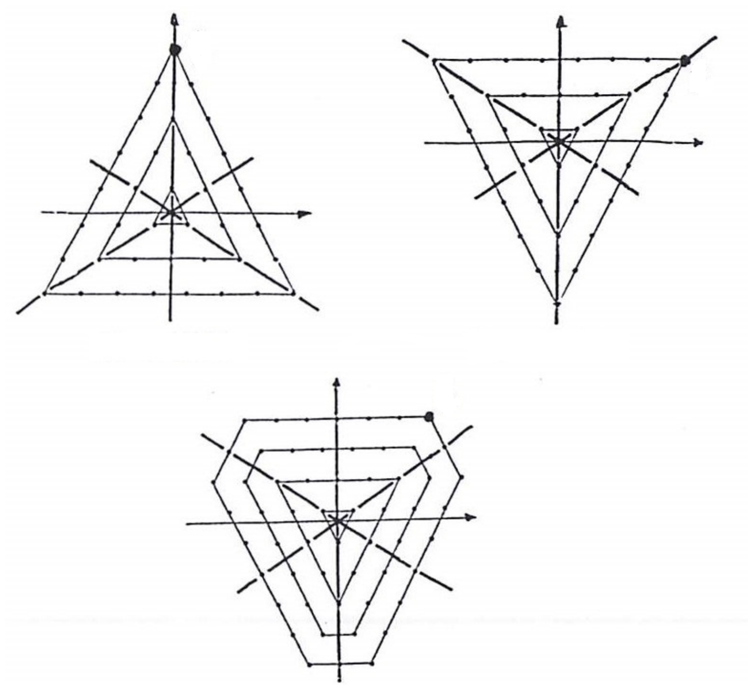

Figure 4.1: Representations of the three planes and of various weights satisfying the relations mentioned in Proposition 4.4.2.

Thus, these values correspond to the eigenvalues of the operator

$$h(H) = h\left(\begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & -\frac{1}{2\sqrt{2}} \end{pmatrix}\right) = \frac{1}{2\sqrt{2}} h(\sigma_3) = \frac{1}{\sqrt{2}} J_3$$

where the notation J_3 is commonly used for $\frac{1}{2}h(\sigma_3)$ in quantum mechanics. It thus follows that the eigenvalues of J_3 belong to $\frac{1}{2}\mathbb{Z}$. It turns out that value μ, of equivalently the maximal eigenvalue of J_3, given by $\sqrt{2}\mu$, determines uniquely an equivalence class of irreducible representation of $\mathrm{su}(2)_{\mathbb{C}}$, or of SU(2). More precisely one has:

Proposition 4.4.3. *For any $n \in \mathbb{N}^*$. there exists a unique (up to equivalence) irreducible representation of* $\mathrm{su}(2)_{\mathbb{C}}$. *In this representation the maximal value of j is $\frac{n}{2}$.*

This result is a very standard result and a proof can be found in many textbooks, as for example in [8, Sec. 4.4].

Let us now come back to the general setting. We consider a complex semi-simple Lie algebra L endowed with its standard basis, and let L_0 denote its Cartan subalgebra. Let also $(\mathcal{V}, h)$ be a finite dimensional irreducible representation. We denote by $\mu_{\max}$ the maximal weight, once $\mathbb{R}^{d_0}$ is endowed with the lexicographic order, starting from the last component. Clearly, if α is a positive root, then $L_{\mu_{\max}+\alpha} = \{0\}$, otherwise one would get a contradiction. Equivalently. $\mathfrak{E}_\alpha v_{\max} = 0$ for the maximal weight vector $v_{\max}$. By collecting the information obtained some far, the following exercise is rather instructing:

Exercise 4.4.4. *In the framework introduced above, show that*

$$\begin{aligned} \mathcal{V} &= \mathrm{Span}\{v_{\max}, \mathfrak{E}_\alpha v_{\max}, \mathfrak{E}_\alpha \mathfrak{E}_\beta v_{\max}, \dots \mid \alpha, \beta, \dots \in \mathcal{R}_-\} \\ &= \mathrm{Span}\{v_{\max}, \mathfrak{E}_{-\alpha} v_{\max}, \mathfrak{E}_{-\alpha} \mathfrak{E}_{-\beta} v_{\max}, \dots \mid \alpha, \beta, \dots \in \{\text{simple roots}\}\}. \end{aligned}$$

More generally, for any weight vector v show that

$$\mathcal{V} = \mathrm{Span}\{v, \mathfrak{E}_\alpha v, \mathfrak{E}_\alpha \mathfrak{E}_\beta v, \ldots \mid \alpha, \beta, \ldots \in \mathcal{R}\}.$$

If we generalize and summarize the content of this section, we end up with the following statement:

Proposition 4.4.5. *Let $(\mathcal{V}, h)$ be an irreducible finite dimensional representation of a complex Lie algebra L, and let $\mu_{\max}$ be its maximal weight. Then,*

1. *All operators $h(H)$, with $H \in L_0$, are simultaneously diagonalizable,*
2. *Any weight is given by $\mu = \mu_{\max} - \sum_{\alpha \in \mathcal{R}_s} n_\alpha a$ with $n_\alpha \in \mathbb{N}$ and $\mathcal{R}_s$ the set of simple roots,*
3. *The sum of all weights, multiplicity counted, is equal to the dimension of $\mathcal{V}$,*
4. *The dimension of $L_{\mu_{\max}}$ is 1.*

Let us conclude this section with a few more deep results. It can be shown that if two irreducible representations share the same maximal weight, then these representations are equivalent. Thus, it is important to know the set of all maximal weights. In fact, an indexation of all maximal weights is possible, and a formula for the dimension in which this representation is taking place exists. Namely, if $\mu_{\max}$ is a maximal weight, then

$$n := \frac{\prod_{\alpha \in \mathcal{R}_+} \alpha \cdot (\mu_{\max} + \delta)}{\prod_{\alpha \in \mathcal{R}_+} \alpha \cdot \delta}, \qquad \text{with } \delta := \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha,$$

provides the dimension of an irreducible representation for which $\mu_{\max}$ is the maximal weight. Note that the multiplicity of a weight can also be computed with the so-called Kostant's formula. Details can be found in the literature.

Chapter 5

Examples

In this final chapter, we illustrate the general theory with two examples which are rather famous.

5.1 Representations of SU(3)

Let us recall that from the end of Section 4.3 that the dimension of su(3) is 8 and that the dimension of its Cartan subalgebra is 2, or equivalently $d = 8$ and $d_0 = 2$. The standard basis of $\mathrm{su}(3)_{\mathbb{C}}$ has also been introduced at the end of this section. We also recall that its positive roots are

$$\alpha = \Big(\frac{1}{2\sqrt{3}}, \frac{1}{2}\Big), \quad \beta = \Big(-\frac{1}{2\sqrt{3}}, \frac{1}{2}\Big), \quad \gamma = \Big(\frac{1}{\sqrt{3}}, 0\Big).$$

We now define a new basis, namely

$$I_3 := \sqrt{3}H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_+ := \sqrt{6}E_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_- := \sqrt{6}E_{-\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.1.1}$$

$$U_3 := \frac{3}{2}H_2 - \frac{\sqrt{3}}{2}H_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U_+ := \sqrt{6}E_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_- := \sqrt{6}E_{-\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.1.2}$$

$$V_3 := -\frac{3}{2}H_2 - \frac{\sqrt{3}}{2}H_1, = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_+ := \sqrt{6}E_{-\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad V_- := \sqrt{6}E_\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.1.3}$$

By looking at the exact form of these matrices one infers that U_3, U_-, U_+ leave the subspace $\{(k, 0, 0)^t \mid k \in \mathbb{R}\}$ invariant, that V_3, V_-, V_+ leave the subspace $\{(0, k, 0)^t \mid k \in \mathbb{R}\}$ invariant, and that I_3, I_-, I_+ leave the subspace $\{(0, 0, k)^t \mid k \in \mathbb{R}\}$ invariant. In addition, by checking the exact form of these matrices, one observes that these three triples generate three representations of $\mathrm{su}(2)_{\mathbb{C}}$ which are not irreducible because of the invariant subspace. In other words, there exist three subgroups of SU(3) which are isomorphic to SU(2).

Consider now a finite dimensional and irreducible representation $(\mathcal{V}, h)$ of $\mathrm{su}(3)_{\mathbb{C}}$, and set $\mathfrak{H}_j := h(H_j)$ and $\mathfrak{E}_\alpha := h(E_\alpha)$ for $j \in \{1, 2\}$ and $\alpha \in \mathcal{R}$, and also $\mathfrak{U}_3 := h(U_3)$, $\mathfrak{V}_3 := h(V_3)$, and $\mathfrak{I}_3 := h(I_3)$. Since U_3, V_3

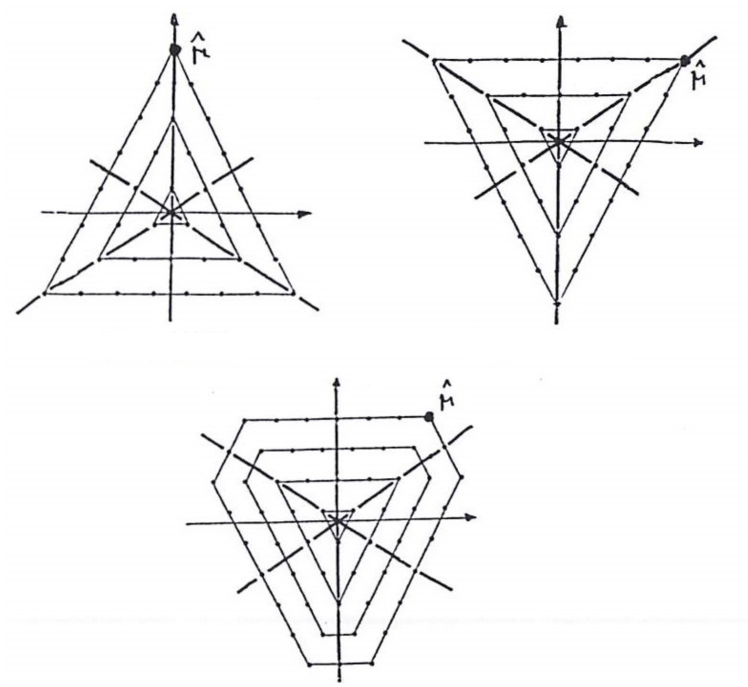

Figure 5.1: Representations of the possible weights, starting from a maximal weight denoted by $\hat{\mu}$.

and I_3 are equal to $\frac{1}{2}\sigma_3 \oplus 0$, it follows as for $\mathrm{su}(2)_\mathbb{C}$, that the eigenvalues of $\mathfrak{U}_3$, $\mathfrak{V}_3$ and $\mathfrak{I}_3$ belong $\frac{1}{2}\mathbb{Z}$. As a consequence, we infer from (5.1.1), (5.1.2), and (5.1.3) that if $\mu = (\mu_1, \mu_2) = (\mu(\mathfrak{H}_1), \mu(\mathfrak{H}_2))$ is a weight, then $\sqrt{3}\mu_1 \in \frac{1}{2}\mathbb{Z}$ and $3\mu_2 \in \frac{1}{2}\mathbb{Z}$. It thus follows that

$$\mu_1 \in \frac{1}{2\sqrt{3}}\mathbb{Z}, \qquad \mu_2 \in \frac{1}{6}\mathbb{Z}.$$

By acting with the operators $\mathfrak{E}_{\pm\alpha}$, $\mathfrak{E}_{\pm\beta}$, and $\mathfrak{E}_{\pm\gamma}$ on the corresponding weight vector, one then gets the new possible weights (as long as the corresponding new weight vector is not 0)

$$\mu \pm \alpha = \Big(\mu_1 \pm \frac{1}{2\sqrt{3}}, \mu_2 \pm \frac{1}{2}\Big), \qquad \mu \pm \beta = \Big(\mu_1 \mp \frac{1}{2\sqrt{3}}, \mu_2 \pm \frac{1}{2}\Big), \qquad \mu \pm \gamma = \Big(\mu_1 \pm \frac{1}{\sqrt{3}}, \mu_2\Big).$$

It remains now to list the maximal weights, leading to the irreducible representations. These maximal weights are indexed by $(\kappa_1, \kappa_2) \in \mathbb{N} \times \mathbb{N}$, and the relations with $\mu_{\max}$ are of the form $\kappa_1 := 2\sqrt{3}\mu_{\max,1}$ and $\kappa_2 := 3\mu_{\max,2} - \sqrt{3}\mu_{\max,1}$ (inspired from (5.1.2)), or equivalently

$$\mu_{\max} := \Big(\frac{\kappa_1}{2\sqrt{3}}, \frac{\kappa_1 + 2\kappa_2}{6}\Big).$$

Starting from different maximal weights, it is possible to represent all possible weights, see Figure 5.1.

From the irreducible representations of $\mathrm{su}(3)_\mathbb{C}$, indexed by (κ_1, κ_2), one gets by exponentiation the irreducible and unitary representations of SU(3), denoted by $D^{(\kappa_1,\kappa_2)}$. The dimension of these representations can be computed explicitly, namely they take place in $\mathbb{C}^n$ with

$$n := \frac{1}{2}(\kappa_1 + 1)(\kappa_2 + 1)(\kappa_1 + \kappa_2 + 2).$$

(κ_1,κ_2)	(0,0)	(1,0)	(0,1)	(2,0)	(0,2)	(1,1)	(3,0)	(0,3)	(2,1)	(1,2)	(2,2)	(4,1)	(1,4)
n	1	3	3	6	6	8	10	10	15	15	27	35	35
notation	**1**	**3**	$\overline{\mathbf{3}}$	**6**	$\overline{\mathbf{6}}$	**8**	**10**	$\overline{\mathbf{10}}$	**15**	$\overline{\mathbf{15}}$	**27**	**35**	$\overline{\mathbf{35}}$

Figure 5.2: Alternative names for the irreducible representations, and their dimension.

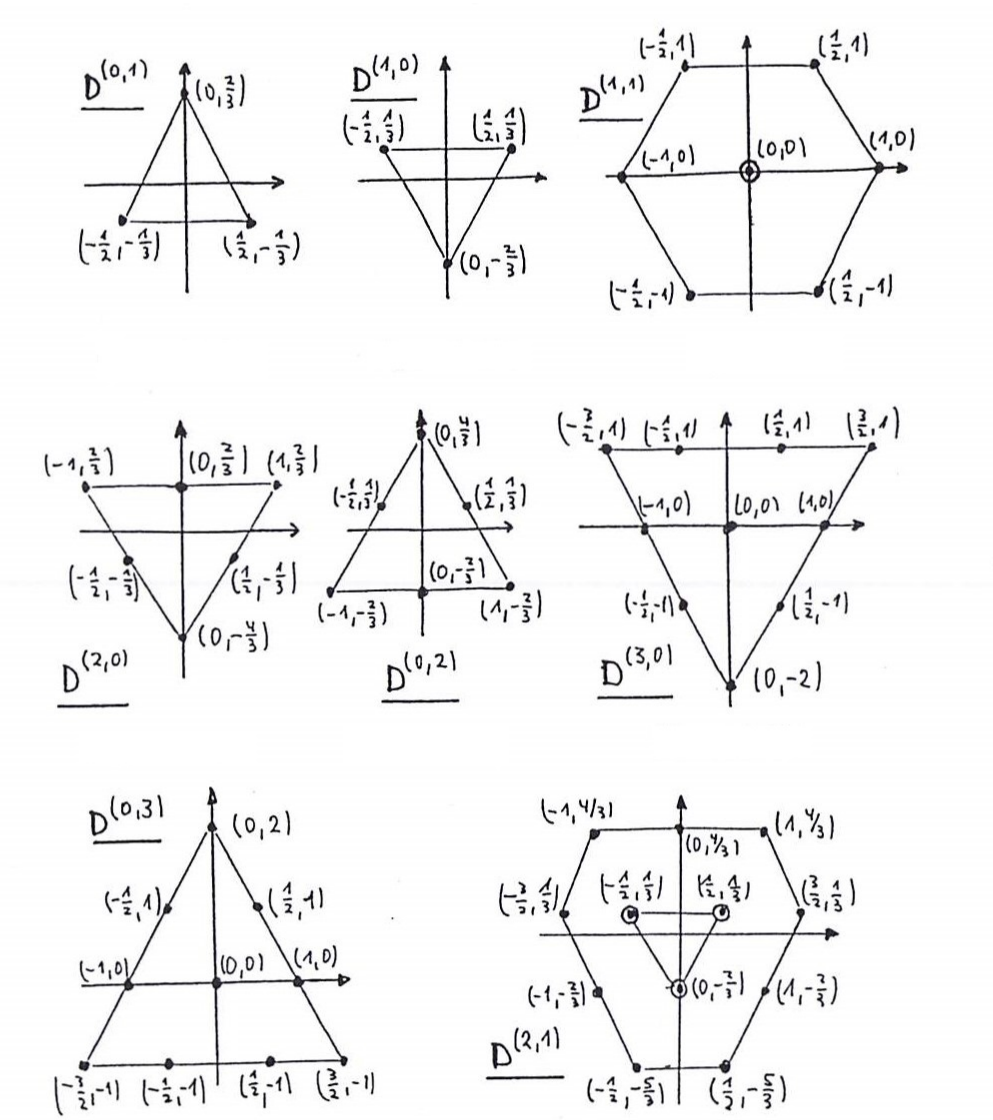

Figure 5.3: Weight diagrams for a few irreducible representations, with the values indicated corresponding to $(\sqrt{3}\mu_1, 2\mu_2)$. The circled points correspond to weights of multiplicity 2.

In Figure 5.2, alternative and shorter names to some irreducible representations are introduced. We also list in Figure 5.3 the weight diagram of a few representations of small dimensions.

Let us recall from Section 2.5 that tensor products of irreducible representations can be decomposed, and often contain other representations. For example, with the notation introduced in Figure 5.2, one can construct the tensor products of irreducible representations, and decompose them, leading to the formulas (which can be computed with the characters):

$$3 \otimes 3 = 6 \oplus \overline{3}, \qquad 3 \otimes \overline{3} = 8 \oplus 1, \qquad 6 \otimes 3 = 10 \oplus 8, \qquad 3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1. \tag{5.1.4}$$

Let us close this section with an important remark which goes beyond the content of this course.

Remark 5.1.1. *Each semi-simple Lie algebra L possesses d_0 independent* Casimir operator(s)*, where d_0 is its rank, or in other words the dimension of its Cartan subalgebra. These operators can be constructed in terms of the elements of L, but they are non-linear expressions. As a consequence, they don't belong to L, and their definitions require the introduction of the notion of the universal enveloping algebra, see*

https://en.wikipedia.org/wiki/Casimir_element

Usually, these Casimir operators are denoted by $C_2, \dots, C_{d_0+1}$ and they satisfy $[C_j, Y] = 0$ for any $Y \in L$, once a suitable meaning to this expression has been given. In addition, in any finite dimensional irreducible representation $(\mathcal{V}, h)$ they satisfy $h(C_j) = c_j \mathbb{1}$ for some $c_j \in \mathbb{C}$. For $\mathrm{su}(3)_\mathbb{C}$ *and if L is endowed with the standard basis one has*

$$C_2 = H_1^2 + H_2^2 + E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha + E_\beta E_{-\beta} + E_{-\beta} E_\beta + E_\gamma E_{-\gamma} + E_{-\gamma} E_\gamma.$$

In the representation $D^{(\kappa_1,\kappa_2)}$ the corresponding constant c_2 can be computed explicitly, and one has

$$c_2 = \frac{1}{9}(\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2) + \frac{1}{3}(\kappa_1 + \kappa_2).$$

Still for $\mathrm{su}(3)_\mathbb{C}$*, the second Casimir operator is a polynomial of order 3 in the generators of L, and in the representation $D^{(\kappa_1,\kappa_2)}$ it taks the values*

$$c_3 = \frac{1}{9}(\kappa_1 - \kappa_2)(2\kappa_1 + \kappa_2 + 3)(\kappa_1 + 2\kappa_2 + 3).$$

5.2 Application of SU(3) in physics

Already a long time ago, it had been observed that particles with similar properties can be organized in families of 1, 8, or 10 members, It was then realized that these numbers appeared in the decomposition of tensor product of representation of SU(3), as shown in (5.1.4). Thus, it has been decided to describe particles with irreducible representations of su(3). In fact, this turns out to be a rather successful approach since some particles predicted with this construction were only discovered later.

By looking at the simplest and non-trivial irreducible representations of SU(3), namely $3 := D^{(1,0)}$ and $\overline{3} := D^{(0,1)}$, see Figures 5.2 and 5.3, it has been decided that each weight of the corresponding representation of su(3) would be associated with one elementary particle, namely the *quarks* u, d, or s, and the anti-quarks $\overline{u}$, $\overline{d}$, and $\overline{s}$, see Figure 5.4. Since the rank of su(3) is 2, each weight is two dimensional and corresponds to the eigenvalues of two matrices which can be simultaneously diagonalized. A standard choice is to use the *isospin* $I_3 := \sqrt{3}H_1$, taking values in $\frac{1}{2}\mathbb{Z}$, and the *hypercharge* $Y := 2H_2$, taking values in $\frac{1}{3}\mathbb{Z}$. Note that the

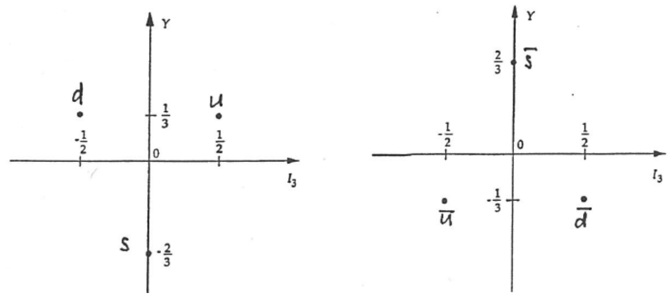

Figure 5.4: The choice of 3 quarks and their anti-quarks, based on the irreducible representations 3 and $\overline{3}$ of su(3).

charge Q can also be used for indexing the particles, and that the relation between these three quantities is $Y = 2(Q - I_3)$.

Based on this idea and by looking at the tensor product $3 \otimes \overline{3} = 8 \oplus 1$, an eightfold of particles, called *mesons*, corresponds to the irreducible representation $8 := D^{(1,1)}$. It turns out that six of these particles can be described with the tensor products of one quark and one anti-quark, as indicated in Figure 5.5. For the meson

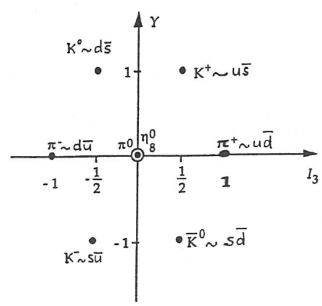

Figure 5.5: The eightfold of mesons, each of them made of one quark and one anti-quark.

with isospin 0 and hypercharge 0, the following relations hold:

$$\pi^0 := \frac{1}{\sqrt{2}}(d \otimes \overline{d} - u \otimes \overline{u}), \quad \eta \equiv \eta_8^0 := \frac{1}{\sqrt{6}}(d \otimes \overline{d} + u \otimes \overline{u} - 2s \otimes \overline{s}).$$

Finally, for the meson which does not belong to the eightfold but generates a trivial representation by itself, the formula is

$$\eta' \equiv \eta_1^0 := \frac{1}{\sqrt{3}}(d \otimes \overline{d} + u \otimes \overline{u} + s \otimes \overline{s}).$$

As it should be, the three vectors π^0, η, and η' are orthogonal and general a subspace of dimension 3, corresponding to an isospin 0 and to a hypercharge 0.

By considering then the decomposition of the tensor product $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ into irreducible representations, it is possible to generate families of particles containing three quarks. The construction is more involved and is not presented here. An illustration for two of these families is provided in Figure 5.6. Note that this initial and simple model has then been further refined and improved.

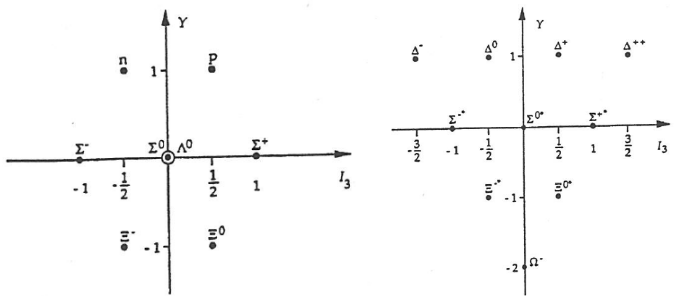

Figure 5.6: Other families of particles, made of three quarks.

5.3 Classification theorem

It has been mentioned in Proposition 4.2.8 that every semi-simple Lie algebra is either simple or the direct sum of a finite set of simple Lie algebras. Thus, the building blocks of the theory are the simple Lie algebras, and we would like to classify them. For that purpose, recall that roots are very special weights, the ones in the adjoint representation. Thus, if L is a simple Lie algebra endowed with the standard basis one infers from Proposition 4.4.2 that for any simple roots α, β

$$-2\frac{\alpha\cdot\beta}{||\alpha||^2}\in\mathbb{Z} \quad \text{and} \quad -2\frac{\alpha\cdot\beta}{||\beta||^2}\in\mathbb{Z}.$$

Equivalently, one has $-2\alpha\cdot\beta = N_1||\alpha||^2$ and $-2\alpha\cdot\beta = N_2||\beta||^2$ for some $N_1, N_2 \in \mathbb{Z}$, or still

$$N_1||\alpha||^2 = N_2||\beta||^2 \quad \text{and} \quad \frac{(\alpha\cdot\beta)^2}{||\alpha||^2||\beta||^2} = \left(\frac{\alpha\cdot\beta}{||\alpha||\,||\beta||}\right)^2 = \frac{N_1N_2}{4}.$$

By observing that the later expression corresponds to $\cos^2(\phi_{\alpha,\beta})$ with $\phi_{\alpha,\beta}$ the angle between the two roots, one obtain that $N_1N_2 \in [0,4]$, which limits drastically the possible choice for N_1 and N_2. In fact, with the other condition one must also choose N_1 and N_2 of the same sign. By one more geometric argument, see [8, Prop. 8.11], it turns out that for simple roots, one has $\alpha\cdot\beta \le 0$, meaning that both N_1 and N_2 are positive. Thus, the only alternatives are provided in Table 5.1

(N_1, N_2)	$(2,2)$	$(1,3)$ or $(3,1)$	$(1,2)$ or $(2,1)$	$(1,1)$	$(1,4)$ or $(4,1)$	$(0,0)$
$\phi_{\alpha,\beta}$	0^o or 180^o	30^o or 150^o	45^o or 135^o	60^o or 120^o	0^o or 180^o	90^o

Table 5.1: Possible angles between roots.

Note that the last option, with $N_1 = N_2 = 0$ leads to an undetermined ratio N_1/N_2. On the other hand, the angles of 0^o or 180^o are not possible, since the roots are assumed to be linearly independent (the case $\beta = -\alpha$ is not possible for simple roots). Also, since $\alpha\cdot\beta < 0$, we have to choose the obtuse angles. It is then possible to have a schematic representation of the possible angles between simple roots, namely if we denote any simple root by a small circle, then the angle between two roots can be represented by the rule presented in Figure 5.7.

The fact that an angle of 90^o is represented by no edge is not surprising: it turns out that for simple Lie algebras, two roots can not have a right angle between them, but this can happen for semi-simple Lie algebras. Thus, a simple Lie algebra is going to correspond to a connected diagram, the so-called *Dynkin diagram*.

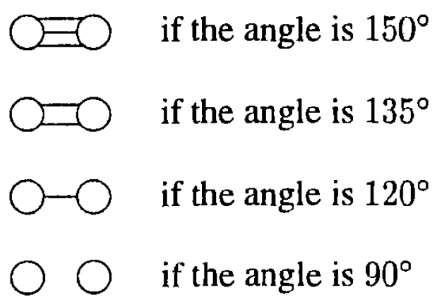

Figure 5.7: Representations of the angles between roots.

By taking these rules into account, it is possible to provide a list of all complex simple Lie algebras, leading to the list of all simple Lie algebras. A presentation of this classification theorem is provided for example in [7, Chap. 20], but many reference books contain a discussion about this result. In fact, there exist four infinite families of such algebras, and 5 exceptional cases. They are schematically represented in Figure 5.8, with their name, the corresponding Dynkin diagram, the dimension of the Lie algebra, and the number of roots. Note that these algebras and their representations have been extensively studied, and have applications

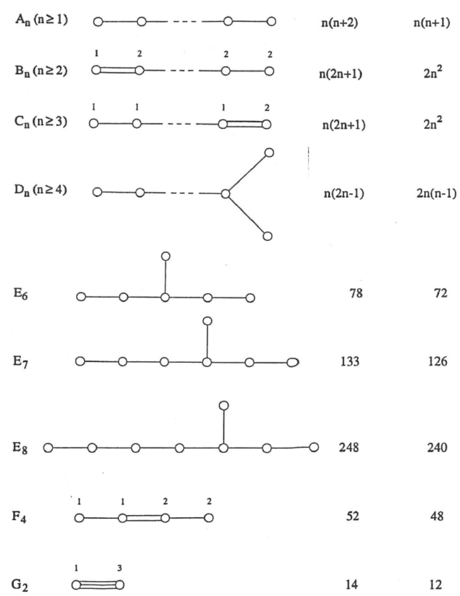

Figure 5.8: Schematic representation of all complex semi-simple Lie algebras with their name, the corresponding Dynkin diagram, the dimension of the Lie algebra, and the number of roots.

in several fields. Some of the most recent investigations were dealing with $E8$, and the following link is quite interesting, and understandable with the notions studied in this course

https://en.wikipedia.org/wiki/E8_(mathematics)

5.4 Induced representations

Given the representation $(\mathcal{V}, U)$ of a group G, the restriction of this representation to any subgroup G_0 of G provides a representation of G_0 in the vector space $\mathcal{V}$. Clearly, even if the initial representation is irreducible, this is not always the case for the representation of G_0. The *induced representation* is a kind of converse to this construction. Given a representation of the subgroup G_0, the induced representation is the "most general" representation of G that extends the initial representation of G_0. Since it is often easier to find representations of the smaller group G_0 than of G, the operation of forming induced representations is an important tool to construct new representations.

The general theory for induced representations is quite involved, and necessitates arguments of measure theory. We shall only present the main ideas and provide the construction for finite groups, since it does not involve any measure theoretical arguments.

Let G be a finite group, and let G_0 be a subgroup of G. We also assume that $(\mathcal{V}.U)$ is a finite dimensional representation of G_0, with $\dim(\mathcal{V}) = n$. We introduce a finite dimensional new vector space $\mathcal{W}$ by

$$\mathcal{W} := \{f : G \to \mathcal{V} \mid f(aa_0) = U(a_0^{-1})f(a) \quad \forall a \in G, a_0 \in G_0\}. \tag{5.4.1}$$

Thus, $\mathcal{W}$ is made of functions from G to $\mathcal{V}$ having a certain property when the variable is multiplied on the right by an element of G_0. It is easy to check that $\mathcal{W}$ is indeed a vector space (stable by addition and by multiplication by a scalar), and that it is finite dimensional, since both G and $\mathcal{V}$ are finite dimensional. For any $f \in \mathcal{W}$ and any $a, b \in G$ we then set

$$[\mathcal{U}(b)f](a) := f(b^{-1}a). \tag{5.4.2}$$

Exercise 5.4.1. *Check that the pair $(\mathcal{W}, \mathcal{U})$ define a representation of G in $\mathcal{W}$, namely check that $\mathcal{U}(b)$ maps $\mathcal{W}$ on $\mathcal{W}$, that $\mathcal{U}(b)$ is a linear map, and that the map $G \ni b \mapsto \mathcal{U}(b) \in \mathcal{L}(\mathcal{W})$ is a homomorphism with $\mathcal{U}(e) = \mathbb{1}$.*

Once this exercise is proved, we can define:

Definition 5.4.2 (Induced representation)**.** *Given a finite group G with a subgroup G_0, and a finite dimensional representation $(\mathcal{V}, U)$ of G_0, the* induced representation *of G corresponds to the representation $(\mathcal{W}, \mathcal{U})$ given by* (5.4.1) *and* (5.4.2)*.*

For any subgroup G_0 of G, recall that the left coset have been introduced in (1.2.1) and correspond to ${}_{G_0}[a] := aG_0$. These equivalence classes form a partition of G into $J := \frac{|G|}{|G_0|}$ classes. We denote these classes by $\{C_1, \dots, C_J\}$, and set $C_1 := G_0$, with no loss of generality. For each $j \in \{1, \dots, J\}$ we also fix a representative b_j such that $C_j = {}_{G_0}[b_j] = b_jG_0$.

With the notation just introduced, let us define for each $j \in \{1, \dots, J\}$ a subspace of $\mathcal{W}$, namely

$$\mathcal{W}_j := \{f \in \mathcal{W} \mid f(b_k) = \delta_{jk} f(b_i) \quad \forall k \in \{1, \dots, J\}\}.$$

Then, observe that for $f \in \mathcal{W}_j$ and any $a_0 \in G_0$ one has $f(b_ja_0) = U(a_0^{-1})f(b_j)$, meaning that f is completely determined on b_jG_0 by its value at b_j. Also, the value of f on any other coset is 0 since $f(b_ka_0) = 0$ if $j \neq k$. Since $f(b_j)$ can be any element of $\mathcal{V}$ one infers that $\dim(\mathcal{W}_j) = \dim(\mathcal{V}) = n$. In addition, one has

$$\mathcal{W} = \oplus_{j=1}^J \mathcal{W}_j \tag{5.4.3}$$

leading to $\dim(\mathcal{W}) = Jn = \frac{|G|}{|G_0|}\dim(\mathcal{V})$.

In the following exercise, we gather a few results which can be easily proved in this framework:

Exercise 5.4.3. *1) Show the equality* (5.4.3)*,*

2) Show that the regular representation introduced in Definition 2.4.7 *corresponds to the induced representation with the subgroup G_0 consisting of the identity only, namely $G_0 = \{e\}$,*

3) Show that the characters for the representation $(\mathcal{W}, \mathcal{U})$ can be computed with the following formula:

$$\chi_{\mathcal{U}}(c) = \frac{1}{|G_0|} \sum_{b\in G,\ b^{-1}cb\in G_0} \chi_U(b^{-1}cb).$$

Let us now look at a special family of groups for which all complex and irreducible representations are equivalent to induced representations. Namely, we shall consider groups G which are semi-direct products with normal subgroups which are Abelian. Recall that the notion of semi-direct groups was introduced in Definition 1.3.4. More precisely, we assume that G is a finite group, A, B are two subgroups of G with A normal and Abelian, $A \cap B = \{e\}$, and any element c of G admits a unique decomposition $c = ab$ with $a \in A$ and $b \in B$.

Since A is Abelian, it follows from Corollary 2.3.8 that all its unitary irreducible representations are of dimension 1. In such a situation, if $(\mathbb{C}, U)$ denotes such a unitary representation, one has

$$U(a) = \chi_U(a) \ \in \mathbb{T}.$$

We can then infer from Theorem 2.4.9 that there exist $|A|$ such irreducible representations. We shall denote them by $\chi_j : A \to \mathbb{C}$ with $j \in \{1, \dots, |A|\}$. Note that the set of all irreducible representations of the Abelian group A is denoted by A^* and is called the *dual group* of A, which is indeed a group ☺. Observe that an action of G on A^* can be defined: For $\chi_j \in A^*$, any $c \in G$ and $a \in A$ we set

$$[c\chi_j](a) := \chi_j(c^{-1}ac) \tag{5.4.4}$$

which is well-defined since A is a normal subgroup.

In this framework the following lemma can be easily proved. We recall that the notion of transformation group has been introduced in Definition 1.4.1.

Lemma 5.4.4. *Let $G = A \rtimes B$ be a finite semi-direct group with A Abelian. The group G and the subgroup B act on A^* by* (5.4.4) *as transformation groups. The action of A on A^* is trivial, namely $a\chi_j = \chi_j$ for any $a \in A$.*

According to Definition 1.4.3 we can now define

1) O_j the orbit of χ_j in A^* under the action of G,

2) G_j the stabilizer of χ_j under the action of G,

3) B_j the stabilizer of χ_j under the action of B.

Clearly, G_j is a subgroup of G, while B_j is a subgroup of B. The groups B_j are often referred to as the *little groups*, and these groups are isomorphic along orbits in A^*, see Lemma 1.4.4. In addition, one can check that $G_j = A \rtimes B_j$, since $(ab)\chi_j = a(b\chi_j) = b\chi_j$, for any $a \in A$ and $b \in B$.

The following statement is the main result in this framework. We refer for example to [1, Thm. 6.9] for its proof.

Theorem 5.4.5. *Let $G = A \rtimes B$ be a finite semi-direct group with A Abelian. Any complex and irreducible representation $(\mathcal{W}, \mathcal{U})$ of G is equivalent to a representation induced by a subgroup $G_j = A \rtimes B_j$ of G. More precisely, there exists $\chi_j \in A^*$ and a irreducible representation $(\mathcal{V}, \mathrm{U})$ of the little group B_j such that $\mathcal{U}$ is equivalent to the induced representation constructed from the representation $(\mathcal{V}, U)$ of G_j defined by $U(ab) = \chi_j(a)\mathrm{U}(b)$ for any $a \in A$ and $b \in B_j$.*

This statement means that there is no restriction in constructing representations with induced representations, we get all representations of G. In fact, the statement can be even strengthened: If the representation of B_j is irreducible, then the representation of G itself is irreducible, see [1, Thm. 6.10]. As a consequence, each irreducible representation of G can be indexed by two parameters: one related to the orbits of the elements χ_j of A^* under the action of G, and one related to the equivalent class of irreducible representation of the little group B_j. By working carefully, one gets a bijective relation between the set of equivalence classes of representations of G and this double indexation.

5.5 Representations of the Poincaré group

In this section we sketch the representations of the Poincaré group $\mathcal{P}$, based on the construction and of the results of the previous section. The Poincaré group was introduced in Definition 1.5.5. It is clearly not a finite group, but it has the structure of a semi-direct product with an Abelian normal subgroup, namely $\mathcal{P} \cong T(4) \rtimes \mathcal{L}$, with $T(4)$ the usual translations in $\mathbb{R}^4$ and $\mathcal{L}$ the Lorentz group, as introduced in Section 1.5. Since $\mathcal{P}$ is not finite, none of the results of the previous section can be directly applied, but the abstract theory presented in Section 5.4 has been extended to semi-direct Lie groups, as the Euclidean group $E(n)$ or the Poincaré group $\mathcal{P}$. In particular, since these Lie groups are non-compact, the representations will not be finite dimensional, but infinite dimensional.

Recall that the Poincaré group is made of pairs (b, Λ), with $b \in T(4)$ and $\Lambda \in \mathcal{L}$, with the product defined in (1.5.5). This Lorentz groups is made of different components. More precisely, any element Λ of the Lorentz group verifies $\mathrm{Det}(\Lambda) = \pm 1$, and also $|\Lambda^0{}_0| \geq 1$, with $\Lambda^0{}_0$ the first entry of the matrix Λ. Thus, $\mathcal{L}$ can be divided into 4 connected components

1) $\mathcal{L}_+^\uparrow := \{\Lambda \in \mathcal{L} \mid \mathrm{Det}(\Lambda) = 1 \text{ and } \Lambda^0{}_0 \geq 1\}$,

2) $\mathcal{L}_-^\uparrow := \{\Lambda \in \mathcal{L} \mid \mathrm{Det}(\Lambda) = -1 \text{ and } \Lambda^0{}_0 \geq 1\}$,

3) $\mathcal{L}_+^\downarrow := \{\Lambda \in \mathcal{L} \mid \mathrm{Det}(\Lambda) = 1 \text{ and } \Lambda^0{}_0 \leq -1\}$,

4) $\mathcal{L}_-^\downarrow := \{\Lambda \in \mathcal{L} \mid \mathrm{Det}(\Lambda) = -1 \text{ and } \Lambda^0{}_0 \leq -1\}$.

Note that $\mathcal{L}_+^\uparrow$ is the component connected to the identity. This normal subgroup is often referred to as the proper, orthochronous Lorentz group or *restricted Lorentz group*.

As mentioned for SO(3) at the end of Section 2.6, it is sometimes useful to consider the universal cover of a group. Then, it turns out that the universal cover of $\mathcal{L}_+^\uparrow$ is given by $\mathrm{SL}(2,\mathbb{C})$, and the map from $\mathrm{SL}(2,\mathbb{C})$ to $\mathcal{L}_+^\uparrow$ is defined for any $A \in \mathrm{SL}(2,\mathbb{C})$ by

$$(\Lambda_A)^\mu{}_\nu := \frac{1}{2}\mathrm{Tr}(\sigma_\mu A \sigma_\nu A^*).$$

Again, the map from $\mathrm{SL}(2,\mathbb{C})$ to $\mathcal{L}_+^\uparrow$ is surjective, with kernel $\{\mathbb{1}, -\mathbb{1}\}$. Then, instead of considering the subgroup $\mathcal{P}_+^\uparrow$ made of $(b, \Lambda)$with matrices Λ in $\mathcal{L}_+^\uparrow$, it is convenient to consider the group $\tilde{\mathcal{P}}_+^\uparrow$ consisting of pairs (a, A) with $a \in T(4)$ and $A \in \mathrm{SL}(2, \mathbb{C})$ together with the composition law

$$(a, A)(a', A') = (a + \Lambda_A a', AA').$$

Then one has $\tilde{\mathcal{P}}_+^\uparrow = T(4) \rtimes \mathrm{SL}(2, \mathbb{C})$, and it is the representations of this group that we shall now consider.

Following the idea of the previous section, we firstly look for all complex and irreducible representations of $T(4)$. These representations are one dimensional and easy to index. Indeed, for any $p := (p^0, p^1, p^2, p^3) \in \mathbb{R}^4$ and for any $a = (a^0, a^1, a^2, a^3) \in T(4)$, one can set

$$\chi_p(a) := \mathrm{e}^{ip\cdot a} = \mathrm{e}^{i(p^0a^0 - p^1a^1 - p^2a^2 - p^3a^3)} \in \mathbb{T}$$

and the map $\chi_p : T(4) \to \mathbb{T}$ is indeed a group morphism, namely an element of $T(4)^*$. It turns out that all unitary irreducible representations of $T(4)$ are described by these morphisms, which means that $T(4)^*$ can be identified with the Minkowski space $\mathbb{M}$, *i.e.* $\mathbb{R}^4$ with the bilinear form introduced in (1.5.1).

Let us now look at the action of $B = \mathrm{SL}(2, \mathbb{C})$ on χ_p, as introduced in (5.4.4). For that purpose, observe that for $A \in \mathrm{SL}(2, \mathbb{C})$, $p \in \mathbb{M}$ and $a \in T(4)$ one has

$$[A\chi_p](a) = \chi_p((0, A^{-1})(a, \mathbb{1})(0, A)) = \chi_p((\Lambda_{A^{-1}}a, \mathbb{1})) = \chi_p(\Lambda_{A^{-1}}a).$$

By using the property (1.5.3) of the elements of $\mathcal{L}$ one infers that

$$\chi_p(\Lambda_{A^{-1}}a) = \mathrm{e}^{ip\cdot\Lambda_{A^{-1}}a} = \mathrm{e}^{i\Lambda_A p\cdot a} = \chi_{\Lambda_A p}(a).$$

Thus, one has obtained that

$$A\chi_p = \chi_{\Lambda_A p},$$

which provides the orbit of χ_p in $\mathbb{T}(4)^*$ under the action of $\mathrm{SL}(2, \mathbb{C})$.

In order to study the little groups, let us still introduce six different types of orbits of $\mathbb{M}$. The subsets are generated by orbits under the action of $\mathcal{L}_+^\uparrow$. For any fixed $M > 0$, set

1) $O_M^+ := \{p \in \mathbb{M} \mid p \cdot p = M^2 \text{ and } p^0 > 0\}$,
2) $O_M^- := \{p \in \mathbb{M} \mid p \cdot p = M^2 \text{ and } p^0 < 0\}$,
3) $O_{iM} := \{p \in \mathbb{M} \mid p \cdot p = -M^2\}$,
4) $O_o^+ := \{p \in \mathbb{M} \mid p \cdot p = 0 \text{ and } p^0 > 0\}$,
5) $O_o^- := \{p \in \mathbb{M} \mid p \cdot p = 0 \text{ and } p^0 < 0\}$,
6) $O_o^o := \{0\}$.

Some of these orbits are represented in Figure 5.9, with the first two components shown.

It is then possible to determine the little group corresponding to any point on these orbits. As mentioned in Lemma 1.4.4, the stabilizers are isomorphic along the orbits, which means that there are only six little groups which have to be studied. We summarize the results in an exercise.

Exercise 5.5.1. *Show that the following subgroups of* $\mathrm{SL}(2, \mathbb{C})$ *are isomorphic to the stabilizers for any point in the mentioned orbit. Note that it is enough to study one specific point of each orbit.*

1) O_M^+ *and* O_M^- : $\mathrm{SU}(2)$,

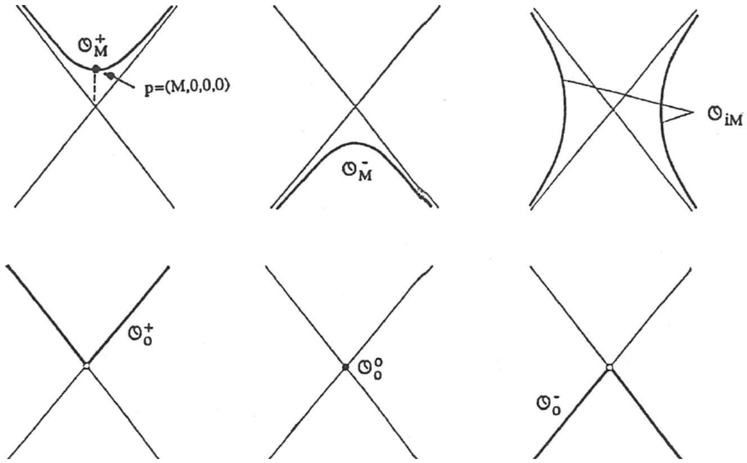

Figure 5.9: Various orbits of $\mathbb{M}$ under the action of $\mathcal{P}_+^\uparrow$, with (p^0, p^1) represented only.

2) O_{iM} : SL$(2,\mathbb{R})$*,*

3) O_o^+ *and* O_o^- : $\left\{\begin{pmatrix} e^{i\theta/2} & e^{-i\theta/2}(\xi + i\eta) \\ 0 & e^{-i\theta/2}\end{pmatrix} \mid \theta \in [0,4\pi), \xi \in \mathbb{R}, \eta \in \mathbb{R}\right\}$*, which is isomorphic to a double cover of* $E(2)$ *and is denoted by* $\hat{E}(2)$*,*

4) O_o^o : SL$(2,\mathbb{C})$.

Based on the representations of these little groups, it will be possible to obtain representations of $\tilde{\mathcal{P}}_+^\uparrow$. Indeed, a refined version of Theorem 5.4.5 also applies in this framework, which means that representations of $\tilde{\mathcal{P}}_+^\uparrow$ can be obtained by irreducible representations of $T(4)$ and representations of the corresponding little group. In fact, the only orbits in Figure 5.9 which are physically interesting are O_M^+ and O_o^+, since free particles with imaginary masses or with negative energy are not known. We do not construct explicitly the representations, but discuss their indexation.

For any $M > 0$ and any $p \in O_M^+$, the corresponding little group is isomorphic to SU(2). As sketched in Exercise 3.4.6 and in Proposition 4.4.3, irreducible representations of SU(2) can be indexed by $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$, and they take place in spaces of dimension $2j+1$. Thus unitary irreducible representations of $\tilde{\mathcal{P}}_+^\uparrow$ can be constructed for any $M > 0$ and for each index $j \in \frac{1}{2}\mathbb{N}$. We say that these representations are associated with a quantum particle of mass $M > 0$ and of *spin* j.

Similarly for any $p \in O_o^+$, the irreducible representation of $\hat{E}(2)$ can be studied. These representations are of two types, but so far only one type has been used in physics. These specific representations are indexed by $s \in \frac{1}{2}\mathbb{Z}$. Thus, the corresponding unitary irreducible representations of $\tilde{\mathcal{P}}_+^\uparrow$ are index by $s \in \frac{1}{2}\mathbb{Z}$, and this number is called the *helicity* of the particle of mass $M = 0$. The absolute value of s is again called the spin of the particle.

As already mentioned, other representations of $\tilde{\mathcal{P}}_+^\uparrow$ can be introduced either with other orbits in $\mathbb{M}$, or with other representations of $\hat{E}(2)$. They are nice mathematical constructions, but have no physical interpretation yet. However, observe that we have only indexed the representations, but not given explicit expressions, similar to the construction of the space $\mathcal{W}$ introduced in (5.4.1) and of the representation $\mathcal{U}$ introduced in (5.4.2). In the present situation, the space $\mathcal{W}$ would be an infinite dimensional Hilbert space, and their construction can be found in several textbooks.

Since $\mathcal{P}_+^\uparrow$ is the identity component of $\mathcal{P}$, let us add some information on its Lie algebra. First of all, it is simpler to recast $\mathcal{P}$ in a matrix format. For this, we use an idea coming from the *affine group* and write any element (b, Λ) of $\mathcal{P}$ in the form of a 5×5 matrix

$$\begin{pmatrix} \Lambda & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 & b^0 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 & b^1 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 & b^2 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 & b^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, the product introduced in (1.5.5) corresponds simply to the product of matrices. Let us emphasize that this representation of $\mathcal{P}$ with 5×5 matrices is a faithful representation.

Based on this representation of $\mathcal{P}$, it can be shown that its Lie algebra L has dimension $d = 10$ and rank $d_0 = 2$. Several bases for L can be defined, and they also consist in families of 5×5 matrices. The commutation relations for these matrices can be computed as well as the structure coefficients c^ℓ_{jk}, see Section 3.3. According to Remark 5.1.1, there exist also two Casimir operators which can be expressed in terms of the elements of L. These operators can be computed explicitly (in terms of the so called *Pauli-Lubanski vector*) and their values in any irreducible representation can be evaluated. It turns out that for the representation indexed by $(M, j) \in (0, \infty) \times \frac{1}{2}\mathbb{N}$, these operators take the values M^2 and $-M^2\, j(j+1)$. For a representation indexed by $(0, s) \in \{0\} \times \frac{1}{2}\mathbb{Z}$, these operators take the values 0 and s. A posteriori, these values confirm the correctness of the indexation for the irreducible representations of $\mathcal{P}_+^\uparrow$ constructed so far.

Let us conclude this section with a few information about the representations of the full group $\mathcal{P}$, and not only $\mathcal{P}_+^\uparrow$ or $\tilde{\mathcal{P}}_+^\uparrow$. Since $\mathcal{L}$ is made of four connected components, the group $\mathcal{P}$ also possesses four connected components, denoted respectively by $\mathcal{P}_+^\uparrow$, $\mathcal{P}_-^\uparrow$, $\mathcal{P}_+^\downarrow$, and $\mathcal{P}_-^\downarrow$. The elements of $\mathcal{P}_-^\uparrow$ can be obtained by the factorization $(b, \Lambda)(0, \Pi)$ with Π the parity operator acting as $\Pi(x^0, x^1, x^2, x^3) = (x^0, -x^1, -x^2, -x^3)$ and (b, Λ) an arbitrary element of $\mathcal{P}_+^\uparrow$. Similarly, the elements of $\mathcal{P}_-^\downarrow$ can be obtained by the factorization $(b, \Lambda)(0, \Theta)$ with Θ the time reversal operator acting as $\Theta(x^0, x^1, x^2, x^3) = (-x^0, x^1, x^2, x^3)$. Finally, the elements of $\mathcal{P}_+^\downarrow$ can be obtained by the factorization $(b, \Lambda)(0, \mathcal{I})$ with $\mathcal{I}$ the total inversion operator acting as $\mathcal{I}(x^0, x^1, x^2, x^3) = (-x^0, -x^1, -x^2, -x^3)$.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Figure 5.10: The Dihedral group D_2.

Thus, in order to describe a representation of $\mathcal{P}$, it is necessary and sufficient to describe the representation of $\mathcal{P}_+^\uparrow$ together with the representation of the operators Π, Θ, and $\mathcal{I}$. In fact, it can be checked that $\mathcal{P}$ is isomorphic to the semi-direct product of $\mathcal{P}_+^\uparrow$ with the group containing the four elements $(0, \mathbb{1})$, $(0, \Pi)$, $(0, \Theta)$, and $(0, \mathcal{I})$. This group corresponds to the dihedral group D_2, with composition table shown in Figure 5.10, once the identification $e := (0, \mathbb{1})$, $a := (0, \Pi)$, $b := (0, \Theta)$, and $c := (0, \mathcal{I})$, is taken into account. The relations between these four elements impose some restrictions on their possible representations. It turns out the representation theory of $\mathcal{P}$ is rich and interesting, and deserves to be further studied. The theory has applications in quantum mechanics, in quantum field theory, in chemistry, and in several other research fields. What about applications in your domain of interest ?

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