Deformation of Kähler Metrics and an Eigenvalue Problem for the Laplacian on a Compact Kähler Manifold

(コンパクトケーラー多様体におけるケーラー計量の変形とラプラシアンの固有値問題)

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Contents

1. Introduction

Let M be a compact manifold (without boundary) of dimension m. Given a Riemannian metric g on M, the volume $\mathrm{Vol}(M, g)$ and the Laplace-Beltrami operator Δ_g are defined. Let $0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \lambda_k(g) \leq \cdots$ be the eigenvalues of Δ_g. The quantity $\lambda_k(g)\mathrm{Vol}(M, g)^{2/m}$ is invariant under scaling of the metric g. Hersch [16] proved that on a 2-dimensional sphere S^2, the scale-invariant quantity $\lambda_1(g)\mathrm{Area}(g)$ is maximized exactly when g is a round metric. Inspired by the work, Berger [2] posed the question whether

$$\Lambda_1(M) := \sup_g \lambda_1(g)\mathrm{Vol}(M, g)^{\frac{2}{m}}$$

is finite for a compact manifold M of dimension m. For a surface M, $\Lambda_1(M)$ is bounded by a constant depending on the genus [41, 17]. Berger [2] also conjectured that for a 2-dimensional torus T^2, the equilateral flat metric attains $\Lambda_1(T^2)$. Nadirashvili [27] settled Berger's conjecture affirmatively. In the same paper, he proved a theorem that if a metric g on a given surface M is extremal for the functional $\lambda_k : g \mapsto \lambda_k(g)$ with respect to all the volume-preserving deformations of the metric, then (M, g) admits an isometric minimal immersion into a Euclidean sphere by first eigenfunctions. After that, El Soufi–Ilias [11] simplified the proof of the theorem and generalized it to a compact manifold M of any dimension for $k = 1$. Later, El Soufi–Ilias [13] improved this result and proved the following:

Theorem 1.1 ([27], [11], [13])**.** (Theorem 3.4) Let (M, g) be a compact m-dimensional Riemannian manifold. If the metric g is extremal for the functional λ_k with respect to all the volume-preserving deformations of the metric, then there exists a finite collection of $\lambda_k(g)$-eigenfunctions $\{f_1, \ldots, f_N\}$ such that $F := (f_1, \cdots, f_N) : (M, g) \to \mathbf{R}^N$ is an isometric minimal immersion into $S^{N-1}(\sqrt{m/\lambda_k(g)}) \subset \mathbf{R}^N$. For $k = 1$, the converse also holds.

For the precise definition of extremality of the functional λ_k, see Definition 3.3 in Section 3. For a surface M, there has been a remarkable progress in the study of $\Lambda_1(M)$ such as [30], [18], [31], [32] and [19].

On the other hand, on any compact manifold M with $m \geq 3$, one can construct a 1-parameter family $\{g_t\}_{t>0}$ such that the quantity $\lambda_1(g_t)\mathrm{Vol}(M, g_t)^{2/m}$ diverges to infinity as t goes to infinity [8]. (See also [6], [26], [25], [37] and [38].) This motivates us to restrict ourselves to studying the functional λ_1 only in a certain class of metrics. For example, for a given Riemannian metric g on M, the restriction of the functional λ_1 to the metrics in its conformal class with fixed volume is bounded [20, 10]. El Soufi-Ilias [12, 13] proved the following:

Theorem 1.2 ([12], [13])**.** (Theorem 3.7) Let (M, g) be a compact m-dimensional Riemannian manifold. If the metric g is extremal for the functional λ_k with respect to all the volume-preserving deformations within its conformal class, then there exists a finite collection of $\lambda_k(g)$-eigenfunctions $\{f_1, \ldots, f_N\}$ such that $F := (f_1, \cdots, f_N) : (M, g) \to \mathbf{R}^N$ is a harmonic map into $S^{N-1}(\sqrt{m/\lambda_1(g)}) \subset \mathbf{R}^N$ with constant energy density $|dF|^2 \equiv m$. For $k = 1$, the converse also holds.

On the other hand, Bourguignon–Li–Yau [7] proved the following result:

Theorem 1.3 ([7])**.** (Theorem 4.1) Let (M, J) be a compact complex n-dimensional manifold that admits a full holomorphic immersion $\Phi : (M, J) \to \mathbf{C}P^N$. Let σ_{FS} be the Fubini-Study form on $\mathbf{C}P^N$ with constant holomorphic sectional curvature 1. Then, for any Kähler form ω on (M, J), the first eigenvalue $\lambda_1(\omega)$ satisfies

$$\lambda_1(\omega) \leq n \frac{N+1}{N} \frac{\int_M \Phi^* \sigma_{FS} \wedge \omega^{n-1}}{\int_M \omega^n}.$$

Stokes theorem implies that $\lambda_1(\omega)$ is bounded by a constant depending on only n, N, Φ and the Kähler class $[\omega]$. The above theorem implies that the Fubini-Study metric on $\mathbf{C}P^N$ is a λ_1-maximizer in its Kähler class. Biliotti–Ghigi [5] generalized this result and showed that the canonical Kähler-Einstein metric on a Hermitian symmetric space of compact type is a λ_1-maximizer in its Kähler class (Theorem 4.2). Motivated by these results, Apostolov–Jakobson–Kokarev [1] proved the following:

Theorem 1.4 ([1]). (Theorem 4.10) Let (M, J, g, ω) be a compact Kähler manifold of complex dimension n. If the Kähler metric g is extremal for the functional λ_k within its Kähler class, then there exists a nontrivial finite collection of $\lambda_k(g)$-eigenfunctions $\{f_1, \dots, f_N\}$ satisfying the equation

$$\lambda_k(g)^2 \sum_{j=1}^{N} f_j^2 - 2\lambda_k(g) \sum_{j=1}^{N} |\nabla f_j|^2 + \sum_{j=1}^{N} |dd^c f_j|^2 = 0. \tag{1.1}$$

For $k = 1$, the converse also holds.

Although we do not explain the precise definition of the extremality in Theorem 1.4 here, we remark that the metric maximizing λ_1 in its Kähler class is extremal for the functional λ_1 within its Kähler class. (See Definition 4.3 for the precise definition of the extremality of a Kähler metric within its Kähler class.) Using (1.1), Apostolov–Jakobson–Kokarev [1] also proved the following:

Proposition 1.5. ([1])(Proposition 4.14) The metric on a compact homogeneous Kähler-Einstein manifold of positive scalar curvature is extremal for the functional λ_1 within its Kähler class.

However, compared to Theorem 1.1 and Theorem 1.2, the geometric meaning of (1.1) is not clear, whence further study should be done.

Let (M, J) be a compact complex manifold satisfying the assumption of Theorem 1.3. Let $H^{1,1}(M, J; \mathbf{R}) := H^{1,1}(M, J) \cap H^2_{dR}(M)$. Then the map

$$H^{1,1}(M, J; \mathbf{R}) \to \mathbf{R}, \quad [\omega] \mapsto \int_M \Phi^* \sigma_{FS} \wedge \omega^{n-1}$$

is a well-defined continuous function. Thus this is bounded on the compact subset $\{[\omega] \in H^{1,1}(M, J; \mathbf{R}) \mid \int_M \omega^n = 1\}$. In other words, the functional λ_1 is bounded on the set of Kähler metrics with fixed volume on (M, J). However, the property of the λ_1-maximizing Kähler metrics has not been studied.

In the main part of this thesis, on a compact complex manifold (M, J), we introduce the notion of λ_k-extremal Kähler metric by considering all volume-preserving deformations of the Kähler metric. Be cautioned that we fix the complex structure J and consider only J-compatible Kähler metrics. (See Definition 5.1 for the precise definition of the λ_k-extremality.) The notion of λ_k-extremality introduced by the author is stronger than the extremality in Theorem 1.4, but weaker than that in Theorem 1.1. The first main theorem is the following:

Theorem 1.6. (Theorem 5.8) Let (M, J, g, ω) be a compact Kähler manifold. The Kähler metric g is λ_1-extremal if and only if there exists a finite collection of eigenfunctions $\{f_j\}_{j=1}^N$ such that the equations

$$(1.2) \qquad \begin{cases} H\left(\sum_{j=1}^{N} f_j dd^c f_j\right) = -\omega, \\ \lambda_1(g)^2\left(\sum_{j=1}^{N} f_j^2\right) - 2\lambda_1(g)\left(\sum_{j=1}^{N} |\nabla f_j|^2\right) + \sum_{j=1}^{N} |dd^c f_j|^2 = 0 \end{cases}$$

hold. Here H is the harmonic projector, which is defined due to the Hodge decomposition on a compact Kähler manifold.

It is obvious that (1.2) implies (1.1). We also give an example of a Kähler metric that is λ_1-extremal within its Kähler class, but not so for all volume-preserving deformations of the Kähler metric (Example 5.14 and Example 5.15).

In the final part of this thesis, we consider flat complex tori. It is known that for a flat tori, the multiplicity of each eigenvalue is even. We have the following proposition, which should be compared with Proposition 1.5.

Proposition 1.7. (Proposition 6.2) Let (T^n_Γ, g) be a flat complex torus determined by a lattice $\Gamma \subset \mathbf{C}^n$. Then the flat metric g is extremal for the functional λ_1 within its Kähler class.

Montiel–Ros [23] showed that among all the real 2-dimensional flat tori, only the square torus $\mathbf{R}^2/\mathbf{Z}^2$ admits an isometric minimal immersion into a 3-dimensional Euclidean sphere by first eigenfunctions. Later, using Theorem 1.1, El Soufi–Ilias [11] improved this result. That is, they proved that a real 2-dimensional flat torus admits an isometric minimal immersion into a Euclidean sphere of some dimension by first eigenfunctions if and only if the torus is the square torus or the equilateral torus. Recently, Lü–Wang–Xie [22] classified all the 3-dimensional tori and 4-dimensional tori that admit an isometric minimal immersion into a Euclidean sphere of some dimension by first eigenfunctions. For (real) dimension higher than 4, the standard torus is the only currently known example that admits an isometric minimal immersion into a Euclidean sphere by the first eigenfunctions. We will prove the following:

Theorem 1.8. (Theorem 6.3) Let (T^n_Γ, g) be a flat n-dimensional complex torus. Let $\{w_\nu\}_{\nu=1}^{l(\lambda_k(g))}$ be linearly independent vectors in Γ^* satisfying $\lambda_k(g) = 4\pi^2|w_\nu|^2$, where $l(\lambda_k(g))$ is half of the multiplicity of $\lambda_k(g)$. If the flat metric g is λ_k-extremal for all the volume-preserving deformations of the Kähler metric then there exists

$\{R_\nu \geq 0\}_{\nu=1}^{l(\lambda_k(g))}$ such that the following equations hold:

$$\begin{cases} \displaystyle\sum_{\nu=1}^{l(\lambda_k(g))} R_\nu \overline{w}_\nu^\alpha w_\nu^\beta = 0 \quad \text{for} \quad 1 \leq \alpha \neq \beta \leq n, \\ \displaystyle\sum_{\nu=1}^{l(\lambda_k(g))} R_\nu |w_\nu^\alpha|^2 = 1 \quad \text{for} \quad 1 \leq \alpha \leq n. \end{cases}$$

For $k = 1$, the converse also holds.

The notion of λ_1-extremality in the above theorem is stronger than the extremality in Theorem 1.4, but weaker than that in Theorem 1.1. Hence Theorem 1.8 gives a necessary condition for a flat complex torus to admit an isometric minimal immersion into a Euclidean sphere by first eigenfunctions. This is the only currently known necessary condition for a flat torus of dimension higher than 5 to admit an isometric minimal immersion into a Euclidean sphere by first eigenfunctions. We also show that if the multiplicity of the first eigenvalue of a flat complex torus is 2, then the flat metric on the torus is not λ_1-extremal in our sense (Corollary 6.4). We also show that the flat torus $\mathbf{R}^m/D_m$ ($m \geq 3$), where D_m is a lattice called the checkerboard lattice, admits an isometric minimal immersion by first eigenfunctions (Proposition 6.7). This result was obtained by Lü–Wang–Xie [22] for $m = 3, 4$. For (real) dimension higher than 4, the standard torus is the only currently known example that admits an isometric minimal immersion into a Euclidean sphere by the first eigenfunctions. Hence $\mathbf{R}^m/D_m$ ($m \geq 3$) are new examples. We consider several examples of 2-dimensional complex tori and see that the notion of λ_1-extremality actually depends on the complex structure on an underlying compact manifold (Example 6.8 and Example 6.9).

The main part of this thesis is based on the author's preprint [29].

Organization of this thesis. In Section 2, we review basic facts in Kähler geometry and convex geometry for later use. In Section 3, we review previous studies on behaviors of eigenvalues with respect to metric deformations. In Section 4, we review the work by Apostolov–Jakobson–Kokarev [1]. Section 5 and Section 6 are the main part of this thesis. In Section 5, we introduce the notion of λ_k-extremal Kähler metric different from that due to Apostolov et al., and prove Theorem 1.6 (Theorem 5.8). In Section 6, we consider whether the flat metric on a complex torus is λ_k-extremal. We prove Proposition 1.7 (Proposition 6.2) and Theorem 1.8 (Theorem 6.3).

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2. Preliminaries

2.1. **Notations and basic facts in Kähler geometry.** In this subsection, we fix some notations and review some basic facts in Kähler geometry for later use. We refer the readers to the textbooks [14], [24] and [34].

Let (M, J) be a complex manifold of complex dimension n. For $0 \le r \le n$, let $\Lambda^r T^*M$ be the exterior cotangent bundle and $\Omega^r(M)$ be the set of smooth sections of $\Lambda^r T^*M$. Let $\Lambda^r T^*M \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of $\Lambda^r T^*M$. We define a complex bundle $\Lambda^{p,q}M$ by

$$\Lambda^{p,q}M := \bigcup_{x\in M} \{x\} \times \mathrm{span}_{\mathbf{C}}\{(dz^{j_1})_x \wedge \cdots \wedge (dz^{j_p})_x \wedge (d\overline{z}^{k_1})_x \wedge \cdots \wedge (d\overline{z}^{k_q})_x \mid j_1 < \cdots < j_p,\ k_1 < \cdots < k_q\},$$

where $\{z^j\}_{j=1}^n$ is a local holomorphic coordinate around $x \in M$. $\Lambda^{p,q}M$ is a well-defined complex subbundle of $\Lambda^{p+q}T^*M \otimes_{\mathbf{R}} \mathbf{C}$. We have

$$\Lambda^r T^*M \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{p+q=r} \Lambda^{p,q}M.$$

Let $\Omega^{p,q}(M)$ be the set of smooth sections of $\Lambda^{p,q}M$, each element of which is called a *(p,q)-form*.

A Riemannian metric g on (M, J) is called *Hermitian* if g satisfies

$$g(X, Y) = g(JX, JY)$$

for all vector fields X, Y. The $\mathbf{C}$-bilinear extension of g to the complexified tangent bundle $TM \otimes_{\mathbf{R}} \mathbf{C}$ is also denoted by g. Set

$$g_{j\overline{k}} := g\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\right)$$

in a local holomorphic coordinate $\{z^j\}_{j=1}^n$. Then the $n \times n$ matrix $(g_{j\overline{k}})$ is a Hermitian matrix. A (p,q)-form α is called *real* if α satisfies the condition that $\overline{\alpha} = \alpha$. It immediately follows that if a (p,q)-form α is real, then α is a (p,p)-form. When a $(1,1)$-form α is locally expressed as $\alpha = \sqrt{-1}\alpha_{j\overline{k}} dz^j \wedge d\overline{z}^k$, α is real if and only if the matrix $(\alpha_{j\bar{k}})$ is Hermitian. For a Hermitian metric g on (M, J), the formula

$$\omega(X, Y) := g(JX, Y)$$

defines a real $(1,1)$-form ω. In a local holomorphic coordinate $\{z^j\}_{j=1}^n$, ω can be locally expressed as

$$\omega = \sqrt{-1} g_{j\bar{k}} dz^j \wedge d\bar{z}^k. \tag{2.1}$$

The Riemannian volume form $d\mu$ on (M, J, g, ω) is given by

$$d\mu = \frac{\omega^n}{n!}. \tag{2.2}$$

If ω is d-closed, then the Hermitian metric g is called a *Kähler metric*, ω is called a *Käher form* and (M, J, g, ω) is called a *Kähler manifold*. The following lemma is very useful in computations:

Lemma 2.1. Let (M, J, g, ω) be a Kähler manifold of complex dimension n. For any $x \in M$, there exists a local holomorphic coordinate $\{z^j\}_{j=1}^n$ around x such that the equations

$$g_{j\bar{k}}(x) = \delta_{jk}, \quad \frac{\partial g_{j\bar{k}}}{\partial z^l}(x) = 0, \qquad \frac{\partial g_{j\bar{k}}}{\partial \bar{z}^l}(x) = 0$$

hold for any $1 \leq j, k, l \leq n$.

Hereinafter, suppose that (M, J, g, ω) is a compact Kähler manifold (without boundary) of complex dimension n. Let $d\mu$ be its volume form. For $\alpha \in \Omega^1(M)$, a vector field $\alpha^\sharp$ is uniquely defined so that

$$\alpha(X) = g(\alpha^\sharp, X)$$

for any vector field X. This correspondence gives an isomorphism between T^*M and TM. We define a fiber metric g^* on T^*M by

$$g^*(\alpha, \beta) := g(\alpha^\sharp, \beta^\sharp).$$

For $\alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^r \in \Omega^1(M)$, we define

$$g_{\Lambda^r}(\alpha^1 \wedge \cdots \wedge \alpha^r, \beta^1 \wedge \cdots \wedge \beta^r) := \det\left(g^*(\alpha^j, \beta^k)\right)_{1 \leq j,k \leq r}.$$

We extend this $\mathbf{R}$-bilinearly and define a fiber metric g_{Λ^r} on $\Lambda^r T^*M$. The metric g_{Λ^r} can also be extended $\mathbf{C}$-bilinearly to $\Lambda^r T^*M \otimes_{\mathbf{R}} \mathbf{C}$ and we use the same notation g_{Λ^r} for the fiber metric. We define a Hermitian fiber metric h_{Λ^r} on $\Lambda^r T^*M \otimes_{\mathbf{R}} \mathbf{C}$ by

$$h_{\Lambda^r}(\alpha, \beta) := g_{\Lambda^r}(\alpha, \overline{\beta}).$$

If β is real, then we clearly have $h_{\Lambda^r}(\alpha,\beta) = g_{\Lambda^r}(\alpha,\beta)$. In what follows, we abbreviate h_{Λ^r} as h when there is no room for confusion. By (2.1), we have

$$|\omega|^2 := h(\omega,\omega) = -g_{j\bar{k}}g_{l\bar{m}} \det \begin{pmatrix} g^{jl} & g^{j\bar{m}} \\ g^{\bar{k}l} & g^{\bar{k}\bar{m}} \end{pmatrix} = g_{j\bar{k}}g_{l\bar{m}}g^{j\bar{m}}g^{l\bar{k}} = \delta_k^m\delta_m^k = n. \tag{2.3}$$

The following lemma will be used later:

Lemma 2.2. ([34, Lemma 4.7], [1]) Let (M, J, g, ω) be a Kähler manifold of complex dimension n. For any pair of real $(1,1)$-forms α and β, one has

$$h(\alpha,\beta)\omega^n = h(\alpha,\omega)h(\beta,\omega)\omega^n - n(n-1)\alpha\wedge\beta\wedge\omega^{n-2}.$$

For any $0 \leq p, q \leq n$, we define a $\mathbf{C}$-linear map $* : \Lambda^{p,q}M \to \Lambda^{n-q,n-p}M$ by

$$\alpha \wedge *\overline{\beta} = h(\alpha,\beta)d\mu \quad \text{for any } \alpha,\beta \in \Omega^{p,q}(M).$$

This map $*$ is called the *Hodge $*$-operator*. We have $(*|_{\Lambda^{p,q}M})^2 = (-1)^{p+q}$ and so $* : \Lambda^{p,q}M \to \Lambda^{n-q,n-p}M$ is an isomorphism. Using (2.2) and (2.3), one obtains

$$\omega \wedge (*\omega) = n \cdot d\mu = \frac{\omega^n}{(n-1)!}$$

and so

$$*\omega = \frac{\omega^{n-1}}{(n-1)!}.$$

This implies that the equation

$$\alpha \wedge \omega^{n-1} = (n-1)!\alpha \wedge *\omega = (n-1)!h(\alpha,\omega)\frac{\omega^n}{n!} = \frac{1}{n}h(\alpha,\omega)\omega^n \tag{2.4}$$

holds for any $(1,1)$-form α.

Set $d^c := \sqrt{-1}(\overline{\partial} - \partial)$. Then we have

$$dd^c = \sqrt{-1}(\partial + \overline{\partial})(\overline{\partial} - \partial) = 2\sqrt{-1}\partial\overline{\partial}.$$

Set

$$\delta := - * d*, \quad \delta^c := - * d^c *,$$
$$\partial^* := - * \overline{\partial} *, \quad \overline{\partial}^* := - * \partial * .$$

Then they are L^2-adjoint operators of d, d^c, ∂ and $\overline{\partial}$ respectively. We define the *Laplacian* Δ_g by $\Delta_g := d\delta + \delta d$. Since g is Kähler, we have

$$\Delta_g = 2(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) = 2(\partial\partial^* + \partial^*\partial). \tag{2.5}$$

If there exists $f \in C^\infty(M)\setminus\{0\}$ such that $\Delta_g f = \lambda f$, then λ is called an *eigenvalue* of the Laplacian Δ_g. It is known that the eigenvalues of the Laplacian Δ_g are nonnegative and form a discrete sequence that diverges to $+\infty$. We denote the eigenvalues by $0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \lambda_k(g) \leq \cdots$. For any $k \in \mathbf{N}$, let $E_k(g)$ be the vector space of real-valued eigenfunctions of Δ_g corresponding to $\lambda_k(g)$. That is, $E_k(g)$ is given by $E_k(g) = \mathrm{Ker}(\Delta_g - \lambda_k(g)I)$, where I is the identity map acting on functions. It is also known that $E_k(g)$ is finite dimensional for any $k \in \mathbf{N}$. The positive integer $\dim E_k(g)$ is called the *multiplicity* of $\lambda_k(g)$. The eigenvalue $\lambda_k(g)$ is called *simple* if its multiplicity is exactly 1. For a complex-valued function f, $\Delta_g f$ can be locally expressed as

$$\Delta_g f = -2g^{j\overline{k}} \frac{\partial^2 f}{\partial z^j \partial \overline{z}^k} \tag{2.6}$$

in a local holomorphic coordinate $\{z^j\}_{j=1}^n$ (see [34, p. 33]). Hence we have

$$\Delta_g f = -2g^{j\overline{k}} \frac{\partial^2 f}{\partial z^j \partial \overline{z}^k} = -2h(\sqrt{-1}\partial\overline{\partial} f, \omega) = -h(dd^c f, \omega). \tag{2.7}$$

Thus (2.4) and (2.7) imply

$$n dd^c f \wedge \omega^{n-1} = h(dd^c f, \omega)\omega^n = -(\Delta_g f)\omega^n. \tag{2.8}$$

A (p,q)-form α is called *harmonic* if it satisfies $\Delta_g \alpha = 0$, which is equivalent to the condition that α satisfies both $\overline{\partial}\alpha = 0$ and $\overline{\partial}^* \alpha = 0$. Set

$$\mathcal{H}^{p,q}(M) := \{\alpha \in \Omega^{p,q}(M) \mid \Delta_g \alpha = 0\}.$$

We state the Hodge–Dolbeault theorem:

Theorem 2.3. Let (M, J, g, ω) be a compact Kähler manifold. Then $\mathcal{H}^{p,q}(M)$ is finite dimensional. Furthermore, there exist unique operators $H : \Omega^{p,q}(M) \to \mathcal{H}^{p,q}(M)$ and $G : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$ such that all of the following hold:

$$G(\mathcal{H}^{p,q}(M)) = 0, \quad \overline{\partial} G = G\overline{\partial}, \quad \overline{\partial}^* G = G\overline{\partial}^* \text{ and}$$
$$\alpha = H(\alpha) + \frac{1}{2}\Delta_g G(\alpha) \quad \text{for any } \alpha \in \Omega^{p,q}(M).$$

Since we have (2.5), any $\alpha \in \Omega^{p,q}(M)$ can be written as

$$\alpha = \overline{\partial}^* \left(\overline{\partial} G(\alpha)\right) + H(\alpha) + \overline{\partial}\left(\overline{\partial}^* G(\alpha)\right).$$

This gives the following L^2-orthogonal decomposition:

$$\Omega^{p,q}(M) = \overline{\partial}^* \Omega^{p,q+1}(M) \oplus \mathcal{H}^{p,q}(M) \oplus \overline{\partial}\Omega^{p,q-1}(M). \tag{2.9}$$

We call the L^2-orthogonal projection H a *harmonic projector*. We have the following lemma:

Lemma 2.4. Let (M, J, g, ω) be a compact Kähler manifold. Let $H : \Omega^{p,p}(M) \to \mathcal{H}^{p,p}(M)$ be the harmonic projector. Then $H(\alpha)$ is real for any real (p,p)-form α.

We state the dd^c-lemma:

Lemma 2.5. (dd^c-lemma) Let (M, J, g, ω) be a compact Kähler manifold and α a real d-closed $(1,1)$-form. If α is d-, $\overline{\partial}$- or ∂-exact, then there exists a real valued smooth function φ such that $\alpha = dd^c\varphi$.

2.2. **Hyperplane separation.** In this subsection, we recall some basic facts about hyperplane separation in a finite dimensional vector space for later use. For details, we refer the readers to the book [15] for example.

Definition 2.6. ([15, p.42, p.46]) Let V be a finite dimensional vector space over $\mathbf{R}$. If $x \in V$ can be expressed as $x = \sum_{j=1}^{n} a_j x_j$ for some $\{x_j\}_{j=1}^{n} \subset V$ and $\{a_j \geq 0\}_{j=1}^{n}$ with $\sum_{j=1}^{n} a_j = 1$, then x is called a *convex combination* of $\{x_j\}_{j=1}^{n}$. Let A be a subset of V. The *convex hull* of A is defined as the set of all convex combinations of points in A. Let $U := \{u_1, \dots, u_n\}$ be a nonempty finite subset of V. The *positive hull* of U is defined as the set

$$\{a_1u_1 + \cdots a_nu_n \in V \mid a_j \geq 0 \text{ for all } 1 \leq j \leq n\}.$$

We have the following propositions:

Proposition 2.7. ([15, p.44]) Let V be a finite dimensional vector space over $\mathbf{R}$. If a subset $A \subset V$ is compact, then the convex hull of A in V is also compact.

Proposition 2.8. ([15, p.46]) Let V be a finite dimensional vector space over $\mathbf{R}$. Let $U := \{u_1, \dots, u_n\}$ be a nonempty finite subset of V. The positive hull of U in V is a closed convex cone.

Definition 2.9. ([15, pp. 53-54]) Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space over $\mathbf{R}$. Let S be a closed subset of V and x its boundary point. $H_S(x)$ is called a *support hyperplane* of S at x if it satisfies the following:

(1) $H_S(x)$ is a hyperplane containing x with some normal vector $u \in V$. That is, $H_S(x)$ is given by

$$H_S(x) = \{y \in V \mid \langle y - x, u\rangle = 0\}$$

for some $u \in V \setminus \{0\}$.

(2) S is contained in $H_S^-(x) := \{y \in V \mid \langle y - x, u\rangle \leq 0\}$.

For a support hyperplane $H_S(x)$, the closed half-space $H_S^-(x)$ is called a *support half-space* of S at x.

Note that $H_S(x)$ is not necessarily unique.

Theorem 2.10. ([15, pp. 54-55]) Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space over $\mathbf{R}$. Let S be a closed convex subset of V that has an interior point. Then there exists a support hyperplane $H_S(x)$ for any boundary point $x \in \partial S$. Furthermore, S is precisely the intersection of all the support half-spaces of S at the boundary points.

We state the hyperplane separation theorem in such a way that it can be used in the proofs of Proposition 4.9 and Theorem 5.7.

Theorem 2.11. Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space over $\mathbf{R}$.

(1) Let S be a closed convex subset of V that has an interior point. Suppose that $0 \notin S$. Then there exists $u \in V$ such that $\langle s, u\rangle < 0$ for all $s \in S$.

(2) Let S be a positive hull of some finite points. For any $v \notin S$, there exists $u \in V$ such that $\langle v, u\rangle > 0$ and $\langle y, u\rangle \leq 0$ for all $y \in S$.

Proof. Suppose that $0 \notin S$. Then by Theorem 2.10, there exists $x \in \partial S$ and a support hyperplane $H_S(x)$ such that $0 \notin H_S^-(x)$. Let u be the normal vector of $H_S(x)$. Then we have $\langle -x, u\rangle > 0$. Hence for any $y \in S$, we have

$$\langle y, u\rangle \leq \langle x, u\rangle < 0.$$

Hence the assertion (1) is proved.

We show the assertion (2). By Proposition 2.8, S is closed and convex. Theorem 2.10 implies that for any $v \notin S$, there exists $x \in \partial S$ and a support hyperplane $H_S(x)$ such that $v \notin H_S^-(x)$. Let u be the normal vector of $H_S(x)$. Then we have $\langle v - x, u\rangle > 0$. Since S is a positive hull, we have $2x, 0 \in S \subset H_S^-(x)$ in particular. Hence we have

$$\langle 2x - x, u\rangle \leq 0 \quad \text{and} \quad \langle 0 - x, u\rangle \leq 0.$$

Hence we obtain $\langle x, u\rangle = 0$. Thus we have $\langle v, u\rangle > 0$ and $H_S^-(x)$ is represented as

$$H_S^-(x) = \{y \in V \mid \langle y, u\rangle \leq 0\}.$$

Since we have $S \subset H_S^-(x)$, the proof is completed. □

3. Behaviors of eigenvalues with respect to metric deformations

Let (M, g) be a compact Riemannian manifold without boundary. As in the previous subsection, the Laplacian is denoted by Δ_g and eigenvalues Laplacian Δ_g are denoted by $0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \cdots \leq \lambda_k(g) \leq \cdots$. For any positive integer k, we regard λ_k as a functional on the set of Riemannian metrics on M. It is classically known that for any 1-parameter family of Riemannian metrics $\{g_t\}$ that is continuous in t, $\lambda_k(g_t)$ is continuous in t. Using the Kato–Rellich perturbation theory of self-adjoint operators, Berger [2] and Bando–Urakawa [3] proved the following:

Theorem 3.1 ([2], [3])**.** Let (M, g) be a compact Riemannian manifold without boundary. Let $I \subset \mathbf{R}$ be an open interval containing 0. Let $\{g_t\}_{t\in I}$ be a 1-parameter family of Riemannian metrics that depends real analytically on t with $g_0 = g$. Set $\tau := \dim E_k(g)$. Then there exist $\{\Lambda_i(t)\}_{i=1}^{\tau} \subset \mathbf{R}$ and $\{u_i(t)\}_{i=1}^{\tau} \subset C^\infty(M)$ satisfying all the following:

(1) For any $1 \leq i \leq \tau$, $\Lambda_i(t)$ and $u_i(t)$ are defined over I and depend real analytically on t.

(2) For any $1 \leq i \leq \tau$ and any $t \in I$, $\Delta_{g_t} u_i(t) = \Lambda_i(t) u_i(t)$ holds.

(3) For any $1 \leq i \leq \tau$, $\Lambda_i(0) = \lambda_k(g)$ holds.

(4) For any $t \in I$, $\{u_i(t)\}_{i=1}^{\tau}$ is $L^2(g_t)$-orthonormal.

Using the above theorem, El Soufi–Ilias [13] proved the following theorem:

Theorem 3.2 ([13])**.** Let (M, g) be a compact Riemannian manifold without boundary. Let $I \subset \mathbf{R}$ be an open interval containing 0. Let $\{g_t\}_t$ be a 1-parameter family of Riemannian metrics that depends real-analytically on t with $g_0 = g$. Let $\Pi_k : L^2(M, g) \to E_k(g)$ be the orthogonal projection onto $E_k(g)$. Define the operator $P_k : E_k(g) \to E_k(g)$ by

$$P_k(f) := \Pi_k \left(\left. \frac{d}{dt} \right|_{t=0} \Delta_{g_t} f \right). \tag{3.1}$$

Then the following hold:

(1) The function $I \ni t \mapsto \lambda_k(g_t)$ admits left and right derivatives at $t = 0$, i.e. $\frac{d}{dt}\big|_{t=0^-} \lambda_k(g_t)$ and $\frac{d}{dt}\big|_{t=0^+} \lambda_k(g_t)$ exist.

(2) $\frac{d}{dt}\big|_{t=0^-} \lambda_k(g_t)$ and $\frac{d}{dt}\big|_{t=0^+} \lambda_k(g_t)$ are eigenvalues of P_k.

(3) If $\lambda_k(g) > \lambda_{k-1}(g)$, then $\frac{d}{dt}\big|_{t=0^-} \lambda_k(g_t)$ and $\frac{d}{dt}\big|_{t=0^+} \lambda_k(g_t)$ are the greatest and the least eigenvalues of P_k.

(4) If $\lambda_k(g) < \lambda_{k+1}(g)$, then $\frac{d}{dt}\big|_{t=0^-} \lambda_k(g_t)$ and $\frac{d}{dt}\big|_{t=0^+} \lambda_k(g_t)$ are the least and the greatest eigenvalues of P_k.

(5) P_k is symmetric with respect to $L^2(g)$-inner product.

Proof. Set $\tau := \dim E_k(g)$. By the assumption, there exist $\{\Lambda_i(t)\}_{i=1}^{\tau} \subset \mathbf{R}$ and $\{u_i(t)\}_{i=1}^{\tau} \subset C^\infty(M)$ as in Theorem 3.1. For any $1 \le i \le \tau$, $\Lambda_i(t)$ is continuous in t and satisfies $\Lambda_i(0) = \lambda_k(g)$. Hence there exist integers $1 \le p, q \le \tau$ such that

$$\lambda_k(g_t) = \begin{cases} \Lambda_p(t) & \text{for } -\delta \le t \le 0 \\ \Lambda_q(t) & \text{for } 0 \le t \le \delta. \end{cases} \tag{3.2}$$

$\Lambda_i(t)$ is real analytic in t for any $1 \le i \le \tau$ and so we obtain

$$\frac{d}{dt}\bigg|_{t=0^-} \lambda_k(g_t) = \Lambda_p'(0)$$

and

$$\frac{d}{dt}\bigg|_{t=0^+} \lambda_k(g_t) = \Lambda_q'(0).$$

This proves the assertion (1).

By Theorem 3.1 (2), $\Delta_{g_t} u_i(t) = \Lambda_i(t) u_i(t)$ holds for any $1 \le i \le \tau$ and any $t \in I$. Differentiating the both sides of this equation at $t = 0$, one obtains

$$\left(\frac{d}{dt}\bigg|_{t=0} \Delta_{g_t} u_i \right) + \Delta_g u_i' = \Lambda_i'(0) u_i + \lambda_k(g) u_i', \tag{3.3}$$

where $u_i := u_i(0)$ and $u_i' := \frac{d}{dt}\big|_{t=0} u_i(t)$. Let $d\mu$ be the Riemannian measure with respect to g. By Theorem 3.1 (4), $\{u_i\}_{i=1}^{\tau}$ is an $L^2(g)$-orthonormal basis of $E_k(g)$.

Hence by (3.3) and Stokes Theorem, one obtains

$$\begin{aligned}\int_M u_j \left(\frac{d}{dt}\bigg|_{t=0} \Delta_{g_t} u_i \right) d\mu &= \Lambda_i'(0) \int_M u_j u_i d\mu + \lambda_k(g) \int_M u_j u_i' d\mu - \int_M u_j \Delta_g u_i' d\mu \\ &= \Lambda_i'(0)\delta_{ij} + \lambda_k(g) \int_M u_j u_i' d\mu - \int_M (\Delta_g u_j) u_i' d\mu \\ &= \Lambda_i'(0)\delta_{ij} + \lambda_k(g) \int_M u_j u_i' d\mu - \lambda_k(g) \int_M u_j u_i' d\mu \\ &= \Lambda_i'(0)\delta_{ij}.\end{aligned}$$

Since $\{u_i\}_{i=1}^{\tau}$ is an $L^2(g)$-orthonormal basis of $E_k(g)$, one has $P_k(u_i) = \Lambda_i'(0)u_i$ for any $1 \leq i \leq \tau$. Hence the assertion (2) follows from this equation and the equation (3.2).

Next we prove the assertion (3). For any $1 \leq i \leq \tau$, we have $\Lambda_i(0) = \lambda_k(g)$ and $\Lambda_i(t)$ is continuous in $t \in I$. Hence there exists $\delta > 0$ such that $\Lambda_i(t) > \lambda_{k-1}(g_t)$ for any $t \in (-\delta, \delta)$ and any $1 \leq i \leq \tau$. Since $\Lambda_i(t)$ is an eigenvalue of Δ_{g_t}, one can deduce that $\Lambda_i(t) \geq \lambda_k(g_t)$ for any $t \in (-\delta, \delta)$ and any $1 \leq i \leq \tau$. This implies that $\lambda_k(g_t) = \min\{\Lambda_1(t), \ldots, \Lambda_\tau(t)\}$. Since we have $\Lambda_i(0) = \lambda_k(g)$ for $1 \leq i \leq \tau$, we conclude that

$$\frac{d}{dt}\bigg|_{t=0^-} \lambda_k(g_t) = \max\{\Lambda_1'(0), \ldots, \Lambda_\tau'(0)\} \text{ and } \frac{d}{dt}\bigg|_{t=0^+} \lambda_k(g_t) = \min\{\Lambda_1'(0), \ldots, \Lambda_\tau'(0)\}.$$

The assertion (3) is proved.

The proof of the assertion (4) is similar to that of (3) and so is omitted.

Take any $u, v \in E_k(g)$. Let $d\mu_t$ be the Riemannian measure with respect to g_t. By Stokes Theorem, we have

$$\begin{aligned}&\int_M u P_k(v) d\mu \\ =& \int_M u \left(\frac{d}{dt}\bigg|_{t=0} \Delta_{g_t} v \right) d\mu \\ =& \frac{d}{dt}\bigg|_{t=0} \left(\int_M u \Delta_{g_t} v d\mu_t \right) - \int_M u(\Delta v) \left(\frac{d}{dt}\bigg|_{t=0} d\mu_t \right) \\ =& \frac{d}{dt}\bigg|_{t=0} \left(\int_M (\Delta_{g_t} u) v d\mu_t \right) - \lambda_k(g) \int_M uv \left(\frac{d}{dt}\bigg|_{t=0} d\mu_t \right) \\ =& \int_M \left(\frac{d}{dt}\bigg|_{t=0} \Delta_{g_t} u \right) v d\mu + \int_M (\Delta u) v \left(\frac{d}{dt}\bigg|_{t=0} d\mu_t \right) - \lambda_k(g) \int_M uv \left(\frac{d}{dt}\bigg|_{t=0} d\mu_t \right) \\ =& \int_M \left(\frac{d}{dt}\bigg|_{t=0} \Delta_{g_t} u \right) v d\mu \\ =& \int_M P_k(u) v d\mu.\end{aligned}$$

The proof is completed. $\square$

We introduce the following notion:

Definition 3.3 ([27], [11], [13])**.** Let (M, g) be a compact Riemannian manifold without boundary. The metric g is said to be λ_k-*extremal* if the inequality

$$\left(\frac{d}{dt}\bigg|_{t=0^-} \lambda_k(g_t)\right) \cdot \left(\frac{d}{dt}\bigg|_{t=0^+} \lambda_k(g_t)\right) \leq 0.$$

holds for any volume-preserving 1-parameter family of Riemannian metrics $\{g_t\}_{t \in I}$ that depends real analytically on t and satisfies $g_0 = g$.

Nadirashvili [27] proved that if a metric g on a given surface M is λ_k-extremal, then there exists a finite collection of $\lambda_k(g)$-eigenfunctions $\{f_j\}_{j=1}^N$ such that $F := (f_1, \cdots, f_N) : (M, g) \to \mathbf{R}^N$ is an isometric minimal immersion into a round sphere in $\mathbf{R}^N$. After that, El Soufi–Ilias [11] simplified the proof of this theorem and generalized it to a compact manifold M of any dimension for $k = 1$. Later, El Soufi–Ilias [13] improved this result and proved the following:

Theorem 3.4 ([27], [11], [13])**.** Let (M, g) be a compact m-dimensional Riemannian manifold without boundary. Let $E_k(g)$ be the space of $\lambda_k(g)$-eigenfunctions. If the metric g is λ_k-extremal, then there exists a finite collection of $\lambda_k(g)$-eigenfunctions $\{f_1, \ldots, f_N\} \subset E_k(g)$ such that $F := (f_1, \cdots, f_N) : (M, g) \to \mathbf{R}^N$ is an isometric minimal immersion into $S^{N-1}(\sqrt{m/\lambda_k(g)}) \subset \mathbf{R}^N$. For $k = 1$, the existence of such a finite collection of $\lambda_k(g)$-eigenfunctions is also a sufficient condition for the metric g to be λ_1-extremal.

Example 3.5 ([35], [11])**.** A homogeneous manifold G/K is said to be *isotropy irreducible* if the linear isotropy representation of the isotropy subgroup K at the point $eK \in G/K$ is irreducible. Takahashi [35] proved that a compact isotropy irreducible homogeneous Riemannian manifold admits an isometric minimal immersion into a Euclidean sphere by first eigenfunctions. Hence the metric on a compact isotropy irreducible homogeneous manifold is λ_1-extremal.

In contrast to the above theorem, we can restrict ourselves to considering volume-preserving metric deformations only in a certain class of metrics. El Soufi–Ilias [12] considered all the volume-preserving deformations of a given metric within its conformal class.

Definition 3.6 ([12])**.** Let (M, g) be a compact Riemannian manifold without boundary. Let $C(g)$ be the conformal class of g. The metric g is said to be

λ_k*-extremal within its conformal class* if the inequality

$$\left(\frac{d}{dt}\bigg|_{t=0^-} \lambda_k(g_t) \right) \cdot \left(\frac{d}{dt}\bigg|_{t=0^+} \lambda_k(g_t) \right) \leq 0.$$

holds for any volume-preserving 1-parameter family of Riemannian metrics $\{g_t\}_{t\in I} \subset C(g)$ that depends real analytically on t and satisfies $g_0 = g$.

Theorem 3.7 ([12])**.** Let (M, g) be a compact m-dimensional Riemannian manifold without boundary. Let $E_k(g)$ be the space of $\lambda_k(g)$-eigenfunctions. If the metric g is λ_k-extremal within its conformal class, then there exists a finite collection of $\lambda_k(g)$-eigenfunctions $\{f_1, \ldots, f_N\} \subset E_k(g)$ such that $F := (f_1, \cdots, f_N) : (M, g) \to \mathbf{R}^N$ is a harmonic map into $S^{N-1}(\sqrt{m/\lambda_k(g)}) \subset \mathbf{R}^N$ with constant energy density $|dF|^2 \equiv m$. For $k = 1$, the existence of such a finite collection of $\lambda_k(g)$-eigenfunctions is also a sufficient condition for the metric g to be λ_1-extremal.

It is well known that a smooth map $\phi = (\phi_1, \ldots, \phi_N) : (M, g) \to S^{N-1}(c) \subset \mathbf{R}^N$ is a harmonic map if and only if the condition $\Delta_g \phi_j = c^{-2}|d\phi|^2\phi_j$ holds for any $1 \leq j \leq N$. Hence an isometric minimal immersion consisting of $\lambda_k(g)$-eigenfunctions $F = (f_1, \ldots, f_N) : (M, g) \to S^{N-1}(\sqrt{m/\lambda_k(g)})$ is in particular a harmonic map with constant energy density $|dF|^2 \equiv m$. This is a natural fact since the assumption of the above theorem is weaker than that of Theorem 3.4.

Example 3.8 ([12])**.** The metric on a compact homogeneous Riemannian manifold is λ_1-extremal within its conformal class.

4. Work by Apostolov–Jakobson–Kokarev

In this section, we review the work by Apostolov–Jakobson–Kokarev [1]. To explain their research backgrounds, we quote the following result due to Bourguignon–Li–Yau [7]:

Theorem 4.1 ([7])**.** Let (M, J) be a compact complex n-dimensional manifold admitting a full holomorphic immersion $\Phi : (M, J) \to \mathbf{C}P^N$. Let σ_{FS} be the Fubini-Study form on $\mathbf{C}P^N$ with constant holomorphic sectional curvature 1. Then, for any Kähler form ω on (M, J), the first eigenvalue $\lambda_1(\omega)$ satisfies

$$\lambda_1(\omega) \leq n\frac{N+1}{N}\frac{\int_M \Phi^*\sigma_{FS} \wedge \omega^{n-1}}{\int_M \omega^n}.$$

Stokes theorem implies that the functional $\lambda_1(\omega)$ is bounded by a constant depending on only n, N, Φ and the Kähler class $[\omega]$. The above theorem implies that the Fubini-Study metric on $\mathbf{C}P^N$ is a λ_1-maximizer in its Kähler class. Biliotti-Ghigi[5] generalized the fact as follows:

Theorem 4.2 ([5]). The Kähler-Einstein metric on an irreducible Hermitian symmetric space of compact type maximizes the functional λ_1 in its Kähler class.

Motivated by these results, Apostolov–Jakobson–Kokarev [1] considered deformations of a given Kähler metric within its Kähler class. Let (M, J, g, ω) be a compact Kähler manifold (without boundary) of complex dimension n. Let $d\mu$ be its volume form. Let $E_k(g)$ be the real vector space of real-valued $\lambda_k(g)$-eigenfunctions. Let $K_{[\omega]}(M, J)$ be the space of Kähler metrics whose Kähler forms are cohomologous to ω. Set

$$C_0^\infty(M; \mathbf{R}) := \left\{ \varphi \in C^\infty(M; \mathbf{R}) \mid \int_M \varphi \, d\mu = 0 \right\}.$$

Then the dd^c-lemma (Lemma 2.5) gives a bijection between $K_{[\omega]}(M, J)$ and the set

$$\{\varphi \in C_0^\infty(M; \mathbf{R} \mid \omega + dd^c\varphi > 0\},$$

where $\omega + dd^c\varphi > 0$ means that the associated J-invariant bilinear form is positive definite and so a Kähler metric. Apostolov–Jakobson–Kokarev [1] introduced the following notion:

Definition 4.3 ([1]). Let (M, J, g, ω) be a compact Kähler manifold (without boundary) of complex dimension n. Let $K_{[\omega]}(M, J)$ be the space of Kähler metrics whose Kähler classes are all equal to $[\omega]$. The Kähler metric g is said to be λ_k-*extremal within its Kähler class* if the inequality

$$\left(\frac{d}{dt}\bigg|_{t=0^-} \lambda_k(g_t) \right) \cdot \left(\frac{d}{dt}\bigg|_{t=0^+} \lambda_k(g_t) \right) \leq 0.$$

holds for any 1-parameter family of Kähler metrics $\{g_t\}_{t \in I} \subset K_{[\omega]}(M, J)$ that depends real analytically on t and satisfies $g_0 = g$.

Fix any 1-parameter family of Kähler metrics $\{g_t\}_{t \in I} \subset K_{[\omega]}(M, J)$ that depends real analytically on t and satisfies $g_0 = g$. Let $P_k : E_k(g) \to E_k(g)$ be the associated operator given by (3.1). P_k is determined by g and $\frac{d}{dt}\big|_{t=0} g_t$. By Theorem 3.2(2), $\frac{d}{dt}\big|_{t=0^-} \lambda_k(g_t)$ and $\frac{d}{dt}\big|_{t=0^+} \lambda_k(g_t)$ are eigenvalues of P_k. Hence when we study λ_k-extremality, we may assume that $\omega_t = \omega + t dd^c\varphi$ with $\varphi \in C_0^\infty(M; \mathbf{R})$. Note that since M is compact, the condition that $\omega_t > 0$ is satisfied for a sufficiently small t. By Theorem 3.2 (5), P_k is symmetric with respect to the $L^2(g)$-inner product.

Hence one can consider the corresponding quadratic form Q_φ on $E_k(g)$, which is given by

$$Q_\varphi(f) := \int_M f P_k(f)\, d\mu = \int_M f \left(\frac{d}{dt}\bigg|_{t=0} \Delta_{g_t} f \right) d\mu. \tag{4.1}$$

The following proposition is an immediate consequence of Theorem 3.2 (2):

Proposition 4.4 ([1])**.** Let (M, J, g, ω) be a compact Kähler manifold. If the metric g is λ_k-extremal within its Kähler class, then the quadratic form Q_φ, defined in (4.1), is indefinite on $E_k(g)$ for any $\varphi \in C_0^\infty(M; \mathbf{R})$.

We also have the following proposition, which follows form Theorem 3.2 (3) and (4):

Proposition 4.5 ([1])**.** Let (M, J, g, ω) be a compact Kähler manifold. Suppose that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$ holds. Then the metric g is λ_k-extremal within its Kähler class if and only if the quadratic form Q_φ is indefinite on $E_k(g)$ for any $\varphi \in C_0^\infty(M; \mathbf{R})$.

The next corollary immediately follows.

Corollary 4.6 ([1])**.** Let (M, J, g, ω) be a compact Kähler manifold. Then the metric g is λ_1-extremal if and only if the quadratic form Q_α is indefinite on $E_1(g)$ for any $\varphi \in C_0^\infty(M; \mathbf{R})$.

The following lemma is important in [1], but we omit the proof since we will prove the generalization of this lemma (Theorem 5.6):

Lemma 4.7 ([1])**.** For any $\varphi \in C_0^\infty(M; \mathbf{R})$, the quadratic form Q_φ, which is given by (4.1), can be expressed as

$$Q_\varphi(f) = \int_M \varphi \delta^c \delta(f dd^c f) d\mu.$$

Motivated by this lemma, Apostolov–Jakobson–Kokarev[1] introduced the fourth order differential operator L defined by $L(f) := \delta^c\delta(f dd^c f)$, whence $Q_\varphi(f)$ can be written as $Q_\varphi(f) = \int_M \varphi L(f) d\mu$. Obviously we have

$$\int_M L(f) d\mu = \int_M h(dd^c 1, dd^c f) d\mu = 0. \tag{4.2}$$

$L(f)$ does not have a simpler expression in general, but it does if f is an eigenfunction:

Lemma 4.8 ([1])**.** Let (M, J, g, ω) be a compact Kähler manifold of complex dimension n. Let $E_k(g)$ be the real vector space of real-valued $\lambda_k(g)$-eigenfunctions. Then for any $f \in E_k(g)$, $L(f)$ is expressed as

$$L(f) = \lambda_k(g)^2 f^2 - 2\lambda_k(g)|\nabla f|^2 + |dd^c f|^2.$$

We give a proof relying on Lemma 2.1, which is different from the original one due to Apostolov–Jakobson–Kokarev [1].

Proof. Fix an arbitrary point $x \in M$. It suffices to prove the lemma at x. We have $dd^c = 2\sqrt{-1}\partial\overline{\partial}$ and $\delta^c\delta = 2\sqrt{-1}\partial^*\overline{\partial}^*$. Hence we have $L(f) = -4\partial^*\overline{\partial}^*(f\partial\overline{\partial}f)$. By Lemma 2.1, there exists a local holomorphic coordinate $\{z^\alpha\}_{\alpha=1}^n$ around x such that the equations

$$g_{\alpha\overline{\beta}}(x) = \delta_{\alpha\beta}, \quad \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma}(x) = 0, \qquad \frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^\gamma}(x) = 0$$

hold for any $1 \le \alpha, \beta, \gamma \le n$. Then we have

$$d\mu = \frac{\omega^n}{n!} = (\sqrt{-1})^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n$$

at x. Since $\overline{\partial}^*$ is a first order differential operator, we may compute $\overline{\partial}^*(f\partial\overline{\partial}f)$ with respect to this coordinate. Then it is straightforward to obtain

$$\begin{aligned}
*(f\partial\overline{\partial}f) = (\sqrt{-1})^n \sum_{\alpha<\beta} f\frac{\partial^2 f}{\partial z^\alpha \partial \overline{z}^\beta} dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge \widehat{d\overline{z}^\alpha} \wedge \cdots \wedge \widehat{dz^\beta} \wedge \cdots \wedge dz^n \wedge d\overline{z}^n \\
- (\sqrt{-1})^n \sum_{\beta\le\alpha} f\frac{\partial^2 f}{\partial z^\alpha \partial \overline{z}^\beta} dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge \widehat{dz^\beta} \wedge \cdots \wedge \widehat{d\overline{z}^\alpha} \wedge \cdots \wedge dz^n \wedge d\overline{z}^n
\end{aligned}$$

at x, where $\widehat{a}$ denotes the omission of a. Hence we have

$$(\partial *)(f\partial\overline{\partial}f) = -(\sqrt{-1})^n \sum_{\alpha,\beta=1}^n \frac{\partial}{\partial z^\beta}\left(f\frac{\partial^2 f}{\partial z^\alpha \partial \overline{z}^\beta}\right) dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge \widehat{d\overline{z}^\alpha} \wedge \cdots \wedge dz^n \wedge d\overline{z}^n$$

at x. Thus we obtain

$$\begin{aligned}
\overline{\partial}^*(f\partial\overline{\partial}f) &= \sum_{j,k=1}^n \frac{\partial}{\partial z^\beta}\left(f\frac{\partial^2 f}{\partial z^\alpha \partial \overline{z}^\beta}\right) dz^\alpha \\
&= \sum_{j,k=1}^n \left(\frac{\partial f}{\partial z^\beta}\frac{\partial^2 f}{\partial z^\alpha \partial \overline{z}^\beta} + f\frac{\partial^3 f}{\partial z^\alpha \partial z^\beta \partial \overline{z}^\beta}\right) dz^\alpha \\
&= \sum_{j,k=1}^n \frac{\partial f}{\partial z^\beta}\frac{\partial^2 f}{\partial z^\alpha \partial \overline{z}^\beta} dz^\alpha - \frac{\lambda_k(g)}{2}\sum_{j=1}^n f\frac{\partial f}{\partial z^\alpha} dz^\alpha \\
&= (\partial\overline{\partial}f)\left(\cdot, (\partial f)^\sharp\right) - \frac{\lambda_k(g)}{2} f\partial f
\end{aligned}$$

at x where (2.7) is used at the third equality. The rightmost side is coordinate-free. Hence we may compute the value of $L(f) = -4\partial^*\overline{\partial}^*(f\partial\overline{\partial}f)$ at x with respect to the coordinate $\{z^\alpha\}_{\alpha=1}^n$. Then we readily have

$$\begin{aligned}
L(f) &= -4\partial^*\overline{\partial}^*(f\partial\overline{\partial}f) \\
&= 4\sum_{\alpha,\beta=1}^n \frac{\partial}{\partial\overline{z}^\alpha}\left(\frac{\partial f}{\partial z^\beta}\frac{\partial^2 f}{\partial z^\alpha\partial\overline{z}^\beta}\right) - 2\lambda_k(g)\sum_{\alpha=1}^n \frac{\partial}{\partial\overline{z}^\alpha}\left(f\frac{\partial f}{\partial z^\alpha}\right) \\
&= 4\sum_{\alpha,\beta=1}^n \left|\frac{\partial^2 f}{\partial z^\alpha\partial\overline{z}^\beta}\right|^2 - 2\lambda_k(g)\sum_{\beta=1}^n \left|\frac{\partial f}{\partial z^\beta}\right|^2 - 2\lambda_k(g)\sum_{\alpha=1}^n \left|\frac{\partial f}{\partial z^\alpha}\right|^2 + \lambda_k(g)^2 f^2 \\
&= |dd^c f|^2 - 2\lambda_k(g)|\nabla f|^2 + \lambda_k(g)^2 f^2.
\end{aligned}$$

The proof is completed. □

Next we prove the following proposition:

Proposition 4.9 ([1])**.** Let (M, J, g, ω) be a compact Kähler manifold of complex dimension n. The following are equivalent:

(1) For any $\varphi \in C_0^\infty(M; \mathbf{R})$, the quadratic form Q_φ given by (4.1) is indefinite on the eigenspace $E_k(g)$.

(2) There exists a nontrivial finite subset $\{f_1, \dots, f_N\} \subset E_k(g)$ such that

$$\sum_{j=1}^N L(f_j) = \lambda_k(g)^2 \sum_{j=1}^N f_j^2 - 2\lambda_k(g)\sum_{j=1}^N |\nabla f_j|^2 + \sum_{j=1}^N |dd^c f_j|^2 = 0. \tag{4.3}$$

Proof. Let K be a convex hull of $\{L(f) \mid f \in E_k(g), \int_M f^2\, d\mu = 1\}$ in $C^\infty(M; \mathbf{R})$. Since $E_k(g)$ is finite dimensional, K is contained in a finite dimensional subspace of $C^\infty(M; \mathbf{R})$. Let $C^\infty(M; \mathbf{R})$ be endowed with the L^2-inner product. We assume that $0 \notin K$. Then by Proposition 2.7 and the hyperplane separation theorem (Theorem 2.11 (1)), there exists $\varphi \in C^\infty(M; \mathbf{R})$ such that

$$\int_M \varphi s\, d\mu > 0 \tag{4.4}$$

for any $s \in K$. Define $\widetilde{\varphi} \in C_0^\infty(M : \mathbf{R})$ by

$$\widetilde{\varphi} := \varphi - \frac{1}{\mathrm{Vol}(M, g)}\int_M \varphi\, d\mu.$$

Then by (4.2) and (4.4), we have

$$Q_{\widetilde{\varphi}}(f) = \int_M \varphi L(f)\, d\mu - \frac{1}{\mathrm{Vol}(M,g)} \left(\int_M \varphi\, d\mu \right) \left(\int_M L(f)\, d\mu \right) > 0$$

for any $f \in E_k(g) \setminus \{0\}$. Hence we have proved (1) $\Rightarrow$ (2).

We show the converse. We assume that there exists a nontrivial finite subset $\{f_1, \ldots, f_N\} \subset E_k(g)$ satisfying (4.3). Then for any $\varphi \in C_0^\infty(M; \mathbf{R})$, we have

$$\sum_{j=1}^N Q_\varphi(f_j) = \sum_{j=1}^N \int_M \varphi L(f_j)\, d\mu = \int_M \varphi \left(\sum_{j=1}^N L(f_j) \right) d\mu = 0$$

and so Q_φ is indefinite on $E_k(g)$. The proof is completed. □

Combining Proposition 4.4, Corollary 4.6 and Proposition 4.9, one concludes the following:

Theorem 4.10 ([1])**.** Let (M, J, g, ω) be a compact Kähler manifold of complex dimension n. If g is λ_k-extremal within its Kähler class, then there exists a nontrivial finite subset $\{f_1, \ldots, f_N\} \subset E_k(g)$ satisfying (4.3). If $k = 1$, the existence of such a finite subset of $E_k(g)$ is also a sufficient condition for the metric g to be λ_1-extremal within its Kähler class.

Corollary 4.11. [1] Let (M, J, g, ω) be a compact Kähler manifold. If the metric g is λ_k-extremal within its Kähler class, then the eigenvalue $\lambda_k(g)$ is not simple.

Proof. Assume that $\lambda_k(g)$ is simple and take $f \in E_k(g) \setminus \{0\}$ arbitrarily. Then $E_k(g)$ is a 1-dimensional space spanned by f. Hence if the metric g is λ_k-extremal within its Kähler class, then we have

$$L(f) = \lambda_k(g)^2 f^2 - 2\lambda_k(g)|\nabla f|^2 + |dd^c f|^2 = 0.$$

Since M is compact, f attains the maximum at some points. Then we have $|\nabla f|^2 = 0$ at such points and so we conclude the maximum must be 0. Similarly, the minimum of f must be 0. This is a contradiction and so the proof is completed. □

For a compact Riemann surface, considering all the volume-preserving deformations of the given metric in its conformal class is equivalent to considering those in its Kähler class. Hence, considering both Theorem 3.7 and Theorem 4.10, one can see that the following proposition is very natural:

Proposition 4.12 ([1])**.** Let (M, J, g, ω) be a compact Riemann surface. A nontrivial finite collection of eigenfunctions $\{f_1, \ldots, f_N\} \subset E_k(g)$ satisfies (4.3) if and only if $F := (f_1, \dots, f_N) : (M, g) \to \mathbf{R}^N$ is a harmonic map into $S^{N-1}(c)$ with constant energy density $|dF|^2 \equiv c\lambda_k(g)$ for some $c > 0$.

Proof. By (2.7), we have $dd^c\psi = -(\Delta_g \psi)\omega$ for any smooth function ψ on a compact Riemann surface (M, J, g, ω). Hence for any nontrivial finite collection of eigenfunctions $\{f_1, \dots, f_N\} \subset E_k(g)$, we have

$$\Delta_g \left(\sum_{j=1}^N f_j^2 \right) = 2\lambda_k(g)^2 \sum_{j=1}^N f_j^2 - \sum_{j=1}^N |\nabla f_j|^2 = \frac{1}{\lambda_k(g)} \sum_{j=1}^N L(f_j). \tag{4.5}$$

Hence $\{f_1, \dots, f_N\} \subset E_k(g)$ satisfies (4.3) if and only if $F := (f_1, \dots, f_N) : (M, g) \to \mathbf{R}^N$ is a map into a Euclidean sphere of some radius $c > 0$. Since $f_j's$ are $\lambda_k(g)$-eigenfunctions, such a map $F : (M, g) \to S^{N-1}(c)$ is a harmonic map with constant energy density $|dF|^2 \equiv c\lambda_k(g)$. □

Apotolov–Jakobson–Kokarev [1] studied the λ_1-extremality of a product Kähler metric within its Kähler class. Before stating the result, we recall basic facts about the first Laplace eigenvalue of a product Riemannian metric. Let (M, g) and (M', g') be compact Riemannian manifolds. For a function $f \in C^\infty(M; \mathbf{R})$, we define the function $f \times 1$ on $M \times M'$ by

$$(f \times 1)(x, y) := f(x), \quad (x, y) \in M \times M'.$$

For a function $h \in C^\infty(M'; \mathbf{R})$, we define the function $1 \times h$ on $M \times M'$ in a similar manner. Suppose that $\lambda_1(g) \le \lambda_1(g')$. Then the first eigenvalue $\lambda_1(g \times g')$ of the product Riemannian manifold $(M \times M', g \times g')$ is equal to $\lambda_1(g)$. $E_1(g \times g')$, the space of $\lambda_1(g \times g')$-eigenfunctions on $M \times M'$, is given by

$$E_1(g \times g') = \begin{cases} \mathrm{span}\{f \times 1, 1 \times h \mid f \in E_1(g), h \in E_1(g')\} & (\text{if } \lambda_1(g) = \lambda_1(g')) \\ \mathrm{span}\{f \times 1 \mid f \in E_1(g)\} & (\text{if } \lambda_1(g) < \lambda_1(g')). \end{cases}$$

For details of the above, see [33, p. 286, pp.336-337]. Then we state the following result:

Proposition 4.13 ([1])**.** Let (M, J, g, ω) and (M', g', J', ω') be compact Kähler manifolds. Suppose that $\lambda_1(g) \le \lambda_1(g')$ and the metric g is λ_1-extremal within its Kähler class on (M, J). Then the product Kähler metric $g \times g'$ is λ_1-extremal within its Kähler class on $(M, J) \times (M', J')$.

Proof. By hypothesis and Theorem 4.10, there exists a nontrivial collection of $\lambda_1(g)$-eigenfunctions $\{f_1, \dots, f_N\}$ on M such that $\sum_{j=1}^N L_M(f_j) = 0$. Since we now assume that $\lambda_1(g) \le \lambda_1(g')$, $\{f_1 \times 1, \dots, f_N \times 1\}$ is a nontrivial finite collection of

$\lambda_1(g\times g')$-eigenfunctions on $M\times M'$. Since we clearly have $L_{M\times M'}(f\times 1) = L_M(f)$ for any smooth function f defined on M, we immediately obtain

$$\sum_{j=1}^{N} L_{M\times M'}(f\times 1) = \sum_{j=1}^{N} L_M(f_j) = 0.$$

Hence Theorem 4.10 concludes the assertion. □

We end this section with the following proposition:

Proposition 4.14 ([1])**.** Let (M, J, g, ω) be a compact homogeneous Kähler-Einstein manifold with positive scalar curvature. Then the metric g is λ_1-extremal within its Kähler class.

5. A λ_k-extremal Kähler metric

In this section, on a compact complex manifold (M, J) that admits a Kähler metric, we introduce the notion of λ_k-extremal Kähler metric by considering all the volume preserving deformations of the given Kähler metric. Be cautioned that we fix the complex structure J and consider only J-compatible Kähler metrics. (See Definition 5.1 for the precise definition of the λ_k-extremality.)

Let (M, J, g, ω) be a compact Kähler manifold (without boundary) of complex dimension n. By scaling the metric, we assume that $\mathrm{Vol}(M, g) = 1$. Let $d\mu = \omega^n/n!$ be the volume form.

Let $Z^{1,1}(M;\mathbf{R})$ be the real vector space of d-closed real $(1,1)$-forms on (M, J). Let $Z_0^{1,1}(M;\mathbf{R})$ be its subspace defined by

$$Z_0^{1,1}(M;\mathbf{R}) := \left\{ \alpha \in Z^{1,1}(M;\mathbf{R}) \mid \int_M h(\alpha,\omega)d\mu = 0 \right\}.$$

Fix an arbitrary element $\alpha \in Z_0^{1,1}(M;\mathbf{R})$. We consider the volume-preserving deformation of ω in the direction of α. The $(1,1)$-form

$$\widetilde{\omega}_t := \omega + t\alpha \tag{5.1}$$

is clearly real and d-closed. Since M is compact, the J-invariant symmetric $(0,2)$-tensor $\widetilde{g}_t$, which is defined by $\widetilde{g}_t(X,Y) = \widetilde{\omega}_t(X, JY)$, is positive definite for a sufficiently small t. Hence $\widetilde{\omega}_t$ is a Kähler form for a sufficiently small t. In particular, if we consider $\alpha = dd^c\varphi$ for a real-valued function φ, then α satisfies

$$\int_M h(\alpha,\omega)d\mu = -\int_M \Delta_g\varphi d\mu = 0,$$

and so we have $\alpha \in Z_0^{1,1}(M;\mathbf{R})$. As we have seen in the previous section, the case where $\alpha = dd^c\varphi$, which is a deformation of ω in its Kähler class $[\omega]$, was studied by Apostolov–Jakobson–Kokarev [1]. Set

$$g_t := \mathrm{Vol}(M,\tilde{g}_t)^{-1/n}\widetilde{g}_t, \quad \omega_t := \mathrm{Vol}(M,\tilde{g}_t)^{-1/n}\widetilde{\omega}_t. \tag{5.2}$$

Then we have $g_0 = g$ and $\omega_0 = \omega$. We also see that $(g_t)_t$ is a volume-preserving 1-parameter family of Kähler metrics that depends analytically on t, and ω_t is the Kähler form associated with g_t. Moreover, we can verify that $\frac{d}{dt}\big|_{t=0}\omega_t = \alpha$. (See (5.4) below.)

Definition 5.1. The Kähler metric g on a compact Kähler manifold (M,J,g,ω) is called *λ_k-extremal (for all the volume-preserving deformations of the Kähler metric)* if the inequality

$$\left(\frac{d}{dt}\bigg|_{t=0^-}\lambda_k(g_t)\right)\cdot\left(\frac{d}{dt}\bigg|_{t=0^+}\lambda_k(g_t)\right) \le 0$$

holds for any 1-parameter family of volume-preserving Kähler metrics $(g_t)_t$ that depends real analytically on t.

Remark 5.2. When we consider whether a Kähler metric g on (M,J) is λ_k-extremal, we may rescale the metric so that $\mathrm{Vol}(M,g) = 1$. Let $(g_t)_t$ be a 1-parameter family of volume-preserving Kähler metrics that depends real analytically on t. Let ω_t be the Kähler form associated with g_t. Then Theorem 3.2 implies that $\frac{d}{dt}\big|_{t=0^-}\lambda_k(g_t)$ and $\frac{d}{dt}\big|_{t=0^+}\lambda_k(g_t)$ depend on only ω and $\frac{d}{dt}\big|_{t=0}\omega_t$. Since $(\omega_t)_t$ is volume-preserving, we have $\frac{d}{dt}\big|_{t=0}\omega_t \in Z_0^{1,1}(M;\mathbf{R})$. Hence it suffices to consider $(\omega_t)_t$ given by (5.2). Thus a Kähler metric g on (M,J) with $\mathrm{Vol}(M,g)=1$ is λ_k-extremal if and only if for any $\alpha \in Z_0^{1,1}(M;\mathbf{R})$, the associated volume-preserving 1-parameter family of Kähler metrics $(g_t)_t$ defined by (5.2) satisfies

$$\left(\frac{d}{dt}\bigg|_{t=0^-}\lambda_k(g_t)\right)\cdot\left(\frac{d}{dt}\bigg|_{t=0^+}\lambda_k(g_t)\right) \le 0.$$

Let (M,J,g,ω) be a compact Kähler manifold of complex dimension n with $\mathrm{Vol}(M,g) = 1$. For any $\alpha \in Z_0^{1,1}(M;\mathbf{R})$, the associated volume-preserving 1-parameter family of Kähler metrics $(g_t)_t$ given by (5.2) defines the associated operator $P_{k,\alpha} : E_k(g) \to E_k(g)$ by (3.1). Since $P_{k,\alpha}$ is symmetric with respect to the $L^2(g)$-inner product by Theorem 3.2 (5), one can consider the corresponding quadratic form on $E_k(g)$, given by

$$Q_\alpha(f) := \int_M f P_{k,\alpha}(f) d\mu = \int_M f\left(\frac{d}{dt}\bigg|_{t=0}\Delta_{g_t} f\right) d\mu. \tag{5.3}$$

The following proposition is an immediate consequence of Theorem 3.2 (2):

Proposition 5.3. Let (M, J, g, ω) be a compact Kähler manifold. If the metric g of (M, J, g, ω) is λ_k-extremal, then the quadratic form Q_α, defined in (5.3), is indefinite on $E_k(g)$ for any $\alpha \in Z_0^{1,1}(M; \mathbf{R})$.

We also have the following proposition, which follows form Theorem 3.2 (3) and (4):

Proposition 5.4. Let (M, J, g, ω) be a compact Kähler manifold. Suppose that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$ holds. Then the metric g of (M, J, g) is λ_k-extremal if and only if the quadratic form Q_α is indefinite on $E_k(g)$ for any $\alpha \in Z_0^{1,1}(M; \mathbf{R})$.

The next corollary immediately follows.

Corollary 5.5. Let (M, J, g, ω) be a compact Kähler manifold. Then the metric g is λ_1-extremal if and only if the quadratic form Q_α is indefinite on $E_1(g)$ for any $\alpha \in Z_0^{1,1}(M; \mathbf{R})$.

Theorem 5.6. Let (M, J, g, ω) be a compact Kähler manifold of complex dimension n with $\mathrm{Vol}(M, g) = 1$. For any $\alpha \in Z_0^{1,1}(M; \mathbf{R})$, the quadratic form Q_α, given by (5.3), can be expressed as

$$Q_\alpha(f) = \int_M h(f dd^c f, \alpha) d\mu.$$

We remark that if $\alpha = dd^c\varphi$, then this theorem immediately implies Lemma 4.7.

Proof. First we calculate $\frac{d}{dt}\big|_{t=0} \mathrm{Vol}(M, \widetilde{g}_t)$ and $\frac{d}{dt}\big|_{t=0} \omega_t$. The volume form $\widetilde{d\mu_t}$, determined by $\widetilde{\omega}_t$, can be written as

$$\widetilde{d\mu_t} = \frac{1}{n!}\widetilde{\omega}_t^n = \frac{1}{n!}\left(\omega^n + tn\alpha \wedge \omega^{n-1}\right) + O(t^2) = [1 + th(\alpha, \omega)]d\mu + O(t^2).$$

Hence one obtains

$$\mathrm{Vol}(M, \widetilde{g}_t) = \int_M \widetilde{d\mu_t} = 1 + t\int_M h(\alpha, \omega) d\mu + O(t^2) = 1 + O(t^2),$$

where in the last equality we have used the the assumptions that $\mathrm{Vol}(M, g) = 1$ and $\alpha \in Z_0^{1,1}(M; \mathbf{R})$. Hence this implies $\frac{d}{dt}\big|_{t=0} \mathrm{Vol}(M, \tilde{g}_t) = 0$. Thus one obtains

$$\frac{d}{dt}\bigg|_{t=0} \omega_t = \frac{d}{dt}\left(\mathrm{Vol}(M, \widetilde{g}_t)^{-1/n}\widetilde{\omega}_t\right) = \mathrm{Vol}(M, \tilde{g}_0)^{-1/n}\frac{d}{dt}\bigg|_{t=0}\widetilde{\omega}_t = \alpha. \tag{5.4}$$

Next we differentiate

$$n dd^c f \wedge \omega_t^{n-1} = -(\Delta_{g_t} f)\omega_t^n, \tag{5.5}$$

which comes from (2.8). Differentiating the left hand side at $t=0$, one obtains

$$n(n-1)dd^cf\wedge\left(\frac{d}{dt}\bigg|_{t=0}\omega_t\right)\wedge\omega^{n-2}=n(n-1)dd^cf\wedge\alpha\wedge\omega^{n-2},$$

where (5.4) is used. On the other hand, differentiating the right hand side of (5.5) at $t=0$, one obtains

$$\begin{aligned}&-\left(\frac{d}{dt}\bigg|_{t=0}\Delta_{g_t}f\right)\omega^n-n(\Delta_g f)\left(\frac{d}{dt}\bigg|_{t=0}\omega_t\right)\wedge\omega^{n-1}\\&=-\left(\frac{d}{dt}\bigg|_{t=0}\Delta_{g_t}f\right)\omega^n-n(\Delta_g f)\alpha\wedge\omega^{n-1}\\&=-\left(\frac{d}{dt}\bigg|_{t=0}\Delta_{g_t}f\right)\omega^n-(\Delta_g f)h(\alpha,\omega)\omega^n,\end{aligned}$$

where (2.4) is used for the last equality. Hence one obtains

$$\left(\frac{d}{dt}\bigg|_{t=0}\Delta_{g_t}f\right)\omega^n=-n(n-1)dd^cf\wedge\alpha\wedge\omega^{n-2}-(\Delta_g f)h(\alpha,\omega)\omega^n. \tag{5.6}$$

Thus using Lemma 2.2 and the equation (2.7), one obtains

$$\left(\frac{d}{dt}\bigg|_{t=0}\Delta_{g_t}f\right)\omega^n=h(dd^cf,\alpha)\omega^n.$$

Hence one concludes

$$\frac{d}{dt}\bigg|_{t=0}\Delta_{g_t}f=h(dd^cf,\alpha)$$

and thus the assertion follows. □

In the above calculations, we consider $\alpha\in Z_0^{1,1}(M;\mathbf{R})$. Hereinafter, we use the Hodge decomposition, and consider both the harmonic part and the exact part of α. Let $\mathcal{H}^{1,1}(M;\mathbf{R})$ be the vector space of real harmonic $(1,1)$-forms. Set

$$\mathcal{H}_0^{1,1}(M;\mathbf{R}):=\left\{\alpha\in\mathcal{H}^{1,1}(M;\mathbf{R})\mid\int_M h(\alpha,\omega)d\mu=0\right\}$$

and

$$C_0^\infty(M;\mathbf{R}):=\left\{\varphi\in C^\infty(M;\mathbf{R})\mid\int_M\varphi d\mu=0\right\}.$$

By the dd^c-lemma, $\alpha\in Z_0^{1,1}(M;\mathbf{R})$ can be decomposed as

$$\alpha=H(\alpha)+dd^c\varphi.$$

By Lemma 2.4, $H(\alpha)$ is real. This decomposition gives an $\mathbf{R}$-linear bijection between $Z_0^{1,1}(M;\mathbf{R})$ and $\mathcal{H}_0^{1,1}(M;\mathbf{R})\times C_0^\infty(M;\mathbf{R})$. We prove the following theorem:

Theorem 5.7. Let (M,J,g,ω) be a compact Kähler manifold of complex dimension n. The following are equivalent:

(1) For any $\alpha \in Z_0^{1,1}(M;\mathbf{R})$, the quadratic form Q_α given by (5.3) is indefinite on the eigenspace $E_k(g)$.

(2) There exists a finite subset $\{f_1, \cdots, f_N\} \subset E_k(g)$ such that the following equations hold:

$$
\begin{cases} H\left(\sum_{j=1}^{N} f_j dd^c f_j\right) = -\omega \\ \sum_{j=1}^{N} L(f_j) = \lambda_k(g)^2 \left(\sum_{j=1}^{N} f_j^2\right) - 2\lambda_k(g)\left(\sum_{j=1}^{N} |\nabla f_j|^2\right) + \sum_{j=1}^{N} |dd^c f_j|^2 = 0. \end{cases} \tag{5.7}
$$

The proof of this theorem is inspired by that of Proposition 4.9.

Proof. We may assume that $\mathrm{Vol}(M,g) = 1$. We assume the condition (1). Let K be the convex hull of $\{(H(fdd^cf), L(f)) \mid f \in E_k(g)\}$ in $\mathcal{H}^{1,1}(M;\mathbf{R}) \times C^\infty(M;\mathbf{R})$. Set $m := \dim E_k(g)$ and let $\{u_a\}_{a=1}^m$ be an $L^2(g)$-orthonormal basis of $E_k(g)$. Then K is a positive hull of the finite points $\{H(u_a dd^c u_a), L(u_a)\}_{a=1}^m$ and $\{\pm H(u_a dd^c u_b), \pm\delta^c\delta(u_a dd^c u_b)\}_{1\le a\neq b\le m}$. In particular, Proposition 2.8 implies that K is a closed convex cone contained in a finite dimensional subspace of $\mathcal{H}^{1,1}(M;\mathbf{R})\times C^\infty(M;\mathbf{R})$. We assume that $(-\omega, 0) \notin K$. Let V be a subspace in $\mathcal{H}^{1,1}(M;\mathbf{R}) \times C^\infty(M;\mathbf{R})$ that contains K and $(-\omega, 0)$. We consider the product L^2-inner metric on V. Then the hyperplane separation theorem (Theorem 2.11(2)) implies that there exists $(\widetilde{\alpha}_H, \widetilde{\varphi}) \in \mathcal{H}^{1,1}(M;\mathbf{R}) \times C^\infty(M;\mathbf{R})$ such that the inequalities

$$
\int_M h(-\omega, \widetilde{\alpha}_H) d\mu < 0 \quad \text{and} \quad \int_M h(\eta, \widetilde{\alpha}_H) d\mu + \int_M s\varphi d\mu \ge 0 \tag{5.8}
$$

hold for all for all $(\eta, s) \in K \setminus \{0\}$. Consider $\alpha_H \in \mathcal{H}_0^{1,1}(M;\mathbf{R})$ and $\varphi \in C_0^\infty(M;\mathbf{R})$ respectively defined by

$$
\alpha_H := \widetilde{\alpha}_H - \frac{1}{n}\left(\int_M h(\omega, \widetilde{\alpha}_H) d\mu\right)\omega. \tag{5.9}
$$

$$
\varphi := \widetilde{\varphi} - \int_M \widetilde{\varphi} d\mu. \tag{5.10}
$$

Set

$$
\alpha := \alpha_H + dd^c\varphi = \alpha_H + dd^c\widetilde{\varphi}.
$$

Then we have $\alpha \in Z_0^{1,1}(M;\mathbf{R})$. For any $f \in E_k(g) \setminus \{0\}$, one has

$$
\begin{aligned}
Q_\alpha(f) &= \int_M h(f dd^c f, \alpha) d\mu \\
&= \int_M h(f dd^c f, \alpha_H) d\mu + \int_M h(f dd^c f, dd^c \widetilde{\varphi}) d\mu \\
&= \int_M h(H(f dd^c f), \widetilde{\alpha}_H) d\mu - \frac{1}{n} \left[\int_M h(f dd^c f, \omega) d\mu \right] \left[\int_M h(\omega, \widetilde{\alpha}_H) d\mu \right] \\
&\quad + \int_M L(f) \widetilde{\varphi} d\mu \\
&= \int_M h(H(f dd^c f), \widetilde{\alpha}_H) d\mu + \int_M L(f) \widetilde{\varphi} d\mu + \frac{\lambda_k(g)}{n} \left[\int_M f^2 d\mu \right] \left[\int_M h(\omega, \widetilde{\alpha}_H) d\mu \right] \\
&> 0.
\end{aligned}
$$

This contradicts the condition (1) and hence one concludes that $(-\omega, 0) \in K$. This implies that (1) $\Rightarrow$ (2).

Conversely, we assume that there exists a finite subset $\{f_1, \cdots, f_N\} \subset E_k(g)$ satisfying (5.7). Take $\alpha \in Z_0^{1,1}(M;\mathbf{R})$ arbitrarily. Then there exist $\alpha_H \in \mathcal{H}_0^{1,1}(M;\mathbf{R})$ and $\varphi \in C_0^\infty(M;\mathbf{R})$ such that $\alpha = \alpha_H + dd^c\varphi$. Then one has

$$
\begin{aligned}
\sum_{j=1}^N Q_\alpha(f_j) &= \sum_{j=1}^N \int_M h(f_j dd^c f_j, \alpha) d\mu \\
&= \int_M h\left(\sum_{j=1}^N f_j dd^c f_j, \alpha_H \right) d\mu + \sum_{j=1}^N \int_M h(f_j dd^c f_j, dd^c \varphi) d\mu \\
&= \int_M h(-\omega, \alpha_H) d\mu + \sum_{j=1}^N \int_M L(f_j) \varphi d\mu \\
&= 0.
\end{aligned}
$$

This implies that Q_α is indefinite on $E_k(g)$. This completes the proof. □

Combining Corollary 5.5 and Theorem 5.7, one concludes the following:

Theorem 5.8. Let (M, J, g, ω) be a compact Kähler manifold. Suppose that the Kähler metric g is λ_k-extremal. Then there exists a finite subset $\{f_1, \cdots, f_N\} \subset E_k(g)$ satisfying (5.7). For $k = 1$, the existence of such a finite collection of eigenfunctions is also a sufficient condition for the Kähler metric g to be λ_1-extremal.

Remark 5.9. For a compact Riemann surface, Theorem 4.10 and Theorem 5.8 are equivalent. In fact, (4.5) shows that if a nontrivial finite collection of $\lambda_k(g)$-eigenfunctions $\{f_j\}$ satisfy $\sum_j L(f_j) = 0$, then we have $\sum_j f_j^2 \equiv c$ for some $c > 0$.

Hence we immediately obtain $H(\sum_j f_j dd^c f_j) = -\lambda_k(g)c\,\omega$. This equivalence is natural since deformations within the Kähler class and general volume-preserving Kähler deformations are equivalent for a compact Riemann surface.

By Corollary 4.11, we immediately have the following:

Corollary 5.10. Let (M, J, g, ω) be a compact Kähler manifold. Suppose that the Kähler metric g is λ_k-extremal. Then the eigenvalue $\lambda_k(g)$ is not simple.

Recalling the remark before Proposition 4.13, one obtains the following from Theorem 5.8:

Corollary 5.11. Let (M, J, g, ω) and (M', J', g', ω') be compact Kähler manifolds. Assume that $\lambda_1(g) = \lambda_1(g')$ and that g and g' are both λ_1-extremal for all the volume-preserving deformations of the Kähler metrics. Then the product Kähler metric $g \times g'$ on $(M, J) \times (M, J)$ is λ_1-extremal for all the volume-preserving deformations of the Kähler metric.

Proof. By hypothesis, there exist finite subsets $\{f_j\} \subset E_1(g)$ and $\{h_k\} \subset E_1(g')$ such that

$$\begin{cases} \sum_j H_M\left(f_j dd^c f_j\right) = -\omega, \quad \sum_j L_M(f_j) = 0 \\ \sum_k H_{M'}\left(h_k dd^c h_k\right) = -\omega', \quad \sum_k L_{M'}(h_k) = 0. \end{cases}$$

Then one has

$$\begin{aligned} &\sum_j L_{M\times M'}(f_j \times 1) + \sum_k L_{M\times M'}(1 \times h_k) \\ &= \sum_j L_M(f_j dd^c f_j) + \sum_k L_{M'}(h_k) \\ &= 0. \end{aligned}$$

We also have

$$\begin{aligned} &\sum_j H_{M\times M'}\big((f_j \times 1) d_{M\times M'} d^c_{M\times M'}(f_j \times 1)\big) \\ &= \sum_j H_{M\times M'}(f_j d_M d^c_M f_j \oplus 0) \\ &= \sum_j H_M(f_j d_M d^c_M f_j) \oplus 0 \\ &= -\omega \oplus 0. \end{aligned}$$

Similarly, we obtain

$$\sum_k H_{M\times M'}\left((1\times h_k)d_{M\times M'}d^c_{M\times M'}(1\times h_k)\right) = 0 \oplus -\omega'.$$

Hence we obtain

$$\sum_j H\left((f_j\times 1)dd^c(f_j\times 1)\right) + \sum_k H\left((1\times h_k)dd^c(1\times h_k)\right) = -\omega \oplus -\omega',$$

where we omit the subscript $M\times M'$. Thus one concludes that $\{f_j\times 1\}_j \cup \{1\times h_k\}_k$ satisfy (5.7). The proof is completed. □

From the above proof, one can immediately obtain the following corollary:

Corollary 5.12. Let (M, J, g, ω) and (M', J', g', ω') be compact Kähler manifolds. Assume that $\lambda_1(g) \neq \lambda_1(g')$. Then the product Kähler metric $g\times g'$ on $(M, J)\times(M', J')$ is not λ_1-extremal for all the volume-preserving deformations of the Kähler metric.

The notion of λ_1-extremality in Example 3.5 is stronger than that in Theorem 5.8. Hence we immediately obtain the following:

Example 5.13. Let G/K be a compact isotropy irreducible homogeneous Kähler manifold. Then the metric is λ_1-extremal for all the volume-preserving deformations of the Kähler metric.

We remark that one can also prove this fact directly from Theorem 5.8, using a similar discussion to that in [35, Section 3]. It is known that the metric of a compact isotropy irreducible homogeneous Kähler manifold is Einstein [40]. An irreducible Hermitian symmetric space of compact type is a compact isotropy irreducible homogeneous Kähler manifold. In fact, the converse also holds. That is, a compact isotropy irreducible homogeneous Kähler manifold is an irreducible Hermitian symmetric space of compact type [39, 21].

When the complex dimension is bigger than 1, the notion of λ_1-extremality in Corollary 5.11 is stronger than that in Proposition 4.13. The following example shows the difference:

Example 5.14. Let (M, J, g, ω), (M', J', g', ω') be irreducible Hermitian symmetric spaces of compact type with $\rho = c\,\omega$ and $\rho' = c'\omega'$ for some $c, c' > 0$, where ρ and ρ' are the Ricci forms on M and M' respectively. By Example 5.13, g is λ_1-extremal for all the volume-preserving deformations of the Kähler metric and so is g'. The result due to Nagano [28] shows that $\lambda_1(g) = 2c$ and $\lambda_1(g') = 2c'$ (see also

[36]). By Proposition 4.13, the product Kähler metric $g \times g'$ on $(M, J) \times (M', J')$ is λ_1-extremal within its Kähler class. However, Corollary 5.11 and 5.12 imply that the metric $g \times g'$ is λ_1-extremal for all the volume-preserving deformations of the Kähler metric if and only if $c = c'$.

The simplest case of this example is the following:

Example 5.15. Let g_{FS} be the Fubini-Study metric on the complex projective space $\mathbf{C}P^n$. Take any $c > 0$. Then the product Kähler metric $g_{FS} \times cg_{FS}$ on $\mathbf{C}P^n \times \mathbf{C}P^n$ is λ_1-extremal in its Kähler class. Nevertheless, $g_{FS} \times cg_{FS}$ is λ_1-extremal for all the volume-preserving deformations of the Kähler metric if and only if $c = 1$.

6. Complex Tori

In view of Theorem 5.8, the harmonic projector H and information about the space of eigenfunctions are important. However, it is hard to find an explicit formula for the harmonic projector H on a general Kähler manifold. It is also hard to determine the space of eigenfunctions explicitly in general. However, the harmonic projector H and the eigenfunctions can be written explicitly for a complex torus. Using Theorem 4.10, we see that the metric on any flat complex torus is λ_1-extremal within its Kähler class. In addition, we use Theorem 5.8 to deduce a condition for the flat metric to be λ_1-extremal for all the volume-preserving deformations of the Kähler metric.

Let $\gamma_1, \dots, \gamma_{2n}$ be vectors in $\mathbf{C}^n$ that are linearly independent over $\mathbf{R}$. We denote by Γ the lattice in $\mathbf{C}^n$ with basis $\{\gamma_1, \dots, \gamma_{2n}\}$. Let $\{z^j\}_{j=1}^n$ be the standard complex coordinates of $\mathbf{C}^n$ and $\{x^j\}_{j=1}^{2n}$ the real coordinates defined by

$$z^k = x^{2k-1} + \sqrt{-1}x^{2k} \quad (k = 1, \dots, n).$$

We remark that one should keep in mind this correspondence between the complex and real coordinates in particular when considering examples that will appear later. The lattice Γ acts on $\mathbf{C}^n$ by translation. Then the quotient space $T^n_\Gamma := \mathbf{C}^n/\Gamma$ becomes a complex manifold in a natural way. T^n_Γ is called a *complex torus*. The standard metric on $\mathbf{C}^n$ is given by $\sum_{j=1}^n dz^j \otimes d\overline{z}^j$ and its associated Kähler form on $\mathbf{C}^n$ is given by

$$\widetilde{\omega} := \frac{\sqrt{-1}}{2} \sum_{j=1}^n dz^j \wedge d\overline{z}^j.$$

The canonical holomorphic projection $\mathbf{C}^n \to \mathbf{C}^n/\Gamma = T^n_\Gamma$ induces a flat Kähler metric g and a Kähler form ω on T^n_Γ. If we express $w_1, w_2 \in \mathbf{C}^n$ as

$$w_k = (w_k^1, \ldots, w_k^n), \quad w_k = (u_k^1, \ldots, u_k^{2n}), \quad w_k^j = u_k^{2j-1} + \sqrt{-1}u_k^{2j} \quad (k = 1, 2)$$

in the complex coordinates $(z^1, \cdots z^n)$ and the real coordinates $(x^1, \cdots, x^{2n})$ respectively, then the standard inner product of w_1 and w_2 is given by

$$\sum_{j=1}^{2n} u_1^j u_2^j = \frac{1}{2}(w_1 \cdot \overline{w}_2 + \overline{w}_1 \cdot w_2).$$

Consider

$$\begin{aligned}\Gamma^* &:= \{w \in \mathbf{C}^n \mid \frac{1}{2}(v \cdot \overline{w} + \overline{v} \cdot w) \in \mathbf{Z} \quad \text{for all } v \in \Gamma\} \\ &= \{w \in \mathbf{C}^n \mid \exp\left(\pi i(v \cdot \overline{w} + \overline{v} \cdot w)\right) = 1 \quad \text{for all } v \in \Gamma\}.\end{aligned}$$

Γ^* is also a lattice in $\mathbf{C}^n$ and called the *dual lattice* of Γ. For any $w \in \Gamma^*$, we define a function $\Phi_w : T^n_\Gamma \to \mathbf{C}$ by

$$\Phi_w(z) := \exp\left(\pi\sqrt{-1}(z \cdot \overline{w} + \overline{z} \cdot w)\right).$$

Then Φ_w is actually a well-defined function on the complex torus T^n_Γ. It is known that λ is an eigenvalue of the Laplacian on (T^n_Γ, g) if and only if there exists $w \in \Gamma^*$ such that $\lambda = 4\pi^2|w|^2$, where $|w|^2 = w \cdot \overline{w}$ (see [33, pp.272-273], for instance). We set

$$S(\lambda) := \{w \in \Gamma^* \mid \lambda = 4\pi^2|w|^2\}.$$

For $\lambda \neq 0$, the number $\#S(\lambda)$ is an even integer and so we define $l(\lambda) \in \mathbf{Z}$ by $l(\lambda) = \#S(\lambda)/2$. $S(\lambda)$ can be written as

$$S(\lambda) = \{\pm w_1, \pm w_2, \cdots, \pm w_{l(\lambda)}\}, \tag{6.1}$$

where each w_ν $(\nu = 1, \cdots, l(\lambda))$ is an element of Γ^* with $\lambda = 4\pi^2|w_\nu|^2$. For $w \in \Gamma^*$, set

$$\varphi_w(z) := \sqrt{\frac{2}{\mathrm{Vol}(T^n_\Gamma)}}\mathrm{Re}\left(\Phi_w(z)\right) = \sqrt{\frac{2}{\mathrm{Vol}(T^n_\Gamma)}}\cos\left(2\pi\sum_{k=1}^{2n} x^k u^k\right), \tag{6.2}$$

$$\psi_w(z) := \sqrt{\frac{2}{\mathrm{Vol}(T^n_\Gamma)}}\mathrm{Im}\left(\Phi_w(z)\right) = \sqrt{\frac{2}{\mathrm{Vol}(T^n_\Gamma)}}\sin\left(2\pi\sum_{k=1}^{2n} x^k u^k\right), \tag{6.3}$$

where we use the real coordinates $z = (x^1, \cdots, x^{2n})$ and $w = (u^1, \cdots, u^{2n})$. We give a proof to the following well-known fact:

Lemma 6.1. Let Γ be a lattice in $\mathbf{C}^n$ and Γ^* its dual lattice. Let T^n_Γ be the flat torus defined by Γ. Let λ be a positive eigenvalue of the Laplacian and $E(\lambda)$ its real eigenspace. Let $S(\lambda)$ be as (6.1). For $w \in \Gamma^*$, define the functions φ_w and ψ_w by (6.2) and (6.3). Then $\{\varphi_{w_\nu}, \psi_{w_\nu} \mid \nu = 1, \cdots, l(\lambda)\}$ is an L^2-orthonormal basis of $E(\lambda)$.

Proof. Let A be an invertible $2n \times 2n$ matrix such that $\Gamma = A\mathbf{Z}^{2n}$. Then we have $\mathrm{Vol}(T^n_\Gamma) = \det A$. We also have $\Gamma^* = (A^T)^{-1}\mathbf{Z}^{2n}$. Hence for each $1 \leq j \leq l(\lambda)$, there exists $y_j \in \mathbf{Z}^{2n}$ such that $w_j = (A^T)^{-1}y_j$, where w_j is regarded as a vector in $\mathbf{R}^{2n}$. Then we have

$$\begin{aligned}
&\int_{T^n_\Gamma} \varphi_{w_j}(x)\varphi_{w_k}(x)dx \\
&= \frac{2}{\mathrm{Vol}(T^n_\Gamma)} \int_{T^n_\Gamma} \cos(2\pi w_j^T x)\cos(2\pi w_k^T x)dx \\
&= \frac{1}{\mathrm{Vol}(T^n_\Gamma)} \int_{T^n_\Gamma} \left(\cos(2\pi(w_j + w_k)^T x) + \cos(2\pi(w_j - w_k)^T x)\right) dx \\
&= \frac{1}{\mathrm{Vol}(T^n_\Gamma)} \int_{T^n_\Gamma} \left(\cos(2\pi(y_j + y_k)^T A^{-1}x) + \cos(2\pi(y_j - y_k)^T A^{-1}x)\right) dx \\
&= \frac{1}{\mathrm{Vol}(T^n_\Gamma)} \frac{1}{|\det A^{-1}|} \int_{T^n_{\mathbf{Z}^{2n}}} \left(\cos(2\pi(y_j + y_k)^T u) + \cos(2\pi(y_j - y_k)^T u)\right) du \\
&= \int_{[0,1]^{2n}} \left(\cos(2\pi(y_j + y_k)^T u) + \cos(2\pi(y_j - y_k)^T u)\right) du \\
&= \delta_{jk}.
\end{aligned}$$

Here be cautioned that $y_j = y_k$ is equivalent to $w_j = w_k$ and the case where $y_j = -y_k$, that is, where $w_j = -w_k$, is impossible by the choice of $\{w_\nu\}_{\nu=1}^{l(\lambda)}$. By similar computations, we also have

$$\begin{aligned}
&\int_{T^n_\Gamma} \psi_{w_j}(x)\psi_{w_k}(x)dx \\
&= \int_{[0,1]^{2n}} \left(-\cos(2\pi(y_j + y_k)^T u) + \cos(2\pi(y_j - y_k)^T u)\right) du \\
&= \delta_{jk}
\end{aligned}$$

and

$$\begin{aligned}&\int_{T^n_\Gamma}\varphi_{w_j}(x)\psi_{w_k}(x)dx\\&=\int_{[0,1]^{2n}}\left(\sin(2\pi(y_j+y_k)^Tu)-\sin(2\pi(y_j-y_k)^Tu)\right)du\\&=0.\end{aligned}$$

The proof is completed. □

Apostolov–Jakobson–Kokarev [1] proved that the metric on a compact homogeneous Kähler-Einstein manifold of positive scalar curvature is λ_1-extremal within its Kähler class (Proposition 4.14). We show that the flat metric on a complex torus is λ_1-extremal within its Kähler class.

Proposition 6.2. Let (T^n_Γ, g) be a flat complex torus. Then the metric g is λ_1-extremal within its Kähler class.

Proof. In the proof, we use the notations introduced above. Take $w\in S(\lambda_1(g))$ arbitarily. By Theorem 4.10, it suffices to show that the equation

$$L(\varphi_w)+L(\psi_w)=0. \tag{6.4}$$

holds. By a straightforward calculation, we have

$$|\nabla\varphi_w|^2=\frac{2}{\mathrm{Vol}(T^n_\Gamma)}\left[4\pi^2\sum_{j=1}^{2n}(u^j)^2\right]\sin^2\left(2\pi\sum_{k=1}^{2n}x^ku^k\right)=\lambda_1(g)\psi_w^2.$$

Similarly, we have

$$|\nabla\psi_w|^2=\lambda_1(g)\varphi_w^2.$$

On the other hand, it is straightforward to obtain

$$dd^c\varphi_w=-2\pi^2\sqrt{-1}\varphi_w\sum_{\alpha,\beta=1}^{n}\overline{w}^\alpha w^\beta dz^\alpha\wedge d\overline{z}^\beta. \tag{6.5}$$

Hence we obtain

$$\begin{aligned}|dd^c\varphi_w|^2&=4\pi^4\varphi_w^2\sum_{\alpha,\beta,\gamma,\delta}g^{\alpha\overline{\delta}}g^{\overline{\beta}\gamma}\overline{w}^\alpha w^\beta\overline{w}^\gamma w^\delta\\&=16\pi^4\varphi_w^2\sum_{\alpha,\beta}|w^\alpha|^2|w^\beta|^2\\&=\lambda_1(g)^2\varphi_w^2.\end{aligned}$$

Similarly, we have

$$dd^c\psi_w = -2\pi^2\sqrt{-1}\psi_w \sum_{\alpha,\beta=1}^{n} \overline{w}^\alpha w^\beta dz^\alpha \wedge d\overline{z}^\beta \tag{6.6}$$

and

$$|dd^c\psi_w|^2 = \lambda_1(g)^2\psi_w^2.$$

Thus we have

(6.7)

$$\begin{aligned} &L(\varphi_w) + L(\psi_w) \\ &= \left(\lambda_1(g)^2\varphi_w - 2\lambda_1(g)^2\psi_w^2 + \lambda_1(g)^2\varphi_w^2\right) + \left(\lambda_1(g)^2\psi_w - 2\lambda_1(g)^2\varphi_w^2 + \lambda_1(g)^2\psi_w^2\right) \\ &= 0. \end{aligned}$$

Hence (6.4) is proved. □

The harmonic projector $H : \Omega^{1,1}(T^n_\Gamma) \to \mathcal{H}^{1,1}(T^n_\Gamma)$ plays an important role in Theorem 5.8. We explain that harmonic forms on a flat complex torus are forms with constant coefficients and find the explicit expression of the harmonic projector H. Let ϕ be a (p,q)-form on T^n_Γ. Then ϕ is expressed as

$$\phi = \sum_{|J|=p,|K|=q} \phi_{J\overline{K}} dz^J \wedge d\overline{z}^K,$$

where J and K are multi-indices, and each $\phi_{J\overline{K}}$ is a Γ-periodic complex-valued function globally defined on $\mathbf{C}^n$. A straightforward calculation shows that ϕ is a harmonic form if and only if each $\phi_{J\overline{K}}$ is a harmonic function. Hence the maximum principle implies that ϕ is a harmonic form if and only if each $\phi_{J\overline{K}}$ is constant. Thus the harmonic projector $H : \Omega^{1,1}(T^n_\Gamma) \to \mathcal{H}^{1,1}(T^n_\Gamma)$ is given by

$$H(\phi) = \frac{1}{\mathrm{Vol}(T^n_\Gamma)} \sum_{\alpha,\beta=1}^{n} \left(\int_{T^n_\Gamma} \phi_{\alpha\overline{\beta}} d\mu \right) dz^\alpha \wedge d\overline{z}^\beta$$

for a $(1,1)$-form $\phi = \sum_{\alpha,\beta=1}^n \phi_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$, where $d\mu$ is the volume form of (T^n_Γ, g). (For details of the Hodge decomposition on a complex torus, see [4, Section 1.4].)

Theorem 6.3. Let (T^n_Γ, g) be a flat n-dimensional complex torus. Let $\{w_\nu\}_{\nu=1}^{l(\lambda_k(g))}$ be linearly independent vectors in Γ^* satisfying $\lambda_k(g) = 4\pi^2|w_\nu|^2$. If the flat metric g is λ_k-extremal for all the volume-preserving deformations of the Kähler metric, then there exists $\{R_\nu \geq 0\}_{\nu=1}^{l(\lambda_k(g))}$ such that the following equations hold:

$$\left\{ \begin{array}{l} \sum_{\nu=1}^{l(\lambda_k(g))} R_\nu \overline{w}_\nu^\alpha w_\nu^\beta = 0 \quad \text{for} \quad 1 \leq \alpha \neq \beta \leq n, \\ \sum_{\nu=1}^{l(\lambda_k(g))} R_\nu |w_\nu^\alpha|^2 = 1 \quad \text{for} \quad 1 \leq \alpha \leq n. \end{array} \right. \tag{6.8}$$

For $k = 1$, the existence of such $\{R_\nu \geq 0\}_{\nu=1}^{l(\lambda_1(g))}$ is also a sufficient condition for the metric g to be λ_1-extremal for all the volume-preserving deformations of the Kähler metric.

Proof. First we prove the former assertion. By theorem 5.8, we must have

$$H\left(\sum_{j=1}^{N} f_j dd^c f_j\right) = -\frac{\sqrt{-1}}{2}\sum_{\alpha=1}^{n} dz^\alpha \wedge d\overline{z}^\alpha \tag{6.9}$$

for some finite collection of eigenfunctions $\{f_j\}_{j=1}^N \subset E_k(g)$. Each eigenfunction f_j is of the form

$$f_j(z) = \sum_{\nu=1}^{l(\lambda_k(g))} a_{j\nu}\varphi_{w_\nu} + b_{j\nu}\psi_{w_\nu} \quad (a_{j\nu}, b_{j\nu} \in \mathbf{R}).$$

Using (6.5) and (6.6), we obtain

$$\begin{aligned} f_j dd^c f_j = -2\pi^2\sqrt{-1} \sum_{\alpha,\beta=1}^{n} \sum_{\nu,\tau=1}^{l(\lambda_k(g))} \overline{w}_\nu^\alpha w_\nu^\beta [&a_{j\nu}a_{j\tau}\varphi_{w_\nu}\varphi_{w_\tau} + a_{j\nu}b_{j\tau}\varphi_{w_\nu}\psi_{w_\tau} \\ &+ a_{j\tau}b_{j\nu}\varphi_{w_\tau}\psi_{w_\nu} + b_{j\nu}b_{j\tau}\psi_{w_\nu}\psi_{w_\tau}] dz^\alpha \wedge d\overline{z}^\beta. \end{aligned}$$

Hence one obtains

$$H(f_j dd^c f_j) = \frac{-2\pi^2\sqrt{-1}}{\mathrm{Vol}(T_\Gamma^n)} \sum_{\alpha,\beta=1}^{n} \sum_{\nu=1}^{l(\lambda_k(g))} \overline{w}_\nu^\alpha w_\nu^\beta (a_{j\nu}^2 + b_{j\nu}^2) dz^\alpha \wedge d\overline{z}^\beta.$$

Thus one obtains

$$\sum_{j=1}^{N} H(f_j dd^c f_j) = \frac{-2\pi^2\sqrt{-1}}{\mathrm{Vol}(T_\Gamma^n)} \sum_{\alpha,\beta=1}^{n} \sum_{\nu=1}^{l(\lambda_k(g))} \overline{w}_\nu^\alpha w_\nu^\beta (\sum_{j=1}^{N} a_{j\nu}^2 + b_{j\nu}^2) dz^\alpha \wedge d\overline{z}^\beta. \tag{6.10}$$

Then (6.9) and (6.10) imply that we must have

$$\left\{ \begin{array}{l} \sum_{\nu=1}^{l(\lambda_k(g))} \overline{w}_\nu^\alpha w_\nu^\beta (\sum_{j=1}^{N} a_{j\nu}^2 + b_{j\nu}^2) = 0 \quad \text{for} \quad 1 \leq \alpha \neq \beta \leq n, \\ \sum_{\nu=1}^{l(\lambda_k(g))} |\overline{w}_\nu^\alpha|^2 (\sum_{j=1}^{N} a_{j\nu}^2 + b_{j\nu}^2) = \frac{\mathrm{Vol}(T_\Gamma^n)}{4\pi^2} \quad \text{for} \quad 1 \leq \alpha \leq n. \end{array} \right.$$

Setting $R_\nu := 4\pi^2(\sum_{j=1}^N a_{j\nu}^2 + b_{j\nu}^2)/\mathrm{Vol}(T_\Gamma^n)$, one concludes the foemer assertion.

Next we prove the latter assertion. We assume the existence of $\{R_\nu \geq 0\}_{\nu=1}^{l(\lambda_1(g))}$ satisfying (6.8). We use Theorem 5.8 to prove the proposition. We show that $\{\sqrt{R_\nu}\varphi_{w_\nu}, \sqrt{R_\nu}\psi_{w_\nu} \mid \nu = 1, \cdots, l(\lambda_1(g))\}$ satisfies (5.7). By (6.7), we immediately have

$$L(\sqrt{R_\nu}\varphi_{w_\nu}) + L(\sqrt{R_\nu}\psi_{w_\nu}) = 0$$

for each ν. Hence we have

$$\sum_{\nu=1}^{l(\lambda_1(g)} L(\sqrt{R_\nu}\varphi_{w_\nu}) + L(\sqrt{R_\nu}\psi_{w_\nu}) = 0.$$

Thus it suffices to prove

$$\sum_{\nu=1}^{l(\lambda_1(g))} R_\nu \left[H(\varphi_{w_\nu} dd^c \varphi_{w_\nu}) + H(\psi_{w_\nu} dd^c \psi_{w_\nu})\right] = -a\omega$$

for some $a > 0$. Using (6.5) and (6.6), one obtains

$$\begin{aligned} &\sum_{\nu=1}^{l(\lambda_1(g))} R_\nu \left[H(\varphi_{w_\nu} dd^c \varphi_{w_\nu}) + H(\psi_{w_\nu} dd^c \psi_{w_\nu})\right] \\ &= \frac{-2\pi^2\sqrt{-1}}{\mathrm{Vol}(T_\Gamma^n)} \sum_{\nu=1}^{l(\lambda_1(g))} \sum_{\alpha,\beta=1}^{n} R_\nu \left(\int_{T_\Gamma^n} (\phi_{w_\nu}^2 + \psi_{w_\nu}^2) d\mu\right) \overline{w}_\nu^\alpha w_\nu^\beta dz^\alpha \wedge d\overline{z}^\beta \\ &= \frac{-4\pi^2\sqrt{-1}}{\mathrm{Vol}(T_\Gamma^n)} \sum_{\nu=1}^{l(\lambda_1(g))} \sum_{\alpha,\beta=1}^{n} R_\nu \overline{w}_\nu^\alpha w_\nu^\beta dz^\alpha \wedge d\overline{z}^\beta. \end{aligned}$$

By hypothesis, the proof is completed. □

The notion of λ_1-extremality in Theorem 6.3 is weaker than that in Theorem 3.4. Hence Theorem 6.3 gives a necessary condition for a flat complex torus to admit an isometric minimal immersion into a Euclidean sphere of some dimension by first eigenfunctions. This is the only currently known necessary condition for a flat complex torus to admit an isometric minimal immersion into a Euclidean sphere by first eigenfunctions. Since the condition (6.8) is not clear, we consider simple cases in what follows. First we consider the case where $\dim E_k(g) = 2$, that is, $l(\lambda_k(g)) = 1$. Then we have the following corollary:

Corollary 6.4. Let (T_Γ^n, g) be a flat n-dimensional complex torus. Suppose that $\dim E_k(g) = 2$ holds. Then the metric g is not λ_k-extremal.

Proof. We prove the assertion by contradiction. Assume that the metric g is λ_k-extremal. By hypothesis, there exists a pair w, $-w \in \Gamma^* \subset \mathbf{C}^n$ uniquely up to sign such that $\lambda_k(g) = 4\pi^2|w|^2$. First we show that the vector w is of the form $w = (0, \cdots, \xi, \cdots, 0)$ for some $\xi \in \mathbf{C}$. The first equation in (6.8) implies $\overline{w}^\alpha w^\beta = 0$ for any pair of distinct integers (α, β). Since $w = (w^1, \cdots, w^n)$ is a nonzero vector, we have $w^\alpha \neq 0$ for some α. Let $w^j = u^{2j-1} + \sqrt{-1}u^{2j}$ for each $1 \leq j \leq n$. Then for any $\beta \neq \alpha$, we have

$$u^{2\alpha-1}u^{2\beta-1} = -u^{2\alpha}u^{2\beta} \tag{6.11}$$

and

$$u^{2\alpha}u^{2\beta-1} = u^{2\alpha-1}u^{2\beta}. \tag{6.12}$$

Assume that $u^{2\alpha} \neq 0$ and $u^{2\beta} \neq 0$. Then by (6.12), there exists $c \in \mathbf{R}$ such that $u^{2\alpha-1} = cu^{2\alpha}$ and $u^{2\beta-1} = cu^{2\beta}$. Substituting these for (6.11), one obtains

$$c^2u^{2\alpha}u^{2\beta} = -u^{2\alpha}u^{2\beta}.$$

This is a contradiction and so we have $u^{2\alpha} = 0$ or $u^{2\beta} = 0$.

If we have $u^{2\alpha} = 0$, then we must have $u^{2\alpha-1} \neq 0$ since we now assume $w^\alpha \neq 0$. Hence by (6.12), we have $u^{2\beta} = 0$. Then (6.11) immediately implies $u^{2\beta-1} = 0$. Thus we have $u^{2\beta-1} = u^{2\beta} = 0$, that is, $w^\beta = 0$.

If we have $u^{2\beta} = 0$, then (6.12) implies that we have $u^{2\alpha} = 0$ or $u^{2\beta-1} = 0$. We have already considered the case where $u^{2\alpha} = 0$. Hence we consider the case where $u^{2\beta-1} = 0$, but this immediately implies $w^\beta = 0$.

Thus we conclude that w is of the form $w = (0, \cdots, \xi, \cdots, 0)$. However, this contradicts the second equation in (6.8). The proof is completed. □

Example 6.5. The standard lattice: $\Gamma = \mathbf{Z}^{2n}$. Consider the standard complex torus $\mathbf{C}^n/\mathbf{Z}^{2n}$ with the flat metric g. Let $\{e_j\}_{j=1}^n$ be the standard orthonormal basis of $\mathbf{C}^n$. Set

$$w_{2k-1} := e_k, \quad w_{2k} := \sqrt{-1}e_k$$

for every $1 \leq k \leq n$. Then we have $S(\lambda_1(g)) = \{\pm w_\nu\}_{\nu=1}^{2n}$ and $l(\lambda_1(g)) = 2n$. It is clear that (6.8) is equivalent to the condition where $R_{2k-1} + R_{2k} = 1$ for any $1 \leq k \leq n$ and so the torus $\mathbf{C}^n/\mathbf{Z}^{2n}$ satisfies the assumption of Proposition 6.3. Hence the metric g is λ_1-extremal for all the volume-preserving deformations of the Kähler metric. This fact is not new since the metric is λ_1-extremal for all the volume-preserving metric deformations. This can be seen from Theorem 3.4 and the classical fact that the standard torus admits an isometric minimal immersion into a unit sphere by first eigenfunctions as follows:

$$\mathbf{C}^n/\mathbf{Z}^{2n} \to S^{4n-1}\left(\sqrt{\frac{n}{2\pi^2}}\right),$$

$$(x^1, \ldots, x^{2n}) \mapsto \left(\frac{1}{2\pi}\cos(2\pi x^1), \frac{1}{2\pi}\sin(2\pi x^1), \ldots, \frac{1}{2\pi}\cos(2\pi x^{2n}), \frac{1}{2\pi}\sin(2\pi x^{2n})\right).$$

Example 6.6. The checkerboard lattice. First we consider the (real) 4-dimensional checkerboard lattice D_4, which is defined by

$$D_4 := \{(x^1, \ldots, x^4) \in \mathbf{Z}^4 \mid x^1 + \cdots + x^4 \in 2\mathbf{Z}\}.$$

The dual lattice D_4^* is known to be the lattice in $\mathbf{C}^2$ with the basis $(1,0)$, $(0,1)$, $(\sqrt{-1}, 0)$, $(\frac{1+\sqrt{-1}}{2}, \frac{1+\sqrt{-1}}{2})$. (See [9, pp.117-120], for instance.) Set

$$w_1 := (1,0), \quad w_2 := (0,1), \quad w_3 := (\sqrt{-1}, 0), \quad w_4 := (0, \sqrt{-1}),$$
$$w_5 := \left(\frac{1+\sqrt{-1}}{2}, \frac{1+\sqrt{-1}}{2}\right), \quad w_6 := \left(\frac{1-\sqrt{-1}}{2}, \frac{1-\sqrt{-1}}{2}\right),$$
$$w_7 := \left(\frac{1+\sqrt{-1}}{2}, -\frac{1+\sqrt{-1}}{2}\right), \quad w_8 := \left(\frac{1-\sqrt{-1}}{2}, -\frac{1-\sqrt{-1}}{2}\right),$$
$$w_9 := \left(\frac{1+\sqrt{-1}}{2}, \frac{1-\sqrt{-1}}{2}\right), \quad w_{10} := \left(\frac{1+\sqrt{-1}}{2}, -\frac{1-\sqrt{-1}}{2}\right),$$
$$w_{11} := \left(\frac{1-\sqrt{-1}}{2}, \frac{1+\sqrt{-1}}{2}\right), \quad w_{12} := \left(\frac{1-\sqrt{-1}}{2}, -\frac{1+\sqrt{-1}}{2}\right).$$

If we set $R_1 = \cdots = R_4 = 1/4$, $R_5 = \cdots = R_{12} = 1/8$, then it is elementary to check that (6.8) holds for $k = 1$. Hence by Theorem 6.3, the flat metric g on the 2-dimensional complex torus $\mathbf{C}^2/D_4$ is λ_1-extremal for all the volume-preserving deformations of the Kähler metric. This fact is not new since the metric is λ_1-extremal for all the volume-preserving metric deformations. This follows from Theorem 3.4 and the fact that there exists a 2-parameter family of isometric minimal immersions by first eigenfunctions from $\mathbf{C}^2/D_4$ into the unit sphere $S^{23} \subset \mathbf{R}^{24}$. (See Example 1.1 in [22].)

In fact, for any $m \geq 3$, the checkerboard lattice D_m is defined as a lattice in $\mathbf{R}^m$ by

$$D_m := \{(x^1, \ldots, x^m) \in \mathbf{Z}^m \mid x^1 + \cdots + x^m \in 2\mathbf{Z}\}.$$

We show the following:

Proposition 6.7. For any $m \geq 3$, the flat torus $\mathbf{R}^m/D_m$ admits an isometric minimal immersion by first eigenfunctions into a Euclidean sphere. Hence the flat metric on $\mathbf{R}^m/D_m$ is λ_1-extremal for all the volume-preserving metric deformations.

Proof. The property of D_m should be considered separately for the case $m = 3$, $m = 4$ and $m \geq 5$. For the case $m = 3$ and $m = 4$, the assertion has been proved by Lü–Wang–Xie [22]. (See Example 4.3 in [22] for $m = 3$ and Example 1.1 in [22] for $m = 4$.) Hence it suffices to consider the case where $m \geq 5$. For $m \geq 5$, D_m^* is a lattice with the basis $\{e_j\}_{j=1}^{m-1} \cup \{\frac{1}{2}\sum_{k=1}^{m} e_k\}$, where $\{e_j\}_{j=1}^{m}$ is the standard basis in $\mathbf{R}^m$. (See [9, p.120], for instance.) The shortest vectors are exactly the $2m$ vectors $\{\pm e_j\}_{j=1}^{m}$. Thus we have $\lambda_1(g) = 4\pi^2$ and $E_1(g)$ is spanned by $\{\cos(2\pi x^j), \sin(2\pi x^j)\}_{j=1}^{m}$, where $\{x^j\}_{j=1}^{m}$ is the standard coordinate in $\mathbf{R}^m$. It is obvious that the map

$$\mathbf{R}^m/D_m \to S^{2m-1}\left(\frac{\sqrt{m}}{2\pi}\right),$$

$$(x^1, \ldots, x^m) \mapsto \left(\frac{1}{2\pi}\cos(2\pi x^1), \frac{1}{2\pi}\sin(2\pi x^1), \ldots, \frac{1}{2\pi}\cos(2\pi x^m), \frac{1}{2\pi}\sin(2\pi x^m)\right)$$

is an isometric minimal immersion. The latter assertion immediately follows from the former one and Theorem 3.4. □

Before [22], only the standard torus (Example 6.5) had been an example of higher dimensional tori that admit an isometric minimal immersion into a Euclidean sphere by first eigenfunctions. Lü–Wang–Xie [22] classified all the 3-dimensional and 4-dimensional tori that admit an isometric minimal immersion into a Euclidean sphere of some dimension by first eigenfunctions. Hence only the standard torus has been an example of tori of dimension higher than 4 that admit an isometric minimal immersion into a Euclidean sphere by first eigenfunctions, but Proposition 6.7 gives new examples.

Example 6.8. For $a, b \in [1, \infty)$, consider the lattice $\Gamma_{a,b}$ in $\mathbf{C}^2$ with the lattice basis $(1, 0)$, $(a^{-1}\sqrt{-1}, 0)$, $(0, 1)$, $(0, b^{-1}\sqrt{-1})$. Let $T^2_{a,b}$ be the 2-dimensional complex torus determined by $\Gamma_{a,b}$ with the flat metric $g_{a,b}$. Let $\Gamma_a \subset \mathbf{C}$ be the lattice with the lattice basis $(1, 0)$, $(a^{-1}\sqrt{-1}, 0)$ and (T^1_a, h_a) the flat 1-dimensional complex torus determined by Γ_a. Then $(T^2_{a,b}, g_{a,b})$ is the product of (T^1_a, h_a) and (T^1_b, h_b). We have $\lambda_1(T^1_a, h_a) = 1 = \lambda_1(T^1_b, h_b)$. Hence Proposition 6.2, Remark 5.9 and Corollary 5.11 imply that the metric $g_{a,b}$ on $T^2_{a,b}$ is λ_1-extremal for all the volume-preserving deformations of the Kähler metric. However, since we have $E_1(g_{a,b}) = \mathrm{span}\{\cos(2\pi x^1), \sin(2\pi x^1), \cos(2\pi x^3), \sin(2\pi x^3)\}$, the flat torus $(T^2_{a,b}, g_{a,b})$ does not admit an isometric minimal immersion into a Euclidean sphere by first eigenfunctions. Thus $g_{a,b}$ is not λ_1-extremal for all the volume-preserving metric deformations.

Example 6.9. For $a, b \in [1, \infty)$, consider the lattice $\widetilde{\Gamma}_{a,b}$ in $\mathbf{C}^2$ with the lattice basis $(1, 0)$, $(\sqrt{-1}, 0)$, $(0, a^{-1})$, $(0, b^{-1}\sqrt{-1})$. Let $\widetilde{T}^2_{a,b}$ be the 2-dimensional complex torus determined by $\widetilde{\Gamma}_{a,b}$ with the flat metric $\widetilde{g}_{a,b}$. Let (T^1_{std}, h_{std}) be the flat 1-dimensional complex torus determined by the lattice with the lattice basis $(1, 0)$, $(\sqrt{-1}, 0)$. Let $(T^1_{a,b}, h_{a,b})$ be the flat 1-dimensional complex torus determined by the lattice with the lattice basis $(0, a^{-1})$, $(0, b^{-1}\sqrt{-1})$. Then $(\widetilde{T}^2_{a,b}, \widetilde{g}_{a,b})$ is the product of (T^1_{std}, h_{std}) and $(T^1_{a,b}, h_{a,b})$. We have $\lambda_1(T^1_{std}, h_{std}) = 1$ and $\lambda_1(T^1_{a,b}, h_{a,b}) = \min\{a, b\}$. Hence if we have $a = 1$ or $b = 1$, then Proposition 6.2, Remark 5.9 and Corollary 5.11 imply that the metric $\widetilde{g}_{a,b}$ on $\widetilde{T}^2_{a,b}$ is λ_1-extremal for all the volume-preserving deformations of the Kähler metric. On the other hand, if we have $a > 1$ and $b > 1$, then by Corollary 5.12, $\widetilde{g}_{a,b}$ is not λ_1-extremal for all the volume-preserving deformations of the Kähler metric.

In Example 6.8 and Example 6.9, if we ignore the complex structure on $\mathbf{C}^2$ and regard $\mathbf{C}^2$ as $\mathbf{R}^4$, then we have $\Gamma_{a,b} = \widetilde{\Gamma}_{a,b}$. However, whether the flat metric is λ_1-extremal is different in Example 6.8 and Example 6.9. Hence Example 6.8 and Example 6.9 show that the notion of λ_1-extremality actually depends on the complex structure.

Finally we give a 1-parameter family of 2-dimensional complex tori whose flat metrics are not λ_1-extremal for all the volume-preserving deformations of the Kähler metric.

Example 6.10. For $\pi/3 < \theta < \pi/2$, we consider the lattice $\Gamma_\theta \subset \mathbf{C}^2$ with the lattice basis $(1, 0)$, $(\cos\theta, \sin\theta)$, $(\sqrt{-1}, 0)$, $(\sqrt{-1}\cos\theta, \sqrt{-1}\sin\theta)$. Let g_θ be the flat metric on $\mathbf{C}^2/\Gamma_\theta$. It is straightforward to check that the dual lattice Γ^*_θ is the lattice with the basis $w_1 := (1, -\cos\theta/\sin\theta)$, $w_2 := (0, 1/\sin\theta)$, $w_3 := (\sqrt{-1}, -\cos\theta/\sin\theta)$, $w_4 := (0, \sqrt{-1}/\sin\theta)$. Then we have $S(\lambda_1(g_\theta)) = \{\pm w_\nu\}^4_{\nu=1}$ and so the multiplicity of $\lambda_1(g_\theta)$ is 8. If g is λ_1-extremal, then Theorem 6.3 implies that there exists $\{R_\nu\}^4_{\nu=1}$ such that

$$-\frac{\cos\theta}{\sin\theta}(R_1 + R_3) = 0, \quad R_1 + R_3 = 1, \quad \frac{1}{\sin^2\theta}(R_2 + R_4) = 1.$$

Since we have $\pi/3 < \theta < \pi/2$, this is a contradiction. Hence g_θ is not λ_1-extremal. $\mathbf{C}^2/\Gamma_\theta$ is not a product of 1-dimensional flat complex tori. In fact, assume that $\mathbf{C}^2/\Gamma_\theta$ is a product of (T_1, h_1) and (T_2, h_2), where each is a 1-dimensional flat complex torus. If we had $\lambda_1(h_1) = \lambda_1(h_2)$, then by Proposition 6.2, Remark 5.9

and Corollary 5.11, g_θ would be λ_1-extremal. Hence we have $\lambda_1(h_1) \neq \lambda_1(h_2)$. We may assume $\lambda_1(h_1) < \lambda_1(h_2)$. Then the multiplicity of of $\lambda_1(h_1)$ is equal to that of $\lambda_1(g_\theta)$, that is, 8. This is a contradiction since the multiplicity of the first eigenvalue of a 1-dimensional flat complex torus is at most 6 (see [11], for example). Thus $\mathbf{C}^2/\Gamma_\theta$ is not a product of 1-dimensional complex tori. Let $\widetilde{\Gamma}_\theta \subset \mathbf{R}^2$ be the lattice with the lattice basis $(1,0)$, $(\cos\theta, \sin\theta)$. Let $(\mathbf{R}^2/\widetilde{\Gamma}_\theta, h_\theta)$ be the flat real 2-dimensional torus. If we ignore the complex structure on $\mathbf{C}^2$ and regard it as $\mathbf{R}^4$, then $(\mathbf{R}^4/\Gamma_\theta, g_\theta)$ is a Riemannian product of $(\mathbf{R}^2/\widetilde{\Gamma}_\theta, h_\theta)$ and $(\mathbf{R}^2/\widetilde{\Gamma}_\theta, h_\theta)$.

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