Some results on cluster algebra theory and its application
(団代数の理論とその応用に関するいくつかの結果)

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Contents

1 Introduction

Cluster algebras are a class of commutative algebras generated by special variables called *cluster variables*, introduced by [FZ02a] in early 2000s. The cluster variables refer to the all variables obtained sequentially by an operation called *mutation*, and it is known that the combinatorial structure of the cluster variable and the mutation appear in various fields such as Teichmüller theory, Poisson geometry, representation theory of quiver, gauge theory, knot theory, etc. Cluster algebra theory is also closely related to number theory. In this thesis, we explain our results on the cluster algebra theory, which were written in our papers [Mat21] and [GM23].

This thesis is divided into two parts.

In the first part, which focuses on the foundations of cluster algebra, our results are related to cluster scattering diagrams. We prove the consistency relations of rank 2 cluster scattering diagrams of affine type using an infinite repetition of pentagon relations. This part is based on [Mat21].

In the second part, which concerns the application of cluster algebras to number theory, we use generalized cluster algebras to solve some Diophantine equations. Positive integer solutions of Markov equations can be obtained in chains by using the Vieta jumping, and it is well-known that this can be explained by using the Laurent phenomenon of a cluster algebra. We generalized this kind of equations by using generalized cluster algebras introduced by [CS14]. This part is based on [GM23].

1.1 General theory of cluster algebra

Let $\mathbb{P}$ be a semifield, i.e., $\mathbb{P}$ has an addition $\oplus$ and a multiplication $\cdot$ which are commutative, associative and distributive, and there are multiplicative inverses for all elements. Remark that it has no unit of addition unless it has only one element. The group ring $\mathbb{ZP}$ is a integral domain, and let $\mathbb{QP}$ be its fields of fraction. Let $\mathcal{F}$ be a field which is isomorphic to the rational function field over $\mathbb{QP}$ whose transcendence degree is n. In the following, we define a cluster algebra as a $\mathbb{ZP}$-subalgebra of $\mathcal{F}$.

Definition 1.1. A matrix B is *skew-symmetrizable* if there is a diagonal matrix D whose diagonal elements are positive rational numbers such that BD is skew-symmetric. A *seed* is a triplet $(\mathbf{x}, \mathbf{y}, B)$ where $\mathbf{x}$ is an n-tuple $(x_1, \dots, x_n)$ of transcendental basis of $\mathcal{F}$ over $\mathbb{QP}$, $\mathbf{y}$ is an n-tuple $(y_1, \dots, y_n)$ of $\mathbb{P}$ and B is an $n \times n$ skew-symmetrizable matrix. Each $\mathbf{x}$ is called *cluster*, and each component x_i of a cluster is called *cluster variable*.

For a real number a, we define $[a]_+ = \max(a, 0)$.

Definition 1.2 (mutation)**.** Let $(\mathbf{x}, \mathbf{y}, B)$ be a seed and k be a element of $\{1, \dots, n\}$. A *mutation* in direction k is an operation making a new seed $(\mathbf{x}', \mathbf{y}', B')$ such that $B' = (b'_{ij})$ is defined by

$$b'_{ij} := \begin{cases} -b_{ij} & (i = k \text{ or } j = k), \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & (\text{otherwise}), \end{cases}$$

$\mathbf{y}' = (y'_1, \dots, y'_n)$ is defined by

$$y'_j = \begin{cases} {y_k}^{-1} & (j = k), \\ y_j {y_k}^{[b_{kj}]_+}(y_k \oplus 1)^{-b_{kj}} & (j \neq k), \end{cases}$$

and $\mathbf{x}' = (x'_1, \dots, x'_n)$ is defined by

$$x'_i = \begin{cases} \dfrac{y_k \prod_{i=1}^n {x_i}^{[b_{ik}]_+} + \prod_{i=1}^n {x_i}^{[-b_{ik}]_+}}{(y_k \oplus 1) x_k} & (j = k), \\ x_i & (j \neq k). \end{cases} \tag{1.1}$$

The relation of the case that $j = k$ of (1.1) is called the *exchange relation*.

Let $\mathbb{T}_n$ be an n-regular tree. For each vertex of $\mathbb{T}_n$, let the edges which are adjacent to the vertex be labeled by elements of $\{1,\dots,n\}$ and all labels be distinct. Abusing a symbol, we denote the vertex set of $\mathbb{T}_n$ as $\mathbb{T}_n$.

Definition 1.3 (cluster algebra)**.** A *cluster pattern* Σ is a map that assigns to each vertex $t\in\mathbb{T}_n$ a seed $\Sigma_t=(\mathbf{x_t},\mathbf{y_t},B_t)$, where $\mathbf{x}_t=(x_{1;t},\dots,x_{n;t}),\mathbf{y}_t=(y_{1;t},\dots,y_{n;t}),B_t=(b_{ij;t})$, and if t, t' are adjacent to each other through an edge labeled by k, then Σ_t and $\Sigma_{t'}$ are related by mutation each other. The *cluster algebra* $\mathcal{A}_\Sigma$ of a cluster pattern Σ is $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\{x_{i;t}\mid i=1,\dots n,\, t\in\mathbb{T}_n\}$.

The following is one of the most significant property of cluster algebras and called the *Laurent phenomenon*:

Theorem 1.1 ([FZ02a])**.** *Let Σ be a cluster pattern and $(x_1,\dots,x_n)$ be cluster variables of a seed of Σ. The cluster algebra $\mathcal{A}_\Sigma$ of the cluster pattern Σ is contained in the Laurent polynomial ring $\mathbb{ZP}[x_1^{\pm1},\dots,x_n^{\pm1}]$.*

For a skew-symmetrizable matrix B^0 and a vertex t_0 of $\mathbb{T}_n$, we recursively define families of $n\times n$ matrices $\{B_t^{B^0;t_0}=(b_{ij;t}^{B^0;t_0})\}_{t\in\mathbb{T}_n}$, $\{C_t^{B^0;t_0}=(c_{ij;t}^{B^0;t_0})\}_{t\in\mathbb{T}_n}$, $\{G_t^{B^0;t_0}=(g_{ij;t}^{B^0;t_0})\}_{t\in\mathbb{T}_n}$ and a family of rational functions $\{F_{i;t}^{B^0;t_0}\}_{t\in\mathbb{T}_n,i=1,\dots,n}$ of n variables $y_1,\dots,y_n$ as follows. For simplicity, we omit superscripts B^0 and t_0 if they are obvious from the context. As initial data, let B_{t_0} be B^0, let C_{t_0} and G_{t_0} be identity matrix and let $F_{i;t_0}$ be 1. For any $i,j=1,\dots,n$ and $t,t'\in\mathbb{T}_n$, if t, t' are adjacent through edges labeled by l, we define

$$b_{ij;t'}:=\begin{cases}-b_{ij;t} & (i=l\text{ or }j=l),\\ b_{ij;t}+[-b_{il;t}]_+b_{lj;t}+b_{il;t}[b_{lj;t}]_+ & (\text{otherwise}),\end{cases}$$

$$c_{ij;t'}=\begin{cases}-c_{il;t} & (j=l),\\ c_{ij;t}+c_{il;t}[b_{lj;t}]_++[c_{il;t}]_+b_{lj;t} & (j\neq l),\end{cases}$$

$$g_{ij;t'}=\begin{cases}g_{il;t} & (j\neq l),\\ g_{il;t}+\displaystyle\sum_{k=1}^n g_{ik;t}[b_{kl;t}]_+-\sum_{k=1}^n b_{ik;t}[c_{kl;t}]_+ & (j=l),\end{cases}$$

$$F_{i;t'}=\begin{cases}F_{i;t} & (i\neq l),\\ \dfrac{\prod_{k=1}^n y_k^{[c_{kl;t}]_+}\prod_{k=1}^n(F_{k;t})^{[b_{kl;t}]_+}+\prod_{k=1}^n y_k^{[-c_{kl;t}]_+}\prod_{k=1}^n(F_{k;t})^{[-b_{kl;t}]_+}}{F_{l;t}} & (i=l).\end{cases}$$

These matrices C_t and G_t are called *C-matrices* and *G-matrices* respectively, and the column vectors of these matrices are called c-vectors and g-vectors, respectively. From these data, we can construct each cluster variable $x_{i;t}$. Before explaining that, a little preparation is required. The *universal semifield* $\mathbb{Q}_{\mathrm{sf}}(y_1,\dots,y_n)$ generated by variables $\{y_1,\dots,y_n\}$ is the multiplicative subgroup of the rational function field $\mathbb{Q}(y_1,\dots,y_n)$ consisting of elements that have subtraction-free expression. The addition is defined by usual one in $\mathbb{Q}(y_1,\dots,y_n)$. For example, $(1-x^3)/(1-x)$ is element of $\mathbb{Q}_{\mathrm{sf}}(x)$ since that has a subtraction-free expression $1+x+x^2$. Note that all $F_{i;t}^{B;t_0}$ are clearly in $\mathbb{Q}_{\mathrm{sf}}(y_1,\dots,y_n)$ because recursion steps are expressed without subtractions. For any semifield $\mathbb{P}$ and its n elements $p_1,\dots,p_n$, there is unique semifield homomorphism from $\mathbb{Q}_{\mathrm{sf}}(y_1,\dots,y_n)$ to $\mathbb{P}$ which assigns p_i to y_i for all $i=1,\dots,n$. We denote the image of $F\in\mathbb{Q}_{\mathrm{sf}}(y_1,\dots,y_n)$ by $F|_{\mathbb{P}}(p_1,\dots,p_n)$. On the other hand, for $F\in\mathbb{Q}_{\mathrm{sf}}(y_1,\dots,y_n)$ and n elements $f_1,\dots,f_n$ of $\mathcal{F}$, we denote the element made by a usual substitution by $F|_{\mathcal{F}}(f_1,\dots,f_n)$.

Theorem 1.2 ([FZ07]). *If cluster pattern Σ has a seed $((x_1,\ldots,x_n),(y_1,\ldots,y_n),B^0)$ at a vertex $t_0\in\mathbb{T}_n$, then each cluster variable $x_{l;t}$ is described as*

$$x_{l;t}=\frac{F_{l;t}^{B^0;t_0}|_{\mathcal{F}}(\hat{y}_1,\ldots,\hat{y}_n)}{F_{l;t}^{B^0;t_0}|_{\mathbb{P}}(y_1,\ldots,y_n)}\prod_{i=1}^{n}x_i^{g_{il;t}^{B^0;t_0}} \tag{1.2}$$

where $\hat{y}_j=y_j\prod_{i=1}^n x_i^{b_{ij}}$.

This formula is called *separation formula* because it separates the usual addition $+$ of $\mathcal{F}$ and the addition $\oplus$ of $\mathbb{P}$ into numerator and denominator. The following was conjectured in [FZ02a] and proved in [GHKK18].

Theorem 1.3 (Positivity of the Laurent phenomenon). *Let Σ be a cluster pattern and t be a vertex of tree $\mathbb{T}_n$. For each cluster algebra $\mathcal{A}_\Sigma$ and any cluster variable x, the Laurent polynomial which expresses x in terms of the cluster variables from a seed Σ_t has coefficients which are nonnegative integer linear combinations of elements in $\mathbb{P}$.*

The following was conjectured in [FZ07] and proved in [GHKK18].

Theorem 1.4 (Sign coherence of c-vector). *For skew-symmetrizable matrix B_0, any $t_0,t\in\mathbb{T}_n$ and $j\in\{1,\ldots,n\}$, a c-vector $(c_{1j;t}^{B_0;t_0},\ldots,c_{nj;t}^{B_0;t_0})$ has either all entries nonnegative or all entries nonpositive.*

These two important theorems were proved by using scattering diagram methods. Scattering diagrams for cluster algebras are characterized by the consistency relations in their structure groups G.

1.2 Overview of Part I

The structure group G of a given cluster scattering diagram is generated by the special elements called dilogarithm elements $\Psi[n]$ parameterized by some lattice points n [GHKK18, Nak23]. (We note that this character 'n' does not mean natural number as in the previous section.) The precise definition of the group G and the family of elements $\Psi[n]$ is given in §3. These elements satisfy the significant relations called the *pentagon relations*:

$$\Psi[n]^c\Psi[n']^c=\Psi[n]^c\Psi[n+n']^c\Psi[n]^c$$

where $\{n,n'\}=c^{-1}$.

In this thesis, we prove the consistency relations of rank 2 cluster scattering diagrams of affine type, namely types $A_1^{(1)}$ and $A_2^{(2)}$. More precisely, we prove the following theorem. For simplicity, let $\begin{bmatrix}n_1\\n_2\end{bmatrix}=\Psi[(n_1,n_2)]$.

Theorem 1.5 ([Mat21, Theorem 1]). *The following relations holds:*

$$\begin{aligned}\begin{bmatrix}0\\1\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix}^2&=\begin{bmatrix}1\\0\end{bmatrix}^2\begin{bmatrix}2\\1\end{bmatrix}^2\begin{bmatrix}3\\2\end{bmatrix}^2\cdots\left(\prod_{j=0}^{\infty}\begin{bmatrix}2^j\\2^j\end{bmatrix}^{2^{2-j}}\right)\cdots\begin{bmatrix}2\\3\end{bmatrix}^2\begin{bmatrix}1\\2\end{bmatrix}^2\begin{bmatrix}0\\1\end{bmatrix}^2\\&=\overrightarrow{\prod_{p\geq 0}}\begin{bmatrix}p+1\\p\end{bmatrix}^2\left(\prod_{j=0}^{\infty}\begin{bmatrix}2^j\\2^j\end{bmatrix}^{2^{2-j}}\right)\overleftarrow{\prod_{p\geq 0}}\begin{bmatrix}p\\p+1\end{bmatrix}^2,\end{aligned} \tag{1.3}$$

$$\begin{aligned}\begin{bmatrix}0\\1\end{bmatrix}^4\begin{bmatrix}1\\0\end{bmatrix}&=\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^4\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}^4\begin{bmatrix}5\\8\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix}^4\cdots\\&\times\left(\begin{bmatrix}1\\2\end{bmatrix}^6\prod_{j=1}^{\infty}\begin{bmatrix}2^j\\2^{j+1}\end{bmatrix}^{2^{2-j}}\right)\cdots\begin{bmatrix}5\\12\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}^4\begin{bmatrix}3\\8\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}^4\begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}^4\end{aligned}$$

$$= \left(\prod_{p\geq 0}^{\longrightarrow} \begin{bmatrix} 2p+1 \\ 4p \end{bmatrix} \begin{bmatrix} p+1 \\ 2p+1 \end{bmatrix}^4 \right) \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \prod_{j=1}^{\infty} \begin{bmatrix} 2^j \\ 2^{j+1} \end{bmatrix}^{2^{2-j}} \right) \left(\prod_{p\geq 0}^{\longleftarrow} \begin{bmatrix} 2p+1 \\ 4p+4 \end{bmatrix} \begin{bmatrix} p \\ 2p+1 \end{bmatrix}^4 \right), \tag{1.4}$$

where the right hand sides of them converge with respect to topology of G, whose details are explained in §3. The symbols $\prod^{\longrightarrow}$ and $\prod^{\longleftarrow}$ are defined by equation (4.1) in §4. Moreover, these formulas can be reduced to trivial relations by iterated applications of the pentagon relations.

We remark that the relations (1.3) first proved by [Rei10] by using quiver representations. Also, the relations (1.3) and (1.4) were proved by cluster mutation technique by [Rea20]. The relations (1.3) and (1.4) are the (unique) consistency relations of type $A_1^{(1)}$ and type $A_2^{(2)}$, respectively.

We say a product of dilogarthm elements is *ordered*, (resp. *anti-ordered*) if, for any adjacent pair $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \begin{bmatrix} n_1' \\ n_2' \end{bmatrix}$, the inequality $n_1/n_2 \geq n_1'/n_2'$ (resp. $n_1/n_2 \leq n_1'/n_2'$) holds. The consistency relations of scattering diagrams in $\mathbb{R}^2$ have the form of

$$\text{“anti-ordered product”} = \text{“ordered product”}.$$

It was shown that the consistency relations are generated by the pentagon relation [Nak23], and the above theorem provides an simplest examples involving the *infinite* product.

In §3, we introduce dilogarithm elements and the pentagon relations. In §4, we prove a generalization of the formula (1.3). In §5, we prove a generalization of the formula (1.4) by using results of §4.

1.3 Overview of Part II

In the second part of this thesis, we deal with some Diophantine equations. One of equations with which we deal has the following form:

$$x^2 + y^2 + z^2 + k_3xy + k_1yz + k_2zx = (3 + k_1 + k_2 + k_3)xyz, \tag{1.5}$$

where $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$. We describe all positive integer solutions to (1.5) in a combinatorial way. We give a rooted tree $\mathbb{T}^{k_1,k_2,k_3}$ with triplets of positive integers as its vertices in the following steps.

(1) The root vertex is $(1, 1, 1)$,

(2) the triplet $(1, 1, 1)$ has three children, $(k_1 + 2, 1, 1), (1, k_2 + 2, 1), (1, 1, k_3 + 2)$,

(3) the generation rule below $(k_1 + 2, 1, 1), (1, k_2 + 2, 1), (1, 1, k_3 + 2)$ is as follows:

  (i) if a is the maximal number in (a, b, c), then (a, b, c) has two children

$$\left(a, \frac{a^2 + k_2ac + c^2}{b}, c\right) \text{ and } \left(a, b, \frac{a^2 + k_3ab + b^2}{c}\right),$$

  (ii) if b is the maximal number in (a, b, c), then (a, b, c) has two children

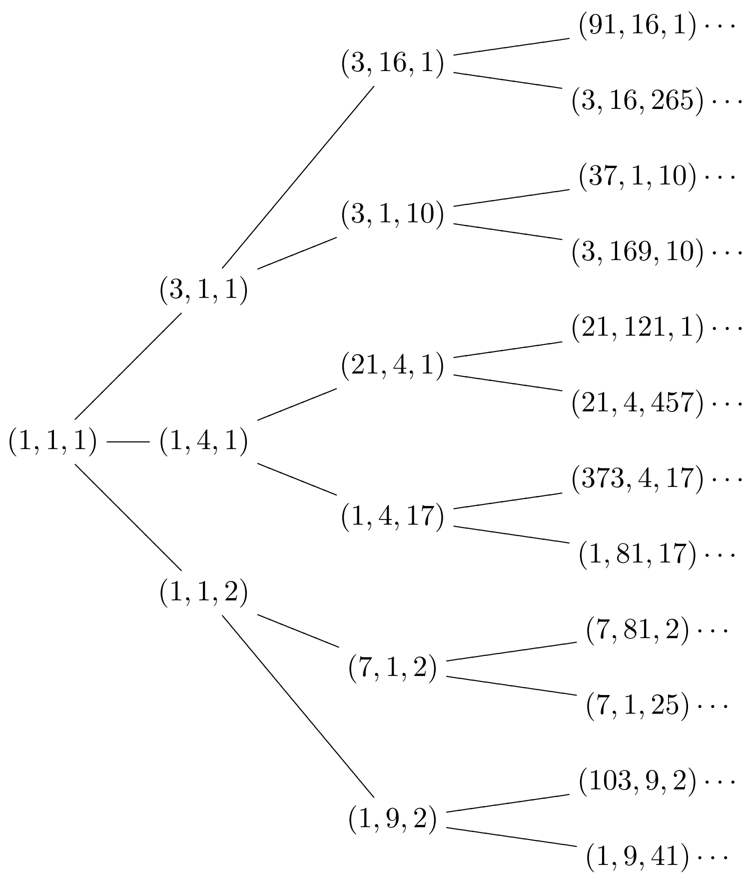
$$\left(\frac{b^2 + k_1bc + c^2}{a}, b, c\right) \text{ and } \left(a, b, \frac{a^2 + k_3ab + b^2}{c}\right),$$

  (iii) if c is the maximal number in (a, b, c), then (a, b, c) has two children

$$\left(\frac{b^2 + k_1bc + c^2}{a}, b, c\right) \text{ and } \left(a, \frac{a^2 + k_2ac + c^2}{b}, c\right).$$

We remark that, for any triplets (a, b, c) in the rooted tree $\mathbb{T}^{k_1,k_2,k_3}$, the case that $a = b > c$ (similarly $b = c > a$, $c = a > b$) does not occur. This can be proved by using Lemma 8.1 and the fact that (a, b, c) is a solution of equation (1.5).

For example, when $k_1 = 1, k_2 = 2, k_3 = 0$, the first few terms of $\mathbb{T}^{1,2,0}$ are as follows:



. (1.6)

The first main result in Part II is the following theorem:

Theorem 1.6 ([GM23, Theorem 1])**.** *Every positive integer solution to* (1.5) *appears exactly once in* $\mathbb{T}^{k_1,k_2,k_3}$*.*

When $k_1 = k_2 = k_3 = 0$, the equation (1.5) is the *Markov Diophantine equation*

$$x^2 + y^2 + z^2 = 3xyz. \tag{1.7}$$

This is an equation that has received much attention since the work on the *Markov spectrum*, and is now being studied in relation to combinatorial objects such as *Christoffel words*, *perfect matchings* of graphs, and *continuous fractions* (for detail, see Aigner's book [Aig13]). The proof of Theorem 1.6 for the case of $k_1 = k_2 = k_3 = 0$ is known, for example, by [Aig13, Section 3.1].

Moreover, when $k_1 = k_2 = k_3 = 1$, the equation (1.5) is a Diophantine equation

$$(x + y)^2 + (y + z)^2 + (z + x)^2 = 12xyz. \tag{1.8}$$

studied in [Gyo22]. The positive integer solutions to this equation, as well as the Markov equation, have been shown to be closely related to perfect matchings of graphs and continuous fractions. The specialized version of Theorem 1.6 for the case of $k_1 = k_2 = k_3 = 1$ is proved by [Gyo22, Theorem 1.1].

Furthermore, Lampe proved specialized version of Theorem 1.6 for the case of $k_1 = 0, k_2 = k_3 = 2$ in [Lam16, Lemma 2.7], that is, the description of all positive integer solutions to

$$x^2 + y^2 + z^2 + 2xy + 2zx = 7xyz. \tag{1.9}$$

In [Lam16], this theorem is used to describe all positive integer solutions to

$$x^2 + y^4 + z^4 + 2xy^2 + 2zx^2 = 7xy^2z^2. \tag{1.10}$$

In this thesis, we also deal with the generalized version of the equation (1.10),

$$x^2 + y^4 + z^4 + ky^2z^2 + 2xy^2 + 2xz^2 = (7+k)xy^2z^2, \tag{1.11}$$

where $k \in \mathbb{Z}_{\geq 0}$.

As in (1.5), we describe all positive integer solutions to (1.11) in a combinatorial way. We give a tree $\mathbb{T}^k$ with triplets of positive integers as its vertices in the following steps.

(1) The root vertex is $(1,1,1)$,

(2) the triplet $(1,1,1)$ has three children, $(k+2,1,1),(1,2,1),(1,1,2)$,

(3) the generation rule below $(k+2,1,1),(1,2,1),(1,1,2)$ is as follows:

 (i) if a is the maximal number in (a,b^2,c^2), then (a,b,c) has two children

$$\left(a, \frac{a+c^2}{b}, c\right) \text{ and } \left(a, b, \frac{a+b^2}{c}\right),$$

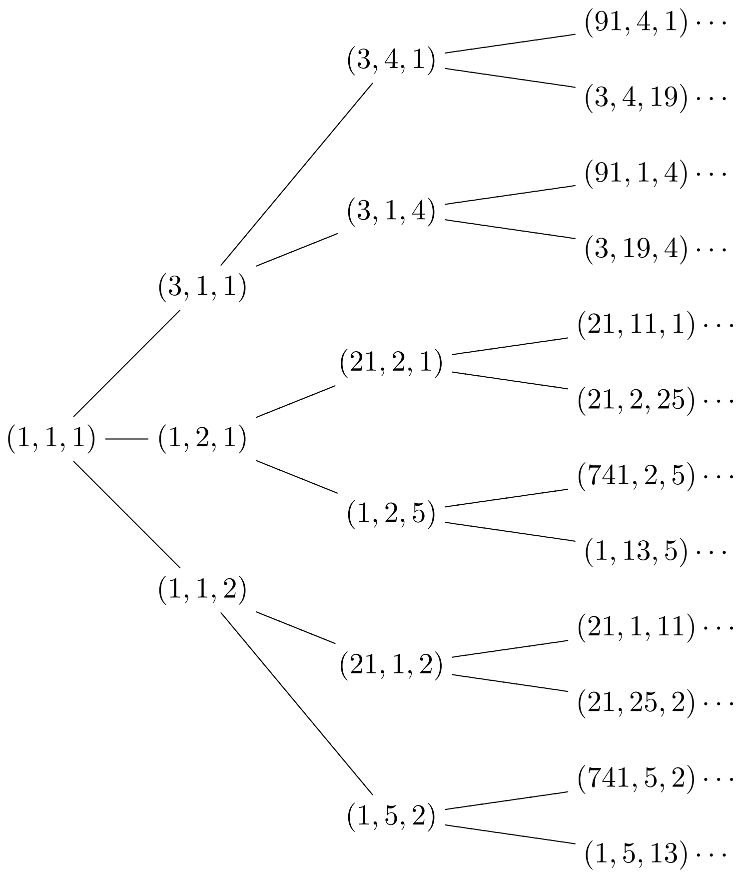
 (ii) if b^2 is the maximal number in (a,b^2,c^2), then (a,b,c) has two children

$$\left(\frac{b^4+kb^2c^2+c^4}{a}, b, c\right) \text{ and } \left(a, b, \frac{a+b^2}{c}\right),$$

 (iii) if c^2 is the maximal number in (a,b^2,c^2), then (a,b,c) has two children

$$\left(\frac{b^4+kb^2c^2+c^4}{a}, b, c\right) \text{ and } \left(a, \frac{a+c^2}{b}, c\right).$$

When $k=1$, the first few terms of $\mathbb{T}^1$ are as follows:



. (1.12)

The second main result in Part II is the following theorem:

Theorem 1.7 ([GM23, Theorem 2])**.** *Every positive integer solution to* (1.11) *appears exactly once in* $\mathbb{T}^k$.

This is a generalization of Lampe's result [Lam16, Theorem 2.6]. In Lampe's paper, the case $k = 0$ was shown as mentioned above. To prove it, the $k_1 = 0, k_2 = k_3 = 2$ case of Theorem 1.6 is used in his paper. In the proof of Theorem 1.7, we use the $k_1 = k, k_2 = k_3 = 2$ case of Theorem 1.6, which is its generalization.

The methods of enumerating the positive integer solutions to the equations mentioned above have one thing in common: it has a structure that can generate three another positive integer solutions from one positive integer solution. This operation is called the *Vieta jumping* and is the key operation of the two main theorems given in this part. We also explain that these Vieta jumpings and positive integer solutions have a structure derived from *cluster algebra theory*. Immediately after the birth of cluster algebra, it was shown that the integer sequences called *Somos-4* and *Somos-5* can be seen as cluster variables, and the recurrence formula that gives it can be seen as a specialization of mutation (see [FZ02b]). In the context of the Diophantine problem, it was first known that the Vieta jumpings and positive integer solutions of the Markov Diophantine equation (1.7) are a specialization of a class of mutations and cluster variables (for example, there is a description of it in [FZW16]), and then it was found by [Lam16] that those of equation (1.10) are given by a specialization of another class of mutation and cluster variables. Recently, it was found by [Gyo22] that those of the equation (1.8) can be given as a specialization of the mutation and cluster variable associated with the *generalized cluster algebra* by [CS14]. In this thesis, we will discuss the generalized cluster algebra structure of equations (1.5) and (1.11), including all of the above mentioned.

At the end of this part, we will consider whether there are any other Diophantine equations with the structure of a generalized cluster algebra like these equations.

Acknowledgements

I would like to thank my supervisor, Tomoki Nakanishi, for reading this thesis and giving me helpful advice. I would like to thank Yasuaki Gyoda for studying with me.

Part I
Consistency relations of rank 2 cluster scattering diagrams of affine type and pentagon relation

The content of this part is based on [Mat21].

2 Cluster scattering diagrams

In this section, we define cluster scattering diagrams following [GHKK18]. We will use only the case that $N = N_{\mathrm{uf}} = \mathbb{Z}^2$, but we now define that in general case. First, we fix the following data:

- A lattice N, i.e., a free abelian group of finite rank.
- A skew-symmetric $\mathbb{Z}$-bilinear form $\{\cdot,\cdot\}\colon N \times N \longrightarrow \mathbb{Q}$. For a subsets $A, B \subset N$, we write $\{A, B\} := \{\{a, b\} \mid a \in A, b \in B\}$.
- A saturated sublattice $N_{\mathrm{uf}} \subset N$, i.e., N/N_{uf} is torsion-free.
- An index set I such that $|I| = \operatorname{rank} N$.
- A subset I_{uf} of I such that $|I_{\mathrm{uf}}| = \operatorname{rank} N_{\mathrm{uf}}$.
- A family of positive numbers d_i $(i \in I)$ whose greatest common divisor is 1.
- A sublattice $N^\circ \subset N$ which satisfies $|N/N^\circ| < \infty$, $\{N_{\mathrm{uf}}, N^\circ\} \subset \mathbb{Z}$ and $\{N, N_{\mathrm{uf}} \cap N^\circ\} \subset \mathbb{Z}$.
- The dual lattices $M = \operatorname{Hom}(N, \mathbb{Z})$, $M^\circ = \operatorname{Hom}(N^\circ, \mathbb{Z})$.

We need to assume the map $p_1^*\colon N_{\mathrm{uf}} \longrightarrow M^\circ, n \mapsto \{n, \cdot\}$ is injective. This injectivity assumption always holds in the case that $N = N_{\mathrm{uf}} = \mathbb{Z}^2$ and the skew-symmetric form $\{\cdot,\cdot\}$ is not zero.

Definition 2.1. A *seed* is a $\mathbb{Z}$-basis $\mathbf{s} = (e_i)_{i\in I}$ of N such that $\{e_i \mid i \in I_{\mathrm{uf}}\}$ is a basis of N_{uf} and $\{d_i e_i \mid i \in I\}$ is a basis of N°.

For seed $\mathbf{s}$, we can define the subset of N by

$$N^+ := N_{\mathbf{s}}^+ := \left\{ \sum_{i\in I_{\mathrm{uf}}} a_i e_i \,\middle|\, a_i \text{ are non-negative integers} \right\} \setminus \{0\}.$$

We fix a $\mathbb{Z}$-linear map $d\colon N \to \mathbb{Z}$ such that $d(n) > 0$ for $n \in N^+$.

We denote $M \otimes \mathbb{R}$ by $M_{\mathbb{R}}$. We can naturally regard M as a subset of $M_{\mathbb{R}}$. The $\mathbb{R}$-linear map $M \otimes \mathbb{R} \to M^\circ \otimes \mathbb{R}$ induced by the $\mathbb{Z}$-linear map $M \to M^\circ$, $f \mapsto f|_{N^\circ}$ is isomorphism. Thus we can also regard M° as a subset of $M_{\mathbb{R}}$.

We review some terms used in arguments about polyhedral cones.

Definition 2.2. Let V be a $\mathbb{R}$-linear space. A subset $\sigma \subset V$ is the *polyhedral cone* if there is a finite subset $\{v_1, \dots v_n\} \subset V$ such that

$$\sigma = \{r_1 v_1 + \cdots + r_n v_n \mid r_i \geq 0, i = 1, \dots, n\}.$$

Definition 2.3. Let σ be a polyhedral cone in V.

- A cone σ is *top dimensional* if $V = \{v - w \mid v, w \in \sigma\}$.
- A cone σ is *strictly convex* if σ includes no one-dimensional vector spaces.
- Let $V = M_{\mathbb{R}}$. A cone σ is *rational* if σ is generated by finite subset of M.

We fix a top dimensional strictly convex polyhedral cone $\sigma \subset M_{\mathbb{R}}$ such that $p_1^*(e_i)$ is in $(\sigma \cap M^\circ) \setminus \{0\}$ for all $i \in I_{\mathrm{uf}}$ In fact, we can take such a convex cone under the injectivity assumption. For example, a convex cone generated by $\{p_1^*(e_i) \mid i \in I_{\mathrm{uf}}\} \cup \{e_i^* \mid i \in I\}$ is such one, where $\{e_i^* \mid i \in I\}$ is the dual basis of the seed $\{e_i \mid i \in I\}$. We next consider a monoid $P := \sigma \cap M^\circ$ and monoid ring $\mathbf{k}[P]$ over a field of characteristic 0. We denote by J the ideal of the monoid ring which is generated by $P \setminus \{0\}$. We denote the completion with respect to J by

$$\widehat{\mathbf{k}[P]} := \varprojlim_k \mathbf{k}[P]/J^k := \left\{ (a_i + J^i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} \mathbf{k}[P]/J^i \;\middle|\; a_i \in \mathbf{k}[P], a_i - a_j \in J^i, i < j \right\}.$$

We identify an element of $\widehat{\mathbf{k}[P]}$ with a formal power series $a = \sum_{m \in P} a_m x^m$ in natural way. For $m \in M^\circ$, $n \in N^\circ$, we denote $m(n)$ by $\langle m, n \rangle$. We define a Lie algebra $\widehat{\Theta(\mathbf{k}[P])} := \widehat{\mathbf{k}[P]} \otimes_{\mathbb{Z}} N^\circ$ whose Lie bracket is defined by

$$[z^p \partial_n, z^{p'} \partial_{n'}] = z^{p+p'} \partial_{\langle p', n \rangle n' - \langle p, n' \rangle n},$$

where we denote $z^p \otimes n$ by $z^p \partial_n$. Consider the Lie subalgebra $J\widehat{\Theta(\mathbf{k}[P])} := (J\widehat{\mathbf{k}[P]}) \otimes N^\circ$ of $\widehat{\Theta(\mathbf{k}[P])}$, in which we define the group structure by using the Baker–Campbell–Hausdorff formula, i.e., for $\xi, \eta \in J\widehat{\Theta(\mathbf{k}[P])}$, we define

$$\xi * \eta = \xi + \eta + \frac{1}{2}[\xi, \eta] + \frac{1}{12}[\xi, [\xi, \eta]] + \cdots .$$

Consider the subspace

$$\mathfrak{g} := \sum_{n \in N^+} \mathfrak{g}_n, \quad \mathfrak{g}_n := \mathbf{k} z^{p_1^*(n)} \partial_n$$

of $\widehat{\Theta(\mathbf{k}[P])}$. Then, for $n, n' \in N^+$, we have

$$\begin{aligned} [z^{p_1^*(n)} \partial_n, z^{p_1^*(n')} \partial_{n'}] &= z^{p_1^*(n) + p_1^*(n')} \partial_{\{n', n\} n' - \{n, n'\} n} \\ &= z^{p_1^*(n) + p_1^*(n')} \partial_{\{n', n\}(n + n')} \\ &= \{n', n\} z^{p_1^*(n + n')} \partial_{n + n'}, \end{aligned}$$

and thus $[\mathfrak{g}_n, \mathfrak{g}_{n'}] = \mathfrak{g}_{n+n'}$. For each natural number k, we consider the ideal $\mathfrak{g}^{>k} := \sum_{d(n) > k} \mathfrak{g}_n$ of the Lie algebra $\mathfrak{g}$ and the nilpotent Lie algebras $\mathfrak{g}^{\leq k} := \mathfrak{g}/\mathfrak{g}^{>k}$. We can define the group multiplication $*$ on each $\mathfrak{g}^{\leq k}$ by using the Baker–Campbell–Hausdorff formula. We denote the groups $(\mathfrak{g}^{\leq k}, *)$ by $G^{\leq k}$ and define $G := \exp(\mathfrak{g}) := \varprojlim_k G^{\leq k}$. We denote the set bijection $\mathfrak{g}^{\leq k} \to G^{\leq k}$ by exp. For any $n_0 \in N^+$, we define

$$\begin{aligned} \mathfrak{g}_{n_0}^{||} &= \sum_{k > 0} \mathfrak{g}_{k \cdot n_0} \subset \mathfrak{g}, \\ G_{n_0}^{||} &= \exp(\mathfrak{g}_{n_0}^{||}) \subset G. \end{aligned}$$

We remark that $\mathfrak{g}_{n_0}^{||}$ are commutative Lie subalgebras of $\mathfrak{g}$, and thus $G_{n_0}^{||}$ are abelian groups.

Definition 2.4. Let $n_0 \in N^+$ and $m_0 := p_1^*(n_0)$. For $f \in 1 + z^{m_0}\mathbf{k}[[z^{m_0}]] \subset \widehat{\mathbf{k}[P]}$, we define $\mathfrak{p}_f \in \mathrm{Aut}(\widehat{\mathbf{k}[P]})$ by

$$\mathfrak{p}_f(z^m) = f^{\langle m, n_0' \rangle} z^m,$$

where n_0' is the generator of the monoid $\mathbb{R}_{\geq 0} n_0 \cap N^\circ$.

Lemma 2.1 ([GHKK18, Lemma 1.3]). Let $n_0 \in N^+$, $m_0 = p_1^*(n_0)$ and $d = \min\{r \in \mathbb{Q} \mid rn_0 \in N^\circ, r > 0\}$. Then, $G_{n_0}^{||}$ is identical with $\{\mathfrak{p}_f \mid f \in 1 + z^{m_0}k[[z^{m_0}]]\}$ as a subgroup of $\mathrm{Aut}(\widehat{\mathbf{k}[P]})$.

An element $n \in N^+$ is *primitive* if $k \cdot n' = n$ implies $n' = n$ for each natural number k and element n' of N^+.

Definition 2.5. A *wall* in $M_\mathbb{R}$ with respect to N^+ and $\mathfrak{g}$ is a pair $\mathfrak{d} = (\operatorname{supp}\mathfrak{d}, g_\mathfrak{d})$ which satisfying

1. there exists $n_0 \in N^+$ satisfying $g_\mathfrak{d} \in G_{n_0}^{||}$.
2. $\operatorname{supp}\mathfrak{d} \subset n_0^\perp = \{m \in M_\mathbb{R} \mid \langle m, n_0 \rangle = 0\}$.
3. $\operatorname{supp}\mathfrak{d}$ is a $(\operatorname{rank} N - 1)$-dimensional rational polyhedral cone.

Definition 2.6. A *scattering diagram* $\mathfrak{D}$ with respect to N^+ and $\mathfrak{g}$ is a collection of walls which satisfy that for every degree k, the set

$$\mathfrak{D}_k := \{\mathfrak{d} \in \mathfrak{D} \mid \text{The image of } g_\mathfrak{d} \text{ in } G^{\leq k} \text{ is not unit}\}$$

is a finite collection. We call $\mathfrak{D}_k$ by reducted scattering diagram of $\mathfrak{D}$ with respect to a degree k.

Here we remark that the term *collection* is used to mean a multiset, i.e., a set in which the multiplicity is considered. For a scattering diagram $\mathfrak{D}$, we define the support $\mathrm{Supp}(\mathfrak{D})$ and the set $\mathrm{Sing}(\mathfrak{D})$ of singular points by

$$\mathrm{Supp}(\mathfrak{D}) = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \mathrm{supp}(\mathfrak{d}), \quad \mathrm{Sing}(\mathfrak{D}) = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial\mathfrak{d} \cup \bigcup_{\substack{\mathfrak{d}_1, \mathfrak{d}_2 \in \mathfrak{D} \\ \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = n-2}} \mathfrak{d}_1 \cap \mathfrak{d}_2$$

We fix a scattering diagram $\mathfrak{D}$ and will define a *path-ordered product* with respect to $\mathfrak{D}$. Consider the smooth embedding

$$\gamma \colon [0,1] \longrightarrow M_\mathbb{R} \setminus \mathrm{Sing}(\mathfrak{D})$$

such that $\gamma(0), \gamma(1) \notin \mathrm{Supp}(\mathfrak{D})$ and γ intersects transversally with any walls crossing to γ. Fix a degree k and we first consider the reducted scattering diagram $\mathfrak{D}_k$ having finite cardinality. Later, we will define the path-ordered product with respect to $\mathfrak{D}$ by taking the limit of reducted ones. To define the path-ordered product of $\mathfrak{D}_k$, consider the finite sequence $0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1$ which satisfies the following:

1. For each $i = 1, \ldots, s$, there exists $\mathfrak{d}_i \in \mathfrak{D}_k$ such that $\gamma(t_i) \in \mathfrak{d}_i$.
2. $t_i = t_j \Rightarrow \mathfrak{d}_i \neq \mathfrak{d}_j$.
3. $0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1$ is the longest among sequences satisfying 1 and 2.

For each i, we define

$$\epsilon_i = \begin{cases} +1 & \text{if } \langle n_0, \gamma'(t_i)\rangle < 0, \\ -1 & \text{if } \langle n_0, \gamma'(t_i)\rangle > 0, \end{cases}$$

where n_0 is the element of N^+ satisfying $\mathfrak{d}_i \subset n_0^\perp$. We define the path-ordered product with respect to γ in $\mathfrak{D}_k$ by

$$\mathfrak{p}^k_{\gamma,\mathfrak{D}} := g^{\epsilon_s}_{\mathfrak{d}_s} \cdots g^{\epsilon_1}_{\mathfrak{d}_1}$$

and we define the path-ordered product with respect to γ in $\mathfrak{D}$ by

$$\mathfrak{p}_{\gamma,\mathfrak{D}} = \varprojlim_k \mathfrak{p}^k_{\gamma,\mathfrak{D}}.$$

Definition 2.7. Two scattering diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ are *equivalent* if for any curve γ such that both $\mathfrak{p}_{\gamma,\mathfrak{D}}$ and $\mathfrak{p}_{\gamma,\mathfrak{D}'}$ are defined, $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ holds.

We define the most important concept in the theory of scattering diagrams:

Definition 2.8. A scattering diagram is *consistent* if the path-ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is only depends on the end points of the curve γ with which $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is defined.

It is difficult to imagine examples of consistent scattering diagrams from the definition, but we can construct scattering diagrams from some sort of given data. In order to explain this, we introduce some terminologies.

Definition 2.9. Let $\mathfrak{d} \subset n_0^\perp$ be a wall.

- A wall $\mathfrak{d}$ is *incoming* if $p_1^*(n_0)$ is in $\mathfrak{d}$.
- A wall $\mathfrak{d}$ is *outgoing* if $p_1^*(n_0)$ is not in $\mathfrak{d}$.

Definition 2.10. For a seed $(e_i)_{i\in I}$, we define as follows.

- $(e_i^*)_{i\in I} \subset M$ is a dual basis of $(e_i)_{i\in I} \subset N$.
- $(f_i)_{i\in I} \subset M^\circ$ is a dual basis of $(d_ie_i)_{i\in I} \subset N^\circ$.
- $\epsilon_{ij} := \{e_i, e_j\}d_j$ $(i, j \in I)$.
- $v_i := p_1^*(e_i) \in P$ $(i \in I_{\mathrm{uf}})$.
- $A_i := z^{f_i} \in \mathbb{Z}[M^\circ]$ $(i \in I)$.

For seed $\mathbf{s}$, we define the scattering diagram that consists of some sort of incoming walls by

$$\mathfrak{D}_{\mathrm{in},\mathbf{s}} := \{(e_i^\perp, 1 + z^{v_i}) \mid i \in I_{\mathrm{uf}}\} = \left\{ (e_i^\perp, 1 + \prod_{j\in I} A_j^{\epsilon_{ij}}) \,\middle|\, i \in I_{\mathrm{uf}} \right\},$$

which is not consistent generally. We can construct the consistent scattering diagram from $\mathfrak{D}_{\mathrm{in},\mathbf{s}}$ by adding only outgoing walls. Namely, the following theorem holds.

Theorem 2.2 ([GHKK18, Theorem 1.12])**.** *For a seed* $\mathbf{s}$*, there is a scattering diagram* $\mathfrak{D}_{\mathbf{s}}$ *satisfying the following:*

1. $\mathfrak{D}_{\mathbf{s}}$ *is consistent.*
2. $\mathfrak{D}_{in,\mathbf{s}} \subset \mathfrak{D}_{\mathbf{s}}$.
3. $\mathfrak{D}_{\mathbf{s}} \setminus \mathfrak{D}_{in,\mathbf{s}}$ *consists of only outgoig walls.*

Moreover, such $\mathfrak{D}_{\mathbf{s}}$ *is determined uniquely up to equivalence.*

We will construct consistent scattering diagrams in heuristic ways in the case of $N = N_{\mathrm{uf}} = \mathbb{Z}^2$ by using the special elements of G called *dilogarithm elements* and their significant relations called *pentagon relations* in the following sections.

3 Dilogarithm elements and pentagon relation

From now, we mainly consider the case of $N = N_{\text{uf}} = N^\circ = \mathbb{Z}^2$. For the sake of the later arguments, we rewrite the above setting in simpler ways. Let N be a rank 2 lattice with a skew-symmetric bilinear form

$$\{\cdot,\cdot\}\colon N \times N \longrightarrow \mathbb{Q}.$$

Let e_1, e_2 be a basis of N, and we define

$$N^+ := \{a_1e_1 + a_2e_2 \mid a_1, a_2 \in \mathbb{Z}_{\geq 0}, a_1 + a_2 > 0\}.$$

Let $\Bbbk$ be a field of characteristic 0, and we define an N^+-graded Lie algebra $\mathfrak{g}$ over $\Bbbk$ with generators X_n such that

$$\mathfrak{g} = \bigoplus_{n \in N^+} \mathfrak{g}_n, \quad \mathfrak{g}_n = \Bbbk X_n, \quad [X_n, X_{n'}] = \{n, n'\} X_{n+n'}.$$

Let $\mathcal{L} := \{L \subset N^+ \mid N^+ + L \subset L, \#(N^+ \setminus L) < \infty\}$. For $L \in \mathcal{L}$, we define a Lie algebra ideal $\mathfrak{g}^L := \bigoplus_{n \in L} \mathfrak{g}_n$ and the quotient of $\mathfrak{g}$ by $\mathfrak{g}^L$

$$\mathfrak{g}_L := \mathfrak{g}/\mathfrak{g}^L = \bigoplus_{n \in N^+ \setminus L} \mathfrak{g}_n \quad \text{(as a vector space)}.$$

Let G_L be a group with a set bijection

$$\exp_L\colon \mathfrak{g}_L \longrightarrow G_L$$

and the product is defined by a Baker–Campbell–Hausdorff (BCH) formula:

$$\exp_L(X)\exp_L(Y) = \exp_L(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots). \tag{3.1}$$

This product formula is well-defined because $\mathfrak{g}_L$ is nilpotent.

For $L, L' \in \mathcal{L}$ such that $L \subset L'$, there exists the canonical Lie algebra homomorphism $\mathfrak{g}_L \longrightarrow \mathfrak{g}'_L$, which induces the group homomorphism $G_L \longrightarrow G'_L$. Thus, by the inverse limit we obtain a Lie algebra $\hat{\mathfrak{g}}$ and a group G:

$$\hat{\mathfrak{g}} := \varprojlim_{L \in \mathcal{L}} \mathfrak{g}_L, \quad G := \varprojlim_{L \in \mathcal{L}} G_L.$$

There is a set bijection

$$\exp\colon \hat{\mathfrak{g}} \longrightarrow G, \quad (X_L)_{L \in \mathcal{L}} \longmapsto (\exp_L(X_L))_{L \in \mathcal{L}}.$$

We use an infinite sum to express an element of $\hat{\mathfrak{g}}$.

We define important elements in G:

Definition 3.1 (Dilogarithm element)**.** For any $n \in N^+$, define

$$[n] := \exp\left(\sum_{j>0} \frac{(-1)^{j+1}}{j^2} X_{jn}\right) \in G.$$

We call $[n]$ the *dilogarithm element* for n.

For $c \in \mathbb{Q}$ and $g = \exp(X) \in G$, we define $g^c := \exp(cX)$.

Proposition 3.1 (Pentagon relation [GHKK18], [Nak23])**.** Let $n, n' \in N^+$. Then, the following relations hold in G:

1. If $\{n', n\} = 0$, then
$$[n'][n] = [n][n'], \tag{3.2}$$
2. If $\{n', n\} = c^{-1}$ $(c \in \mathbb{Q} \setminus \{0\})$, then
$$[n']^c[n]^c = [n]^c[n+n']^c[n']^c \quad \text{(pentagon relation)}. \tag{3.3}$$

We will use the following formulas later.

Lemma 3.2 ([Mat21, Lemma 1]). If $\{n', n\} = c^{-1}$ $(c \in \mathbb{Q} \setminus \{0\})$, we obtain
$$[n']^c[n]^{2c} = [n]^{2c}[2n+n']^c[n+n']^{2c}[n']^c, \tag{3.4}$$
$$[n']^{2c}[n]^c = [n]^c[n+n']^{2c}[n+2n']^c[n']^{2c}. \tag{3.5}$$

Proof. The equality (3.4) can be proved by repeatedly applying the pentagon relation:
$$\begin{aligned}[n']^c[n]^{2c} &= [n']^c[n]^c[n]^c = [n]^c[n+n']^c[n']^c[n]^c \\ &= [n]^c[n+n']^c[n]^c[n+n']^c[n']^c = [n]^{2c}[2n+n']^c[n+n']^{2c}[n']^c.\end{aligned}$$
The equality (3.5) can be proved in the same way:
$$\begin{aligned}[n']^{2c}[n]^c &= [n']^c[n']^c[n]^c = [n']^c[n]^c[n+n']^c[n']^c \\ &= [n]^c[n+n']^c[n']^c[n+n']^c[n']^c = [n]^c[n+n']^{2c}[n+2n']^c[n']^{2c}.\end{aligned}$$
□

4 Proof of formula (1.3)

For a subset $I = \{i_1 < i_2 < i_3 < \cdots\}$ of $\mathbb{Z}$ and a sequence $(a_i)_{i\in I}$ of elements of G, we write
$$\overrightarrow{\prod_{i\in I}} a_i := a_{i_1}a_{i_2}a_{i_3}\cdots, \quad \overleftarrow{\prod_{i\in I}} a_i := \cdots a_{i_3}a_{i_2}a_{i_1}. \tag{4.1}$$
For example, $\overrightarrow{\prod}_{i\geq 0} a_i = a_0a_1a_2\cdots$ and $\overleftarrow{\prod}_{i\geq 0} a_i = \cdots a_2a_1a_0$.

The following is the main theorem of this section:

Theorem 4.1 ([Mat21, Theorem 2]). *If* $\{n', n\} = c^{-1}$ $(c \in \mathbb{Q} \setminus \{0\})$, *then*
$$[n']^{2c}[n]^{2c} = \overrightarrow{\prod_{p\geq 0}}[n+p(n+n')]^{2c}\prod_{p\geq 0}[2^p(n+n')]^{4c/2^p}\overleftarrow{\prod_{p\geq 0}}[n'+p(n+n')]^{2c}. \tag{4.2}$$

The case of $c = 1$, $n = [(1,0)]$, $n' = [(0,1)]$ is nothing but the formula (1.3). We remark that the factors of $\prod_{p\geq 0}[2^p(n+n')]^{4c/2^p}$ are mutually commutative by equation (3.2) in Proposition 3.1 since $\{2^p(n+n'), 2^q(n+n')\} = 0$ for any $p, q \geq 0$.

To prove this theorem, we introduce some notations and lemmas.

Let $L \in \mathcal{L}$. For two elements g_1, g_2 of G, let us denote $g_1 \equiv g_2 \bmod L$ if their images in G_L are identical. For example, if $n \in N^+$ is in L, then $[n] \equiv \exp(0) = 1_G \bmod L$. By the definition of G, two elements g_1, g_2 of G are identical if and only if $g_1 \equiv g_2 \bmod L$ for all $L \in \mathcal{L}$.

The following is a key lemma:

Lemma 4.2 ([Mat21, Lemma 2]). Let l be a non-negative integer, and let $n, n' \in N^+$. If $\{n', n\} = c^{-1}$,
$$[n']^{2c}\left(\overrightarrow{\prod_{0\leq p\leq l}}[n+2pn']^c\right) = [n]^c\left(\overrightarrow{\prod_{1\leq p\leq 2l+1}}[n+pn']^{2c}\right)[n+(2l+2)n']^c[n']^{2c} \tag{4.3}$$

Proof. We will prove it by induction on l.

If $l = 0$, the equality (4.3) is nothing but (3.5).

Let $l > 0$. Suppose that the claim is true in the case of $l-1$, then by the induction hypothesis,

$$\begin{aligned}[n']^{2c}\left(\overrightarrow{\prod_{0\le p\le l}}[n+2pn']^{c}\right) &= [n']^{2c}\left(\overrightarrow{\prod_{0\le p\le l-1}}[n+2pn']^{c}\right)[n+2ln']^{c} \\ &= [n]^{c}\left(\overrightarrow{\prod_{1\le p\le 2l-1}}[n+pn']^{2c}\right)[n+2ln']^{c}[n']^{2c}[n+2ln']^{c} \\ &= [n]^{c}\left(\overrightarrow{\prod_{1\le p\le 2l+1}}[n+pn']^{2c}\right)[n+(2l+2)n']^{c}[n']^{2c}.\end{aligned}$$

In the last equality, we use

$$[n']^{2c}[n+2ln']^{c} = [n+2ln']^{c}[n+(2l+1)n']^{2c}[n+(2l+2)n']^{c}[n']^{2c},$$

which is a specialization of (3.5). □

Now we consider the limit of Lemma 4.2:

Lemma 4.3 ([Mat21, Lemma 3])**.** If $\{n', n\} = c^{-1}$, then

$$[n']^{2c}\left(\overrightarrow{\prod_{p\ge 0}}[n+2pn']^{c}\right) = [n]^{c}\left(\overrightarrow{\prod_{p\ge 1}}[n+pn']^{2c}\right)[n']^{2c}, \tag{4.4}$$

$$\left(\overleftarrow{\prod_{p\ge 0}}[n'+2pn]^{c}\right)[n]^{2c} = [n]^{2c}\left(\overleftarrow{\prod_{p\ge 1}}[n'+pn]^{2c}\right)[n']^{c}. \tag{4.5}$$

Proof. Let $L \in \mathcal{L}$. Then, there exists some positive integer l such that $n + 2ln' \in L$. Then, by Lemma 4.2, we obtain

$$\begin{aligned}[n']^{2c}\left(\overrightarrow{\prod_{p\ge 0}}[n+2pn']^{c}\right) &\equiv [n']^{2c}\left(\overrightarrow{\prod_{0\le p\le l}}[n+2pn']^{c}\right) \bmod L \\ &= [n]^{c}\left(\overrightarrow{\prod_{1\le p\le 2l+1}}[n+pn']^{2c}\right)[n+(2l+2)n]^{c}[n']^{2c} \\ &\equiv [n]^{c}\left(\overrightarrow{\prod_{p\ge 1}}[n+pn']^{2c}\right)[n']^{2c} \bmod L.\end{aligned}$$

Thus, the equality (4.4) holds.

Since $\{n, n'\} = (-c)^{-1}$, by (4.4), we obtain

$$[n]^{-2c}\left(\overrightarrow{\prod_{p\ge 0}}[n'+2pn]^{-c}\right) = [n']^{-c}\left(\overrightarrow{\prod_{p\ge 1}}[n'+pn]^{-c}\right)[n]^{-2c}. \tag{4.6}$$

The the equality (4.5) is obtained by taking the inverse of the both sides of (4.6). □

Proof of Theorem 4.1. For $L \in \mathcal{L}$ and $k \in \mathbb{Z}_{>0}$, let $P_L(k)$ be the following assertion: for $n, n' \in N^+$ and $c \in \mathbb{Q} \setminus \{0\}$, if $\{n', n\} = c^{-1}$ and $k(n+n') \in L$, then

$$[n']^{2c}[n]^{2c} \equiv \overrightarrow{\prod_{p\geq 0}}[n+p(n+n')]^{2c} \prod_{p\geq 0}[2^p(n+n')]^{4c/2^p} \overleftarrow{\prod_{p\geq 0}}[n'+p(n+n')]^{2c} \,\mathsf{mod}\, L. \tag{4.7}$$

For any $L \in \mathcal{L}$, there exists some positive integer k such that $k(n+n') \in L$. Thus, if $P_L(k)$ is true for any $k \in \mathbb{Z}_{>0}$ and $L \in \mathcal{L}$, then a relation (4.7) holds for any $L \in \mathcal{L}$, and Theorem 4.1 is proved. Fix $L \in \mathcal{L}$, and we prove $P_L(k)$ by induction on k.

If $k = 1$, the right hand side of (4.7) is equivalent to $[n]^{2c}[n']^{2c}$ because $ln + l'n' \in L$ for any $l, l' \in \mathbb{Z}_{\geq 1}$. Since

$$[n']^c[n]^c = [n]^c[n+n']^c[n']^c \equiv [n]^c[n']^c \,\mathsf{mod}\, L,$$

we obtain $[n']^{2c}[n']^{2c} \equiv [n']^{2c}[n']^{2c} \,\mathsf{mod}\, L$.

Let $k \geq 2$, and we suppose a proposition $P_L(k-1)$ is true. By the equality (3.5),

$$\begin{aligned}[n']^{2c}[n]^{2c} &= ([n']^{2c}[n]^c)[n]^c \\ &= [n]^c[n+n']^{2c}[n+2n']^c([n']^{2c}[n]^c) \\ &= [n]^c[n+n']^{2c}([n+2n']^c[n]^c)[n+n']^{2c}[n+2n']^c[n']^{2c}.\end{aligned}$$

Since $(k-1)(n+(n+2n')) \in L$ and $\{n+2n', n\} = (c/2)^{-1}$, by the induction hypothesis,

$$\begin{aligned}[n+2n']^c[n]^c &= [n+2n']^{(c/2)\cdot 2}[n]^{(c/2)\cdot 2} \\ &\equiv \overrightarrow{\prod_{p\geq 0}}[n+p(2n+2n')]^{2\cdot(c/2)} \prod_{p\geq 0}[2^p(2n+2n')]^{4\cdot(c/2)/2^p} \\ &\quad \times \overleftarrow{\prod_{p\geq 0}}[(n+2n')+p(2n+2n')]^{2\cdot(c/2)} \quad \mathsf{mod}\, L \qquad \text{(by equation (4.7))} \\ &= \overrightarrow{\prod_{p\geq 0}}[n+2p(n+n')]^c \prod_{p\geq 1}[2^p(n+n')]^{4c/2^p} \overleftarrow{\prod_{p\geq 0}}[(n+2n')+2p(n+n')]^c.\end{aligned}$$

Since $\{n+n', n\} = c^{-1}$ and $\{n+2n', n+n'\} = c^{-1}$, by Lemma 4.3,

$$\begin{aligned}[n']^{2c}[n]^{2c} &\equiv [n]^c[n+n']^{2c} \overrightarrow{\prod_{p\geq 0}}[n+2p(n+n')]^c \prod_{p\geq 1}[2^p(n+n')]^{4c/2^p} \\ &\quad \times \overleftarrow{\prod_{p\geq 0}}[(n+2n')+2p(n+n')]^c \\ &\quad \times [n+n']^{2c}[n+2n']^c[n']^{2c} \,\mathsf{mod}\, L \\ &= [n]^c[n]^c \left(\overrightarrow{\prod_{p\geq 1}}[n+p(n+n')]^c\right) [n+n']^{2c} \prod_{p\geq 1}[2^p(n+n')]^{4c/2^p} \\ &\quad \times [n+n']^{2c} \overleftarrow{\prod_{p\geq 1}}[(n+2n')+p(n+n')]^c \\ &\quad \times [n+2n']^c[n+2n']^c[n']^{2c} \\ &= \overrightarrow{\prod_{p\geq 0}}[n+p(n+n')]^{2c} \prod_{p\geq 0}[2^p(n+n')]^{4c/2^p} \overleftarrow{\prod_{p\geq 0}}[n'+p(n+n')]^{2c}.\end{aligned}$$

This completes the proof of Theorem 4.1. ☐

5 Proof of formula (1.4)

The formula (1.4) is the case that $c = -1$, $n = [(0,1)]$, $n' = [(1,0)]$ of the following theorem:

Theorem 5.1 ([Mat21, Theorem 3]). *If $\{n', n\} = c^{-1}$ $(c \in \mathbb{Q})$, then we obtain*

$$\begin{aligned}
[n']^c[n]^{4c} = &\overrightarrow{\prod_{p\geq 0}}([(2p+1)n + pn']^{4c}[(4p+4)n + (2p+1)n']^c) \\
&\times [2n+n']^{2c} \prod_{p\geq 0} [2^p(2n+n')]^{4c/2^p} \\
&\times \overleftarrow{\prod_{p\geq 0}}([(2p+1)n + (p+1)n']^{4c}[4pn + (2p+1)n']^c).
\end{aligned}$$

To prove this theorem, we consider some lemmas.

Lemma 5.2 ([Mat21, Lemma 4]). If $\{n', n\} = c^{-1}$ $(c \in \mathbb{Q} \setminus \{0\})$, then we obtain

$$\begin{aligned}
[n']^c \left(\overrightarrow{\prod_{0\leq p\leq l}} [n+pn']^{2c} \right) = &[n]^{2c} \left(\overrightarrow{\prod_{1\leq p\leq l}} ([2n+(2p-1)n']^c[n+pn']^{4c}) \right) \\
&\times [2n+(2l+1)n']^c[n+(l+1)n']^{2c}[n']^c.
\end{aligned}$$

Proof. We prove it by induction on l. The case of $l = 0$ is nothing less than the equality (3.4).
Let $l > 0$. Suppose that the claim is true in the case of $l-1$, then

$$\begin{aligned}
[n']^c \left(\overrightarrow{\prod_{0\leq p\leq l}} [n+pn']^{2c} \right) &= [n']^c \left(\overrightarrow{\prod_{0\leq p\leq l-1}} [n+pn']^{2c} \right) [n+ln']^{2c} \\
&= [n]^{2c} \left(\overrightarrow{\prod_{1\leq p\leq l-1}} ([2n+(2p-1)n']^c[n+pn']^{4c}) \right) \\
&\quad \times [2n+(2l-1)n']^c[n+ln']^{2c}[n']^c[n+ln']^{2c} \\
&= [n]^{2c} \left(\overrightarrow{\prod_{1\leq p\leq l}} ([2n+(2p-1)n']^c[n+pn']^{4c}) \right) \\
&\quad \times [2n+(2l+1)n']^c[n+(l+1)n']^{2c}[n']^c.
\end{aligned}$$

In the last equality, we use

$$[n']^c[n+ln']^{2c} = [n+ln']^{2c}[2n+(2l+1)n']^c[n+(l+1)n']^{2c}[n']^2,$$

which is a specialization of (3.4). □

Now we consider the limit of Lemma 5.2:

Lemma 5.3. If $\{n', n\} = c^{-1}$, then we obtain

$$[n']^c \left(\overrightarrow{\prod_{p\geq 0}} [n+pn']^{2c} \right) = [n]^{2c} \left(\overrightarrow{\prod_{p\geq 1}} [2n+(2p-1)n']^c[n+pn']^{4c} \right) [n']^c, \tag{5.1}$$

$$\left(\overleftarrow{\prod_{p\geq 0}} [n'+pn]^{2c} \right) [n]^c = [n]^c \left(\overleftarrow{\prod_{p\geq 1}} [n'+pn]^{4c}[2n'+(2p-1)n]^c \right) [n']^{2c}. \tag{5.2}$$

Proof. Let $L \in \mathcal{L}$. Then, there exist some positive integer l such that $n + ln' \in L$. Then, by Lemma 5.2, we obtain

$$\begin{aligned}
[n']^c \overrightarrow{\prod_{p\geq 0}}[n+pn']^{2c} &\equiv [n']^c \left(\overrightarrow{\prod_{0\leq p\leq l}} [n+pn']^{2c} \right) \operatorname{mod} L \\
&= [n]^{2c} \left(\overrightarrow{\prod_{1\leq p\leq l}} ([2n+(2p-1)n']^c[n+pn']^{4c}) \right) \\
&\quad \times [2n+(2l+1)n']^c[n+(l+1)n']^{2c}[n']^c \\
&\equiv [n]^{2c} \left(\overrightarrow{\prod_{p\geq 1}}[2n+(2p-1)n']^c[n+pn']^{4c} \right) [n']^c \operatorname{mod} L.
\end{aligned}$$

Thus, the equality (5.1) holds.

Since $\{n, n'\} = (-c)^{-1}$, by (5.1), we obtain

$$[n]^{-c} \left(\overrightarrow{\prod_{p\geq 0}}[n'+pn]^{-2c} \right) = [n']^{-2c} \left(\overrightarrow{\prod_{p\geq 1}}[2n'+(2p-1)n]^{-c}[n'+pn]^{-4c} \right) [n]^{-c}.$$

By taking the inverse of both sides of this equality, we have the equality (5.2). □

Proof of Theorem 5.1. Using the pentagon relations, Theorem 4.1 and Lemma 5.3, we can calculate as follows:

$$\begin{aligned}
[n']^c[n]^{4c} &= [n']^c[n]^{2c}[n]^{2c} = [n]^{2c}[2n+n']^c[n+n']^{2c}[n']^c[n]^{2c} \qquad \text{(by equation (3.4))} \\
&= [n]^{2c}[2n+n']^c[n+n']^{2c}[n]^{2c}[2n+n']^c[n+n']^{2c}[n]^c \qquad \text{(by equation (3.4))} \\
&= [n]^{2c}[2n+n']^c \\
&\quad \times \left(\overrightarrow{\prod_{p\geq 0}}[n+p(2n+n')]^{2c} \right) \left(\prod_{p\geq 0}[2^p(2n+n')]^{4c/2^p} \right) \left(\overleftarrow{\prod_{p\geq 0}}[n+n'+p(2n+n')]^{2c} \right) \\
&\quad \times [2n+n']^c[n+n']^{2c}[n']^c \qquad \text{(by equation (4.2))} \\
&= [n]^{2c} \times [n]^{2c} \left(\overrightarrow{\prod_{p\geq 1}}([2n+(2p-1)(2n+n')]^c[n+p(2n+n')]^{4c}) \right) [2n+n']^c \\
&\quad \times \prod_{p\geq 0}[2^p(2n+n')]^{4c/2^p} \\
&\quad \times [2n+n']^c \left(\overleftarrow{\prod_{p\geq 1}}[(n+n')+p(2n+n')]^{4c}[2(n+n')+(2p-1)(2n+n')]^c \right) [n+n']^{2c} \\
&\quad \times [n+n']^{2c}[n']^c \qquad \text{(by equations (5.1), (5.2))} \\
&= \left(\overrightarrow{\prod_{p\geq 0}}[n+p(2n+n')]^{4c}[2n+(2p+1)(2n+n')]^c \right) \\
&\quad \times [2n+n']^{2c} \prod_{p\geq 0}[2^p(2n+n')]^{4c/2^p} \\
&\quad \times \left(\overleftarrow{\prod_{p\geq 0}}[(n+n')+p(2n+n')]^{4c}[2(n+n')+(2p-1)(2n+n')]^c \right)
\end{aligned}$$

$$
\begin{aligned}
&= \overrightarrow{\prod_{p\geq 0}}([(2p+1)n+pn']^{4c}[(4p+4)n+(2p+1)n']^{c}) \\
&\quad \times [2n+n']^{2c}\prod_{p\geq 0}[2^p(2n+n')]^{4c/2^p} \\
&\quad \times \overleftarrow{\prod_{p\geq 0}}([(2p+1)n+(p+1)n']^{4c}[4pn+(2p+1)n']^{c}).
\end{aligned}
$$

In the second equality from the last, we used commutativity of $[2n+n']^{2c}$ and $\prod_{p\geq 0}[2^p(2n+n')]^{4c/2^p}$. This completes the proof. □

Part II

Generalization of Markov Diophantine equation via generalized cluster algebra

6 Markov equation

Consider the equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{6.1}$$

This equation has a property that if (x_0, y_0, z_0) is a solution of the equation (6.1) then

$$\left(\frac{y_0^2 + z_0^2}{x_0}, y_0, z_0\right), \quad \left(x_0, \frac{x_0^2 + z_0^2}{y_0}, z_0\right), \quad \left(x_0, y_0, \frac{x_0^2 + y_0^2}{z_0}\right) \tag{6.2}$$

are also solutions of the equation (6.1). This is checked by direct calculation. But we can also calculate it as follows. First, we calculate sum and products of x_0 and $(y_0^2 + z_0^2)/x_0$:

$$x_0 + \frac{y_0^2 + z_0^2}{x_0} = \frac{x_0^2 + y_0^2 + z_0^2}{x_0} = \frac{3x_0y_0z_0}{x_0} = 3y_0z_0,$$
$$x_0 \cdot \frac{y_0^2 + z_0^2}{x_0} = y_0^2 + z_0^2.$$

Then, by the relation between roots and coefficients of a quadratic equation, x_0 and $(y_0^2+z_0^2)/x_0$ are two solutions of the quadratic equation

$$X^2 - 3y_0z_0X + y_0^2 + z_0^2 = 0 \tag{6.3}$$

on a variable X. Comparing (6.1) and (6.3), we see that (X, y_0, z_0) is also a solution of (6.1). We call the operation obtaining three solutions (6.2) from (x_0, y_0, z_0) the *first Vieta jumping*, the *second Vieta jumping* and the *third Vieta jumping*, respectively. These Vieta jumpings assign positive integer solutions to positive integer solutions. In fact, if x_0, y_0, z_0 are integers, then the rational number obtained by the first Vieta jumping

$$\frac{y_0^2 + z_0^2}{x_0} = \frac{3x_0y_0z_0 - x_0^2}{x_0} = 3y_0z_0 - x_0$$

is also integers. In the same way, the solutions obtained by the second and third Vieta jumpings are also integer solutions. Furthermore, if x_0, y_0, z_0 are positive numbers, then all components of solutions obtained by the Vieta jumpings are positive numbers. This is clear because expressions (6.2) contains no subtractions. Since $(x, y, z) = (1, 1, 1)$ is a positive integer solutions of (6.1), we may obtain infinitely many solutions of the equation (6.1) by the repeating Vieta jumping from $(1, 1, 1)$.

This phenomenon can be explained by using a cluster algebra. Let the initial exchange matrix be a matrix

$$B_0 = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

Then the mutations of B_0 at any directions are $-B_0$. Thus, in the case that the coefficients semifield $\mathbb{P}$ is $\{1\}$, the exchange relations of all seeds are the same forms. Namely, for an arbitrary seed $((x, y, z), (1, 1, 1), B)$, since the exchange matrix B is B_0 or $-B_0$, the exchange relations of the mutations at direction 1, 2 and 3 are

$$xx' = y^2 + z^2,$$

$$yy' = x^2 + z^2,$$
$$zz' = x^2 + y^2,$$

respectively, where x', y' and z' are cluster variable obtained by the mutations. Comparing to the expression (6.2), you can see three clusters (x', y, z), (x, y', z) and (x, y, z') are obtained from the cluster (x, y, z) by the first, second and third Vieta jumping, respectively. Then the phenomenon that any triplets obtained by the repeated Vieta jumping from $(1, 1, 1)$ are positive integers is explained by using Laurent positivity of cluster variables. To explain this, let $((x_0, y_0, z_0), (1, 1, 1), B_0)$ be the initial seed. Then an arbitrary cluster variable is expressed by the form of $P(x_0, y_0, z_0)$, where P is a Laurent polynomial with positive integer coefficients. Thus $P(1, 1, 1)$ is positive integer. Since numbers obtained by the repeated Vieta jumpings from $(1, 1, 1)$ are of the forms $P(1, 1, 1)$, these are positive integers.

7 Generalized cluster pattern

We can generalize the above procedure by using generalized cluster patterns by [CS14]. From now, we explain generalised cluster patterns. Let $\mathbb{P}$ be the semifield $\{1\}$ (trivial semifield) and $\mathcal{F}$ be a field isomorphic to a rational function field over $\mathbb{QP} = \mathbb{Q}$ whose transcendence degree is n as mentioned in the introduction. A *labeled seed* is a triplet $(\mathbf{x}, B, \mathbf{Z})$, where

- $\mathbf{x} = (x_1, \dots, x_n)$ is an n-tuple of elements of $\mathcal{F}$ forming a free generating set of $\mathcal{F}$,
- $B = (b_{ij})$ is an $n \times n$ integer matrix which is *skew-symmetrizable*, that is, there exists a positive integer diagonal matrix S such that SB is skew-symmetric. We call S a *skew-symmetrizer* of B,
- $\mathbf{Z} = (Z_1, \dots, Z_n)$ is an n-tuple of polynomials with the coefficient in $\mathbb{N}$ $(= \mathbb{NP})$

  $$Z_i(u) = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i}u^{d_i}$$

  satisfying $z_{i,0} = z_{i,d_i} = 1$.

In [CS14], [Nak23], the coefficients of $Z_i(u)$ are $\mathbb{P}$. However the coefficients can be extended to $\mathbb{NP}$. We say that $\mathbf{x}$ is a *cluster*, and we refer to x_i, B and Z_i as the *cluster variable*, the *exchange matrix* and the *exchange polynomial*, respectively. Furthermore, we set $D = \mathrm{diag}(d_1, \dots, d_n)$, that is a positive integer diagonal matrix of rank n. We remark that $\deg Z_i = d_i \geq 1$ for $i = 1, \dots, n$.

For an integer b, we use the notation $[b]_+ = \max(b, 0)$. Let $(\mathbf{x}, B, \mathbf{Z})$ be a labeled seed, and let $k \in \{1, \dots, n\}$. The *seed mutation* μ_k *in direction* k transforms $(\mathbf{x}, B, \mathbf{Z})$ into another labeled seed $\mu_k(\mathbf{x}, B, \mathbf{Z}) = (\mathbf{x}', B', \mathbf{Z}')$ defined as follows:

- The entries of $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + d_k \left([b_{ik}]_+ b_{kj} + b_{ik} [-b_{kj}]_+ \right) & \text{otherwise.} \end{cases} \tag{7.1}$$

- The cluster variables $\mathbf{x}' = (x'_1, \dots, x'_n)$ are given by

$$x'_j = \begin{cases} \dfrac{\left(\prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \right)^{d_k} Z_k \left(\prod_{i=1}^{n} x_i^{b_{ik}} \right)}{x_k} & \text{if } j = k, \\ x_j & \text{otherwise.} \end{cases} \tag{7.2}$$

- The exchange polynomials $\mathbf{Z}' = (Z'_1, \ldots, Z'_n)$ are given by

$$Z'_j(u) = \begin{cases} u^{d_k} Z_k(u^{-1}) & \text{if } j = k, \\ Z_j(u) & \text{otherwise.} \end{cases} \tag{7.3}$$

The formula of the case that $j = k$ in (7.2) is called the *exchange relation.* In [CS14], y-variables and their mutaions are also considered. However we omit them since they are trivial when $\mathbb{P} = \{1\}$.

When $D = \mathrm{diag}(1, \ldots, 1)$, labeled seeds and their mutations are nothing but ones of ordinary cluster algebras explained in §1.1. (In the definitions of ordinary cluster algebras, skew-symmetrizer of B is denoted by D instead of S. In the case of generalised cluster algebras, however, we use a symbol D for denoting a diagonal matrix whose entries are degrees of polynomials Z_i.) Note that in the case that $u^{d_k} Z_k(u^{-1}) = Z(u)$, if exchange matrix B were replaced by $-B$, the exchange relations would still have the same form. In fact,

$$\begin{aligned} \left(\prod_{i=1}^n x_i^{[-b_{ik}]_+}\right)^{d_k} Z_k\left(\prod_{i=1}^n x_i^{b_{ik}}\right) &= \left(\prod_{i=1}^n x_i^{[-b_{ik}]_+}\right)^{d_k} \left(\prod_{i=1}^n x_i^{b_{ik}}\right)^{d_k} Z_k\left(\prod_{i=1}^n x_i^{-b_{ik}}\right) \\ &= \left(\prod_{i=1}^n x_i^{[b_{ik}]_+}\right)^{d_k} Z_k\left(\prod_{i=1}^n x_i^{-b_{ik}}\right). \end{aligned}$$

In the last equality, we used the formula

$$[-b]_+ + b = [b]_+ \quad (b \in \mathbb{R}).$$

Let $\mathbb{T}_n$ be the *n-regular tree.*

Definition 7.1 (generalized cluster pattern)**.** A *generalized cluster pattern* Σ is a map that assigns to each vertex $t \in \mathbb{T}_n$ a seed $\Sigma_t = (\mathbf{x_t}, B_t, \mathbf{Z}_t)$ and if t, t' are adjacent to each other through an edge labeled by k then Σ_t and $\Sigma_{t'}$ are related by mutation each other.

The Laurent phenomenon holds in generalized cluster algebra similarly to ordinary cluster algebra.

Theorem 7.1 ([CS14])**.** *Let Σ be any generalized cluster pattern. Let $t_0, t \in \mathbb{T}_n$ be any vertices. Then, any cluster variable $x_{i;t}$ is expressed as a Laurent polynomial in x_{t_0} with coefficients in $\mathbb{ZP}$.*

Consider the case that $D = \mathrm{diag}(2, 2, 2)$ and the initial seed is $(\mathbf{x}, B_0, \mathbf{Z})$ where $\mathbf{Z} = (Z_1, Z_2, Z_3)$,

$$B_0 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad Z_i(u) = 1 + k_i u + u^2 \quad (i = 1, 2, 3)$$

and k_1, k_2 and k_3 are positive integers. In this case, $\mu_k(B_0) = -B_0$, $\mu_k(Z_i(u)) = Z_i(u)$ and $u^{d_k} Z_k(u^{-1}) = Z(u)$ $(k, i = 1, 2, 3)$ hold. Thus the exchange relations of any seeds have the same form. Namely, for an arbitrary cluster (x_1, x_2, x_3), the exchange relations are

$$\begin{aligned} x_1 x'_1 &= x_2^2(1 + k_1 x_2^{-1} x_3 + x_2^{-2} x_3^2) = x_2^2 + k_1 x_2 x_3 + x_3^2, \\ x_2 x'_2 &= x_3^2(1 + k_2 x_3^{-1} x_1 + x_3^{-2} x_1^2) = x_3^2 + k_2 x_3 x_1 + x_1^2, \\ x_3 x'_3 &= x_1^2(1 + k_3 x_1^{-1} x_2 + x_1^{-2} x_2^2) = x_1^2 + k_3 x_1 x_2 + x_3^2, \end{aligned}$$

where x'_1, x'_2 and x'_3 are new cluster variable obtained by a mutation at direction 1, 2 and 3, respectively.

We now consider if there exists an equation $F(x, y, z) = 0$ in three variables which satisfies the following two conditions:

1. If (x_1, x_2, x_3) is a solution of it, then

$$\left(\frac{x_2^2 + k_1x_2x_3 + x_3^2}{x_1}, x_2, x_3\right), \left(x_1, \frac{x_3^2 + k_2x_3x_1 + x_1^2}{x_2}, x_3\right), \left(x_1, x_2, \frac{x_1^2 + k_3x_1x_2 + x_3^2}{x_3}\right)$$

are also solutions of the same equations, and

2. $(1, 1, 1)$ is solution of it.

We call the equation satisfying these properties as *generalised Markov equations*. For example, in [Gyo22], the equation

$$(x + y)^2 + (y + z)^2 + (z + x)^2 = 12xyz \tag{7.4}$$

is considered. This is a generalized Markov equation in the case of $k_1 = k_2 = k_3 = 1$. We fortunately found out more generalised one

$$x^2 + y^2 + z^2 + k_1yz + k_2zx + k_3xy = (3 + k_1 + k_2 + k_3)xyz,$$

which is already mentioned in introduction as the equation (1.5).

Proposition 7.2 ([GM23, Proposition 3])**.** If $(x, y, z) = (a, b, c)$ is a positive integer solution to (1.5), then so are $\left(\frac{b^2 + k_1bc + c^2}{a}, b, c\right)$, $\left(a, \frac{a^2 + k_2ac + c^2}{b}, c\right)$, and $\left(a, b, \frac{a^2 + k_3ab + b^2}{c}\right)$. In particular, the equation (1.5) is a generalised Markov equation.

Proof. Let (a, b, c) be a solution of (1.5). We will prove

$$\left(\frac{b^2 + k_1bc + c^2}{a}, b, c\right) \tag{7.5}$$

is also a solution of (1.5). This can be verified by direct calculation, but it is a somewhat complicated calculation. It is easier to understand if we first consider the sum and product of

$$a \quad \text{and} \quad \frac{b^2 + k_1bc + c^2}{a},$$

as in the discussion when Markov equation is considered:

$$a + \frac{b^2 + k_1bc + c^2}{a} = \frac{a^2 + b^2 + c^2 + k_1bc}{a} = (3 + k_1 + k_2 + k_3)bc - k_2c - k_3b, \tag{7.6}$$

$$a \times \frac{b^2 + k_1bc + c^2}{a} = b^2 + k_1bc + c^2.$$

In the equality (7.6), we used the fact that (a, b, c) is a solution of (1.5). By the relaion between roots and coefficients, a and $(b^2 + k_1bc + c^2)/a$ are two solutions of quadratic equation

$$X^2 - \{(3 + k_1 + k_2 + k_3)bc - k_2c - k_3b\}X + b^2 + k_1bc + c^2 = 0.$$

This equation can be rewritten as

$$X^2 + b^2 + c^2 + k_1bc + k_2Xc + k_3Xb = (3 + k_1 + k_2 + k_3)Xbc.$$

Since this is obtained by substituting $(x, y, z) = (X, b, c)$ in (1.5), (7.5) is solution of (1.5). By the equation (7.6), $\frac{b^2 + k_1bc + c^2}{a}$ is an integer. □

The above argument even works for the equation (1.11) and we obtain the following theorem:

Theorem 7.3 ([GM23, Theorem 17]). *For each equation given in* (1.5) *and* (1.11), B *and* $\mathbf{Z}$ *(and* D*) are set as in Table 1. Then, the generalized cluster pattern* $CP_{(\mathbf{x},B,\mathbf{Z})}$ *with a substitution* $x_1 = x_2 = x_3 = 1$ *gives the tree* $\mathbb{T}^{k_1,k_2,k_3}$ *or* $\mathbb{T}^k$ *giving positive integer solutions to the corresponding equation (where we ignore exchange matrices in cluster pattern and consider only clusters).*

The top four rows in Table 1 are the case that $k_1 = k_2 = k_3 = 0$, the case that $k_1 = k_2 = 0, k_3 \neq 0$, the case that $k_2 = 0, k_1 \neq 0, k_3 \neq 0$ and the case that k_1, k_2, k_3 are non zero of the equation (1.5) from the top. The remaining two rows are the case that $k = 0$ and $k \neq 0$ of the equation (1.11), respectively.

Equation	B	$\mathbf{Z}$	D
$x^2+y^2+z^2=3xyz$	$\begin{bmatrix}0&2&-2\\-2&0&2\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$x^2+y^2+z^2+k_3xy=(3+k_3)xyz$	$\begin{bmatrix}0&2&-1\\-2&0&1\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+k_3u+u^2\end{cases}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$
$x^2+y^2+z^2+k_3xy+k_1yz$ $=(3+k_3+k_1)xyz$	$\begin{bmatrix}0&2&-1\\-1&0&1\\1&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+k_1u+u^2\\Z_2(u)=1+u\\Z_3(u)=1+k_3u+u^2\end{cases}$	$\begin{bmatrix}2&0&0\\0&1&0\\0&0&2\end{bmatrix}$
$x^2+y^2+z^2+k_3xy+k_1yz+k_2zx$ $=(3+k_1+k_2+k_3)xyz$	$\begin{bmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+k_1u+u^2\\Z_2(u)=1+k_2u+u^2\\Z_3(u)=1+k_3u+u^2\end{cases}$	$\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$
$x^2+y^4+z^4+2xy^2+2z^2x$ $=7xy^2z^2$	$\begin{bmatrix}0&1&-1\\-4&0&2\\4&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$x^2+y^4+z^4+2xy^2+ky^2z^2+2z^2x$ $=(7+k)xy^2z^2$	$\begin{bmatrix}0&1&-1\\-2&0&2\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+ku+u^2\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Table 1: Equations and corresponding triplets $(B, \mathbf{Z}, D)$

8 Proof of Theorem 1.6 and its corollaries

We will prove the first main theorem, Theorem 1.6, in Part II. We first determine the solutions that contain two or more of the same number.

Lemma 8.1 ([GM23, Lemma 4]). In the positive integer solutions to (1.5), the only solutions that contain repeated numbers are $(1,1,1), (k_1+2,1,1), (1,k_2+2,1)$ and $(1,1,k_3+2)$.

Proof. Let (a,b,c) be a positive integer solution to (1.5) that contains repeated numbers. We prove the case of $a = b$. Then, by substituting (a,a,c) for (x,y,z) in (1.5), we have

$$(2+k_3)a^2 + c^2 + (k_1+k_2)ac = (3+k_1+k_2+k_3)a^2c.$$

Therefore, we have

$$c = \frac{a^2k_3 + a^2k_1 + a^2k_2 + 3a^2 - ak_1 - ak_2 \pm a\sqrt{(ak_3 + (a-1)k_1 + (a-1)k_2 + 3a)^2 - 4(k_3+2)}}{2}.$$

We set $k = ak_3 + (a-1)k_1 + (a-1)k_2 + 3a > 0$. In order for c to be an integer, the inside of the square root must be a square number. Therefore, there exists an positive integer l such that $l^2 = k^2 - 4(k_3+2)$. Since $a \geq 1$, we have $k \geq k_3 + 3$. Therefore, $k + l > k_3 + 2$ holds. Since $(k+l)(k-l) = 4(k_3+2)$, we have $1 \leq k - l \leq 3$, and $(k-l, k+l)$ must be one of $(1, 4(k_3+2)), (2, 2(k_3+2)), \left(3, \frac{4(k_3+2)}{3}\right)$. Of the three, it cannot be $(1, 4(k_3+2))$ and $\left(3, \frac{4(k_3+2)}{3}\right)$ because $k = \frac{(k+l)+(k-l)}{2}$ is an integer. When $(k-l, k+l) = (2, 2(k_3+2))$, we have $k = k_3 + 3$ and $l = k_3 + 1$. Thus we have $(a, a, c) = (1, 1, 1)$ or $(1, 1, k_3+2)$. The cases that $a = c$ and $b = c$ can be proved in the same way. □

The triples $(1,1,1), (k_1+2,1,1), (1,k_2+2,1)$ and $(1,1,k_3+2)$ are said to be *singular*, and other positive integer solutions to (1.5) are said to be *nonsingular*.

Proposition 8.2 ([GM23, Proposition 5])**.** Let $(x,y,z) = (a,b,c)$ be a nonsingular positive integer solution to (1.5), and we assume $a > b > c$. Then we have

(1) $\dfrac{a^2 + k_2ac + c^2}{b} > a(> c)$,

(2) $\dfrac{a^2 + k_3ab + b^2}{c} > a(> b)$,

(3) $b > \dfrac{b^2 + k_1bc + c^2}{a}$.

Proof. We prove (1). We have

$$\frac{a^2 + k_2ac + c^2}{b} - a = \frac{a^2 + k_2ac + c^2 - ab}{b} > \frac{a^2 + k_2ac + c^2 - a^2}{b} = \frac{c^2 + k_2ac}{b} > 0.$$

We can show (2) in the same way as (1). We will show (3). We set

$$\begin{aligned} f(x) &:= (x-a)\left(x - \frac{b^2 + k_1bc + c^2}{a}\right) \\ &= x^2 - ((3 + k_1 + k_2 + k_3)bc - k_3b - k_2c)x + (b^2 + c^2 + k_1bc) \end{aligned}$$

It suffices to show that

$$f(b) = (2 + k_3)b^2 - (3 + k_1 + k_2 + k_3)b^2c + (k_1 + k_2)bc + c^2 < 0.$$

We consider a function from $\mathbb{R}^2$ to $\mathbb{R}$

$$g(y,z) = (2 + k_3)y^2 - (3 + k_1 + k_2 + k_3)y^2z + (k_1 + k_2)yz + z^2. \tag{8.1}$$

We remark that $g(b,c) = f(b)$. By considering the partial derivative of g in the y direction, we have

$$\frac{\partial g}{\partial y} = 2(2 + k_3)y - 2(3 + k_1 + k_2 + k_3)yz + (k_1 + k_2)z.$$

When $y > z \geq 1$, we have

$$\begin{aligned}\frac{\partial g}{\partial y}(y,z) &< 2(2+k_3)y - 2(3+k_1+k_2+k_3)yz + (k_1+k_2)y \\ &= -y((6z-4) + k_3(2z-2) + k_1(2z-1) + k_2(2z-1)) \\ &< -((6z-4) + k_3(2z-2) + k_1(2z-1) + k_2(2z-1)) < 0.\end{aligned}$$

Moreover, by considering the partial derivative of g in the z direction, we have

$$\frac{\partial g}{\partial z} = -(3+k_1+k_2+k_3)y^2 + (k_1+k_2)y + 2z.$$

When $y > z \geq 1$, we have

$$\begin{aligned}\frac{\partial g}{\partial z}(y,z) &< -(3+k_1+k_2+k_3)y^2 + (k_1+k_2)y + 2y \\ &< -(3+k_1+k_2+k_3)y^2 + (k_1+k_2)y^2 + 2y^2 < -y^2(1+k_3) < 0.\end{aligned}$$

Therefore, $g(y,z)$ is strictly monotonically decreasing in the y and z directions in the range $y > z \geq 1$. Since $g(1,1) = 0$, we have $g(b,c) = f(b) < 0$. □

In Proposition 8.2, we assume that $a > b > c$, but this assumption is not essential:

Corollary 8.3 ([GM23, Corollary 6])**.** Let $(x,y,z) = (a,b,c)$ be a nonsingular positive integer solution to (1.5). We set (a',b,c) (resp. $(a,b',c),(a,b,c')$) as the first (resp. second, third) Vieta jumping.

(1) If a is the maximal in (a,b,c), then a' is not maximal in (a',b,c), b' is maximal in (a,b',c), and c' is maximal in (a,b,c'),

(2) if b is the maximal in (a,b,c), then a' is maximal in (a',b,c), b' is not maximal in (a,b',c), and c' is maximal in (a,b,c'),

(3) if c is the maximal in (a,b,c), then a' is maximal in (a',b,c), b' is maximal in (a,b',c), and c' is not maximal in (a,b,c').

Proof. When $a > b > c$, it is proved by Proposition 8.2. The other cases are proved in the same way as the proof of Proposition 8.2. □

Remark 8.1. By Corollary 8.3, for a nonsingular triplet (a,b,c) in $\mathbb{T}^{k_1,k_2,k_3}$,

(i) if a is the maximal number in (a,b,c), then the parent of (a,b,c) is $\left(\dfrac{b^2+k_1bc+c^2}{a}, b, c\right)$,

(ii) if b is the maximal number in (a,b,c), then the parent of (a,b,c) is $\left(a, \dfrac{a^2+k_2ac+c^2}{b}, c\right)$,

(iii) if c is the maximal number in (a,b,c), then the parent of (a,b,c) is $\left(a, b, \dfrac{a^2+k_3ab+b^2}{c}\right)$.

Moreover, each non-singular triplet in $\mathbb{T}^{k_1,k_2,k_3}$ have a smaller maximum than its children. Therefore, singular triplets cannot be children of non-singular triplets in $\mathbb{T}^{k_1,k_2,k_3}$. Hence, each singular triplet appears in $\mathbb{T}^{k_1,k_2,k_3}$ once, and the above facts (i),(ii),(iii) are also true for singular triplets other than $(1,1,1)$. Thus, in $\mathbb{T}^{k_1,k_2,k_3}$, the three vertices adjacent to each vertex (a,b,c) are respectively the one where a in (a,b,c) is replaced by another number, the one where b in (a,b,c) is replaced by another number, and the one where c in (a,b,c) is replaced by another number.

Now, we will show Theorem 1.6.

Proof of Theorem 1.6. By Proposition 7.2 and the fact that $(x, y, z) = (1, 1, 1)$ is a positive integer solution to (1.5), all vertices in $\mathbb{T}^{k_1,k_2,k_3}$ are positive integer solutions to (1.5). Suppose that $(x, y, z) = (a, b, c)$ is a nonsingular positive integer solution to (1.5). Then, by Corollary 8.3, there is one of the Vieta jumpings of (a, b, c) whose maximal number is smaller than that of (a, b, c). This process can be continued as long as the solution is nonsingular. Since the solutions that appear in this operation are always positive integer solutions, a singular solution will appear in a finite number of the operations. By Lemma 8.1, when a nonsingular solution changes to a singular solution, the singular solution is $(k_1 + 2, 1, 1), (1, k_2 + 2, 1)$ or $(1, 1, k_3 + 2)$. Since any Vieta jumping of a triplet in $\mathbb{T}^{k_1,k_2,k_3}$ is again in $\mathbb{T}^{k_1,k_2,k_3}$ by Remark 8.1, we see that (a, b, c) is contained in the vertices of the tree $\mathbb{T}^{k_1,k_2,k_3}$ by following above operations in reverse. We prove the uniqueness. If not, we see that $(1, 1, 1)$ is not unique by repeating above operations. This is a contradiction. □

As in the Markov Diophantine equation (1.7) or the Gyoda's equation (1.8), there are several corollaries that can be established.

Corollary 8.4 ([GM23, Corollary 8])**.** For any positive integer solution $(x, y, z) = (a, b, c)$ to (1.5), all pairs in a, b, c are relatively prime.

Proof. The claim is true for $(a, b, c) = (1, 1, 1)$. We prove only that a and b are relatively prime. By transforming (1.5) as

$$z^2 = (3 + k_1 + k_2 + k_3)xyz - x^2 - y^2 - k_3xy - k_1yz - k_2zx,$$

and substituting $(x, y, z) = (a, b, c)$, if a, b have a common divisor $d \neq 1$, then we see that c can be divided by a prime divisor d' of d. Thus, d' is a common divisor of a, b, c. Therefore, by Proposition 7.2, the neighbor (a', b', c') of (a, b, c) on the tree $\mathbb{T}$ whose maximal number is smaller than $\max\{a, b, c\}$ has the common divisor d'. By repeating this operation, we see that d' is a common divisor of $(1, 1, 1)$. Thus, we must $d' = 1$. This is a contradiction. Therefore, we have $d = 1$. □

Corollary 8.5 ([GM23, Corollary 9])**.** Every number appearing in the tree $\mathbb{T}^{k_1,k_2,k_3}$ appears as the maximal number of some positive integer solution to (1.5).

Proof. Let n be a number appearing in $\mathbb{T}^{k_1,k_2,k_3}$. When $n = 1, k_1 + 2, k_2 + 2, k_3 + 2$, they are the maximal numbers of $(1, 1, 1), (k_1+2, 1, 1), (1, k_2+2, 1)$ and $(1, 1, k_3+2)$, respectively. We assume $n \neq 1, k_1 + 2, k_2 + 2, k_3 + 2$. We take a positive integer solution $(x, y, z) = (a, b, c)$ containing n. We assume that $a > b > c$. If $n = a$, then we are done. If $n = b$, then n is the maximal number in the neighbor of (a, b, c) in the tree $\mathbb{T}^{k_1,k_2,k_3}$ obtained by swapping a by Proposition 7.2. If $n = c$, as we traverse the neighbors with smaller maximal number, n becomes the second largest. Therefore, this case is attributed to the $n = b$ case. Even if the magnitude correlation of a, b, and c are different, it is proved in the same way. □

Remark 8.2. In the Markov case, that is, $k_1 = k_2 = k_3 = 0$, there is a conjecture that triplets with a common maximum will coincide if the order of the components is reordered (the *Markov Conjecture*). However, in the general case, there can be essentially different triplets with a common maximum (i.e., they will not coincide if the order of the components is reordered). Actually, when $k_1 = 1, k_2 = 2, k_3 = 0$, $(1, 81, 17)$ and $(7, 81, 2)$ are both solutions to (1.5), as seen in (1.6). The Markov Conjecture is proved to be true when the largest number in a triplet can be written as p^n using the prime number p ([Sch96, But98]), but this counterexample shows that even that does not hold in the general case.

Let us consider the case of $k_1 = k_2 = k_3 = 2$, that is, the equation

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 9xyz. \tag{8.2}$$

In this situation, we have the following theorem:

Theorem 8.6 ([GM23, Theorem 11])**.** *If positive integer triplet (a,b,c) is one of solutions to the Markov equation (1.7), then we have (a^2,b^2,c^2) is one of solutions to (8.2). Conversely, if positive integer triplet (A,B,C) is one of solutions to (8.2), then $(\sqrt{A},\sqrt{B},\sqrt{C})$ is one of positive integer solutions to (1.7).*

Proof. We prove the former statement. When $(a,b,c)=(1,1,1)$, it is clear. We assume that (a^2,b^2,c^2) is an integer solution to (8.2). It suffices to show that the Vieta jumpings of (a^2,b^2,c^2) in (8.2) are given by

$$\left(\left(\frac{b^2+c^2}{a}\right)^2,b^2,c^2\right),\left(a^2,\left(\frac{a^2+c^2}{b}\right)^2,c^2\right),\left(a^2,b^2,\left(\frac{a^2+b^2}{c}\right)^2\right).$$

We only prove the case of the first Vieta jumping. The first Vieta jumping of (a^2,b^2,c^2) in (8.2) is

$$\left(\frac{(b^2)^2+2b^2c^2+(c^2)^2}{a^2},b^2,c^2\right)=\left(\left(\frac{b^2+c^2}{a}\right)^2,b^2,c^2\right),$$

as desired. We will show the latter statement. By Theorem 1.6, each positive solution to (8.2) has the form (a^2,b^2,c^2), where (a,b,c) is a solution to (1.7). This finishes the proof. □

9 Proof of Theorem 1.7 and its corollaries

To prove Theorem 1.7, we consider the following equation:

$$X^2+Y^2+Z^2+2XY+kYZ+2ZX=(7+k)XYZ. \tag{9.1}$$

This is the equation substituted (1.11) with $X=x,Y=y^2,Z=z^2$ and a specialization of (1.5) with $k_3=2,k_1=k,k_2=2$. Then, the Vieta jumpings of (A,B,C) are

$$\left(\frac{B^2+kBC+C^2}{A},B,C\right),\left(A,\frac{(A+C)^2}{B},C\right),\left(A,B,\frac{(A+B)^2}{C}\right).$$

Lemma 9.1 ([GM23, Lemma 12])**.** Every positive integer solution to (9.1) appears exactly once in $\mathbb{T}^{2,k,2}$. Moreover, for any positive integer solution (A,B,C) to (9.1), there exist positive integers b and c such that $b^2=B$ and $c^2=C$.

Proof. The former statement follows from Theorem 1.6. We prove the latter statement. When $(X,Y,Z)=(1,1,1)$, it is clear. We assume that (A,b^2,c^2) is a solution to (9.1). The second Vieta jumping of (A,b^2,c^2) in (9.1) is $\left(A,\left(\frac{A+c^2}{b}\right)^2,c^2\right)$. Since $\left(\frac{A+c^2}{b}\right)^2$ is an integer, so is $\frac{A+c^2}{b}$. In the same way, we obtain $\left(A,b^2,\left(\frac{A+b^2}{c}\right)^2\right)$ from (A,b^2,c^2) by the third Vieta jumping. These facts finish the proof. □

Proposition 9.2 ([GM23, Proposition 13])**.** If a positive integer triplet (a,b,c) is one of solutions to (9.1), then (a,b^2,c^2) is one of solutions to (1.11). Conversely, if a positive integer triplet (A,B,C) is one of solutions to (1.11), then $(A,\sqrt{B},\sqrt{C})$ is one of positive integer solutions to (9.1).

Proof. The former statement is clear. The latter follows from Lemma 9.1. □

Now, we prove Theorem 1.7.

Proof of Theorem 1.7. By Lemma 9.1 and Proposition 9.2, all positive integer solutions to (1.11) are obtained from $(1, 1, 1)$ by repeating the Vieta jumpings

$$(a, b, c) \mapsto \left(\frac{b^4 + kb^2c^2 + c^4}{a}, b, c\right), (a, b, c) \mapsto \left(a, \frac{a + c^2}{b}, c\right), (a, b, c) \mapsto \left(a, b, \frac{a + b^2}{c}\right).$$

Since the three vertices adjacent to (a, b, c) in $\mathbb{T}^k$ are triplets replacing different components of (a, b, c), respectively, as in $\mathbb{T}^{2,k,2}$, any Vieta jumping of a triplet in $\mathbb{T}^k$ is again in $\mathbb{T}^k$. Therefore, all positive integer solutions to (1.11) appear in $\mathbb{T}^k$. The uniqueness follows from the uniqueness of any triplet in $\mathbb{T}^{2,k,2}$. Thus we obtain Theorem 1.7. □

Next, we prove an analogue of Corollary 8.4.

Corollary 9.3 ([GM23, Corollary 14])**.** For any positive integer solution $(x, y, z) = (a, b, c)$ to (1.11), all pairs in a, b, c are relatively prime.

Proof. By Corollary 8.4, (a, b^2, c^2) is relatively prime. Thus (a, b, c) is relatively prime. □

Remark 9.1. An analogue of Corollary 8.5 does not hold in (1.11). Actually, when $k = 1$, 11 appears in positive integer solutions in (1.11) (for example, $(21, 11, 1)$ is one of solutions). However, 11 is not maximal number in any solutions containing it. See (1.12).

10 Questions and consideration of class of rank 2

The cluster patterns corresponding to the equations listed in Table 1 satisfy the following two conditions:

Condition 1.

(1) The exchange matrix is multiplied by -1 for a mutation in any direction,

(2) the exchange polynomials are mutation invariant.

Moreover, these six triplets $(B, \mathbf{Z}, D)$ in the table in Theorem 7.3 can be divided into two types: for the top four in the Table 1, $BD = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ is satisfied, and for the remaining two, $BD = \begin{bmatrix} 0 & 1 & -1 \\ -4 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix}$ is satisfied. At present, we know a cluster pattern that satisfies these two conditions and $BD = \begin{bmatrix} 0 & 1 & -1 \\ -4 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix}$ but for which no corresponding equation has been found. It is the cluster pattern determined by

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}, \begin{cases} Z_1(u) = 1 + k_3u + k_1u^2 + k_3u^3 + u^4, \\ Z_2(u) = 1 + u, \\ Z_3(u) = 1 + u, \end{cases} D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10.1}$$

Therefore, the following question can be considered.

Question 1. Is there a Diophantine equation corresponding to (10.1)?

As a more general question, the following problems are considered.

Question 2.

(1) What kind of a triplet $(B, \mathbf{Z}, D)$ satisfying the two condition in Condition 1 such that BD is neither $\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ nor $\begin{bmatrix} 0 & 1 & -1 \\ -4 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix}$?

(2) Is there a general way to construct a Diophantine equation from information in $(B, \mathbf{Z}, D)$?

We will now consider Question 2. All of cluster patterns of rank 2 satisfy Condition 1 (1). Therefore, there are infinitely many cluster patterns of rank 2 which is the answer to Question 2 (1). In this class, there are cluster patterns whose corresponding equations are derived from Theorem 7.3. Since all cluster patterns treated in Theorem 4.2 are of rank 3, each seed can be mutated in three directions. We consider prohibiting mutaions in one of these directions. By substituting 1 to the cluster variable corresponding to the direction in which the mutation was prohibited, the cluster pattern that was originally rank 3 can be viewed as that of rank 2. In this case, the exchange matrix corresponding to the cluster pattern is a submatrix of the original one that removes the row and column corresponding to the direction in which mutation is prohibited. The equation corresponding to this cluster pattern is the equation that substitute 1 to the variable corresponding to the direction in which mutation is prohibited. Therefore, the following theorem holds.

Theorem 10.1 ([GM23, Theorem 21]). *We set equations, B and $\mathbf{Z}$ (and D) as in Table 2. Then, the generalized cluster pattern $CP_{(\mathbf{x},B,\mathbf{Z})}$ with a substitution $x_1 = x_2 = 1$ gives the tree giving all positive integer solutions to the corresponding equation (where we ignore exchange matrices in cluster pattern and consider only clusters).*

Equation	B	$\mathbf{Z}$	D
$x^2 + y^2 + 1 = 3xy$	$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u) = 1 + u \\ Z_2(u) = 1 + u \end{cases}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$x^2 + y^2 + k_3x + 1 = (3 + k_3)xy$	$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u) = 1 + u \\ Z_2(u) = 1 + k_3u + u^2 \end{cases}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
$x^2 + y^2 + k_3x + k_1y + 1 = (3 + k_3 + k_1)xy$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u) = 1 + k_1u + u^2 \\ Z_2(u) = 1 + k_3u + u^2 \end{cases}$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
$x^2 + y^4 + 2x + 1 = 5xy^2$	$\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u) = 1 + u \\ Z_2(u) = 1 + u \end{cases}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$x^2 + y^4 + ky^2 + 2x + 1 = (5 + k)xy^2$	$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u) = 1 + ku + u^2 \\ Z_2(u) = 1 + u \end{cases}$	$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Table 2: Equations and corresponding triplets $(B, \mathbf{Z}, D)$

Apart from the cluster pattern induced by Theorem 7.3, we give the equation induced by the cluster pattern of type A_2, i.e., the cluster pattern determined by

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{cases} Z_1(u) = 1 + u, \\ Z_2(u) = 1 + u, \end{cases} D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This cluster pattern differs from the ones we have considered above in that it has finitely many cluster variables. By computing the cluster variable according to the mutation rule

$$\mu_1(x_1, x_2) = \left(\frac{x_2 + 1}{x_1}, x_2\right) \text{ and } \mu_2(x_1, x_2) = \left(x_1, \frac{x_1 + 1}{x_2}\right),$$

we see that clusters in the cluster pattern of type A_2 are

$$\begin{aligned}&(x_1,x_2), \left(x_1, \frac{x_1+1}{x_2}\right), \left(\frac{x_1+x_2+1}{x_1x_2}, \frac{x_1+1}{x_2}\right), \left(\frac{x_1+x_2+1}{x_1x_2}, \frac{x_2+1}{x_1}\right), \left(x_2, \frac{x_2+1}{x_1}\right), \\ &(x_2,x_1), \left(\frac{x_1+1}{x_2}, x_1\right), \left(\frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_1x_2}\right), \left(\frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}\right), \left(\frac{x_2+1}{x_1}, x_2\right)\end{aligned} \tag{10.2}$$

in total. Therefore, we want to find an equation such that the five pairs

$$(1,1),(1,2),(2,1),(2,3),(3,2)$$

where 1 is substituted for x_1 and x_2 of (10.2), are all positive integer solutions.

The set consisting of cluster variables

$$\left\{x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}, \frac{x_1+1}{x_2}\right\}$$

appearing in (10.2) (it is called the *Lyness 5-cycle*) is invariant by the substitutions

$$(x_1,x_2) \mapsto \left(\frac{x_2+1}{x_1}, x_2\right) \text{ and } (x_1,x_2) \mapsto \left(x_1, \frac{x_1+1}{x_2}\right).$$

Therefore, we have the following proposition.

Proposition 10.2 ([GM23, Proposition 22])**.** Let $f(a_1,a_2,a_3,a_4,a_5)$ be a symmetric polynomial of five variables. Then, the equation

$$f\left(x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y}\right) = f\left(x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y}\right)\Bigg|_{x=y=1} \tag{10.3}$$

has positive integer solutions

$$(x,y) = (1,1),(1,2),(2,1),(2,3),(3,2).$$

Furthermore, the following proposition also holds:

Proposition 10.3 ([GM23, Proposition 23])**.** If $f(a_1,a_2,a_3,a_4,a_5) = a_1+a_2+a_3+a_4+a_5$ in Proposition 10.2, then all positive integer solutions to

$$x^2+y^2+2x+2y+x^2y+xy^2+1 = 9xy, \tag{10.4}$$

which corresponds to (10.3), are

$$(x,y) = (1,1),(1,2),(2,1),(2,3),(3,2).$$

Proof. We set

$$g(x,y) = x^2+y^2+2x+2y+x^2y+xy^2+1-9xy.$$

It suffices to show that positive integer pairs (x,y) satisfying $g(x,y)=0$ are given by

$$(x,y) = (1,1),(1,2),(2,1),(2,3),(3,2).$$

First, we will consider the case of $x \geq y$. If $x = y$, then we can see that the positive solution to $g(x,y)=0$ is only $(x,y)=(1,1)$ immediately. We assume that $x > y$. By considering the partial derivative of g in x direction, we have

$$\frac{\partial g}{\partial x} = y^2+2xy+2x+2-9y.$$

Equation	B	$\mathbf{Z}$	D
$x^2+y^2+2x+2y+x^2y+xy^2+1=9xy$	$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\end{cases}$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
$x^2+y^2+k_3x+k_1y+1=(3+k_3+k_1)xy$	$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+k_1u+u^2\\Z_2(u)=1+k_3u+u^2\end{cases}$	$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$
$x^2+y^2+k_3x+1=(3+k_3)xy$	$\begin{bmatrix}0 & 1\\-2 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+k_3u+u^2\end{cases}$	$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$
$x^2+y^4+ky^2+2x+1=(5+k)xy^2$	$\begin{bmatrix}0 & 1\\-2 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+ku+u^2\\Z_2(u)=1+u\end{cases}$	$\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$
$x^2+y^4+2x+1=5xy^2$	$\begin{bmatrix}0 & 1\\-4 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\end{cases}$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
$x^2+y^2+1=3xy$	$\begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\end{cases}$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Table 3: Equations and corresponding triplets $(B, \mathbf{Z}, D)$

By using $x > y$, we have

$$\frac{\partial g}{\partial x}(x,y) > 3y^2 - 7y + 2.$$

Therefore, if $x > y \geq 3$, then we have $\frac{\partial g}{\partial x}(x,y) > 0$. On the other hand, by considering the partial derivative of g in x direction, we have

$$\frac{\partial g}{\partial y} = x^2 + 2xy + 2y + 2 - 9x = \left(x - \frac{9}{2}\right)^2 + 2xy + 2x - \frac{73}{4}.$$

Therefore, if $x > y \geq 3$, then we have $\frac{\partial g}{\partial y}(x,y) > 0$. Now, since $g(4,3) = 16 > 0$, we have $g(x,y) > 0$ when x and y are integer and $x > y \geq 3$. Second, we will consider the case of $y > x$. By symmetry of $g(x,y)$ for x and y, we have $g(x,y) > 0$ if x and y are integer and $y > x \geq 3$. Therefore, the only possible pairs of integers that satisfy $g(x,y) = 0$ are

$$(1,1), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2).$$

Of these, $g(3,1) = g(1,3) = 4$, thus $(x,y) = (1,3), (3,1)$ are not solutions. The other five are all solutions according to Proposition 10.2. □

Remark 10.1. In Proposition 10.3, even though $f(a_1, a_2, a_3, a_4, a_5) = a_1a_2a_3a_4a_5$, the equation corresponding to (10.3) is the same as (10.4).

From the above, the cluster pattern of rank 2 for which the corresponding equation is known is given in Table 3. In order to find the answer to Question 2 (2) about the cluster patterns of rank 2, the first thing to do is to consider the following question:

Question 3. Are there any laws between the triplets $(B, \mathbf{Z}, D)$ and the equations given in Table 3?

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