Multifractal Analysis for Birkhoff Sums and Word Appearance in Symbolic Dynamics

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Abstract

Given a topologically mixing subshift of finite type, the Hausdorff dimension of the level sets given by the limit of the quotient of the Birkhoff sums of two observables has been well studied. When the limit of the quotient is $\alpha \in \mathbb{R}$, the level set is called the α-level set. At the points in the α-level set, the difference between the n-th Birkhoff sum $S_n\psi$ of one observable ψ and the n-th Birkhoff sum $\alpha S_n v$ of another observable v scaled by α is $o(n)$ as n tends to infinity. This thesis studies the α-uniform level set, where this difference is bounded.

We shall first present the known results which describe the Hausdorff dimension of the level sets using thermodynamic formalism in ergodic theory. Then, we shall show that the α-uniform level set has the same Hausdorff dimension as the α-level set for all $\alpha \in \mathbb{R}$. Furthermore, we consider sequences in the α-uniform level set which satisfy some conditions on the words appearing in them. We shall show that the set of these sequences also has the same Hausdorff dimension as the α-level set, for all but two $\alpha \in \mathbb{R}$.

One of our results will be applied to the study of the Hölder regularity of a Gibbs measure on the real line $\mathbb{R}$. To be more precise, we will study the set of points in $\mathbb{R}$ at which the upper and lower α-Hölder derivatives of the cumulative distribution function of the Gibbs measure are positive and finite. We will show that, for all but two $\alpha \in \mathbb{R}$, the Hausdorff dimension of this set is equal to the Hausdorff dimension of the $-\alpha$-level set of the quotient of the Birkhoff sums of two observables which are chosen in terms of the Gibbs measure.

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Chapter 1

Introduction

This thesis presents the results the author obtained in the preprint [Liu23]. Some of the claims are weakened so that we can avoid some technical arguments in the original proofs.

Hausdorff Dimension of Uniform Level Sets and Their Subsets

Symbolic dynamical systems are well-adopted prototypes for dynamical systems. Historically, symbolic dynamics arose naturally in the study on the geodesics on a surface of negative curvature [Mor21; MH38]. Nowadays it has become clear that a wide range of dynamical systems can be modelled by a symbolic dynamical system using a Markov partition. For instance, the basic sets of axiom A diffeomorphisms admit Markov partitions and thus have symbolic representations [Sma67; Bow08]. Another example is the dynamical system given by an open, expanding continuous mapping on a compact metric space. Chapter 4 of [PU10] shows the existence of Markov partitions and gives the resulting symbolic representations of the dynamical system. In fractal geometry, symbolic dynamics also plays an important role in many situations. For instance, there are various interesting fractals generated by iterated function systems as their limit sets; see e.g. [Fal03]. When an iterated function system satisfies a separation condition called the open set condition, the limit set of this iterated function system has a natural symbolic representation [Fal03]. A more flexible notion generalizing iterated function systems, called a graph directed Markov system (GDMS), was introduced in [MU03]. When a GDMS satisfies the open set condition and some regularity conditions, it is called a conformal graph directed Markov system (CGDMS) [MU03]. The limit set of a CGDMS also can also be encoded into a symbolic dynamical system [MU03]. There is a special class of CGDMSs, called conformal graph directed systems (CGDSs). In Chapter 5, we shall study a Gibbs measure in $\mathbb{R}$. The support of this measure will be the limit set of a CGDS.

In this thesis, by a symbolic dynamical system, we mean a one-sided subshift of finite type (SFT) [Kit98]. An SFT consists of a compact set Σ called the shift space and a continuous map $\sigma : \Sigma \to \Sigma$ called the left shift. The shift space Σ is specified by a finite set of symbols A and a matrix $\mathbb{M} : A \times A \to \{ 0, 1 \}$ called the incidence matrix. The elements of Σ are infinite sequences $\xi = \xi_1\xi_2 \cdots$ over A satisfying that $\mathbb{M}(\xi_k, \xi_{k+1}) = 1$ for any positive integer k. A sequence in Σ is called an admissible sequence. The map $\sigma : \Sigma \to \Sigma$ is called the left shift, which sends $\xi_1\xi_2 \cdots \in \Sigma$ to $\xi_2\xi_3 \cdots \in \Sigma$. Endowed with a topology to be defined in Subsection 2.1.1, Σ is compact. Moreover, the left shift σ is continuous. Therefore, an SFT is a topological dynamical system. For simplicity, we shall basically focus on topologically mixing SFTs in this thesis when we state our main results; the precise definition of topological mixing will be given in Subsection 2.1.1. The theorems in [Liu23] are proved for topologically transitive SFTs, which are more general than topologically mixing SFTs. Until the end of this introduction, we shall always assume that the SFT we consider is topologically mixing.

From a perspective of mechanics, Σ can be thought of as the phase space, and a sequence in Σ can be regarded as a state. Then, naturally, $\sigma : \Sigma \to \Sigma$, as a mapping sending one state to another state, describes the evolution of the states. If the initial state is ξ, then for any non-negative integer k, the state at time k will be $\sigma^k(\xi)$. Viewing a function $f : \Sigma \to \mathbb{R}$ as an observable, $f(\xi)$ is the number produced by the measurement for the observable f when the current state is ξ. Thus, for a positive integer n, the n-th Birkhoff sum of f, which is $S_n f = \sum_{k=0}^{n-1} f \circ \sigma^k$, can be interpreted as the sum of the values that the observable f takes from the time 0 until the time $n - 1$. By convention, we set $S_0 f$ to be the constant function 0.

For a pair of continuous functions $\psi : \Sigma \to \mathbb{R}$ and $v : \Sigma \to (0, +\infty)$, we would like to compare the asymptotic growth of Birkhoff sums of ψ and v. We thus define

$$\mathcal{L}^{\alpha}_{\psi,v} = \left\{ \xi \in \Sigma \;\middle|\; \lim_{n\to+\infty} \frac{S_n\psi(\xi)}{S_n v(\xi)} = \alpha \right\};$$
$$\mathcal{UL}^{\alpha}_{\psi,v} = \left\{ \xi \in \Sigma \;\middle|\; \sup_{n\to+\infty} |S_n\psi(\xi) - \alpha S_n v(\xi)| < +\infty \right\},$$

for any $\alpha \in \mathbb{R}$. The set $\mathcal{L}^{\alpha}_{\psi,v}$ is a well-studied set [PW97], which is usually called a level set of quotients of Birkhoff sums. The set $\mathcal{UL}^{\alpha}_{\psi,v}$ is a set which is rarely considered in the existing literature. Indeed, [FS03] and [GJK22] introduced sets similar to $\mathcal{UL}^{\alpha}_{\psi,v}$, but not exactly the same; in both [FS03] and [GJK22], v was taken to be the constant function 1. In this thesis, we shall call $\mathcal{UL}^{\alpha}_{\psi,v}$ a uniform level set. Note that a sequence in $\mathcal{UL}^{\alpha}_{\psi,v}$ is a sequence $\xi \in \Sigma$ for which $S_n\psi(\xi) = \alpha S_n v(\xi) + O(1)$. From the compactness of Σ and the continuity of v, we see that a sequence in $\mathcal{L}^{\alpha}_{\psi}$ is a sequence ξ for which $S_n\psi(\xi) = \alpha S_n v(\xi) + o(n)$. Hence, clearly, we have $\mathcal{UL}^{\alpha}_{\psi,v} \subseteq \mathcal{L}^{\alpha}_{\psi,v}$ for any $\alpha \in \mathbb{R}$.

We are interested in the Hausdorff dimension of $\mathcal{L}^{\alpha}_{\psi,v}$ and $\mathcal{UL}^{\alpha}_{\psi,v}$. For that purpose, we need to define a metric on Σ. For a Hölder continuous function $u : \Sigma \to (0, +\infty)$, we will define a metric d_u in Subsection 2.1.2; the meaning of Hölder continuity of a function on Σ will also be made clear in Subsection 2.1.2. With the assumption of Hölder continuity of u, we shall see in Proposition 3.3 that the Hausdorff dimension with respect to the metric d_u coincides with the u-dimension, which is a notion introduced by Barreira and Schmeling in [BS00]. Hence, for a subset E of Σ, the Hausdorff dimension of E with respect to d_u will be simply called the u-dimension of E and denoted by $\dim_u(E)$.

Henceforth assume that ψ and v are Hölder continuous. For any $\alpha \in \mathbb{R}$, the u-dimension of $\mathcal{L}^{\alpha}_{\psi,v}$ is a well-studied topic [PW97; Sch99]. Define

$$\alpha^{-}_{\psi,v} = \inf_{\xi\in\Sigma} \liminf_{n\to+\infty} \frac{S_n\psi(\xi)}{S_n v(\xi)}; \quad \alpha^{+}_{\psi,v} = \sup_{\xi\in\Sigma} \limsup_{n\to+\infty} \frac{S_n\psi(\xi)}{S_n v(\xi)}.$$

Then, $\mathcal{L}^{\alpha}_{\psi,v}$ is non-empty if and only if $\alpha^{-}_{\psi,v} \le \alpha \le \alpha^{+}_{\psi,v}$ [Sch99]. Pesin and Weiss gave an expression for $\dim_u(\mathcal{L}^{\alpha}_{\psi,v})$ when $v = u$ and $\alpha \in (\alpha^{-}_{\psi,v}, \alpha^{+}_{\psi,v})$ in [PW96; PW97]. Also when $v = u$, for $\alpha \in \{\, \alpha^{-}_{\psi,v}, \alpha^{+}_{\psi,v} \,\}$, a formula for $\dim_u(\mathcal{L}^{\alpha}_{\psi,v})$ can be found in [Sch99]. Using the ideas in [PW97; Sch99] and some arguments handling the case for $\alpha \in \{\, \alpha^{-}_{\psi,v}, \alpha^{+}_{\psi,v} \,\}$, we extend the theorems in [PW97; Sch99] to the case where v is not necessarily equal to u. The corresponding theorems will be Theorem 4.1 and Theorem 4.2. We remark that the setting in [Cli13] is much more general than ours. In the setup of [Cli13], the dynamical system is not necessarily an SFT, and the level sets are defined in terms of any finitely many pairs of observables $(\psi_1, v_1, \cdots, \psi_n, v_n)$. For $\alpha \in (\alpha^{-}_{\psi,v}, \alpha^{+}_{\psi,v})$, [Cli13] also gives the u-dimension of $\mathcal{L}^{\alpha}_{\psi,v}$ in a sense weaker than what we shall claim in Theorem 4.1 and Theorem 4.2. The existence of maximizing measures ν^{α} in Theorem 4.1 can be shown in our setting, but not in the setting of [Cli13]. In turn, we can use this to show the real analyticity of the spectrum in Theorem 4.2, which was not contained in [Cli13]. The existence of the maximizing measures can also be employed to show the continuity of the dimension spectrum on the closed interval $[\alpha^{-}_{\psi,v}, \alpha^{+}_{\psi,v}]$, as we shall see in Theorem 4.2.

We have seen that the u-dimension of $\mathcal{L}^{\alpha}_{\psi,v}$ is well understood. It is thus natural to ask whether

$$\dim_u(\mathcal{UL}^{\alpha}_{\psi,v}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v}) \tag{1.1}$$

for all $\alpha \in \mathbb{R}$. If (1.1) always holds, then the formula for $\dim_u(\mathcal{L}^{\alpha}_{\psi,v})$ also serves as a formula for $\dim_u(\mathcal{UL}^{\alpha}_{\psi,v})$. Hence, $\dim_u(\mathcal{UL}^{\alpha}_{\psi,v})$ can be well understood as well.

Indeed, if one adds some extra assumptions, (1.1) has been shown. Fan and Schmeling showed (1.1) in [FS03], when u is a constant function, σ is a full shift and $\alpha^{-}_{\psi,v} < \alpha < \alpha^{+}_{\psi,v}$. In a recent article [GJK22] by Gröger, Jaerisch and Kesseböhmer, the authors essentially showed the same assertion when σ is a full shift and $\alpha^-_{\psi,v} < \alpha < \alpha^+_{\psi,v}$. In other words, the proof in [GJK22] does not need the function $u : \Sigma \to (0, +\infty)$ to be constant. In this thesis, we shall see in Proposition 4.6 that (1.1) *is* valid for all $\alpha \in \mathbb{R}$ in general. Therefore, we immediately get a formula for the u-dimension of $\mathcal{UL}^\alpha_{\psi,v}$.

Now we turn our attention to two types of subsets of $\mathcal{UL}^\alpha_{\psi,v}$. A word over A is said to be admissible if it is a subword of some $\xi \in \Sigma$. Let $\mathcal{W}$ be a finite set of admissible words. Define for each positive integer k,

$$\mathcal{F}_{\mathcal{W},k} = \bigcap_{n=0}^{\infty} \{ \xi \in \Sigma \mid \text{all the words in } \mathcal{W} \text{ are the subwords of } \xi_{n+1} \cdots \xi_{n+k} \},$$

and then set $\mathcal{F}_{\mathcal{W}} = \bigcup_{k=1}^{\infty} \mathcal{F}_{\mathcal{W},k}$. In words, an element of $\mathcal{F}_{\mathcal{W}}$ is an admissible sequence in which all words from $\mathcal{W}$ appears regularly. For every non-negative integer n, we define the n-th power of a word ω as $\omega^n = \omega \cdots \omega$, where the right-hand side is the n-fold concatenation of ω. More formally, define ω^0 as the empty word, and for each positive integer n, define $\omega^n = \omega^{n-1}\omega$. Define $\mathcal{F}'_{\mathcal{W}} = \bigcup_{k=1}^{\infty} \mathcal{F}'_{\mathcal{W},k}$, where for any positive integer k,

$$\mathcal{F}'_{\mathcal{W},k} = \bigcap_{\omega \in \mathcal{W}} \left\{ \xi \in \Sigma \;\middle|\; \xi \text{ does not contain } \omega^k \text{ as a subword} \right\}.$$

In words, the sequences in $\mathcal{F}'_{\mathcal{W}}$ are those in which none of the words in $\mathcal{W}$ appears with arbitrarily high power. For the subsets $\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}_{\mathcal{W}}$ and $\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}'_{\mathcal{W}}$, we shall show in Theorem 4.7 and Theorem 4.8 that

$$\dim_u(\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}_{\mathcal{W}}) = \dim_u(\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}'_{\mathcal{W}}) = \dim_u(\mathcal{L}^\alpha_{\psi,v}), \tag{1.2}$$

for $\alpha \notin \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$.

Remark 1.1. It might happen that ω is admissible but ω^2 is not. Indeed, all the powers of an admissible word ω are admissible if and only if ω^2 is admissible. If we define

$$\tilde{\mathcal{W}} = \{ \omega \in \mathcal{W} \mid \omega^2 \text{ is admissible} \},$$

we will clearly have $\mathcal{F}'_{\tilde{\mathcal{W}}} = \mathcal{F}'_{\mathcal{W}}$. Hence, when we prove $\dim_u(\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}'_{\mathcal{W}}) = \dim_u(\mathcal{L}^\alpha_{\psi,v})$ for $\alpha \notin \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$, we may assume, without loss of generality, that $\mathcal{W}$ is a (possibly empty) finite set of words whose powers are all admissible.

For $\alpha \notin \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$, clearly, (1.2) is stronger than (1.1) because

$$\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}_{\mathcal{W}} \subseteq \mathcal{UL}^\alpha_{\psi,v} \subseteq \mathcal{L}^\alpha_{\psi,v};$$
$$\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}'_{\mathcal{W}} \subseteq \mathcal{UL}^\alpha_{\psi,v} \subseteq \mathcal{L}^\alpha_{\psi,v}.$$

We shall show (1.2) for $\alpha \notin \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$ in Subsection 4.2.2 by improving an argument in [GJK22]. For $\alpha \in \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$, we shall see in Subsection 4.2.3 that $\dim_u(\mathcal{U}\mathcal{L}^\alpha_{\psi,v} \cap \mathcal{F}_{\mathcal{W}}) < \dim_u(\mathcal{L}^\alpha_{\psi,v})$ can happen. On the other hand, it remains unclear to me whether $\dim_u(\mathcal{U}\mathcal{L}^\alpha_{\psi,v} \cap \mathcal{F}'_{\mathcal{W}}) = \dim_u(\mathcal{L}^\alpha_{\psi,v})$ holds or not for $\alpha \in \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$. As we claimed before, $\dim_u(\mathcal{U}\mathcal{L}^\alpha_{\psi,v}) = \dim_u(\mathcal{L}^\alpha_{\psi,v})$ even for $\alpha \in \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$. This will be proved using some facts from the theory of ergodic optimization.

Hölder Regularity of Gibbs Measure in $\mathbb{R}$

As an application of the results we have claimed for symbolic dynamics, we study the Hölder regularity of a Gibbs measure on $\mathbb{R}$. It is natural to consider Gibbs measures from the viewpoint of thermodynamic formalism. In the context of thermodynamic formalism, a Gibbs measure is usually defined on a shift space; see e.g. [Bow08]. The precise definition of a Gibbs measure on a shift space is given in Subsection 2.2.2. The definition of a Gibbs measure on $\mathbb{R}$ will be explained as follows. The full details will be given in Section 5.1.

In order to define a Gibbs measure on $\mathbb{R}$, we first describe its support. The support of a Gibbs measure is generated by a finitely generated conformal graph directed system (CGDS) in $\mathbb{R}$. Roughly speaking, a finitely generated CGDS in $\mathbb{R}$ mainly consists of a finite family of compact intervals $\mathcal{I} = \{ I_p \mid p \in \mathcal{V} \}$ with pairwise disjoint interiors and a finite family of contractions $\Phi = \{ g_e \mid e \in \mathcal{E} \}$. Each contraction $g_e \in \Phi$ maps one $I_{p_+(e)} \in \mathcal{I}$ into $I_{p_-(e)} \in \mathcal{I}$, where p_- and p_+ are mappings from $\mathcal{E}$ to $\mathcal{V}$. An incidence matrix $\mathbb{M} : \mathcal{E} \times \mathcal{E} \to \{ 0, 1 \}$ is then defined by $\mathbb{M}(e, e') = 1$ if and only if $p_+(e) = p_-(e')$ for any $e, e' \in \mathcal{E}$. Thus, we have an SFT $\sigma : \Sigma \to \Sigma$, for which Σ contains admissible sequences over $\mathcal{E}$.

For any $\xi \in \Sigma$, the composition $\xi_1 \circ \cdots \circ \xi_n : I_{p_+(\xi_n)} \to I_{p_-(\xi_1)}$ is well-defined for any positive integer n. By the assumption that the maps in Φ are all contractions,

$$I_{\xi_1 \cdots \xi_n} = (\xi_1 \circ \cdots \circ \xi_n)(I_{p_+(\xi_n)})$$

descends to a singleton contained in $I_{p_-(\xi_1)}$as n approaches infinity. Hence, we can define a map $\pi : \Sigma \to \mathbb{R}$ by letting

$$\{ \pi(\xi) \} = \bigcap_{n=1}^{\infty} I_{\xi_1 \cdots \xi_n},$$

for any $\xi \in \Sigma$. The map π is called the coding map, and the limit set of Φ is defined as $\Lambda = \pi(\Sigma) \subseteq \mathbb{R}$. The precise definitions will be given in Section 5.1, in which we basically follow [MU03]. The limit set Λ is compact, and it will be the support of the Gibbs measure in $\mathbb{R}$ we shall consider.

Let $\psi : \Sigma \to \mathbb{R}$ be a Hölder continuous function satisfying that the topological pressure of ψ is zero. We refer to Subsection 2.2.2 for the definition of the topological pressure. A Gibbs measure ν_ψ for ψ is a Borel probability measure on Σ for which there is a constant $C_{\nu_\psi} > 1$ such that for any positive integer n and any admissible word ω of length n,

$$C_{\nu_\psi}^{-1} \exp(\inf_{\xi\in[\omega]} S_n\psi(\xi)) \le \nu([\omega]) \le C_{\nu_\psi} \exp(\sup_{\xi\in[\omega]} S_n\psi(\xi)),$$

where $[\omega] = \{\, \xi \in \Sigma \mid \xi_1 \cdots \xi_n = \omega \,\}$. A Gibbs measure in $\mathbb{R}$ is then the pushforward measure $\pi_*\nu_\psi$, which means that for any Borel subset E of $\mathbb{R}$,

$$\pi_*\nu_\psi(E) = \nu_\psi(\pi^{-1}(E)). \tag{1.3}$$

As we shall see in Proposition 5.4, the coding map $\pi : \Sigma \to \mathbb{R}$ is continuous, so $\pi^{-1}(E)$ in (1.3) is Borel.

For a continuous function $f : \mathbb{R} \to \mathbb{R}$, $\alpha \ge 0$ and $x \in \mathbb{R}$, we define

$$\begin{aligned}
\underline{D}^\alpha f(x) &= \liminf_{y\to x} \frac{|f(x) - f(y)|}{|x-y|^\alpha};\\
\overline{D}^\alpha f(x) &= \limsup_{y\to x} \frac{|f(x) - f(y)|}{|x-y|^\alpha};\\
\mathcal{UD}_f^\alpha &= \left\{\, x \in \mathbb{R} \;\middle|\; 0 < \underline{D}^\alpha f(x) \le \overline{D}^\alpha f(x) < +\infty \,\right\}.
\end{aligned}$$

When f is non-decreasing, we call $\underline{D}^\alpha f(x)$ the lower α-Hölder derivative of f at x. Likewise, $\overline{D}^\alpha f(x)$ is called the upper α-Hölder derivative of f at x. The points in $\mathcal{UD}_f^\alpha$ can be interpreted as the points at which $f(y)$ changes neither too rapidly nor too slowly compared with $|y-x|^\alpha$ as y varies in a small neighbourhood of x. The α-Hölder regularity of a Gibbs measure $\pi_*\nu_\psi$ we shall consider in this thesis is reflected by the set $\mathcal{UD}_F^\alpha$, where F is the cumulative distribution function of $\pi_*\nu_\psi$.

Our main result for this application is Theorem 5.5, which relates the Hausdorff dimension of $\mathcal{UD}_F^\alpha$, denoted by $\dim_H(\mathcal{UD}_F^\alpha)$, with the u-dimension of $\mathcal{L}_{\psi,u}^{-\alpha}$, where $u : \Sigma \to (0, +\infty)$ is the volume potential of the CGDS Φ to be defined in Subsection 5.1.3. More precisely, we have

$$\dim_H(\mathcal{UD}_F^\alpha) = \dim_u(\mathcal{L}_{\psi,u}^{-\alpha}),$$

for any $\alpha \notin \{\, -\alpha_{\psi,u}^-, -\alpha_{\psi,u}^+ \,\}$. Hence, the formula for $\dim_u(\mathcal{L}_{\psi,u}^{-\alpha})$ we mentioned in the previous section also can be used to express $\dim_H(\mathcal{UD}_F^\alpha)$ for $\alpha \notin \{\, -\alpha_{\psi,u}^-, -\alpha_{\psi,u}^+ \,\}$.

The key observation for showing Theorem 5.5 is the following inclusion in Lemma 5.8:

$$\mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W}} \subseteq \pi^{-1}\left(\mathcal{UD}_F^\alpha\right) \subseteq \mathcal{UL}_{\psi,u}^{-\alpha}, \tag{1.4}$$

where $\alpha \geq 0$ and $\mathcal{W}$ is a finite set of admissible words to be defined in Section 5.2. This suggests that Theorem 4.8, which gives that

$$\dim_u(\mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W}}) = \dim_u(\mathcal{L}_{\psi,u}^{-\alpha}) \tag{1.5}$$

for $\alpha \notin \{-\alpha_{\psi,u}^{-}, -\alpha_{\psi,u}^{+}\}$, can be applied to prove Theorem 5.5. Following this idea, we shall give a proof in Section 5.2.

There have been many known results on the Hölder regularity of Gibbs measures on $\mathbb{R}$ in different senses. Some articles studying the Hölder differentiability of F, but with a focus on sets different from $\mathcal{UD}_F^{\alpha}$. For instance, Kesseböhmer and Stratmann evaluated in [KS09] the Hausdorff dimension of the set

$$\{\, x \in \mathbb{R} \mid \underline{D}^{\alpha} F(x) < \overline{D}^{\alpha} F(x) = +\infty \,\}.$$

At these points, F fails to be α-Hölder differentiable.

There are also a series of articles studying pointwise Hölder exponents of F, which are closely related to our result. Here, by pointwise Hölder exponents of F, we roughly mean the limiting behavior of

$$\frac{\log |F(x) - F(y)|}{\log |x - y|}$$

as y approaches x. For instance, one may consider the limit

$$\lim_{y \to x} \frac{\log |F(x) - F(y)|}{\log |x - y|}$$

at x where this limit exists. One may also consider the limit inferior, the limit superior and accumulation points of $\log |F(x) - F(y)| / \log |x - y|$ as y approaches x. In [Pat97], Patzschke showed that the Hausdorff dimension of

$$\mathcal{E}^{\alpha} = \left\{ x \in \mathbb{R} \;\middle|\; \lim_{\varepsilon \to 0^+} \frac{\log |F(x+\varepsilon) - F(x-\varepsilon)|}{\log \varepsilon} = \alpha \right\} \tag{1.6}$$

is equal to $\dim_u(\mathcal{L}_{\psi,u}^{-\alpha})$. As a continuation of this result, Jaerisch and Sumi studied extensively in [JS20] various types of sets that contain $\mathcal{E}^{\alpha}$ as a subset, including

$$\left\{ x \in \mathbb{R} \;\middle|\; \liminf_{y \to x} \frac{\log |F(x) - F(y)|}{\log |x - y|} = \alpha \right\};$$

$$\left\{ x \in \mathbb{R} \;\middle|\; \limsup_{y \to x} \frac{\log |F(x) - F(y)|}{\log |x - y|} = \alpha \right\};$$

$$\left\{ x \in \mathbb{R} \;\middle|\; \exists (y_n)_{n=1}^{\infty} \text{ converging to } x \text{ such that } \lim_{n \to +\infty} \frac{\log |F(x) - F(y_n)|}{\log |x - y_n|} = \alpha \right\}.$$

They proved that all these sets have the same Hausdorff dimension as $\mathcal{E}^\alpha$. The Hausdorff dimension and also the packing dimension of some sets even larger than the sets in [JS20] are given in [BOS07]. Different from [BOS07] and [JS20], in which the sets larger than $\mathcal{E}^\alpha$ are considered, the set $\mathcal{UD}_F^\alpha$ we consider is actually a subset of $\mathcal{E}^\alpha$. Hence, our result is a continuation of the previous works on the pointwise Hölder exponent.

It is also worth noticing that the results in [JS20] give the Hausdorff dimension of

$$\mathcal{H}_F^\alpha = \left\{ x \in \mathbb{R} \;\middle|\; \sup\left\{ \gamma > 0 \;\middle|\; \overline{D}^\gamma F(x) < +\infty \right\} = \alpha \right\}.$$

This set is interesting in that at points in $\mathcal{H}_F^\alpha$, α can be regarded as the exact Hölder exponent locally. Indeed, the Hausdorff dimension of $\mathcal{H}_F^\alpha$ is also equal to the Hausdorff dimension of $\mathcal{E}^\alpha$ [JS20]. Note that for a point $x \in \mathcal{H}_F^\alpha$, we do not know any information about the values of $\underline{D}^\alpha F(x)$ and $\overline{D}^\alpha F(x)$, while for a point x in the set

$$\mathcal{UD}_F^\alpha = \left\{ x \in \mathbb{R} \;\middle|\; 0 < \underline{D}^\alpha F(x) \le \overline{D}^\alpha F(x) < +\infty \right\}$$

which we consider, we know that $\underline{D}^\alpha F(x)$ must be positive and $\overline{D}^\alpha F(x)$ must be finite. Hence, our result complements [JS20] by showing that $\mathcal{UD}_F^\alpha$, in which the conditions on the values $\underline{D}^\alpha F(x)$ and $\overline{D}^\alpha F(x)$ are imposed, has the same Hausdorff dimension as $\mathcal{H}_F^\alpha$.

Finally, we mention that there are also many articles studying some sort of Hölder regularity of a self-affine function; see [Dub18; All20] for the definition of a self-affine function. Firstly, the Hausdorff dimension of $\mathcal{H}_f^\alpha$ for f in a certain class of self-affine functions was studied in [All18]. There is one sort of Hölder regularity different from any other Hölder regularity we have seen. For a compact interval I, $\alpha > 0$ and $x \in I$, a continuous function $f : I \to \mathbb{R}$ is said to be in $C^\alpha(x)$ if there exists a polynomial h of degree less than α such that

$$\sup_{x \in I} \frac{|f(y) - h(y)|}{|y - x|^\alpha} < +\infty, \tag{1.7}$$

and define $\alpha_f(x) = \sup\left\{ \alpha' \ge 0 \mid f \in C^{\alpha'}(x) \right\}$ [Jaf97; All20]. This $\alpha_f(x)$ is called the pointwise Hölder exponent of f at x in [Jaf97; All20], and called the Hölder cut of f at x in [Dub18]. As we have used the expression *pointwise Hölder exponent*, we shall follow [Dub18] to call $\alpha_f(x)$ the Hölder cut of f at x. The multifractal analysis for the Hölder cut of a self-affine function studies the Hausdorff dimension of the set

$$\mathcal{C}_f^\alpha = \left\{ x \in I \mid \alpha_f(x) = \alpha \right\}.$$

Generally speaking, the set $\mathcal{C}_f^\alpha$ is different from any other set we have seen so far, but for $\alpha \le 1$, we have $\mathcal{C}_f^\alpha = \mathcal{H}_f^\alpha$, because when $\alpha \le 1$, the polynomial h in (1.7) must be of degree 0 and thus equal to the constant function $f(x)$.

Unsolved Questions

The author here raises two questions which are not answered by our results. The first one, which we have already asked, is whether

$$\dim_u(\mathcal{U}\mathcal{L}^{\alpha}_{\psi,v} \cap \mathcal{F}'_{\mathcal{W}}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v})$$

holds for $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$. The second question is whether

$$\dim_H(\mathcal{U}\mathcal{D}^{\alpha}_F) = \dim_u(\mathcal{L}^{-\alpha}_{\psi,u}),$$

for $\alpha \in \{\, -\alpha^-_{\psi,u}, -\alpha^+_{\psi,u} \,\}$. These two questions are actually closely related. Suppose that the first question can be answered affirmatively. Then, (1.5) is true for $\alpha \in \{\, -\alpha^-_{\psi,u}, -\alpha^+_{\psi,u} \,\}$. Hence, combining this with (1.4), we can readily see that the answer to the second question is positive as well.

Organization of Thesis

This thesis is organized as follows. In Chapter 2 and Chapter 3, we lay the background for our discussions in subsequent chapters. Chapter 4 is the central part of this thesis. Section 4.1 presents simplified versions of the known results on the u-dimension of level sets $\mathcal{L}^{\alpha}_{\psi,v}$, following the ideas in [PW97] and [Sch99]. Section 4.2 presents the original results in the preprint [Liu23] of the author, which gives the u-dimension of $\mathcal{U}\mathcal{L}^{\alpha}_{\psi,v}$ for all $\alpha \in \mathbb{R}$ in Proposition 4.6, and the u-dimension of $\mathcal{U}\mathcal{L}^{\alpha}_{\psi,v} \cap \mathcal{F}_{\mathcal{W}}$ and $\mathcal{U}\mathcal{L}^{\alpha}_{\psi,v} \cap \mathcal{F}'_{\mathcal{W}}$ for $\alpha \notin \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$ in Theorem 4.7 and 4.8. Chapter 5 is also based on [Liu23]; it studies the Hölder regularity of a Gibbs measure in $\mathbb{R}$.

Conventions

The following notations will be used throughout the dissertation.

- $\mathbb{Z}_{>0}$ denotes the set of all positive integers.
- $\mathbb{Z}_{\geq 0}$ denotes the set of all non-negative integers.
- Int(E) denotes the interior of a subset E of a topological space.
- $\overline{E}$ denotes the closure of a subset E of a topological space.
- diam$(E) = \sup_{x,y \in E} d_X(x, y)$ denotes the diameter of a subset E of a metric space (X, d_X).

Chapter 2

Preliminaries on Dynamics

2.1 Symbolic Dynamics

2.1.1 Subshifts of Finite Type

In this subsection, we define the subshifts of finite type. The definitions in this subsection can be found in many textbooks; see e.g. [Kit98].

Definition. *Let A be a finite set containing at least two elements. Let $\mathbb{M} : A \times A \to \{ 0, 1 \}$ satisfy that for each $a \in A$, there exists some $b \in A$ such that $\mathbb{M}(a, b) = 1$. We call $\mathbb{M}$ satisfying this condition an* incidence matrix. *Define the* shift space *as*

$$\Sigma = \{ \xi = \xi_1 \xi_2 \cdots \in A^{\mathbb{Z}_{>0}} \mid \forall k \geq 1, \, \mathbb{M}(\xi_k, \xi_{k+1}) = 1 \} .$$

A sequence ξ in $A^{\mathbb{Z}_{>0}}$ is said to be admissible *if and only if $\xi \in \Sigma$. The* left shift *on Σ is the map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(\xi_1 \xi_2 \cdots) = \xi_2 \xi_3 \cdots$, for any $\xi = \xi_1 \xi_2 \cdots \in \Sigma$.*

The ordered pair (Σ, σ) is called a *subshift of finite type* (SFT). Besides the ordered pair (Σ, σ), it is also widely accepted to denote this SFT by $\sigma : \Sigma \to \Sigma$.

For every $n \in \mathbb{Z}_{>0}$,

$$A^n = \{ \omega = \omega_1 \cdots \omega_n \mid \forall k \in \{ 1, \cdots , n \} , \omega_k \in A \}$$

is the set of all words of length n over A. For $n = 0$, A^n is a singleton containing one element called the empty word. The empty word is not in any of A^n, for $n \geq 1$. Hence, $A^m \cap A^n = \varnothing$ for any two distinct non-negative integers m and n. The length of a word ω will be denoted by $|\omega|$; the empty word has length 0. The set of all words over A will be denoted by A^*. More formally, $A^* = \bigcup_{n=0}^{\infty} A^n$.

Among all the words over A, we are especially interested in the words appearing in some sequence $\xi \in \Sigma$. Such words are said to be admissible. The precise definition is given as follows.

Definition. *Define* $A^0_{\mathbb{M}} = A^0 = \{ \text{\textit{empty word}} \}$ *and* $A^1_{\mathbb{M}} = A^1 = A$. *For every integer* $n \geq 2$, *define*

$$A^n_{\mathbb{M}} = \{ \omega = \omega_1 \cdots \omega_n \in A^n \mid \forall k \in \{ 1, \cdots, n-1 \}, \mathbb{M}(\omega_k, \omega_{k+1}) = 1 \}.$$

Define $A^*_{\mathbb{M}} = \bigcup_{n=0}^{\infty} A^n_{\mathbb{M}}$, *whose elements are called* admissible words.

Definition. *The* cylinder set *of a word* ω *over* A *is*

$$[\omega] = \{ \xi \in \Sigma \mid \xi_1 \cdots \xi_{|\omega|} = \omega \}.$$

The length of $[\omega]$ *is defined as* $|\omega|$, *namely the length of the word* ω.

Clearly, the cylinder set $[\omega]$ of ω is non-empty if and only if ω is admissible.

The set of all the cylinder sets generates a topology, thus turning Σ into a topological space.

Proposition 2.1 ([Kit98]). *For any SFT* $\sigma : \Sigma \to \Sigma$, *we have that* Σ *is compact and* σ *is continuous.*

A (discrete-time) topological dynamical system consists of a topological space X and a continuous map $T : X \to X$. As the notation we adopted for an SFT, a topological dynamical system given by X and $T : X \to X$ will be denoted simply by $T : X \to X$. Thus, by Proposition 2.1, an SFT $\sigma : \Sigma \to \Sigma$ is a topological dynamical system.

We will often consider topologically mixing dynamical systems, whose definition is given as follows.

Definition. *A topological dynamical system* $T : X \to X$ *is said to be* topologically mixing *if and only if for any two non-empty open subsets* O_1, O_2 *of* X, *there exists a positive integer* M *such that* $\sigma^m(O_1) \cap O_2 \neq \varnothing$ *for any integer* $m \geq M$.

For SFTs, there is a criterion for the topological mixing condition.

Proposition 2.2 ([Bow08, Lemma 1.3]). *Let* $\sigma : \Sigma \to \Sigma$ *be an SFT, given by the set of symbols* A *and the incidence matrix* $\mathbb{M}$. *Then,* $\sigma : \Sigma \to \Sigma$ *is topologically mixing if and only if there exists some non-negative integer* l *such that for any two symbols* $a, b \in A$, *there exists some* $\rho \in A^l_{\mathbb{M}}$ *such that the word* $a\rho b$ *is admissible.*

Definition. *An SFT* $\sigma : \Sigma \to \Sigma$ *is called a* full shift *if* $\mathbb{M}(a, b) = 1$ *for any two symbols* $a, b \in A$.

From Proposition 2.2, we see that a full shift is topologically mixing; indeed, l in Proposition 2.2 can be taken to be any non-negative integer.

2.1.2 Metrics on Shift Space

The shift space Σ, endowed with the topology we introduced in the previous subsection, is metrizable. In this subsection, we shall define a family of metrics, any of which induces this topology.

We first define one metric that induces the topology of Σ as follows.

Definition. *For any two $\xi, \xi' \in \Sigma$, we write $\xi \wedge \xi'$ to denote the longest common initial block of ξ and ξ'. If $\xi = \xi'$, then $\xi \wedge \xi' = \xi = \xi'$. Define $d_1 : \Sigma \times \Sigma \to \mathbb{R}$ by $d_1(\xi, \xi) = 0$ and $d_1(\xi, \xi') = \exp(-|\xi \wedge \xi'|)$ for any two distinct $\xi, \xi' \in \Sigma$.*

As we shall see from Proposition 2.4, d_1 is an ultrametric. The open balls taken with respect to d_1 are precisely the cylinder sets, so d_1 induces the topology we defined previously for Σ.

Recall that the α-Hölder continuity of a map between two metric spaces means the following.

Definition. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Then, for any $\alpha > 0$, a map $f : X \to Y$ is said to be* α-Hölder continuous *if and only if*

$$\sup \left\{ \frac{d_Y(f(x), f(x'))}{d_X(x, x')^\alpha} \,\middle|\, x, x' \in X,\ x \neq x' \right\} < +\infty.$$

Commonly, 1-Hölder continuous maps are also called Lipschitz continuous *maps. A Hölder continuous map is a map which is α-Hölder continuous for some $\alpha > 0$.*

Given a function $\phi : \Sigma \to \mathbb{R}$, when we say that ϕ is Hölder continuous, we mean that ϕ is Hölder continuous with the metric of Σ taken to be d_1 and the metric of $\mathbb{R}$ taken to be the Euclidean metric.

Definition. *Let $\sigma : \Sigma \to \Sigma$ be an SFT, and $\phi : \Sigma \to \mathbb{R}$ be a continuous function. Then, the* n-th Birkhoff sum *of ϕ is $S_n\phi = \sum_{k=0}^{n-1} \phi \circ \sigma^k$, for any $n \in \mathbb{Z}_{>0}$. We also define $S_0\phi$ to be the constant function 0. Moreover, we define $S_\omega\phi = \sup_{\xi \in [\omega]} S_{|\omega|}\phi(\xi)$, for any non-empty word $\omega \in A_{\mathbb{M}}^*$; for ω being the empty word, we define $S_\omega\phi = 0$.*

Definition. *Let $\sigma : \Sigma \to \Sigma$ be an SFT. Then, for any Hölder continuous $\phi : \Sigma \to \mathbb{R}$, we define the* distortion constant *of ϕ to be*

$$V_\phi = \sup_{\omega \in A_{\mathbb{M}}^*} \sup_{\xi, \xi' \in [\omega]} \left| S_{|\omega|}\phi(\xi) - S_{|\omega|}\phi(\xi') \right|.$$

The bounded distortion property below asserts that the distortion constant of any Hölder continuous function on Σ is finite.

Proposition 2.3 ([MU03, Lemma 2.3.1]). *Let $\sigma : \Sigma \to \Sigma$ be an SFT. For any Hölder continuous $\phi : \Sigma \to \mathbb{R}$, we have $V_\phi < +\infty$.*

Now we introduce a family of metrics, all of which induce the same topology.

Definition. *Let $u : \Sigma \to (0, +\infty)$ be a Hölder continuous function. Define $d_u : \Sigma \times \Sigma \to \mathbb{R}$ by $d_u(\xi, \xi) = 0$ and $d_u(\xi, \xi') = \exp(S_{\xi \wedge \xi'}(-u))$, for any two distinct $\xi, \xi' \in \Sigma$.*

An even broader family of metrics is defined in [KS04].

Proposition 2.4. *For any Hölder continuous function $u : \Sigma \to (0, +\infty)$, d_u is an ultrametric.*

Proof. It is clear that for any $\xi, \xi' \in \Sigma$, $d_u(\xi, \xi') \geq 0$ with equality if and only if $\xi = \xi'$, and $d_u(\xi, \xi') = d_u(\xi', \xi)$. Therefore, it suffices to show that for any $\xi, \xi', \xi'' \in \Sigma$,

$$d_u(\xi, \xi') \leq \max\left\{ d_u(\xi, \xi''), d_u(\xi', \xi'') \right\}. \tag{2.1}$$

Clearly, we only need to deal with the situation where ξ, ξ', ξ'' are distinct. Note that either $\xi \wedge \xi''$ is an initial block of $\xi' \wedge \xi''$ or $\xi' \wedge \xi''$ is an initial block of $\xi \wedge \xi''$. Due to the symmetry of (2.1), without loss of generality, we may further assume that $\xi \wedge \xi''$ is an initial block of $\xi' \wedge \xi''$.

If $\xi \wedge \xi'' = \xi' \wedge \xi''$, then we have that $\xi \wedge \xi''$ must be an initial block of $\xi \wedge \xi'$. If $\xi \wedge \xi'' \neq \xi' \wedge \xi''$, since we have assumed that $\xi \wedge \xi''$ is an initial block of $\xi' \wedge \xi''$, we have $\xi \wedge \xi' = \xi \wedge \xi''$. Therefore, in any case, we always have that $\xi \wedge \xi''$ is an initial block of $\xi \wedge \xi'$. Hence, we have $[\xi \wedge \xi'] \subseteq [\xi \wedge \xi'']$ and $|\xi \wedge \xi'| \geq |\xi \wedge \xi''|$, implying that

$$\begin{aligned} S_{\xi \wedge \xi'}(-u) &= \sup_{\zeta \in [\xi \wedge \xi']} -S_{|\xi \wedge \xi'|} u(\zeta) \\ &\leq \sup_{\zeta \in [\xi \wedge \xi'']} -S_{|\xi \wedge \xi'|} u(\zeta) \leq \sup_{\zeta \in [\xi \wedge \xi'']} -S_{|\xi \wedge \xi''|} u(\zeta) = S_{\xi \wedge \xi''}(-u). \end{aligned}$$

Thus, we conclude that $d_u(\xi, \xi') \leq d_u(\xi, \xi'') \leq \max\left\{ d_u(\xi, \xi''), d_u(\xi', \xi'') \right\}$. $\square$

When u is the constant function 1, then d_u is precisely the metric d_1 we defined at the beginning of this subsection. Therefore, the definition of d_u is consistent with our definition of d_1.

Proposition 2.5 ([Liu23]). *Let $u : \Sigma \to (0, +\infty)$ be a Hölder continuous function. Then, for any $f : \Sigma \to \mathbb{R}$, f is Hölder continuous if and only if f is Hölder continuous with the metric of Σ being replaced by d_u.*

Hence, when we say that a function is Hölder continuous, we can alternatively say that it is Hölder continuous with respect to another d_u rather than d_1, provided that u is Hölder continuous.

Proof. Since u is a positive continuous function on the compact space Σ, we have

$$0 < \min_{\zeta\in\Sigma} u(\zeta) \le \max_{\zeta\in\Sigma} u(\zeta) < +\infty.$$

Also note that, for any $\xi, \xi' \in \Sigma$, by definition, we have

$$\max_{\zeta\in\Sigma} u(\zeta) \log d_1(\xi,\xi') \le \log d_u(\xi,\xi') \le \min_{\zeta\in\Sigma} u(\zeta) \log d_1(\xi,\xi').$$

From these facts and the definition of Hölder continuity, our claim follows. $\square$

2.2 Facts From Ergodic Theory

2.2.1 Theorems for Measure-Preserving Dynamical Systems

In this subsection, we shall state several theorems for measure-preserving dynamical systems. In order to avoid lengthy discussions for the general cases, in what follows, we shall state all the theorems only for a topologically mixing SFT $\sigma : \Sigma \to \Sigma$.

Definition. *A Borel probability measure μ on Σ is said to be* σ-invariant*, if μ equals the pushforward measure $\sigma_*\mu$, meaning that for any Borel $E \subseteq \Sigma$, $\mu(E) = \sigma_*\mu(E) = \mu(\sigma^{-1}(E))$.*

A σ-invariant Borel probability measure μ on Σ is said to be ergodic*, if for any Borel set $E \subseteq \Sigma$, $\sigma^{-1}(E) = E$ implies $\mu(E) \in \{0, 1\}$.*

Theorem 2.6 (Birkhoff's ergodic theorem, [Wal82, Theorem 1.14])**.** *Let $\sigma : \Sigma \to \Sigma$ be an SFT, and μ be a σ-invariant Borel probability measure on Σ. Then, for any μ-integrable function $f : \Sigma \to \mathbb{R}$, we have that*

$$\lim_{n\to+\infty} \frac{1}{n} S_n f(\xi) \text{ exists, for } \mu\text{-a.e. } \xi \in \Sigma,$$

and that for any Borel set E satisfying $\sigma^{-1}E = E$,

$$\int_E \lim_{n\to+\infty} \frac{1}{n} S_n f \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu.$$

If μ is ergodic, then we further have $\lim_{n\to+\infty} n^{-1} S_n f = \int_\Sigma f \,\mathrm{d}\mu$, μ-a.e.

For the ergodic case above, if $\int_\Sigma f \,\mathrm{d}\mu = 0$, then we have

$$\lim_{n\to+\infty} \frac{1}{n} S_n f(\xi) = 0, \text{ for } \mu\text{-a.e. } \xi \in \Sigma.$$

Atkinson showed in [Atk76] that in this case, one also has

$$\liminf_{n\to+\infty} |S_n f(\xi)| = 0, \text{ for } \mu\text{-a.e. } \xi \in \Sigma. \tag{2.2}$$

For the convenience of later reference, we state it as a theorem.

Theorem 2.7 ([Atk76]). *Let μ be an ergodic σ-invariant Borel probability measure on Σ. Then, for any μ-integrable function $f : \Sigma \to \mathbb{R}$, we have $\int_\Sigma f \,\mathrm{d}\mu = 0$ if and only if f satisfies (2.2).*

One may regard (Σ, μ) as the underlying probability space of a random walk on $\mathbb{R}$. In this sense, (2.2) means the recurrence of this random walk on $\mathbb{R}$, and Theorem 2.7 provides a criterion for the recurrence.

Lastly, we state Shannon-McMillan-Breiman theorem. For this purpose, we need to introduce the notion of measure-theoretic entropy, which is also known as the Kolmogorov-Sinai entropy.

Definition. *The* Kolmogorov-Sinai entropy *of a σ-invariant Borel probability measure μ on Σ is defined as*

$$h_{KS}(\mu) = \lim_{n\to+\infty} -\frac{1}{n} \sum_{\omega\in A^n_{\mathbb{M}}} \mu([\omega]) \log(\mu([\omega])).$$

It is a convention that $0 \cdot \log(0) = 0$. The limit in the defining equation of the Kolmogorov-Sinai entropy exists, because

$$n \mapsto -\sum_{\omega\in A^n_{\mathbb{M}}} \mu([\omega]) \log(\mu([\omega]))$$

is in fact a subadditive sequence.

We calculate the Kolmogorov-Sinai entropy of a particular family of measures as follows.

Example 2.8 (Kolmogorov-Sinai entropy of Bernoulli measure, [Wal82, p. 102]). Let m be a positive integer no less than 2. Let $\sigma : \Sigma \to \Sigma$ be the full shift for which the set of symbols is $A = \{1, \cdots, m\}$. Let $(\lambda_1, \cdots, \lambda_m)$ be a non-negative of positive integers satisfying $\sum_{k=1}^m \lambda_k = 1$. The *Bernoulli measure* associated with $(\lambda_1, \cdots, \lambda_m)$ is then the unique Borel probability measure ν satisfying

1. $\nu([k]) = \lambda_k$ for any $k \in A$;
2. $\nu([\omega]) = \prod_{j=1}^{|\omega|} \nu([\omega_j])$ for any word ω over A.

It is not hard to see that ν is σ-invariant [Wal82, p. 21]. Note that for any $n \in \mathbb{Z}_{>0}$,

$$\begin{aligned}\sum_{\omega\in A^n} \nu([\omega])\log(\nu([\omega])) &= \sum_{j=1}^{n} \sum_{\omega_1,\cdots,\omega_n\in A} \nu([\omega])\log(\nu([\omega_j])) \\ &= \sum_{j=1}^{n} \sum_{\omega_j\in A} \nu(\sigma^{-(j-1)}[\omega_j])\log(\nu([\omega_j])) \\ &= n\sum_{k\in A} \nu([k])\log(\nu([k])) = n\sum_{k=1}^{m} \lambda_k \log(\lambda_k).\end{aligned}$$

Therefore, we have $h_{KS}(\nu) = -\sum_{k=1}^{m} \lambda_k \log(\lambda_k)$.

Now we can state the Shannon-McMillan-Breiman theorem as follows.

Theorem 2.9 (Shannon-McMillan-Breiman theorem, [PU10, Theorem 2.5.4 & Theorem 2.5.5])**.** *Let μ be a σ-invariant Borel probability measure on Σ. For any $n \in \mathbb{Z}_{>0}$, define a function $\mathcal{I}_n : \Sigma \to \mathbb{R}$ by*

$$\mathcal{I}_n(\xi) = -\log(\mu([\xi_1 \cdots \xi_n])),$$

for every $\xi \in \Sigma$. Then, $\mathcal{I}_n/n$ converges to a μ-integrable function both μ-a.e. and in L^1-norm, and

$$\int_\Sigma \lim_{n\to+\infty} \frac{\mathcal{I}_n}{n}\, \mathrm{d}\mu = h_{KS}(\mu).$$

Moreover, if μ is ergodic, then $\lim_{n\to+\infty} \mathcal{I}_n/n = h_{KS}(\mu)$, μ-a.e.

2.2.2 Thermodynamic Formalism

In some situations, there is no Borel probability measure specified on the topological dynamical system we consider. Hence, one needs to pick a suitable measure so as to apply the powerful theorems we stated in the previous subsection. Thermodynamic formalism provides one way to pick such measures, which are called equilibrium states. For simplicity, we will still state all the assertions for SFTs.

Definition ([Wal82])**.** *Let $\sigma : \Sigma \to \Sigma$ be an SFT. The* topological pressure *of a continuous function $\phi : \Sigma \to \mathbb{R}$ is*

$$P(\phi) = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{\omega\in A^n_{\mathbb{M}}} \exp(S_\omega\phi).$$

As an example, we calculate the topological pressure of a function of the following type.

Example 2.10. As in Example 2.8, let the SFT $\sigma : \Sigma \to \Sigma$ be a full shift, for which the set of symbols is denoted by A. Suppose that $\phi : \Sigma \to \mathbb{R}$ is a function which is constant on every cylinder of length 1. For every $k \in A$, let ϕ_k be the real number for which $\phi([k]) = \{ \phi_k \}$. Then, for any $n \in \mathbb{Z}_{>0}$,

$$\sum_{\omega \in A^n} \exp(S_\omega \phi) = \sum_{\omega_1, \cdots, \omega_n \in A} \prod_{j=1}^{n} \exp(\phi_{\omega_j}) = \left(\sum_{k \in A} \exp(\phi_k) \right)^n .$$

Thus, we have $P(\phi) = \log \sum_{k \in A} \exp(\phi_k)$.

Although no measure appears in the definition of the topological pressure, the topological pressure turns out to be the infimum of a functional on $\mathfrak{M}_\sigma(\Sigma)$, where $\mathfrak{M}_\sigma(\Sigma)$ denotes the set of all σ-invariant Borel probability measures on Σ.

Theorem 2.11 ([PU10, Chapter 3 & Chapter 5]). *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Suppose that $\phi : \Sigma \to \mathbb{R}$ is a Hölder continuous function. Then,*

$$P(\phi) = \sup \left\{ h_{KS}(\mu) + \int_\Sigma \phi \, \mathrm{d}\mu \;\middle|\; \mu \in \mathfrak{M}_\sigma(\Sigma) \right\},$$

and there exists a unique σ-invariant Borel probability measure ν on Σ such that $P(\phi) = h_{KS}(\nu) + \int_\Sigma \phi \, \mathrm{d}\nu$. Furthermore, this measure ν is ergodic.

Definition. *For any continuous function $\phi : \Sigma \to \mathbb{R}$, an* equilibrium state *for ϕ is a σ-invariant Borel probability measure ν on Σ satisfying that $P(\phi) = h_{KS}(\nu) + \int_\Sigma \phi \, \mathrm{d}\nu$.*

Remark 2.12. Theorem 2.11 asserts that the topological pressure of ϕ is the supremum of the sum of two functionals on $\mathfrak{M}_\sigma(\Sigma)$. One functional is $\mu \mapsto \int_\Sigma \phi \, \mathrm{d}\mu$ and the other is the entropy map $h_{KS} : \mu \mapsto h_{KS}(\mu)$. Endow $\mathfrak{M}_\sigma(\Sigma)$ with the weak* topology. Then, $\mu \mapsto \int_\Sigma \phi \, \mathrm{d}\mu$ is clearly affine and continuous. The entropy map is also known to be affine [Wal82, Theorem 8.1]. In addition, the entropy map is upper semi-continuous because $\sigma : \Sigma \to \Sigma$ is expansive; see Theorem 3.5.6 in [PU10] for details.

When $\phi : \Sigma \to \mathbb{R}$ is Hölder continuous, the unique equilibrium state for ϕ is known to be a Gibbs measure for ϕ [Bow08]. The definition of a Gibbs measure is given as follows.

Definition. *Let $\sigma : \Sigma \to \Sigma$ be an SFT, and $\phi : \Sigma \to \mathbb{R}$ be a Hölder continuous function. Then, a Borel probability measure ν_ϕ is called a Gibbs measure for ϕ if and only if there is some $C_{\nu_\phi} > 1$ such that for any $\omega \in A^*_{\mathbb{M}}$,*

$$C_{\nu_\phi}^{-1} \leq \frac{\nu_\phi([\omega])}{\exp(S_\omega \phi - |\omega| P(\phi))} \leq C_{\nu_\phi}.$$

As we shall see in Theorem 2.13, Gibbs measures for a Hölder continuous ϕ exist. Gibbs measures for ϕ are not unique. To see this, suppose that ν_ϕ is a Gibbs measure for ϕ and $f : \Sigma \to (0, +\infty)$ is an arbitrary measurable function satisfying that $0 < \inf_{\xi\in\Sigma} f(\xi) \leq \sup_{\xi\in\Sigma} f(\xi) < +\infty$ and $\int_\Sigma f \,\mathrm{d}\nu_\phi = 1$. Define a new Borel probability measure ν'_ϕ by $\mathrm{d}\nu'_\phi = f \,\mathrm{d}\nu_\phi$. Then, ν'_ϕ is also a Gibbs measure for ϕ. Take f satisfying $\nu_\phi(\{\, \xi \in \Sigma \mid f(\xi) \neq 1 \,\}) > 0$, and we will have $\nu'_\phi \neq \nu_\phi$.

The following theorem gives a characterization of the equilibrium state for a Hölder continuous function. It is a consequence of Ruelle's Perron-Frobenius theorem; see e.g. Chapter 1 of [Bow08] for details.

Theorem 2.13 ([Bow08])**.** *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT, and $\phi : \Sigma \to \mathbb{R}$ be a Hölder continuous function. Then, the unique equilibrium state ν_ϕ for ϕ is the unique σ-invariant Gibbs measure for ϕ.*

Finally, we need the following properties of the topological pressure.

Theorem 2.14 ([PP90; PU10])**.** *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let m be a positive integer and $\psi, \phi_1, \cdots, \phi_m$ be real-valued Hölder continuous functions on Σ. Then, the multivariate function $(t_1, \cdots, t_m) \mapsto P(\psi + \sum_{k=1}^m t_k\phi_k)$ is real analytic and convex. Moreover, for any $j \in \{\, 1, \cdots, m \,\}$,*

$$\frac{\partial}{\partial t_j} P(\psi + \sum_{k=1}^m t_k\phi_k) = \int_\Sigma \phi_j \,\mathrm{d}\nu_{\psi + \sum_{k=1}^m t_k\phi_k}, \tag{2.3}$$

where $\nu_{\psi+\sum_{k=1}^m t_k\phi_k}$ is the equilibrium state for $\psi + \sum_{k=1}^m t_k\phi_k$.

Now let us consider the case where m in Theorem 2.14 is equal to 1. Denote ϕ_1 by simply ϕ. Then, by Theorem 2.14, the function $t \mapsto P(\psi + t\phi)$ is convex and analytic, which implies that the second derivative of $t \mapsto P(\psi + t\phi)$ exists and is non-negative. Indeed, there is a formula for the second derivative of $t \mapsto P(\psi + t\phi)$ due to D. Ruelle [Rue04]. We will not use this formula in this thesis, so we shall omit it and refer to [PU10, Theorem 5.7.4] for precise statements. However, in the subsequent discussions, we do need a criterion for the second derivative of $t \mapsto P(\psi + t\phi)$ being zero. Towards this end, we introduce the following definition.

Definition. *Let $\sigma : \Sigma \to \Sigma$ be an SFT. Let $\phi_1 : \Sigma \to \mathbb{R}$ and $\phi_2 : \Sigma \to \mathbb{R}$ be two continuous functions. We say that ϕ_1 is* cohomologous *to ϕ_2 if there exists a continuous function $f : \Sigma \to \mathbb{R}$ such that $\phi_1 = \phi_2 + f \circ \sigma - f$.*

Clearly, the cohomology relation is an equivalence relation. The importance of this relation is well captured by the following simple observation. For any two mutually cohomologous continuous function ϕ_1 and ϕ_2, the difference between their Birkhoff sums of the same degree is uniformly bounded. To be more precise, for any topological space X, the supremum norm of a bounded continuous function $f : X \to \mathbb{R}$ is defined to be $\|f\| = \sup_{x \in X} |f(x)|$. Then, for mutually cohomologous continuous functions $\phi_1, \phi_2 : \Sigma \to \mathbb{R}$ and $f : \Sigma \to \mathbb{R}$ satisfying $\phi_1 = \phi_2 + f \circ \sigma - f$, we have

$$\sup_{n \in \mathbb{Z}_{>0}} \sup_{\xi \in \Sigma} |S_n\phi_1(\xi) - S_n\phi_2(\xi)| \leq \sup_{n \in \mathbb{Z}_{>0}} \|f \circ \sigma^n(\xi) - f\| \leq 2\|f\| < +\infty.$$

From this observation, we can easily see that any two mutually cohomologous Hölder continuous functions on Σ have the same topological pressure, the same equilibrium states and the same family of Gibbs measures.

Now we state the necessary and sufficient condition for the second derivative of the function $t \mapsto P(\psi + t\phi)$ being zero.

Theorem 2.15 ([PP90, Proposition 4.12])**.** *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let $\psi : \Sigma \to \mathbb{R}$ and $\phi : \Sigma \to \mathbb{R}$ be Hölder continuous functions. Then, ϕ is cohomologous to a constant function if and only if*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} P(\psi + t\phi)\big|_{t=0} = 0.$$

2.2.3 Ergodic Optimization

Consider a topologically mixing SFT $\sigma : \Sigma \to \Sigma$ and a Hölder continuous function $\psi : \Sigma \to \mathbb{R}$. If we pick a σ-invariant Borel probability measure μ on Σ, then Birkhoff's ergodic theorem guarantees the almost everywhere convergence of the Birkhoff averages $S_n\psi/n$ of ψ. Slightly extending the scope of our discussion, suppose that we have another Hölder continuous function $v : \Sigma \to (0, +\infty)$, and we are interested in the limit $\lim_{n\to+\infty} S_n\psi/S_nv$. By Birkhoff's ergodic theorem, $S_n\psi/n$ and S_nv/n both converge μ-a.e., so

$$\lim_{n\to+\infty} \frac{S_n\psi}{S_nv} = \lim_{n\to+\infty} \frac{S_n\psi/n}{S_nv/n}$$

exists μ-a.e. as well. Now our concern is the range of $\lim_{n\to+\infty} S_n\psi/S_nv$. For this purpose, we introduce the following two quantities.

Definition. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. For any two continuous functions $\psi : \Sigma \to \mathbb{R}$ and $v : \Sigma \to (0, +\infty)$, define*

$$\alpha^-_{\psi,v} = \inf_{\xi\in\Sigma} \liminf_{n\to+\infty} \frac{S_n\psi(\xi)}{S_nv(\xi)};\ \alpha^+_{\psi,u} = \sup_{\xi\in\Sigma} \limsup_{n\to+\infty} \frac{S_n\psi(\xi)}{S_nv(\xi)}.$$

It is clear that for every $\xi \in \Sigma$,

$$\alpha^-_{\psi,v} \leq \lim_{n\to+\infty} \frac{S_n\psi(\xi)}{S_nv(\xi)} \leq \alpha^+_{\psi,v},$$

whenever the limit in the middle exists. In Section 4.1, we will study the set

$$\mathcal{L}^\alpha_{\psi,v} = \left\{ \xi \in \Sigma \,\middle|\, \lim_{n\to+\infty} \frac{S_n\psi(\xi)}{S_nv(\xi)} = \alpha \right\},$$

for $\alpha \in [\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$. For the case where $\alpha \in \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$, the theory of ergodic optimization gives some useful facts. In what follows, we shall give the statements and some implications of these facts.

Now, we can summarize the assertions we need from the theory of ergodic optimization into one single theorem as follows.

Theorem 2.16 ([Sav99; Gar17; Jen19])**.** *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT, and $\phi : \Sigma \to \mathbb{R}$ be a Hölder continuous function. Then, the following statements are equivalent:*

1. *there is a Hölder continuous function $v : \Sigma \to (0, +\infty)$ such that $\alpha^+_{\phi,v} \leq 0$;*
2. $\alpha^+_{\phi,1} \leq 0$*;*
3. $\alpha^+_{\phi,v} \leq 0$ *for any Hölder continuous function $v : \Sigma \to (0, +\infty)$;*
4. $\sup_{n\in\mathbb{Z}_{>0}} \sup_{\xi\in\Sigma} S_n\phi(\xi) < +\infty$*;*
5. *ϕ is cohomologous to some Hölder continuous $\phi_- : \Sigma \to (-\infty, 0]$.*

Similarly, the following statements are equivalent as well:

1. *there is a Hölder continuous function $v : \Sigma \to (0, +\infty)$ such that $\alpha^-_{\phi,v} \geq 0$;*
2. $\alpha^-_{\phi,1} \geq 0$*;*
3. $\alpha^-_{\phi,v} \geq 0$ *for any Hölder continuous function $v : \Sigma \to (0, +\infty)$;*
4. $\inf_{n\in\mathbb{Z}_{>0}} \inf_{\xi\in\Sigma} S_n\phi(\xi) > -\infty$*;*

5. *ϕ is cohomologous to some Hölder continuous $\phi_+ : \Sigma \to [0, +\infty)$.*

Proof. The equivalence among the first five statements implies the equivalence among the last five statements, so we only need to show the equivalence among the first five statements.

Clearly, the second statement implies the first, the third implies the second, the fourth implies the third, and the fifth implies the fourth. Therefore, the only non-trivial implication is that the first statement implies the fifth.

First note that the first statement also implies the second. To see this, suppose that $\alpha^+_{\phi,1} > 0$. Then, there exists some $\eta \in \Sigma$ such that $\limsup_{n\to+\infty} S_n\phi(\eta)/n > 0$. Thus, for any Hölder continuous function $v : \Sigma \to (0, +\infty)$, we have

$$\alpha^+_{\phi,v} \geq \limsup_{n\to+\infty} \frac{S_n\phi(\eta)}{S_n v(\eta)} \geq \frac{1}{\|v\|} \limsup_{n\to+\infty} \frac{S_n\phi(\eta)}{n} > 0.$$

Therefore, the first statement implies the second. Now it only remains to show that the second statement implies the fifth. This is a well-known fact. We refer to [Sav99] for the proof. Besides, there is a different approach in [Gar17; Jen19]. The proof is thus complete. $\square$

An immediate consequence of Theorem 2.16 is given as follows.

Corollary 2.17. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT, and $\phi : \Sigma \to \mathbb{R}$ be a Hölder continuous function. Then, the following statements are equivalent:*

1. $\alpha^-_{\phi,1} = \alpha^+_{\phi,1} = 0$*;*
2. $\alpha^-_{\phi,v} = \alpha^+_{\phi,v} = 0$*, for any Hölder continuous $v : \Sigma \to (0, +\infty)$;*
3. $\sup_{n\in\mathbb{Z}_{>0}} \|S_n\phi\| < +\infty$.

For a topologically mixing SFT $\sigma : \Sigma \to \Sigma$, if a Hölder continuous function $\phi : \Sigma \to \mathbb{R}$ satisfies one and thus all of the statements in Corollary 2.17, then we also have that ϕ is cohomologous to the constant function 0. See Proposition 4.4.5 in [PU10].

The following proposition will be used later in Section 4.1. We state it here because its proof only uses the facts we stated in this section.

Proposition 2.18. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let $\phi : \Sigma \to \mathbb{R}$ and $v : \Sigma \to (0, +\infty)$ be Hölder continuous. Let μ be a σ-invariant Borel probability measure on Σ satisfying that $\mu(O) > 0$ for any non-empty open set $O \subseteq \Sigma$. Suppose that $0 \in \{\alpha^-_{\phi,v}, \alpha^+_{\phi,v}\}$. Then, if $\int_\Sigma \phi \, \mathrm{d}\mu = 0$, we have $\alpha^-_{\phi,v} = \alpha^+_{\phi,v} = 0$.*

Proof. Suppose that $\alpha^-_{\phi,v} = 0$. Then, by Theorem 2.16, ϕ is cohomologous to a Hölder continuous $\phi_- : \Sigma \to [0, +\infty)$. Let $f : \Sigma \to \mathbb{R}$ be a continuous function such that $\phi_- = \phi + f \circ \sigma - f$. Then, by the σ-invariance of μ, we have

$$\int_\Sigma \phi_- \, \mathrm{d}\mu = \int_\Sigma \phi \, \mathrm{d}\mu + \int_\Sigma f \circ \sigma \, \mathrm{d}\mu - \int_\Sigma f \, \mathrm{d}\mu = 0.$$

As $\phi_- \geq 0$, we have $\phi_- = 0$ μ-a.e. Thus, we see that $\{ \xi \in \Sigma \mid \phi_-(\xi) > 0 \}$ is an open and μ-null set. By our assumption on μ, we have $\{ \xi \in \Sigma \mid \phi_-(\xi) > 0 \} = \varnothing$. As a result, $\phi_- = 0$, so $\alpha^+_{\phi,v} = 0$. Therefore, we have shown that if $\alpha^-_{\phi,v} = 0$, then $\alpha^+_{\phi,v} = 0$.

Conversely, if $\alpha^+_{\phi,v} = 0$, applying what we have proved to $-\phi$ and v, we can easily see that $\alpha^-_{\phi,v} = 0$ as well. This completes our proof. $\square$

Chapter 3

Preliminaries on Dimension Theory

In this chapter, we state some preliminary facts related to the dimension theory. The facts in this chapter are not new, but some of them take a form slightly different from the corresponding theorems in our reference. For such facts, we will provide their proofs.

Given a metric space (X, d_X), $B(x, r) = \{ y \in X \mid d_X(x, y) < r \}$ is the open ball with centre $x \in X$ and radius $r > 0$. For any $E \subseteq X$, the diameter of E is $\operatorname{diam}(E) = \sup \{ d_X(x, y) \mid x, y \in E \}$.

3.1 Hausdorff Dimension and u-Dimension

In this section, we shall give the definition of the Hausdorff dimension of a subset of a metric space. In a similar manner, we shall also define the u-dimension of a subset of the shift space of an SFT.

We begin with the definition of the Hausdorff dimension.

Definition. *Let (X, d_X) be a metric space. For any $\delta > 0$, a* δ-covering *of an arbitrary $E \subseteq X$ is a covering $\mathcal{U}$ of E satisfying that for any $U \in \mathcal{U}$,* $\operatorname{diam}(U) \leq \delta$. *Fix some $s \geq 0$, and define*

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^s \;\middle|\; \mathcal{U} \text{ is a countable } \delta\text{-covering of } E \right\} \tag{3.1}$$

and $\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E)$. The quantity $\mathcal{H}^s(E)$ is called the s-dimensional Hausdorff (outer) measure *of E.*

Remark 3.1. Note that for every $U \subseteq X$, $\operatorname{diam}(U) = \operatorname{diam}(\overline{U})$. Therefore, the infimum in (3.1) can be taken in a smaller range, namely the range of all countable *closed* δ-coverings of E. Here we say a covering is closed if all its members are closed. To put it more formally, we have

$$\mathcal{H}^s_\delta(E) = \inf\left\{ \sum_{U\in\mathcal{U}} \operatorname{diam}(U)^s \,\middle|\, \mathcal{U} \text{ is a countable closed } \delta\text{-covering of } E \right\}.$$

Note that by construction, for any $\delta > 0$ and any two positive numbers $s_1 > s_2$,

$$0 \leq \mathcal{H}^{s_1}_\delta(E) \leq \delta^{s_1 - s_2}\mathcal{H}^{s_2}_\delta(E). \tag{3.2}$$

From this observation, the following proposition follows.

Proposition 3.2 ([Fol99, Proposition 11.19])**.** *There exists one* $\dim_H(E) \in [0, +\infty]$ *such that*

1. *for all* $s < \dim_H(E)$*,* $\mathcal{H}^s(E) = +\infty$*;*
2. *for all* $s' > \dim_H(E)$*,* $\mathcal{H}^{s'}(E) = 0$*.*

The graph of $s \mapsto \mathcal{H}^s(E)$ is illustrated by Figure 3.1. Clearly, this $\dim_H(E)$ is unique.

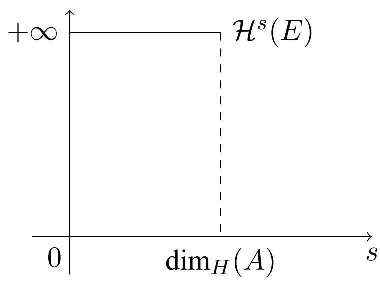

Figure 3.1: Change of $\mathcal{H}^s(E)$ as the dimension s changes

Definition. *The* Hausdorff dimension *of* $E \subseteq X$ *is* $\dim_H(E)$ *satisfying the conditions in Proposition 3.2.*

Evidently, $\dim_H(E)$ satisfies

$$\begin{aligned}\dim_H(E) &= \inf\{\, s \in [0, +\infty] \mid \mathcal{H}^s(E) = 0 \,\}\\ &= \sup\{\, s \in [0, +\infty] \mid \mathcal{H}^s(E) = +\infty \,\}.\end{aligned}$$

Here we set $\inf(\varnothing) = +\infty$ and $\sup(\varnothing) = 0$.

Now we shall turn our attention to subshifts of finite type. For any subset E of the shift space Σ and any continuous function $u : \Sigma \to (0, +\infty)$, the u-dimension of E is defined in [BS00] as follows.

Definition. *Let $\sigma : \Sigma \to \Sigma$ be an SFT, and $u : \Sigma \to (0, +\infty)$ be a continuous function. For every $E \subseteq \Sigma$, $n \in \mathbb{Z}_{>0}$ and $s \in \mathbb{R}$, define*

$$\mathcal{S}_n(E, s, u) = \inf\left\{ \sum_{\omega \in \mathcal{W}} \exp(-sS_\omega u) \;\middle|\; \bigcup_{\omega \in \mathcal{W}} [\omega] \supseteq E \text{ and } \forall \omega \in \mathcal{W},\ |\omega| \geq n \right\},$$

*and $\mathcal{S}(E, s, u) = \lim_{n \to +\infty} \mathcal{S}_n(E, s, u)$. The u-*dimension *of E is then $\dim_u(E) \in [0, +\infty]$ defined by*

$$\begin{aligned} \dim_u(E) &= \inf \{ s \in [0, +\infty] \mid \mathcal{S}(E, s, u) = 0 \} \\ &= \sup \{ s \in [0, +\infty] \mid \mathcal{S}(E, s, u) = +\infty \}. \end{aligned}$$

As before, we set $\inf(\varnothing) = +\infty$ and $\sup(\varnothing) = 0$.

The following proposition shows that, if $u : \Sigma \to (0, +\infty)$ is Hölder continuous, then the u-dimension of a set is exactly its Hausdorff dimension with respect to the metric d_u.

Proposition 3.3. *Suppose that $u : \Sigma \to (0, +\infty)$ is Hölder continuous. Take d_u as the metric of Σ. Then, for any $E \subseteq \Sigma$, $\dim_u(E) = \dim_H(E)$.*

Proof. Until the end of this proof, the metric of Σ will always be taken to be d_u. Fix $E \subseteq \Sigma$ and $s \geq 0$. Then, for any $n \in \mathbb{Z}_{>0}$ and any set $\mathcal{W}$ of words of length no less than n satisfying $\bigcup_{\omega \in \mathcal{W}} [\omega] \supseteq E$, we have

$$\sum_{\omega \in \mathcal{W}} \exp(-sS_\omega u) \geq \sum_{\omega \in \mathcal{W}} \exp(sS_\omega(-u) - sV_u) = \exp(-sV_u) \sum_{\omega \in \mathcal{W}} \operatorname{diam}([\omega])^s,$$

and for every $\omega \in \mathcal{W}$, $\operatorname{diam}([\omega]) \leq \exp(S_\omega(-u)) \leq \exp(-n \inf_{\xi \in \Sigma} u)$. Thus, on the one hand, we have

$$\mathcal{S}(E, s, u) \geq \exp(-sV_u)\mathcal{H}^s(E). \tag{3.3}$$

On the other hand, let $\delta > 0$ and $\mathcal{U}$ be an arbitrary δ-covering of E. For every non-empty $U \in \mathcal{U}$, take an arbitrary $\xi(U) \in U$, and $\omega(U)$ be the longest initial block of $\xi(U)$ such that $[\omega(U)] \supseteq U$. Denote by ω' be the initial block of $\xi(U)$ whose length is $|\omega(U)| + 1$. Then, we have

$$\operatorname{diam}([\omega(U)]) \leq \exp(\|u\|)\operatorname{diam}([\omega']) \leq \exp(\|u\|)\operatorname{diam}(U). \tag{3.4}$$

A lower bound of $|\omega(U)|$ can thus be given as follows:

$$|\omega(U)| \geq \frac{S_{\omega(U)}(-u)}{-\|u\|} = \frac{-\log(\operatorname{diam}([\omega(U)]))}{\|u\|} \geq \frac{-\log(\delta)}{\|u\|} - 1. \tag{3.5}$$

Define $\mathcal{W} = \{\omega(U) \mid U \in \mathcal{U}\}$. Clearly, $\bigcup_{\omega\in\mathcal{W}}[\omega] \supseteq E$. The estimate in (3.5) indicates that the lengths of the words in $\mathcal{W}$ diverge to $+\infty$ uniformly, as δ approaches 0. Moreover, by (3.4),

$$\sum_{U\in\mathcal{U}} \operatorname{diam}(U)^s \geq \sum_{\omega\in\mathcal{W}} \exp(-s\|u\|)\operatorname{diam}([\omega])^s \geq \exp(-s\|u\|)\sum_{\omega\in\mathcal{W}} \exp(-sS_\omega u).$$

Therefore, we have $\mathcal{H}^s(E) \geq \exp(-s\|u\|)\mathcal{S}(E,s,u)$. Combining this with (3.3), we conclude that $\dim_u(E) = \dim_H(E)$. $\square$

When we study the Hausdorff dimension of a subset of the shift space, it is sometimes unclear which metric we choose. In order to eliminate this ambiguity, we shall use the notion of the u-dimension when we consider a subset of the shift space.

3.2 Basic Properties of Hausdorff Dimension

Some basic properties of the Hausdorff dimension are given in the following proposition.

Proposition 3.4 ([Fal03, Proposition 2.2 & pp. 32-33])**.** *Let (X, d_X) be a metric space. Then, the following statements hold.*

1. *If $E_1 \subseteq E_2 \subseteq X$, then we have* $\dim_H(E_1) \leq \dim_H(E_2)$.
2. *For any sequence $(E_k)_{k\in\mathbb{Z}_{>0}}$ of subsets of X, we have* $\dim_H(\bigcup_{k=1}^\infty E_k) = \sup_{k\in\mathbb{Z}_{>0}} \dim_H(E_k)$.
3. *For any countable set $E \subseteq X$, we have* $\dim_H(E) = 0$.
4. *Let (Y, d_Y) be a metric space, and $f : X \to Y$ be Lipschitz. Then, we have* $\dim_H(f(X)) \leq \dim_H(X)$. *If f is further bi-Lipschitz, then we have* $\dim_H(f(X)) = \dim_H(X)$.

The second assertion in Proposition 3.4 is usually called the countable stability of Hausdorff dimension.

3.3 Estimating Hausdorff Dimension With Measures

Given a metric space, the measures on this space carry rich information about its Hausdorff dimension. The analysis of the measures is especially effective when one attempts to obtain a lower bound for the Hausdorff dimension. The following theorem, called the mass distribution principle, shows one way to give a lower bound.

Theorem 3.5 (Mass distribution principle, [Fal03, p. 60]). *Let μ be a Borel measure on a metric space X. Suppose that for some $s \geq 0$, there are $C > 0$ and $\delta > 0$ such that $\mu(U) \leq C\mathrm{diam}(U)^s$ for all closed sets U with $\mathrm{diam}(U) \leq \delta$. Then for any Borel set $E \subseteq X$ with $\mu(E) > 0$, we have $\mathcal{H}^s(E) \geq C^{-1}\mu(E)$ and $\dim_H(E) \geq s$.*

The Hausdorff dimension of a Borel probability measure on a metric space is defined in the following way.

Definition. *For any Borel probability measure μ on the metric space (X, d), define*

$$\overline{\dim}_H(\mu) = \inf\{\, \dim_H(E) \mid E \subseteq X \textit{ is Borel and } \mu(E) = 1 \,\};$$
$$\underline{\dim}_H(\mu) = \inf\{\, \dim_H(E) \mid E \subseteq X \textit{ is Borel and } \mu(E) > 0 \,\}.$$

When $\overline{\dim}_H(\mu) = \underline{\dim}_H(\mu)$, this common value is called the Hausdorff dimension *of μ, and denoted by $\dim_H(\mu)$.*

When X is the shift space Σ of an SFT $\sigma : \Sigma \to \Sigma$ and the metric on X is d_u for some Hölder continuous $u : \Sigma \to (0, +\infty)$, the Hausdorff dimension of μ will be called the u-dimension of μ, and will be denoted by $\dim_u(\mu)$ rather than $\dim_H(\mu)$.

Definition. *Let μ be a Borel probability measure on a metric space (X, d_X). Then, the* lower and upper pointwise dimensions *of μ at $x \in X$ are defined as*

$$\underline{\dim}_\mu(x) = \liminf_{r \to 0^+} \frac{\log(\mu(B(x,r)))}{\log(r)};$$
$$\overline{\dim}_\mu(x) = \limsup_{r \to 0^+} \frac{\log(\mu(B(x,r)))}{\log(r)},$$

respectively. When they coincide at some $x \in X$, their common value $\dim_\mu(x)$ is called the pointwise dimension *of μ at x.*

When X is the shift space Σ of an SFT $\sigma : \Sigma \to \Sigma$ and d_X is d_u for some Hölder continuous $u : \Sigma \to (0, +\infty)$, we shall write $\underline{\dim}_{\mu,u}$, $\overline{\dim}_{\mu,u}$ and $\dim_{\mu,u}$ to denote $\underline{\dim}_\mu$, $\overline{\dim}_\mu$ and $\dim_\mu$ respectively.

Proposition 3.6. *Let (X, d_X) be a metric space, and μ be a Borel probability measure on X. Then, $\underline{\dim}_\mu : X \to \mathbb{R} \cup \{\pm\infty\}$ and $\overline{\dim}_\mu : X \to \mathbb{R} \cup \{\pm\infty\}$ are measurable functions.*

Proof. We shall only prove that $\underline{\dim}_\mu$ is measurable. The measurability of $\overline{\dim}_\mu$ can be shown in a similar way.

By the monotone convergence theorem, for every $x \in X$, $r \mapsto \mu(B(x,r))$ is a left continuous function, so we have

$$\inf_{r\in\mathbb{Q}\cap(0,1/n)} \frac{\log(\mu(B(x,r)))}{\log(r)} = \inf_{r\in(0,1/n)} \frac{\log(\mu(B(x,r)))}{\log(r)}.$$

Therefore, we have for every $x \in X$,

$$\underline{\dim}_\mu(x) = \sup_{n\in\mathbb{Z}_{>0}} \inf_{r\in(0,1/n)} \frac{\log(\mu(B(x,r)))}{\log(r)} = \sup_{n\in\mathbb{Z}_{>0}} \inf_{r\in\mathbb{Q}\cap(0,1/n)} \frac{\log(\mu(B(x,r)))}{\log(r)}.$$

As both the infimum and the supremum of countably many measurable functions are also measurable, we only need to show that $x \mapsto \mu(B(x,r))$ is a measurable for any $r > 0$.

Fix $r > 0$ arbitrarily. For the rest of this proof, we will show that the function $x \mapsto \mu(B(x,r))$ is lower-semicontinuous, which implies the measurability. Let x be an arbitrary point in X. Note that for any sequence $(x_n)_{n\in\mathbb{Z}_{>0}}$ in X converging to x, we have $B(x,r) \subseteq \bigcup_{k=1}^{\infty}\bigcap_{l=k}^{\infty} B(x_n,r)$. By Fatou's lemma, we thus have

$$\mu(B(x,r)) \le \mu\left(\bigcup_{k=1}^{\infty}\bigcap_{l=k}^{\infty} B(x_n,r)\right) \le \liminf_{n\to+\infty} \mu(B(x_n,r)).$$

As this holds for any x and any sequence $(x_n)_{n\in\mathbb{Z}_{>0}}$ converging to x, we conclude that $x \mapsto \mu(B(x,r))$ is lower-semicontinuous. $\square$

In this dissertation, the lower pointwise dimension is more important than the upper pointwise dimension, in that the lower pointwise dimension can be used to estimate the Hausdorff dimensions of measures and sets. This will be made clear in Theorem 3.7. The upper pointwise dimension is known to be related to the packing dimension [PU10], which we will not consider in this dissertation.

Before stating Theorem 3.7, let us recall two notions from the measure theory. Let (X,μ) be a measure space; the σ-algebra is omitted here. Then, for any measurable function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, the essential infimum of f and the essential supremum of f are defined as

$$\operatorname*{ess\,inf}_{x\in X} f(x) = \sup\{\, s \in \mathbb{R} \mid \mu(f^{-1}([-\infty,s))) = 0 \,\};$$
$$\operatorname*{ess\,sup}_{x\in X} f(x) = \inf\{\, s \in \mathbb{R} \mid \mu(f^{-1}((s,+\infty])) = 0 \,\},$$

respectively.

Theorem 3.7 (cf. [PU10, Theorem 8.6.3 & Theorem 8.6.5]). *Let (X, d_X) be a separable metric space. Let μ be a Borel probability measure on X. Then, we have*

$$\underline{\dim}_H(\mu) = \operatorname*{ess\,inf}_{x \in X} \underline{\dim}_\mu(x);$$
$$\overline{\dim}_H(\mu) = \operatorname*{ess\,sup}_{x \in X} \underline{\dim}_\mu(x).$$

Moreover, for any Borel subset E of X, we have $\dim_H(E) \leq \sup_{x \in E} \underline{\dim}_\mu(x)$.

The corresponding theorems in [PU10] are stated for X being a Borel subset of a Euclidean space. As we shall apply Theorem 3.7 to the shift space Σ, we need Theorem 3.7 to be in this slightly more general form. As Theorem 3.7 does not directly follow from the assertions in [PU10], we provide a proof in Appendix A.

Now we turn our focus from the general case to the symbolic case. The following theorem is called the volume lemma. Theorem 9.1.11 in [PU10] gives similar statements in a slightly different setup; the underlying space of the dynamical system is not the shift space but a conformal repeller in a Euclidean space. Therefore, we shall give a proof.

We first prove an elementary fact in real analysis.

Lemma 3.8. *Let $(X, \mathcal{A}, \mu)$ be a probability space. Let $f_1 : X \to \mathbb{R}$ and $f_2 : X \to (0, +\infty)$ be μ-integrable functions. Then,*

$$\mu\left(\left\{ x \in X \;\middle|\; \frac{f_1(x)}{f_2(x)} \leq \frac{\int_X f_1 \, \mathrm{d}\mu}{\int_X f_2 \, \mathrm{d}\mu} \right\}\right) > 0;$$
$$\mu\left(\left\{ x \in X \;\middle|\; \frac{f_1(x)}{f_2(x)} \geq \frac{\int_X f_1 \, \mathrm{d}\mu}{\int_X f_2 \, \mathrm{d}\mu} \right\}\right) > 0.$$

Moreover, the following statements are equivalent.

1. $f_1/f_2 \leq \int_X f_1 \, \mathrm{d}\mu / \int_X f_2 \, \mathrm{d}\mu$, *$\mu$-a.e.;*
2. $f_1/f_2 \geq \int_X f_1 \, \mathrm{d}\mu / \int_X f_2 \, \mathrm{d}\mu$, *$\mu$-a.e.;*
3. $f_1/f_2 = \int_X f_1 \, \mathrm{d}\mu / \int_X f_2 \, \mathrm{d}\mu$, *$\mu$-a.e.*

Proof. Note that

$$\int_X f_1 \, \mathrm{d}\mu = \int_X \frac{\int_X f_1 \, \mathrm{d}\mu}{\int_X f_2 \, \mathrm{d}\mu} f_2 \, \mathrm{d}\mu.$$

All our claims follow from this observation. □

Theorem 3.9 (cf. [PU10, Theorem 9.1.11]). *Let $\sigma : \Sigma \to \Sigma$ be an SFT, $u : \Sigma \to (0, +\infty)$ be a Hölder continuous function and μ be a σ-invariant Borel probability measure on Σ. Then, $\underline{\dim}_{\mu,u} = \overline{\dim}_{\mu,u}$ μ-a.e., and*

$$\underline{\dim}_u(\mu) \leq \frac{h_{KS}(\mu)}{\int_\Sigma u \, \mathrm{d}\mu} \leq \overline{\dim}_u(\mu), \tag{3.6}$$

where the first inequality is an equality if and only if the second is an equality. When μ is ergodic, we further have

$$\dim_u(\mu) = \underline{\dim}_u(\mu) = \overline{\dim}_u(\mu) = \frac{h_{KS}(\mu)}{\int_\Sigma u \, \mathrm{d}\mu}. \tag{3.7}$$

Proof. Define $r_0 = \exp(-\sup_{\xi \in \Sigma} u(\xi))$. Then, for any $\xi \in \Sigma$ and any $r \in (0, r_0)$, there is a unique positive integer $n(r, \xi)$ such that

$$[\xi_1 \cdots \xi_{n(r,\xi)+1}] \subseteq B(\xi, r) \subseteq [\xi_1 \cdots \xi_{n(r,\xi)}].$$

From the inclusion relations above and the definition of d_u, we have for every $\xi \in \Sigma$ and every $r \in (0, r_0)$,

$$\frac{\mathcal{I}_{n(r,\xi)}(\xi)}{S_{n(r,\xi)+1}u(\xi)} \leq \frac{\log(\mu(B(x,r)))}{\log(r)} \leq \frac{\mathcal{I}_{n(r,\xi)+1}(\xi)}{S_{n(r,\xi)}u(\xi) + V_u}, \tag{3.8}$$

where $\mathcal{I}_n(\xi) = -\log(\mu([\xi_1 \cdots \xi_n]))$ for any $n \in \mathbb{Z}_{>0}$ and any $\xi \in \Sigma$, as in the statement of Shannon-McMillan-Breiman theorem. Since u is Hölder continuous, the distortion constant V_u is finite. By Shannon-McMillan-Breiman theorem and Birkhoff's ergodic theorem, we have

$$\lim_{n \to +\infty} \frac{\mathcal{I}_{n+1}(\xi)}{S_n u(\xi) + V_u} = \frac{\lim_{n\to+\infty} \mathcal{I}_{n+1}(\xi)/n}{\lim_{n\to+\infty}(S_n u(\xi) + V_u)/n} = \frac{\lim_{n\to+\infty} \mathcal{I}_n(\xi)/n}{\lim_{n\to+\infty} S_n u(\xi)/n}$$

for μ-a.e. $\xi \in \Sigma$. Therefore, on the one hand, we have that for any $\xi \in \Sigma$ at which the previous equality holds,

$$\begin{aligned}
\overline{\dim}_{\mu,u}(\xi) &= \limsup_{r \to 0^+} \frac{\log(\mu(B(x,r)))}{\log(r)} \\
&\leq \limsup_{r \to 0^+} \frac{\mathcal{I}_{n(r,\xi)+1}(\xi)}{S_{n(r,\xi)}u(\xi) + V_u} \\
&= \lim_{n \to +\infty} \frac{\mathcal{I}_{n+1}(\xi)}{S_n u(\xi) + V_u} = \frac{\lim_{n\to+\infty} \mathcal{I}_n(\xi)/n}{\lim_{n\to+\infty} S_n u(\xi)/n}.
\end{aligned}$$

The second to last equality here holds because for any $\xi \in \Sigma$, the image of $n(\cdot, \xi)$ contains all sufficiently large integers and $\lim_{r\to 0^+} n(r,\xi) = +\infty$. On the other hand, using a similar argument, we have for μ-a.e. $\xi \in \Sigma$,

$$\underline{\dim}_{\mu,u}(\xi) \geq \frac{\lim_{n\to+\infty} \mathcal{I}_n(\xi)/n}{\lim_{n\to+\infty} S_n u(\xi)/n}.$$

Therefore, for μ-a.e. $\xi \in \Sigma$, $\dim_{\mu,u}(\xi)$ exists and

$$\dim_{\mu,u}(\xi) = \frac{\lim_{n\to+\infty} \mathcal{I}_n(\xi)/n}{\lim_{n\to+\infty} S_n u(\xi)/n}. \tag{3.9}$$

By Shannon-McMillan-Breiman theorem and Birkhoff's ergodic theorem, we have

$$\int_\Sigma \lim_{n\to+\infty} \frac{1}{n}\mathcal{I}_n(\xi)\,\mathrm{d}\mu = h_{KS}(\mu) \text{ and } \int_\Sigma \lim_{n\to+\infty} \frac{1}{n} S_n u(\xi)\,\mathrm{d}\mu = \int_\Sigma u\,\mathrm{d}\mu.$$

Hence, by Theorem 3.7 and Lemma 3.8, we have

$$\begin{aligned}
\underline{\dim}_u(\mu) &= \operatorname*{ess\,inf}_{\xi\in\Sigma} \dim_{\mu,u}(\xi) = \operatorname*{ess\,inf}_{\xi\in\Sigma} \frac{\lim_{n\to+\infty} \mathcal{I}_n(\xi)/n}{\lim_{n\to+\infty} S_n u(\xi)/n} \leq \frac{h_{KS}(\mu)}{\int_\Sigma u\,\mathrm{d}\mu};\\
\overline{\dim}_u(\mu) &= \operatorname*{ess\,sup}_{\xi\in\Sigma} \dim_{\mu,u}(\xi) = \operatorname*{ess\,sup}_{\xi\in\Sigma} \frac{\lim_{n\to+\infty} \mathcal{I}_n(\xi)/n}{\lim_{n\to+\infty} S_n u(\xi)/n} \geq \frac{h_{KS}(\mu)}{\int_\Sigma u\,\mathrm{d}\mu}.
\end{aligned}$$

Hence, (3.6) is proven. Lemma 3.8 also implies that the first inequality in (3.6) is an equality if and only if the second inequality in (3.6) is an equality.

Lastly, if μ is ergodic, then we can see from the Shannon-McMillan-Breiman theorem and Birkhoff's ergodic theorem that

$$\dim_{\mu,u}(\xi) = \frac{\lim_{n\to+\infty} \mathcal{I}_n(\xi)/n}{\lim_{n\to+\infty} S_n u(\xi)/n} = \frac{h_{KS}(\mu)}{\int_\Sigma u\,\mathrm{d}\mu},$$

for μ-a.e. $\xi \in \Sigma$. Therefore, by Theorem 3.7, we conclude that (3.7) holds when μ is ergodic. $\square$

Remark 3.10. If the measure μ in Theorem 3.9 is further the equilibrium state ν_ψ for some Hölder continuous $\psi : \Sigma \to \mathbb{R}$, then the lower and upper pointwise dimensions of ν_ψ can be written in the following way:

$$\underline{\dim}_{\nu_\psi,u}(\xi) = \liminf_{n\to+\infty} \frac{S_n\psi(\xi) - nP(\psi)}{-S_n u(\xi)}; \tag{3.10}$$

$$\overline{\dim}_{\nu_\psi,u}(\xi) = \limsup_{n\to+\infty} \frac{S_n\psi(\xi) - nP(\psi)}{-S_n u(\xi)}, \tag{3.11}$$

for every $\xi \in \Sigma$. Note that different from (3.9) which is stated for μ-a.e. $\xi \in \Sigma$, (3.10) and (3.11) are state for all $\xi \in \Sigma$. Let $\mathcal{I}_n(\xi) = -\log(\nu_\psi([\xi_1 \cdots \xi_n]))$ for any $n \in \mathbb{Z}_{>0}$ and $\xi \in \Sigma$. Then, the key ingredients to show (3.10) and (3.11) are the inequalities in (3.8) and the observation that

$$\sup_{n \in \mathbb{Z}_{>0}} \sup_{\xi \in \Sigma} \left| -\mathcal{I}_n(\xi) - (S_n\psi(\xi) - nP(\psi)) \right| < +\infty,$$

which is a consequence of the fact that ν_ϕ is a Gibbs measure for ϕ.

3.4 u-Dimension of Shift Space

In this section, we describe the u-dimension of the shift space of an SFT, using the notions from thermodynamic formalism. The formula for the u-dimension of the shift space will be given in Theorem 3.12.

Lemma 3.11. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let $\phi : \Sigma \to \mathbb{R}$ and $u : \Sigma \to (0, +\infty)$ be Hölder continuous functions. Then, there exists a unique $\beta \in \mathbb{R}$ such that $P(\phi - \beta u) = 0$.*

Proof. Since u is a positive function, $P(\phi - tu)$ monotonically decreases as t increases. Therefore, there is at most one $\beta \in \mathbb{R}$ such that $P(\phi - \beta u) = 0$.

To show the existence of β, take real numbers $t_1 < t_2$ such that $\phi - t_1 u \geq 0$ and $\phi - t_2 u \leq -\log \# A$. Then, we have

$$P(\phi - t_1 u) \geq P(0) = \lim_{n \to +\infty} \frac{1}{n} \log(\# A_{\mathbb{M}}^n \cdot \exp(0)) \geq 0;$$
$$P(\phi - t_2 u) \leq P(-\log \# A) = \lim_{n \to +\infty} \frac{1}{n} \log(\# A_{\mathbb{M}}^n \cdot (\# A)^{-n}) \leq 0.$$

Since the function $t \mapsto P(\phi - tu)$ is continuous, we can thus conclude from the intermediate value theorem that there is one $\beta \in [t_1, t_2]$ such that $P(\phi - \beta u) = 0$. This completes the proof. □

The u-dimension of the shift space can be characterized by the topological pressure in the following manner.

Theorem 3.12 ([BS00, Proposition 6.4])**.** *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT, and $u : \Sigma \to (0, +\infty)$ be a Hölder continuous function. Then, there exists a unique $\beta > 0$ such that $P(-\beta u) = 0$, and we have* $\dim_u(\Sigma) = \beta$.

This type of result was first discovered by Bowen in his study on the Hausdorff dimension of quasi-circles [Bow79].

Proof. By Lemma 3.11, there is a unique $\beta \in \mathbb{R}$ such that $P(-\beta u) = 0$. The number t_1 in the proof of Lemma 3.11 can be taken as 0, where ϕ is set as the constant function 0. Then, as in the proof of Lemma 3.11, we have $\beta \geq t_1 = 0$.

In order to prove that β cannot be zero, it suffices to show that $P(0) > 0$. Since $\sigma : \Sigma \to \Sigma$ is topologically mixing, by Proposition 2.2, there is some positive integer l such that for any $a, b \in A$, there exists some admissible word ρ of length l such that $a\rho b$ is admissible. This means that for any integer $k \geq 2$ and any $a_1, \cdots, a_k \in A$, there are $\rho^{(1)}, \cdots, \rho^{(k-1)} \in A^l_{\mathbb{M}}$ such that the word $a_1\rho^{(1)}a_2 \cdots a_{k-1}\rho^{(k-1)}a_k$, whose length is clearly $kl + k - l$, is admissible. Hence, we have $\#A^{kl+k-l}_{\mathbb{M}} \geq (\#A)^k$. It thus follows that

$$P(0) = \lim_{k\to+\infty} \frac{\log(\#A^{kl+k-l}_{\mathbb{M}})}{kl + k - l} \geq \lim_{k\to+\infty} \frac{k\log(\#A)}{kl + k - l} = \frac{\log(\#A)}{l + 1}.$$

As we always assume that A contains at least two elements, we have $P(0) > 0$.

Take $\nu_{-\beta u}$ as the equilibrium state for $-\beta u$. Then, using the fact that $\nu_{-\beta u}$ is a Gibbs measure, we have for any $\xi \in \Sigma$,

$$\dim_{\nu_{-\beta u}}(\xi) = \lim_{n\to+\infty} \frac{-\beta S_n u(\xi)}{-S_n u(\xi)} = \beta.$$

Thus, from Theorem 3.7, we can conclude that $\dim_u(\Sigma) = \beta$. □

Chapter 4

Multifractal Analysis for Symbolic Dynamics

In this chapter, we shall conduct the multifractal analysis of level sets and uniform level sets in symbolic dynamics. To be more precise, fix an arbitrary real number α. We shall estimate the u-dimensions of the α-level set $\mathcal{L}^{\alpha}_{\psi,v}$, the uniform α-level set $\mathcal{UL}^{\alpha}_{\psi,v}$ and two types of subsets of $\mathcal{UL}^{\alpha}_{\psi,v}$. We handle $\mathcal{L}^{\alpha}_{\psi,v}$ in Section 4.1, following the ideas in [PW97; Sch99]. In Section 4.2, we treat the u-dimensions of $\mathcal{UL}^{\alpha}_{\psi,v}$ and two types of subsets of it. Section 4.2 contains the work of the author in [Liu23].

4.1 Multifractal Analysis of Level Sets

4.1.1 u-Dimensions of Level Sets

In this section, we study the u-dimension of

$$\mathcal{L}^{\alpha}_{\psi,v} = \left\{ \xi \in \Sigma \;\middle|\; \lim_{n\to+\infty} \frac{S_n\psi(\xi)}{S_n v(\xi)} = \alpha \right\},$$

for any $\alpha \in \mathbb{R}$. For any $\alpha \in \mathbb{R}$ and $q \in \mathbb{R}$, Lemma 3.11 guarantees that there is a unique $\beta_\alpha(q) \in \mathbb{R}$ such that $P(q(\psi - \alpha v) - \beta_\alpha(q)u) = 0$. For each α, the u-dimension of $\mathcal{L}^{\alpha}_{\psi,v}$ can be characterized by the function $\beta_\alpha : \mathbb{R} \to \mathbb{R}$ in the following way.

Theorem 4.1 (cf. [PW97; Sch99]). *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let $\psi : \Sigma \to \mathbb{R}$, $v : \Sigma \to (0, +\infty)$ and $u : \Sigma \to (0, +\infty)$ be Hölder continuous functions. Then, $\mathcal{L}^{\alpha}_{\psi,v}$ is non-empty if and only if $\alpha^{-}_{\psi,v} \le \alpha \le \alpha^{+}_{\psi,v}$.*

If $\alpha^{-}_{\psi,v} = \alpha^{+}_{\psi,v}$, then $\mathcal{L}^{\alpha}_{\psi,v} = \varnothing$ for $\alpha \neq \alpha^{-}_{\psi,v}$, and $\mathcal{L}^{\alpha}_{\psi,v} = \Sigma$ for $\alpha = \alpha^{-}_{\psi,v}$.

If $\alpha^{-}_{\psi,v} < \alpha^{+}_{\psi,v}$, then the following statements hold.

1. *For any $\alpha \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$, there exists a unique $q_\alpha \in \mathbb{R}$ such that*

$$\beta'_\alpha(q_\alpha) = \frac{\int_\Sigma (\psi - \alpha v)\,\mathrm{d}\nu^\alpha}{\int_\Sigma u\,\mathrm{d}\nu^\alpha} = 0,$$

where ν^α is the equilibrium state for the potential $q_\alpha(\psi - \alpha v) - \beta_\alpha(q_\alpha)u$. We have that $\nu^\alpha(\mathcal{L}^\alpha_{\psi,v}) = 1$ and

$$\dim_u(\mathcal{L}^\alpha_{\psi,v}) = \dim_u(\nu^\alpha) = \min_{q\in\mathbb{R}} \beta_\alpha(q) = \beta_\alpha(q_\alpha).$$

2. *For any $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$, there is a σ-invariant Borel probability measure ν^α such that $\nu^\alpha(\mathcal{L}^\alpha_{\psi,v}) = 1$ and*

$$\dim_u(\mathcal{L}^\alpha_{\psi,v}) = \dim_u(\nu^\alpha) = \inf_{q\in\mathbb{R}} \beta_\alpha(q).$$

Theorem 4.2 (cf. [Sch99])**.** *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let $\psi : \Sigma \to \mathbb{R}$, $v : \Sigma \to (0, +\infty)$ and $u : \Sigma \to (0, +\infty)$ be Hölder continuous functions. Then, the function $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$ is real analytic on the open interval $(\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ and continuous on the compact interval $[\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$.*

The proofs of Theorem 4.1 and Theorem 4.2 will be given in the next subsection.

When $v = u$, in addition to all the properties stated in Theorem 4.1, we also have the concavity of the dimension spectrum $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$ on $[\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$.

Corollary 4.3 ([Sch99, Theorem 2.1])**.** *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let $\psi : \Sigma \to \mathbb{R}$ and $u : \Sigma \to (0, +\infty)$ be Hölder continuous functions. Suppose that $\alpha^-_{\psi,u} < \alpha^+_{\psi,u}$. Define $\beta_0 : \mathbb{R} \to \mathbb{R}$ by letting $P(q\psi - \beta_0(q)u) = 0$ for each $q \in \mathbb{R}$. Then, we have*

1. *for any $\alpha \in (\alpha^-_{\psi,u}, \alpha^-_{\psi,u})$, there exists a unique $q_\alpha \in \mathbb{R}$ such that $\beta'_0(q_\alpha) = \alpha$; for this q_α,*

$$\dim_u(\mathcal{L}^\alpha_{\psi,u}) = \min_{q\in\mathbb{R}} \beta_0(q) - \alpha q = \beta_0(q_\alpha) - \alpha q_\alpha;$$

2. *for any $\alpha \in \{\, \alpha^-_{\psi,u}, \alpha^+_{\psi,u} \,\}$, $\dim_u(\mathcal{L}^\alpha_{\psi,u}) = \inf_{q\in\mathbb{R}} \beta_0(q) - \alpha q$;*

3. *the function $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,u})$ is concave on $[\alpha^-_{\psi,u}, \alpha^+_{\psi,u}]$.*

Proof. As in Theorem 4.1, for each $\alpha \in [\alpha^-_{\psi,u}, \alpha^+_{\psi,u}]$, define $\beta_\alpha : \mathbb{R} \to \mathbb{R}$ by letting $P(q(\psi - \alpha u) - \beta_\alpha(q)u) = 0$ for every $q \in \mathbb{R}$. For each α, comparing the definitions of β_α and β_0, we immediately see that $\beta_\alpha(q) = \beta_0(q) - \alpha q$ for every $q \in \mathbb{R}$. Hence, the first two items are direct consequences of Theorem 4.1. The last item follows from the properties of Legendre transform; see Theorem 12.2 in [Roc70] for the details. $\square$

We conclude this section with the following proposition which gives information on α for which $\dim_u(\mathcal{L}^{\alpha}_{\psi,v})$ attains $\dim_u(\Sigma)$.

Proposition 4.4. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let $\psi : \Sigma \to \mathbb{R}$, $v : \Sigma \to (0,+\infty)$ and $u : \Sigma \to (0,+\infty)$ be Hölder continuous functions. Then there is a unique $\alpha \in \mathbb{R}$ such that $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) = \dim_u(\Sigma)$. Moreover, this α satisfies either $\alpha^-_{\psi,v} = \alpha = \alpha^+_{\psi,v}$ or $\alpha^-_{\psi,v} < \alpha < \alpha^+_{\psi,v}$.*

The proof of this proposition is also postponed to the next subsection.

4.1.2 Proofs

We begin with the proof of Theorem 4.1. As we shall see in the proof of Theorem 4.1, the properties of the function β_α we need can be deduced from the following lemma.

Lemma 4.5. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT, and $\phi : \Sigma \to \mathbb{R}$ and $u : \Sigma \to (0,+\infty)$ be Hölder continuous functions. For every $q \in \mathbb{R}$, set $\beta(q)$ as the unique real number such that $P(q\phi - \beta(q)u) = 0$. Then, we have*

1. *$\beta : \mathbb{R} \to \mathbb{R}$ is real analytic and*
$$\beta'(q) = \frac{\int_\Sigma \phi \, \mathrm{d}\nu_{q\phi-\beta(q)u}}{\int_\Sigma u \, \mathrm{d}\nu_{q\phi-\beta(q)u}} \tag{4.1}$$
for any $q \in \mathbb{R}$, where $\nu_{q\phi-\beta(q)u}$ is the unique equilibrium state for $q\phi-\beta(q)u$;
2. *$\beta : \mathbb{R} \to \mathbb{R}$ is convex;*
3. *$\lim_{q\to\pm\infty} \beta(q) = +\infty$ if $\alpha^-_{\phi,1} < 0 < \alpha^+_{\phi,1}$.*

Proof. By (2.3) in Theorem 2.14, we have

$$\frac{\partial}{\partial b} P(q\phi - bu) = -\int_\Sigma u \, \mathrm{d}\nu_{q\phi-bu} \leq -\min_{\xi\in\Sigma} u(\xi) < 0,$$

for any $q, b \in \mathbb{R}$. Hence, by Theorem 2.14 and the analytic implicit function theorem, we have the real analyticity of the function $\beta : \mathbb{R} \to \mathbb{R}$. Again by Theorem 2.14,

$$\int_\Sigma (\phi - \beta'(q)u) \, \mathrm{d}\nu_{q\phi-\beta(q)u} = \frac{\mathrm{d}}{\mathrm{d}q} P(q\phi - \beta(q)u) = 0.$$

Rearranging the terms above, we obtain (4.1). Therefore, the first item is true.

To show the second item, take $q_1, q_2 \in \mathbb{R}$ and $s \in [0,1]$ arbitrarily. Let $q_0 = sq_1 + (1-s)q_2$. Then, we need to show that $\beta(q_0) \leq s\beta(q_1) + (1-s)\beta(q_2)$. By the convexity in Theorem 2.14, we have

$$\begin{aligned}&P(q_0\phi - (s\beta(q_1) + (1-s)\beta(q_2))u)\\&\leq sP(q_1\phi - \beta(q_1)u) + (1-s)P(q_2\phi - \beta(q_2)u) = 0 = P(q_0\phi - \beta(q_0)u).\end{aligned}$$

Since $u > 0$, we have $\beta(q_0) \leq s\beta(q_1) + (1-s)\beta(q_2)$. This shows the second item.

Now we show the last item. Suppose that $\alpha^-_{\phi,1} < 0 < \alpha^+_{\phi,1}$. Fix an arbitrary $q \in \mathbb{R}$. We shall show that $\beta(q) > 0$. Equivalently, we may instead prove $P(q\phi) > 0$. When $q = 0$, we have seen in the proof of Theorem 3.12 that $P(0) > 0$. For $q \neq 0$, as $\alpha^-_{\phi,1} < 0 < \alpha^+_{\phi,1}$, we have $\alpha^-_{q\phi,1} < 0 < \alpha^+_{q\phi,1}$. It thus follows that

$$\begin{aligned}P(q\phi) &= \lim_{n\to+\infty} \frac{1}{n} \log \sum_{\omega\in A^n_{\mathbb{M}}} \exp(S_\omega(q\phi))\\&\geq \sup_{\zeta\in\Sigma} \limsup_{n\to+\infty} \frac{qS_n\phi(\zeta)}{n} = \alpha^+_{q\phi,1} > 0.\end{aligned}$$

This shows our claim.

Since $\beta(q) \geq 0$, we have $q\phi - \beta(q)u \geq q\phi - \beta(q)\|u\|$. Hence,

$$\begin{aligned}\sup_{\xi\in\Sigma} \limsup_{n\to+\infty} \frac{qS_n\phi(\xi)}{n} - \beta(q)\|u\| &\leq P\left(q\phi - \beta(q)\|u\|\right)\\&\leq P\left(q\phi - \beta(q)u\right) = 0.\end{aligned}$$

Therefore, we have $\beta(q) \geq q\alpha^+_{\phi,1}/\|u\|$ for $q \geq 0$, and $\beta(q) \geq q\alpha^-_{\phi,1}/\|u\|$ for $q \leq 0$. As $\alpha^-_{\phi,1} < 0 < \alpha^+_{\phi,1}$, we deduce that $\lim_{q\to\pm\infty} \beta(q) = +\infty$. □

Now we move on to the proof of Theorem 4.1. We shall basically follow the proof ideas in [PW97] and [Sch99].

Proof of Theorem 4.1. Fix an arbitrary $\alpha \in \mathbb{R}$. Define $\psi_\alpha = \psi - \alpha v$.

It is clear from the definitions of $\alpha^-_{\psi,v}$ and $\alpha^+_{\psi,v}$ that $\mathcal{L}^\alpha_{\psi,v}$ is empty for any $\alpha \notin [\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$. For $\alpha \in [\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$, we shall show that there is a probability measure ν^α such that $\nu^\alpha(\mathcal{L}^\alpha_{\psi,v}) = 1$, which in particular implies that $\mathcal{L}^\alpha_{\psi,v}$ is non-empty.

The statements for the case where $\alpha^-_{\psi,v} = \alpha^+_{\psi,v}$ are straightforward, so we only need to handle the case where $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$. For the rest of the proof, assume that $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$. Let α be an arbitrary number in the closed interval $[\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$.

Consider the case where $\alpha^-_{\psi,v} < \alpha < \alpha^+_{\psi,v}$. Then, $\alpha^-_{\psi_\alpha,v} < 0 < \alpha^+_{\psi_\alpha,v}$. Hence, by Lemma 4.5 with ϕ taken to be ψ_α, β_α is real analytic and convex and

$\lim_{q\to\pm\infty} \beta_\alpha(q) = +\infty$. Consequently, by the mean value theorem, there is some $q_\alpha \in \mathbb{R}$ such that $\beta'_\alpha(q_\alpha) = 0$. This q_α is further unique. To see this, assume that there is another $q'_\alpha \neq q_\alpha$ satisfying $\beta'_\alpha(q'_\alpha) = 0$. Without loss of generality, we may assume that $q_\alpha < q'_\alpha$. Then, by the convexity of β_α, for all $q \in [q_\alpha, q'_\alpha]$, $\beta'_\alpha(q) = 0$. By the real analyticity of β_α, this implies that β_α is a constant function, which contradicts the fact that $\lim_{q\to\pm\infty} \beta_\alpha(q) = +\infty$. Therefore, there is only one $q_\alpha \in \mathbb{R}$ satisfying that $\beta'_\alpha(q_\alpha) = 0$. By (4.1) and Birkhoff's ergodic theorem, we then have

$$\lim_{n\to+\infty} \frac{S_n\psi_\alpha}{S_n u} = \frac{\int_\Sigma (\psi - \alpha v)\, \mathrm{d}\nu^\alpha}{\int_\Sigma u\, \mathrm{d}\nu^\alpha} = \beta'_\alpha(q_\alpha) = 0,\ \nu^\alpha\text{-a.e.}, \tag{4.2}$$

where ν^α is the equilibrium state for $q_\alpha\psi_\alpha - \beta_\alpha(q_\alpha)u$, which is ergodic. It is not hard to see that for any $\xi \in \Sigma$,

$$\lim_{n\to+\infty} \left| \frac{S_n\psi_\alpha(\xi)}{S_n u(\xi)} \right| = 0 \text{ if and only if } \lim_{n\to+\infty} \left| \frac{S_n\psi(\xi)}{S_n v(\xi)} - \alpha \right| = 0.$$

This implies that

$$\mathcal{L}^\alpha_{\psi,v} = \left\{ \xi \in \Sigma \;\middle|\; \lim_{n\to+\infty} \frac{S_n\psi_\alpha(\xi)}{S_n u(\xi)} = 0 \right\}. \tag{4.3}$$

Combining this observation with (4.2), we have $\nu^\alpha(\mathcal{L}^\alpha_{\psi,v}) = 1$. Note that from the observation in Remark 3.10, we have

$$\dim_{\nu^\alpha,u}(\xi) = \lim_{n\to+\infty} \frac{q_\alpha S_n\psi_\alpha(\xi) - \beta_\alpha(q_\alpha)S_n u(\xi)}{-S_n u(\xi)} = \beta_\alpha(q_\alpha),$$

for any $\xi \in \mathcal{L}^\alpha_{\psi,v}$. By Theorem 3.7, we thus have

$$\dim_u(\nu^\alpha) = \dim_u(\mathcal{L}^\alpha_{\psi,v}) = \beta_\alpha(q_\alpha).$$

As β_α is convex and $\beta'_\alpha(q_\alpha) = 0$, we further have $\min_{q\in\mathbb{R}} \beta_\alpha(q) = \beta_\alpha(q_\alpha)$.

Now let us consider the case where $\alpha = \alpha^-_{\psi,v}$. For $\alpha = \alpha^+_{\psi,v}$, one can make arguments similar to what we shall give below for $\alpha = \alpha^-_{\psi,v}$. For each $q \in \mathbb{R}$, let $\nu^{\alpha,q}$ denote the equilibrium state for $q\psi_\alpha - \beta_\alpha(q)u$. Then, by the weak* compactness of the space $\mathfrak{M}_\sigma(\Sigma)$ of all σ-invariant Borel probability measures on Σ, there exists a monotonically decreasing sequence $(q_k)_{k\in\mathbb{Z}_{>0}}$ diverging to $-\infty$ such that ν^{α,q_k} weak* converges to some $\nu^\alpha \in \mathfrak{M}_\sigma(\Sigma)$ as k approaches infinity.

We claim that $\nu^\alpha(\mathcal{L}^\alpha_{\psi,v}) = 1$. Towards this end, we first show that

$$\lim_{k\to+\infty} \beta'_\alpha(q_k) = \lim_{k\to+\infty} \frac{\int_\Sigma \psi_\alpha\, \mathrm{d}\nu^{\alpha,q_k}}{\int_\Sigma u\, \mathrm{d}\nu^{\alpha,q_k}} = 0. \tag{4.4}$$

The first equality is given by Lemma 4.5; the limits exist because β_α is convex. Since $\alpha = \alpha^-_{\psi,v}$, we have $\alpha^-_{\psi_\alpha,v} = 0$. Hence, by Theorem 2.16, we have $\int_\Sigma \psi_\alpha \,\mathrm{d}\nu^{\alpha,q} \geq 0$ for any $q \in \mathbb{R}$, so on the one hand, we have

$$\lim_{k\to+\infty} \beta'_\alpha(q_k) = \lim_{k\to+\infty} \frac{\int_\Sigma \psi_\alpha \,\mathrm{d}\nu^{\alpha,q_k}}{\int_\Sigma u \,\mathrm{d}\nu^{\alpha,q_k}} \geq 0.$$

We here remark that combining $\int_\Sigma \psi_\alpha \,\mathrm{d}\nu^{\alpha,q} \geq 0$ and Proposition 2.18, we further have

$$\int_\Sigma \psi_\alpha \,\mathrm{d}\nu^{\alpha,q} > 0, \tag{4.5}$$

for any $q \in \mathbb{R}$. On the other hand, note that by Theorem 2.16, $\alpha^-_{\psi_\alpha,v} = 0$ implies that $\alpha^-_{\psi_\alpha,1} = 0$, which further implies that $\alpha^+_{q\psi_\alpha,1} = 0$ for any $q < 0$. As in the proof of the last item in Lemma 4.5, we can show that $P(q\psi_\alpha) \geq \alpha^+_{q\psi_\alpha,1} = 0$ for any $q \in \mathbb{R}$. Therefore, $\inf_{q<0} \beta_\alpha(q) \geq 0$. As a consequence, we have

$$\lim_{k\to+\infty} \beta'_\alpha(q_k) = \lim_{q\to-\infty} \frac{\beta_\alpha(q)}{q} \leq 0.$$

Therefore, (4.4) is true. Using Birkhoff's ergodic theorem and (4.4), we have

$$\frac{\int_\Sigma \lim_{n\to+\infty} S_n\psi/n \,\mathrm{d}\nu^\alpha}{\int_\Sigma \lim_{n\to+\infty} S_n v/n \,\mathrm{d}\nu^\alpha} = \frac{\int_\Sigma \psi \,\mathrm{d}\nu^\alpha}{\int_\Sigma v \,\mathrm{d}\nu^\alpha} = \lim_{k\to+\infty} \frac{\int_\Sigma \psi \,\mathrm{d}\nu^{\alpha,q_k}}{\int_\Sigma v \,\mathrm{d}\nu^{\alpha,q_k}} = \alpha.$$

Note that

$$\frac{\lim_{n\to+\infty} S_n\psi/n}{\lim_{n\to+\infty} S_n v/n} = \lim_{n\to+\infty} \frac{S_n\psi}{S_n v} \geq \alpha^-_{\psi,v} = \alpha,\ \nu^\alpha\text{-a.e.}$$

By Lemma 3.8, we thus obtain that

$$\lim_{n\to+\infty} \frac{S_n\psi}{S_n v} = \frac{\lim_{n\to+\infty} S_n\psi/n}{\lim_{n\to+\infty} S_n v/n} = \alpha,\ \nu^\alpha\text{-a.e.}$$

This is equivalent to $\nu^\alpha(\mathcal{L}^\alpha_{\psi,v}) = 1$.

Next we evaluate the u-dimension of ν^α as well as the u-dimension of $\mathcal{L}^\alpha_{\psi,v}$. By Theorem 3.9 and the upper semi-continuity of the entropy map, we have, on the one hand,

$$\begin{aligned}\dim_u(\mathcal{L}^\alpha_{\psi,v}) \geq \overline{\dim}_u(\nu^\alpha) &\geq \frac{h_{KS}(\nu^\alpha)}{\int_\Sigma u \,\mathrm{d}\nu^\alpha} \\ &\geq \limsup_{k\to+\infty} \frac{h_{KS}(\nu^{\alpha,q_k})}{\int_\Sigma u \,\mathrm{d}\nu^{\alpha,q_k}} \\ &= \limsup_{k\to+\infty} \beta_\alpha(q_k) - \frac{q_k \int_\Sigma \psi_\alpha \,\mathrm{d}\nu^{\alpha,q_k}}{\int_\Sigma u \,\mathrm{d}\nu^{\alpha,q_k}} \geq \inf_{q\in\mathbb{R}} \beta_\alpha(q).\end{aligned}$$

On the other hand, by Remark 3.10, for any $\xi \in \mathcal{L}^{\alpha}_{\psi,v}$ and any $q \in \mathbb{R}$,

$$\dim_{\nu^{\alpha,q},u}(\xi) = \lim_{n\to+\infty} \frac{qS_n\psi_\alpha(\xi) - \beta_\alpha(q)S_n u(\xi)}{-S_n u(\xi)} = \beta_\alpha(q).$$

Therefore, by the last claim of Theorem 3.7, we have $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) \le \inf_{q\in\mathbb{R}} \beta_\alpha(q)$. Hence, $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) = \overline{\dim}_u(\nu^\alpha) = h_{KS}(\nu^\alpha)/\int_\Sigma u\,\mathrm{d}\nu^\alpha = \inf_{q\in\mathbb{R}} \beta_\alpha(q)$. Combining this with Theorem 3.9, we also have $\underline{\dim}_u(\nu^\alpha) = \overline{\dim}_u(\nu^\alpha)$, so the proof of the second item for the case where $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$ is complete. □

Proof of Theorem 4.2. Without loss of generality, we assume that $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$. Throughout this proof, for any $\alpha \in \mathbb{R}$, we still use ψ_α to denote the function $\psi - \alpha v$. We shall begin with the proof of the real analyticity. An argument similar to the proof we shall present below can be found in [IJ15].

Consider the function $(q, \alpha, b) \mapsto P(q\psi_\alpha - bu)$. As in the proof of the first claim of Lemma 4.5, we deduce from (2.3) in Theorem 2.14 that for any $q, \alpha, b \in \mathbb{R}$,

$$\frac{\partial}{\partial b} P(q\psi_\alpha - bu) = -\int_\Sigma u\,\mathrm{d}\nu_{q\psi_\alpha - bu} < 0,$$

where $\nu_{q\psi_\alpha - bu}$ is the equilibrium state for $q\psi_\alpha - bu$. Hence, by Theorem 2.14 and the analytic implicit function theorem, $(q, \alpha) \mapsto \beta_\alpha(q)$ is real analytic.

Now we claim that the function $\alpha \mapsto q_\alpha$ with its domain being $(\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$, which is defined in Theorem 4.1, is real analytic as well. Towards this end, we first show

$$\frac{\partial^2}{\partial q^2} P(q\psi_\alpha - bu) > 0, \tag{4.6}$$

for any $\alpha \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$, any $q \in \mathbb{R}$ and any $b \in \mathbb{R}$. Suppose otherwise. By the convexity stated in Theorem 2.14, the second derivative above cannot be negative. Hence, there exist $\alpha^* \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ and $q^*, b^* \in \mathbb{R}$ such that

$$\frac{\partial^2}{\partial q^2} P(q\psi_{\alpha^*} - b^* u)\big|_{q=q^*} = 0.$$

Combining this with Theorem 2.15, we have that ψ_{α^*} is cohomologous to a constant, say $C \in \mathbb{R}$. Thus, by (4.1) in Lemma 4.5, for any $q \in \mathbb{R}$,

$$\frac{\partial \beta_{\alpha^*}(q)}{\partial q} = \frac{\int_\Sigma \psi_{\alpha^*}\,\mathrm{d}\nu_{q\psi_{\alpha^*} - \beta_{\alpha^*}(q)}}{\int_\Sigma u\,\mathrm{d}\nu_{q\psi_{\alpha^*} - \beta_{\alpha^*}(q)}} = \frac{C}{\int_\Sigma u\,\mathrm{d}\nu_{q\psi_{\alpha^*} - \beta_{\alpha^*}(q)}}.$$

By taking $q = q_{\alpha^*}$, we have

$$\frac{C}{\int_\Sigma u\,\mathrm{d}\nu_{q_{\alpha^*}\psi_{\alpha^*} - \beta_{\alpha^*}(q_{\alpha^*})}} = \frac{\partial \beta_{\alpha^*}(q)}{\partial q}\Big|_{q=q_{\alpha^*}} = 0.$$

Consequently, $C = 0$. It thus follows that for any $\xi \in \Sigma$,

$$\lim_{n\to+\infty} \frac{S_n\psi(\xi)}{S_n v(\xi)} = \alpha^*.$$

This means that $\alpha^-_{\psi,v} = \alpha^+_{\psi,v} = \alpha^*$, which contradicts the assumption that $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$. Therefore, (4.6) is proved.

Fix an arbitrary $\alpha_0 \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$. Note that, on the one hand, since

$$P(q\psi_{\alpha_0} - \beta_{\alpha_0}(q)u) = 0$$

for any $q \in \mathbb{R}$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}q^2} P(q\psi_{\alpha_0} - \beta_{\alpha_0}(q)u) = 0 \tag{4.7}$$

for any $q \in \mathbb{R}$. On the other hand, let $\tilde{P}(q, b) = P(q\psi_{\alpha_0} - bu)$ be a function of two variables q and b. Then, for any $q \in \mathbb{R}$,

$$\begin{aligned}
\frac{\mathrm{d}^2 \tilde{P}(q, \beta_{\alpha_0}(q))}{\mathrm{d}q^2} &= \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\partial \tilde{P}}{\partial q}(q, \beta_{\alpha_0}(q)) + \frac{\partial \tilde{P}}{\partial b}(q, \beta_{\alpha_0}(q)) \cdot \beta'_{\alpha_0}(q) \right) \\
&= \frac{\partial^2 \tilde{P}}{\partial q^2}(q, \beta_{\alpha_0}(q)) + \frac{\partial^2 \tilde{P}}{\partial q \partial b}(q, \beta_{\alpha_0}(q)) \cdot \beta'_{\alpha_0}(q) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\partial \tilde{P}}{\partial b} \right)(q, \beta_{\alpha_0}(q)) \cdot \beta'_{\alpha_0}(q) + \frac{\partial \tilde{P}}{\partial b}(q, \beta_{\alpha_0}(q)) \cdot \beta''_{\alpha_0}(q).
\end{aligned}$$

Take $q = q_{\alpha_0}$. Then, since $\beta'_{\alpha_0}(q_{\alpha_0}) = 0$, we have

$$\frac{\mathrm{d}^2 \tilde{P}(q, \beta_{\alpha_0}(q))}{\mathrm{d}q^2}\Big|_{q=q_{\alpha_0}} = \frac{\partial^2 \tilde{P}}{\partial q^2}(q_{\alpha_0}, \beta_{\alpha_0}(q_{\alpha_0})) + \frac{\partial \tilde{P}}{\partial b}(q_{\alpha_0}, \beta_{\alpha_0}(q_{\alpha_0})) \cdot \beta''_{\alpha_0}(q_{\alpha_0}).$$

Combining this with (4.6) and (4.7), we have

$$\beta''_{\alpha_0}(q_{\alpha_0}) \cdot \frac{\partial \tilde{P}}{\partial b}(q_{\alpha_0}, \beta_{\alpha_0}(q_{\alpha_0})) = -\frac{\partial^2 \tilde{P}}{\partial q^2}(q_{\alpha_0}, \beta_{\alpha_0}(q_{\alpha_0})) < 0.$$

Therefore, $\beta''_{\alpha_0}(q_{\alpha_0}) \neq 0$. Since α_0 is taken arbitrary from $(\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$, we deduce that

$$\frac{\partial^2 \beta_\alpha(q)}{\partial q^2}\Big|_{q=q_\alpha} \neq 0$$

for any $\alpha \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$. As $(q, \alpha) \mapsto \beta_\alpha(q)$ is real analytic on $\mathbb{R}^2$, by the analytic implicit function theorem, we have that $\alpha \mapsto q_\alpha$ is a real analytic function on the open interval $(\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$.

Now that we have shown that $\alpha \mapsto q_\alpha$ is a real analytic function on $(\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ and $(q, \alpha) \mapsto \beta_\alpha(q)$ is a real analytic function on $\mathbb{R}^2$, we have that the function $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v}) = \beta_\alpha(q_\alpha)$ is real analytic on $(\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ as well.

The analyticity of $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$ on $(\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ implies the continuity on this open interval. Therefore, we only need to show the continuity of the function $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$, with its domain restricted to $[\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$, at $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$. This means that we need to show the right continuity of $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$ at $\alpha^-_{\psi,v}$ and the left continuity of $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$ at $\alpha^+_{\psi,v}$. In what follows, we shall only prove the right continuity at $\alpha^-_{\psi,v}$; the proof of the left continuity at $\alpha^+_{\psi,v}$ is similar. For the rest of this proof, we shall also use α^- to denote $\alpha^-_{\psi,v}$ for simplicity.

We divide the proof of the right continuity of $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$ at α^- into two steps. In this paragraph, we shall show the lower semi-continuity, and leave the proof of the upper semi-continuity to the next paragraph. As $(\alpha, q) \mapsto \beta_\alpha(q)$ is real analytic on $\mathbb{R}^2$, we have that $(\alpha, q) \mapsto \beta'_\alpha(q)$ is continuous on $\mathbb{R}^2$. From (4.5), we see that $\beta'_{\alpha^-}(q) > 0$ for any $q \in \mathbb{R}$. Thus, by the continuity of $(\alpha, q) \mapsto \beta'_\alpha(q)$, we have that for $\alpha' \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ sufficiently close to $\alpha^- = \alpha^-_{\psi,v}$, $\beta'_{\alpha'}(-1) \geq 0 = \beta'_{\alpha'}(q_{\alpha'})$. By the convexity of $\beta_{\alpha'}$, we have $q_{\alpha'} \leq -1$. Hence, for such α' and any $\xi \in \mathcal{L}^{\alpha^-}_{\psi,v}$, we have

$$\begin{aligned}
\underline{\dim}_{\nu^{\alpha'},u}(\xi) &= \liminf_{n\to+\infty} \frac{q_{\alpha'} S_n\psi_{\alpha'}(\xi) - \beta_{\alpha'}(q_{\alpha'}) S_n u(\xi)}{-S_n u(\xi)} \\
&= \beta_{\alpha'}(q_{\alpha'}) + \liminf_{n\to+\infty} \frac{-q_{\alpha'}(\alpha^- - \alpha') S_n v(\xi)}{S_n u(\xi)} \\
&\leq \beta_{\alpha'}(q_{\alpha'}).
\end{aligned}$$

By the last assertion of Theorem 3.7, we can thus deduce that $\dim_u(\mathcal{L}^{\alpha^-}_{\psi,v}) \leq \beta_{\alpha'}(q_{\alpha'}) = \dim_u(\mathcal{L}^{\alpha'}_{\psi,v})$, for $\alpha' \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ sufficiently close to $\alpha^- = \alpha^-_{\psi,v}$. Therefore, $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$ is right lower semi-continuous at α^-.

Suppose that $\alpha \mapsto \dim_u(\mathcal{L}^\alpha_{\psi,v})$ is not right upper semi-continuous at α^-. Then, by the weak* compactness of $\mathfrak{M}_\sigma(\Sigma)$, there is a decreasing sequence $(\alpha_l)_{l\in\mathbb{Z}_{>0}}$ converging to α^- such that $\lim_{l\to+\infty} \dim_u(\mathcal{L}^{\alpha_l}_{\psi,v}) > \dim_u(\mathcal{L}^{\alpha^-}_{\psi,v})$ and ν^{α_l} weak* converges to some $\nu \in \mathfrak{M}_\sigma(\Sigma)$ as l tends to infinity. Then,

$$\frac{\int_\Sigma \psi \,\mathrm{d}\nu}{\int_\Sigma v \,\mathrm{d}\nu} = \frac{\lim_{l\to+\infty} \int_\Sigma \psi \,\mathrm{d}\nu^{\alpha_l}}{\lim_{l\to+\infty} \int_\Sigma v \,\mathrm{d}\nu^{\alpha_l}} = \lim_{l\to+\infty} \alpha_l = \alpha^-.$$

As before, we have $\lim_{m\to+\infty} S_m\psi / S_m v \geq \alpha^-_{\psi,v} = \alpha^-$, ν-a.e., so by Lemma 3.8, we deduce that

$$\lim_{m\to+\infty} \frac{S_m\psi}{S_m v} = \alpha^-, \ \nu\text{-a.e.},$$

or equivalently, $\nu(\mathcal{L}^{\alpha^-}_{\psi,v}) = 1$. Again, by Theorem 3.9 and the upper semi-continuity of the entropy map,

$$\dim_u(\mathcal{L}^{\alpha^-}_{\psi,v}) \geq \overline{\dim}_u(\nu) \geq \limsup_{l\to+\infty} \frac{h_{KS}(\nu^{\alpha_l})}{\int_\Sigma u \,\mathrm{d}\nu^{\alpha_l}} = \limsup_{l\to+\infty} \dim_u(\mathcal{L}^{\alpha_l}_{\psi,v}).$$

However, the sequence $(\alpha_l)_{l\in\mathbb{Z}_{>0}}$ we took satisfies that

$$\limsup_{l\to+\infty} \dim_u(\mathcal{L}^{\alpha_l}_{\psi,v}) > \dim_u(\mathcal{L}^{\alpha^-}_{\psi,v}).$$

From this contradiction, we can conclude that $\alpha \mapsto \dim_u(\mathcal{L}^{\alpha}_{\psi,v})$ is right upper semi-continuous at α^-. Our proof is thus complete. $\square$

Proof of Proposition 4.4. When $\alpha^-_{\psi,v} = \alpha^+_{\psi,v}$, our claim is a direct consequence of Theorem 4.1. For the rest of the proof, we assume that $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$.

As in Theorem 3.12, set $\beta = \dim_u(\Sigma)$ and $\nu_{-\beta u}$ to be the unique equilibrium state for $-\beta u$. By Birkhoff's ergodic theorem and the ergodicity of $\nu_{-\beta u}$, we have

$$\lim_{n\to+\infty} \frac{S_n\psi}{S_n v} = \frac{\int_\Sigma \psi \,\mathrm{d}\nu_{-\beta u}}{\int_\Sigma v \,\mathrm{d}\nu_{-\beta u}},\ \nu_{-\beta u}\text{-a.e.}$$

Thus, by either the first or the second assertion of Theorem 3.7 and Theorem 3.12, we have $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) = \dim_u(\Sigma)$ for $\alpha = \int_\Sigma \psi \,\mathrm{d}\nu_{-\beta u} / \int_\Sigma v \,\mathrm{d}\nu_{-\beta u}$.

For $\alpha \notin [\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$, we have $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) = 0 < \dim_u(\Sigma)$, where the last inequality was shown in Theorem 3.12. For $\alpha = \alpha^-_{\psi,v}$, note that $\beta_\alpha(0) = \beta$. In addition, (4.5) implies that $\beta'_\alpha(q) > 0$ for any $q \in \mathbb{R}$. Consequently, we have

$$\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) = \inf_{q\in\mathbb{R}} \beta_\alpha(q) < \beta_\alpha(0) = \beta.$$

By symmetry, we also have $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) < \beta$ for $\alpha = \alpha^+_{\psi,v}$.

Now, it only remains to show that for any $\alpha \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ satisfying

$$\alpha \neq \frac{\int_\Sigma \psi \,\mathrm{d}\nu_{-\beta u}}{\int_\Sigma v \,\mathrm{d}\nu_{-\beta u}},$$

we have $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) < \dim_u(\Sigma)$. Fix such an α, and define q_α and ν^α as in Theorem 4.1. In addition, we shall still use ψ_α to denote $\psi - \alpha v$. Observe that $\nu_{-\beta u}(\mathcal{L}^{\alpha}_{\psi,v}) = 0$ while $\nu^\alpha(\mathcal{L}^{\alpha}_{\psi,v}) = 1$, so $\nu^\alpha \neq \nu_{-\beta u}$. Therefore, by the uniqueness of the equilibrium state, we have on the one hand,

$$h_{KS}(\nu^\alpha) - \int_\Sigma \beta u \,\mathrm{d}\nu^\alpha < P(-\beta u) = 0.$$

On the other hand, by Theorem 4.1, we have $\int_\Sigma \psi_\alpha \, \mathrm{d}\nu^\alpha = 0$, which implies

$$\begin{aligned} h_{KS}(\nu^\alpha) - \int_\Sigma \beta u \, \mathrm{d}\nu^\alpha &= h_{KS}(\nu^\alpha) + \int_\Sigma (q_\alpha \psi_\alpha - \beta u) \, \mathrm{d}\nu^\alpha \\ &= h_{KS}(\nu^\alpha) + \int_\Sigma (q_\alpha \psi_\alpha - \beta_\alpha(q_\alpha) u) \, \mathrm{d}\nu^\alpha \\ &\quad + (\beta_\alpha(q_\alpha) - \beta) \int_\Sigma u \, \mathrm{d}\nu^\alpha \\ &= P(q_\alpha \psi_\alpha - \beta_\alpha(q_\alpha) u) + (\beta_\alpha(q_\alpha) - \beta) \int_\Sigma u \, \mathrm{d}\nu^\alpha \\ &= (\beta_\alpha(q_\alpha) - \beta) \int_\Sigma u \, \mathrm{d}\nu^\alpha. \end{aligned}$$

Consequently, $\dim_u(\mathcal{L}^\alpha_{\psi,v}) = \beta_\alpha(q_\alpha) < \beta = \dim_u(\Sigma)$ if $\alpha \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$. Our proof is thus complete. □

4.2 Multifractal Analysis of Uniform Level Sets

In this section, we shall explore the u-dimensions of the uniform α-level set

$$\mathcal{UL}^\alpha_{\psi,v} = \left\{ \xi \in \Sigma \;\middle|\; \sup_{n \in \mathbb{Z}_{>0}} |S_n\psi(\xi) - \alpha S_n v(\xi)| < +\infty \right\}$$

and two types of subsets of it.

4.2.1 Main Theorems and Related Discussions

In this subsection, we shall state our main theorems, namely Proposition 4.6 and, more importantly, Theorem 4.7 and Theorem 4.8. The proofs of these results will be postponed to the next subsection. After that, we shall give an immediate corollary. At the end of this subsection, we shall point out that the results in this subsection cannot be proved directly by the arguments using equilibrium states as we presented in the previous section.

Firstly, we claim that whichever α we take, the u-dimension of $\mathcal{UL}^\alpha_{\psi,v}$ is always equal to the u-dimension of $\mathcal{L}^\alpha_{\psi,v}$.

Proposition 4.6 ([Liu23]). *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT and let the functions $\psi : \Sigma \to \mathbb{R}$, $v : \Sigma \to (0, +\infty)$ and $u : \Sigma \to (0, +\infty)$ be Hölder continuous. Then, for any $\alpha \in \mathbb{R}$, we have*

1. *$\mathcal{UL}^\alpha_{\psi,v} = \varnothing$ if and only if $\alpha \in [\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$;*

2. $\dim_u(\mathcal{U}\mathcal{L}^{\alpha}_{\psi,v}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v})$.

Now we proceed to the two main theorems, namely Theorem 4.7 and Theorem 4.8. In order to give the precise statements, we need to define some notations for each of them.

Let $\mathcal{W}$ be a finite set of admissible words. Define for each positive integer k,

$$\mathcal{F}_{\mathcal{W},k} = \bigcap_{n=0}^{\infty} \{\, \xi \in \Sigma \mid \text{all the words in } \mathcal{W} \text{ are the subwords of } \xi_{n+1} \cdots \xi_{n+k} \,\},$$

and then set $\mathcal{F}_{\mathcal{W}} = \bigcup_{k=1}^{\infty} \mathcal{F}_{\mathcal{W},k}$. When $\mathcal{W}$ contains only one word, say ω, then $\mathcal{F}_{\mathcal{W}}$ will also be denoted by $\mathcal{F}_{\omega}$.

Theorem 4.7 ([Liu23]). *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT and let the functions $\psi : \Sigma \to \mathbb{R}$, $v : \Sigma \to (0,+\infty)$ and $u : \Sigma \to (0,+\infty)$ be Hölder continuous. Suppose that $\mathcal{W}$ is a finite set of admissible words over A. Then, for any $\alpha \notin \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$, we have*

$$\dim_u(\mathcal{U}\mathcal{L}^{\alpha}_{\psi,v} \cap \mathcal{F}_{\mathcal{W}}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v}). \tag{4.8}$$

For every non-negative integer l, we define the l-th power of a word ω as $\omega^l = \omega \cdots \omega$, where the right-hand side is the l-fold concatenation of ω. More formally, define ω^0 as the empty word, and for each positive integer l, define $\omega^l = \omega^{l-1}\omega$. Define $\mathcal{F}'_{\mathcal{W}} = \bigcup_{l=1}^{\infty} \mathcal{F}'_{\mathcal{W},l}$, where for any positive integer l,

$$\mathcal{F}'_{\mathcal{W},l} = \bigcap_{\omega \in \mathcal{W}} \left\{\, \xi \in \Sigma \;\middle|\; \xi \text{ does not contain } \omega^l \text{ as a subword} \,\right\}.$$

Theorem 4.8 ([Liu23]). *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT and let the functions $\psi : \Sigma \to \mathbb{R}$, $v : \Sigma \to (0,+\infty)$ and $u : \Sigma \to (0,+\infty)$ be Hölder continuous. Suppose that $\mathcal{W}$ is a finite set of admissible words over A. Then, for any $\alpha \notin \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$, we have*

$$\dim_u(\mathcal{U}\mathcal{L}^{\alpha}_{\psi,v} \cap \mathcal{F}'_{\mathcal{W}}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v}). \tag{4.9}$$

The proofs of Proposition 4.6, Theorem 4.7 and Theorem 4.8 will be postponed to the next subsection, namely Subsection 4.2.2.

The two main theorems above in particular imply that $\mathcal{F}_{\mathcal{W}}$ and $\mathcal{F}'_{\mathcal{W}}$ both have full u-dimension for any finite $\mathcal{W} \subseteq A^*_{\mathbb{M}}$.

Corollary 4.9 ([Liu23]). *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT and let $u : \Sigma \to (0,+\infty)$ be a Hölder continuous function. Suppose that $\mathcal{W}$ is a finite set of admissible words over A. Then,* $\dim_u(\mathcal{F}_{\mathcal{W}}) = \dim_u(\mathcal{F}'_{\mathcal{W}}) = \dim_u(\Sigma)$.

Proof. Take Hölder continuous $\psi : \Sigma \to \mathbb{R}$ and $v : \Sigma \to (0, +\infty)$ such that $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$. Then, by Proposition 4.4, there is a unique $\alpha \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$ such that $\dim_u(\mathcal{L}^\alpha_{\psi,v}) = \dim_u(\Sigma)$. For this α, by Theorem 4.7 and Theorem 4.8, we have that for any finite set $\mathcal{W}$ of admissible words,

$$\dim_u(\Sigma) = \dim_u(\mathcal{L}^\alpha_{\psi,v}) = \dim_u(\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}_\mathcal{W}) \leq \dim_u(\mathcal{F}_\mathcal{W}) \leq \dim_u(\Sigma);$$
$$\dim_u(\Sigma) = \dim_u(\mathcal{L}^\alpha_{\psi,v}) = \dim_u(\mathcal{UL}^\alpha_{\psi,v} \cap \mathcal{F}'_\mathcal{W}) \leq \dim_u(\mathcal{F}'_\mathcal{W}) \leq \dim_u(\Sigma).$$

This completes the proof. □

At the end of this subsection, we remark that the previous results do not follow from Theorem 4.1. To make it more precise, we state the following proposition.

Proposition 4.10. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT, and μ be an ergodic Borel probability measure on Σ satisfying that $\mu(O) > 0$ for every non-empty open subset O of Σ. Let the functions $\psi : \Sigma \to \mathbb{R}$ and $v : \Sigma \to (0, +\infty)$ be Hölder continuous. Let $\mathcal{W}$ be a non-empty finite set of admissible words. Then,*

1. *$\mu(\mathcal{F}_\mathcal{W}) = 0$, if $\mathcal{W}$ contains at least one admissible word ω whose length is no less than $\#A + 1$;*

2. *$\mu(\mathcal{F}'_\mathcal{W}) = 0$, if $\mathcal{W}$ contains at least one word ω' satisfying that $(\omega')^2 = \omega'\omega'$ is admissible;*

3. *$\mu(\mathcal{UL}^\alpha_{\psi,v}) = 0$, for any $\alpha \notin \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$.*

Remark 4.11. When showing Theorem 4.1, we used the equilibrium state ν^α to estimate the u-dimension of $\mathcal{L}^\alpha_{\psi,v}$. If $\mathcal{UL}^\alpha_{\psi,v}$ had positive ν^α measure, then by the mass distribution principle, we would have $\dim_u(\mathcal{L}^\alpha_{\psi,v}) = \dim_u(\mathcal{UL}^\alpha_{\psi,v})$ immediately. Proposition 4.10 shows that $\nu^\alpha(\mathcal{UL}^\alpha_{\psi,v}) = 0$ for $\alpha \notin \{ \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \}$, so we cannot deduce Proposition 4.6, Theorem 4.7 and Theorem 4.8 from Theorem 4.1 directly. It suggests that we need to construct a new measure for which $\mathcal{UL}^\alpha_{\psi,v}$ has positive measure, so that we can use the mass distribution principle to give a lower bound for $\dim_u(\mathcal{UL}^\alpha_{\psi,v})$. In a similar sense, Proposition 4.10 also suggests that $\dim_u(\mathcal{F}_\mathcal{W})$ and $\dim_u(\mathcal{F}'_\mathcal{W})$ cannot be estimated by one single equilibrium state in general.

In order to show the first item of Proposition 4.10, we introduce the following notation. For any $a \in A$, define $\mathcal{P}(a)$ to be the set of all non-empty admissible words ω satisfying that

1. ωa is admissible;

2. $\omega_j = a$ if and only if $j = 1$, for all $j \in \{ 1, \cdots, |\omega| \}$.

Lemma 4.12. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Then, we have $\#\mathcal{P}(a) \geq 2$ for each $a \in A$.*

Proof. Fix an arbitrary $a \in A$. It is clear from Proposition 2.2 that $\mathcal{P}(a)$ is non-empty. Suppose that $\mathcal{P}(a)$ is a singleton, and let l be the length of the only word in $\mathcal{P}(a)$. Then, for any word ρ satisfying that $a\rho a$ is admissible, $a\rho$ must be a power of the only word in $\mathcal{P}(a)$, thus implying that l divides $|\rho| + 1$. As the SFT $\sigma : \Sigma \to \Sigma$ is topologically mixing, we deduce from Proposition 2.2 that $l = 1$, which means that $\mathcal{P}(a) = \{ a \}$. As we always assume that $\#A \geq 2$, take $b \neq a$ from A. Let $\rho^{a\to b}$ and $\rho^{b\to a}$ be the shortest words, possibly empty, such that $a\rho^{a\to b}b$ and $b\rho^{b\to a}a$ are admissible. By construction, $a\rho^{a\to b}b\rho^{b\to a}$ is an element of $\mathcal{P}(a)$ and is obviously different from a, hence contradicting the claim $\mathcal{P}(a) = \{ a \}$ we have shown previously. Therefore, $\mathcal{P}(a)$ contains at least two words. $\square$

Proof of Proposition 4.10. We begin with the proof of the first claim. Take $\omega \in \mathcal{W}$ satisfying $|\omega| \geq \#A + 1$. By the pigeonhole principle, there is at least one $a \in A$ appearing in ω at least twice. This further implies the existence of $\tilde{\omega} \in \mathcal{P}(a)$ which satisfies that $\tilde{\omega}a$ is a subword of ω. By Lemma 4.12, there is some $\tilde{\omega}' \in \mathcal{P}(a)$ other than $\tilde{\omega}$. Since $\tilde{\omega}' \in \mathcal{P}(a)$, the powers of $\tilde{\omega}'$ are all admissible. Moreover, $\tilde{\omega}a$ is by construction not a subword of any of the powers of $\tilde{\omega}'$. From this, we can immediately see that ω is not a subword of any of the powers of $\tilde{\omega}'$ either.

Fix an arbitrary positive integer l. By Birkhoff's ergodic theorem, we have

$$\lim_{n\to+\infty} \frac{1}{n} S_n \mathbb{1}_{[(\tilde{\omega}')^l]} = \mu([(\tilde{\omega}')^l]) > 0,\ \mu\text{-a.e.},$$

where $\mathbb{1}_E : \Sigma \to \mathbb{R}$ is the function satisfying $\mathbb{1}_E(E) = \{ 1 \}$ and $\mathbb{1}_E(\Sigma \backslash E) = \{ 0 \}$ for every $E \subseteq \Sigma$. This in particular indicates that for μ-a.e. $\xi \in \Sigma$, $(\tilde{\omega}')^l$ is a subword of ξ. As we have seen that ω is not a subword of $(\tilde{\omega}')^l$, we deduce $\mu(\mathcal{F}_{\mathcal{W},l|\tilde{\omega}'|}) = 0$ for each $l \in \mathbb{Z}_{>0}$. Since $|\tilde{\omega}'| \geq |a| = 1$, $l|\tilde{\omega}'|$ diverges to the positive infinity as l tends to the positive infinity. Note that $(\mathcal{F}_{\mathcal{W},j})_{j\in\mathbb{Z}_{>0}}$ is an ascending set sequence and $\mathcal{F}_{\mathcal{W}} = \bigcup_{j=1}^{\infty} \mathcal{F}_{\mathcal{W},j}$, we have

$$\mu(\mathcal{F}_{\mathcal{W}}) = \sup_{l\in\mathbb{Z}_{>0}} \mu(\mathcal{F}_{\mathcal{W},l|\tilde{\omega}'|}) = 0,$$

showing our first claim.

Now we prove the second claim. Let ω' be a word in $\mathcal{W}$ satisfying that $(\omega')^2 = \omega'\omega'$ is admissible. Then, for any $k \in \mathbb{Z}_{>0}$, following the same arguments as in the previous paragraph, we have that for μ-a.e. $\xi \in \Sigma$, $(\omega')^k$ is a subword of ξ. Therefore, $\mu(\mathcal{F}'_{\mathcal{W},k}) = 0$ for any positive integer k. This shows our second claim.

Finally, we prove the last claim. For $\alpha \notin [\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$, we have $\mathcal{UL}^\alpha_{\psi,v} \subseteq \mathcal{L}^\alpha_{\psi,v} = \varnothing$, so $\mu(\mathcal{UL}^\alpha_{\psi,v}) = 0$. For the rest of the proof, we assume that $\alpha^-_{\psi,v} <$

$\alpha < \alpha^+_{\psi,v}$. For every $M > 0$, there is then an admissible word τ such that $S_\tau \psi_\alpha > 2M + V_{\psi_\alpha}$. Again by Birkhoff's ergodic theorem, we see that for μ-a.e. $\xi \in \Sigma$, τ is a subword of ξ. For each $\xi \in \Sigma$ satisfying that there is a non-negative integer m such that $\sigma^m(\xi) \in [\tau]$, note that either $S_m\psi_\alpha(\xi) < -M$ or $S_{m+|\tau|}\psi_\alpha(\xi) > M$. Therefore, we have

$$\begin{aligned}\mu\left(\left\{ \xi \in \Sigma \,\middle|\, \sup_{n\geq 0} |S_n\psi_\alpha(\xi)| \leq M \right\}\right) &\leq \mu\left(\{ \xi \in \Sigma \mid \tau \text{ is not a subword of } \xi \}\right) \\ &= 0\end{aligned}$$

for any $M > 0$, from which the third claim follows. □

4.2.2 Proofs

We first prove Theorem 4.7, then Theorem 4.8 and lastly Proposition 4.6.

Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT, with A and $\mathbb{M}$ being the set of symbols and the incidence matrix. Let $\psi : \Sigma \to \mathbb{R}$, $v : \Sigma \to (0, +\infty)$ and $u : \Sigma \to (0, +\infty)$ be Hölder continuous functions. Let $\mathcal{W}$ be a finite subset of $A^*_{\mathbb{M}}$. For each α, we still use ψ_α to denote the Hölder continuous function $\psi - \alpha v$. As $\sigma : \Sigma \to \Sigma$ is topologically mixing, by Proposition 2.2, there is some integer l such that for any two symbols $a, b \in A$, there is some $\rho \in A^l_{\mathbb{M}}$ such that $a\rho b$ is admissible. We fix this l, and will use it in the subsequent proofs. Finally, for any $M > 0$, any $\alpha \in \mathbb{R}$ and any $n \in \mathbb{Z}_{>0}$, define

$$\mathcal{B}^n_{\alpha,M} = \left\{ \omega \in A^n_{\mathbb{M}} \,\middle|\, \sup_{\xi \in [\omega]} |S_n\psi_\alpha(\xi)| \leq M \right\},$$

and $\mathcal{B}_{\alpha,M} = \bigcup_{n=1}^{\infty} \mathcal{B}^n_{\alpha,M}$.

The proof of Theorem 4.7 needs the following lemma.

Lemma 4.13 ([Liu23]). *Suppose that $\alpha^-_{\psi,v} < \alpha < \alpha^+_{\psi,v}$. Then, for any two constants $M > 2V_{\psi_\alpha} + l \cdot \|\psi_\alpha\|$ and $M' > 0$, there exists a finite family $\mathcal{S}$ of admissible words such that for any $\omega \in \mathcal{B}_{\alpha,M'}$, there is some $\tau \in \mathcal{S}$ for which $\omega\tau \in \mathcal{B}_{\alpha,M}$.*

A similar lemma was shown in [GJK22] for $\sigma : \Sigma \to \Sigma$ being a full shift. In [Liu23], it was extended to topologically transitive, and in particular topologically mixing, SFTs. The proof in [Liu23] was also more elementary than the original proof in [GJK22]. The proof we are to give below is the one in [Liu23].

Proof. As $\alpha^-_{\psi,v} < \alpha < \alpha^+_{\psi,v}$, there exist $\xi^-, \xi^+ \in \Sigma$ and $n^-, n^+ \in \mathbb{Z}_{>0}$ such that

$$S_{n^-}\psi_\alpha(\xi^-) < -M' - 2V_{\psi_\alpha} - l \cdot \|\psi_\alpha\| < M' + 2V_{\psi_\alpha} + l \cdot \|\psi_\alpha\| < S_{n^+}\psi_\alpha(\xi^+).$$

Define

$$\mathcal{S}_0 = \{ \xi_1^- \cdots \xi_k^- \mid 1 \le k \le n^- \} \cup \{ \xi_1^+ \cdots \xi_k^+ \mid 1 \le k \le n^+ \} \cup \{ \text{empty word} \},$$

and $\mathcal{S} = \{ \rho\tau \in A_{\mathbb{M}}^* \mid \rho \in A_{\mathbb{M}}^l,\ \tau \in \mathcal{S}_0 \}$. We claim that $\mathcal{S}$ is the family of words we want. Clearly, $\mathcal{S}$ is a finite set of admissible words, so we only need to show that $\mathcal{S}$ meets the last requirement.

Take an arbitrary $\omega \in \mathcal{B}_{\alpha,M'}$. Let $\xi \in [\omega]$. Then, $\left|S_{|\omega|}\psi_\alpha(\xi)\right| \le M'$. Consider the case where $S_{|\omega|}\psi_\alpha(\xi) \ge 0$. Take one $\rho^- \in A_{\mathbb{M}}^l$ such that $\omega\rho^-\xi^-$ is admissible. Note that

$$S_{|\omega|+l}\psi_\alpha(\omega\rho^-\xi^-) \ge S_{|\omega|}\psi_\alpha(\xi) - V_{\psi_\alpha} - l \cdot \|\psi_\alpha\| \ge -V_{\psi_\alpha} - l \cdot \|\psi_\alpha\|.$$

If $S_{|\omega|+l}\psi_\alpha(\omega\rho^-\xi^-) \le 0$, then the inequality above indicates that $\omega\rho^- \in \mathcal{B}_{\alpha,M}$. Otherwise, note that

$$S_{|\omega|+l+n^-}\psi_\alpha(\omega\rho^-\xi^-) \le S_{|\omega|}\psi_\alpha(\xi) + V_{\psi_\alpha} + l \cdot \|\psi_\alpha\| + S_{n^-}\psi_\alpha(\xi^-) < -V_{\psi_\alpha}.$$

Hence, we can take the smallest $k^- \in \{ 1, \cdots, n^- \}$ such that

$$S_{|\omega|+l+k^-}\psi_\alpha(\omega\rho^-\xi^-) < -V_{\psi_\alpha}. \tag{4.10}$$

This means that $S_{|\omega|+l+k^--1}\psi_\alpha(\omega\rho^-\xi^-) \ge -V_{\psi_\alpha}$, and therefore,

$$S_{|\omega|+l+k^-}\psi_\alpha(\omega\rho^-\xi^-) \ge S_{|\omega|+l+k^--1}\psi_\alpha(\omega\rho^-\xi^-) - \|\psi_\alpha\| \ge -V_{\psi_\alpha} - l \cdot \|\psi_\alpha\|.$$

Combining this fact with (4.10), we can thus deduce that $\omega\rho^-\xi_1^- \cdots \xi_{k^-}^- \in \mathcal{B}_{\alpha,M}$, when $S_{|\omega|+|\rho^-|}\psi_\alpha(\omega\rho^-\xi^-) > 0$. We have thus shown that if $S_{|\omega|}\psi_\alpha(\xi) \ge 0$, then either $\omega\rho^-$ or $\omega\rho^-\xi_1^- \cdots \xi_{k^-}^-$, for some $k^- \in \{ 1, \cdots, n^- \}$, is in $\mathcal{B}_{\alpha,M}$. This completes the proof for the case where $S_{|\omega|}\psi_\alpha(\xi) \ge 0$. The case where $S_{|\omega|}\psi_\alpha(\xi) < 0$ can be treated in the same way. □

Now we give the proof of Theorem 4.7. The proof is taken from [Liu23]. As Theorem 4.7 is weaker than the theorem claimed in [Liu23], the proof will be slightly simpler than the one in [Liu23]. The proof which the author gave in [Liu23] is greatly inspired by [GJK22].

Proof of Theorem 4.7. When $\alpha \notin [\alpha_{\psi,v}^-, \alpha_{\psi,v}^+]$, we have $\mathcal{UL}_{\psi,v}^\alpha \cap \mathcal{F}_{\mathcal{W}} = \mathcal{L}_{\psi,v}^\alpha = \varnothing$, so our claim holds trivially. Thus, we only need to handle the case where $\alpha_{\psi,v}^- < \alpha < \alpha_{\psi,v}^+$. Assume that $\alpha_{\psi,v}^- < \alpha < \alpha_{\psi,v}^+$ for the rest of the proof.

Let $\tilde{\omega}$ be an admissible word in which all words from $\mathcal{W}$ appear at least once. Fix $M > 2V_{\psi_\alpha} + l \cdot \|\psi_\alpha\|$ and let $M' = 2M + (2l + |\tilde{\omega}|)\|\psi_\alpha\|$. For these M and M', let $\mathcal{S} \subseteq A_{\mathbb{M}}^*$ satisfy the conditions in Lemma 4.13, and define $\|\mathcal{S}\| = \max_{\tau\in\mathcal{S}} |\tau|$.

Let $q_\alpha \in \mathbb{R}$ and the measure ν^α be defined as in Theorem 4.1. Then, ν^α is ergodic and $\int_\Sigma \psi_\alpha \, \mathrm{d}\nu^\alpha = 0$. Hence, by Theorem 2.7, $\liminf_{n\to+\infty} |S_n\psi_\alpha| = 0$, ν^α-a.e. Thus, for any $M > 2V_{\psi_\alpha} + l \cdot \|\psi_\alpha\| \geq V_{\psi_\alpha}$, by the Borel-Cantelli lemma, we have

$$\sum_{\omega\in\mathcal{B}_{\alpha,M}} \nu^\alpha([\omega]) = +\infty.$$

Combining the divergence of this series with the fact that ν^α is a Gibbs measure for $q_\alpha\psi_\alpha - \beta_\alpha(q_\alpha)u$ with $P(q_\alpha\psi_\alpha - \beta_\alpha(q_\alpha)u) = 0$, we have

$$\sum_{\omega\in\mathcal{B}_{\alpha,M}} \exp(S_\omega(q_\alpha\psi_\alpha - \beta_\alpha(q_\alpha)u)) = +\infty.$$

As $\sup_{\omega\in\mathcal{B}_{\alpha,M}} |S_\omega(q_\alpha\psi_\alpha)| < +\infty$, this in turn gives

$$\sum_{\omega\in\mathcal{B}_{\alpha,M}} \exp(-\beta_\alpha(q_\alpha)S_\omega u) = +\infty.$$

For any positive $s < \beta_\alpha(q_\alpha) = \dim_u(\mathcal{L}^\alpha_{\psi,v})$, pick a positive integer m such that

$$\frac{1}{s}\log \sum_{\omega\in\mathcal{B}^m_{\alpha,M}} \exp(-sS_\omega u) > (2l + \|\mathcal{S}\| + |\tilde{\omega}|)\,\|u\|. \tag{4.11}$$

To ease the notation, we shall write C_0 to denote $(2l + \|\mathcal{S}\| + |\tilde{\omega}|)\,\|u\|$.

We claim that $\dim_u(\mathcal{U}\mathcal{L}^\alpha_{\psi,v} \cap \mathcal{F}_{\tilde{\omega}}) \geq s$. To show this claim, we construct a Borel probability measure μ_s for which $\mu_s(\mathcal{U}\mathcal{L}^\alpha_{\psi,v} \cap \mathcal{F}_{\tilde{\omega}}) = 1$, and then apply the mass distribution principle to this μ_s. Define by induction a sequence $(\mathcal{A}_k)_{k\geq1}$ of subsets of $\mathcal{B}_{\alpha,M}$ as follows. Let $\mathcal{A}_1 = \mathcal{B}^m_{\alpha,M}$. For $k \geq 2$, fix $\omega^{(k-1)} \in \mathcal{A}_{k-1}$ and $\omega \in \mathcal{A}_1$, and choose $\rho, \lambda \in A^l_{\mathbb{M}}$ satisfying that $\omega^{(k-1)}\rho\omega\lambda\tilde{\omega}$ is admissible. Then, since $\mathcal{A}_{k-1}$ and $\mathcal{A}_1$ are, by the induction hypothesis, both subsets of $\mathcal{B}_{\alpha,M}$, we have

$$\sup_{\xi\in[\omega^{(k-1)}\rho\omega\lambda\tilde{\omega}]} \left|S_{|\omega^{(k-1)}\rho\omega\lambda\tilde{\omega}|}\psi_\alpha(\xi)\right| \leq M'.$$

Therefore, there is some $\tau \in \mathcal{S}$ such that $\omega^{(k-1)}\rho\omega\lambda\tilde{\omega}\tau \in \mathcal{B}_{\alpha,M}$. Note that the word $\omega^{(k-1)}\rho\omega\lambda\tilde{\omega}\tau$ is constructed from $\omega^{(k-1)} \in \mathcal{A}_{k-1}$ and $\omega \in \mathcal{A}_1$, so we may denote it by $\theta_k(\omega^{(k-1)}, \omega)$. For every $\omega^{(k-1)} \in \mathcal{A}_{k-1}$, define $\mathcal{A}_k(\omega) = \{\, \theta_k(\omega,\omega') \mid \omega' \in \mathcal{A}_1 \,\}$ and $\mathcal{A}_k = \bigcup_{\omega\in\mathcal{A}_{k-1}} \mathcal{A}_k(\omega)$. It is clear from our discussion above that $\mathcal{A}_k \subseteq \mathcal{B}_{\alpha,M}$.

Now we are ready to construct the Borel probability measure μ_s from the set sequence $(\mathcal{A}_k)_{k\geq1}$. Set

$$\mu_s([\omega^{(1)}]) = \frac{\exp(-sS_{\omega^{(1)}}u)}{\sum_{\omega\in\mathcal{A}_1}\exp(-sS_\omega u)}$$

for any $\omega^{(1)} \in \mathcal{A}_1$ and

$$\mu_s([\omega^{(k)}]) = \frac{\exp(-sS_{\omega^{(k)}}u)\mu_s([\omega^{(k-1)}])}{\sum_{\omega \in \mathcal{A}_k(\omega^{(k-1)})} \exp(-sS_\omega u)}$$

for any $\omega^{(k)} \in \mathcal{A}_k(\omega^{(k-1)})$, where $k \geq 2$ and $\omega^{(k-1)} \in \mathcal{A}_{k-1}$. Clearly, for every $\omega^{(k-1)} \in \mathcal{A}_{k-1}$ and every $\omega^{(k)} \in \mathcal{A}_k(\omega)$, $\omega^{(k)}$ is a continuation of $\omega^{(k-1)}$, or equivalently, we can say $[\omega^{(k)}] \subseteq [\omega^{(k-1)}]$. Moreover, for any $\omega^{(k-1)} \in \mathcal{A}_{k-1}$, and any two distinct $\omega, \omega' \in \mathcal{A}_1$, we claim that

$$[\theta_k(\omega^{(k-1)}, \omega)] \cap [\theta_k(\omega^{(k-1)}, \omega')] = \varnothing. \tag{4.12}$$

Once we manage to prove (4.12), we can immediately deduce the existence and uniqueness of the measure μ_s from Kolmogorov consistency theorem. The proof of (4.12) is given as follows. As before, we write $\theta_k(\omega^{(k-1)}, \omega) = \omega^{(k-1)}\rho\omega\lambda\tilde{\omega}\tau$. For ω', we write $\theta_k(\omega^{(k-1)}, \omega') = \omega^{(k-1)}\rho'\omega'\lambda'\tilde{\omega}\tau'$ in the same way as we wrote $\theta_k(\omega^{k-1}, \omega)$. This means that ρ and ρ' are both in $A^l_{\mathbb{M}}$. If $\rho \neq \rho'$, then we have

$$[\theta_k(\omega^{(k-1)}, \omega)] \cap [\theta_k(\omega^{(k-1)}, \omega')] \subseteq [\omega^{(k-1)}\rho] \cap [\omega^{k-1}\rho'] = \varnothing,$$

because ρ and ρ' have the same length. If $\rho = \rho'$, then we have

$$[\theta_k(\omega^{(k-1)}, \omega)] \cap [\theta_k(\omega^{(k-1)}, \omega')] \subseteq [\omega^{(k-1)}\rho\omega] \cap [\omega^{k-1}\rho'\omega'] = \varnothing,$$

because ω and ω' are distinct words with the same length m. Therefore, in any of the two possible cases above, (4.12) always holds.

By construction, μ_s is supported on $\bigcap_{k=1}^{\infty} \bigcup_{\omega \in \mathcal{A}_k} [\omega]$. Also note that for any $\xi \in \bigcap_{k=1}^{\infty} \bigcup_{\omega \in \mathcal{A}_k} [\omega]$,

$$\sup_{n \geq 1} |S_n \psi_\alpha(\xi)| \leq M' + \|\mathcal{S}\| \cdot \|\psi_\alpha\|.$$

Moreover, all subwords of ξ with length no less than $2(m + l + |\tilde{\omega}|) + \|\mathcal{S}\|$ contain $\tilde{\omega}$ as a subword, thus containing all the words from $\mathcal{W}$ as subwords. Therefore, we have $\mu_s(\mathcal{UL}^{\alpha}_{\psi,v} \cap \mathcal{F}_{\mathcal{W}}) = 1$.

Observe that for any integer $k \geq 2$ and $\omega \in \mathcal{A}_{k-1}$,

$$\begin{aligned}\sum_{\omega' \in \mathcal{A}_k(\omega)} \exp(-sS_{\omega'}u) &\geq \exp(-sS_\omega u - sC_0) \sum_{\omega'' \in \mathcal{A}_1 = \mathcal{B}^m_K} \exp(-sS_{\omega''}u) \\ &\geq \exp(-sS_\omega u),\end{aligned}$$

where the second inequality is due to (4.11). As a result, for any integer $k \geq 2$,

$$\max_{\omega \in \mathcal{A}_k} \frac{\mu_s([\omega])}{\exp(-sS_\omega u)} = \max_{\omega' \in \mathcal{A}_{k-1}} \frac{\mu_s([\omega'])}{\sum_{\omega'' \in \mathcal{A}_k(\omega')} \exp(-sS_{\omega''}u)} \leq \max_{\omega' \in \mathcal{A}_{k-1}} \frac{\mu_s([\omega'])}{\exp(-sS_{\omega'}u)},$$

thus implying that

$$\max_{\omega\in\mathcal{A}_k}\frac{\mu_s([\omega])}{\exp(-sS_\omega u)}\le\max_{\omega\in\mathcal{A}_1}\frac{\mu_s([\omega])}{\exp(-sS_\omega u)}.$$

By the mass distribution principle, $\dim_u(\mathcal{UL}^\alpha_{\psi,v}\cap\mathcal{F}_\mathcal{W})\ge s$. Since s is an arbitrary positive number smaller than $\dim_u(\mathcal{L}^\alpha_{\psi,v})$, we conclude that $\dim_u(\mathcal{UL}^\alpha_{\psi,v}\cap\mathcal{F}_\mathcal{W})\ge\dim_u(\mathcal{L}^\alpha_{\psi,v})$. Since $\mathcal{UL}^\alpha_{\psi,v}\cap\mathcal{F}_\mathcal{W}\subseteq\mathcal{L}^\alpha_{\psi,v}$, the proof is complete. □

Now we use Theorem 4.7 to show Theorem 4.8.

Proof of Theorem 4.8. We construct an admissible word ω^* as follows. Let $\mathcal{W}^\infty=\{\,\omega\omega\cdots\mid\omega\in\mathcal{W}\,\}\subseteq\Sigma$. Note that $\bigcup_{n=0}^\infty\sigma^n\mathcal{W}^\infty$ is finite. Thus, we can take one admissible word ω^* whose cylinder set $[\omega^*]$ is disjoint from $\bigcup_{n=0}^\infty\sigma^n\mathcal{W}^\infty$. For this ω^*, we claim that

$$\mathcal{F}_{\omega^*}\subseteq\mathcal{F}'_\mathcal{W}. \tag{4.13}$$

To see this, take an arbitrary $\xi\in\mathcal{F}_{\omega^*}$. Then, there exists some $k\in\mathbb{Z}_{>0}$ such that every subword of ξ with length no less than k has ω^* appearing therein. For any $\omega\in\mathcal{W}$, note that the word ω^k does not have ω^* as a subword, and has its length being $k\cdot|\omega|\ge k$. Hence, ω^k cannot appear in ξ, so $\xi\in\mathcal{F}'_\mathcal{W}$. This proves (4.13).

Combining (4.13) with Theorem 4.7, we immediately have

$$\dim_u(\mathcal{UL}^\alpha_{\psi,v}\cap\mathcal{F}'_\mathcal{W})\ge\dim_u(\mathcal{UL}^\alpha_{\psi,v}\cap\mathcal{F}_{\omega^*})=\dim_u(\mathcal{L}^\alpha_{\psi,v}),$$

for any $\alpha\notin\{\,\alpha^-_{\psi,v},\alpha^+_{\psi,v}\,\}$. Since $\mathcal{UL}^\alpha_{\psi,v}\cap\mathcal{F}'_\mathcal{W}\subseteq\mathcal{L}^\alpha_{\psi,v}$, the proof is complete. □

We conclude this subsection by showing Proposition 4.6 as follows.

Proof of Proposition 4.6. Without loss of generality, we may assume that $\alpha^-_{\psi,v}\le\alpha\le\alpha^+_{\psi,v}$, for otherwise $\mathcal{UL}^\alpha_{\psi,v}$ and $\mathcal{L}^\alpha_{\psi,v}$ would both be empty. When $\alpha^-_{\psi,v}=\alpha^+_{\psi,v}=\alpha$, by Corollary 2.17, we have $\mathcal{UL}^\alpha_{\psi,v}=\mathcal{L}^\alpha_{\psi,v}=\Sigma$, so the claim holds trivially. Hence, we shall henceforth assume that $\alpha^-_{\psi,v}<\alpha^+_{\psi,v}$. If $\alpha^-_{\psi,v}<\alpha<\alpha^+_{\psi,v}$, note that $\mathcal{UL}^\alpha_{\psi,v}\cap\mathcal{F}_\mathcal{W}\subseteq\mathcal{UL}^\alpha_{\psi,v}\subseteq\mathcal{L}^\alpha_{\psi,v}$, where $\mathcal{W}$ is an arbitrary finite set of admissible words. Therefore, by Theorem 4.7, we have $\dim_u(\mathcal{UL}^\alpha_{\psi,v})=\dim_u(\mathcal{L}^\alpha_{\psi,v})$. In order to show that $\mathcal{UL}^\alpha_{\psi,v}$ is non-empty, we shall prove that $\dim_u(\mathcal{L}^\alpha_{\psi,v})>0$ for u taken to be v as follows. Suppose that there exists some $\alpha_0\in(\alpha^-_{\psi,v},\alpha^+_{\psi,v})$ such that $\dim_u(\mathcal{L}^{\alpha_0}_{\psi,v})>0$ for $u=v$. For any $\alpha\in[\alpha^-_{\psi,v},\alpha^+_{\psi,v}]$ not equal to α_0, clearly there exists some $\alpha'\in[\alpha^-_{\psi,v},\alpha^+_{\psi,v}]$ and $s\in(0,1)$ such that $\alpha_0=s\alpha+(1-s)\alpha'$. Recall that Corollary 4.3 asserts that when $u=v$, $\alpha\mapsto\dim_u(\mathcal{L}^\alpha_{\psi,v})$ is a concave function on the closed interval $[\alpha^-_{\psi,v},\alpha^+_{\psi,v}]$. Therefore, we have

$$s\dim_u(\mathcal{L}^\alpha_{\psi,v})+(1-s)\dim_u(\mathcal{L}^{\alpha'}_{\psi,v})\le\dim_u(\mathcal{L}^{\alpha_0}_{\psi,v})=0$$

provided that $u = v$. Since $0 < s < 1$ and the u-dimension is always non-negative, we have

$$\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) = \dim_u(\mathcal{L}^{\alpha'}_{\psi,v}) = 0.$$

From this, we have that $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) = 0$ for any $\alpha \in [\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$ and for $u = v$. However, Proposition 4.4 states that there is some $\alpha \in [\alpha^-_{\psi,v}, \alpha^+_{\psi,v}]$ satisfying $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) = \dim_u(\Sigma)$, which is positive by Theorem 3.12. This contradiction shows that $\dim_u(\mathcal{UL}^{\alpha}_{\psi,v}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v}) > 0$ for u taken to be v and for any $\alpha \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$. This in particular implies that $\mathcal{UL}^{\alpha}_{\psi,v}$ is non-empty for any $\alpha \in (\alpha^-_{\psi,v}, \alpha^+_{\psi,v})$.

Now it only remains to handle the case where $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$ and α is equal to either $\alpha^-_{\psi,v}$ or $\alpha^+_{\psi,v}$. Suppose that $\alpha = \alpha^-_{\psi,v}$. By Theorem 4.1, there exists a σ-invariant Borel probability measure ν^{α} such that $\nu^{\alpha}(\mathcal{L}^{\alpha}_{\psi,v}) = 1$ and $\dim_u(\nu^{\alpha}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v})$. In addition, Theorem 2.16 guarantees the existence of a continuous $f : \Sigma \to \mathbb{R}$ satisfying that $\psi_\alpha + f \circ \sigma - f \geq 0$. As $\nu^{\alpha}(\mathcal{L}^{\alpha}_{\psi,v}) = 1$, by Birkhoff's ergodic theorem, we have $\int_\Sigma \psi_\alpha \,\mathrm{d}\nu^{\alpha} = 0$. The σ-invariance of ν^{α} then yields

$$\int_\Sigma S_n(\psi_\alpha + f \circ \sigma - f)\,\mathrm{d}\nu^{\alpha} = n \int_\Sigma (\psi_\alpha + f \circ \sigma - f)\,\mathrm{d}\nu^{\alpha} = n \int_\Sigma \psi_\alpha \,\mathrm{d}\nu^{\alpha} = 0,$$

for any $n \in \mathbb{Z}_{>0}$. For any $n \in \mathbb{Z}_{>0}$, since $S_n(\psi_\alpha + f \circ \sigma - f) \geq 0$, we have

$$S_n\psi_\alpha + f \circ \sigma^n - f = S_n(\psi_\alpha + f \circ \sigma - f) = 0,\ \nu^{\alpha}\text{-a.e.}$$

It follows that $|S_n\psi_\alpha| \leq 2\|f\|$, ν^{α}-a.e. Therefore, we have $\nu^{\alpha}(\mathcal{UL}^{\alpha}_{\psi,v}) = 1$. Hence, $\mathcal{UL}^{\alpha}_{\psi,v}$ is non-empty. Moreover, since $\dim_u(\nu^{\alpha}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v})$, we deduce that $\dim_u(\mathcal{UL}^{\alpha}_{\psi,v}) = \dim_u(\mathcal{L}^{\alpha}_{\psi,v})$ when $\alpha = \alpha^-_{\psi,v}$. By symmetry, we also have the same result for $\alpha = \alpha^+_{\psi,v}$. $\square$

4.2.3 Remarks on Boundary of Dimension Spectrum

We assumed in both Theorem 4.7 and Theorem 4.8 that $\alpha \notin \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$. It is thus natural to ask whether these two theorems remains true for $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$. It is unclear if Theorem 4.8 holds for $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$. On the other hand, Theorem 4.7, as we shall see below, fails in general for $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$.

Proposition 4.14. *Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing SFT. Let $\psi : \Sigma \to \mathbb{R}$ and $v : \Sigma \to (0, +\infty)$ be Hölder continuous functions satisfying that $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$. Then, for any $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$, there is an admissible word ω satisfying $\mathcal{UL}^{\alpha}_{\psi,v} \cap \mathcal{F}_\omega = \varnothing$.*

Remark 4.15. Pick ψ such that $\alpha^-_{\psi,v} < \alpha^+_{\psi,v}$, $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$ and $\dim_u(\mathcal{L}^{\alpha}_{\psi,v}) > 0$. Such a Hölder continuous function ψ exists [Sch99]. Then, we immediately get a counterexample for the claim of Theorem 4.7 when $\alpha \in \{\, \alpha^-_{\psi,v}, \alpha^+_{\psi,v} \,\}$.

Proof of Proposition 4.14. We only prove the proposition for the case where $\alpha = \alpha^-_{\psi,v}$. When $\alpha = \alpha^+_{\psi,v}$, one can use a similar argument to prove the same claim. We still let l be a positive integer satisfying that for any two symbols $a, b \in A$, there is some $\rho \in A^l_{\mathbb{M}}$ such that $a\rho b$ is admissible.

Assume $\alpha = \alpha^-_{\psi,v}$. Since $\alpha^+_{\psi,v} > \alpha^-_{\psi,v} = \alpha$, there is some $\xi^* \in \Sigma$ such that $\inf_{n\geq 1} S_n\psi_\alpha(\xi^*) = -\infty$. Take a positive integer n such that

$$S_n\psi_\alpha(\xi^*) < -V_{\psi_\alpha} - l \cdot \|\psi_\alpha\| - C_- - 1, \tag{4.14}$$

where $C_- = \sup_{m\geq 1}\sup_{\zeta\in\Sigma} S_m\psi_\alpha(\zeta)$ is finite due to Theorem 2.16. Set

$$\omega = \xi^*_1 \cdots \xi^*_n \rho,$$

where $\rho \in A^l_{\mathbb{M}}$ satisfies that $\omega^2 = \omega\omega$ is admissible. It then follows from (4.14) that

$$\sup_{\zeta\in[\omega]} S_{|\omega|}\psi_\alpha(\zeta) \leq S_{|\omega|}\psi_\alpha(\xi^*) + V_{\psi_\alpha} < -C_- - 1. \tag{4.15}$$

Given any $\xi \in \mathcal{F}_\omega$, we shall show that $\xi \notin \mathcal{UL}^\alpha_{\psi,v}$, which will complete the proof.

Since $\xi \in \mathcal{F}_\omega$, we can find a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that $\xi \in \bigcap_{k=1}^\infty \sigma^{-n_k}[\omega]$ and $n_j - n_{j-1} > |\omega|$ for each positive integer j, where we set $n_0 = 0$. Then, for any integer $k \geq 2$, from (4.15) and the definition of C_-, we see that

$$\begin{aligned}
S_{n_k}\psi_\alpha(\xi) &\leq \sum_{j=1}^{k} S_{n_j - n_{j-1}}\psi_\alpha(\sigma^{n_{j-1}}(\xi)) \\
&\leq C_- + \sum_{j=2}^{k} S_{|\omega|}\psi_\alpha(\sigma^{n_{j-1}}(\xi)) + S_{n_j-n_{j-1}-|\omega|}\psi_\alpha(\sigma^{n_{j-1}+|\omega|}(\xi)) \\
&< C_- + \sum_{j=2}^{k} (-C_- - 1 + C_-) = C_- - k + 1.
\end{aligned}$$

We can thus conclude that $\xi \notin \mathcal{UL}^\alpha_{\psi,v}$. $\square$

Chapter 5

Multifractal Analysis of Hölder Regularity of Gibbs Measures in $\mathbb{R}$

In this chapter, we use Theorem 4.8 we proved in Section 4.2 to study the Hölder regularity of the cumulative distribution function of a Gibbs measure in $\mathbb{R}$.

In Section 5.1, we define a conformal graph directed system, as well as its limit set and the coding map. The limit set will be the support of the Gibbs measure in $\mathbb{R}$ which we consider. In Section 5.2, we shall give some results on the Hausdorff dimension of $\mathcal{UD}_F^\alpha$, where F is the cumulative distribution function of a Gibbs measure in $\mathbb{R}$.

5.1 Conformal Graph Directed Systems

In this section we basically follow [MU03].

5.1.1 Directed Multigraph and Edge Shift

In this subsection, we define the notion of a directed multigraph and the edge shift associated to it.

We begin with the definition of directed multigraphs. Let $\mathcal{V}$ and $\mathcal{E}$ be two disjoint sets. The set $\mathcal{V}$ is a non-empty set, whose elements are called *vertices*. The set $\mathcal{E}$ is a set disjoint from $\mathcal{V}$, whose elements are called *edges*. For each edge $e \in \mathcal{E}$, there are two vertices associated to e. One is the initial vertex $p_-(e)$ of e, and the other is the terminal vertex $p_+(e)$ of e. This defines two mappings $p_-, p_+ : \mathcal{E} \to \mathcal{V}$. A *directed multigraph* is then the tuple $(\mathcal{V}, \mathcal{E}, p_-, p_+)$. Henceforth, we always make the following assumptions for every directed multigraph that will appear in the subsequent discussions:

1. $\mathcal{V}$ is finite;

2. $\mathcal{E}$ contains at least two and at most countably many edges;

3. for any vertex $p \in \mathcal{V}$, there is at least one edge $e \in \mathcal{E}$ such that $p = p_-(e)$.

If an edge $e \in \mathcal{E}$ satisfies that $p_-(e) = p_+(e)$, it is called a self-loop. Note that according to our definition, a directed graph may possibly have self-loops. In addition, there might be two distinct edges $e, e' \in \mathcal{E}$ such that $p_-(e) = p_-(e')$ and $p_+(e) = p_+(e')$. In some literature in graph theory, a directed multigraph can have multiple edges with the same initial and terminal vertices, but is not allowed to have a self-loop; see e.g. [BG10, p. 4]. Here we use the terminology in the same way as [MU03], in which directed multigraphs are allowed to have self-loops as well.

Given a directed multigraph $(\mathcal{V}, \mathcal{E}, p_-, p_+)$ for which $\mathcal{V}$ and $\mathcal{E}$ is finite, a natural SFT arises as follows. We call it the edge shift associated with the directed multigraph as in Section 3.2 of [BS02].

Definition. *Let $(\mathcal{V}, \mathcal{E}, p_-, p_+)$ be a directed multigraph with finitely many vertices and edges. Set $A = \mathcal{E}$. Define $\mathbb{M} : A \times A \to \{ 0, 1 \}$ by letting $\mathbb{M}(e, e') = 1$ if and only if $p_+(e) = p_-(e')$, for any $e, e' \in \mathcal{E}$. Then, the* edge shift *associated with the directed multigraph $(\mathcal{V}, \mathcal{E}, p_-, p_+)$ is the SFT $\sigma : \Sigma \to \Sigma$ for which the set of symbols is A and the incidence matrix is $\mathbb{M}$.*

Note that the matrix $\mathbb{M}$ is an incidence matrix because we assumed that for every vertex $p \in \mathcal{V}$, there is at least one edge $e \in \mathcal{E}$ such that $p = p_-(e)$.

A *path* in the directed multigraph is an admissible word $e_1 \cdots e_n$ over $A = \mathcal{E}$, with the incidence matrix taken to be $\mathbb{M}$ defined above. For $p = p_-(e_1)$ and $p' = p_+(e_n)$, the path $e_1 \cdots e_n$ is said to be a path from p to p'. Then, for SFTs given by directed multigraphs, Proposition 2.2 can be rewritten as follows.

Proposition 5.1. *The SFT $\sigma : \Sigma \to \Sigma$ associated with a directed multigraph $(\mathcal{V}, \mathcal{E}, p_-, p_+)$ is topologically mixing if and only if there exists a positive integer $l \in \mathbb{Z}_{>0}$ such that for any two vertices $p, p' \in \mathcal{V}$, there is a path $e_1 \cdots e_l$ from p to p'.*

5.1.2 Conformal Graph Directed Systems in $\mathbb{R}$

In this subsection, we define the conformal graph directed systems in $\mathbb{R}$.

Definition. *A* conformal graph directed system (CGDS) *in $\mathbb{R}$ consists of a directed multigraph $(\mathcal{V}, \mathcal{E}, p_-, p_+)$, a family of compact intervals $\mathcal{I} = \{ I_p \mid p \in \mathcal{V} \}$ and a family of contractions $\Phi = \{ g_e : I_{p_+(e)} \to I_{p_-(e)} \mid e \in \mathcal{E} \}$ satisfying the following conditions:*

1. for each $p \in \mathcal{V}$, I_p has positive length;

2. *for any $p \in \mathcal{V}$, there is an open neighbourhood U_p of I_p such that for every $e \in \mathcal{E}$ with $p_+(e) = p$, there is a C^1 diffeomorphism $\tilde{g}_e : U_p \to \tilde{g}_e(U_p) \subseteq U_{p_-(e)}$ such that $\tilde{g}_e\big|_{I_p} = g_e$;*

3. *for every $e \in \mathcal{E}$, $|\tilde{g}_e'|$ is Hölder continuous on $I_{p_+(e)}$;*

4. *Φ satisfies the* open set condition*, i.e. for any two distinct $a, b \in \mathcal{E}$,*

$$g_a(\mathrm{Int}(I_{p_+(a)})) \cap g_b(\mathrm{Int}(I_{p_+(b)})) = \varnothing.$$

We shall simply use $\Phi = \{\, g_e \mid e \in \mathcal{E} \,\}$ to denote the CGDS as in [MU03].

The original definition of a CGDS in a Euclidean space of an arbitrary finite dimension was introduced in Section 4.2 of [MU03]. For higher dimensions, the contractions in Φ are assumed to be conformal, which explains its name. In this chapter, we shall only consider the CGDS in $\mathbb{R}$, so the conformality of the contractions trivially holds.

We shall only consider a CGDS for which the directed multigraph has merely finitely many edges.

Definition. *We say that a CGDS $\Phi = \{\, g_e \mid e \in \mathcal{E} \,\}$ in $\mathbb{R}$ is* finitely generated *if and only if $\mathcal{E}$ is finite.*

For a finitely generated CGDS $\Phi = \{\, g_e \mid e \in \mathcal{E} \,\}$ in $\mathbb{R}$, we define

$$\lambda_\Phi = \max_{e \in \mathcal{E}} \sup \left\{ \frac{|g_e(x) - g_e(y)|}{|x - y|} \;\middle|\; x, y \in I_{p_+(e)},\ x \neq y \right\}.$$

Clearly, $\lambda_\Phi < 1$ because every $g_e \in \Phi$ is a contraction. Similarly, for a finitely generated CGDS $\Phi = \{\, g_e \mid e \in \mathcal{E} \,\}$ in $\mathbb{R}$, as $|\tilde{g}_e'|$ is Hölder continuous for each $e \in \mathcal{E}$, there exist $s_\Phi > 0$ and $M_\Phi \geq 1$ such that for any $e \in \mathcal{E}$ and any $x, y \in I_{p_+(e)}$, we have

$$\big||\tilde{g}_e'(x)| - |\tilde{g}_e'(y)|\big| \leq M_\Phi |x - y|^{s_\Phi}.$$

5.1.3 Limit Set and Coding Map of CGDS

Given a finitely generated CGDS $\Phi = \{\, g_e \mid e \in \mathcal{E} \,\}$ in $\mathbb{R}$, we construct a set called the limit set of Φ.

In what follows, the edge shift of Φ means the edge shift of the directed multigraph of Φ. Recall that when defining the edge shift $\sigma : \Sigma \to \Sigma$, we took the set of symbols A to be $\mathcal{E}$. For any non-empty admissible word ω over $A = \mathcal{E}$, define

$$g_\omega = g_{\omega_1} \circ \cdots \circ g_{\omega_{|\omega|}}.$$

Then, g_ω is clearly a contraction defined on $I_{p_+(\omega_{|\omega|})}$. Define $I_\omega = g_\omega(I_{p_+(\omega_{|\omega|})})$. Similarly, for any non-empty admissible word ω over $A = \mathcal{E}$, define

$$\tilde{g}_\omega = \tilde{g}_{\omega_1} \circ \cdots \circ \tilde{g}_{\omega_{|\omega|}},$$

where for each $e \in \mathcal{E}$, $\tilde{g}_e$ is the extension of g_e to U_e in the definition of CGDS. With the definitions above defined, we have the following proposition.

Proposition 5.2 ([MU03, Lemma 4.2.2]). *For any finitely generated CGDS Φ in $\mathbb{R}$, we have*

$$\sup_{\omega \in A^*_{\mathbb{M}}} \sup \left\{ \frac{\big|\log|\tilde{g}'_\omega(x)| - \log|\tilde{g}'_\omega(y)|\big|}{|x-y|^{s_\Phi}} \;\middle|\; x, y \in I_{p_+(\omega_{|\omega|})},\ x \neq y \right\} < +\infty.$$

In particular, we have

$$\sup_{\omega \in A^*_{\mathbb{M}}} \sup_{x,y \in I_{p_+(\omega_{|\omega|})}} \big|\log|\tilde{g}'_\omega(x)| - \log|\tilde{g}'_\omega(y)|\big| < +\infty.$$

We continue with our construction of the limit set of the CGDS Φ. Note that for any non-empty admissible word ω over $A = \mathcal{E}$ and any positive integer $k < |\omega|$, we have $I_{\omega_1 \cdots \omega_k} \supsetneq I_\omega$ [MU03, p. 2]. For any $\xi \in \Sigma$ and any $k \in \mathbb{Z}_{>0}$, we have

$$\operatorname{diam}(I_{\xi_1 \cdots \xi_k}) \leq \lambda_\Phi^{k-1} \max_{e \in \mathcal{E}} \operatorname{diam}(I_e).$$

As $\lambda_\Phi < 1$, we see that $\lim_{k \to +\infty} \operatorname{diam}(I_{\xi_1 \cdots \xi_k}) = 0$. Hence, we can see that there is a unique element in $\bigcap_{k=1}^\infty I_{\xi_1 \cdots \xi_k}$. Denote this element by $\pi(\xi)$. Thus, we have a map $\pi : \Sigma \to \bigcup_{p \in \mathcal{V}} I_p$. This map π is called the *coding map* of Φ and the *limit set* of Φ is

$$\Lambda = \pi(\Sigma) = \bigcup_{\xi \in \Sigma} \bigcap_{k=1}^\infty I_{\xi_1 \cdots \xi_k}.$$

From the definition, we can readily see that

$$\pi(\xi) = g_{\xi_1 \cdots \xi_n}(\pi(\sigma^n(\xi))), \tag{5.1}$$

for any $n \in \mathbb{Z}_{>0}$ and any $\xi \in \Sigma$.

The coding map is continuous; indeed we can say more about the coding map. For this purpose, we introduce a metric on Σ, which will be given by the volume potential defined as follows.

Definition ([MU03, Section 8.2]). *The* volume potential *of a finitely generated CGDS $\Phi = \{ g_e \mid e \in \mathcal{E} \}$ in $\mathbb{R}$ is the function $u : \Sigma \to \mathbb{R}$ defined by*

$$u(\xi) = -\log|\tilde{g}'_{\xi_1}(\pi(\sigma(\xi)))|$$

for every $\xi \in \Sigma$.

Proposition 5.3. *For any finitely generated CGDS Φ in $\mathbb{R}$, its volume potential u is well-defined, positive and Hölder continuous.*

Proof. For any $\xi \in \Sigma$, note that $\pi(\sigma(\xi)) \in I_{\xi_2} \subseteq I_{p_-(\xi_2)} = I_{p_+(\xi_1)}$, so $\pi(\sigma(\xi))$ is in the domain of $\tilde{g}_{\xi_1}$. As $\tilde{g}_{\xi_1}$ is a C^1 diffeomorphism, $\tilde{g}'_{\xi_1}(\pi(\sigma(\xi)))$ cannot be zero. Therefore, u is well-defined. Since every member of Φ is a contraction, we also have $u > 0$.

Finally, we shall show the Hölder continuity of u. Take $\xi, \zeta \in \Sigma$ satisfying $\xi_1 = \zeta_1$ arbitrarily. Define $n = |\xi \wedge \zeta| \geq 1$. Then, $p_-(\xi_{n+1}) = p_-(\zeta_{n+1})$. Hence, by (5.1), we have

$$|\pi(\sigma(\xi)) - \pi(\sigma(\zeta))| \leq \lambda_\Phi^{n-1} |\pi(\sigma^n(\xi)) - \pi(\sigma^n(\zeta))| \leq \lambda_\Phi^{n-1} \max_{p \in \mathcal{V}} \operatorname{diam}(I_p).$$

Combining this with Proposition 5.2, we have the Hölder continuity of u. □

Note that by the chain rule, we have

$$S_n u(\xi) = -\log |\tilde{g}'_{\xi_1 \cdots \xi_n}(\pi(\sigma^n(\xi)))|,$$

for any $\xi \in \Sigma$ and $n \in \mathbb{Z}_{>0}$. Hence, as a consequence of Proposition 5.2, we have

$$\sup_{n \in \mathbb{Z}_{>0}} \sup_{\xi \in \Sigma} \sup_{x \in I_{p_+(\xi_n)}} \left| -S_n u(\xi) - \log \left| \tilde{g}'_{\xi_1 \cdots \xi_n}(x) \right| \right| < +\infty. \tag{5.2}$$

Proposition 5.3 enables us to endow the shift space Σ with the metric d_u. For this metric, we claim the following for the coding map π.

Proposition 5.4. *The coding map $\pi : \Sigma \to \bigcup_{p \in \mathcal{V}} I_p$ is Lipschitz continuous. Moreover, for any compact $X \subseteq \Sigma$ satisfying that $\sigma(X) \subseteq X$, we have that $\pi\big|_X : X \to \pi(X)$ is bi-Lipschitz if and only if it is injective.*

Proof. By the mean value theorem and (5.1), we have that for any two distinct $\xi, \xi' \in \Sigma$ with $\xi_1 = \xi'_1$,

$$|\pi(\xi) - \pi(\xi')| \leq \sup \left\{ \left| \tilde{g}'_{\xi \wedge \xi'}(x) \right| \;\middle|\; x \in I_{p_+(\xi_n)} \right\} \cdot \max_{p \in \mathcal{V}} \operatorname{diam}(I_p),$$

where $n = |\xi \wedge \xi'|$. Hence, combining this observation with (5.2) and the Hölder continuity of u, we can deduce that π is Lipschitz continuous.

Regarding the second claim, we only need to show that $\pi\big|_X^{-1}$ is Lipschitz continuous under the assumption that $\pi\big|_X$ is injective and thus invertible. Again, using the mean value theorem and (5.1), we have

$$|\pi(\zeta) - \pi(\zeta')| \geq \inf \left\{ \left| \tilde{g}'_{\zeta \wedge \zeta'}(x) \right| \;\middle|\; x \in I_{p_+(\zeta_n)} \right\} \cdot |\pi(\sigma^m(\zeta)) - \pi(\sigma^m(\zeta'))|,$$

for any two distinct $\zeta, \zeta' \in X$ with $\zeta_1 = \zeta_1'$ and $m = |\zeta \wedge \zeta'|$. Since $\sigma(X) \subseteq X$, we have that $\sigma^m(\zeta)$ and $\sigma^m(\zeta')$ are both in X. Hence, $|\pi(\sigma^m(\zeta)) - \pi(\sigma^m(\zeta'))| \geq c_0$, where

$$c_0 = \min\left\{ \inf_{\eta \in [e]\cap X} \inf_{\eta' \in [e']\cap X} |\pi(\eta) - \pi(\eta')| \;\middle|\; e, e' \in \mathcal{E},\, e \neq e' \right\}.$$

As long as we can show that the constant c_0 is positive, then as in the previous paragraph, we can show that $\pi\big|_X^{-1}$ is Lipschitz continuous. Suppose that $c_0 = 0$. Then, as $[a] \cap X$ is compact for any $a \in \mathcal{E}$, there exist distinct $e, e' \in \mathcal{E}$, $\eta \in [e] \cap X$ and $\eta' \in [e'] \cap X$ such that $\pi(\eta) = \pi(\eta')$. This violates the injectivity of π, so $c_0 > 0$. $\square$

5.2 Multifractal Analysis of Hölder Regularity

Suppose that we are given a Gibbs measure ν on Σ of some Hölder continuous function $\psi : \Sigma \to \mathbb{R}$, whose topological pressure $P(\psi)$ equals zero. The Gibbs measure ν for ψ, which is a measure supported on Σ, gives rise to a measure supported on the limit set Λ, namely the pushforward measure $\pi_*\nu$ through the coding map π. Such a measure on Λ is called a Gibbs measure in $\mathbb{R}$.

5.2.1 Main Results

Recall that for any continuous function $f : \mathbb{R} \to \mathbb{R}$, any $\alpha \geq 0$ and any $x \in \mathbb{R}$,

$$\begin{aligned}
\underline{D}^\alpha f(x) &= \liminf_{y \to x} \frac{|f(y) - f(x)|}{|y - x|^\alpha}; \\
\overline{D}^\alpha f(x) &= \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|^\alpha}; \\
\mathcal{UD}_f^\alpha &= \left\{ x \in \mathbb{R} \;\middle|\; 0 < \underline{D}^\alpha f(x) \leq \overline{D}^\alpha f(x) < +\infty \right\}.
\end{aligned}$$

The main theorem of this chapter is then given as follows.

Theorem 5.5 ([Liu23]). *Let $\Phi = \{ g_e \mid e \in \mathcal{E} \}$ be a finitely generated CGDS in $\mathbb{R}$. Let $\sigma : \Sigma \to \Sigma$ be the edge shift associated with Φ. We assume that $\sigma : \Sigma \to \Sigma$ is topologically mixing. Let $\psi : \Sigma \to \mathbb{R}$ be a Hölder continuous function satisfying $P(\psi) = 0$. Let ν be a Gibbs measure for ψ and F be the cumulative distribution function of $\pi_*\nu$, where $\pi : \Sigma \to \mathbb{R}$ denotes the coding map. Let $u : \Sigma \to (0, +\infty)$ denote the volume potential of Φ. Then, we have*

$$\dim_H(\mathcal{UD}_F^\alpha) \leq \dim_u(\mathcal{L}_{\psi,u}^{-\alpha})$$

for any $\alpha \geq 0$*, and*

$$\dim_H(\mathcal{UD}_F^{\alpha}) = \dim_u(\mathcal{L}_{\psi,u}^{-\alpha})$$

if $\alpha \notin \{ -\alpha_{\psi,u}^-, -\alpha_{\psi,u}^+ \}$.

Similar to Proposition 4.4, the following corollary of Theorem 5.5 holds.

Corollary 5.6. *There is a unique* $\alpha_0 \in \mathbb{R}$ *such that* $\dim_H(\mathcal{UD}_F^{\alpha_0}) = \dim_H(\Lambda)$. *Moreover, we have either* $\alpha_0 = -\alpha_{\psi,u}^- = -\alpha_{\psi,u}^+$ *or* $-\alpha_{\psi,u}^+ < \alpha_0 < -\alpha_{\psi,u}^-$.

5.2.2 Proofs

In this subsection, we prove Theorem 5.5 and Corollary 5.6.

Our proof of Theorem 5.5 uses Theorem 4.8. It means that we need to first construct a finite set $\mathcal{W}$ of admissible words.

For every edge $e \in \mathcal{E}$, define $P_e = \{ \min(I_e \cap \Lambda), \max(I_e \cap \Lambda) \}$ and $P = \bigcup_{e\in\mathcal{E}} P_e$.

Lemma 5.7 ([Liu23])**.** *The set* $\pi^{-1}(P)$ *is finite. Moreover, there exists a non-negative integer* N *such that for every* $\xi \in \pi^{-1}(P)$*,* $\sigma^N(\xi)$ *is a periodic sequence.*

Proof. Note that for any two distinct admissible words ω, τ of the same length, the open set condition implies that $\mathrm{Int}(I_\omega) \cap \mathrm{Int}(I_\tau) = \varnothing$. It follows that, for any $x \in \Lambda$, $\pi^{-1}\{ x \}$ has no more than two elements. Therefore, we have

$$\#\pi^{-1}(P) \leq 2 \cdot \#P \leq 4 \cdot \#\mathcal{E} < +\infty.$$

In order to show the second claim, we prove the following claim:

$$\sigma(\pi^{-1}P) \subseteq \pi^{-1}(P). \tag{5.3}$$

Take $\xi \in \pi^{-1}(P)$ arbitrarily. As $g_{\xi_1} : I_{\xi_2} \to I_{\xi_1\xi_2}$ is a C^1 diffeomorphism, it either preserves or reverses the ordering. Note that $g_{\xi_1}(\pi(\sigma(\xi))) = \pi(\xi)$ and $\pi(\xi) \in P \cap I_{\xi_1} = P_{\xi_1}$, so $g_{\xi_1}(\pi(\sigma(\xi))$ is either the minimum or the maximum of $I_{\xi_1} \cap \Lambda$. Therefore, by the monotonicity of g_{ξ_1} we have $\pi(\sigma(\xi)) \in P_{\xi_2} \subseteq P$. Since ξ is arbitrarily taken from $\pi^{-1}(P)$, we conclude that $\sigma(\pi^{-1}(P)) \subseteq \pi^{-1}(P)$.

Combining (5.3) with the fact that $\pi^{-1}(P)$ is a finite set, we see that for each $\xi \in \pi^{-1}(P)$, we can take a non-negative integer N_ξ such that $\sigma^{N_\xi}(\xi)$ is periodic. Take an arbitrary $N \geq \max_{\xi\in\pi^{-1}(P)} N_\xi$. This N then satisfies the requirement in the second claim. □

Take a non-negative integer N satisfying the condition in Lemma 5.7. Then, for each $\xi \in \pi^{-1}(P)$, we are able to take one word $\omega(\xi)$ such that $\sigma^N(\xi) = \omega(\xi)\omega(\xi)\cdots$. Henceforth, we always set

$$\mathcal{W} = \{ \omega(\xi) \mid \xi \in \pi^{-1}(P) \} .$$

Clearly, $\mathcal{W}$ is a finite set of admissible words.

Now we state the key lemma for the proof of Theorem 5.5. A similar assertion for a more restricted situation appears in [KS09, Proposition 2.3].

Lemma 5.8 ([Liu23]). *For any $\alpha > 0$, we have*

$$\mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W}} \subseteq \pi^{-1}\left(\mathcal{UD}_F^{\alpha}\right) \subseteq \mathcal{UL}_{\psi,u}^{-\alpha}. \tag{5.4}$$

Proof. Fix $\alpha > 0$. Define $\psi_{-\alpha} = \psi + \alpha u$. In this proof, given any two real numbers $x < y$, the closed interval $[x, y]$ can be denoted by $[x, y]$ itself or $[y, x]$.

We begin with the first inclusion in the proposition. Without loss of generality, assume that $\mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W}}$ is non-empty and take an arbitrary $\xi \in \mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W}}$. Now that $\xi \in \mathcal{F}'_{\mathcal{W}}$, we have $\sigma^n \xi \notin \pi^{-1}(P)$ for any non-negative integer n. In particular, $\pi(\xi)$ must lie in the interior of every $I_{\xi_1 \cdots \xi_n}$.

Now fix an arbitrary $y \neq \pi(\xi)$ in I_{ξ_1}. Since the set sequence $(I_{\xi_1 \cdots \xi_n})_{n \geq 1}$ is descending and $\bigcap_{n=1}^{\infty} I_{\xi_1 \cdots \xi_n} = \{ \pi(\xi) \}$, there exists a unique positive integer m such that $y \in I_{\xi_1 \cdots \xi_m} \setminus I_{\xi_1 \cdots \xi_{m+1}}$.

Note that $I_{\xi_1 \cdots \xi_m}$ is connected, so on the one hand, we can readily see that

$$[\pi(\xi), y] \subseteq I_{\xi_1 \cdots \xi_m}. \tag{5.5}$$

On the other hand, take l to be a positive integer such that $\xi \in \mathcal{F}'_{\mathcal{W},l}$. Let N be a non-negative integer such that for any $\xi' \in \pi^{-1}(P)$, $\sigma^N(\xi')$ is periodic. The existence of this N is guaranteed by Lemma 5.7. Define

$$L = (l+1)\|\mathcal{W}\| + N.$$

We claim that there is an admissible word τ over $\mathcal{E}$ of length L such that $\xi_{m+1}\tau$ is admissible and

$$I_{\xi_1 \cdots \xi_{m+1} \tau} \subseteq [\pi(\xi), y]. \tag{5.6}$$

To prove this claim, first note that $g_{\xi_1 \cdots \xi_{m-1}} : I_{\xi_m} \to I_{\xi_1 \cdots \xi_m}$ is invertible. Set $y' = g_{\xi_1 \cdots \xi_{m-1}}^{-1}(y)$ and $\xi' = \sigma^{m-1}(\xi)$. Define

$$z = \max(\Lambda \cap I_{\xi_m \xi_{m+1}})$$

if $\pi(\xi') \leq y'$. Otherwise, define

$$z = \min(\Lambda \cap I_{\xi_m \xi_{m+1}}).$$

Take $\zeta \in [\xi_m \xi_{m+1}]$ satisfying $\pi(\zeta) = z$. By the monotonicity of g_{ξ_m}, we see that $\pi(\sigma(\zeta))$ is in $P_{\xi_{m+1}}$. It follows that $\sigma^{N+1}(\zeta)$ is periodic, and furthermore, there exists an integer $k \in \{ 1, \cdots, \|\mathcal{W}\| \}$ and a word $\omega \in \mathcal{W}$ such that $\sigma^{N+1+k}(\zeta) = \omega\omega\cdots$. Take τ as the unique admissible word of length L such that $\zeta \in [\xi_m \xi_{m+1} \tau]$.

Then, on the one hand, we have that τ contains ω^l as a subword. On the other hand, since $\xi \in \mathcal{F}'_{\mathcal{W},l}$, the word $\xi_{m+2} \cdots \xi_{m+L+1}$ does not contain ω^l as a subword. Therefore, $\xi_m \xi_{m+1} \tau$ and $\xi_m \cdots \xi_{m+L+1}$ are distinct words of length $L+2$, so their cylinder sets are disjoint. As a result, we have

$$\pi(\xi') \notin I_{\xi_m \xi_{m+1} \tau}.$$

As $I_{\xi_m \xi_{m+1} \tau}$ is connected, $I_{\xi_m \xi_{m+1} \tau}$ lies on one side of $\pi(\xi')$. By the same reason, because $y' \notin I_{\xi_m \xi_{m+1}}$, the interval $I_{\xi_m \xi_{m+1} \tau}$ also lies on only one side of y'. By our definition of z, we have $z \in [y', \pi(\xi')]$, so $[y', \pi(\xi')] \cap I_{\xi_m \xi_{m+1} \tau} \neq \varnothing$. Combining this with the fact that $I_{\xi_m \xi_{m+1} \tau}$ is on one side of $\pi(\xi')$ and also on one side of y', we deduce that

$$I_{\xi_m \xi_{m+1} \tau} \subseteq [y', \pi(\xi')].$$

Applying $g_{\xi_1 \cdots \xi_{m-1}}$ on both sides, we have

$$\begin{aligned} I_{\xi_1 \cdots \xi_{m+1} \tau} &= g_{\xi_1 \cdots \xi_{m-1}}(I_{\xi_m \xi_{m+1} \tau}) \\ &\subseteq [g_{\xi_1 \cdots \xi_{m-1}}(y'), g_{\xi_1 \cdots \xi_{m-1}}(\pi(\xi'))] = [y, \pi(\xi)]. \end{aligned}$$

Note that the monotonicity of $g_{\xi_1 \cdots \xi_{m-1}}$ is used here. Therefore, τ satisfies (5.6).

By (5.5) and (5.6), we have

$$\pi_* \nu([y, \pi(\xi)]) \leq \pi_* \nu(I_{\xi_1 \cdots \xi_m}) \leq C_F \exp(S_{\xi_1 \cdots \xi_m} \psi) \leq C_F \exp(V_\psi) \exp(S_m \psi(\xi)),$$

and

$$\begin{aligned} \pi_* \nu([y, \pi(\xi)]) \geq \pi_* \nu(I_{\xi_1 \cdots \xi_m \tau}) &\geq C_F^{-1} \exp(S_{\xi_1 \cdots \xi_m \tau} \psi) \\ &\geq C_F^{-1} \exp(-L\|\psi\| - V_\psi) \exp(S_m \psi(\xi)). \end{aligned}$$

Then, by the mean value theorem and (5.2), there exists a constant $C_1 \geq 1$ such that

$$\begin{aligned} \operatorname{diam}(I_{\xi_1 \cdots \xi_m}) &\leq C_1 \exp(-S_m u(\xi)); \\ \operatorname{diam}(I_{\xi_1 \cdots \xi_m \tau}) &\geq C_1^{-1} \exp(-S_m u(\xi) - L\|u\|). \end{aligned}$$

Combining these inequalities with (5.5) and (5.6), we have

$$C_1^{-1} \exp(-S_n u(\xi) - L\|v\|) \leq |\pi(\xi) - y| \leq C_1 \exp(-S_n u(\xi)).$$

Consequently, there exists a constant $C \geq 1$ such that for $y \neq \pi(\xi)$ sufficiently close to $\pi(\xi)$,

$$C^{-1} \exp(S_n \psi(\xi)) \leq \pi_* \nu([\pi(\xi), y]) \leq C \exp(S_n \psi(\xi)); \tag{5.7}$$

$$C^{-1} \exp(-S_n u(\xi)) \leq |\pi(\xi) - y| \leq C \exp(-S_n u(\xi)). \tag{5.8}$$

From this we see that

$$C^{-2}\exp(S_n\psi_{-\alpha}(\xi)) \le \underline{D}^\alpha F(\pi(\xi)) \le \overline{D}^\alpha F(\pi(\xi)) \le C^2\exp(S_n\psi_{-\alpha}(\xi)).$$

As ξ is taken from $\mathcal{UL}_{\psi,u}^{-\alpha}$, we have $\pi(\xi) \in \mathcal{UD}_F^\alpha$. This shows the first half of (5.4).

Now it only remains to show the second half of (5.4). If $\pi^{-1}(\mathcal{UD}_F^\alpha)$ is empty, then the claim is trivial. Otherwise, take $\zeta \in \pi^{-1}(\mathcal{UD}_F^\alpha)$ arbitrarily.

We define a sequence $(y_n)_{n\ge1}$ of points in I in the following manner. For any $n \ge 1$, let $\omega^{(n)}$ be an admissible word of length $m(n) \le \#A$ satisfying that $\omega_1^{(n)} = \zeta_{n+1}$ and $\omega_{m(n)}^{(n)} \ne \zeta_{n+m(n)}$. By construction, we have $\pi(\zeta) \notin I_{\zeta_1\cdots\zeta_n\omega^{(n)}}$, so there is a unique endpoint y_n of $I_{\zeta_1\cdots\zeta_n\omega^{(n)}}$ satisfying that $I_{\zeta_1\cdots\zeta_n\omega^{(n)}} \subseteq [\pi(\zeta), y_n]$. Since $\pi(\zeta)$ and y_n are both in $I_{\zeta_1\cdots\zeta_n}$, we also have $[\pi(\zeta), y_n] \subseteq I_{\zeta_1\cdots\zeta_n}$. Therefore, there is some constant $C' \ge 1$ such that for all $n \in \mathbb{N}$,

$$\frac{1}{C'}\exp(S_n\psi_{-\alpha}(\zeta)) \le \frac{\pi_*\nu([\pi(\zeta), y_n])}{|\pi(\zeta) - y_n|^\alpha} \le C'\exp(S_n\psi_{-\alpha}(\zeta)).$$

Clearly $\lim_{n\to\infty} y_n = \pi(\zeta)$, so the inequality above implies that $\zeta \in \mathcal{UL}_{\psi,u}^{-\alpha}$. Hence, we may conclude that the second inclusion in (5.4) holds as well. □

Proof of Theorem 5.5. First note that $\mathcal{UD}_F^\alpha \subseteq \Lambda$ for $\alpha > 0$, because for any $x \notin \Lambda$, $\underline{D}^\alpha F(x) = \overline{D}^\alpha F(x) = 0$.

Endow Σ with the metric d_u given by u. Then, by Proposition 5.4, the coding map $\pi : \Sigma \to \Lambda$ is Lipschitz continuous. Hence, on the one hand, by Proposition 4.6, Lemma 5.8 and Proposition 3.4, we have that for any $\alpha \ge 0$,

$$\dim_u(\mathcal{L}_{\psi,u}^{-\alpha}) = \dim_u(\mathcal{UL}_{\psi,u}^{-\alpha}) \ge \dim_u(\pi^{-1}(\mathcal{UD}_F^\alpha)) \ge \dim_H(\mathcal{UD}_F^\alpha).$$

On the other hand, for any integer $l \ge 1$, it is clear that $\mathcal{F}'_{\mathcal{W},l}$ is compact and $\sigma(\mathcal{F}'_{\mathcal{W},l}) \subseteq \mathcal{F}'_{\mathcal{W},l}$. Also note that π is injective on $\mathcal{F}'_{\mathcal{W}}$. Hence, by Proposition 5.4, Proposition 3.4 and Lemma 5.8, we have that

$$\dim_u(\mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W},l}) = \dim_H(\pi(\mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W},l})) \le \dim_H(\mathcal{UD}_F^\alpha),$$

for any non-negative $\alpha \notin \{-\alpha_{\psi,u}^-, -\alpha_{\psi,u}^+\}$ and any positive integer l. Therefore, by the countable stability of Hausdorff dimension,

$$\dim_u\left(\mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W}}\right) = \sup_{l\ge1}\dim_u\left(\mathcal{UL}_{\psi,u}^{-\alpha} \cap \mathcal{F}'_{\mathcal{W},l}\right) \le \dim_H\left(\mathcal{UD}_F^\alpha\right).$$

By Theorem 4.8, when $\alpha \notin \{-\alpha_{\psi,u}^-, -\alpha_{\psi,u}^+\}$, all the inequalities above are equalities, so $\dim_H(\mathcal{UD}_F^\alpha) = \dim_u(\mathcal{L}_{\psi,u}^{-\alpha})$. □

Finally we prove Corollary 5.6.

Proof of Corollary 5.6. First consider the case where $\alpha^-_{\psi,u} < \alpha^+_{\psi,u}$. Then by Proposition 4.4. there is a unique $\alpha_0 \in (-\alpha^+_{\psi,u}, -\alpha^-_{\psi,u})$ such that $\dim_u(\mathcal{L}^{-\alpha_0}_{\psi,u}) = \dim_u(\Sigma)$. By Theorem 5.5, we thus have

$$\dim_H(\mathcal{UD}^{\alpha_0}_F) = \dim_u(\mathcal{L}^{-\alpha_0}_{\psi,u}) = \dim_u(\Sigma) = \dim_H(\Lambda),$$

and for any $\alpha \neq \alpha_0$,

$$\dim_H(\mathcal{UD}^{\alpha}_F) \leq \dim_u(\mathcal{L}^{-\alpha}_{\psi,u}) < \dim_u(\Sigma) = \dim_H(\Lambda).$$

This completes the proof for the case where $\alpha^-_{\psi,u} < \alpha^+_{\psi,u}$.

When $\alpha^-_{\psi,u} = \alpha^+_{\psi,u}$, take α_0 to be $-\alpha^-_{\psi,u} = -\alpha^+_{\psi,u}$. Then we only need to show that

$$\dim_H(\mathcal{UD}^{\alpha_0}_F) = \dim_H(\Lambda), \tag{5.9}$$

because for any non-negative $\alpha \neq \alpha_0$, by Lemma 5.8, we have $\mathcal{UD}^{\alpha}_F \subseteq \pi(\mathcal{L}^{-\alpha}_{\psi,u}) = \varnothing$. To show (5.9), first note that by Lemma 5.8 and Corollary 4.9, we have

$$\mathcal{F}'_{\mathcal{W}} = \Sigma \cap \mathcal{F}'_{\mathcal{W}} = \mathcal{UL}^{-\alpha_0}_{\psi,u} \cap \mathcal{F}'_{\mathcal{W}} \subseteq \pi^{-1}(\mathcal{UD}^{\alpha_0}_F).$$

Also recall that we have seen in Corollary 4.9 that $\dim_u(\mathcal{F}'_{\mathcal{W}}) = \dim_u(\Sigma)$, so

$$\dim_H(\mathcal{UD}^{\alpha_0}_F) = \dim_u(\pi^{-1}(\mathcal{UD}^{\alpha_0}_F)) \geq \dim_u(\Sigma) = \dim_H(\Lambda).$$

Our proof is thus complete. $\square$

5.2.3 Case Study

In this section, we apply our results to one family of Borel probability measures, each of which is supported on $[0, 1]$.

Let $\mathcal{V} = \{ p \}$ and $\mathcal{E} = \{ e_1 = 0, e_2 = 1 \}$. Then there is only one directed multigraph for which the set of vertices is $\mathcal{V}$ and the set of edges is $\mathcal{E}$, for the initial and terminal vertices of e_1 and e_2 must be p. It is easy to check using Proposition 5.1 that the associated edge shift $\sigma : \Sigma \to \Sigma$ is topologically mixing.

Let $I_p = [0, 1]$. Define $g_{e_1}, g_{e_2} : I_p \to I_p$ by letting

$$g_{e_1}(x) = x/2;$$
$$g_{e_2}(x) = (x + 1)/2,$$

for any $x \in I_p$. Then, it is clear that $\Phi = \{ g_{e_1}, g_{e_2} \}$ is a CGDS in $\mathbb{R}$. For any arbitrary non-empty word ω over $\mathcal{E}$, we have

$$I_\omega = \left[\sum_{j=1}^{|\omega|} \omega_j 2^{-j}, \sum_{j=1}^{|\omega|} \omega_j 2^{-j} + 2^{-|\omega|} \right]$$

This can be shown by induction on the length of ω. Recall that any $x \in [0,1]$ has a binary expansion

$$x = \sum_{j=1}^{\infty} \xi_j 2^{-j},$$

where $\xi_j \in \{0,1\}$ for any positive integer j. It is then clear that $x \in I_{\xi_1 \cdots \xi_k}$ for any positive integer k. It thus follows that $\pi(\xi) = x$. In particular, the limit set of Φ is $I_p = [0,1]$.

Now for any $\lambda \in [0,1]$, we define a Borel probability measure μ_λ on I_p by letting

$$\mu_\lambda(I_\omega) = \lambda^{\#\{k \in \{1,\cdots,|\omega|\} | \omega_k = e_1\}} (1-\lambda)^{\#\{k \in \{1,\cdots,|\omega|\} | \omega_k = e_2\}}, \tag{5.10}$$

for any non-empty word ω over $\mathcal{E}$. When $\lambda = 1$, μ_λ is supported on $\{0\}$. When $\lambda = 0$, μ_λ is supported on $\{1\}$. When $\lambda = 1/2$, μ_λ is precisely the Lebesgue measure on I_p. In what follows, we will focus on the non-trivial case, namely the case where $\lambda \in (0,1/2) \cup (1/2,1)$. Let $F_\lambda : \mathbb{R} \to \mathbb{R}$ be the cumulative distribution function of μ_λ.

We define the binary entropy function $H_2 : [0,1] \to \mathbb{R}$ by

$$H_2(t) = \frac{-t\log(t) - (1-t)\log(1-t)}{\log(2)}$$

for $t \in (0,1)$ and $H_2(t) = 0$ for $t \in \{0,1\}$. Then, for $\lambda \in (0,1/2) \cup (1/2,1)$, we will show the following properties about F_λ.

Proposition 5.9. *For any $\lambda \in (0,1/2) \cup (1/2,1)$, the following statements hold:*

1. *F_λ is strictly increasing on $I_p = [0,1]$, and also continuous on $\mathbb{R}$;*
2. *$\mathcal{UD}^\alpha_{F_\lambda}$ is non-empty if and only if $\alpha_- < \alpha < \alpha_+$, where*

$$\begin{aligned} \alpha_- &= \min\{-\log(1-\lambda)/\log(2), -\log(\lambda)/\log(2)\}; \\ \alpha_+ &= \max\{-\log(1-\lambda)/\log(2), -\log(\lambda)/\log(2)\}; \end{aligned}$$

3. *for any $\alpha \in [\alpha_-, \alpha_+]$, we have*

$$\dim_H(\mathcal{UD}^\alpha_{F_\lambda}) = H_2\left(\frac{\alpha\log(2) + \log(1-\lambda)}{\log(1-\lambda) - \log(\lambda)}\right). \tag{5.11}$$

As an example, the graph of $\alpha \mapsto \dim_H(\mathcal{UD}^\alpha_{F_{1/3}})$ is plotted in Figure 5.1.

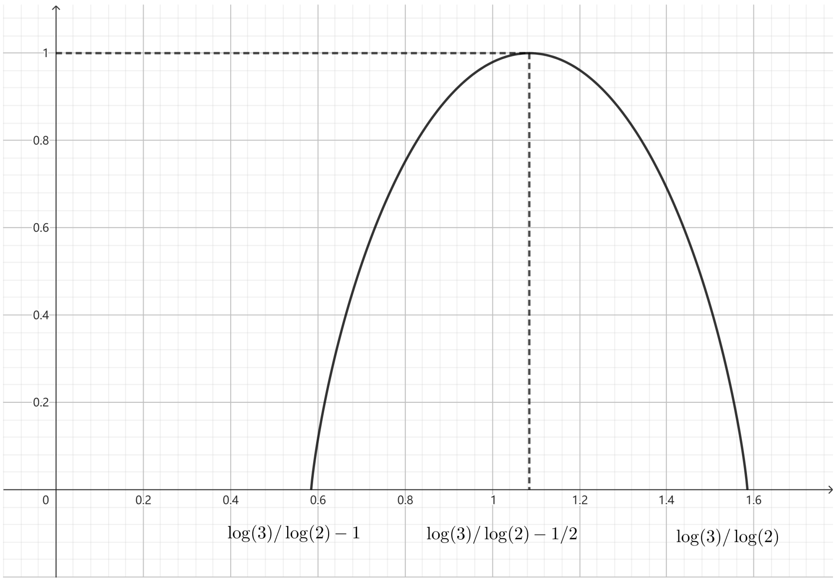

Figure 5.1: The dimension spectrum of $\mathcal{UD}^{\alpha}_{F_{1/3}}$.

Proof. Fix $\lambda \in (0, 1/2) \cup (1/2, 1)$. We first show that μ_λ is a Gibbs measure in $\mathbb{R}$, which is supported on I_p. Define $\psi : \Sigma \to \mathbb{R}$ by letting $\psi([e_1]) = \{ \log(\lambda) \}$ and $\psi([e_2]) = \{ \log(1 - \lambda) \}$. By the formula we gave in Example 2.10, we have

$$P(\psi) = \log(\lambda + 1 - \lambda) = 0.$$

Note that the Bernoulli measure ν_ψ associated with the 2-tuple $(\lambda, 1 - \lambda)$, which was defined in Example 2.8, is a Gibbs measure for ψ, so ν_ψ is the unique equilibrium state of ψ. Let $\pi : \Sigma \to I_p$ be the coding map for the CGDS we defined. Then, we have $\mu_\lambda = \pi_* \nu_\psi$, so μ_λ is a Gibbs measure in $\mathbb{R}$. As ν_ψ is a Gibbs measure for ψ, we see that $\nu_\psi(O) > 0$ for any non-empty open subset of Σ. Hence, by the continuity of the coding map π, we have that the support of μ_λ is precisely I_p.

Since the support of μ_λ is I_p, F_λ is strictly increasing. For any $x \in \mathbb{R}$, the cardinality of $\pi^{-1}(\{ x \})$ is no greater than 2. Thus, from the fact that $\nu_\psi(E) = 0$ for any finite subset E of Σ, we obtain that $\mu_\lambda(\{ x \}) = 0$, implying the continuity of F_λ.

In our case, the volume potential u of our CGDS Φ is clearly constantly equal to $\log(2)$. Therefore, we have

$$-\alpha^+_{\psi,u} = -\sup_{\xi \in \Sigma} \limsup_{n \to +\infty} \frac{S_n \psi(\xi)}{S_n u(\xi)} = -\frac{\max_{j \in \{1,2\}} \psi([e_j])}{\log(2)} = \alpha_-;$$

$$-\alpha^-_{\psi,u} = -\inf_{\xi \in \Sigma} \liminf_{n \to +\infty} \frac{S_n \psi(\xi)}{S_n u(\xi)} = -\frac{\min_{j \in \{1,2\}} \psi([e_j])}{\log(2)} = \alpha_+.$$

Therefore, the second claim for $\alpha \notin \{ \alpha_-, \alpha_+ \}$ can be proved from Theorem 4.1, Theorem 4.8 and Lemma 5.8. For the rest of this paragraph, we assume that $\alpha \in \{ \alpha_-, \alpha_+ \}$. In view of Lemma 5.8, we can see that the proof for our second item will be complete if we can show that $\pi(\xi) \notin \mathcal{UD}^{\alpha}_{F_\lambda}$ for any $\xi \in \mathcal{UL}^{-\alpha}_{\psi,u}$. When $\pi(\xi) \in \{ 0, 1 \}$, as $F_\lambda = 0$ outside $[0,1]$, we have $\pi(\xi) \notin \mathcal{UD}^{\alpha}_{F_\lambda}$. When $\pi(\xi) \notin \{ 0, 1 \}$, as $\xi \in \mathcal{UL}^{-\alpha}_{\psi,u}$, there exist non-empty words ω^1 and ω^2 over $\mathcal{E}$ such that $\pi(\xi) = \pi(\omega^1 e_1 e_1 \cdots) = \pi(\omega^2 e_2 e_2 \cdots)$. Note that

$$\lim_{n\to+\infty} \frac{S_n\psi(\omega^1 e_1 e_1 \cdots)}{S_n u(\omega^1 e_1 e_1 \cdots)} = \frac{\log(\lambda)}{\log(2)};$$
$$\lim_{n\to+\infty} \frac{S_n\psi(\omega^2 e_2 e_2 \cdots)}{S_n u(\omega^2 e_2 e_2 \cdots)} = \frac{\log(1-\lambda)}{\log(2)}.$$

Therefore, if $\pi(\xi)$ were in $\mathcal{UD}^{\alpha}_{F_\lambda}$, then by Lemma 5.8 we would have

$$\xi \in \mathcal{L}^{-\log(\lambda)/\log(2)}_{\psi,u} \cap \mathcal{L}^{-\log(1-\lambda)/\log(2)}_{\psi,u}.$$

In particular, the intersection of the two level sets on the right would be non-empty, thus indicating that $\lambda = 1/2$. This contradicts our assumption that $\lambda \neq 1/2$. Hence, we see that $\pi(\xi) \notin \mathcal{UD}^{\alpha}_{F_\lambda}$. The proof of the second claim is thus complete.

For $\alpha \in \{ \alpha_-, \alpha_+ \}$, we have

$$\frac{\alpha\log(2) + \log(1-\lambda)}{\log(1-\lambda) - \log(\lambda)} \in \{ 0, 1 \}.$$

As $H_2(0) = H_2(1) = 0$, we see that the right-hand side of (5.11) is zero. We have seen that $\mathcal{UD}^{\alpha}_{F_\lambda} = \varnothing$ for $\alpha \in \{ \alpha_-, \alpha_+ \}$, so the left-hand side of (5.11) is zero as well. Hence, (5.11) holds for $\alpha \in \{ \alpha_-, \alpha_+ \}$. For $\alpha \in (\alpha_-, \alpha_+)$, there is a unique $t \in (0,1)$ such that

$$\alpha = \frac{-t\log(\lambda) - (1-t)\log(1-\lambda)}{\log(2)},$$

or equivalently,

$$t = \frac{\alpha\log(2) + \log(1-\lambda)}{\log(1-\lambda) - \log(\lambda)}. \tag{5.12}$$

Let ν be the Bernoulli measure associated with the 2-tuple $(t, 1-t)$. Then, as ν_ψ, we also have that ν is the equilibrium state for some function $\phi : \Sigma \to \mathbb{R}$, which is constant on every cylinder of length 1. This in particular implies that ν is ergodic, so by Birkhoff's ergodic theorem, we have

$$\lim_{n\to+\infty} \frac{S_n\psi}{S_n u} = \frac{\int_\Sigma \psi \,\mathrm{d}\nu}{\int_\Sigma u \,\mathrm{d}\nu} = \frac{t\log(\lambda) + (1-t)\log(1-\lambda)}{\log(2)} = -\alpha,\ \nu\text{-a.e.}$$

Hence, $\nu(\mathcal{L}_{\psi,u}^{-\alpha}) = 1$. Note that the the vector space of all functions on σ which are constant on each cylinder of length 1 is spanned by $\{\, \psi, u \,\}$. Therefore, ϕ is a linear combination of ψ and u. We have also seen in Remark 3.10 that the pointwise dimension of ν at $\xi \in \Sigma$ is

$$\dim_{\nu,u}(\xi) = \lim_{n \to +\infty} \frac{S_n\phi(\xi) - nP(\phi)}{S_n u(\xi)},$$

if the limit exists. Hence, $\dim_{\nu,u}$ exists and is a constant function on $\mathcal{L}_{\psi,u}^{-\alpha}$. By Theorem 3.7 and Theorem 3.9, we thus have

$$\dim_{\nu,u}(\xi) = \dim_u(\nu) = \frac{h_{KS}(\nu)}{\int_\Sigma u \, \mathrm{d}\nu} = \frac{-t\log(t) - (1-t)\log(1-t)}{\log(2)} = H_2(t),$$

for any $\xi \in \Sigma$, where the Kolmogorov-Sinai entropy of the Bernoulli measure ν has been calculated in Example 2.8. As we have seen that $\nu(\mathcal{L}_{\psi,u}^{-\alpha}) = 1$, by Theorem 3.7, we have $\dim_u(\mathcal{L}_{\psi,u}^{-\alpha}) = H_2(t)$. Finally, combining this with Theorem 5.5 and (5.12), we can conclude that (5.11) holds for $\alpha \in (\alpha_-, \alpha_+)$ as well. $\square$

Appendix A

Proof of Theorem 3.7

In this appendix, we shall prove Theorem 3.7.

When one attempts to show the last inequality in Theorem 3.7, which gives the upper bound of the Hausdorff dimension, it is standard to use a covering lemma. The covering lemma we shall need here is asserted as follows.

Lemma A.1. *Assume that (X, d_X) is a separable metric space. Let $\mathcal{C}$ be a non-empty family of open balls. For each open ball $O \in \mathcal{C}$, take $x_O \in X$ and $r_O > 0$ satisfying $O = B(x_O, r_O)$. Suppose that $\sup_{O\in\mathcal{C}} r_O < \infty$. Then, there is a countable (possibly finite) subcollection $\mathcal{C}' \subseteq \mathcal{C}$ of pairwise disjoint open balls such that*

$$\bigcup_{O\in\mathcal{C}'} B(x_O, 9r_O) \supseteq \bigcup_{O\in\mathcal{C}} O. \tag{A.1}$$

This lemma will play a role similar to Lemma 4.8 in [Fal03]. Lemma 4.8 in [Fal03] is stated for a subset of a Euclidean space, so we cannot deduce Lemma A.1 from it directly. Therefore, we give a proof of Lemma A.1 as follows.

Proof. Let $\mathcal{C}_0$ be the collection of open balls $O \in \mathcal{C}$ satisfying $r_O > r_{O'}/2$ for any $O' \in \mathcal{C}$ that intersects with O. We first show that $\mathcal{C}_0$ is non-empty. If $\mathcal{C}_0$ were empty, there would exist a sequence of elements in $\mathcal{C}$, denoted by $(O^{(k)})_{k\geq 1}$, satisfying $r_{O^{(k+1)}} \geq 2r_{O^{(k)}}$ for any positive integer k. Consequently, we have

$$\sup_{k\in\mathbb{Z}_{>0}} r_{O^{(k)}} \geq \sup_{k\in\mathbb{Z}_{>0}} 2^{k-1} r_{O^{(1)}} = +\infty,$$

contradicting the assumption that $\sup_{O\in\mathcal{C}} r_O < +\infty$. Therefore, $\mathcal{C}_0$ is non-empty.

By Zorn's lemma, we may take a maximal pairwise disjoint subcollection $\mathcal{C}'$ of $\mathcal{C}_0$. Here, the maximality means there is no pairwise disjoint subcollection of $\mathcal{C}_0$ that properly contains $\mathcal{C}'$. We shall show that this $\mathcal{C}' \subseteq \mathcal{C}$ is what we desire.

The members of $\mathcal{C}'$ are by definition pairwise disjoint. Since X is separable, there exists some countable dense subset E of X. Associate to each $O \in \mathcal{C}'$ one element $p_O \in O \cap E$. As the members of $\mathcal{C}'$ are pairwise disjoint, the map $O \mapsto p_O$ is an injection which maps $\mathcal{C}'$ to E. Hence, we deduce that $\mathcal{C}'$ is countable.

Now it only remains to show that (A.1) hold. Clearly, it suffices to show that every $O \in \mathcal{C}$ satisfies

$$O \subseteq \bigcup_{O' \in \mathcal{C}'} B(x_{O'}, 9r_{O'}). \tag{A.2}$$

Note that any $O \in \mathcal{C}$ must be a member of $\mathcal{C}'$, or $\mathcal{C}_0 \setminus \mathcal{C}'$, or $\mathcal{C} \setminus \mathcal{C}_0$. If $O \in \mathcal{C}'$, note that $O = B(x_O, r_O) \subseteq B(x_O, 9r_O)$, so (A.2) holds. If $O \in \mathcal{C}_0 \setminus \mathcal{C}'$, due to the maximality of $\mathcal{C}'$, there exists some $O' \in \mathcal{C}'$ such that $O \cap O' \neq \varnothing$. Since $O' \in \mathcal{C}_0$, we have $r_{O'} > r_O/2$. Take $z \in O \cap O'$ arbitrarily. Then, for any $y \in O$, we have

$$d_X(x_{O'}, y) \leq d_X(x_{O'}, z) + d_X(z, x_O) + d_X(x_O, y) < r_{O'} + 2r_O < 5r_{O'}.$$

As $O' \in \mathcal{C}'$, (A.2) also holds for $O \in \mathcal{C}_0 \setminus \mathcal{C}'$.

Lastly, we assume that $O \in \mathcal{C} \setminus \mathcal{C}_0$, and we shall show that (A.2) still holds. Let $O_1 = O$. For any positive integer k, if we have defined O_k and $O_k \in \mathcal{C} \setminus \mathcal{C}_0$, then take one $O_{k+1} \in \mathcal{C}$ satisfying that $O_{k+1} \cap O_k \neq \varnothing$ and $r_{O_{k+1}} \geq 2r_{O_k}$. This process to take open balls must terminate in finitely many steps because $\sup_{O \in \mathcal{C}} r_O < +\infty$. Therefore, we finally get finitely many open balls $O_1, \cdots, O_n$, for some integer $n \geq 2$. It is clear that $O_n \in \mathcal{C}_0$. If $O_n \notin \mathcal{C}'$, take $O_{n+1} \in \mathcal{C}'$ such that $O_n \cap O_{n+1} \neq \varnothing$. Otherwise, set $O_{n+1} = O_n$. Then, again since $O_{n+1} \in \mathcal{C}_0$, we have $r_{O_n} < 2r_{O_{n+1}}$. Therefore, we have for any $k \in \{1, \cdots, n\}$,

$$r_{O_{n+1-k}} < 2^{2-k} r_{O_{n+1}}.$$

Now for every $j \in \{1, \cdots, n\}$, take $z_j \in O_j \cap O_{j+1}$. Then, for any $y \in O$, we have

$$\begin{aligned} d_X(y, x_{O_{n+1}}) &\leq d_X(y, x_{O_1}) + \sum_{j=1}^{n} d_X(x_{O_j}, z_j) + d_X(z_j, x_{O_{j+1}}) \\ &< r_{O_{n+1}} + 2\sum_{j=1}^{n} r_{O_j} \\ &= r_{O_{n+1}} + 2\sum_{k=1}^{n} r_{O_{n+1-k}} < r_{O_{n+1}} \left(1 + 2\sum_{k=1}^{\infty} 2^{2-k}\right) = 9r_{O_{n+1}}. \end{aligned}$$

Recall that O_{n+1} is taken from $\mathcal{C}'$, so (A.2) holds for $O \in \mathcal{C} \setminus \mathcal{C}_0$ as well. $\square$

Now we are able to give a proof of Theorem 3.7 as follows.

Proof of Theorem 3.7. We first prove the last inequality using Lemma A.1. Fix an arbitrary number $s > \sup_{x\in E} \underline{\dim}_\mu(x)$. Take $\delta \in (0,1)$ arbitrarily. Then, for each $x \in E$, there exists some $r_x \in (0, \delta/9)$ such that

$$\frac{\log(\mu(B(x, r_x)))}{\log(r_x)} < s,$$

from which we have $\mu(B(x, r_x)) > r_x^s$. Then, by Lemma A.1, there is a countable (possibly finite) subset E_0 of E satisfying that $\{ B(x, r_x) \mid x \in E_0 \}$ is a family of pairwise disjoint open balls and

$$\bigcup_{x\in E_0} B(x, 9r_x) \supseteq \bigcup_{x\in E} B(x, r_x) \supseteq E.$$

Consequently, we have

$$\sum_{x\in E_0} \mathrm{diam}(B(x, 9r_x))^s \le 18^s \sum_{x\in E_0} r_x^s < 18^s \sum_{x\in E_0} \mu(B(x, r_x)) \le 18^s \mu(E) \le 18^s.$$

Since $\{ B(x, 9r_x) \mid x \in E_0 \}$ is a countable δ-covering of E, we have $\mathcal{H}^s_\delta(E) \le 18^s$. As δ is independent of s, we have $\mathcal{H}^s(E) \le 18^s$. From Proposition 3.2, we have that $\dim_H(E) \le s$. As this holds for any $s > \sup_{x\in E} \underline{\dim}_\mu(x)$, we can see that the last inequality in Theorem 3.7 holds.

Define $E^+ = \{ y \in X \mid \underline{\dim}_\mu(y) \le \operatorname{ess\,sup}_{x\in X} \underline{\dim}_\mu(x) \}$. Clearly, E^+ is a Borel set and $\mu(E^+) = 1$. From the last inequality in Theorem 3.7 we have just shown, we obtain

$$\overline{\dim}_H(\mu) \le \dim_H(E^+) \le \sup_{y\in E^+} \underline{\dim}_\mu(y) \le \operatorname*{ess\,sup}_{x\in X} \underline{\dim}_\mu(x). \tag{A.3}$$

Similarly, define $E^-_\varepsilon = \{ y \in X \mid \underline{\dim}_\mu(y) \le \operatorname{ess\,inf}_{x\in X} \underline{\dim}_\mu(x) + \varepsilon \}$ for any $\varepsilon > 0$. Then, for any $\varepsilon > 0$, we have E^-_ε is a Borel set, $\mu(E^-_\varepsilon) > 0$ and

$$\underline{\dim}_H(\mu) \le \dim_H(E^-_\varepsilon) \le \sup_{y\in E^-_\varepsilon} \underline{\dim}_\mu(y) \le \operatorname*{ess\,inf}_{x\in X} \underline{\dim}_\mu(x) + \varepsilon.$$

As ε is an arbitrary positive number, we have

$$\underline{\dim}_H(\mu) \le \operatorname*{ess\,inf}_{x\in X} \underline{\dim}_\mu(x). \tag{A.4}$$

Now it only remains to show the reverse inequalities of (A.3) and (A.4). We shall only show the reverse inequality of (A.3), namely

$$\overline{\dim}_H(\mu) \ge \operatorname*{ess\,sup}_{x\in X} \underline{\dim}_\mu(x);$$

the proof of the reverse inequality of (A.4) is easier. Without loss of generality, assume that $\operatorname{ess\,sup}_{x\in X} \underline{\dim}_\mu(x) > 0$. Take an arbitrary positive number $s' < \operatorname{ess\,sup}_{x\in X} \underline{\dim}_\mu(x)$. Define $\varepsilon' = \operatorname{ess\,sup}_{x\in X} \underline{\dim}_\mu(x) - s'$ and

$$E^+_{\varepsilon'} = \{\, y \in X \mid \underline{\dim}_\mu(y) \geq \operatorname*{ess\,sup}_{x\in X} \underline{\dim}_\mu(x) - \varepsilon'/2 \,\}\,.$$

Then, by Egorov's theorem, there are a Borel set $E_1 \subseteq E^+_{\varepsilon'}$ satisfying $\mu(E_1) > 0$ and some $N \in \mathbb{Z}_{>0}$ such that for any integer $n \geq N$ and any $y \in E_1$,

$$\inf_{r\leq 1/n} \frac{\log \mu(B(y,r))}{\log(r)} \geq \underline{\dim}_\mu(y) - \varepsilon'/2 \geq \operatorname*{ess\,sup}_{x\in X} \underline{\dim}_\mu(x) - \varepsilon' = s'.$$

This means that for any $r \leq 1/N$ and any $y \in E_1$, $\mu(B(y,r)) \leq r^{s'}$.

Let X_0 be an arbitrary Borel subset of X satisfying $\mu(X_0) = 1$. It is clear that $\mu(X_0 \cap E_1) = \mu(E_1) > 0$. Let U be an arbitrary non-empty Borel subset of $X_0 \cap E_1$ with $\operatorname{diam}(U) \leq (2N)^{-1}$. We claim that

$$\mu(U) \leq 2^{s'} \operatorname{diam}(U)^{s'}. \tag{A.5}$$

Take an arbitrary $y_U \in U$. If $U = \{\, y_U \,\}$, then $\mu(U) = \inf_{r\leq 1/N} \mu(B(y_U, r)) = 0$, so (A.5) holds in this case. If U is not a singleton, we have $U \subseteq B(y_U, 2\operatorname{diam}(U))$. It then follows that

$$\mu(U) \leq \mu(B(y_U, 2\operatorname{diam}(U))) \leq 2^{s'} \operatorname{diam}(U)^{s'},$$

so (A.5) is true in this case as well. Applying the mass distribution principle, we then have $\dim_H(E_1 \cap X_0) \geq s'$. As s' and X_0 are taken arbitrarily, we have $\overline{\dim}_H(\mu) \geq \operatorname{ess\,sup}_{x\in X} \underline{\dim}_\mu(x)$. $\square$

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