Scattering theory for unitary operators

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Warning: These notes correspond to a first attend of a course on this topic. For that reason, they contain mistakes and will no more be updated (after July 2023). Feel free to use them, but with a grain of salt.

Scattering theory for unitary operators

Motivation :

1) Scattering theory is a comparison theory : one complicated operator, and one simpler operator which describes the large time behavior.

2) The theory is well-developed for 2 self-adjoint operators H and H_0, and the comparison is taking place through the unitary evolution groups $\{e^{-itH}\}_{t\in\mathbb{R}}$ and $\{e^{-itH_0}\}_{t\in\mathbb{R}}$. Two objects of major importance are $W_\pm := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} e^{-itH_0}$ (wave operators), and also the scattering operator $S := W_+^* W_-$.

3) The discrete time analog is defined by 2 unitary operators U and U_0, and the study of $\operatorname*{s-lim}_{n\to\pm\infty} U^{-n} U_0^{n}$ ($n \in \mathbb{Z}$). This theory is much less developed.

4) Both theories can be applied in many contexts, but they become quickly very technical.

5) Japanese researchers have played a key role in the early development of these theories, and important names are Kato, Kuroda, Ikebe, Yajima, Nakamura, ...

6) The continuous time theory has gained new interest with applications in non-commutative geometry, group theory, number theory, dynamical systems, ...

I : Hilbert space and linear operators

I.1 : $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$

Def : A Hilbert space $\mathcal{H}$ is a complex vector space with a strictly positive inner product (or scalar product) $\langle \cdot, \cdot \rangle$, complete for the norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$, and separable.

countable orthonormal basis

linearity in second argument

Convention : $\langle f, g + \lambda h \rangle = \langle f, g \rangle + \lambda \langle f, h \rangle$,

$\forall f, g, h \in \mathcal{H}, \ \lambda \in \mathbb{C}$

Examples : $\mathbb{C}^n$, $\ell^2(\mathbb{Z}^d)$, $L^2(\mathbb{R}^d)$.

Convergence : Consider $(f_n)_{n \in \mathbb{N}} \subset \mathcal{H}$.

strong

1) $s\text{-}\lim f_n = f_\infty \in \mathcal{H}$ iff $\lim_{n \to \infty} \| f_n - f_\infty \| = 0$,

weak

2) $w\text{-}\lim f_n = f_\infty \in \mathcal{H}$ iff $\lim_{n \to \infty} \langle g, f_n - f_\infty \rangle = 0$,

$\forall g \in \mathcal{H}$. Clearly 1) $\Rightarrow$ 2 but

2) + $\lim_{n \to \infty} \| f_n \| = \| f_\infty \|$ $\Rightarrow$ 1).

There are 2 topologies on $\mathcal{H}$.

Def: A map $T : \mathcal{H} \to \mathcal{H}$ is a <u>bounded linear map</u> (or bounded linear operator) if T is linear and if $\exists c \geq 0$ s.t. $\|Tf\| \leq c\|f\|$ $\forall f \in \mathcal{H}$.

Then $\|T\| := \sup_{\substack{f \in \mathcal{H} \\ f \neq 0}} \frac{\|Tf\|}{\|f\|} = \ldots$ other expressions

The set of all bounded linear operators is denoted by <u>$\mathcal{B}(\mathcal{H})$</u>.

adjoint of T.

Lemma: $\forall\ T \in \mathcal{B}(\mathcal{H})$, $\exists\ T^* \in \mathcal{B}(\mathcal{H})$ s.t.

$$\langle T^* f, g \rangle = \langle f, Tg \rangle, \quad \forall f, g \in \mathcal{H}.$$

Proof based on Riesz lemma

called C^*-property

Properties: 1) $\|T^*\| = \|T\|$ and $\|T^* T\| = \|T\|^2$

2) $(T^*)^* = T$

3) $(ST)^* = T^* S^*$

the map $T \mapsto T^*$ is an involution

<u>Special element of $\mathcal{B}(\mathcal{H})$</u> : Let $T \in \mathcal{B}(\mathcal{H})$.

- T is <u>self-adjoint</u> if $T^* = T$,
- T is <u>normal</u> if $T^* T = T T^*$,
- T is a <u>projection</u> if $T^2 = T$,
- T is an <u>orthogonal projection</u> if $T^2 = T = T^*$,
- T is an <u>isometry</u> if $T^* T = \mathbb{1}$,
- T is <u>unitary</u> if $T^* T = \mathbb{1} = T T^*$,
- T is a <u>partial isometry</u> if $T^* T$ is an orthogonal projection.

<u>Convergence</u> : Consider $(B_n)_{n \in \mathbb{N}} \subset \mathcal{B}(\mathcal{H})$

1) $u\text{-}\lim_{n \to \infty} B_n = B_\infty \in \mathcal{B}(\mathcal{H})$ iff $\lim_{n \to \infty} \| B_n - B_\infty \| = 0$,

$\Downarrow$

2) $s\text{-}\lim_{n \to \infty} B_n = B_\infty \in \mathcal{B}(\mathcal{H})$ iff $\lim_{n \to \infty} \| (B_n - B_\infty) f \| = 0$

$\Downarrow$

3) $w\text{-}\lim_{n \to \infty} B_n = B_\infty \in \mathcal{B}(\mathcal{H})$ iff $\lim_{n \to \infty} \langle g, (B_n - B_\infty) f \rangle = 0$

$\forall f, g \in \mathcal{H}$. Other topologies on $\mathcal{B}(\mathcal{H})$ exist.

$\mathcal{B}(\mathcal{H})$ is complete for these 3 topologies.

⚠ not always closed ideals

I.2 : Ideals in $B(\mathcal{H})$

Finite rank : For $\{f_i, g_i\}_{i=1}^N \subset \mathcal{H}$, set

bra-ket notation

$$Tf := \sum_{j=1}^N |g_j\rangle\langle f_j| f = \sum_{j=1}^N \langle f_j, f\rangle g_j$$

T is called a finite rank operator, and the set of all finite rank operators is denoted by $\mathcal{F}(\mathcal{H})$.

$\mathcal{F}(\mathcal{H})$ is an ideal in $B(\mathcal{H})$.

Compact operators : $\mathcal{K}(\mathcal{H}) := \overline{\mathcal{F}(\mathcal{H})}^{B(\mathcal{H})}$. Ideal in $B(\mathcal{H})$. If $T \in \mathcal{K}(\mathcal{H})$ and $w\text{-}\lim_{n\to\infty} f_n = f_\infty$ then $s\text{-}\lim Tf_n = Tf_\infty$. If $s\text{-}\lim_{n\to\infty} B_n = B_\infty$, then $u\text{-}\lim_{n\to\infty} B_n T = B_\infty T$ and $u\text{-}\lim_{n\to\infty} T B_n^* = T B_\infty^*$.

Hilbert-Schmidt : $T \in B(\mathcal{H})$ is Hilbert-Schmidt if $\sum_{j\in\mathbb{N}} \|Te_j\|^2 < \infty$, $\{e_j\}_{j\in\mathbb{N}}$ orthonormal basis.

The set of all H.S. op. is denoted by $B_2(\mathcal{H})$.

independent of the basis

One has $\|T\|_{B_2}^2 := \sum_{j\in\mathbb{N}} \|Te_j\|^2 \geq \|T\|^2$, and $B_2(\mathcal{H})$ is an ideal in $B(\mathcal{H})$. $B_2(\mathcal{H})$ is a Hilbert space with scalar product

$$\langle S, T\rangle = \sum_{j\in\mathbb{N}} \langle Se_j, Te_j\rangle.$$

Remarks: 1) For any $T\in B(\mathcal{H})$ with $T\geq 0$ (meaning $\langle f, Tf\rangle \geq 0 \ \forall f\in\mathcal{H}$), $\exists!\, T^{1/2}\geq 0$ s.t. $T^{1/2}T^{1/2} = T$.

2) For any $T\in B(\mathcal{H})$, $T^*T \geq 0$ and we set $|T| = (T^*T)^{1/2}$

Trace-class: $T\in B(\mathcal{H})$ is trace class iff $|T|^{1/2}\in B_2(\mathcal{H})$. The set of all trace class op. is denoted by $B_1(\mathcal{H})$. $\|T\|_{B_1} := \||T|^{1/2}\|_{B_2}^2$

$$= \sum_{j\in\mathbb{N}} \||T|^{1/2}e_j\|^2 = \sum_{j\in\mathbb{N}} \langle e_j, |T|e_j\rangle \geq \|T\|.$$

$B_1(\mathcal{H})$ is an ideal in $B(\mathcal{H})$, and $\forall\, S,T \in B_2(\mathcal{H})$ one has $ST \in B_1(\mathcal{H})$. For $T \in B_1(\mathcal{H})$ we set

$$\mathrm{Tr}(T) := \sum_{j\in\mathbb{N}} \langle e_j, Te_j\rangle .$$

<u>Lemma</u>: If $ST \in B_1(\mathcal{H})$, then

$$\mathrm{Tr}(ST) = \mathrm{Tr}(TS).$$

check if this is sufficient ?

I.3 : <u>General linear operators</u>

operator

domain

<u>Def</u>: A <u>linear operator</u> is a pair $(A, D(A))$ with $D(A)$ a linear subspace of $\mathcal{H}$, and $A : D(A) \to \mathcal{H}$ a linear map. $(A, D(A))$ is <u>densely defined</u> if $D(A)$ is dense in $\mathcal{H}$.

⚠ For 2 linear operators $(A, D(A))$, $(B, D(B))$, $D(A+B) = D(A) \cap D(B)$ and $D(AB) = \{ f \in D(B) \mid Bf \in D(A)\}$.

Remark: If $T \in \mathcal{B}(\mathcal{H})$, then $(T, \mathcal{H})$ is a linear operator.

Exercise: Exhibit some concrete linear operators which are not bounded, namely which satisfy

$$\sup_{f \in D(A)} \frac{\|Af\|}{\|f\|} = \infty .$$

Def: $(A, D(A))$ is a <u>closed</u> linear operator if $\forall (f_n)_{n\in\mathbb{N}} \subset D(A)$ s.t. $\lim_{n\to\infty} f_n = f \in \mathcal{H}$ and $(Af_n)_{n\in\mathbb{N}}$ is Cauchy, then $f \in D(A)$ and $\lim_{n\to\infty} Af_n = Af$.

We usually consider only closed operators (or if they are not closed, we consider their closure).

Remark: Any bounded operator is closed.

we assume it automatically in the sequel

Def: A closed linear operator $(A, D(A))$ is <u>invertible</u>

kernel

if $\ker(A) = \{f \in D(A) \mid Af = 0\} = \{0\}$. Then $\exists!$

range

$(A^{-1}, D(A^{-1}))$ defined by $D(A^{-1}) = \mathrm{Ran}(A) =$

$= \{Af \mid f \in D(A)\}$ and satisfying

$AA^{-1} = \mathbb{1}$ on $D(A^{-1})$ and $A^{-1}A = \mathbb{1}$ on $D(A)$.

boundedly invertible operator

Thm: If $R(A) = \mathcal{H}$, then $A^{-1} \in \mathcal{B}(\mathcal{H})$.

More generally, if $D(B) = \mathcal{H}$, then $B \in \mathcal{B}(\mathcal{H})$.

Def: The <u>resolvent set</u> of $(A, D(A))$ is defined by

$\rho(A) := \{z \in \mathbb{C} \mid (A - z\mathbb{1}) \text{ is boundedly invertible}\}$.

The <u>spectrum</u> of $(A, D(A))$ is defined by

$\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Thm: $\sigma(A)$ is a closed set in $\mathbb{C}$.

Def: A pair $(z, f) \in \mathbb{C} \times D(A)$ is called an <u>eigenvalue - eigenfunction</u> if $Af = zf$. The set of all eigenvalues is denoted by $\sigma_p(A)$.

Clearly, $\sigma_p(A) \subset \sigma(A)$.
not equal in general

Def: For $(A, D(A))$ densely defined, its <u>adjoint</u> $(A^*, D(A^*))$ is defined by

$$D(A^*) = \{ f \in \mathcal{H} \mid \exists f^* \in \mathcal{H} \text{ with } \langle f, Ag \rangle = \langle f^*, g \rangle \ \forall g \in D(A) \}$$

and $A^* f = f^* \quad \forall f \in D(A^*)$.

Observe that $\langle f, Ag \rangle = \langle A^* f, g \rangle \quad \forall f \in D(A^*), g \in D(A)$.

Lemma: $(A^*, D(A^*))$ is closed, and

$$\operatorname{Ker}(A^*) = \operatorname{Ran}(A)^{\perp}.$$

I.4 Self-adjoint operators

<u>Def</u> : A densely defined linear operator $(A, D(A))$ is <u>self-adjoint</u> if $(A, D(A)) = (A^*, D(A^*))$.

Observe that $\langle Af, g\rangle = \langle f, Ag\rangle \quad \forall f, g \in D(A)$ if $(A, D(A))$ is self-adjoint, but this condition is not sufficient.

<u>Thm</u> : If $(A, D(A))$ is self-adjoint, then $\sigma(A) \subset \mathbb{R}$.

<u>Def</u> : A <u>spectral family</u> (or <u>resolution of the identity</u>) is a family of orthogonal projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ satisfying

1) $E_\mu E_\lambda = E_{\min\{\mu, \lambda\}}$ → implying that these projection commute

2) $E_\lambda = E_{\lambda+0} = \underset{\varepsilon \searrow 0}{s\text{-}\lim}\, E_{\lambda+\varepsilon}$

3) $\underset{\lambda \to -\infty}{s\text{-}\lim}\, E_\lambda = 0$, $\underset{\lambda \to \infty}{s\text{-}\lim}\, E_\lambda = 1\!\!1$.

By setting $E((a,b]) := E_b - E_a$ and by extending
obtained by countable union, intersection and complement
this to the Borel algebra of $\mathbb{R}$, one gets
a spectral measure : $\mathcal{A}_B \ni v \mapsto E(v) \in \mathcal{P}(\mathcal{H})$
↑ Borel algebra
↑ set of orthogonal projections

Thm : [Spectral theorem] $\exists$ a bijective relation between 1) self-adjoint operators, 2) spectral families, 3) spectral measures s.t.

self-adj op :
$$\begin{cases} D(A) = \{ f \in \mathcal{H} \mid \int_{\mathbb{R}} \lambda^2 \langle E(d\lambda) f, f \rangle < \infty \} \\ A f = \int_{\mathbb{R}} \lambda \, E(d\lambda) f . \end{cases}$$
spectral measure
integral in the strong topology.

Thm : [Lebesgue's decomposition theorem]

Any Borel measure μ on $\mathbb{R}$ admits a decomposition

$$\mu = \mu_{ac} + \mu_{sc} + \mu_p$$
pure point
absolutely continuous
singular continuous
$\left(\begin{array}{l} \exists V \subset \mathcal{A}_B \text{ with } \int_V 1 \, dx = 0 \\ \mu_{sc}(\mathbb{R} \setminus V) = 0 \end{array} \right)$

Thm : [Spectral decomposition]

For any self-adjoint operator $(A, D(A))$, there exits a decomposition $\mathcal{H} = \underbrace{\mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A)}_{=: \mathcal{H}_c(A)} \oplus \mathcal{H}_p(A)$

and $A = \underbrace{A_{ac} \oplus A_{sc}}_{=: A_c} \oplus A_p$ s.t.

1) $\forall f \in \mathcal{H}_{ac}$, $\mathcal{A}_B \ni V \mapsto \langle E(V) f, f \rangle$ is a.c

$\forall f \in \mathcal{H}_{sc}$, $\mathcal{A}_B \ni V \mapsto \langle E(V) f, f \rangle$ is s.c

$\forall f \in \mathcal{H}_p$, $\mathcal{A}_B \ni V \mapsto \langle E(V) f, f \rangle$ is p.p.

will respect to the Lebesgue measure on $\mathbb{R}$.

2) $\mathcal{H}_p(A) = \bigoplus_{\nu \in \sigma_p(A)} E(\{\nu\}) \mathcal{H}$

$\sigma(A_p) = \overline{\sigma_p(A)}$

3) $\sigma(A) = \sigma(A_{ac}) \cup \sigma(A_{sc}) \cup \sigma(A_p)$

Remark : $\sigma(A) = \sigma_d(A) \cup \sigma_{ess}(A)$

another decomposition

discrete spectrum

essential spectrum

with $\lambda \in \sigma_d(A)$ is λ is an eigenvalue of finite multiplicity and isolated from the rest of the spectrum.

II : Spectral theory for unitary operators

II. 1 Spectral theorem

Let us firstly recall the spectral theorem for normal operators ($T^*T = TT^*$). We follow [Weid. Sec. 7.5]

Def: A function $G : \mathbb{C} \to \mathcal{P}(\mathcal{H})$ set of projections is called a complex spectral family if $G_{t+is} = E_t F_s = F_s E_t$ with $\{E_t\}_{t\in\mathbb{R}}$, $\{F_s\}_{s\in\mathbb{R}}$ 2 (real) spectral families. For $J = (a_1, b_1] \times i(a_2, b_2] \subset \mathbb{C}$, we set $G(J) := E((a_1, b_1])F((a_2, b_2])$, and extend it to all Borel sets of $\mathbb{C}$. Then one has

Thm [W. Thm 7.31] Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then there exists a unique complex spectral family for which $T = \int_{\mathbb{C}} z\, G(dz)$ in the strong sense. In the decomposition $G_{t+is} = E_t F_s = F_s E_t$, $\{E_t\}_{t\in\mathbb{R}}$ correspond to the spectral family of $\mathrm{Re}(T) = \frac{T+T^*}{2}$ and $\{F_s\}_{s\in\mathbb{R}}$ corresponds to the spectral family of $\mathrm{Im}\, T = \frac{T-T^*}{2i}$.

Note that the complex spectral family fully determines the spectrum of the normal operator T [Weid, Thm 7.4.(a)]. Indeed, $z \in \sigma(T)$ if and only if

$$G(z+\varepsilon+i\varepsilon) + G(z-\varepsilon-i\varepsilon) - G(z+\varepsilon-i\varepsilon) - G(z-\varepsilon+i\varepsilon) \neq 0$$

$\forall\, \varepsilon > 0$.

Let us now consider a unitary operator U. Since $U^*U = 1\!\!1 = UU^*$, one infers that U is normal, and that $U^* = U^{-1}$. It is also known that $\sigma(U) \subset \mathbb{S}^1$, see for example [W. Ex 2 in Sec. 5.2]. Based on this property, a real spectral family can be defined for U. Consider the parametrization $\mathbb{S}^1 = \{e^{i\theta}\}_{\theta \in [0,2\pi)}$ and set

$$E^U_\theta := \begin{cases} 0 & \text{if } \theta < 0 \\ G_{\cos(\theta)+i\sin(\theta)} & \text{if } \theta \in [0,2\pi) \\ 1 & \text{if } \theta \geq 2\pi. \end{cases}$$

We then set $E^U((a,b]) := E^U_b - E^U_a$ and extends this to the Borel algebra of $\mathbb{R}$. We then get a spectral measure with suport $\mathrm{supp}(E^U) \subset [0, 2\pi]$.

In addition one has

<u>Thm</u> [Weid, thm 7.36]

The map $\mathcal{A}_B \ni v \longmapsto E^U(v) \in \mathcal{B}(\mathcal{H})$ is a (real) spectral measure, and one has $U = \int_0^{2\pi} e^{i\theta} E^U(d\theta)$.

By the theorem of spectral decomposition, there exist 3 subspaces $\mathcal{H}_{ac}(U) \oplus \mathcal{H}_{sc}(U) \oplus \mathcal{H}_p(U) = \mathcal{H}$, and accordingly 3 orthogonal projections $P_{ac}(U)$, $P_{sc}(U)$, and $P_p(U)$, with the maps

$\mathcal{A}_B \ni v \longmapsto \langle f, E^U(v) f \rangle \in \mathbb{R}$ either ac if $f \in \mathcal{H}_{ac}(U)$, sc if $f \in \mathcal{H}_{sc}(U)$ or p if $f \in \mathcal{H}_p(U)$.

absolutely continuous spectrum

Accordingly: $\sigma_{ac}(U) := \sigma\left(U|_{\mathcal{H}_{ac}(U)}\right)$

singular continuous spectrum $\sigma_{sc}(U) := \sigma\left(U|_{\mathcal{H}_{sc}(U)}\right)$

$\Bigg\} = \sigma_c(U)$

point spectrum $\overline{\sigma_p(U)} := \sigma\left(U|_{\mathcal{H}_p(U)}\right)$.

It is also convenient to define

$$\sigma_d(U) := \{e^{i\theta} \in \sigma_p(U) \mid \exists \varepsilon > 0 \text{ with } E^U((\theta-\varepsilon, \theta+\varepsilon)) = E^U(\{\theta\}) \text{ and } \dim E^U(\{\theta\}) < \infty\}.$$

This is called the <u>discrete spectrum</u>, and one sets

$$\sigma_{ess}(U) := \sigma(U) \setminus \sigma_d(U)$$

for the <u>essential spectrum</u> of U.

II. 2 Spectral properties

In this section, we list a few results related to the spectral part of the operator U.

Lemma If $f \in \mathcal{H}_{ac}(U)$, then $\displaystyle w\text{-}\lim_{n \to \pm\infty} U^n f = 0$.

Proof as an exercise.

The following result is equivalent to von Neumann ergodic theorem, and is borrowed from [Si4, Thm 5.5.4]

Proposition: Let $\underset{\overset{\|}{E^U(\{0\})}}{P}$ be the orthogonal projection on $\{f \in \mathcal{H} \mid Uf = f\}$. Then $\displaystyle s\text{-}\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} U^j = P$.

In the same direction, one has the celebrated RAGE theorem
Ruelle, Amrein, Georgescu, Enss.

Thm [Si, Thm 5.5.6]

For any $K \in \mathcal{K}(\mathcal{H})$ one has for any $f \in \mathcal{H}$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} \| K U^{-j} f \|^2 = \| K P_p(U) f \|^2 .$$

Corollary : For $K \in \mathcal{H}$,

$$s\text{-}\lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} U^j K U^{-j} = \sum_{\{e^{i\theta} \text{ eigenvalue of } U\}} E^U(\{e^{i\theta}\}) K E^U(\{e^{i\theta}\}) .$$

Proof in paper [RT 2023, Thm 3.2].

Proof of the Proposition (p 19)

Set $F_N : \mathbb{S} \to \mathbb{C}$, $F_N(e^{i\theta}) = \frac{1}{N}\sum_{j=0}^{N-1} e^{ij\theta} = \begin{cases} \frac{1}{N}\frac{e^{iN\theta}-1}{e^{i\theta}-1} & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0. \end{cases}$

Then $\|F_N\|_\infty \le 1$ for any N and $\lim_{N\to\infty} F_N(e^{i\theta}) = 0$ for $\theta \neq 0$. By writing

$$\frac{1}{N}\sum_{j=0}^{N-1} U^j f = \int_0^{2\pi} \frac{1}{N}\sum_{j=0}^{N-1} e^{ij\theta} E^U(d\theta) f$$

$$= \int_0^{2\pi} F_N(e^{i\theta}) E^U(d\theta) f ,$$ one gets the result by the dominated convergence theorem. □

III Scattering theory

In this chapter, we deal with 2 unitary operators U and U_0. They could live in the same Hilbert space, but it is sometimes useful to consider them in 2 distinct Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_0$. In this case, one needs an additional bounded operator $J : \mathcal{H}_0 \to \mathcal{H}$, called the <u>identification operator</u>. Constraints on J will be imposed when necessary.

Thus, the framework is 2 Hilbert spaces $\mathcal{H}_0$ and J, one linear identification operator $J : \mathcal{H}_0 \to \mathcal{H}$, 2 unitary operators U_0 and U in $\mathcal{H}_0$ and $\mathcal{H}$, respectively.

III.1 Wave operators: time dependent approach

<u>Aim</u>: For $f \in \mathcal{H}$, understand $U^n f$ for $|n| \longrightarrow \infty$. Clearly, if $Uf = \lambda f$, $U^n f = \lambda^n f$, which means $\lim_{n\to\pm\infty} U^n f = \lim_{n\to\pm\infty} \lambda^n f$ does not

exist if $Uf = \lambda f$ with $\lambda \in \mathbb{S}$ except if $\lambda = 1$.

Thus, we shall mainly work in $\mathcal{H}_c(U) = \mathcal{H}_p(U)^\perp$.

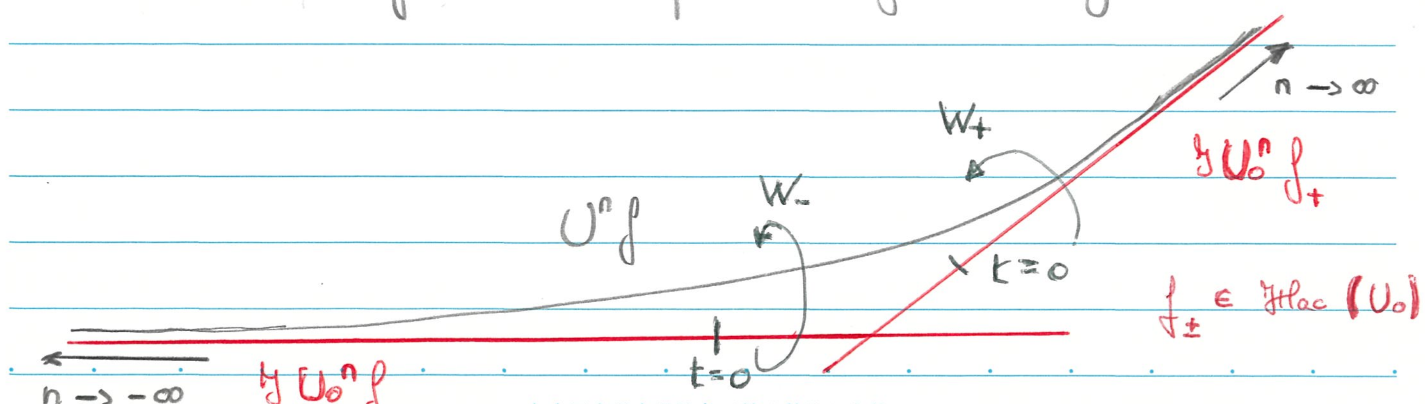
<u>Leading idea</u>: Find a simpler asymptotic evolution, and new vector $f_\pm$ such that

$$\| U^n f - J U_0^n f_\pm \|_{\mathcal{H}} \xrightarrow{n \to \pm\infty} 0$$

$$\iff \| f - U^{-n} J U_0^n f_\pm \|_{\mathcal{H}} \xrightarrow{n \to \pm\infty} 0$$

<u>Def</u>: The <u>wave operators</u> $W_\pm \equiv W_\pm(U, U_0, J)$ are defined by $\operatorname*{s-lim}_{n \to \pm\infty} U^{-n} J U_0^n P_{ac}(U_0)$ if these limits exist.

An illustration of this concept is given by



If the wave operators exist, the relations read $f = W_- f_-$ and $f = W_+ f_+$. Thus, this approach is possible only for those $f \in \mathrm{Ran}(W_-)$ or $f \in \mathrm{Ran}(W_+)$.

Remark: Usually, requiring the existence of $W_\pm$ is too strong, a local version (local in the spectrum of U_0) is preferable. For that purpose, set $E^{U_0}_{ac}(\cdot) = P_{ac}(U_0)\, E^{U_0}(\cdot)$, the a.c. spectral measure of U_0.

Def: For any Borel set $\Theta \subset [0, 2\pi)$, the local wave operators are defined by

$$W_\pm(\Theta) \equiv W_\pm(U, U_0, J, \Theta) := \underset{n \to \pm\infty}{s\text{-}\lim}\; U^{-n} J\, U_0^{\,n} E^{U_0}_{ac}(\Theta).$$

Clearly, the global ones are obtained for $\Theta = [0, 2\pi)$, and the standard theory for $\mathcal{H} = \mathcal{H}_0$ and $J = 1$.

The first property is called the <u>intertwining property</u>:

<u>Lemma</u>: For any bounded Borel function $\eta : \mathbb{S}^1 \to \mathbb{C}$, one has $W_\pm(\Theta)\, \eta(U_0) = \eta(U)\, W_\pm(\Theta)$.

<u>Proof</u>: A direct computation, with a change of variable, leads to $W_\pm(\Theta)\, U_0^k = U^k\, W_\pm(\Theta)$, $\forall k \in \mathbb{Z}$. Using Stone-Weierstrass theorem, we then get the statement for any $\eta(e^{i\cdot}) \in C([0,2\pi])$. Finally, by a standard approximation argument in the weak topology, the result can be extended to all (bounded) Borel $\eta : \mathbb{S}^1 \to \mathbb{C}$. $\square$

In particular, observe that $W_\pm(\Theta)\, E^{U_0}(\Theta') = E^U(\Theta')\, W_\pm(\Theta)$ for any Borel set $\Theta' \subset [0, 2\pi)$.

$\Rightarrow$ $\underline{\operatorname{Ran}(W_\pm(\Theta)) \subset E^U(\Theta)\,\mathcal{H}}$.

In fact a stronger result holds:

<u>Lemma</u>: In the framework introduced above,

$\mathrm{Ran}\,(W_\pm(\theta)) \subset E^U_{ac}(\theta)\,\mathcal{H}$.

<u>Proof</u>: Let us consider the polar decomposition of

$$W_\pm(\theta) = F_\pm(\theta)\,\underbrace{|W_\pm(\theta)|}_{= (W_\pm(\theta)^* W_\pm(\theta))^{1/2}}$$

$= \mathrm{sgn}(W_\pm(\theta)) : \mathcal{H}_0 \to \mathcal{H}$ a partial isometry vanishing on $\mathrm{Ker}\,(W_\pm(\theta))$ and having range equal to $\overline{\mathrm{Ran}\,(W_\pm(\theta))}$.

Then $W_\pm(\theta)^* W_\pm(\theta)\,\eta(U_0) = \eta(U_0)\,W_\pm(\theta)^* W_\pm(\theta)$ ← bounded and Borel

$\Rightarrow$ any function of $W_\pm(\theta)^* W_\pm(\theta)$ commute with $\eta(U_0)$

$\Rightarrow |W_\pm(\theta)|\,\eta(U_0) = \eta(U_0)\,|W_\pm(\theta)|$

$\Rightarrow \eta(U)\,F_\pm(\theta)\,|W_\pm(\theta)| = F_\pm(\theta)\,\eta(U_0)\,|W_\pm(\theta)|$

$\Rightarrow \eta(U)\,F_\pm(\theta) \overset{(*)}{=} F_\pm(\theta)\,\eta(U_0)$ on $\overline{\mathrm{Ran}\,(|W_\pm(\theta)|)}$.

Since $\mathcal{H}_0 \ominus \overline{\mathrm{Ran}\,(|W_\pm(\theta)|)} = \ker\,(W_\pm(\theta))$, the equality ⊛ also holds on this space, and both terms are 0: this is clear for the l.h.s.

For the r.h.s., recall that if $W_\pm(\Theta)f = 0$, then $W_\pm(\Theta)\eta(U_0)f = \eta(U)W_\pm(\Theta)f = 0 \Rightarrow \eta(U_0)f$ belongs to $\ker(W_\pm(\Theta))$ if $f \in \ker(W_\pm(\Theta))$.

As a consequence of the equality

$$\eta(U)F_\pm(\Theta) = F_\pm(\Theta)\eta(U_0), \qquad \forall \eta \text{ bounded and Borel on } \mathbb{S}^1,$$

one gets that $\eta(U_0)\big|_{\ker(W_\pm(\Theta))^\perp}$ is unitarily equivalent to $\eta(U)\big|_{\overline{\mathrm{Ran}(W_\pm(\Theta))}}$.

In particular, since $\mathcal{H}_0 \ominus \ker(W_\pm(\Theta)) \subset \mathcal{H}_{ac}(U_0)$, then

$$\begin{aligned}\mathrm{Ran}(W_\pm(\Theta)) &\subset \mathcal{H}_{ac}(U) \cap E^U(\Theta)\mathcal{H} \\ &= E^U_{ac}(\Theta)\mathcal{H}. \qquad \square\end{aligned}$$

Let us now define the closed subspaces of $\mathcal{H}$: $\mathcal{N}_\pm(\Theta) \equiv$

$\mathcal{N}_\pm(U, J, \Theta) := \{ f \in \mathcal{H} \mid \lim_{n \to \pm\infty} \| J^* U^n E_{ac}^U(\Theta) f \|_{\mathcal{H}_0} = 0 \}$.

Clearly $(E_{ac}^U(\Theta)\mathcal{H})^\perp \subset \mathcal{N}_\pm(U, J, \Theta)$.

U is reduced by the decomposition $\mathcal{N}_\pm \oplus \mathcal{N}_\pm^\perp$ (it leaves these subspaces invariant), and

$\mathrm{Ran}(W_\pm(\Theta)) \perp \mathcal{N}_\pm(\Theta)$ by the previous lemma

$\subset E_{ac}^U(\Theta)\mathcal{H}$

Proof: Let $f \in \mathrm{Ran}(W_\pm(\Theta)) \iff \exists f_\pm \in E_{ac}^{U_0}(\Theta)\mathcal{H}_0$ s.t. $\| U^n f - J U_0^n f_\pm \|_{\mathcal{H}} \longrightarrow 0$ as $n \to \pm\infty$.

For $g \in \mathcal{N}_\pm(\Theta)$ one has

$$
\begin{aligned}
|\langle f, g \rangle_{\mathcal{H}}| = |\langle E_{ac}^U(\Theta) f, g \rangle_{\mathcal{H}}| &= |\lim_{n\to\infty} \langle U^n f, U^n E_{ac}^U(\Theta) g \rangle_{\mathcal{H}}| \\
&= |\lim_{n \to \pm\infty} \langle J U_0^n f_\pm , U^n E_{ac}^U(\Theta) g \rangle_{\mathcal{H}}| \\
&= |\lim_{n \to \pm\infty} \langle U_0^n f_\pm , J^* U^n E_{ac}^U(\Theta) g \rangle_{\mathcal{H}_0}| \\
&\leq \lim_{n \to \pm\infty} \| U_0^n f_\pm \|_{\mathcal{H}_0} \| J^* U^n E_{ac}^U(\Theta) g \|_{\mathcal{H}_0} \\
&= 0 \text{, by assumption.}
\end{aligned}
$$

□

As a consequence of the previous observations one has

$$\overline{\mathrm{Ran}\,(W_\pm(\Theta))} \subset E^U_{ac}(\Theta)\mathcal{H} \ominus N_\pm(\Theta).$$

<u>Def</u>: The operators $W_\pm(\Theta)$ are <u>J-complete</u> on Θ if $\overline{\mathrm{Ran}\,(W_\pm(\Theta))} = E^U_{ac}(\Theta)\mathcal{H} \ominus N_\pm(\Theta)$.

<u>Remark</u>: If J is unitary, then $N_\pm(\Theta) = \left(E^U_{ac}(\Theta)\mathcal{H}\right)^\perp$ and the wave operators have closed ranges since they are partial isometries. In this case the J-completeness reduces to completeness, namely to $\mathrm{Ran}\,(W_\pm(\Theta)) = E^U_{ac}(\Theta)\mathcal{H}$. If so, $W_\pm(\Theta)$ is unitary from $E^{U_0}_{ac}(\Theta)\mathcal{H}_0$ to $E^U_{ac}(\Theta)\mathcal{H}$.

<u>Remark</u>: With J-completeness or completeness, we get the information about the possible f for which $U^n f$ can be well approximated.

standard result. usually more complicated

Lemma: If $W_\pm(U, U_0, J, \Theta)$ and $W_\pm(U_0, U, J^*, \Theta)$ exist, then $W_\pm(U, U_0, J, \Theta)$ are J-complete on Θ.

Proof: Since $\mathrm{Ran}(W_\pm(U, U_0, J, \Theta)) \subset E^U_{ac}(\Theta)\mathcal{H}$, $\forall f \in \mathcal{H}$, $g_0 \in \mathcal{H}_0$

$$\langle W_\pm(U, U_0, J, \Theta)^* f, g_0\rangle_{\mathcal{H}_0} = \langle f, E^U_{ac}(\Theta) W_\pm(U, U_0, J, \Theta) g_0\rangle_{\mathcal{H}}$$

$$= \lim_{n\to\pm\infty} \langle E^U_{ac}(\Theta) f, U^{-n} J U_0^n E^{U_0}_{ac}(\Theta) g_0\rangle_{\mathcal{H}}$$

$$= \lim_{n\to\pm\infty} \langle E^{U_0}_{ac}(\Theta) U_0^{-n} J^* U^n E^U_{ac}(\Theta) f, g_0\rangle_{\mathcal{H}_0}$$

$$= \langle W_\pm(U_0, U, J^*, \Theta) f, g_0\rangle_{\mathcal{H}_0}.$$

because we have assumed the existence

Thus $W_\pm(U_0, U, J^*, \Theta) = W_\pm(U, U_0, J, \Theta)^*$.

Since $\mathrm{Ker}(W_\pm(U_0, U, J^*, \Theta)) \overset{\text{by def of } N_\pm(\Theta)}{=} N_\pm(\Theta)$ and $(E^U_{ac}(\Theta)\mathcal{H})^\perp \subset N_\pm(\Theta)$, it follows that

$$\mathrm{Ran}(W_\pm(U, U_0, J, \Theta)) = \mathcal{H} \ominus \mathrm{Ker}(W_\pm(U, U_0, J, \Theta)^*)$$

$\mathrm{Ran}(A) = \mathrm{Ker}(A^*)^\perp$

$$= \mathcal{H} \ominus N_\pm(\Theta) = E^U_{ac}(\Theta)\mathcal{H} \oplus (E^U_{ac}(\Theta)\mathcal{H})^\perp \ominus N_\pm(\Theta)$$

$$= E^U_{ac}(\Theta)\mathcal{H} \ominus N_\pm(\Theta).$$ which is the J-completeness

$\square$

Later, we shall have a criterion for showing the existence in the previous lemma. In the sequel, we weaken partially the condition on the adjoint. We need a preliminary standard result:

Lemma (Chain rule)

For $j \in \{1, 2, 3\}$ let $\mathcal{H}_j$ be Hilbert spaces, and U_j unitary operators in $\mathcal{H}_j$. Let $J_{23} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$, $J_{31} \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$, and let $\Theta \subset [0, 2\pi)$ be a Borel set. Assume that $W_\pm(U_3, U_2, J_{23}, \Theta)$ and $W_\pm(U_1, U_3, J_{31}, \Theta)$ exist. Then $W_\pm(U_1, U_2, J_{31} J_{23}, \Theta)$ exists, and one has

$$W_\pm(U_1, U_2, J_{31} J_{23}, \Theta) \underset{(*)}{=} W_\pm(U_1, U_3, J_{31}, \Theta)\, W_\pm(U_3, U_2, J_{23}, \Theta).$$

Remark: The choice $\Theta = [0, 2\pi)$ corresponds to the standard statement.

<u>Proof</u>: For any $n \in \mathbb{Z}$, the following equality holds: $U_1^{-n} J_{31} J_{23} U_2^n E_{ac}^{U_2}(\Theta) =$

$= U_1^{-n} J_{31} U_3^n \left[E_{ac}^{U_3}(\Theta) + (1 - E_{ac}^{U_3}(\Theta)) \right] U_3^{-n} J_{23} U_2^n E_{ac}^{U_2}(\Theta).$

$= \left(U_1^{-n} J_{31} U_3^n E_{ac}^{U_3}(\Theta) \right) U_3^{-n} J_{23} U_2^n E_{ac}^{U}(\Theta)$

$\quad + U_1^{-n} J_{31} U_3^n \left(1 - E_{ac}^{U_3}(\Theta) \right) U_3^{-n} J_{23} U_2^n E_{ac}^{U_2}(\Theta)$.

The first term converges to the r.h.s. of ⊛.

On the other hand, for any $f \in \mathcal{H}_2$,

$$\lim_{n \to \pm\infty} \| U_1^{-n} J_{31} U_3^n (1 - E_{ac}^{U_3}(\Theta)) U_3^{-n} J_{23} U_2^n E_{ac}^{U_2}(\Theta) f \|$$

$$\leq \| J_{31} \|_{\mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)} \lim_{n \to \pm\infty} \| (1 - E_{ac}^{U_3}(\Theta)) U_3^{-n} J_{23} U_2^n E_{ac}^{U_2}(\Theta) f \|$$

$$= \| J_{31} \|_{\mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)} \underbrace{\| (1 - E_{ac}^{U_3}(\Theta)) W_\pm(U_3, U_2, J_{23}, \Theta) f \|}_{=0}$$

since $\mathrm{Ran}(W_\pm(U_3, U_2, J_{23}, \Theta)) \subset E_{ac}^{U_3}(\Theta)\mathcal{H}_3$.

The claim then follows.

<u>Thm</u> (completeness on Θ)

For a Borel set Θ, assume that $W_\pm(U, U_0, J, \Theta)$ exists, and assume that there exists $J' \in B(\mathcal{H}, \mathcal{H}_0)$ s.t. $W_\pm(U_0, U, J', \Theta)$ exists, and

$$\underset{n \to \pm\infty}{s\text{-}\lim} \, (JJ' - 1) U^n E^U_{ac}(\Theta) = 0 .$$

Then

$$\mathrm{Ran}(W_\pm(U, U_0, J, \Theta)) = E^U_{ac}(\Theta)\mathcal{H} .$$

<u>Proof</u> : By the previous lemma with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $\mathcal{H}_3 = \mathcal{H}_0$ and $U_1 = U_2 = U$, $U_3 = U_0$, $J_{23} = J'$, $J_{31} = J$ one gets

$$W_\pm(U, U, JJ', \Theta) = W_\pm(U, U_0, J, \Theta)\, W_\pm(U_0, U, J', \Theta) .$$

Then, observe that

$$\| W_\pm(U, U, JJ', \Theta) f - E^U_{ac}(\Theta) f \|$$

$$= \lim_{n \to \pm\infty} \| U^{-n} JJ' U^n E^U_{ac}(\Theta) f - E^U_{ac}(\Theta) f \|$$

$$= \lim_{n \to \pm\infty} \| U^{-n} (JJ' - 1) U^n E^U_{ac}(\Theta) f \| = 0$$

by assumption.

Thus $E^U_{ac}(\Theta) = W_\pm(U, U, J'J, \Theta)$

$$= W_\pm(U, U_0, J, \Theta)\, W_\pm(U_0, U, J', \Theta).$$

Since $\operatorname{Ran}(W_\pm(U, U_0, J, \Theta)) \subset E^U_{ac}(\Theta)\mathcal{H}$, one infers that the equality holds. $\square$

<u>Remark</u>: There is a partial converse to this statement, but we do not study it now.

III.2 Resolvent and smooth operators

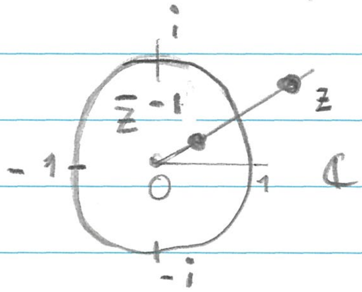
U unitary operator in a separable Hilbert space.

Set $R(z) := (1 - zU^*)^{-1}$ $\quad z \in \mathbb{C} \setminus S^1$.

resolvent

Then $$R(z) = \begin{cases} \sum_{n \geq 0} (zU^*)^n & \text{if } |z| < 1 \\ -\sum_{n \geq 1} (z^{-1}U)^n & \text{if } |z| > 1 \end{cases}$$

and satisfies $R(\bar{z}^{-1})^* = -zU^* R(z)$, which relates the resolvent inside the unit circle, with outside the unit circle



1^{st} resolvent equation : for $z_1, z_2 \in \mathbb{C} \setminus S^1$

$$R(z_1) - R(z_2) = (z_1 - z_2)\, R(z_1)\, U^*\, R(z_2).$$

By setting $z_1 = r e^{i\theta}$, $z_2 = r^{-1} e^{i\theta}$, $r \in (0,\infty)\setminus\{1\}$

and $\theta \in [0, 2\pi)$ one gets

$R(re^{i\theta}) - R(r^{-1}e^{i\theta})$ ⊛

$= (re^{i\theta} - \frac{1}{r}e^{i\theta})(1 - re^{i\theta}U^*)^{-1}U^*(1 - \frac{1}{r}e^{i\theta}U^*)^{-1}$

$= (1-r^2)R(re^{i\theta})(-)\frac{1}{r}e^{i\theta}U^*(1 - \frac{1}{r}e^{i\theta}U^*)^{-1}$

$= (1-r^2)R(re^{i\theta})\underbrace{\left[(-)re^{-i\theta}U(1 - \frac{1}{r}e^{i\theta}U^*)\right]^{-1}}$

$= (1 - re^{-i\theta}U)^{-1} = R(re^{i\theta})^*$

$= (1-r^2)R(re^{i\theta})R(re^{i\theta})^*$

$= (1-r^2)\,|R(re^{i\theta})|^2$. going to be very useful

We then set $\delta(r,\theta) := \frac{1}{2\pi}(1-r^2)\,|R(re^{i\theta})|^2$

Clearly : $\delta(r^{-1},\theta) = -\delta(r,\theta)$ (from ⊛)

and $\delta(r,\theta) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}(U^*)^n$, for $r\in(0,1)$.

Exercise : By using the Poisson kernel, show that

$\int_0^{2\pi} d\theta\ \delta(r,\theta) = 1$ for $r\in(0,\infty)\setminus\{1\}$.

integral in the strong sense

Definition : A linear operator $T \in B(\mathcal{H}, \mathcal{G})$ is called U-smooth on a Borel set $\Theta \subset [0, 2\pi)$ if $\exists\, C_\Theta > 0$ s.t. for any $f \in \mathcal{H}$,

$$\sum_{n \in \mathbb{Z}} \| T U^n E^U(\Theta) f \|_{\mathcal{G}}^2 \leq C_\Theta \| f \|_{\mathcal{H}}^2 .$$

This condition has various possible reformulations.

Thm : The following conditions are equivalent :

1) $C_1 := \frac{1}{2\pi} \sup_{\substack{f \in \mathcal{H} \\ \|f\| = 1}} \sum_{n \in \mathbb{Z}} \| T U^n E^U(\Theta) f \|_{\mathcal{G}}^2 < \infty$

2) $C_2 := \sup_{\substack{g \in \mathcal{G} \\ \|g\| = 1}} \sup_{\substack{r \in (0,1) \\ \theta \in [0, 2\pi)}} \langle T^* g, \delta(r, \theta) E^U(\Theta) T^* g \rangle_{\mathcal{H}} < \infty$

3) $C_3 := \sup_{\substack{f \in \mathcal{H} \\ \|f\| = 1}} \sup_{\substack{a, b \in [0, 2\pi) \\ a < b}} \frac{\| T E^U((a,b)) E^U(\Theta) f \|_{\mathcal{G}}^2}{b - a} < \infty$

4) $C_4 := \sup_{\substack{g \in \mathcal{G} \\ \|g\| = 1}} \sup_{\substack{a, b \in [0, 2\pi) \\ a < b}} \frac{\| E^U((a,b)) E^U(\Theta) T^* g \|_{\mathcal{H}}^2}{b - a} < \infty$

5) $C_5 := \frac{1}{2\pi} \sup_{\substack{g \in \mathcal{G} \\ \|g\| = 1}} \sup_{\substack{r \in (0,1) \\ \theta \in [0, 2\pi)}} (1 - r^2) \| R(r e^{i\theta}) E^U(\Theta) T^* g \|_{\mathcal{H}}^2 < \infty .$

Also, $C_1 = C_2 = C_3 = C_4 = C_5 .$

Let us jointly show the interest of the notion of U-smooth operators for the proof of the existence and J-completeness of the wave operators.

<u>Thm</u>: Let $\Theta \subset [0, 2\pi)$ be an open set, and assume that $(JU_0 - UJ) = T^* T_0$, with $T_0 \in \mathcal{B}(\mathcal{H}_0, \mathcal{G})$ U_0-smooth on any closed subset $\Theta' \subset \Theta$, and with $T \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ U-smooth on any closed subset $\Theta' \subset \Theta$. Then $W_\pm(U, U_0, J, \Theta)$ and $W_\pm(U_0, U, J^*, \Theta)$ exist.

lemma p 29 of the lecture notes

As seen before, the existence of $W_\pm(U_0, U, J^*, \Theta)$ implies the J-completeness of the wave operators $W_\pm(U, U_0, J, \Theta)$.

Proof:

Consider $f_0 \in \mathcal{H}_0$ such that $E^{U_0}(\Theta')f_0 = f_0$ for some closed subset Θ' of Θ. Let also $\eta \in C_c^\infty(e^{i\Theta}, \mathbb{R})$ with $\eta = 1$ on $e^{i\Theta'}$. We shall show that

open ↑ subset of S^1

$$\underset{n \to \pm\infty}{s\text{-}\lim}\ \eta(U) U^{-n} J U_0^n f_0 \quad \text{exist, and}$$

$$\underset{n \to \pm\infty}{s\text{-}\lim}\ (1-\eta(U)) U^{-n} J U_0 f_0 = 0. \ (*)$$

As a consequence, $\underset{n \to \pm\infty}{s\text{-}\lim}\ U^{-n} J U_0^n E^{U_0}(\Theta)$ exist.

↑ even stronger than with $E^{U_0}_{ac}(\Theta)$

1) Set $W_n := \eta(U) U^{-n} J U_0^n$ and observe that for $f \in \mathcal{H}$, one has for $m \leq n-1$

$$\left| \langle f, (W_n - W_m) f_0 \rangle_{\mathcal{H}} \right|$$

$$= \left| \langle \eta(U) f, (U^{-n} J U_0^n - U^{-m} J U_0^m) f_0 \rangle_{\mathcal{H}} \right|$$

telescoping series

$$= \left| \sum_{j=m+1}^{n} \langle \eta(U) f, U^{-j} \underbrace{(J U_0 - U J)}_{T^* T_0} U_0^{j-1} f_0 \rangle_{\mathcal{H}} \right|$$

$$= \left| \sum_{j=m+1}^{n} \langle T U^j \eta(U) f, T_0 U_0^{j-1} f_0 \rangle_{\mathcal{G}} \right|$$

Cauchy-Schwarz

$$\leq \left(\sum_{j=m+1}^{n} \| T U^j \eta(U) g \|_g^2 \right)^{1/2} \left(\sum_{j=m+1}^{n} \| T_0 U_0^{j-1} f_0 \|_g^2 \right)^{1/2}$$

$$\leq c_2 \| g \|_{\mathcal{H}} \left[\sum_{j=m}^{n-1} \| T_0 U_0^j f_0 \|_g^2 \right]^{1/2} .$$

Since T_0 is U_0-smooth on Θ, then

$\|(W_n - W_m) f_0 \|_{\mathcal{H}} \longrightarrow 0$ as $m \to \infty$ or as $n \to -\infty$.

Since any strong Cauchy sequence converges in $\mathcal{H}$, we infer that $\lim\limits_{n \to \pm\infty} W_n f_0$ exist.

2) Take $\eta_0 \in C_c^\infty(e^{i\Theta}, \mathbb{R})$ with $\eta_0 = 1$ on $e^{i\Theta'}$ and s.t. $\eta \eta_0 = \eta_0$. Then $\eta_0(U_0) f_0 = f_0$ and

$$(1 - \eta(U)) J \eta_0(U_0) = (1 - \eta(U)) \big(J \eta_0(U_0) - \eta_0(U) J \big) .$$

Let us show that $\operatorname*{s-lim}\limits_{n \to \pm\infty} \big(J \eta_0(U_0) - \eta_0(U) J \big) U_0^n f_0 = 0$, which implies $\circledast$.

Since the monomials $\mathbb{S}^1 \ni z \longmapsto z^k$ for $k \in \mathbb{Z}$ are total in $C(\mathbb{S}^1)$ for the sup norm, it is sufficient to show that $\operatorname*{s-lim}\limits_{n \to \pm\infty} \big(J U_0^k - U^k J \big) U_0^n f_0 = 0$, $\forall k \in \mathbb{Z}$.

By the telescoping formula

$$\mathfrak{J} U_0^k - U^k \mathfrak{J} = \sum_{j=1}^{k} U^{j-1} (\mathfrak{J} U_0 - U \mathfrak{J}) U_0^{k-j}$$

one infers that for $k \geq 1$:

$$\lim_{n \to \pm\infty} \| (\mathfrak{J} U_0^k - U^k \mathfrak{J}) U_0^n f_0 \|_{\mathcal{H}}$$

$$\leq \lim_{n \to \pm\infty} \sum_{j=1}^{k} \| (\underbrace{\mathfrak{J} U_0 - U \mathfrak{J}}_{T^* T_0}) U_0^{n+k-j} f_0 \|_{\mathcal{H}}$$

$$\leq \| T^* \|_{\mathcal{B}(\mathcal{G}, \mathcal{H})} \sum_{j=1}^{k} \lim_{m \to \pm\infty} \| T_0 U_0^m f_0 \|_{\mathcal{G}}$$

$$= \| T^* \|_{\mathcal{B}(\mathcal{G}, \mathcal{H})} \, k \lim_{m \to \pm\infty} \| T_0 U_0^m f_0 \|_{\mathcal{G}}$$

$= 0$ otherwise, it would not be square summable.

For $k = -\ell \leq 0$, one has

$$\mathfrak{J} U_0^k - U^k \mathfrak{J} = - U^{-\ell} (\mathfrak{J} U_0^{\ell} - U^{\ell} \mathfrak{J}) U_0^{-\ell},$$

and the same argument holds. This concludes the existence of $s\text{-}\lim_{n \to \pm\infty} U^{-n} \mathfrak{J} U_0^n E^{U_0}(\Theta)$.

By a similar argument, using $U_0^* \mathfrak{J}^* - \mathfrak{J}^* U^* = T_0^* T$, one gets that $s\text{-}\lim_{n \to \pm\infty} U_0^{-n} \mathfrak{J}^* U^n E^{U}(\Theta)$ exist.

From the existence of $s\text{-}\lim_{n\to\pm\infty} U^{-n} J U_0^{-n} E^{U_0}(\Theta)$ and of $s\text{-}\lim_{n\to\pm\infty} U_0^{-n} J^* U^n E^U(\Theta)$, one infers the existence of $W_\pm(U, U_0, J, \Theta)$ and $W_\pm(U_0, U, J^*, \Theta)$. $\square$

they contain $E^{U_0}_{ac}(\Theta)$ and $E^{U}_{ac}(\Theta)$

<u>Remark</u>: The proof leads to a slightly stronger result ($E^{U_0}(\Theta)$, $E^U(\Theta)$ and not the a.c. projections). However, the existence of U-smooth operators is very much related to the a.c. subspace. We shall come back to this.

For the proof of the equivalent definition of U-smooth operators, we need a vector valued version of Plancherel thm.

<u>Lemma</u>: Let $\phi : [0, 2\pi) \to \mathcal{H}$ be weakly measurable, and assume that $\phi \in L^2([0,2\pi), \mathcal{H})$. Set $\hat{\phi} : \mathbb{Z} \to \mathcal{H}$ by

$$\hat{\phi}(n) := \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{-in\theta} \phi(\theta).$$

integral in the weak sense.

Then, $\forall B \in \mathcal{B}(\mathcal{H})$ one has

$$\sum_{n \in \mathbb{Z}} \| B \hat{\phi}(n) \|_{\mathcal{H}}^2 = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \| B \phi(\theta) \|_{\mathcal{H}}^2 .$$

(Proof as an exercise)

<u>Proof of the theorem</u>

1) For any $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ one has

$$\| A \|^2_{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)} = \| A^* \|^2_{\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)} = \| A^* A \|_{\mathcal{B}(\mathcal{H}_1)} .$$

From this, one infers that $c_3 = c_4$. Similarly, the equality $c_2 = c_5$ also directly holds.

2) $c_2 \le c_4$: Assume $c_4 < \infty$, choose $g \in \mathcal{G}$, then

$\forall a, b \in [0, 2\pi)$, $a < b$, one has

$$\langle T^* g, E^U((a,b))\, E^U(\Theta)\, T^* g \rangle_{\mathcal{H}} \le c_4 \, (b-a) \, \|g\|_{\mathcal{G}}^2$$

$\Rightarrow$ the measure $v \mapsto \langle T^* g, E^U(v)\, E^U(\Theta)\, T^* g \rangle$ is absolutely continuous on $[0, 2\pi)$, and

$$\left| \frac{d}{d\theta'} \langle T^* g, E^U([0, \theta'))\, E^U(\Theta)\, T^* g \rangle_{\mathcal{H}} \right| \le c_4 \, \|g\|_{\mathcal{G}}^2 .$$

Then, by the spectral theorem

$$\left| \langle T^* g, \delta(r, \theta)\, E^U(\Theta)\, T^* g \rangle_{\mathcal{H}} \right|$$

derivative of the spectral density

$$= \left| \int_0^{2\pi} d\theta' \, \frac{1}{2\pi} (1 - r^2) \left| 1 - r e^{i(\theta - \theta')} \right|^{-2} \frac{d}{d\theta'} \langle T^* g, E^U([0, \theta'))\, E^U(\Theta)\, T^* g \rangle_{\mathcal{H}} \right|$$

$$\le c_4 \, \|g\|_{\mathcal{G}}^2 \int_0^{2\pi} d\theta' \, \frac{1}{2\pi} (1 - r^2) \left| 1 - r e^{i(\theta - \theta')} \right|^{-2}$$

$= c_4 \, \|g\|_{\mathcal{G}}^2$, indep. of $r \in (0,1)$ and $\theta \in [0, 2\pi)$.

Thus $c_2 \le c_4$.

3) $c_1 \le c_5$: Assume $c_5 < \infty$, choose $r \in (0,1)$, and consider the map $\phi : [0, 2\pi) \ni \theta \mapsto T \delta(r, \theta)\, E^U(\Theta) f$ for $f \in \mathcal{H}$.

Then $\hat{\phi}(n) := \frac{1}{2\pi} \int_0^{2\pi} d\theta \; e^{-in\theta} \; \phi(\theta)$

$= \frac{1}{2\pi} |r|^n \, T \, U^n E^U(\Theta) f$ take the series p35 into account.

and by Plancherel lemma (for $\mathcal{H} = \mathcal{G}$ and $B = 1\!\!1$):

$$\sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi}\right)^2 r^{2|n|} \| T U^n E^U(\Theta) f \|_{\mathcal{G}}^2$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \| T \, \delta(r,\theta) \, E^U(\Theta) f \|_{\mathcal{G}}^2$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{1}{2\pi} (1-r^2) \, \| T R(re^{i\theta})^* E^U(\Theta) \|_{\mathcal{B}(\mathcal{H},\mathcal{G})}^2$$

$$\cdot \frac{1}{2\pi} (1-r^2) \, \| R(re^{i\theta}) E^U(\Theta) f \|_{\mathcal{H}}^2$$

$$\leq C_5 \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \langle f, \delta(r,\theta) E^U(\Theta) f \rangle_{\mathcal{H}}$$

$= C_5 \frac{1}{2\pi} \| f \|_{\mathcal{H}}^2$, indep. of $r \in (0,1)$.

By considering a sup on r, one infers that $C_1 \leq C_5$.

5) $\underline{C_4 \leq C_1}$: Assume $C_1 < \infty$, let $g \in \mathcal{G}$, $f \in \mathcal{H}$, $a, b \in [0, 2\pi)$, $a < b$. Then

$$\left| \langle g, T \tfrac{1}{2}\left(E^U((a,b)) + E^U([a,b])\right) E^U(\Theta) f \rangle_{\mathcal{G}} \right|^2$$

$$= \left| \langle T^* g, \tfrac{1}{2}\left(E^U((a,b)) + E^U([a,b])\right) E^U(\Theta) f \rangle_{\mathcal{H}} \right|^2$$

Stone's formula not shown in this course

$$= \left| \lim_{r \nearrow 1} \int_a^b d\theta \, \langle T^* g, \delta(r,\theta) E^U(\Theta) f \rangle_{\mathcal{H}} \right|^2$$

$$\leq \lim_{r \nearrow 1} \left(\int_a^b d\theta \, \left| \langle g, T \delta(r,\theta) E^U(\Theta) f \rangle_{\mathcal{G}} \right| \right)^2$$

$$\leq \lim_{r \nearrow 1} \left(\int_a^b d\theta \, \|g\|_{\mathcal{G}}^2 \right) \left(\int_a^b d\theta \, \| T \delta(r,\theta) E^U(\Theta) f \|_{\mathcal{G}}^2 \right)$$

$$= \|g\|_{\mathcal{G}}^2 \, (b-a) \lim_{r \nearrow 1} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{2|n|} \| T U^n E^U(\Theta) f \|_{\mathcal{G}}^2$$

Plancherel, as on page 44

$$\leq \|g\|_{\mathcal{G}}^2 \, (b-a) \, C_1 \|f\|_{\mathcal{H}}^2$$

Since f is arbitrary, one gets

$$\| E^U((a,b)) E^U(\Theta) T^* g \|_{\mathcal{H}}^2$$

orthogonality

$$\leq \| \tfrac{1}{2}\left(E^U((a,b)) + E^U([a,b])\right) E^U(\Theta) T^* g \|_{\mathcal{H}}^2$$

$$\leq C_1 (b-a) \|g\|_{\mathcal{G}}^2 \implies C_4 \leq C_1.$$

Summary:

1)

2) $\Leftrightarrow$ 5)

3) $\Leftrightarrow$ 4)

$\square$

As already mentioned on page 41, smooth operators are very much related to absolutely continuous subspaces. More precisely, one has:

<u>Lemma</u>: If T is U-smooth on Θ, then $\overline{E^U(\Theta)T^*\mathcal{G}} \subset E^U_{ac}(\Theta)\mathcal{H}$. In particular, if $\ker\big(TE^U(\Theta)\big) = \big(E^U(\Theta)\mathcal{H}\big)^\perp$, then $E^U(\Theta)\mathcal{H} \subset \mathcal{H}_{ac}(U)$.

<u>Proof</u>: As already mentioned on p. 43, the characterization 4) of U-smoothness implies that the measure

$$\vartheta \longmapsto \langle E^U(\Theta)T^*g, E^U(\vartheta)E^U(\Theta)T^*g\rangle_{\mathcal{H}}$$

is absolutely continuous. This implies that

$$E^U(\Theta)T^*g \in E^U_{ac}(\Theta)\mathcal{H}, \quad \forall g \in \mathcal{G}.$$

From the equality $\overline{\operatorname{Ran}\big(E^U(\Theta)T^*\big)} \oplus \ker\big(TE^U(\Theta)\big) = \mathcal{H}$ and from $\overline{\operatorname{Ran}\big(E^U(\Theta)T^*\big)} \subset E^U_{ac}(\Theta)\mathcal{H}$,

$(P_{sc}(U) + P_p(U))\,E^U(\Theta)$

it follows that $E_s^U(\Theta)\mathcal{H} \subset \underset{\text{by assumption}}{\underbrace{\ker(TE^U(\Theta))}_{\shortparallel}}$. However, since $E_s^U(\Theta)\mathcal{H} \subset E^U(\Theta)\mathcal{H} \perp (E^U(\Theta)\mathcal{H})^\perp$, one infers that $E_s^U(\Theta)\mathcal{H} = \{0\}$, meaning

$E^U(\Theta) = E_{ac}^U(\Theta)$, $\Rightarrow$ $E^U(\Theta)\mathcal{H} \subset \mathcal{H}_{ac}(U)$. $\square$

Corollary: If $\ker(T) = \{0\}$ and T is U-smooth on Θ, then U is ac on Θ.

Let us conclude this section on a standard result.

Lemma: Let $f, g \in \mathcal{H}$ be fixed, and consider $S((1-\varepsilon)^{\pm 1}, \theta)$ for the operator U. Then

$$\lim_{\varepsilon \searrow 0} \langle S((1-\varepsilon)^{\pm 1}, \theta) f, g\rangle_{\mathcal{H}} = \pm \frac{d}{d\theta} \langle E^U([0,\theta)) f, g\rangle_{\mathcal{H}}$$

$$= \pm \frac{d}{d\theta} \langle E_{ac}^U([0,\theta)) f, g\rangle_{\mathcal{H}}$$

depends on f and g.

for a.e. $\theta \in [0, 2\pi)$.

Remark: ⚠ The statement depends on f and g, the existence of the limit is a consequence of Fatou's theorem. The equality with the r.h.s. follows from Privalov theorem.

Exercise: Provide a proof of this statement, together with Stone's formula used on p 45.

III. 3 Wave operators : time independent approach

We present another approach of scattering theory. Later, condition for the equality of the two approaches will be shown.

For $\varepsilon \in (0,1)$, $f_0 \in \mathcal{H}_0$, $f \in \mathcal{H}$ and

$g_\pm(\varepsilon) := \frac{1}{2\pi}\left(1-(1-\varepsilon)^{\pm 2}\right)$, one sets

$$\langle \omega_\pm(U, U_0, J, \varepsilon) f_0, f\rangle_{\mathcal{H}} :=$$

$$= \pm g_\pm(\varepsilon)\int_0^{2\pi} d\theta \, \langle J \underbrace{R_0((1-\varepsilon)^{\pm 1} e^{i\theta})}_{\text{resolvent of } U_0} f_0, \underbrace{R((1-\varepsilon)^{\pm 1} e^{i\theta})}_{\text{resolvent of } U} f\rangle_{\mathcal{H}}.$$

Let us firstly observe the boundedness independently of ε:

$$|\langle \omega_\pm(U, U_0, J, \varepsilon) f_0, f\rangle_{\mathcal{H}}|^2 \quad \Rightarrow \text{Cauchy-Schwarz}$$

$$\leq \|J\|^2_{\mathcal{B}(\mathcal{H}_0, \mathcal{H})} \left(g_\pm(\varepsilon)\int_0^{2\pi} d\theta \, \|R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0\|^2_{\mathcal{H}_0}\right)$$

$$\cdot \left(g_\pm(\varepsilon)\int_0^{2\pi} d\theta \, \|R((1-\varepsilon)^{\pm 1} e^{i\theta}) f\|^2_{\mathcal{H}}\right)$$

for U_0

$$= \| \mathcal{J} \|^2_{B(\mathcal{H}_0, \mathcal{H})} \int_0^{2\pi} \langle \delta_0((1-\varepsilon)^{\pm 1}, \theta) f_0, f_0 \rangle_{\mathcal{H}_0} \, d\theta$$

$$\cdot \int_0^{2\pi} \langle \delta((1-\varepsilon)^{\pm 1}, \theta) f, f \rangle_{\mathcal{H}} \, d\theta$$

for U.

$$= \| \mathcal{J} \|^2_{B(\mathcal{H}_0, \mathcal{H})} \| f_0 \|^2_{\mathcal{H}_0} \| f \|^2_{\mathcal{H}} .$$

This implies that

independent of ε.

$$\| \omega_\pm (U, U_0, \mathcal{J}, \varepsilon) \|_{B(\mathcal{H}_0, \mathcal{H})} \leq \| \mathcal{J} \|_{B(\mathcal{H}_0, \mathcal{H})} .$$

In the sequel, we are interested in the limit $\varepsilon \searrow 0$.

A first result, with χ the characteristic function.

<u>Lemma:</u> Let $\Theta_0, \Theta \subset [0, 2\pi)$ be Borel sets, $f_0 \in \mathcal{H}_0$, $f \in \mathcal{H}$, and assume that

$$a_\pm (f_0, f, \theta) := \pm \lim_{\varepsilon \searrow 0} g_\pm(\varepsilon) \langle \mathcal{J} R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0, R((1-\varepsilon)^{\pm 1} e^{i\theta}) f \rangle_{\mathcal{H}}$$

exist for a.e. $\theta \in \Theta_0 \cap \Theta$. Then:

1) $a_\pm (E^{U_0}(\Theta_0) f_0, E^U(\Theta) f, \theta) = 0$ for a.e. $\theta \in [0, 2\pi) \setminus (\Theta_0 \cap \Theta)$,

and $a_\pm\left(E^{U_0}(\Theta_0)f_0,\ E^{U}(\Theta)f,\ \theta\right) =$

$= \chi_{\Theta_0\cap\Theta}(\theta)\ a_\pm(f_0, f, \theta)$ for a.e. $\theta\in[0,2\pi)$.

2) If $a_\pm(f_0, f, \theta)$ exist for a.e. $\theta\in[0,2\pi)$, then the same holds for $P_{ac}(U_0)f_0$ instead of f_0, and for $P_{ac}(U)f$ instead of f. In addition, for a.e. $\theta\in[0,2\pi)$,

$$a_\pm(f_0, f, \theta) = a_\pm\left(P_{ac}(U_0)f_0, f, \theta\right)$$
$$= a_\pm\left(f_0, P_{ac}(U)f, \theta\right)$$
$$= a_\pm\left(P_{ac}(U_0)f_0, P_{ac}(U)f, \theta\right).$$

3) $$\lim_{\varepsilon\searrow 0}\left\langle \omega_\pm(U, U_0, J, \varepsilon)\, E^{U_0}_{ac}(\Theta_0)f_0,\ E^{U}(\Theta)f\right\rangle_{\mathcal{H}}$$
$$= \int_{\Theta_0\cap\Theta} a_\pm(f_0, f, \theta)\, d\theta$$

it is not true if ac is not imposed.

<u>Proof</u>:

1) One has

$$\lim_{\varepsilon\searrow 0}\left|\, g_\pm(\varepsilon)\left\langle JR_0\left((1-\varepsilon)^{\pm}e^{i\theta}\right)E^{U_0}(\Theta_0)f_0,\ R\left((1-\varepsilon)^{\pm 1}e^{i\theta}\right)E^{U}(\Theta)f\right\rangle_{\mathcal{H}}\right|^2$$

(as before) Cauchy-Schwarz

$$\leq \|\mathfrak{J}\|^2_{\mathcal{B}(\mathcal{H}_0,\mathcal{H})} \lim_{\varepsilon \searrow 0} \Big[\langle \delta_0((1-\varepsilon)^{\pm 1}, \theta) E^{U_0}(\Theta_0) f_0, f_0 \rangle_{\mathcal{H}_0}$$

$$\cdot \langle \delta((1-\varepsilon)^{\pm 1}, \theta) E^U(\Theta) f, f \rangle_{\mathcal{H}} \Big]$$

$$= \|\mathfrak{J}\|^2_{\mathcal{B}(\mathcal{H}_0,\mathcal{H})} \frac{d}{d\theta} \langle E^{U_0}([0,\theta)) E^{U_0}(\Theta_0) f_0, f_0 \rangle_{\mathcal{H}_0} \cdot$$

$$\cdot \frac{d}{d\theta} \langle E^U([0,\theta)) E^U(\Theta) f, f \rangle_{\mathcal{H}}$$

$$= \|\mathfrak{J}\|^2_{\mathcal{B}(\mathcal{H}_0,\mathcal{H})} \chi_{\Theta_0 \cap \Theta}(\theta) \cdot \frac{d}{d\theta} \langle E^{U_0}([0,\theta)) f_0, f_0 \rangle_{\mathcal{H}_0} \cdot$$

$$\cdot \frac{d}{d\theta} \langle E^U([0,\theta)) f, f \rangle_{\mathcal{H}}$$

which implies the first part of 1). For the second part one has because of the sesquilinear expression

$$a_\pm(E^{U_0}(\Theta_0) f_0, E^U(\Theta) f, \theta) = a_\pm(f_0, f, \theta)$$

$$- a_\pm(E^{U_0}([0,2\pi) \setminus \Theta_0) f_0, E^U(\Theta) f, \theta) - a_\pm(f_0, E^U([0,2\pi) \setminus \Theta) f, \theta).$$

Then, if $\theta \in \Theta_0 \cap \Theta$, the last 2 terms vanish, as above. This result and the first part lead to the second part of 1).

singular continuous and pure point subspaces

2) Recall that $\mathcal{H}_s(U) = E^U(\Theta_s)\mathcal{H}$ for some Borel set $\Theta_s \subset [0, 2\pi)$ of Lebesgue measure 0. By the second statement of 1), the 4 equalities hold for a.e. $\theta \in [0, 2\pi)$.

3) Set $g_0 := E^{U_0}_{ac}(\Theta_0) f_0$, $g := E^U(\Theta) f$. Then,

$$\lim_{\varepsilon \searrow 0} \langle W_\pm(U, U_0, J, \varepsilon) g_0, g\rangle_{\mathcal{H}} \quad = \quad \text{by def}$$

$$= \pm \lim_{\varepsilon \searrow 0} \int_0^{2\pi} d\theta \, \langle J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) g_0, R((1-\varepsilon)^{\pm 1} e^{i\theta}) g\rangle_{\mathcal{H}}$$

to be proved

$$= \int_0^{2\pi} d\theta \, a_\pm(g_0, g, \theta)$$

by 1) and 2)

$$= \int_{\Theta_0 \cap \Theta} d\theta \, a_\pm(f_0, f, \theta).$$

Thus, only the exchange of the limit and of the integral remains, and this can be done with Vitali's theorem, see [Rudin, real + complex anal., p 133]. The argument is rather long, we do not provide it. $\square$

Fix $\Theta \subset [0, 2\pi)$ a Borel set.

Corollary: Let $\mathcal{D}_0 \subset E^{U_0}(\Theta)\mathcal{H}_0$, $\mathcal{D} \subset E^{U}(\Theta)\mathcal{H}$, dense subsets, and assume that $\forall f_0 \in \mathcal{D}$ $\forall f \in \mathcal{D}$, the expressions $a_\pm(f_0, f, \theta)$ exist for a.e. $\theta \in \Theta$. Then the following weak limits exist

$$\omega_\pm(U, U_0, J, \Theta) := w\text{-}\lim_{\varepsilon \searrow 0} \omega_\pm(U, U_0, J, \varepsilon)\, E^{U_0}_{ac}(\Theta).$$

Proof: just apply the previous statement 3) and use the uniform bound $\|\omega_\pm(U, U_0, J, \varepsilon)\|_{\mathcal{B}(\mathcal{H}_0, \mathcal{H})} \leq \|J\|_{\mathcal{B}(\mathcal{H}_0, \mathcal{H})}$ for a density argument. $\square$

Def: The weak limits $\omega_\pm(U, U_0, J, \Theta)$ are called the <u>local stationary wave operators</u> for the triple (U, U_0, J) and the Borel set Θ.

Observe that only the resolvents are involved, no evolution groups.

Remarks :

1) In the corollary, the r.h.s. can be replaced by $E^U_{ac}(\Theta)\, \omega_\pm(U, U_0, J, \mathcal{E})$ or by $E^U_{ac}(\Theta)\, \omega_\pm(U, U_0, J, \mathcal{E})\, E^{U_0}_{ac}(\Theta)$ and the same expression is obtained, but one can not remove both projections.

2) We infer from $\circledast$ that $\mathrm{Ran}\,(\omega_\pm(U, U_0, J, \Theta)) \subset E^U_{ac}(\Theta)\, \mathcal{H}$.

3) Observe that if $\omega_\pm(U, U_0, J, \Theta)$ exist, then $\omega_\pm(U_0, U, J^*, \Theta)$ exist as well, and are the adjoint of $\omega_\pm(U, U_0, J, \Theta)$.

4) The relation $\omega_\pm(U, U_0, J, \Theta)\, E^{U_0}(\Theta')$ $= E^U(\Theta')\, \omega_\pm(U, U_0, J, \Theta)$ hold, for any Borel set $\Theta' \subset \Theta$. Thus the intertwining relation holds also for the local stationary wave operators.

In order to show the existence of the local stationary wave operator, let us introduce a weaker version of U-smoothness. Recall that $g_\pm(\varepsilon) := \frac{1}{2\pi}\left(1-(1-\varepsilon)^{\pm 2}\right)$

<u>Def</u>: A linear operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ is called <u>weakly U-smooth</u> on a Borel set $\Theta \subset [0, 2\pi)$ if one of the following equivalent conditions hold for some functions $c_j : [0, 2\pi) \mapsto \mathbb{R}_+$, all $\varepsilon \in (0,1)$, and a.e. $\theta \in [0, 2\pi)$:

1) $\| T\, \delta(1-\varepsilon, \theta)\, E^U(\Theta)\, T^* \|_{\mathcal{B}(\mathcal{G})} \le c_1(\theta)$,

with or without adjoint

2) $|g_\pm(\varepsilon)|^{1/2} \, \| T R((1-\varepsilon)^{\pm 1} e^{i\theta})^{(*)} E^U(\Theta) \|_{\mathcal{B}(\mathcal{H}, \mathcal{G})} \le c_2(\theta)$,

4) $\| T E^U((\theta - \eta, \theta + \eta))\, E^U(\Theta)\, T^* \|_{\mathcal{B}(\mathcal{G})} \le c_3(\theta)\, \eta$,

5) $\| T E^U((\theta - \eta, \theta + \eta))\, E^U(\Theta) \|_{\mathcal{B}(\mathcal{H}, \mathcal{G})} \le c_4(\theta)\, \eta^{1/2}$,

6) $\displaystyle \operatorname*{w-lim}_{\eta \searrow 0} \frac{1}{2\eta}\, T E^U((\theta - \lambda, \theta + \lambda))\, E^U(\Theta)\, T^*$ exists, for a.e. θ.

3) $\displaystyle \operatorname*{w-lim}_{\varepsilon \searrow 0} T\, \delta(1-\varepsilon, \theta)\, E^U(\Theta)\, T^*$ exists for a.e. θ.

Clearly, an operator which is U-smooth on Θ is also weakly U-smooth on Θ, but the converse is not true (lack of uniformity in Θ). Nevertheless, it is the right concept for the stationary approach.

In the next statement, we again assume that

$$JU_0 - UJ = T^* T_0 \quad , \text{ with } T_0 \in \mathcal{B}(\mathcal{H}_0, \mathcal{G}), T \in \mathcal{B}(\mathcal{H}, \mathcal{G}).$$

Note also that the following equivalence holds:

$$s\text{-}\lim_{\varepsilon \searrow 0} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0 \quad \text{exist for a.e. } \theta \in [0, 2\pi)$$

$$\Longleftrightarrow \quad s\text{-}\lim_{\varepsilon \searrow 0} T_0 U_0^* R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0 \quad \text{exist for a.e. } \theta \in [0, 2\pi)$$

the proof is left as an exercise, only based on properties of the resolvent: $R_0(z) = 1 + z U^* R_0(z)$.

<u>Thm</u>: Assume that for each $f_0 \in \mathcal{D}_0 \subset E^{U_0}(\Theta)\mathcal{H}_0$ (dense)

$$s\text{-}\lim_{\varepsilon \searrow 0} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0 \quad \text{exist for a.e. } \theta \in \Theta,$$

and assume that T is weakly U-smooth on Θ.

Then the local stationary wave operators $\omega_\pm(U, U_0, J, \Theta)$ exist and satisfy the representation formulas

$$\langle \omega_\pm(U, U_0, J, \Theta) f_0, f \rangle = \int_\Theta d\theta \; a_\pm(f_0, f, \theta).$$

which also exist

(The assumptions can be exchanged, and the same result holds.

Proof: Observe firstly that the second resolvent equation takes the form (for $z \notin S^1$)

$$J R_0(z) - R(z) J = -z R(z) U^* T^* T_0 U_0^* R_0(z)$$

$$= -z \left(T U R(z)^*\right)^* T_0 U_0^* R_0(z).$$

Then one has (expression motivated by the def of $a_\pm(f_0, f, \theta)$)

$$\pm g_\pm(\varepsilon) R\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right)^* J R_0\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right)$$

$$= \pm g_\pm(\varepsilon) R\left((1-\varepsilon)^{\pm} e^{i\theta}\right)^* R\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right) J$$

$$\mp g_\pm(\varepsilon) (1-\varepsilon)^{\pm 1} e^{i\theta} \left(T U R\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right) R\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right)^*\right)^* \cdot$$

$$\cdot \; T_0 U_0^* R_0\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right)$$

$= \delta(1-\varepsilon,\theta)J - (1-\varepsilon)^{\pm 1} e^{i\theta} \left(T U \delta(1-\varepsilon,\theta)\right)^* T_0 U_0^* R_0\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right).$

For $f_0 \in \mathcal{D}_0$ and $f \in \mathcal{H}$, this leads to

$\pm g_\pm(\varepsilon) \langle J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0, R((1-\varepsilon)^{\pm 1} e^{i\theta}) f \rangle_{\mathcal{H}}$

$= \langle \delta(1-\varepsilon,\theta) J f_0, f \rangle$

$- (1-\varepsilon)^{\pm 1} e^{i\theta} \langle T_0 U_0^* R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0, T U \delta(1-\varepsilon,\theta) f \rangle_{\mathcal{G}}.$ ⊛

By the lemma on p 47, the first term has a limit as $\varepsilon \searrow 0$ for a.e. $\theta \in [0, 2\pi)$. By the assumption and by the equivalence mentioned before the statement, $\operatorname{s-lim}_{\varepsilon \searrow 0} T_0 U_0^* R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0$ exist for almost every $\theta \in \Theta$.

Since T is weakly U-smooth on Θ, TU is also weakly U-smooth on Θ (see for example expression 3) in the def. of weak smooth). It then follows that $\operatorname{w-lim}_{\varepsilon \searrow 0} T U \delta(1-\varepsilon,\theta) f$ exists for a.e. $\theta \in \Theta$ (proof below). Thus, the terms ⊛ admit a limit as $\varepsilon \searrow 0$ for a.e. $\theta \in \Theta$. As a consequence, the limits $a_\pm(f_0, f, \theta)$ exist for a.e. $\theta \in \Theta$. Then, the statement follows from the corollary on p 54, and on the statement 3) of Lemma p. 51. $\square$

Let us now provide a few results about weak smoothness. 1. the equivalence in the definition p56, 2. the existence of $w-\lim_{\varepsilon \searrow 0} T\delta(1-\varepsilon, \theta)$ for a.e. $\theta \in \Theta$.

1. : Equivalence for weak smoothness :

1) $\Leftrightarrow$ 2) from the equality $\|BB^*\|_{B(\mathcal{G})} = \|B\|^2_{B(\mathcal{H}, \mathcal{G})}$, for 2 different choices of B. By the same argument, 4) $\Leftrightarrow$ 5).

3) $\Rightarrow$ 1) and 6) $\Rightarrow$ 4) are clear. The proof of 1) $\Rightarrow$ 3) and 4) $\Rightarrow$ 6) are similar, we present only the first one.

Consider a basis $\{g_i\}_{i\in\mathbb{N}}$ of $\mathcal{G}$ and the set $\mathcal{D}$ of finite linear combinations of elements of this basis. Then $\mathcal{D}$ is dense in $\mathcal{G}$. By lemma on p. 47,

$\lim_{\varepsilon \searrow 0} \langle \delta(1-\varepsilon, \theta) E^U(\Theta) T^* g_i , T^* g_j \rangle_{\mathcal{H}}$ exists

for $\theta \in [0, 2\pi) \setminus \Lambda_{ij}$ with Λ_{ij} of Lebesgue measure 0.

Then, $\lim_{\varepsilon \searrow 0} \langle \delta(1-\varepsilon, \theta) E^U(\Theta) T^* \varphi, T^* \psi \rangle$

exist for all $\varphi, \psi \in \mathcal{D}$ and for $\theta \in [0, 2\pi) \setminus \bigcup_{i,j} \Lambda_{ij}$.

Note that $\bigcup_{i,j} \Lambda_{ij}$ has still measure 0. We further remove the set of measure 0 on which 1) does not hold. Thus, on the remaining set of full measure and by a density argument using the bound 1), one infers that

$w-\lim_{\varepsilon \searrow 0} \; T \delta(1-\varepsilon, \theta) E^U(\Theta) T^*$ exist.

Thus, it only remains to show that 1)-3) $\Leftrightarrow$ 4)-6).

Assume that 2) holds for some $\Theta \in [0, 2\pi)$, and let us prove 5) for the same Θ. Observe firstly that

$$1 \leq \frac{\eta^2+\varepsilon^2}{\varepsilon^2} \leq (\eta^2+\varepsilon^2) \frac{4}{(1-(1-\varepsilon)^{\pm 1})^2} \quad \text{for any } \varepsilon \in (0, \tfrac{1}{2}).$$

By the spectral thm it follows that

$$\langle E^U((\Theta-\eta, \Theta+\eta)) f, f \rangle_{\mathcal{H}}$$

$$\leq 4(\eta^2+\varepsilon^2) \langle R((1-\varepsilon)^{\pm 1} e^{i\Theta})^* R((1-\varepsilon)^{\pm 1} e^{i\Theta}) f, f \rangle_{\mathcal{H}}$$

$$\Rightarrow \| E^U((\Theta-\eta, \Theta+\eta)) f \|_{\mathcal{H}} \leq 2(\eta^2+\varepsilon^2)^{\frac{1}{2}} \| R((1-\varepsilon)^{\pm 1} e^{i\Theta}) f \|_{\mathcal{H}}.$$

Set $f := E^U(\Theta) T^* g$, one gets

$$\| E^U((\Theta-\eta, \Theta+\eta)) E^U(\Theta) T^* g \|_{\mathcal{H}}$$

$$\leq 2(\eta^2+\varepsilon^2)^{1/2} \| R((1-\varepsilon)^{\pm 1} e^{i\Theta}) E^U(\Theta) T^* g \|_{\mathcal{H}}$$

$$\leq 2 \frac{(\eta^2+\varepsilon^2)^{1/2}}{|g_\pm(\varepsilon)|^{1/2}} |g_\pm(\varepsilon)|^{1/2} \| R((1-\varepsilon)^{\pm 1} e^{i\Theta}) E^U(\Theta) T^* \|_{\mathcal{B}(\mathcal{G}, \mathcal{H})} \|g\|_{\mathcal{G}}$$

$$= 2 \frac{(\eta^2+\varepsilon^2)^{1/2}}{|g_\pm(\varepsilon)|^{1/2}} \underbrace{|g_\pm(\varepsilon)|^{1/2} \| T R((1-\varepsilon)^{\pm 1} e^{i\Theta})^* E^U(\Theta) \|_{\mathcal{B}(\mathcal{H}, \mathcal{G})}}_{\leq C_2(\Theta)} \|g\|_{\mathcal{G}}$$

no loss of generality thus $\eta \in (0, \tfrac{1}{2})$

But $$4 \frac{\eta^2+\varepsilon^2}{|g_\pm(\varepsilon)|} = \frac{\eta^2+\varepsilon^2}{\varepsilon} O(1) = 2\eta\, O(1)$$ if we choose $\varepsilon = \eta$.

Thus, $\| E^U((\Theta-\eta, \Theta+\eta)) E^U(\Theta) T^* \| \leq \underbrace{\text{cst}\; C_2(\Theta)}_{= C_5(\Theta)} \eta^{1/2}$.

Conversely, let us assume 6) and prove 3). By the spectral Theorem one has

$$\langle \delta((1-\varepsilon),\theta)\, E^U(\Theta)\, T^* g,\ T^* g' \rangle_{\mathcal{H}}$$

$$= g_\pm(\varepsilon) \int_0^{2\pi} d\theta' \left| 1-(1-\varepsilon)^{\pm 1} e^{i(\theta-\theta')} \right|^{-2} \cdot$$

$$\cdot \langle E^U(d\theta')\, E^U(\Theta)\, T^* g,\ T^* g' \rangle .$$

By 6) the functions $\theta \longmapsto \langle E^U([0,\theta))\, E^U(\Theta)\, T^* g,\ T^* g' \rangle$ have (symmetric) derivatives on a set Λ of full measure (independent of g and g'). Then, by an application of Fatou's lemma (see [Yaf, Thm I.2.7]), the limit $\varepsilon \searrow 0$ exists for the r.h.s. whenever a symmetric derivative exists, namely on Λ. Thus, $\lim_{\varepsilon \searrow 0}$ exists for all $\theta \in \Lambda$, which means that $\text{w-}\lim_{\varepsilon \searrow 0} T\, \delta(1-\varepsilon,\theta)\, E^U(\Theta)\, T^*$ exists for all $\theta \in \Lambda$. $\square$

↙ the missing argument on p 59.

<u>Lemma</u> : If T is weakly U-smooth on Θ, then

$\omega\text{-}\lim_{\varepsilon \searrow 0} T\,\delta(1-\varepsilon, \theta)\, E^U(\Theta) f$ exists for a.e. $\theta \in [0, 2\pi)$.

<u>Proof</u> : As on page 61, one easily gets that

$\lim_{\varepsilon \searrow 0} \langle T\,\delta(1-\varepsilon, \theta)\, E^U(\Theta) f, g \rangle_{\mathcal{G}}$ exists for all g

on a dense subset $\mathcal{D} \subset \mathcal{G}$ and for all θ on a set $\Lambda \subset [0, 2\pi)$ of full measure. For the density argument, we need an upper bound independent of ε. Indeed one has $\| T\,\delta(1-\varepsilon, \theta)\, E^U(\Theta) f \|_{\mathcal{G}}$

$\leq g_+(\varepsilon)^{1/2} \| T R((1-\varepsilon)e^{i\theta})^* E^U(\Theta) \|_{\mathcal{B}(\mathcal{H}, \mathcal{G})}$

$\cdot\, g_+(\varepsilon)^{1/2} \| R((1-\varepsilon)e^{i\theta}) f \|_{\mathcal{H}}$.

The first factor is smaller than $c_2(\theta)$, by assumption. For the second, we that $g_+(\varepsilon) \langle R((1-\varepsilon)e^{i\theta})^* R((1-\varepsilon)e^{i\theta}) f, f \rangle_{\mathcal{H}}$

$= \langle \delta((1-\varepsilon), \theta) f, f \rangle$ admits a limit as $\varepsilon \searrow 0$ for a.e. θ, and thus is uniformly bounded in ε. This uniform bound leads to the result. □

For p 59, it only remains to observe that

$$\omega\text{-}\lim_{\varepsilon \searrow 0} T\,\delta(1-\varepsilon,\theta)\,E^U(\Theta)\,f \text{ exists for a.e. } \theta \in [0,2\pi)$$

$$\Leftrightarrow \omega\text{-}\lim_{\varepsilon \searrow 0} T\,\delta(1-\varepsilon,\theta)\,f \text{ exists for a.e. } \theta \in \Theta,$$

which is an easy observation, since $\lim_{\varepsilon \searrow 0} \delta(1-\varepsilon,\theta)$ is linked to the derivative of the spectral measure.

Our next aim is to say something about the range of $\omega_\pm(U, U_0, J, \Theta)$. For that purpose, we shall introduce the auxiliary operators $\omega_\pm(U_0, U_0, J^*J, \varepsilon) \in \mathcal{B}(\mathcal{H}_0)$ by

$$\langle \omega_\pm(U_0,U_0,J^*J,\varepsilon) f_0, f_0' \rangle :=$$

$$:= \pm g_\pm(\varepsilon) \int_0^{2\pi} d\theta \, \langle J^*J\, R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0, R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0' \rangle_{\mathcal{H}_0}.$$

The last main result of this section can now be stated. The assumptions are similar to the thm p 57.

dense

Thm: Assume that for each $f_0 \in \mathcal{D}_0 \subset E^{U_0}(\Theta)\mathcal{H}_0$

$s\text{-}\lim_{\varepsilon \searrow 0} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0$ exist for a.e. $\theta \in \Theta$,

and assume that T is weakly U-smooth on Θ.

a) For a.e. $\theta \in \Theta$, the following equality holds

$$w\text{-}\lim_{\varepsilon \searrow 0} T\,\delta(1-\varepsilon, \theta)\, w_\pm(U, U_0, J, \Theta)\, U_0 f_0 =$$

$$= \pm\, w\text{-}\lim_{\varepsilon \searrow 0} g_\pm(\varepsilon)\, T R((1-\varepsilon)^{\pm 1} e^{i\theta})^* U J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0 .$$

b) $w_\pm(U_0, U_0, J^*J, \Theta) := w\text{-}\lim_{\varepsilon \searrow 0} W_\pm(U_0, U_0, J^*J, \varepsilon) E^{U_0}_{ac}(\Theta)$

exist and satisfy:

$$W_\pm(U, U_0, J, \Theta)^* W_\pm(U, U_0, J, \Theta) = W_\pm(U_0, U_0, J^*J, \Theta).$$

Proof: a) For $f \in \mathcal{H}$, $f_0 \in \mathcal{D}_0$, $\Theta' \subset \Theta$, one has

$w_\pm(U, U_0, J, \Theta) f_0 \subset E^U_{ac}(\Theta)\mathcal{H}$, and then

$$\langle E^U(\Theta') w_\pm(U, U_0, J, \Theta) f_0, f \rangle_{\mathcal{H}} =$$

$$= \int_{\Theta'} d\theta \frac{d}{d\theta} \langle E^U([0,\theta))\, w_\pm(U, U_0, J, \Theta) f_0, f \rangle_{\mathcal{H}}$$

$$= \int_{\Theta'} d\theta \lim_{\varepsilon \searrow 0} \langle \delta(1-\varepsilon, \theta)\, w_\pm(U, U_0, J, \Theta) f_0, f \rangle_{\mathcal{H}} .$$

On the other hand one also has:

$$\langle \overset{E^{U}(\Theta)}{\omega_\pm(U,U_0,\mathfrak{J},\Theta)}^{E^{U_0}(\Theta)} f_0 , E^{U}(\Theta') f \rangle_{\mathcal{H}} =$$ by def of $\omega_\pm$ + properties shown

$$= \pm \int_{\Theta'} d\theta \lim_{\varepsilon\searrow 0} g_\pm(\varepsilon) \langle R((1-\varepsilon)^{\pm 1} e^{i\theta})^* \mathfrak{J} R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0 , \overset{E^{U}(\Theta)}{f} \rangle_{\mathcal{H}}$$

Since Θ' is arbitrary, it follows that for a.e. $\theta \in \Theta$

$$\boxed{} \begin{cases} \lim_{\varepsilon\searrow 0} \langle \delta(1-\varepsilon,\theta)\, \omega_\pm(U,U_0,\mathfrak{J},\Theta) E^{U_0}(\Theta) f_0 , f \rangle_{\mathcal{H}} \\ = \pm \lim_{\varepsilon\searrow 0} \langle g_\pm(\varepsilon) R((1-\varepsilon)^{\pm 1} e^{i\theta})^* \mathfrak{J} R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0 , \overset{E^{U}(\Theta)}{f} \rangle_{\mathcal{H}} \end{cases}$$

By setting $f = U^* T^* g$ with $g \in \mathcal{G}$, $\|g\|_{\mathcal{G}} = 1$ one gets by the intertwining relation for a.e. $\theta \in \Theta$

$$\lim_{\varepsilon\searrow 0} \langle T \delta(1-\varepsilon,\theta)\, \omega_\pm(U,U_0,\mathfrak{J},\Theta) U_0 f , g \rangle_{\mathcal{G}}$$

$$= \pm \lim_{\varepsilon\searrow 0} \langle g_\pm(\varepsilon) T R((1-\varepsilon)^{\pm 1} e^{i\theta})^* \overset{E^{U}(\Theta)}{U} \mathfrak{J} R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0 , g \rangle_{\mathcal{H}} \quad (*)$$

⚠ this is not the weak limit yet, because of a.e.

The existence of the w-lim for the l.h.s. has been shown in the lemma p 64. For the r.h.s., we get that the limits exist for a dense set in $\mathcal{G}$

and for all θ on a set of full measure.
Thus, it only remains to get an upper bound indep. of ε, and one concludes by a density argument.
Observe that ⊛ satisfies (before taking the limit)

$$|⊛| \leq \|\mathcal{J}\|_{\mathcal{B}(\mathcal{H}_0,\mathcal{H})} \Big(|g_\pm(\varepsilon)|^{1/2} \| T R((1-\varepsilon)^{\pm 1} e^{i\theta})^* E^U(\Theta) \|_{\mathcal{B}(\mathcal{H},\mathcal{G})}$$

$$\cdot \Big(|g_\pm(\varepsilon)|^{1/2} \| R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0 \|_{\mathcal{H}_0}.$$

The first factor is smaller than $c_2(\theta)$, since T is weakly U-smooth on Θ. For the second factor, recall that it is equal to

$|\langle \delta_0(1-\varepsilon,\theta) f_0, f_0 \rangle|^{1/2}$, which is bounded for a.e. θ. Thus, $|⊛| < c(\theta)$ for a.e. θ, indep. of ε.

b) Let us now consider $f = \omega_\pm(U, U_0, \mathcal{J}) f_0'$, with $f_0' \in \mathcal{D}_0$. Then one has

$$\langle \omega_\pm(U, U_0, J, \Theta) f_0, \ \omega_\pm(U, U_0, J, \Theta) f_0' \rangle_{\mathcal{H}}$$

$$= \int_\Theta d\theta \ a_\pm\big(f_0, \ \omega_\pm(U, U_0, J, \Theta) f_0', \ \theta\big)$$

def

$$= \pm \int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_\pm(\varepsilon) \langle J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0, \ R((1-\varepsilon)^{\pm 1} e^{i\theta}) \, \omega_\pm(U, U_0, J, \Theta) f_0' \rangle_{\mathcal{H}}$$

resolvent eq.
see p 58-59

$$= + \int_\Theta d\theta \lim_{\varepsilon \searrow 0} \langle J f_0, \ \delta(1-\varepsilon, \theta) \, \omega_\pm(U, U_0, J, \Theta) f_0' \rangle_{\mathcal{H}}$$

$$- \int_\Theta d\theta \lim_{\varepsilon \searrow 0} \langle (1-\varepsilon)^{\pm 1} e^{i\theta} T_0 U_0^* R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0, \ T \delta(1-\varepsilon, \theta) \, \omega_\pm(U, U_0, J, \Theta) U_0 f_0' \rangle_{\mathcal{G}}$$

by $\boxed{\ }$

by a)

$$= \pm \int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_\pm(\varepsilon) \langle J f_0, \ R((1-\varepsilon)^{\pm 1} e^{i\theta})^* J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0' \rangle_{\mathcal{H}}$$

$$\mp \int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_\pm(\varepsilon) \langle (1-\varepsilon)^{\pm 1} e^{i\theta} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) U_0^* f_0, \ T R((1-\varepsilon)^{\pm 1} e^{i\theta})^* U J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0' \rangle_{\mathcal{H}}$$

exists by assumption

resolvent equation

$$= \pm \int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_\pm(\varepsilon) \langle J f_0, \ R((1-\varepsilon)^{\pm 1} e^{i\theta})^* J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0' \rangle_{\mathcal{H}}$$

$$\pm \int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_\pm(\varepsilon) \langle \big(J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) - R((1-\varepsilon)^{\pm 1} e^{i\theta}) J\big) f_0, \ J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0' \rangle_{\mathcal{H}}$$

$$= \pm \int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_\pm(\varepsilon) \langle J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0, \ J R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) E^{U_0}(\Theta) f_0' \rangle_{\mathcal{H}}$$

By applying the corollary on p. 54, one infers that this last expression is equal to

$$\underset{\varepsilon \searrow 0}{\text{w-lim}}\ \omega_\pm(U_0, U_0, J^*J, \varepsilon)\, E_{ac}^{U_0}(\Theta) = \omega_\pm(U_0, U_0, J^*J, \Theta).$$

$\square$

Our final aim is to compare $W_\pm(U, U_0, J, \Theta)$ and $\omega_\pm(U, U_0, J, \Theta)$. For this, we need a few lemmas, and then the main statement. For the next statement, recall that $\text{s-}\lim_{n\to\infty} f_n = f_\infty \iff \text{w-}\lim_{n\to\infty} f_n = f_\infty$ and $\lim_{n\to\infty} \|f_n\| = \|f_\infty\|$.

Back to a time dependent statement

Lemma: $W_\pm(U, U_0, J, \Theta) := \text{s-}\lim_{n\to\pm\infty} U^{-n} J U_0^n E_{ac}^{U_0}(\Theta)$

strong wave operator

exist if and only if

weak wave operator

$$\tilde W_\pm(U, U_0, J, \Theta) := \text{w-}\lim_{n\to\infty} E_{ac}^{U}(\Theta)\, U^{-n} J U^n E_{ac}^{U_0}(\Theta)$$

$$\tilde W_\pm(U_0, U_0, J^*J, \Theta) := \text{w-}\lim_{n\to\pm\infty} E_{ac}^{U_0}(\Theta)\, U_0^{-n} J^*J U_0^n E_{ac}^{U_0}(\Theta)$$ exist

and $\tilde W_\pm(U, U_0, J, \Theta)^* \tilde W_\pm(U, U_0, J, \Theta) = \tilde W_\pm(U_0, U_0, J^*J, \Theta)$, in which case $W_\pm(U, U_0, J, \Theta) = \tilde W_\pm(U, U_0, J, \Theta)$.

We use the simpler notation $W_\pm$ and $\tilde{W}_\pm$.

Proof: Assume $\tilde{W}_\pm$ exist, then for $f_0 \in \mathcal{H}_0$

$\| U^{-n} J U_0^n E_{ac}^{U_0}(\Theta) f_0 - \tilde{W}_\pm f_0 \|^2_{\mathcal{H}}$

$= \langle E_{ac}^{U_0}(\Theta) U_0^{-n} J^* J U_0^n E_{ac}^{U_0}(\Theta) f_0, f_0 \rangle_{\mathcal{H}_0} + \| \tilde{W}_\pm f_0 \|^2_{\mathcal{H}}$

$- \underline{2 \operatorname{Re} \langle U^{-n} J U_0^n E_{ac}^{U_0}(\Theta) f_0, \tilde{W}_\pm f_0 \rangle_{\mathcal{H}}}_{(*)}$.

Since $\tilde{W}_\pm = E_{ac}^{U}(\Theta) \tilde{W}_\pm$, then $(*)$ converges to $-2 \| \tilde{W}_\pm f_0 \|^2$. Thus, the l.h.s. converges to 0 as $n \to \pm\infty$ is equivalent to the existence of $w\text{-lim}\, E_{ac}^{U_0}(\Theta) U_0^{-n} J^* J U_0^n E_{ac}^{U_0}(\Theta)$ and the equality $\langle \tilde{W}_\pm(U_0, U_0, J^*J, \Theta) f_0, f_0 \rangle_{\mathcal{H}_0} = \| \tilde{W}_\pm f_0 \|^2_{\mathcal{H}}$

$$\Updownarrow$$

$$\tilde{W}_\pm^* \tilde{W}_\pm = \tilde{W}_\pm(U_0, U_0, J^*J, \Theta). \qquad \square$$

It has been shown in the lemma p 64 that if T is weakly U-smooth on Θ, then

$$\underset{\varepsilon \searrow 0}{w\text{-}\lim}\; T\,\delta(1-\varepsilon, \theta)\, E^U(\Theta) f \quad \text{exist for a.e. } \theta \in [0,2\pi).$$

We shall start now from this property.

dense

Lemma: Assume that $\forall f \in \mathcal{D} \subset E^U(\Theta)\mathcal{H}$,

$$\underset{\varepsilon \searrow 0}{w\text{-}\lim}\; T\,\delta(1-\varepsilon, \theta)\, E^U(\Theta) f \quad \text{exist for a.e. } \theta \in [0,2\pi).$$

dense

Then $\exists\, \mathcal{D}' \subset E^U_{ac}(\Theta)\mathcal{H}$ such that

$$\sum_{n\in\mathbb{Z}} \| T\,U^n f\|^2 < \infty \qquad \forall f \in \mathcal{D}'.$$

Proof: $\forall f \in \mathcal{D}$, set $F_f(\theta) := \underset{\varepsilon \searrow 0}{w\text{-}\lim}\; T\,\delta(1-\varepsilon, \theta) f$ for a.e. $\theta \in [0,2\pi)$, and define for $N \in \mathbb{N}$.

$$\Theta_{f,N} := \{\theta \in [0,2\pi) \mid \|F_f(\theta)\|_{\mathcal{G}} \leq N\}.$$

Since $|[0,2\pi) \setminus \Theta_{f,N}| \to 0$ as $N \to \infty$, the set

$$\mathcal{D}' := \bigwedge_{\text{finite linear span}} \{E^U_{ac}(\Theta_{f,N}) f \mid f \in \mathcal{D},\, N \in \mathbb{N}\} \subset E^U_{ac}(\Theta)\mathcal{H}$$

because $f \subset E^U(\Theta)\mathcal{H}$

is dense in $E^U(\Theta)\mathcal{H}$.

Consider now $E^U_{ac}(\Theta_{g,N})f =: \tilde{f}$, $f \in \mathcal{D}$ and $N \in \mathbb{N}$, and

$\phi(\theta) := w-\lim_{\varepsilon \searrow 0} T\,\delta(1-\varepsilon, \theta)\,\tilde{f}$. Then one has

$\hat{\phi}(n) = \frac{1}{2\pi} T U^n \tilde{f}$ (see p 44) and by

Plancherel lemma:

$$\frac{1}{4\pi^2} \sum_{n\in\mathbb{Z}} \| T U^n \tilde{f} \|_g^2 \overset{\circledast}{=} \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \| \phi(\theta) \|_g^2$$

$$= \frac{1}{2\pi} \int_{\Theta_{g,N}} d\theta \, \| F_f(\theta) \|_g^2$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} d\theta \, N^2 = N^2,$$

or in other words $\sum_{n\in\mathbb{Z}} \| T U^n \tilde{f} \|_g^2 \leq 4\pi^2 N^2 < \infty$.

The result can then be extended to $\mathcal{D}'$ from $\circledast$.

finite linear combinations.

It does not mean that T is U-smooth since there is no uniformity in the statement (no constant c_0 on the r.h.s.).

Lemma : Assume that for each $f_0 \in \mathcal{D}_0 \subset E^{U_0}(\Theta)\mathcal{H}_0$ (dense)

and each $f \in \mathcal{D} \subset E^U(\Theta)\mathcal{H}$ (dense), one has

$\omega\text{-}\lim_{\varepsilon \searrow 0} T_0\, \delta(1-\varepsilon, \theta) f_0$ and $\omega\text{-}\lim_{\varepsilon \searrow} T\, \delta(1-\varepsilon,\theta) f$

exist for a.e. $\theta \in [0, 2\pi)$. Then $\tilde{W}_\pm(U, U_0, J, \Theta)$ exist.

Proof : Set $W(n) := E^U_{ac}(\Theta)\, U^{-n} J\, U_0^n\, E^{U_0}_{ac}(\Theta)$, $n \in \mathbb{Z}$.

Let $\mathcal{D}_0' \subset E^{U_0}_{ac}(\Theta)\mathcal{H}_0$ and $\mathcal{D}' \subset E^U_{ac}(\Theta)\mathcal{H}$ be the subsets introduced in the previous lemma. Since

$\| W(n) \|_{\mathcal{B}(\mathcal{H}_0, \mathcal{H})} \leq \| J \|_{\mathcal{B}(\mathcal{H}_0, \mathcal{H})}$ (independent of n)

it is sufficient to show that $\forall f_0 \in \mathcal{D}_0'$, $\forall f \in \mathcal{D}'$,

$$\lim_{n \to \pm\infty} \langle W(n) f_0, f \rangle_{\mathcal{H}} = \lim_{n \to \infty} \langle U^{-n} J\, U_0^n f_0, f \rangle_{\mathcal{H}}$$

exist. For $n_1 < n_2$ and by a telescopic formula one has : Quite similar to the proof of Thm p 37

$$U^{-n_2} J\, U_0^{n_2} - U^{-n_1} J\, U_0^{n_1} = \sum_{n=n_1+1}^{n_2} \left(U^{-n} J\, U_0^n - U^{-(n-1)} J\, U_0^{n-1} \right)$$

$$= -\sum_{n=n_1+1}^{n_2} U^{-n} (UJ - JU_0)\, U_0^{n-1} = -\sum_{n=n_1+1}^{n_2} (T U^n)^* T_0\, U_0^{n-1}.$$

Then $\left| \langle W(n_2) f_0, f \rangle_{\mathcal{H}} - \langle W(n_1) f_0, f \rangle_{\mathcal{H}} \right|$

$$= \left| \sum_{n=n_1+1}^{n_2} \langle T_0 U_0^{n-1} f_0, T U^n f \rangle_g \right|$$

$$\leq \left(\sum_{n=n_1+1}^{n_2} \| T_0 U_0^{n-1} f_0 \|_g^2 \right)^{1/2} \left(\sum_{n=n_1+1}^{n_2} \| T U^n f \|_g^2 \right)^{1/2},$$

and both factors go to 0 as $n_1, n_2 \to \pm\infty$.
$\Rightarrow$ Cauchy sequence which converges. $\square$

<u>Proposition</u>: Assume 1) $\forall f_0 \in D_0 \subset E^{U_0}(\Theta)\mathcal{H}_0$ (dense) the limit $s\text{-}\lim_{\varepsilon \searrow 0} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0$ exist for a.e. $\theta \in [0, 2\pi)$

2) T is weakly U-smooth on Θ.

Assumptions similar to thm p 57.

Assume also that $\tilde{W}_\pm(U_0, U_0, J^*J, \Theta)$ exist. Then $W_\pm(U, U_0, J, \Theta)$ exist and coincide with $\omega_\pm(U, U_0, J, \Theta)$.

the def of $\omega_\pm$ was not the correct one !!! ⚠

Recall that these assumptions were sufficient for proving the existence of $\omega_\pm(U, U_0, J, \Theta)$ + equality

$$\omega_\pm(U, U_0, J, \Theta)^* \, \omega_\pm(U, U_0, J, \Theta) = \omega_\pm(U_0, U_0, J^*J, \Theta)$$

(see p 57 and 66).

Proof: For $f_0 \in \mathcal{D}_0$ and $f_0' \in \mathcal{H}_0$ one has

p57

$$\langle \omega_+(U_0, U_0, \mathfrak{z}^*\mathfrak{z}, \Theta) f_0, f_0' \rangle_{\mathcal{H}_0} =$$

$$= \frac{1}{2\pi} \lim_{\varepsilon \searrow 0} \left(1-(1-\varepsilon)^2\right) \int_0$$

can be added in the range

in the def

p34

$$\cdot \int_0^{2\pi} d\theta \, \langle E_{ac}^{U_0}(\Theta) R_0((1-\varepsilon)e^{i\theta})^* \mathfrak{z}^*\mathfrak{z} R_0((1-\varepsilon)e^{i\theta}) E_{ac}^{U_0}(\Theta) f_0, f_0' \rangle_{\mathcal{H}_0}$$

$$= \frac{1}{2\pi} \lim_{\varepsilon \searrow 0} \left(1-(1-\varepsilon)^2\right) \sum_{n,m \geq 0} (1-\varepsilon)^{n+m} \cdot$$

$$\cdot \int_0^{2\pi} d\theta \, e^{i\theta(m-n)} \langle E_{ac}^{U_0}(\Theta) U_0^n \mathfrak{z}^*\mathfrak{z} U_0^{-m} E_{ac}^{U_0}(\Theta) f_0, f_0' \rangle_{\mathcal{H}_0}$$

$$= \lim_{\varepsilon \searrow 0} \left(1-(1-\varepsilon)^2\right) \sum_{n \geq 0} (1-\varepsilon)^{2n} \langle E_{ac}^{U_0}(\Theta) U_0^n \mathfrak{z}^*\mathfrak{z} U_0^{-n} E_{ac}^{U_0}(\Theta) f_0, f_0' \rangle_{\mathcal{H}_0}.$$

Similarly, $\langle \omega_-(U_0, U_0, \mathfrak{z}^*\mathfrak{z}, \Theta) f_0, f_0' \rangle = \ldots =$

$$= \frac{1}{2\pi} \lim_{\varepsilon \searrow 0} \left(1-(1-\varepsilon)^{-2}\right) \sum_{m,n \geq 1} (1-\varepsilon)^{n+m} \cdot$$

factor $(1-\varepsilon)^{-2}$

$$\cdot \int_0^{2\pi} d\theta \, e^{i\theta(n-m)} \langle E_{ac}^{U_0}(\Theta) U_0^{-n} \mathfrak{z}^*\mathfrak{z} U_0^{m} E_{ac}^{U_0}(\Theta) f_0, f_0' \rangle_{\mathcal{H}_0}$$

$$= \lim_{\varepsilon \searrow 0} \left(1-(1-\varepsilon)^2\right) \sum_{n \geq 1} (1-\varepsilon)^{2(n-1)} \langle E_{ac}^{U_0}(\Theta) U_0^{-n} \mathfrak{z}^*\mathfrak{z} U_0^{n} E_{ac}^{U_0}(\Theta) f_0, f_0' \rangle_{\mathcal{H}_0}.$$

Now, observe that $\left(1-(1-\varepsilon)^2\right) \sum_{n \geq 0} \left((1-\varepsilon)^2\right)^n = 1$. By a general Tauber theorem, these limits converge

a reference is needed?

to $\lim_{n \to \pm\infty} \langle E_{ac}^{U_0}(\Theta) U_0^n \mathfrak{z}^*\mathfrak{z} U_0^{-n} E_{ac}^{U_0}(\Theta) f_0, f_0' \rangle_{\mathcal{H}_0} =$

$= \langle \widetilde{W}_{\mp}(U_0, U_0, \mathfrak{z}^*\mathfrak{z}, \Theta) f_0, f_0' \rangle_{\mathcal{H}_0}$, which exist by assumption.

Thus, $\omega_\pm(U_0, U_0, g^*g, \Theta) = \tilde{W}_\mp(U_0, U_0, g^*g, \Theta)$,

but the convention for was not good!

Now, since T is weakly U-smooth on Θ, then

$\omega\text{-}\lim_{\varepsilon \searrow 0} T\,\delta(1-\varepsilon, \theta)\,E^U(\Theta) f$ exists for a.e. $\theta \in [0, 2\pi)$,

see lemma p 64. Observe also that the assumption

$s\text{-}\lim_{\varepsilon \searrow 0} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0$ exist implies that

$\omega\text{-}\lim_{\varepsilon \searrow 0} T_0\,\delta_0(1-\varepsilon, \theta) f_0$ exists, for a.e. $\theta \in [0, 2\pi)$.

By the lemma on p 74, it follows that

$\tilde{W}_\pm(U, U_0, g, \Theta)$ exist. As above, it can be

shown that they correspond to $\omega_\mp(U, U_0, g, \Theta)$.

One then deduces from these equalities and from

Thm p. 66 that

$$\omega_\pm(U, U_0, g, \Theta)^* \, \omega_\pm(U, U_0, g, \Theta) = \omega_\pm(U_0, U_0, g^*g, \Theta)$$

$$\Leftrightarrow \quad \tilde{W}_\mp(U, U_0, g, \Theta)^* \, \tilde{W}_\mp(U, U_0, g, \Theta) = \tilde{W}_\mp(U_0, U_0, g^*g, \Theta).$$

lemma p 70

$\Rightarrow \; W_\mp(U, U_0, g, \Theta)$ exist and coincide with $\omega_\pm(U, U_0, g, \Theta)$. $\square$

<u>Remark</u> : The main 2 assumptions of the previous statement were the same as in the thm p 57 about the existence of $\omega_\pm(U, U_0, \mathcal{J}, \Theta)$, and in the thm p 66 about the equality

$$\omega_\pm(U, U_0, \mathcal{J}, \Theta)^* \, \omega_\pm(U, U_0, \mathcal{J}, \Theta) = \omega_\pm(U_0, U_0, \mathcal{J}^*\mathcal{J}, \Theta).$$

The additional assumption about the existence of $\tilde{W}_\pm(U_0, U_0, \mathcal{J}^*\mathcal{J}, \Theta)$ is automatically satisfied if

1) $\mathcal{J} = 1\!\!1$, 2) $\mathcal{J}^*\mathcal{J} = 1$, 3) $(\mathcal{J}^*\mathcal{J} - 1)E^{U_0}(\Theta) \in \mathcal{K}(\mathcal{H}_0)$. Indeed, one has

$$E_{ac}^{U_0}(\Theta)\, U_0^{-n}\, \mathcal{J}^*\mathcal{J}\, U_0^n\, E_{ac}^{U_0}(\Theta)$$

$$= E_{ac}^{U_0}(\Theta)\, U_0^{-n}\, 1\!\!1\, U_0^n\, E_{ac}^{U_0}(\Theta)$$

$$+ E_{ac}^{U_0}(\Theta)\, U_0^{-n}\, (\mathcal{J}^*\mathcal{J} - 1)\, U_0^n\, E_{ac}^{U_0}(\Theta)$$

$$= E_{ac}^{U_0}(\Theta) + E_{ac}^{U_0}(\Theta)\, U_0^{-n}\, \underbrace{(\mathcal{J}^*\mathcal{J} - 1)E^{U_0}(\Theta)}_{\in \mathcal{K}(\mathcal{H}_0)}\, U_0^n\, E_{ac}^{U_0}(\Theta)$$

but $\operatorname{w-lim}_{n\to\infty} U_0^n E_{ac}^{U_0}(\Theta) f = 0$ cf Lemma p 19, and any compact operator improves a weak convergence into a strong convergence. Thus

$$\lim_{n\to\infty} \left| \langle g, E_{ac}^{U_0}(\theta)\, U_0^{-n} J^* J U_0^n E_{ac}^{U_0}(\theta) f \rangle - \langle g, E_{ac}^{U_0}(\theta) f \rangle \right|$$

$$\leq \underbrace{\| U_0^n E_{ac}^{U_0}(\theta) g \|}_{\leq \|g\|} \; \underbrace{\| (J^* J - 1) E^{U_0}(\theta)\, U_0^n E_{ac}^{U_0}(\theta) f \|}_{\xrightarrow[n\to\infty]{} 0} .$$

Thus, $\underbrace{\tilde{W}_\pm(U_0, U_0, J^*J, \theta)}_{= W_\pm(U_0, U_0, J^*J, \theta)} = E_{ac}^{U_0}(\theta)$, and one infer from

$$W(U, U_0, J, \theta)^* W_\pm(U, U_0, J, \theta) = W_\pm(U_0, U_0, J^*J, \theta)$$
$$= E_{ac}^{U_0}(\theta)$$

that the wave operators are <u>isometries</u>.

The next statement is sometimes useful in the application, but its proof is also somewhat artificial, since one imposes further conditions on U in order to check a property related to U_0.

For the next statement we introduce for $z \notin S^1$

$$B(z) = T R(z) T^* \in \mathcal{B}(\mathcal{G}) .$$

dense

<u>Thm</u> Assume that 1) $\forall f_0 \in \mathcal{D}_0 \subset E^{U_0}(\Theta)\mathcal{H}_0$,

$s-\lim\limits_{\varepsilon \searrow 0} T_0 R_0\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right) f_0$ exist for a.e. $\theta \in [0,2\pi)$

2) $B_\pm(\theta) := w-\lim\limits_{\varepsilon \searrow 0} B\left((1-\varepsilon)^{\pm 1} e^{i\theta}\right)$ exist for

a.e. $\theta \in \Theta$. Then $W_\pm(U, U_0, J, \Theta)$ exist

and coincide with $\omega_\mp(U, U_0, J, \Theta)$.

<u>Proof</u>: Since 2) implies that T is weakly U-smooth on Θ

it is enough to show that $\widetilde{W}_\pm(U_0, U_0, J^*J, \Theta)$ exist,

and the statement follows then from the proposition p 75.

Consider $J^*JU_0 - U_0J^*J = J^*(JU_0 - UJ) + \underline{J^*UJ}$

$-(JU_0^* - U^*J)^*J - \underline{J^*UJ}$ Cancel

$= J^* T^* T_0 + \left[U^{-1}(JU_0 - UJ)U_0^{-1}\right]^* J$

$= \mathcal{J}^* T^* T_0 + U_0 (T^* T_0)^* U \mathcal{J}$

$= (T\mathcal{J})^* T_0 + U_0 T_0^* T U \mathcal{J}$

$= (T\mathcal{J})^* T_0 + (T_0 U_0^{-1})^* (TU\mathcal{J})$.

By following the content of lemma p 74, it is sufficient to get the existence for a.e. $\theta \in \Theta$ of

1) $\underset{\varepsilon \searrow 0}{w\text{-}\lim}\, T_0 \delta_0(1-\varepsilon,\theta) f_0$, 2) $\underset{\varepsilon \searrow 0}{w\text{-}\lim}\, TU\mathcal{J}\, \delta_0(1-\varepsilon,\theta) f_0$

3) $\underset{\varepsilon \searrow 0}{w\text{-}\lim}\, T\mathcal{J}\, \delta_0(1-\varepsilon,\theta) f_0$, 4) $\underset{\varepsilon \searrow 0}{w\text{-}\lim}\, T_0 U_0^{-1} \delta_0(1-\varepsilon,\theta) f_0$.

1) exists by assumption, and 4) similarly, by taking the trick of p 57 into account.

For 3), it is enough to get that $\| T\mathcal{J}\, \delta_0(1-\varepsilon,\theta) f_0 \|_{\mathcal{G}} \leq c_{\theta, f_0}$ for a.e. $\theta \in [0, 2\pi)$, with a constant independent of ε. Then, the density argument already used in Lemma p 64 can be used, and 3) would follow.

From the resolvent equation (see p 58) one has

$$\mathfrak{J} R_0(z) = R(z)\mathfrak{J} - \underbrace{z R(z) U^* T^*}_{= R(\bar{z}^{-1}) \text{ see p 34}} T_0 U_0^* R_0(z)$$

multiply by T on the left

$$\Rightarrow \quad T\mathfrak{J} R_0(z) = TR(z)\mathfrak{J} + B(\bar{z}^{-1})\, T_0 U_0^* R_0(z)$$

set $z=(1-\varepsilon)e^{i\theta}$ and apply to $R_0((1-\varepsilon)e^{i\theta})^* f_0$ and $g_+(\varepsilon)$

$$\Rightarrow \quad T\mathfrak{J}\delta_0(1-\varepsilon,\theta) f_0 =$$

$$= g_+(\varepsilon)\, T R((1-\varepsilon)e^{i\theta})\mathfrak{J} R_0((1-\varepsilon)e^{i\theta})^* f_0$$

$$+ B((1-\varepsilon)^{-1}e^{i\theta})\, T_0 U_0^* \delta_0(1-\varepsilon,\theta) f_0$$

take the norm

$$\Rightarrow \quad \| T\mathfrak{J}\,\delta_0(1-\varepsilon,\theta) f_0 \|$$

$$\leq \|\mathfrak{J}\|_{\mathcal{B}(\mathcal{H}_0,\mathcal{H})} \Big(|g_+(\varepsilon)|^{1/2} \underbrace{\| T R((1-\varepsilon)e^{i\theta})\|_{\mathcal{B}(\mathcal{H},\mathcal{G})}}_{①} \Big)\Big(g(\varepsilon)^{\frac{1}{2}} \underbrace{\| R_0((1-\varepsilon)e^{i\theta})^* f_0\|_{\mathcal{H}_0}}_{②} \Big)$$

$$+ \underbrace{\| B((1-\varepsilon)^{-1}e^{i\theta})\|_{\mathcal{B}(\mathcal{G})}}_{③} \underbrace{\| T_0 U_0^* \delta_0(1-\varepsilon,\theta) f_0\|_{\mathcal{G}}}_{④}.$$

① is bounded indep. of ε because T is weakly U-smooth on Θ,

② $= \langle \delta_0(1-\varepsilon,\theta) f_0, f_0\rangle_{\mathcal{H}_0}^{1/2}$ which is bounded, indep. of ε,

③ is bounded, because it has a weak limit,

④ was already considered in 4).

It only remains to consider 2), as for 3):

$$T\mathcal{U}\mathfrak{J}R_0(z) = T\mathcal{U}R(z)\mathfrak{J} - z\,T\cancel{\mathcal{U}}R(z)\cancel{\mathcal{U}^*}\,T^*\,T_0\,\mathcal{U}_0^*\,R_0(z)$$

$$= T\mathcal{U}R(z)\mathfrak{J} - z\,B(z)\,T_0\,\mathcal{U}_0^*\,R_0(z)$$

$$\Rightarrow T\mathcal{U}\mathfrak{J}\,\delta_0(1-\varepsilon,\theta)f_0 = g_+(\varepsilon)\,T\mathcal{U}R((1-\varepsilon)e^{i\theta})\,\mathfrak{J}\,R_0((1-\varepsilon)e^{i\theta})^* f_0$$

$$- (1-\varepsilon)e^{i\theta}\,B((1-\varepsilon)e^{i\theta})\,T_0\,\mathcal{U}_0^*\,\delta_0(1-\varepsilon,\theta)f_0$$

$$\Rightarrow \|T\mathcal{U}\mathfrak{J}\,\delta_0(1-\varepsilon,\theta)f_0\|$$

$$\leq \|\mathfrak{J}\|_{\mathcal{B}(\mathcal{H}_0,\mathcal{H})}\Big(|g_+(\varepsilon)|^{1/2}\,\underbrace{\|T\mathcal{U}R((1-\varepsilon)e^{i\theta})\|_{\mathcal{B}(\mathcal{H},\mathcal{G})}}_{1}\Big)\Big(|g_+(\varepsilon)|^{1/2}\,\underbrace{\|R_0((1-\varepsilon)e^{i\theta})^* f_0\|_{\mathcal{H}_0}}_{2}\Big)$$

$$+ |1-\varepsilon|\,\underbrace{\|B((1-\varepsilon)e^{i\theta})\|_{\mathcal{B}(\mathcal{G})}}_{3}\,\underbrace{\|T_0\,\mathcal{U}_0^*\,\delta_0(1-\varepsilon,\theta)f_0\|_{\mathcal{G}}}_{4}$$

Only the term 1) is different, but one can use the trick of p57, already used for 4). $\square$

III.4 Hilbert-Schmidt theory

The aim of this section is to show that Hilbert-Schmidt operators are useful and natural for checking the assumptions of the previous section.

Recall that a linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is Hilbert-Schmidt iff $\|T\|_{HS}^2 := \sum_{j\in\mathbb{N}} \|Te_j\|_{\mathcal{H}_2}^2 < \infty$ for any orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $\mathcal{H}_1$. Note that the convergence on 1 basis ensures it on all bases. Also, if $\mathcal{H}_1 = L^2(M_1, dm_1)$, $\mathcal{H}_2 = L^2(M_2, dm_2)$

Hilbert-Schmidt kernel.

then $[Af](x) = \int_{M_1} a(x,y) f(y)\, dm_1(y)$ for a.e. $x \in M_2$

and $\|A\|_{HS}^2 = \int_{M_1} dm_1 \int_{M_2} dm_2\, |a(x,y)|^2$.

The set of Hilbert-Schmidt operators is denoted by $S_2(\mathcal{H}_1, \mathcal{H}_2)$.

The first result is about self-adjoint operators, the link will then be made with the Cayley transform.

Lemma: Let $T \in S_2(\mathcal{H}, \mathcal{G})$, $\varepsilon \in (0,1)$, $\theta \in [0, 2\pi)$ and H self-adjoint in $\mathcal{H}$. Then

$$\lim_{\varepsilon \searrow 0} T\left(H - i\,\frac{1+(1-\varepsilon)^{\pm 1}e^{i\theta}}{1-(1-\varepsilon)^{\pm 1}e^{i\theta}}\right)^{-1} T^* \in S_2(\mathcal{G})$$

exits for a.e. $\theta \in [0, 2\pi)$. The convergence holds in the H.S. norm

We do not provide the proof of this lemma (which is quite involved) and refer to [Yaf, Thm 6.1.9] for a similar proof.

Lemma: For $T_1, T_2 \in S_2(\mathcal{H}, \mathcal{G})$ the following limits exist for a.e. $\theta \in [0, 2\pi)$: the convergence holds in the H.S. norm

1) $\lim_{\varepsilon \searrow 0} T_1 R\big((1-\varepsilon)^{\pm 1} e^{i\theta}\big) T_2^* \in S_2(\mathcal{G})$,

2) $s\text{-}\lim_{\varepsilon \searrow 0} T_1 R\big((1-\varepsilon)^{\pm 1} e^{i\theta}\big) f \in \mathcal{G}$, $\forall f \in \mathcal{H}$.

resolvent of the operator U

Proof: 1) Since $\mathcal{H}$ is separable, $\exists\, \phi \in [0, 2\pi)$ s.t. $1 \notin \sigma_p(e^{i\phi}U)$. Then we can apply the Cayley transform and get a self-adjoint operator:

$$H := i(1+e^{i\phi}U)(1-e^{i\phi}U)^{-1} \quad \text{and} \quad D(H) := \operatorname{Ran}(1-e^{i\phi}U).$$

Then
$$R(z) = (1-e^{i\phi}z)^{-1} + \frac{2ie^{i\phi}z}{(1-e^{i\phi}z)^2}\left(H - i\,\frac{1+e^{i\phi}z}{1-e^{i\phi}z}\right)^{-1}$$
for $z \notin S^1$. In particular for $z = (1-\varepsilon)^{\pm 1} e^{i\theta}$ one gets

$$T_1 R\big((1-\varepsilon)^{\pm 1} e^{i\theta}\big) T_1^* = \big(1-(1-\varepsilon)^{\pm 1} e^{i(\phi+\theta)}\big)^{-1} T_1 T_1^*$$
$$+ \frac{2i(1-\varepsilon)^{\pm 1} e^{i(\phi+\theta)}}{\big(1-(1-\varepsilon)^{\pm 1} e^{i(\phi+\theta)}\big)^2}\, T_1 \left(H - i\,\frac{1+(1-\varepsilon)^{\pm 1} e^{i(\phi+\theta)}}{1-(1-\varepsilon)^{\pm 1} e^{i(\phi+\theta)}}\right)^{-1} T_1^*$$

which has a limit for a.e. $\theta \in [0, 2\pi)$ by the previous lemma. The general case is treated by polarization:

$$4\, T_1 R(z) T_2^* = (T_1+T_2) R(z) (T_1+T_2)^* - (T_1-T_2) R(z) (T_1-T_2)^*$$
$$- i (T_1 - iT_2) R(z) (T_1 - iT_2)^* + i (T_1 + iT_2) R(z) (T_1 + iT_2)^*.$$

2) For $f \in \mathcal{H}$, find any $T_2 \in S_2(\mathcal{H}, \mathcal{G})$ and $g \in \mathcal{G}$ satisfying $f = T_2^* g$. Then the claim follows from the statement 1). $\square$

By the previous lemma, one can now easily check the assumptions of the theorem p. 80. Indeed, if $T \in S_2(\mathcal{H}_0, \mathcal{G})$ and $T \in S_2(\mathcal{H}, \mathcal{G})$, then the assumptions of that Theorem are satisfied for $D_0 = \mathcal{H}$, $\Theta = [0, 2\pi)$. Note that this corresponds to $JU_0 - UJ = T^* T_0$ which is trace class, the simplest version of perturbation theory.

<u>Remark</u>: $T \in S_2(\mathcal{H}, \mathcal{G})$ corresponds to a "universal" weakly U-smooth operator, and does not depend on U. For a given U, more specific weakly U-smooth can be found.

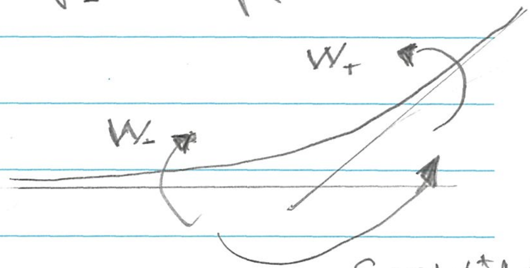
IV Scattering operator

After the wave operators, the second main object of scattering theory is the <u>scattering operator</u>.

In the time dependent approach we set

$$S(U,U_0,J,\Theta) := W_+(U,U_0,J,\Theta)^* \, W_-(U,U_0,J,\Theta) \in \mathcal{B}(\mathcal{H}_0).$$

In the picture already introduced on page 22, it corresponds to the map from f_- to f_+.



$S := W_+^* W_-$

Our aim in this chapter is to provide some properties of S and some explicite formula for it.

It is clear that $S(U,U_0,J,\Theta)\big|_{\mathcal{H}_s(U_0)} = 0$, and that $\mathrm{Ran}\big(S(U,U_0,J,\Theta)\big) \subset E_{ac}^{U_0}(\Theta)\mathcal{H}_0$.

If $W_\pm(U,U_0,J,\Theta)$ are isometries on $E_{ac}^{U_0}(\Theta)\mathcal{H}_0$, then $S(U,U_0,J,\Theta)$ is an <u>isometry on $E_{ac}^{U_0}(\Theta)\mathcal{H}_0$</u> iff $\mathrm{Ran}\big(W_-(U,U_0,J,\Theta)\big) \subset \mathrm{Ran}\big(W_+(U,U_0,J,\Theta)\big)$.

Indeed, for any isometries W, $WW^* = \text{Proj}$ on its range. Thus $\|Sf\|^2 = \langle W_+^* W_- f, W_+^* W_- f\rangle$

$= \langle \underbrace{W_+ W_+^* W_-}_{= W_- \text{ by assumption}} f, W_- f\rangle = \langle W_- f, W_- f\rangle = \|W_- f\|^2$

$= \|f\|^2$ by assumption, $\forall f \in E_{ac}^{U_0}(\Theta)\mathcal{H}_0$.

Also, S is unitary on $E_{ac}^{U_0}(\Theta)\mathcal{H}_0$ if and only if $\text{Ran}(W_-) = \text{Ran}(W_+)$. see p 24

It also follows from the intertwining property that $S(U, U_0, J, \Theta)$ commute with U_0. This property has several consequences, as developed subsequently. Before this, a few abstract elements are necessary.

IV. 1 Spectral representation

For simplicity, we present the theory for U in $\mathcal{H}$, but it will be applied to U_0 in $\mathcal{H}_0$.

Def: A core for U in a Borel set $\hat{U} \subset [0, 2\pi)$ supporting the spectral measure of U, namely $E^U([0,2\pi) \setminus \hat{U}) = 0$, and such that for any other Borel set $\hat{U}'$ supporting the measure of U, the difference $\hat{U} \setminus \hat{U}'$ has Lebesgue measure 0.

By the spectral theorem, $\exists$ for a.e. $\theta \in \hat{U}$ a separable Hilbert space $\mathcal{H}(\theta)$ and a map $\mathcal{F} : \mathcal{H} \to \int_{\hat{U}}^{\oplus} \mathcal{H}(\theta)\, d\theta$ such that $\mathcal{F}|_{\mathcal{H}_s(U)} = 0$ and $\mathcal{F}_{ac} := \mathcal{F}|_{\mathcal{H}_{ac}(U)}$ is unitary and verifies $\mathcal{F}_{ac}\, U|_{\mathcal{H}_{ac}(U)}\, \mathcal{F}_{ac}^* = \int_{\hat{U}}^{\oplus} e^{i\theta}\, d\theta$.

In particular, for any Borel set $\Theta \subset [0, 2\pi)$, and any $f, f' \in \mathcal{H}$,

$$\langle E^U_{ac}(\Theta) f, f' \rangle_{\mathcal{H}} = \int_{\hat\sigma \cap \Theta} d\theta \, \langle [\mathcal{F} f](\theta), [\mathcal{F} f'](\theta) \rangle_{\mathcal{H}(\theta)}$$

We then infer from the Lemma p 47 that

$$\lim_{\varepsilon \searrow 0} \langle \delta((1-\varepsilon)^{\pm 1}, \theta) f, f' \rangle = \pm \langle [\mathcal{F} f](\theta), [\mathcal{F} f'](\theta) \rangle_{\mathcal{H}(\theta)}$$

for a.e. $\theta \in \hat\sigma$. first link between δ and $\mathcal{F}$.

depends on f, f'.

We now present a few standard results:

<u>Lemma</u> Let $a(\theta) \in \mathcal{B}(\mathcal{H}(\theta))$ for a.e. $\theta \in \hat\sigma$, and let

dense

$\mathcal{D} \subset \mathcal{H}$. Suppose that $\forall f, f' \in \mathcal{D}$,

can depend on f, f'

(*) $\langle a(\theta) [\mathcal{F} f](\theta), [\mathcal{F} f'](\theta) \rangle_{\mathcal{H}(\theta)} = 0$ for a.e. $\theta \in \hat\sigma$,

Then $a(\theta) = 0$ for a.e. $\theta \in \hat\sigma$.

independent of any vectors

The proof consists in exhibiting a dense subset $\mathcal{D}'$ of $\mathcal{D}$ and a set $\Lambda \subset \hat\sigma$ s.t. (*) holds $\forall f, g \in \mathcal{D}'$ and $\forall \theta \in \Lambda$, see Yafaev Lemma 1.5.1.

<u>Lemma</u>: Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ be weakly U-smooth on Θ, and set for a.e. $\theta \in [0, 2\pi)$

$$\Delta(\theta, T) := w\text{-}\lim_{\varepsilon \searrow 0} T\, \delta(1-\varepsilon, \theta)\, E^U(\Theta)\, T^* \in \mathcal{B}(\mathcal{G}).$$

Set also $\quad Z(\theta, T)g := [\mathcal{F} T^* g](\theta)$.

This def. should be done on a countable and dense set of vectors

Then on a set of full measure in $\hat{\Theta}$, one has

$Z(\theta, T) \in \mathcal{B}(\mathcal{G}, \mathcal{H}(\theta))$ and

$$(*) \quad Z(\theta, T)^* Z(\theta, T) = \Delta(\theta, T).$$

second link between δ and $\mathcal{F}$, with U-smooth operators.

<u>Proof</u>: Let $\Lambda \subset \hat{\Theta}$ be the set of full measure on which $\Delta(\theta, T)$ exists. Then, for $\theta \in \Lambda$ and $g, g' \in \mathcal{G}$ one has $\quad \langle \Delta(\theta, T) g, g' \rangle_{\mathcal{G}} =$

$$= \lim_{\varepsilon \searrow 0} \langle \delta(1-\varepsilon, \theta) T^* g, T^* g' \rangle_{\mathcal{H}}$$

$$= \langle [\mathcal{F} T^* g](\theta), [\mathcal{F} T^* g'](\theta) \rangle_{\mathcal{H}(\theta)}$$

$$= \langle Z(\theta, T) g, Z(\theta, T) g' \rangle_{\mathcal{H}(\theta)}$$

$$= \langle Z(\theta, T)^* Z(\theta, T) g, g' \rangle_{\mathcal{G}},$$

Since $\Delta(\theta, T) \in \mathcal{B}(\mathfrak{g})$ and g, g' are arbitrary, one infer that $Z(\theta, T) \in \mathcal{B}(\mathfrak{g}, \mathcal{H}(\theta))$ and the equality ⊛. $\square$

<u>Lemma</u> If T is weakly U-smooth on Θ, then for any $f \in \mathcal{H}$, the weak derivative satisfies:

$$\frac{d}{d\theta}\left(T E^U([0,\theta)) f\right) = Z(\theta, T)^*[\mathcal{F} f](\theta).$$

<u>Proof</u>: From

$$\langle E^U_{ac}([0,\theta)) f, T^* g\rangle_{\mathcal{H}} = \int_0^\theta \langle [\mathcal{F} f](\theta'), \underbrace{[\mathcal{F} T^* g](\theta')}_{=[Z(\theta', T) g](\theta')}\rangle_{\mathcal{H}(\theta')} d\theta'$$

$$= \int_0^\theta \langle Z(\theta', T)^*[\mathcal{F} f](\theta'), g\rangle_{\mathfrak{g}} \, d\theta'.$$

By taking the derivative (which is equal to the one without ac), one gets

$$\frac{d}{d\theta}\langle T E^U([0,\theta)) f, g\rangle_{\mathfrak{g}} = \frac{d}{d\theta}\langle E^U([0,\theta)) f, T^* g\rangle_{\mathcal{H}}$$

$$= \frac{d}{d\theta}\langle E^U_{ac}([0,\theta)) f, T^* g\rangle_{\mathcal{H}} = \langle Z(\theta, T)^*[\mathcal{F} f](\theta), g\rangle_{\mathfrak{g}}$$

$\Rightarrow$ the statement, since g is arbitrary. $\square$

Let us now consider the operator $T_1^* A T_2$ with T_1, T_2 weakly U-smooth on Θ and $A \in \mathcal{B}(\mathcal{G})$. Then $T_1^* A T_2 \in \mathcal{B}(\mathcal{H})$ and one has

$$\langle E_{ac}^U(\Theta)\, T_1^* A T_2\, E_{ac}^U(\Theta) f, f' \rangle_{\mathcal{H}}$$

$$= \int_{\Theta \cap \hat{\sigma}} \langle [\mathcal{F} T_1^* (A T_2 E_{ac}^U(\Theta) f)](\mu), [\mathcal{F} f'](\mu) \rangle_{\mathcal{H}(\mu)}\, d\mu$$

$$= \int_{\Theta \cap \hat{\sigma}} d\mu\, \langle A T_2 E_{ac}^U(\Theta) f, Z(\mu, T_1)^* [\mathcal{F} f'](\mu) \rangle_{\mathcal{G}}$$

$$= \int_{\Theta \cap \hat{\sigma}} d\mu\, \langle E_{ac}^U(\Theta) f, T_2^* A^* Z(\mu, T_1)^* \hat{f}'(\mu) \rangle_{\mathcal{G}}$$

$$= \int_{\Theta \cap \hat{\sigma}} d\mu \int_{\Theta \cap \hat{\sigma}} d\nu\, \langle \hat{f}(\nu), [\mathcal{F} T_2^* A^* Z(\mu, T_1)^* \hat{f}'(\mu)](\nu) \rangle_{\mathcal{H}(\nu)}$$

$$= \int_{\Theta \cap \hat{\sigma}} d\mu \int_{\Theta \cap \hat{\sigma}} d\nu\, \langle Z(\mu, T_1)\, A\, Z(\nu, T_2)^* \hat{f}(\nu), \hat{f}'(\mu) \rangle_{\mathcal{H}(\mu)}$$

Thus $E_{ac}^U(\Theta)\, T_1^* A T_2\, E_{ac}^U(\Theta)$ is an integral operator kernel in $\int_{\hat{\sigma}}^{\oplus} \mathcal{H}(\theta)\, d\theta$, with kernel $\underbrace{(\mu, \nu)}_{\in (\hat{\sigma} \cap \Theta)^2} \to$

$$a(\mu, \nu) := Z(\mu, T_1)\, A\, Z(\nu, T_2)^* : \mathcal{H}(\nu) \to \mathcal{H}(\mu).$$

More precisely:

$$\langle E_{ac}^{U}(\Theta)\, T_1^* A T_2 E_{ac}^{U}(\Theta) f, f' \rangle_{\mathcal{H}} =$$

$$= \int_{\Theta\cap\hat{\sigma}} d\mu \int_{\Theta\cap\hat{\sigma}} d\nu \ \langle Z(\mu, T_1) A\, Z(\nu, T_2)^* [\mathcal{F}f](\nu), [\mathcal{F}f'](\mu) \rangle_{\mathcal{H}(\mu)}.$$

Remark: On a set of full measure on

$(\hat{\sigma}\cap\Theta)\times(\hat{\sigma}\cap\Theta)$ one has (by the previous lemma)

$$\langle a(\mu,\nu)\hat{f}(\nu), \hat{f}'(\mu)\rangle_{\mathcal{H}(\mu)} =$$

$$= \langle A \tfrac{d}{d\nu}(T_2 E^U([0,\nu))f), \tfrac{d}{d\mu}(T_1 E^U([0,\mu))f') \rangle_{\mathcal{G}}$$

⚠ We can not consider this expression with 2 simultaneous weak limits, but as iterated limits, namely:

see lemmas p 93 and 47.

$$\langle a(\mu,\nu)\hat{f}(\nu), \hat{f}'(\mu)\rangle_{\mathcal{H}(\mu)} =$$

$$= \lim_{\eta\searrow 0} \lim_{\varepsilon\searrow 0} \langle A T_2\, \delta((1-\varepsilon), \nu) f, T_1\, \delta(1-\eta, \mu) f' \rangle_{\mathcal{G}}.$$

↑ can be interchanged but not considered simultaneously.

In the next statement, we improve the previous observation and generalize it.

<u>Proposition</u>: Let $\{A(\tau)\}_{\tau\in(0,1)} \subset \mathcal{B}(\mathcal{G})$ with $w\text{-}\lim_{\tau\searrow 0} A(\tau) = A \in \mathcal{B}(\mathcal{G})$. Assume that $T_1, T_2 : \mathcal{H} \to \mathcal{G}$ are weakly U-smooth on Θ, and assume that one of the following strong limit exists for a.e. $\theta \in \Theta$:

$$s\text{-}\lim_{\varepsilon\searrow 0} T_2\,\delta(1-\varepsilon,\theta)f \quad , \quad s\text{-}\lim_{\eta\searrow 0} T_1\,\delta(1-\eta,\theta)f' \,. \circledast$$

Then

$$\lim_{\eta\searrow 0,\,\varepsilon\searrow 0} \langle A T_2\,\delta(1-\varepsilon,\nu)f, T_1\,\delta(1-\eta,\mu)f'\rangle_{\mathcal{G}} =$$

$$= \langle \underbrace{Z(\mu,T_1)\,A\,Z(\nu,T_2)^*}_{\equiv\, a(\mu,\nu)}\hat{f}(\nu), \hat{f}(\mu)\rangle_{\mathcal{H}(\mu)}$$

If both limits $\circledast$ exist, then

$$\lim_{\eta\searrow 0,\,\varepsilon\searrow 0,\,\tau\searrow 0} \langle A(\tau) T_2\,\delta(1-\varepsilon,\nu)f, T_1\,\delta(1-\eta,\mu)f'\rangle_{\mathcal{G}} =$$

$$= \langle a(\mu,\nu)\hat{f}(\nu), \hat{f}'(\mu)\rangle_{\mathcal{H}(\mu)} \,.$$

IV. 2 Scattering matrix

Recall that $S(U, U_0, J, \Theta) = W_+(U, U_0, J, \Theta)^* W_-(U, U_0, J, \Theta)$
$\overset{\shortparallel}{S}$ coming from time dependent approach

and that S is reduced by $E^{U_0}_{ac}(\Theta)\mathcal{H}_0$. Since $U_0|_{\mathcal{H}_{ac}(U_0)}$

is scalar in $\int_{\hat{\sigma}}^{\oplus} \mathcal{H}_0(\theta)\, d\theta$, the operator S

is decomposable in this representation, namely

$$\mathcal{F}_{ac}\, S\, E^{U_0}_{ac}(\Theta)\, \mathcal{F}_{ac}^* \overset{*}{=} \int_{\hat{\sigma}\cap\Theta}^{\oplus} S(\theta)\, d\theta$$ and 0 on $\hat{\sigma}\setminus\Theta$

with $S(\theta)$ acting on $\mathcal{H}_0(\theta)$. Clearly, $S(\theta)$

is an isometry on $\mathcal{H}_0(\theta)$ for a.e. θ iff $W_\pm$ are isometries and

$\operatorname{Ran}(W_-) \subset \operatorname{Ran}(W_+)$, and $S(\theta)$ is unitary

on $\mathcal{H}_0(\theta)$ for a.e. θ iff $W_\pm$ are isometries and $\operatorname{Ran}(W_-) = \operatorname{Ran}(W_+)$

* We refer to [Birman, Solomjak Thm 7.2.3 p 166].

coming from time independent approach

Similarly, $s(U, U_0, J, \Theta) := \omega_-^*(U, U_0, J, \Theta)\, \omega_+(U, U_0, J, \Theta)$

and $s = \int_{\hat{\sigma}\cap\Theta}^{\oplus} s(\theta)\, d\theta$, the decomposition of $s\, E^{U_0}_{ac}(\Theta)$.

Note that $\omega_\pm(U_0, U_0, \mathfrak{I}^*\mathfrak{I}, \Theta)$ is also reduced by $E_{ac}^{U_0}(\Theta)\mathcal{H}$, and thus is decomposable, with

$$\mathcal{F}_{ac}\, \omega_\pm(U_0, U_0, \mathfrak{I}^*\mathfrak{I}, \Theta)E_{ac}^{U_0}(\Theta) = \int^{\oplus}_{\hat{\sigma}\cap\Theta} u_\pm(\theta)\, d\theta .$$

Remark that if $\omega_\pm(U_0, U_0, \mathfrak{I}^*\mathfrak{I}, \Theta) = E_{ac}^{U_0}(\Theta)$, then $u_\pm(\theta) = 1\!\!1_{\mathcal{H}(\theta)}$, for a.e. θ. Our final aim is to get some representation formulas for $S(\Theta)$ or $s(\theta)$.

<u>Remark</u>: Even if $\omega_\pm$ have been defined with the wrong convention, we keep the original meaning for S, namely $S = W_+^* W_-$ and $s := \omega_-^* \omega_+$. In the good situation these 2 expressions coincide.

<u>Proposition</u> Let $\Theta \subset [0,2\pi)$ Borel, let $\mathcal{D}_0 \subset E^{U_0}(\Theta)\mathcal{H}_0$ be a dense set, and assume that $\forall f_0 \in \mathcal{D}_0$

$\text{s-}\lim_{\varepsilon \searrow 0} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0$ exist for a.e. $\theta \in [0,2\pi)$,

Assume also that T is weakly U-smooth on Θ.

1) Then, $\forall f_0, f_0' \in \mathcal{D}_0$,

$$\langle E^U(\Theta)\, \omega_+(U,U_0,J,\Theta) f_0,\ \omega_-(U,U_0,J,\Theta) f_0' \rangle_{\mathcal{H}}$$

$$= \langle E^{U_0}(\Theta)\, \omega_\pm(U_0,U_0,J^*J,\Theta) f_0,\ f_0' \rangle_{\mathcal{H}_0}$$

$$+\ 2\pi \int_\Theta d\theta \lim_{\varepsilon \searrow 0} \langle T_\pm((1-\varepsilon)e^{i\theta})\, \delta_0(1-\varepsilon,\theta) f_0,\ \delta_0((1-\varepsilon),\theta) f_0' \rangle_{\mathcal{H}_0}$$

with $T_-(z) = U_0^* J^* V - V^* R(\bar{z}^{-1}) V \in \mathcal{B}(\mathcal{H}_0)$

$V = J U_0 - U_0 J = T^* T_0$ $\quad\Big\{\ T_-(z) = T_+(\bar{z}^{-1})^*$

$T_+(z) = V^* J U_0 - V^* R(z)^* V$

2) If $\text{w-}\lim_{\varepsilon \searrow 0} T R((1-\varepsilon)^{\pm 1} e^{i\theta}) T^*$ exist for a.e. $\theta \in \Theta$,

then $\langle (S(\theta) - U_\mp(\theta))\, \hat{f}_0(\theta),\ \hat{f}_0'(\theta) \rangle_{\mathcal{H}(\theta)} =$

$$= 2\pi \lim_{\varepsilon \searrow 0} \langle E_{ac}^{U_0}(\Theta)\, T_\pm((1-\varepsilon)e^{i\theta})\, E_{ac}^{U_0}(\Theta)\, \delta_0(1-\varepsilon,\theta) f_0,\ \delta_0(1-\varepsilon,\theta) f_0' \rangle_{\mathcal{H}_0}$$

for a.e. $\theta \in \hat{\sigma} \cap \Theta$.

Remark: The first set of assumptions was used on page 75 to show that $W_\pm = \omega_\mp$ if

$\tilde{}$ ← weak time dependent

$\tilde W_\pm(U_0, U_0, J^*J, \Theta)$ exist. The additional assumption was used on p. 80 to remove the additional condition. Note that the following relations hold: ⚠ signs!

$$\omega_\pm(U, U_0, J, \Theta)^* \, \omega_\pm(U, U_0, J, \Theta) = \omega_\pm(U_0, U_0, J^*J, \Theta) \quad ①$$

$$W_\pm(U, U_0, J, \Theta)^* \, W_\pm(U, U_0, J, \Theta) = W_\pm(U_0, U_0, J^*J, \Theta).$$

Proof: 1) The assumptions of 1) are sufficient for the existence of $\omega_\pm(U, U_0, J, \Theta)$ and $\omega_\pm(U_0, U_0, J^*J, \Theta)$ + equality ①, see Thm p 66. Then,

$$\langle E^U(\Theta)\, \omega_+(U, U_0, J, \Theta) f_0 , \omega_-(U, U_0, J, \Theta) f_0' \rangle_{\mathcal{H}}$$

$$= -\int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_-(\varepsilon) \langle R((1-\varepsilon)^{-1} e^{i\theta})\, \omega_+(U, U_0, J, \Theta) f_0 ,$$

$$J R_0((1-\varepsilon)^{-1} e^{i\theta}) f_0' \rangle_{\mathcal{H}}$$

$$\left(\mathfrak{J} R_0(z) = R(z)\, \mathfrak{J} - z R(z)\, U^* V U_0^* R_0(z) \quad \text{resolvent equation} \right)$$

$$= -\int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_-(\varepsilon) \langle R((1-\varepsilon)^{-1} e^{i\theta})\, \omega_+(U, U_0, \mathfrak{J}, \Theta) f_0 ,$$

$$R((1-\varepsilon)^{-1} e^{i\theta})\, \mathfrak{J} f_0' \rangle_{\mathcal{H}}$$

$$+ \int_\Theta d\theta \lim_{\varepsilon \searrow 0} g_-(\varepsilon) \langle R((1-\varepsilon)^{-1} e^{i\theta})\, \omega_+(U, U_0, \mathfrak{J}, \Theta) f_0 ,$$

$$(1-\varepsilon)^{-1} e^{i\theta} R((1-\varepsilon)^{-1} e^{i\theta})\, U^* V U_0^* R_0((1-\varepsilon)^{-1} e^{i\theta}) f_0' \rangle_{\mathcal{H}}$$

compensate

$$= -\int_\Theta d\theta \lim_{\varepsilon \searrow 0} \langle \delta((1-\varepsilon)^{-1}, \theta)\, \omega_+(U, U_0, \mathfrak{J}, \Theta) f_0 , \mathfrak{J} f_0' \rangle_{\mathcal{H}}$$

$$- \int_\Theta d\theta \lim_{\varepsilon \searrow 0} \langle (1-\varepsilon)^{-1} e^{-i\theta} R_0((1-\varepsilon)^{-1} e^{i\theta})^* U_0 \underbrace{V^*}_{T_0^* T} U\, \delta((1-\varepsilon)^{+1}, \theta) \cdot$$

$$\cdot\, \omega_+(U, U_0, \mathfrak{J}, \Theta) f_0 , f_0' \rangle_{\mathcal{H}_0}$$

where T is weakly U-smooth on Θ

integrate 1st term

$$= + \int_\Theta \lim_{\varepsilon \searrow 0} g_+(\varepsilon) \langle \mathfrak{J} R_0((1-\varepsilon) e^{i\theta}) f_0 , R((1-\varepsilon) e^{i\theta}) \mathfrak{J} f_0' \rangle_{\mathcal{H}} \quad (*)$$

see p. 66

$$- \int_\Theta d\theta \lim_{\varepsilon \searrow 0} \langle (1-\varepsilon)^{-1} e^{-i\theta} R_0((1-\varepsilon)^{-1} e^{i\theta})^* U_0 T_0^*$$

$$g_+(\varepsilon)\, T R((1-\varepsilon) e^{i\theta})^* U \mathfrak{J} R_0((1-\varepsilon) e^{i\theta}) f_0 , f_0' \rangle_{\mathcal{H}_0} . \quad (**)$$

For the first term, use $R(z)\mathfrak{J} = \mathfrak{J} R_0(z) + z R(z) U^* V U_0^* R_0(z)$

$$(*) = \int_\Theta \lim_{\varepsilon \searrow 0} g_+(\varepsilon) \langle \mathfrak{J}^* \mathfrak{J} R_0((1-\varepsilon) e^{i\theta}) f_0 , R_0((1-\varepsilon) e^{i\theta}) f_0' \rangle_{\mathcal{H}_0}$$

$$+ \int_\Theta \lim_{\varepsilon \searrow 0} g_+(\varepsilon) \langle (1-\varepsilon) e^{-i\theta} U_0 V^* U R((1-\varepsilon) e^{i\theta})^* \mathfrak{J} R_0((1-\varepsilon) e^{i\theta}) f_0 ,$$

$$R_0((1-\varepsilon) e^{i\theta}) f_0' \rangle_{\mathcal{H}_0}$$

Thus, we have obtained that

see p 66

$$\langle E^{U}(\Theta)\, \omega_+(U,U_0,\mathcal{J},\Theta) f_0 ,\ \omega_-(U,U_0,\mathcal{J},\Theta) f_0' \rangle_{\mathcal{H}_0}$$

$$= + \langle E^{U_0}(\Theta)\ \omega_+(U_0,U_0,\mathcal{J}^*\mathcal{J},\Theta) f_0 ,\ f_0' \rangle_{\mathcal{H}_0}$$

$$- \int_\Theta d\theta \lim_{\varepsilon\searrow 0} g_+(\varepsilon) \langle (1-\varepsilon)^{-1} e^{-i\theta} U_0 V^* U R((1-\varepsilon)e^{i\theta})^* \mathcal{J} R_0((1-\varepsilon)e^{i\theta}) f_0 ,$$
$$R_0((1-\varepsilon)^{-1} e^{i\theta}) f_0' \rangle_{\mathcal{H}_0}$$

$$+ \int_\Theta d\theta \lim_{\varepsilon\searrow 0} g_+(\varepsilon) \langle (1-\varepsilon) e^{-i\theta} U_0 V^* U R((1-\varepsilon)e^{i\theta})^* \mathcal{J} R_0((1-\varepsilon)e^{i\theta}) f_0 ,$$
$$R_0((1-\varepsilon) e^{i\theta}) f_0' \rangle_{\mathcal{H}_0}$$

$$= + \int_\Theta d\theta \lim_{\varepsilon\searrow 0} g_+(\varepsilon) \langle V^* U R((1-\varepsilon)e^{i\theta})^* \mathcal{J} R_0((1-\varepsilon)e^{i\theta}) f_0 ,$$
$$e^{i\theta} U_0^* [(1-\varepsilon) R_0((1-\varepsilon)e^{i\theta}) - (1-\varepsilon)^{-1} R_0((1-\varepsilon)^{-1} e^{i\theta})] f_0' \rangle$$

$$(\ldots) = e^{i\theta} U_0^* (1-\varepsilon)(1-(1-\varepsilon)e^{i\theta}U_0^*)^{-1} - e^{i\theta} U_0^* (1-\varepsilon)^{-1} (1-(1-\varepsilon)^{-1} e^{i\theta} U_0^*)^{-1}$$
$$= ((1-\varepsilon)^{-1} e^{-i\theta} U_0 - 1)^{-1} - ((1-\varepsilon) e^{-i\theta} U_0 - 1)^{-1}$$
$$= (1-(1-\varepsilon) e^{-i\theta} U_0)^{-1} - (1-(1-\varepsilon)^{-1} e^{-i\theta} U_0)^{-1}$$
$$= \sum_{n\geq 0} ((1-\varepsilon) e^{-i\theta} U_0)^n + \sum_{n\geq 1} ((1-\varepsilon) e^{i\theta} U_0^*)^n$$
$$= \sum_{n\leq 0} (1-\varepsilon)^{|n|} e^{i\theta n} (U_0^*)^n + \sum_{n\geq 1} ((1-\varepsilon) e^{i\theta} U_0^*)^n$$

see p 35

$$= 2\pi\ \delta_0(1-\varepsilon, \theta).$$

$$= + \int_\Theta d\theta \lim_{\varepsilon\searrow 0} 2\pi g_+(\varepsilon) \langle V^* U R((1-\varepsilon)e^{i\theta})^* \mathcal{J} R_0((1-\varepsilon)e^{i\theta}) f_0 ,\ \delta_0(1-\varepsilon,\theta) f_0' \rangle_{\mathcal{H}_0}$$

Observe also that

$$V^* U R((1-\varepsilon)e^{i\theta})^* J \left[g_+(\varepsilon) R_0((1-\varepsilon)e^{i\theta}) \right]$$

$$= \delta_0(1-\varepsilon,\theta)\,(1-(1-\varepsilon)e^{i\theta}U_0^*)^*$$

$$= (1-(1-\varepsilon)e^{-i\theta}U_0)\,\delta_0(1-\varepsilon,\theta)$$

$$= V^* \underbrace{U R((1-\varepsilon)e^{i\theta})^*}_{=-e^{i\theta}(1-\varepsilon)^{-1}R((1-\varepsilon)^{-1}e^{i\theta})} J\,(1-(1-\varepsilon)e^{-i\theta}U_0)\,\delta_0(1-\varepsilon,\theta)$$

see p 34

$$= -V^*(1-\varepsilon)^{-1}e^{i\theta} R((1-\varepsilon)^{-1}e^{i\theta}) J\,(1-(1-\varepsilon)e^{-i\theta}U_0)\,\delta_0(1-\varepsilon,\theta)$$

$$= -V^*\Big[J R_0((1-\varepsilon)^{-1}e^{i\theta}) + \underbrace{(1-\varepsilon)^{-1}e^{i\theta} R((1-\varepsilon)^{-1}e^{i\theta})\,U^*}_{=-R((1-\varepsilon)e^{i\theta})^*} V U_0^* R_0((1-\varepsilon)^{-1}e^{i\theta}) \Big]$$

$$\cdot \left[(1-\varepsilon)^{-1}e^{i\theta} - U_0\right] \delta_0(1-\varepsilon,\theta)$$

$$= -\left[-V^* J U_0 + V^* R((1-\varepsilon)e^{i\theta})^* V \right] \delta_0(1-\varepsilon,\theta)$$

$$= +\left[V^* J U_0 - V^* R((1-\varepsilon)e^{i\theta})^* V \right] \delta_0(1-\varepsilon,\theta)$$

$$= +T_+((1-\varepsilon)e^{i\theta})\,\delta_0(1-\varepsilon,\theta).$$

Thus $\langle E^U(\Theta)\, \omega_+(U,U_0,J,\Theta) f_0,\ \omega_-(U,U_0,J,\Theta) f_0' \rangle$

$$= +2\pi \int_\Theta d\theta \lim_{\varepsilon \searrow 0} \langle T_+((1-\varepsilon)e^{i\theta})\,\delta_0(1-\varepsilon,\theta) f_0,\ \delta_0(1-\varepsilon,\theta) f_0' \rangle_{\mathcal{H}_0}$$

$$+ \langle E^{U_0}(\Theta)\, \omega_+(U_0,U_0,J^*J,\Theta) f_0,\ f_0' \rangle_{\mathcal{H}_0}.$$

Similarly :

$$\langle E^U(\Theta)\,\omega_+(U,U_0,\mathcal{J},\Theta)f_0,\ \omega_-(U,U_0,\mathcal{J},\Theta)f_0'\rangle_{\mathcal{G}\oplus\mathcal{H}}$$

$$= \int_\Theta d\theta \lim_{\varepsilon\searrow 0} g_+(\varepsilon)\langle \mathcal{J}R_0((1-\varepsilon)e^{i\theta})f_0,\ R((1-\varepsilon)e^{i\theta})\omega_-(U,U_0,\mathcal{J},\Theta)f_0'\rangle_{\mathcal{G}\oplus\mathcal{H}}$$

$$= \int_\Theta d\theta \lim_{\varepsilon\searrow 0} g_+(\varepsilon)\langle R((1-\varepsilon)e^{i\theta})\mathcal{J}f_0,\ R((1-\varepsilon)e^{i\theta})\omega_-(U,U_0,\mathcal{J},\Theta)f_0'\rangle_{\mathcal{G}\oplus\mathcal{H}}$$

$$-\int_\Theta d\theta \lim_{\varepsilon\searrow 0} g_+(\varepsilon)\langle (1-\varepsilon)e^{i\theta} R((1-\varepsilon)e^{i\theta})U^* V U_0^* R_0((1-\varepsilon)e^{i\theta})f_0,$$

$$R((1-\varepsilon)e^{i\theta})\omega_-(U,U_0,\mathcal{J},\Theta)f_0'\rangle_{\mathcal{G}\oplus\mathcal{H}}$$

$$= \int_\Theta d\theta \lim_{\varepsilon\searrow 0}\langle \delta(1-\varepsilon,\theta)\mathcal{J}f_0,\ \omega_-(U,U_0,\mathcal{J},\Theta)f_0'\rangle_{\mathcal{G}\oplus\mathcal{H}}$$

$$-\int_\Theta d\theta \lim_{\varepsilon\searrow 0}\langle (1-\varepsilon)e^{i\theta} T_0 U_0^* R_0((1-\varepsilon)e^{i\theta})f_0,$$

$$T\delta(1-\varepsilon,\theta)\,\omega_-(U,U_0,\mathcal{J},\Theta)U_0 f_0'\rangle_{\mathcal{G}}$$

p66

$$= -\int_\Theta d\theta \lim_{\varepsilon\searrow 0} g_-(\varepsilon)\langle R((1-\varepsilon)^{-1}e^{i\theta})\mathcal{J}f_0,\ \mathcal{J}R_0((1-\varepsilon)^{-1}e^{i\theta})f_0'\rangle_{\mathcal{G}\oplus\mathcal{H}}$$

$$-\int_\Theta d\theta \lim_{\varepsilon\searrow 0}\langle T_0 R_0((1-\varepsilon)^{-1}e^{i\theta})^* f_0,$$

$$g_-(\varepsilon)\,T R((1-\varepsilon)^{-1}e^{i\theta})^* U\mathcal{J}R_0((1-\varepsilon)^{-1}e^{i\theta})f_0'\rangle_{\mathcal{G}}$$

$$= -\int_\Theta d\theta \lim_{\varepsilon\searrow 0} g_-(\varepsilon)\langle \mathcal{J}R_0((1-\varepsilon)^{-1}e^{i\theta})f_0,\ \mathcal{J}R_0((1-\varepsilon)^{-1}e^{i\theta})f_0'\rangle_{\mathcal{G}\oplus\mathcal{H}}$$

$$- \int_\theta d\theta \lim_{\varepsilon \searrow 0} g_-(\varepsilon) \langle (1-\varepsilon)^{-1} e^{i\theta} R((1-\varepsilon)^{-1} e^{i\theta}) U^* V U_0^* R_0((1-\varepsilon)^{-1} e^{i\theta}) f_0,$$

$$J R_0((1-\varepsilon)^{-1} e^{i\theta}) f_0' \rangle$$

$$- \int_\theta d\theta \lim_{\varepsilon \searrow 0} g_-(\varepsilon) \langle T_0 R_0((1-\varepsilon)^{-1} e^{i\theta})^* f_0,$$

$$T R((1-\varepsilon)^{-1} e^{i\theta})^* U J R_0((1-\varepsilon)^{-1} e^{i\theta}) f_0' \rangle_{\mathcal{G}}$$

$$= - \int_\theta d\theta \lim_{\varepsilon \searrow 0} g_-(\varepsilon) \langle [- R_0((1-\varepsilon) e^{i\theta})^* + R_0((1-\varepsilon)^{-1} e^{i\theta})^*] f_0,$$

$$V^* U R((1-\varepsilon)^{-1} e^{i\theta})^* J R_0((1-\varepsilon)^{-1} e^{i\theta}) f_0' \rangle_{\mathcal{H}_0}$$

$$= 2\pi \int_\theta d\theta \lim_{\varepsilon \searrow 0} \langle V \delta_0(1-\varepsilon, \theta) f_0,$$

$$(-(1-\varepsilon) e^{-i\theta}) R((1-\varepsilon) e^{i\theta}) J g_-(\varepsilon) R_0((1-\varepsilon)^{-1} e^{i\theta}) f_0' \rangle_{\mathcal{H}}$$

$$= -2\pi \int_\theta d\theta \lim_{\varepsilon \searrow 0} \langle V \delta_0(1-\varepsilon, \theta) f_0,$$

$$((1-\varepsilon) e^{i\theta}) [J R_0((1-\varepsilon) e^{i\theta}) + (1-\varepsilon) e^{i\theta} R((1-\varepsilon) e^{i\theta}) U^* V U_0^* R_0((1-\varepsilon) e^{i\theta})] \cdot$$

$$(1 - (1-\varepsilon)^{-1} e^{-i\theta} U_0)^* \delta_0(1-\varepsilon, \theta) f_0' \rangle$$

$$= -2\pi \int_\theta d\theta \lim_{\varepsilon \searrow 0} \langle V \delta_0(1-\varepsilon, \theta) f_0,$$

$$[J R_0((1-\varepsilon) e^{i\theta}) - R((1-\varepsilon)^{-1} e^{i\theta})^* V R_0((1-\varepsilon) e^{i\theta}) U_0^*] \cdot$$

$$\cdot ((1-\varepsilon) e^{i\theta} - U_0) \delta_0(1-\varepsilon, \theta) f_0' \rangle$$

$$= -2\pi \int_\theta d\theta \lim_{\varepsilon \searrow 0} \langle V \delta_0(1-\varepsilon,\theta) f_0 ,$$

$$[-\mathfrak{J} U_0 + R((1-\varepsilon)^{-1} e^{i\theta})^* V]\, \delta_0(1-\varepsilon,\theta) f_0' \rangle$$

$$= 2\pi \int_\theta d\theta \lim_{\varepsilon \searrow 0} \langle (U_0^* \mathfrak{J}^* V - V^* R((1-\varepsilon)^{-1} e^{i\theta})\, V) \delta_0(1-\varepsilon,\theta) f_0,$$

$$\delta_0(1-\varepsilon,\theta) f_0' \rangle$$

$$= 2\pi \int_\theta d\theta \lim_{\varepsilon \searrow 0} \langle (U_0^* \mathfrak{J}^* V - V^* R((1-\varepsilon)^{-1} e^{i\theta})\, V) \delta_0(1-\varepsilon,\theta) f_0,\ \delta_0(1-\varepsilon,\theta) f_0' \rangle.$$

$$\Rightarrow \langle E^U(\theta)\, \omega_+(U, U_0, \mathfrak{J}, \theta) f_0,\ \omega_-(U, U_0, \mathfrak{J}, \theta) f_0' \rangle =$$

$$= \langle E^{U_0}(\theta)\, \omega_-(U_0, U_0, \mathfrak{J}^* \mathfrak{J}, \theta) f_0,\ f_0' \rangle$$

$$+ 2\pi \int_\theta d\theta \lim_{\varepsilon \searrow 0} \langle T_-((1-\varepsilon) e^{i\theta})\, \delta_0(1-\varepsilon,\theta) f_0,\ \delta_0(1-\varepsilon,\theta) f_0' \rangle.$$

2) With the stronger assumptions, $W_\pm = \omega_\mp$,

and $S = \omega_-^* \omega_+$ ⚠ always the strange def. of $\omega_\pm$.

The operators S and $\omega_\pm$ vanish on $\mathcal{H}_s(U_0)$, have range in $\mathcal{H}_{ac}(U_0)$, and S commutes with U_0.

From the theory of direct integral one infers that

$$\langle E^U(\Theta)\, \omega_+(U, U_0, J, \Theta) f_0 , \omega_-(U, U_0, J, \Theta) f_0' \rangle_{\mathcal{H}}$$

$$- \langle E^{U_0}(\Theta)\, \omega_\pm(U_0, U_0, J^*J, \Theta) f_0 , f_0' \rangle_{\mathcal{H}_0}$$

$$= \int_{\hat{\sigma}\cap\Theta}^{\oplus} d\theta \,\langle (S(\theta) - U_\pm(\theta)) [\mathcal{F} f_0](\theta), [\mathcal{F} f_0'](\theta) \rangle_{\mathcal{H}(\theta)} .$$

By comparing this expression with the result of 1) and on proposition on p. 96, one gets that

$$\langle (S(\theta) - U_\pm(\theta)) \hat{f}_0(\theta), \hat{f}_0'(\theta) \rangle_{\mathcal{H}(\theta)} =$$

$$= 2\pi \lim_{\varepsilon \searrow 0} \langle \overset{E_{ac}^{U_0}(\Theta)}{\vee} T_\pm((1-\varepsilon)e^{i\theta}) \overset{E_{ac}^{U_0}(\Theta)}{\vee} S_0(1-\varepsilon, \theta) f_0 , S_0(1-\varepsilon, \theta) f_0' \rangle_{\mathcal{H}(\theta)}$$

for a.e. $\theta \in \hat{\sigma} \cap \Theta$.

$\square$

<u>Remark:</u> $T_+(z) = V^* J U_0 - V^* R(z)^* V$

$$= T_0^* \underline{T J U_0} - T_0^* (T R(z)^* T^*) T_0$$

In the last step of the previous proof, we have implicitly used that $T \mathcal{J} U_0$ is weakly U_0 - smooth on Θ. A result of this type has been proved for the Thm p 80, namely p 81-82, when showing that the existence of $\tilde{W}_\pm(U_0, U_0, \mathcal{J}^*\mathcal{J}, \Theta)$ is obtained with the stronger assumption of 2).

We can finally state the main representation formula for $S(\Theta)$. Other formulations exist, but we present only the simplest one.

<u>Thm</u>: Let $\Theta \subset [0, 2\pi)$ Borel set, $D_0 \subset E^{U_0}(\Theta)\mathcal{H}_0$ dense, assume that $\forall f_0 \in D_0$, $s\text{-}\lim_{\varepsilon \searrow 0} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0$ exist for a.e. $\theta \in [0, 2\pi)$. Assume T_0 weakly U_0-smooth on Θ, and $w\text{-}\lim_{\varepsilon \searrow 0} T R((1-\varepsilon)^{\pm 1} e^{i\theta}) T^*$ exist for a.e. $\theta \in \Theta$.

Then, for a.e. $\theta \in \hat{\sigma} \cap \Theta$ one has index 0 because related to U_0

$$S(\theta) = U_+(\theta) + 2\pi \, Z_0(\theta, T\mathfrak{J}U_0)\, Z_0(\theta, T_0)^* - Z_0(\theta, T_0)\, B_+^*(\theta)\, Z_0(\theta, T_0)^*$$

$$= U_-(\theta) + 2\pi \, Z_0(\theta, T_0)\, Z_0(\theta, T\mathfrak{J}U_0)^* - Z_0(\theta, T_0)\, B_-(\theta)\, Z_0(\theta, T_0)^*$$

with $B_\pm(\theta) := w\text{-}\lim_{\varepsilon \searrow 0} B((1-\varepsilon)^{\pm 1} e^{i\theta})$, and

$B(z) = T R(z) T^*$.

<u>Proof</u>: From the previous proposition, it is sufficient to show that (for +)

$$\lim_{\varepsilon \searrow 0} \langle E_{ac}^{U_0}(\theta)\, T_+((1-\varepsilon)e^{i\theta})\, E_{ac}^{U_0}(\theta)\, \delta_0(1-\varepsilon, \theta) f_0, \ \delta_0(1-\varepsilon, \theta) f_0' \rangle_{\mathcal{H}_0}$$

$$= \langle \left(Z_0(\theta, T\mathfrak{J}U_0) Z_0(\theta, T_0)^* - Z_0(\theta, T_0) B_+(\theta)^* Z_0(\theta, T_0)^*\right) \hat{f_0}(\theta), \hat{f_0'}(\theta) \rangle_{\mathcal{H}(\theta)}$$

and similarly for $-$. For that purpose, observe that

$$T_+((1-\varepsilon)e^{i\theta}) = T_0^*(T\mathfrak{J}U_0) - T_0^* T R(z)^* T^* T_0$$

$$= T_0^* \underline{(T\mathfrak{J}U_0)} - T_0^* B(z)^* T_0 .$$

weakly U-smooth on Θ

satisfy $s\text{-}\lim_{\varepsilon \searrow 0} T_0 R_0((1-\varepsilon)^{\pm 1} e^{i\theta}) f_0$ exist

We can thus apply the content of the proposition p 96. □

Conclusion :

Scattering theory for unitary operators deserves to be studied on its own. Many results are similar to the self-adjoint theory, but a self-adjoint generator does not always exist.

We have only presented the main results, additional investigations and some applications are now necessary!

Thank you.