Proceedings of the Research Institute of Atmospherics,
Nagoya University, vol.35(1988)

RESEARCH REPORT

THEORY OF WAVE-TRAPPING IN SPACE PLASMAS

Haruichi Washimi

Abstract

A new method of perturbation for the study of nonlinear wave propagation in a dielectric medium such as a magnetoplasma is proposed. By this method the wave equations which include the effects due to a nonlinearity or an inhomogeneity of the transverse waves propagating along magnetic field lines are rederived and summarized. Among many kinds of wave modulations described by these equations, the condition for wave-trapping in space plasmas is especially discussed in detail.

1. Introduction

Nonlinear wave propagation is one of the central problems in studies of wave phenomena in plasmas. Due to the ponderomotive force, which represents time-averaged and long-lived nonlinear wave effect, the quasi-monochromatic waves are self-modulated. It is known that the reductive perturbation method [Taniuti and Wei, 1969] is an effective means to study nonlinear wave propagation. By this method many kinds of nonlinear wave equations have been derived such as the one-dimensional nonlinear Schrodinger equations [Taniuti and Washimi, 1968; Taniuti and Yajima 1969], the derivative nonlinear Schrodinger equation for Alfven wave [Mio et al., 1976] and the nonlinear Schrodinger equation in two-dimensional space, which describes the wave-focusing and the wave-trapping [Taniuti and Washimi 1969, Washimi 1973]. In the latter method the nonlinear wave equation and the ponderomotive force can be estimated simultaneously. But for this reason, the calculation is very extensive and sometimes physically complicated because all varying components in plasma fluid should be estimated up to the third or fourth order of the perturbation. On the other hand, in the course of the study for the self-focusing and the self-trapping, both the perpendicular and parallel directions of the ponderomotive force in a magnetoplasma were derived by Washimi [1973], while the general expression of the ponderomotive force was derived independently by the thermodynamic approach [Washimi and Karpman, 1976]. Thus, with the help of the general expression of the ponderomotive force, we can now develop a systematic and more comprehensive method of perturbation.

Our starting equation is the following well-known equation

$$\nabla \times \nabla \times \mathbf{E} + (1/c^2)(\partial^2/\partial t^2)\, \mathbf{D} = 0 \tag{1}$$

and

$$\mathbf{D} = \mathbf{K}\, \mathbf{E} \tag{2}$$

where $\mathbf{E}$ is the electric field, $\mathbf{D}$ the displacement vector and $\mathbf{K}$ the dielectric tensor. A basic assumption for our method is the deviation of the dielectric tensor, $\delta\mathbf{K}$, due to a nonlinearity or an inhomogeneity from the linear and homogeneous part, $\mathbf{K}_0$, i.e.,

$$\delta\mathbf{K} = \mathbf{K} - \mathbf{K}_0 \tag{3}$$

which does not include a fast-varying part in time. This assumption is applied for the case of the ponderomotive force. The present study deals with the new formalism, and we rederive the wave equations systematically.

Another purpose of this paper is to discuss the wave-trapping of transverse waves propagating along magnetic field lines due to the nonlinearity of the waves or the inhomogeneity of the plasmas. Recently, the wave-trapping due to nonlinear Alfven waves has been computer-simulated by Hoshino [1987]. Wave-trapping phenomena are expected in space plasmas, so, for the application of trapping phenomena to space plasma, it is worth summarizing the trapping conditions for transverse waves propagating along magnetic field lines.

It is well-known that the ducting propagation of whistler waves in the terrestrial magnetosphere along magnetic field lines is explained by the ray-theory [Smith, 1961; Helliwell, 1965]. According to this theory both for $\omega < \omega_{ce}/2$ in the region of locally enhanced plasma density and for $\omega > \omega_{ce}/2$ in the region of depressed density the whistler waves are trapped. On the other hand, by the reductive perturbation method, the wave-trapping of the whistler waves in an inhomogeneous plasma has been discussed using wave-theory [Washimi, 1976]. In the present study, the physical meaning of the wave-trapping is also reconfirmed in our new formalism.

The wave equations are derived by the dielectric tensor formalism in section 2, and the physical condition of the wave-trapping is discussed in section 3. The nonlinear wave modulation is discussed in section 4 in which the condition for self-trapping is shown. In the last section 5, a summary is given.

2. Method of wave equation derivation

We consider a transverse wave of slab-shape propagation in z direction oriented along an applied magnetic field $\mathbf{B}_0$ and distributed in the x direction. The transverse wave is considered to be modulated in space and time due to weak inhomogeneity or weak nonlinearity, so that we may put

$$E = (1/2)\{\delta E(x,z,t)\exp i(kz-\omega t) + c.c.\} \qquad (4)$$

The linear and homogeneous part of the dielectric tensor, $\mathbf{K}_0$, is given by (Stix, 1962)

$$\mathbf{K}_0 = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \qquad (5)$$

where

$$S = 1 - \{\omega_{pe}^2/(\omega^2-\omega_{ce}^2) + \omega_{pi}^2/(\omega^2-\omega_{ci}^2)\} \qquad (6a)$$

$$D = -\{\omega_{pe}^2/(\omega^2-\omega_{ce}^2)\}(\omega_{ce}/\omega) + \{\omega_{pi}^2/(\omega^2-\omega_{ci}^2)\}(\omega_{ci}/\omega) \qquad (6b)$$

and

$$P = 1 - (\omega_{pe}^2/\omega^2) - (\omega_{pi}^2/\omega^2) \qquad (6c)$$

Here ω_{pj} and ω_{cj} are the angular plasma frequency and the angular cyclotron frequency, respectively.

Now we estimate the term $\partial^2 \mathbf{D}/\partial t^2$ in eq.(1) by referring to the method of estimation of $\partial \mathbf{D}/\partial t$ given by Landau and Lifshitzs [1960]. Let us put

$$\partial \mathbf{D}/\partial t = \hat{f}\ \mathbf{E}\ , \quad \hat{f} = \partial \mathbf{K}/\partial t \qquad (7)$$

Here the symbol $\wedge$ denotes operator. Because $\delta\mathbf{E}$ is slowly modulated in time, the Fourier component of $\delta\mathbf{E}$ is

$$\delta\mathbf{E}\ \exp -i\alpha t \quad (|\alpha| << |\omega|).$$

Accordingly,

$$\hat{f}\ \delta\mathbf{E}\ \exp\{-i(\omega+\alpha)t\} = f(\omega+\alpha)\ \delta\mathbf{E}\ \exp\{-i(\omega+\alpha)t\} = \{f(\omega)$$

$$+ \alpha\ \partial f(\omega)/\partial\omega + (\alpha^2/2)\ \partial^2 f(\omega)/\partial\omega^2 + ..\}\ \delta\mathbf{E}\ \exp\{-i(\omega+\alpha)t\} \qquad (8)$$

Then we have

$$(\partial/\partial t)[\hat{f}\ \delta\mathbf{E}\ \exp\{-i(\omega+\alpha)t\}] = \{-i\omega f - i\alpha(f+\omega\partial f/\partial\omega)$$

$$- i(\alpha^2/2)\ (2\partial f/\partial\omega + \omega\ \partial^2 f/\partial\omega^2)\}\ \delta\mathbf{E}\ \exp\{-i(\omega+\alpha)t\} \qquad (9)$$

By summing up the Fourier component of α, $\partial^2\mathbf{D}/\partial t^2$ is reduced to

$$\partial^2\mathbf{D}/\partial t^2 = (1/2)\ \{(\partial/\partial t)\hat{f}\ \delta\mathbf{E}\ \exp\ i\ (kz-\omega t) + c.c.\}$$

$$= (1/2)[\{-i\omega f + (f + \omega\partial f/\partial\omega)(\partial/\partial t) + (i/2)(2\partial f/\partial\omega$$

$$+ \omega\partial^2 f/\partial\omega^2)(\partial^2/\partial t^2) - 4i\omega\ \partial(\delta\mathbf{K})/\partial t\ \}$$

$$\delta\mathbf{E}\ \exp\ i(kz - \omega t) + c.c.]$$

$$= -(\omega^2/2)[\{\mathbf{K} + i\,(1/\omega^2)\partial(\omega^2\mathbf{K})/\partial\omega\,(\partial/\partial t) - (1/2\omega^3)(\partial/\partial\omega)$$

$$\{\omega^2\partial(\omega\mathbf{K})/\partial\omega\}(\partial^2/\partial t^2) + (4i/\omega)\,\partial(\delta\mathbf{K})/\partial t\,\}$$

$$\delta E \exp i(kz - \omega t) + c.c.] \tag{10}$$

Thus, the tensor form expression of the starting equation is reduced to

$$[\varepsilon(\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2) - \{\mathbf{K} + (i/\omega^2)\partial(\omega^2\mathbf{K})/\partial\omega(\partial/\partial t)$$

$$-(1/2\omega^3)(\partial/\partial\omega)\{\omega^2\partial(\omega\mathbf{K})/\partial\omega\}(\partial^2/\partial t^2) + (4i/\omega)\partial(\delta\mathbf{K})/\partial t\}]\ \delta\mathbf{E} = 0 \tag{11}$$

where ε is the dielectric constant,

$$\varepsilon = k^2c^2 / \omega^2 \tag{12}$$

and the tensors $\mathbf{A}_0$, $\mathbf{A}_1$ and $\mathbf{A}_2$ are

$$\mathbf{A}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

$$\mathbf{A}_1 = \begin{pmatrix} -(2i/k)\partial/\partial z & 0 & (i/k)\partial/\partial x \\ 0 & -(2i/k)\partial/\partial z & 0 \\ (i/k)\partial/\partial x & 0 & 0 \end{pmatrix} \tag{14}$$

and

$$\mathbf{A}_2 = \begin{pmatrix} -(1/k^2)\partial^2/\partial z^2 & 0 & (1/k^2)\partial^2/\partial x\partial z \\ 0 & -(1/k^2)(\partial^2/\partial x^2+\partial^2/\partial z^2) & 0 \\ (1/k^2)\partial^2/\partial x\partial z & 0 & -(1/k^2)\partial^2/\partial x^2 \end{pmatrix} \tag{15}$$

respectively. The lowest order of eq.(11) leads to

$$\mathbf{W}\ \delta\mathbf{E}^{(1)} = 0, \quad \mathbf{W} = \varepsilon\ \mathbf{A}_0 - \mathbf{K}_0 \tag{16}$$

where $\delta\mathbf{E}^{(1)}$ is the lowest order of $\delta\mathbf{E}$, i.e.,

$$\delta\mathbf{E} = \delta\mathbf{E}^{(1)} + \delta\mathbf{E}^{(2)} + \delta\mathbf{E}^{(3)} + \ldots \tag{17}$$

From the condition det($\mathbf{W}$) = 0, we have the dispersion relation

$$\textbf{(I)}\quad \varepsilon = \varepsilon_0 = S + D = 1 - \omega_{pe}{}^2/\omega(\omega-\omega_{ce}) - \omega_{pi}{}^2/\omega(\omega+\omega_{ci}) \tag{18a}$$

$$\textbf{(II)}\quad \varepsilon = \varepsilon_0 = S - D = 1 - \omega_{pe}{}^2/\omega(\omega+\omega_{ce}) - \omega_{pi}{}^2/\omega(\omega-\omega_{ci}) \tag{18b}$$

By using Wright eigen vector, $\mathbf{R}$, $\delta\mathbf{E}^{(1)}$ is expressed as follows:

$$\delta\mathbf{E}^{(1)} = \begin{pmatrix} \delta E_x{}^{(1)} \\ \delta E_y{}^{(1)} \\ \delta E_z{}^{(1)} \end{pmatrix} = \mathbf{R}\ \phi(x,z,t)\ , \quad \mathbf{R} = \begin{pmatrix} (-/+)\ i \\ 1 \\ 0 \end{pmatrix} \tag{19}$$

For the next order of eq.(11), we have

$$\mathbf{W}\ \delta\mathbf{E}^{(2)} + [\mathbf{W} + \varepsilon\ \mathbf{A}_1 - (i/\omega^2)\{\partial(\omega^2\mathbf{K}_0)/\partial\omega\}(\partial/\partial t)]\ \delta\mathbf{E}^{(1)} = 0 \tag{20}$$

By operating Left eigen vector

$$\mathbf{L} = (\ (+/-)\ i\ ,\ 1\ ,\ 0) \tag{21}$$

to eq.(20) and by noting that

$$\mathbf{L}\ A_1\ \mathbf{R} = -\ i\ (4/k)\ \partial/\partial z \tag{22}$$

and

$$\mathbf{L}\ \mathbf{K}_0\ \mathbf{R} = 2\varepsilon_0 \tag{23}$$

we have

$$- i(4\varepsilon/k)\ \{(1/\lambda_g)\partial/\partial t + \partial/\partial z\ \}\ \phi(x,z,t) = 0 \qquad (24)$$

where λ_g is the group velocity,

$$\lambda g = \partial\omega/\partial k \qquad (25)$$

Eq.(24) means that ϕ propagates with the group velocity within second-order accuracy. From eq.(20), $\delta E^{(2)}$ is given by

$$\delta E^{(2)} = \begin{pmatrix} \delta E_x{}^{(2)} \\ \delta E_y{}^{(2)} \\ \delta E_z{}^{(2)} \end{pmatrix} = \mathbf{R}\ \phi^{(2)}(x,z,t) + \mathbf{V}\ \{\varepsilon/(kP)\}\ \partial\phi/\partial x \qquad (26)$$

where

$$\mathbf{V} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (27)$$

For the third-order equation of (11), we have

$$\mathbf{W}\ \delta E^{(3)} + \{\mathbf{W} + \varepsilon\ \mathbf{A}_1 - (i/\omega^2)\partial(\omega^2\mathbf{K}_0)/\partial\omega(\partial/\partial t)\}\ \delta E^{(2)} + [\mathbf{W} + \varepsilon\mathbf{A}_1$$

$$+\varepsilon\mathbf{A}_2 - (i/\omega^2)\partial(\omega^2\mathbf{K}_0)/\partial\omega(\partial/\partial t) + (1/2\omega^3)(\partial/\partial\omega)\{\omega^2\partial(\omega\mathbf{K}_0)/\partial\omega\}(\partial^2/\partial t^2)$$

$$- \delta\mathbf{K} - (4i/\omega)\ \partial(\delta\mathbf{K})/\partial t]\ \delta E^{(1)} = 0 \qquad (28)$$

By operating $\mathbf{L}$ to eq.(28) from the left-hand side, and by noting that

$$(1/2\omega^3)(\partial/\partial\omega)\{\omega^2\partial(\omega\varepsilon_0)/\partial\omega\} = (c^2/2\omega^3)(\partial/\partial\omega)[\omega^2\{\partial(k^2/\omega)/\partial\omega\}]$$

$$= (\varepsilon/k^2)\ \{\lambda_g{}^{-2} + k\partial^2 k/\partial\omega^2\} \qquad (29)$$

$$\mathbf{L}\ \mathbf{A}_1\ \delta E^{(2)} = -\ i\ (4/k)\ \partial\phi^{(2)}/\partial z - (\varepsilon/k^2P)\ \partial^2\phi/\partial x^2 \qquad (30)$$

and

$$\mathbf{L}\ \mathbf{A}_2\ \mathbf{R} = -\ (1/k^2)\partial^2/\partial x^2 - (2/k^2)\partial^2/\partial z^2 \tag{31}$$

we obtain

$$\begin{aligned}&(2\varepsilon/k^2)\ \{(1/\lambda_g)^2\ \partial^2\phi/\partial z^2 - \partial^2\phi/\partial z^2\ \}\\&-\ (4\varepsilon/k)[i\{(1/\lambda_g)\ \partial\phi^{(2)}/\partial t + \partial\phi^{(2)}/\partial z\} - (1/2)(\partial^2 k/\partial\omega^2)(\partial^2\phi/\partial t^2)\\&+\ i\ \{(1/\lambda_g)\partial\phi/\partial t + \partial\phi/\partial z\} + p\ \partial^2\phi/\partial x^2 - U\ \phi - (4i/\omega)(\partial U/\partial t)\phi]\\&= 0\end{aligned} \tag{32}$$

Here, by using the asymptotic relations

$$(1/\lambda_g)^2\ \partial^2\phi/\partial z^2 - \partial^2\phi/\partial z^2 \simeq 0 \tag{33}$$

and

$$(1/\lambda_g)\ \partial\phi^{(2)}/\partial t + \partial\phi^{(2)}/\partial z \simeq 0 \tag{34}$$

we obtain a general wave equation

$$\begin{aligned}&-\ (1/2)\ (\partial^2 k/\partial\omega^2)(\partial^2\phi/\partial t^2)\\&+\ i\ \{(1/\lambda_g)\partial\phi/\partial t + \partial\phi/\partial z\ \} + p\ \partial^2\phi/\partial x^2 - U\ \phi - (4i/\omega)(\partial U/\partial t)\ \phi\\&= 0\end{aligned} \tag{35}$$

where the potential U is

$$\begin{aligned}U &= -\ \{\mathbf{L}\ (\delta\mathbf{K})\ \mathbf{R}\}\ /\ (4\varepsilon/k) = -\ (k/2)\ (\delta\varepsilon/\varepsilon)\\&= -\ (k/2\varepsilon)\ \{(\partial\varepsilon/\partial\rho)\ \delta\rho + (\partial\varepsilon/\partial B)\ \delta B\}\end{aligned} \tag{36}$$

Now because

$$(\partial\varepsilon/\partial\rho)\ \delta\rho = (\varepsilon - 1)\ (\delta\rho\ /\ \rho_0) \tag{37}$$

and

$$(\partial \varepsilon / \partial B)\ \delta B = -\ [\varepsilon - 1 + \{\omega_{pe}{}^2/((+/-)\omega_{ce}-\omega)^2\}$$

$$+ \{\omega_{pi}{}^2/((+/-)\omega_{ci}+\omega)^2\}]\ (\delta B\ /\ B_0) \qquad (38)$$

U is expressed by

$$U = -\ (k/2\varepsilon)\ [(\varepsilon - 1)\ (\delta\rho\ /\ \rho_0) - [\varepsilon - 1 + \{\omega_{pe}{}^2/((+/-)\omega_{ce}-\omega)^2\}$$

$$+ \{\omega_{pi}{}^2/((+/-)\omega_{ci}+\omega)^2\}]\ (\delta B\ /\ B_0)] \qquad (39)$$

The coefficient of the diffraction term, p, is

$$p = \nu\ /\ 2k \qquad (40)$$

and

$$\nu = 1 + (1/2)(\varepsilon/P - 1)$$

$$=1 + (1/2)[-\ (\omega_{pe}{}^2/\omega^2)\{\omega_{ce}/(\omega-\omega_{ce})\} + (\omega_{pi}{}^2/\omega^2)\{\omega_{ci}/(\omega+\omega_{ci})\}]$$

$$/\ \{1 - (\omega_{pe}{}^2/\omega^2) - (\omega_{pi}{}^2/\omega^2)\} \qquad (41)$$

Eq.(35) involves all terms in one-dimensional Schrodinger equations, derivative nonlinear Schrodinger equation and the two-dimensional Schrodinger equation.

If a time-stationary state is assumed, i.e.,

$$\partial/\partial t \simeq 0 \qquad (42)$$

The wave-equation (35) reduces to the two-dimensional Schrodinger equation

$$i\ \partial\phi/\partial z + p\ \partial^2\phi/\partial x^2 - U(x,z)\ \phi = 0 \qquad (43)$$

3. Physical condition for wave-trapping

In this section the physical condition for wave-trapping is discussed by comparing the wave-theory and the ray-theory. When the wave normal is inclined from z-axis by θ, the angle between the ray and the wave-normal, α, is given by (e.g., Stix [1962])

$$\alpha = \tan^{-1}\{-(1/n)\ \partial n/\partial\theta\} \tag{44}$$

where $n = kc/\omega$ is the refractive index (Figure 1). When θ is suffily small, α is reduced to

$$\alpha = \tan^{-1}\{-(1/\varepsilon)\ \partial\varepsilon/\partial(\theta^2)\ \theta\} \qquad (\text{for } \theta \ll 1) \tag{45}$$

To discuss the wave-trapping or detrapping, the term of the energy flow across the z-axis, which is shown by the diffraction term in eq.(38), is important.
When the plane wave E_{obl} is

$$E_{obl} = (1/2)[\delta E_{obl} \exp\{i(k \cos\theta\ z + k \sin\theta\ x - \omega\ t)\} + c.c.]$$

$$(\text{for } \theta \ll 1) \tag{46}$$

Putting this into the linearized system of basic equations in homogeneous plasma, we get

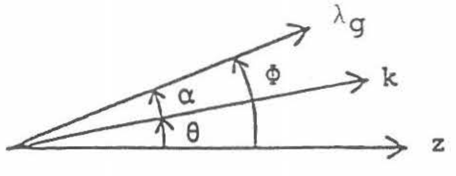

Fig. 1. Directions of wave normal and ray.

$$\begin{pmatrix} (1-\theta^2/2)\varepsilon-S & -iD & i\theta\varepsilon \\ iD & \varepsilon-S & 0 \\ i\theta\varepsilon & 0 & (1-\theta^2/2)P \end{pmatrix} \delta E_{obl} = 0 \tag{47}$$

In view of eqs.(26) and (47) we have

$$\delta E_{obl} = \{\mathbf{R} + \mathbf{V}\ (i\theta)\ (\varepsilon/P)\}\ \phi_{obl} \tag{48}$$

where ϕ_{obl} is a constant and the dispersion relation is

$$\varepsilon = \varepsilon_0\ \{1 - (2kp - 1)\theta^2\} \tag{49}$$

Then p is expressed by

$$p = (1/2k)\ [1 - \{(1/\varepsilon)\ \partial\varepsilon/\partial(\theta^2)\}_{=0}] \tag{50}$$

In view of eqs.(45) and (50), the coefficient of the diffraction, p, is found to be expressed by the ray direction as follows,

$$p = \nu\ /\ (2k), \quad \nu = (\Phi\ /\ \theta)_{=0}, \quad \Phi = \theta + \alpha \tag{51}$$

Therefore the physical condition for wave-trapping is expressed as follows: Because eq.(40) is a Schrodinger-type wave equation, the wave is trapped when

$$p\ U < 0 \tag{52}$$

which means

$$p > 0 \quad \text{and} \quad U < 0 \tag{53a}$$

or

$$p < 0 \quad \text{and} \quad U > 0 \tag{53b}$$

Because the wave-normal is refracted to the region of larger dielectric constant ($\delta\varepsilon > 0$), which corresponds to the region U < 0, (as shown in Figure 2), the wave energy piles up in the same region if the ray is refracted in the same sense as the wave-normal ($p > 0$). This is the case of eq.(53a). On the other hand, if the ray and the wave-normal are refracted in the opposite direction, respectively ($p < 0$), the wave energy accumulates in the region where $\delta\varepsilon < 0$, which corresponds to the region $U > 0$. This is the case of eq.(53b).

For high-frequency electromagnetic waves ($\omega \gg \omega_{ce}$, $\omega \gg \omega_{pe}$), because p is

$$p \simeq 1/2k \tag{54}$$

and U is expressed by

$$U = (k/2)\ (\omega_{pe}^2/\omega^2)\ (\delta\rho/\rho_0) \tag{55}$$

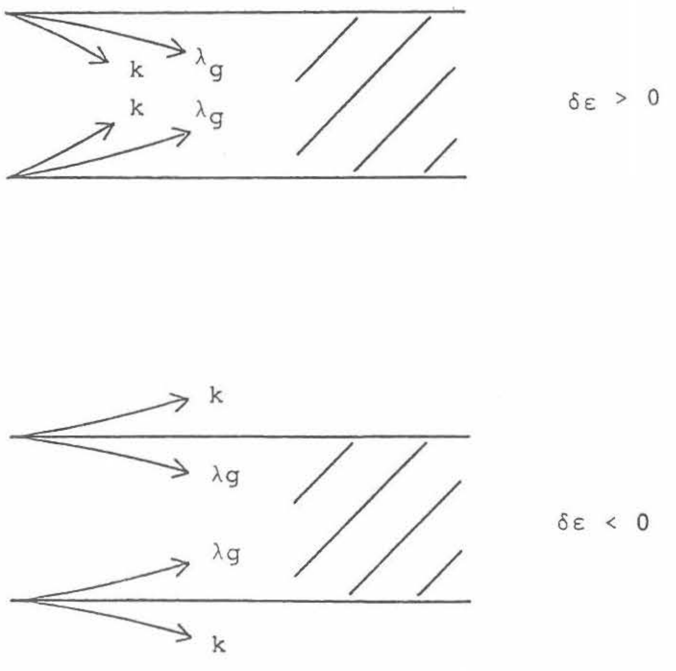

Fig. 2. Conditions of wave-trapping for $\delta\varepsilon > 0$ (a) and $\delta\varepsilon < 0$ (b).

which result in the wave-trapping in the depressed density region,

$$\delta\rho < 0 \qquad (56)$$

For the whistler waves, p is

$$p \simeq [\{(1/2)\omega_{ce} - \omega\}/(\omega_{ce}-\omega)] \ / \ 2k \qquad (57)$$

and U is

$$U \simeq -\ (k/2)\ [(\delta\rho/\rho_0) - \{\omega_{ce}/(\omega_{ce}-\omega)\}(\delta B/B_0)] \qquad (58)$$

Then, with the increase of ω, p changes its sign from positive to negative at about $\omega \simeq \omega_{ce}/2$, and the wave trapping occurs in the enhanced region when $\omega < \omega_{ce}/2$

$$\delta\rho > 0 \qquad (\text{for} \quad \omega < \omega_{ce}/2) \qquad (59)$$

and in the depressed region when $\omega > \omega_{ce}/2$

$$\delta\rho < 0 \qquad (\text{for} \quad \omega > \omega_{ce}/2) \qquad (60)$$

This result is quite consistent with the ray-theory [Smith, 1961].

For Alfven waves, p is

$$p \simeq 1 \ / \ 4k \qquad (61)$$

and U is

$$U \simeq -\ (k/2)\{(\delta\rho/\rho_0) - 2(\delta B/B_0)\} \qquad (62)$$

Then the wave-trapping occurs where

$$\delta\rho > 0 \qquad (63)$$

4. Nonlinear wave modulation

In the general expression of the wave equation (35) in which the potential U is given by eq. (36) and the coefficient of the diffraction term, p, is given by eqs. (43) or (52), we consider here the cases when $\delta\rho$ and δB are derived by the ponderomotive force. As is the case for whistler waves shown by Karpman and Washimi [1977], $\delta\rho$ and δB are generally described as perturbations driven by the ponderomotive force in the following MHD system,

$$\partial\rho / \partial t + \nabla (\rho \mathbf{V}) = 0 \tag{64}$$

$$\rho \{\partial \mathbf{V} / \partial t + (\mathbf{V}\ \nabla)\mathbf{V}\} = - \nabla P + (1/4\pi)\ (\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{F} \tag{65}$$

and

$$\partial \mathbf{B} / \partial t + \nabla \times (\mathbf{V} \times \mathbf{B}) = 0 \tag{66}$$

where the ponderomotive force due to the transverse wave is generally expressed by Washimi and Karpman [1976] as follows:

$$\mathbf{F} = (1/16\pi)\ (K_{ij}-\delta_{ij})\ \nabla(E_i \cdot E_j) + (1/c)\ \mathbf{J}^{\ell} \times \mathbf{B}$$

$$+(1/16c\pi)[(\partial/\partial t)[\{(\varepsilon-I)\mathbf{E}\}\mathbf{H}^{*} + \omega[\{(\partial\varepsilon/\partial\omega)(\partial \mathbf{E}/\partial t)\}\mathbf{H}^{*}] + \text{c.c.}] \tag{67}$$

where $\mathbf{H}$ is the magnetic field and $\mathbf{J}^{\ell}$ the nonlinear current. Due to the ponderomotive force of the waves, slowly varying components $\delta\rho$ and δB are driven, which results in the self-modulation of the waves. Therefore eqs.(35) and (36) together with eqs.(64)-(67) constitute a closed system of wave equations.

When the wave propagates along the field lines, which is the present case, eq.(67) is reduced to [Washimi, 1973]

$$F_x = -(1/16\pi)[\partial\{\omega(\varepsilon-1)\} / \partial\omega]\ \partial|\mathbf{E}|^2 / \partial x \tag{68}$$

and

$$F_z = (1/16\pi)[(\varepsilon-1)\partial/\partial z + (k/\omega^2)[\partial\{\omega^2(\varepsilon-1)\}/\partial\omega](\partial/\partial t)]\ |\mathbf{E}|^2 \tag{69}$$

The coefficient of $\partial|E|^2 / \partial x$ in eq.(68) is negative for all waves which means that the ponderomotive force in the x direction acts as a wave-pressure force. This force induces the depression of the magnetic pressure for the total pressure balance. On the other hand, the ponderomotive force in the z direction acts as a negative pressure for the whistler and the Alfven waves because the coefficient of $\partial|E|^2/\partial z$ in eq.(69) is positive for these waves. This enhances the plasma density at the intense wave region.

When $\partial/\partial t \simeq 0$: in eq.(35) and above equations (64)-(69), we have an equation of modulation in x-z space (43)

$$i\ \partial\phi/\partial z + p\ \partial^2\phi/\partial x^2 - U(x,z)\ \phi = 0$$

and U (eq.(36)) is

$$U(x,z) = -\ (k/2\varepsilon)\ \{(\partial\varepsilon/\partial\rho)\ \delta\rho + (\partial\varepsilon/\partial B)\ \delta B\}$$

where

$$\delta\rho\ /\ \rho_0 = (1/2)\ (\gamma\beta/2)^{-1}\ (\varepsilon - 1)\ \{|\phi|^2\ /\ B_0{}^2\} \tag{70}$$

and

$$\delta B\ /\ B_0 = -\ (1/2)\ [\varepsilon - 1 + \omega_{pe}{}^2\ /\{(+/-)\omega_{ce}-\omega\}^2 + \omega_{pi}{}^2\ /\{(+/-)\omega_{ci}+\omega\}^2]\ \{|\phi|^2\ /\ B_0{}^2\} \tag{71}$$

in which β is

$$\beta = (2/\gamma)(V_s\ /\ V_A)^2 \tag{72}$$

where γ is the adiabatic constant, V_s $(=(\gamma P_{ress}/\rho_0)^{1/2})$ the sound velocity and V_A $(=B_0/(4\pi\rho_0)^{1/2})$ the Alfven velocity. In view of eq.(70) $\delta\rho$ is found to be positive for the whistler and the Alfven waves which results in the self-focusing and the self-trapping. On the other hand, δB is negative for these waves. Finally, we have

$$i\ \partial\phi/\partial z + p\ \partial^2\phi/\partial x^2 + q\ |\phi|^2\ \phi = 0 \tag{73}$$

$$q = (k/4\varepsilon)\ [(\gamma\beta/2)^{-1}(\varepsilon-1)^2 + [\varepsilon-1+\{\omega_{pe}{}^2/((+/-)\omega-\omega_{ce})^2\}$$

$$+ \{\omega_{pi}^2/((+/-)\omega+\omega_{ci})^2\}]^2] \;/\; B_0^2 \tag{74}$$

Because q is positive for all cases, the self-focusing and self-trapping occur when p is positive. The soliton solution is, by assuming $\phi \sim \exp(i\mu^2\kappa z)$ where μ is a smallness parameter,

$$\phi = \mu(2\kappa/q)^{1/2}\ \mathrm{sech}\{\mu(\kappa/p)^{1/2}\ x\}\ \exp(i\mu^2\kappa z) \tag{75}$$

The above analysis is quite parallel for a cylindrical symmetric system, and we have a wave equation

$$i\ \partial\psi/\partial z + p\ \{\partial^2\psi/\partial r^2 + (1/r)\partial\psi/\partial r\} + q\ |\psi|^2\ \psi = 0 \tag{76}$$

The profile of the soliton for this case is given in Chiao et al. (1964), and the threshold power for the wave-trapping is given by

$$P_c = (c/4\pi)\int \mathbf{E} \times \mathbf{B}\ 2\pi r\ dr = 3.7 \times \{c^2/(\omega/k)\}\ p/q \tag{77}$$

For high-frequency electromagnetic waves, p is given by eq.(54) and q by:

$$q = (k/4\varepsilon)\ \{1/(4\pi\Gamma P_{ress})\}\ (\omega_{pe}/\omega)^4 \tag{78}$$

Then the threshold power for self-trapping is

$$P_c = 2.36 \times (\omega/k)\ \lambda^2\ \{\Gamma\varepsilon^2(\omega/\omega_{pe})^4\}\ P_{ress} \tag{79}$$

where λ is the wave-length.

For whistler waves

$$q \simeq (k/4)\ (\Gamma\beta/2)^{-1}\ \varepsilon\ /\ B_0^2 \tag{80}$$

The self-focusing and self-trapping occurs when p (eq.(57)) is positive, which corresponds to $\omega < \omega_{ce}/2$, and the threshold power is

$$P_c \simeq 2.36 \times (\omega/k)\ \lambda^2\ [\Gamma\{(1/2)\omega_{ce}-\omega\}/(\omega_{ce}-\omega)]\ P_{ress} \tag{81}$$

Let us estimate $\delta\rho$ and δB. The power of the wave beam of the diameter, d, is

$$P = (c/4\pi) \int E \times B \; 2\pi r \, dr = (\pi/2) \, (\omega/k) \, d^2 \, (B_{ave}^2 \, / \, 8\pi) \qquad (82)$$

The condition $P = P_c$ gives

$$(B_{ave}/B_0)^2 = 1.5 \times (\lambda/d)^2 \, [\Gamma\{(1/2)\omega_{ce}-\omega\}/(\omega_{ce}-\omega)] \; \beta \qquad (83)$$

$$\delta\rho \, / \, \rho_0 = (1/4) \, (\Gamma\beta/2)^{-1} \, (B_{ave} \, / \, B_0)^2$$

$$= 0.75 \times (\lambda/d)^2 \, [\{(1/2)\omega_{ce}-\omega\}/(\omega_{ce}-\omega)] \qquad (84)$$

For Alfven waves, p is given by eq.(61) and q is

$$q \simeq (k/4) \, (\Gamma\beta/2)^{-1} \, \varepsilon \, / \, B_0^2 \qquad (85)$$

Then the threshold power is

$$P_c \simeq 1.18 \times V_A \, \lambda^2 \, \Gamma \, P_{ress} \qquad (86)$$

The power of the Alfven wave beam is

$$P = (c/4\pi) \int E \times B \; 2\pi r \, dr = (\pi/2) \, V_A \, d^2 \, (B_{ave}^2 \, / \, 8\pi) \qquad (87)$$

and from the condition $P = P_c$ we have

$$(B_{ave} \, / \, B_0)^2 = 0.75 \times (\lambda/d)^2 \, \Gamma \, \beta \qquad (88)$$

$$\delta\rho \, / \, \rho_0 = (1/4) \, (\Gamma\beta/2)^{-1} \, (B_{ave} \, / \, B_0)^2 = 0.38 \times (\lambda/d)^2 \qquad (89)$$

5. Summary

1. With the help of the general expression of the ponderomotive force eq.(67), it was shown by dielectric tensor formalism that the wave equation is rederived.
2. The physical meaning of the wave-trapping condition (53) was clarified in respect to the relation between the ray direction and the wave diffraction. Both for inhomogeneous and nonlinear effects, it was clarified that the sign of the density deviation $\delta\rho$ should be negative for the high-frequency electromagnetic wave and for the

whistler wave of $\omega > \omega_{ce}/2$, and positive for the whistler wave of $\omega < \omega_{ce}/2$ and the Alfven waves for the wave-trapping.

3. The ponderomotive force due to the whistler and the Alfven waves acts outwardly across the field lines but also inwardly along the field lines. This enhances the plasma density, which results in the self-focusing and the self-trapping in the enhanced plasma region.

4. Recently, the coronal loop in the solar atmosphere due to the Alfven wave trapping is under discussion by Y. Chiu and the present author (personal communication, 1988). The whistler wave trapping in the terrestrial magnetosphere is also under discussion by T. Okada and the writer (personal communication, 1988). Because q in eq.(74) involves the factor β, the threshold power becomes very low for the self-trapping in space plasmas. The analysis in the prsent study provides a basis for these studies.

Acknowledgements

The author thanks N. Yajima and M. Hayakawa for their critical reading of this paper. This work was supported by a Grant-in-Aid from the Ministry of Education, Science and Culture of Japan.

References

Chiao, R. Y., E. Garmire, and C. H. Towns, Self-trapping of optical beams, **Phys. Rev. Letters**, **13**, 479-482, 1964.

Helliwell, R. A., Whistlers and related ionospheric phenomena, **Stanford Univ. Press**, Stanford, California, 1965.

Hoshino, M., Evolution of polarization in localized nonlinear Alfven waves, **Phys. Rev. Letters**, **59**, 2639-2642, 1987.

Karpman, V. I., and H. Washimi, Two-dimensional self-modulation of a whistler wave propagation along the magnetic field in a plasma, **J. Plasma Phys.** **18-1**, 173-187, 1977.

Landau, L. D., and E. M. Lifshitz, Electrodynamics of continuous media, **Pergamon Press**, 1960.

Mio, K., T. Ogino, K. Minami, and S. Takeda, Modified nonlinear Schrodinger equation for Alfven waves propagating along the magnetic field in cold plasma, **J. Phys. Soc. Japan**, **41**, 265-271, 1976.

Smith, R. L., Propagation characteristics of whistlers trapped in field aligned columns of enhanced ionization, **J. Geophys. Res.**, **66**, 3699-3707, 1961.

Stix, T. H., The theory of plasma waves, **McGraw-Hill**, New York, 1962.

Taniuti, T., and H. Washimi, Self-trapping and instability of hydromagnetic waves along the magnetic field in a cold plasma, **Phys. Rev. Letters, 21**, 209-212, 1968.

Taniuti, T., and H. Washimi, Self-focusing of a plasma wave along a magnetic field, **Phys. Rev. Letters, 22**, 454-457, 1969; **22**, 752, 1969.

Taniuti, T., and C. C. Wei, Reductive perturbation method in nonlinear wave propagation, I., **J. Phys. Soc. Japan, 24**, 941-946, 1968.

Taniuti, T., and N. Yajima, Perturbation method for a nonlinear wave modulation, I., **J. Math. Phys. 10**, 1369-1372, 1969.

Washimi, H., Self-focusing of transvere waves in a magnetoplasma, **J. Phys. Soc. Japan, 34**, 1373-1383, 1973.

Washimi, H., Wave-trapping in an inhomogeneous magnetoplasma, **J. Phys, Soc. Japan, 41**, 2098-2104, 1976.

Washimi H., and V. I. Karpman, The ponderomotive force of a high-frequency electromagnetic field in a dispersive medium, **Zh. Eksp. Teor. Fiz. 71**, 1010-1016, 1976. [**Soviet Phys. JETP, 44**, 528-531, 1976.]