PROPAGATION OF VLF AND ELF RADIO WAVES WITH MODE COUPLING IN THE INHOMOGENEOUS STRATIFIED IONOSPHERE

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It is well established that the use of the wave theory is very essential for a rigorous treatment of the propagation problem for VLF and ELF radio waves in the ionosphere, but unfortunately the equations that appear in the wave theory are, in general, nonlinear, and so difficult to be solved that the essential point is always how to find the approximate solutions which can be applied to the problem under consideration. One of the approaches to the approximation is to divide the ionosphere into stratified layers of homogeneous media and to use the linearised wave equations which are individually dealt with in each of them. It is noted here that the layer thickness must be taken thin enough in accordance with the amount of height variation of the refractive index for VLF and ELF radio waves. The approximation which devides into the stratified layer of homogeneous media eliminates the mode coupling which results from the inhomogeneity of media. It is very expected that the mode coupling for VLF and ELF radio waves will get strong but it inevitably introduces the troublesome calculations in some region of the ionosphere.

It is the objective of this paper to discuss another approach to the propagation problem of VLF and ELF radio waves taking the height variation of the media in the stratified layers into consideration. In this approach, the layer thickness may be taken thicker than the former and the effect of the mode couplings is allowed to exist, though the wave equations in each layer become to be slightly complicated.

It is well known (Budden, 1961) that the following relations for the electomagnetic waves in the plasma are derived from Maxwell's eqs, together with the constitutive relations of the medium

$$\frac{d\mathbf{e}}{dz} = -j k_0 \mathbf{T} \mathbf{e} \quad \cdots\cdots (1)$$

where,

$$\mathbf{e}=\begin{pmatrix} E_x \\ -E_y \\ \mathscr{H}_x \\ \mathscr{H}_y \end{pmatrix},\quad \mathbf{T}=\begin{pmatrix} -\frac{SM_{zx}}{1+M_{zz}} & \frac{SM_{zy}}{1+M_{zz}} & 0 & \frac{C^2+M_{zz}}{1+M_{zz}} \\ 0 & 0 & 1 & 0 \\ \frac{M_{yz}M_{zx}}{1+M_{zz}}-M_{yx} & C^2+M_{yy}-\frac{M_{yz}M_{zy}}{1+M_{zz}} & 0 & \frac{SM_{yz}}{1+M_{zz}} \\ 1+M_{xx}-\frac{M_{xz}M_{zx}}{1+M_{zz}} & \frac{M_{xz}M_{zy}}{1+M_{zz}}-M_{xy} & 0 & \frac{-SM_{xz}}{1+M_{zz}} \end{pmatrix} \quad \cdots\cdots\cdots\cdots (2)$$

Here, it is assumed that the medium changes its property only in the z-direction which is perpendicular to the horizontally stratified layer, and that the direction of the incident wave propagating into the layer makes the angle θ_i with z-axis. The notations of $\sin\theta_i=S$, $\cos\theta_i=C$ are used; M_{ij} (i, j =x, y, z,) is an element of susceptibility matrix which is a function of z in general; $\mathscr{H}=(\mu_0/\varepsilon_0)^{1/2}\mathbf{H}$; ε_0, μ_0 are respectively the dielectric constant and magnetic permeability in free space, **E** and **H** are the electric and magnetic wave field vectors respectively; k_0 the wave number in free space.

To secure the existence of non-zero solution of the equation it is necessary to make the determinant of the coefficients to vanish, hence matrix **T** must have four characteristic roots or eigenvalues Q_i ($i=1, 2, 3, 4,$) which satisfy the characteristic equation

$$\det\ (\mathbf{T}-Q\,\mathbf{I})=0 \cdots\cdots\cdots\cdots (3)$$

where **I** is the unity matrix. At any level in the ionosphere, the four values of Q are the four characteristic roots of the matrix **T**. Corresponding to any root Q_i there exists a column matrix $\mathbf{e}^{(i)}$ which is an eigenvector of the matrix **T** and satisfies the following equations.

$$\mathbf{T}\mathbf{e}^{(i)}=Q_i\,\mathbf{e}^{(i)} \cdots\cdots\cdots\cdots (4)$$

The problems of propagation in the ionosphere can be discussed by the use of this matrix with given boundary conditions. However it is difficult, in general, to solve the column matrix $\mathbf{e}^{(i)}$ accurately because Eq. (1) nonlinear. Since the matrix **T** for this case is a function of z, so that the characteristic root Q_i also cannot be solved accurately. Physical interpretation of the mathematical representation is that the four waves given by the solution — two of them compose an ordinary, and an extraordinary waves which propagate upwards, and the other two compose an ordinary, and an extraordinary waves which propagate downwards — cannot propagate independent with each other in an inhomogeneous medium but make mutual coupling. If the medium becomes homogeneous, the mode coupling comes to be neglected and the waves propagate independently. The characteristic roots Q_i then coincide with Booker's quartic roots q_i (Budden, 1961), and the column matrix $\mathbf{e}^{(i)}$ in Eq. (4) can be written as

$$\mathbf{e}^{(i)} = \mathbf{e}_0^{(i)} \exp(-j k_0 q_i z) \quad \cdots\cdots (5)$$

Assume that the exponentia expression, Eq. (5) for a homegeneous medium, if modified, is also valid to an inhomogeneous medium, then we get

$$\mathbf{e}^{(i)} = \mathbf{e}_0^{(i)} \exp\{-j k_0 \lambda_i(z) z\} \quad \cdots\cdots (6)$$

Because Eq. (6) must satisfy Eq. (1), we get

$$\mathbf{T}\,\mathbf{e}^{(i)} - \left(\frac{d\lambda_i}{dz} z + \lambda_i\right)\mathbf{e}^{(i)} = 0 \quad \cdots\cdots (7)$$

To let Eq. (7) to have a non-zero solution as before, we must have the determinant of the right hand side of Eq. (7) equated to zero,

$$\det\left[\mathbf{T} - \left(\frac{d\lambda_i}{dz} z + \lambda_i\right)\mathbf{I}\right] = 0 \quad \cdots\cdots (8)$$

It is easy to see that Booker's quartic root q_i in a homogeneous medium is replaced with $(d\lambda_i/dz)\, z + \lambda_i = Q_i$. This means that the characteristics of wave propagation in an inhomogeneous medium are obtained with this replacement. The formulas for wave propagation in the inhomogeneous ionosphere can be derived easily from the those of homogeneous medium, if the value of $(d\lambda_i / dz)\, z + \lambda_i$ are obtained. For example, the reflection coefficients of waves in the inhomogeneous medium are represented as

$$_{\parallel}R_{\parallel} = \left[\frac{_{\parallel}K_{\parallel}^{(T)} - {_{\parallel}K_{\parallel}^{(I)}}}{_{\parallel}K_{\parallel}^{(R)} - {_{\parallel}K_{\parallel}^{(T)}}}\right]_{Z=Z_H}, \quad _{\parallel}R_{\perp} = \left[{_{\parallel}K_{\perp}^{(R)}}\, \frac{_{\parallel}K_{\perp}^{(T)} - {_{\parallel}K_{\perp}^{(I)}}}{_{\parallel}K_{\perp}^{(R)} - {_{\parallel}K_{\perp}^{(T)}}}\right]_{Z=Z_H},$$
$$_{\perp}R_{\parallel} = \left[\frac{1}{_{\perp}K_{\parallel}^{(I)}} \cdot \frac{_{\perp}K_{\parallel}^{(T)} - {_{\perp}K_{\parallel}^{(I)}}}{_{\perp}K_{\parallel}^{(R)} - {_{\perp}K_{\parallel}^{(T)}}}\right]_{Z=Z_H}, \quad _{\perp}R_{\perp} = \left[\frac{_{\perp}K_{\perp}^{(I)} - {_{\perp}K_{\perp}^{(T)}}}{_{\perp}K_{\perp}^{(T)} - {_{\perp}K_{\perp}^{(R)}}}\right]_{Z=Z_H}. \quad \cdots\cdots (9)$$

where, (I), (R), and (T) refer to the incident, reflected, and transmitted waves respectively, $Z = Z_H$ denote a particular height of the stratified layer, and let us define the following quantities as

$$_{\parallel}K_{\parallel} = \frac{(T_{11}-Q_i)(-Q_i^2+T_{32})T_{14} - T_{12}^2 T_{42}}{(T_{11}-Q_i)(-Q_i^2+T_{32})T_{41} - T_{12}T_{31}T_{42}},$$

$$_{\parallel}K_{\perp} = {_{\perp}K_{\parallel}} = \frac{(T_{11}-Q_i)(T_{14}T_{31} - T_{12}T_{41})}{(T_{11}-Q_i)(-Q_i^2+T_{32})T_{42} + T_{12}T_{31}^2}, \quad _{\perp}K_{\perp} = Q_i = \frac{d\lambda_i}{dz} z + \lambda_i\,;$$

$$T_{11}=\frac{SGlnY^2}{1+G(U^2-n^2Y^2)},\quad T_{12}=\frac{SGjlYU}{1+G(U^2-n^2Y^2)},\quad T_{14}=\frac{C^2+G(U^2-n^2Y^2)}{1+G(U^2-n^2Y^2)},$$

$$T_{31}=\frac{G^2jl^2nY^3U}{1+G(U^2-n^2Y^2)}-GjnYU,\quad T_{32}=C^2+GU^2-\frac{G^2l^2Y^2U^2}{1+G(U^2-n^2Y^2)},$$

$$T_{41}=1+G(U^2-lY^2)-\frac{G^2l^2n^2Y^4}{1+G(U^2-n^2Y^2)};\qquad \cdots\cdots\cdots(10)$$

$$G=-\frac{X}{U(U^2-Y^2)},\quad U=I-jZ,\quad X=\frac{\omega_P^2}{\omega^2},\quad Y=\frac{\omega_H}{\omega},\quad Z=\frac{\nu}{\omega}$$

In the coordinate system used here, it is noted that the earth's magnetic field is parallel to the x-z plane; $T_{\alpha\beta}(\alpha,\ \beta=1,2,3,4,\ \alpha\neq2,\ \beta\neq3)$ is an element of the matrix **T** given by Eq. (2); l and n are x- and z- direction cosine of the vector **Y** respectively, whose signs are opposite to the earth's magnetic field.

In the above results, the term of $Q_i=(d\lambda_i/dz)\,z+\lambda_i$ represents exclusively the effect of the inhomogenity, and if the inequality $(d\lambda_i/dz)\,z\ll\lambda_i$ is valid, in other words, if the effect of the inhomogeneity is small, it is easy to see that Q_i coincide with Booker's quartic root q_i of homogeneous medium.

This is a preliminary report for an approximative approach to the full wave solution which describes the VLF and ELF radio wave propagation in the ionosphere. Futher studies are required as regards the derivation of Q_i and λ_i in the stratified layer.

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