Some properties of numbers and functions in analytic number theory

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1. Introduction

In this paper, we discuss some properties of real numbers and some functions in analytic number theory. This paper is divided into the following four sections.

1. Introduction
2. Double analogue of Hamburger's theorem
3. On the lcm-sum function
4. A new construction of the real numbers by alternating series

Sections 2, 3 and 4 are devoted to the explanation of our main results. Section 1 (this section) is divided into the following five parts.

1.1. Zeta-functions and arithmetic functions
1.2. Irrational numbers and Transcendental numbers
1.3. Summary of Section 2
1.4. Summary of Section 3
1.5. Summary of Section 4

Since the theory of zeta-functions and arithmetic functions and the theory of irrational numbers and transcendental numbers are important in analytic number theory, we show some facts in these theories in Sections 1.1 and 1.2. The contents of Sections 2 and 3 relate to the theory of zeta-functions and arithmetic functions. The contents of Section 4 relate to the theory of irrational numbers and transcendental numbers. In Sections 1.3 - 1.5, we give a summary of Sections 2 - 4. These summaries are not long, because we give a detailed introduction at the beginning of each section.

1.1. **Zeta-functions and arithmetic functions.** Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. The Riemann zeta-function $\zeta(s)$ is one of the most important function in analytic number theory. The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $\sigma > 1$. For $\sigma > 1$ the Riemann zeta-function can also be written by the Euler product

$$\zeta(s) = \prod_{p:\text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

By this fact, we can see a relationship between prime numbers and $\zeta(s)$. The function $\zeta(s)$ can be continued meromorphically to $\mathbb{C}$ and satisfies the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)$ is the Gamma function. Now we can state the Riemann Hypothesis as follows :

Riemann Hypothesis. The real part of the zeros of $\zeta(s)$ in the region $0 < \sigma < 1$ is equal to $1/2$.

At present, the Riemann Hypothesis is an unsolved problem and one of the most important problem in mathematics.

Let $k \in \mathbb{N}$ and $s_1 = \sigma_1 + it_1, \ldots, s_k = \sigma_k + it_k$ with $\sigma_1, \ldots, \sigma_k, t_1, \ldots, t_k \in \mathbb{R}$. Recently, multiple zeta-functions are studied by various authors. The Euler-Zagier multiple zeta-function is one of the most important multiple zeta-functions and defined by

$$\zeta_k(s_1, \ldots, s_k) = \sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}} \sum_{n_2=n_1+1}^{\infty} \frac{1}{n_2^{s_2}} \cdots \sum_{n_k=n_{k-1}+1}^{\infty} \frac{1}{n_k^{s_k}},$$

where the sum is absolutely convergent in $\sigma_k > 1, \sigma_{k-1}+\sigma_k > 2, \ldots, \sigma_1 + \cdots + \sigma_k > k$. The function $\zeta_k(s_1, \ldots, s_k)$ is continued meromorphically to $\mathbb{C}^k$ (see [1]). We note that $\zeta_1(s_1) = \zeta(s_1)$. In Section 2, we discuss analytic properties of $\zeta_2(s_1, s_2)$.

An arithmetic function f is a function $f : \mathbb{N} \to \mathbb{C}$. An arithmetic function f is multiplicative if and only if

$$f(mn) = f(m)f(n) \qquad \text{for all } \gcd(m, n) = 1$$

holds, where $\gcd(m, n)$ is the greatest common divisor of m and n. If arithmetic functions f and g are multiplicative, then the Dirichlet convolution product

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d)$$

is also multiplicative. The relationship between the Riemann zeta-function and the arithmetic functions are important. We show some examples.

Example 1. The Möbius function $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n = p_1 p_2 \cdots p_k \\ 0 & \text{otherwise,} \end{cases}$$

where $p_1, p_2, \ldots, p_k$ are distinct prime numbers. The function μ is multiplicative and satisfies

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\sigma > 1$ (see, for example, p. 3 in [26]). The Riemann Hypothesis is equivalent to the statement that the estimate

$$\sum_{1 \le n \le x} \mu(n) = O(x^{1/2+\epsilon})$$

holds for all $\epsilon > 0$ (see, for example, p. 47 in [14]).

Example 2. The Euler function $\varphi(n)$ is defined by

$$\varphi(n) = |\{k \in \mathbb{N} \mid k < n \text{ and } \gcd(k, n) = 1.\}|.$$

The function φ is multiplicative and satisfies

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$$

for $\sigma > 2$ (see, for example, p. 6 in [26]). We note that we use an estimate of

$$\sum_{1 \le n \le x} \varphi(n)$$

in Section 3.

1.2. **Irrational numbers and Transcendental numbers.** An algebraic number is a root of equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

where $n \ge 0$, $a_0 \ldots, a_n \in \mathbb{Z}$ and $a_n \ne 0$. We denote the set of all algebraic numbers by $\overline{\mathbb{Q}}$. A transcendental number α is a number $\alpha \in \mathbb{C} \setminus \overline{\mathbb{Q}}$. If $\alpha \in \mathbb{R}$ is a transcendental number, then it is an irrational number. Since $\overline{\mathbb{Q}}$ (resp. $\mathbb{Q}$) is a countable set, almost all numbers are transcendental (resp. irrational). However, in general, it is difficult to prove the transcendence (or irrationality) of a given number. The following are well-known facts in the theory of irrational numbers and transcendental numbers.

(1). The numbers e and π are transcendental.

(2). Let $a, b \in \overline{\mathbb{Q}}$ with $a \notin \{0, 1\}$ and $b \in \mathbb{Q}$. Then a^b is transcendental, which was proved by Gelfond and Schneider independently (see, for example, p. 102–106 in [8]). By taking $a = i$ and $b = -2i$, we see that e^{π} is transcendental.

(3). For $n \in \mathbb{N}$, $\zeta(2n)$ is transcendental, because

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}$$

holds (see, for example, p. 98 in [24]), where B_n are Bernoulli numbers defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi)$$

(see, for example, p. 59 in [24]). For all $n \in \mathbb{N} \cup \{0\}$, B_n is rational. Apéry proved that $\zeta(3)$ is irrational (see [3] and [28]). However, whether $\zeta(5), \zeta(7), \ldots$ are irrational or not is not known. We note that Rivoal [23] proved that there exist infinitely many irrational numbers in the set $\{\zeta(2n+1) \mid n \in \mathbb{N}\}$ and Zudilin [30] proved that one of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

(4). We can conclude the irrationality of certain numbers by their decimal expansion. For example,

$$\sum_{n=1}^{\infty} \frac{1}{2^{n^2}}$$

is an irrational number. In Section 4, we discuss an analogue of this fact and prove the irrationality of certain numbers.

1.3. **Summary of Section 2.** Hamburger's theorem states that the Riemann zeta-function $\zeta(s)$ is characterized by its functional equation. Recently, Matsumoto [20] found a functional equation for Euler(-Zagier) double zeta function $\zeta_2(s_1, s_2)$, that is

$$\frac{g(s_1, s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} = \frac{g(1-s_2, 1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)} + 2i \sin\left(\frac{\pi}{2}(s_1+s_2-1)\right) F_+(s_1, s_2),$$

where F_+ is a meromorphic function defined in Section 2 and

$$g(s_1, s_2) = \zeta_2(s_1, s_2) - \frac{\Gamma(1-s_1)}{\Gamma(s_2)} \Gamma(s_1 + s_2 - 1)\zeta(s_1 + s_2 - 1).$$

The purpose of Section 2 is to study a characterization of $\zeta_2(s_1, s_2)$ and $\zeta(s)$ by Matsumoto's functional equation and the well-known relation

$$\zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2).$$

The contents in Section 2 are a joint work with Kaneaki Matsuoka (Graduate School of Mathematics, Nagoya University) and based on [12].

1.4. **Summary of Section 3.** Pillai [22] first defined the gcd-sum function

$$g(n) = \sum_{j=1}^{n} \gcd(j, n)$$

and studied this function. The function $g(n)$ was defined again by Broughan [5]. The function $g(n)$ and its analogue have been studied by various authors. In Section 3, we consider the function

$$L_a(n) := \sum_{j=1}^{n} (\operatorname{lcm}(j, n))^a$$

$$T_a(x) := \sum_{n \le x} L_a(n)$$

for $a \in \mathbb{Z}$ and $x \geq 1$. These functions were studied by Bordellès [4] and Alladi [2]. In particular, the function $L_1(n)$ is called the lcm-sum function and is a natural analogue of the gcd-sum function. In Section 3, we give two kinds of asymptotic formulas for $T_a(x)$. Our results include a generalization of Bordellès's results and a refinement of the error estimate in Alladi's result. We prove these results by the method similar to those of Bordellès.

The contents in Section 3 are a joint work with Kaneaki Matsuoka (Graduate School of Mathematics, Nagoya University) and based on [11].

1.5. **Summary of Section 4.** A. Knopfmacher and J. Knopfmacher constructed the complete ordered field of real numbers by the Sylvester expansion and the Engel expansion in [17] and by the alternating-Sylvester expansion and the alternating-Engel expansion in [18]. These series expansions for the real numbers are similar to the decimal expansion and the simple continued fraction expansion. The advantages of these constructions are the fact that those are concrete and do not depend on the notion of equivalence classes.

In Section 4, we define the generalized alternating-Sylvester expansion for the real numbers and present a new method of constructing the complete ordered field of real numbers from the ordered field of rational numbers. Our method is a generalization of that of A. Knopfmacher and J. Knopfmacher [18]. Our result implies that there exist infinitely many ways of constructing the complete ordered field of real numbers. As an application of our results, we prove the irrationality of certain numbers by certain properties of the generalized alternating-Sylvester expansion.

The contents of this section are based on [10].

2. Double analogue of Hamburger's theorem

The contents in this section are a joint work with Kaneaki Matsuoka (Graduate School of Mathematics, Nagoya University) and based on [12].

2.1. **Introduction.** Let $s = \sigma + it$, $s_1 = \sigma_1 + it_1$, $s_2 = \sigma_2 + it_2$ with $\sigma, \sigma_1, \sigma_2, t, t_1, t_2 \in \mathbb{R}$. The Riemann zeta function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{2.1}$$

where

$$\chi(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$$

and $\Gamma(s)$ is the gamma function. The following theorem is well-known as a characterization of $\zeta(s)$.

Hamburger's theorem (see, for example, p. 31 in [26])**.** *Let $G(s)$ be an integral function of finite order, $P(s)$ a polynomial, and $f(s) = G(s)/P(s)$, and let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be absolutely convergent for $\sigma > 1$. Let $\alpha > 0$ and

$$f(s) = \chi(s)g(1-s),$$

where

$$g(1-s) = \sum_{n=1}^{\infty} \frac{b_n}{n^{1-s}},$$

the series being absolutely convergent for $\sigma < -\alpha$. Then $f(s) = C\zeta(s)$, where C is a constant.

The purpose of this paper is to give an analogue of Hamburger's theorem for the Euler double zeta function.

The Euler double zeta function $\zeta_2(s_1, s_2)$ is defined by

$$\zeta_2(s_1, s_2) = \sum_{1 \le m < n} \frac{1}{m^{s_1} n^{s_2}} \quad (\sigma_1 + \sigma_2 > 2,\ \sigma_2 > 1)$$

and continued meromorphically on $\mathbb{C}^2$ (see [1]). The functions $\zeta(s)$ and $\zeta_2(s_1, s_2)$ satisfy the functional relation

$$\zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2) \tag{2.2}$$

for $s_1, s_2 \in \mathbb{C}$. On the other hand Matsumoto obtained the following result in [20].

Let

$$g(s_1, s_2) = \zeta_2(s_1, s_2) - \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1).$$

Let

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)}\int_0^{\infty e^{i\phi}} e^{-xy}y^{a-1}(1+y)^{c-a-1}dy$$

be the confluent hypergeometric function, where $\Re a > 0$, $-\pi < \phi < \pi$, $|\phi + \arg x| < \pi/2$. We use the notation $\sigma_l(k) = \sum_{d|k} d^l$.

Matsumoto's theorem. *We have*

(2.3)
$$\frac{g(s_1, s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} = \frac{g(1-s_2, 1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)} + 2i\sin\left(\frac{\pi}{2}(s_1+s_2-1)\right)F_+(s_1, s_2),$$

where $i = \sqrt{-1} = \exp(\pi i/2)$ *and* $F_+(u, v)$ *is the series defined by*

$$F_+(u, v) = \sum_{k=1}^{\infty}\sigma_{u+v-1}(k)\Psi(v, u+v; 2\pi ik). \tag{2.4}$$

The series (2.4) is convergent only in the region $\Re u < 0$, $\Re v > 1$, *but it can be continued meromorphically to the whole* $\mathbb{C}^2$ *space.*

The equation (2.3) is a functional equation for $\zeta_2(s_1, s_2)$.

Moreover, Komori, Matsumoto and Tsumura obtained the following result in [19].

Let $\omega_1, \omega_2 \in \mathbb{C}$ and

$$\zeta_2(s_1, s_2; \omega_1, \omega_2) = \sum_{m=1}^{\infty}\frac{1}{(m\omega_1)^{s_1}}\sum_{n=1}^{\infty}\frac{1}{(m\omega_1+n\omega_2)^{s_2}},$$

where $z^s = \exp(s\log z)$, $\log z = \log|z| + i\arg z$ and $-\pi < \arg z \leq \pi$ for $z \in \mathbb{C}$. Note that $\zeta_2(s_1, s_2; 1, 1) = \zeta_2(s_1, s_2)$. Let

$$g_0(s_1, s_2; \omega_1, \omega_2) = \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1)\omega_1^{-1}\omega_2^{1-s_1-s_2}.$$

Theorem (Komori, Matsumoto and Tsumura)**.** *For* $\omega_1, \omega_2 \in \mathbb{C}$ *with* $\Re\omega_1 > 0$, $\Re\omega_2 > 0$, *the hyperplane*

$$\Omega_{2k+1} = \{(s_1, s_2) \in \mathbb{C}^2 \mid s_1 + s_2 = 2k+1\} \quad (k \in \mathbb{Z} \setminus \{0\})$$

is not a singular locus of $\zeta_2(s_1, s_2; \omega_1, \omega_2)$*. On this hyperplane the following functional equation holds:*

(2.5)

$$\left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{1-s_1-s_2}{2}} \Gamma(s_2)\{\zeta_2(s_1, s_2; \omega_1, \omega_2) - g_0(s_1, s_2; \omega_1, \omega_2)\}$$
$$= \left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{s_1+s_2-1}{2}} \Gamma(1-s_1)\{\zeta_2(1-s_2, 1-s_1; \omega_1, \omega_2) - g_0(1-s_2, 1-s_1; \omega_1, \omega_2)\}$$

for $(s_1, s_2) \in \Omega_{2k+1}$ $(k \in \mathbb{Z} \setminus \{0\})$*.*

The equation (2.5) is a functional equation for $\zeta_2(s_1, s_2; \omega_1, \omega_2)$ on the hyperplane Ω_{2k+1} $(k \in \mathbb{Z} \setminus \{0\})$. In the case $\omega_1 = \omega_2 = 1$ we have

$$\frac{g(s_1, s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} = \frac{g(1-s_2, 1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)}$$

on the hyperplane Ω_{2k+1} $(k \in \mathbb{Z} \setminus \{0\})$. Therefore we see that

$$2i \sin\left(\frac{\pi}{2}(s_1 + s_2 - 1)\right) F_+(s_1, s_2) = 0 \tag{2.6}$$

on the hyperplane Ω_{2k+1} $(k \in \mathbb{Z} \setminus \{0\})$.

The following is our main result. The cardinal number of the set A is denoted by $|A|$.

Theorem 2.1. *Let* $G(s)$ *be an integral function of finite order,* $P(s)$ *a polynomial, and* $f(s) = G(s)/P(s)$*, and let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be absolutely convergent for $\sigma > 1$*. Let* $f_2(s_1, s_2)$ *be a meromorphic function on* $\mathbb{C}^2$*. Let*

$$f_2(s_1, s_2) + f_2(s_2, s_1) = f(s_1)f(s_2) - f(s_1 + s_2) \tag{2.7}$$

and

(2.8)

$$\frac{1}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)}\left(f_2(s_1, s_2) - \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)f(s_1+s_2-1)\right)$$
$$= \frac{1}{i^{s_1+s_2-1}\Gamma(s_2)}\left(f_2(1-s_2, 1-s_1) - \frac{\Gamma(s_2)}{\Gamma(1-s_1)}\Gamma(1-s_1-s_2)f(1-s_1-s_2)\right) +$$
$$+ 2i \sin\left(\frac{\pi}{2}(s_1+s_2-1)\right) F_+(s_1, s_2)$$

in the $\mathbb{C}^2$ *space. Let* $f(2) = -2\pi^2 f(-1)$ *and*

$$\lim_{s\to -2} \Gamma(s)f(s) = -\frac{f(3)}{8\pi^2} = -\frac{\zeta(3)}{8\pi^2}. \tag{2.9}$$

Assume that at least one of the following conditions (a) *or* (b) *holds.*

(a) *In the closed vertical strip* $D = \{s \in \mathbb{C} \mid 2 \le \sigma \le 4\}$, $\zeta(1-s) \ll |f(1-s)|$ *and* $|\{s \in D \mid f(1-s) = 0\}| \le 1$.

(b) *There exists a constant* $c \in \mathbb{C} \setminus \{0\}$ *such that*

$$c = \lim_{s\to +\infty} \chi(s) f(1-s),$$

where $s \in \mathbb{R}$.

Then $f(s) = \zeta(s)$ *and* $f_2(s_1, s_2) = \zeta_2(s_1, s_2)$.

Note that both f and f_2 are unknown functions in Theorem 2.1. This implies that by using (2.2) and (2.3) we can obtain a characterization of both ζ and ζ_2.

We do not assume $f(s) = \chi(s)f(1-s)$ in Theorem 2.1. However, we can obtain $f(s) = \chi(s)f(1-s)$ from functional equations (2.7) and (2.8). This is a key step of the proof of Theorem 2.1.

It seems that the choice of special values of $f(s)$ in the assumptions of Theorem 2.1 can be replaced by other special values. In some sense, it is indeed possible, but there is a problem. We will explain this point after the proof of Theorem 2.1 (see Remark 2.2).

2.2. **Lemmas for the proof of Theorem 2.1.** In this section, we collect some auxiliary results.

Lemma 2.1. *Let* f *and* g *be meromorphic functions on* $\mathbb{C}$. *Assume that the functions* f *and* g *satisfy the functional equations*

$$f(s)f(1-s) = g(s)g(1-s) = 1 \tag{2.10}$$

and

$$f(s)f(k-s) = g(s)g(k-s) \tag{2.11}$$

for some $k \in \mathbb{R} \setminus \{1\}$. *If there exists a* $\sigma_0 \in \mathbb{R}$ *such that* $f(s)/g(s)$ *is bounded in the closed vertical strip* $D = \{s \in \mathbb{C} \mid \sigma_0 \le \Re s \le \sigma_0 + |k-1|\}$, *then* $f(s) = \pm g(s)$.

Proof. We define $r(s) = f(s)/g(s)$. By using (2.10) and (2.11) we have

$$r(s) = \frac{g(k-s)}{f(k-s)} = \frac{f(1-(k-s))}{g(1-(k-s))} = r(s-(k-1)),$$

namely, $r(s)$ is a periodic function with period $|k-1|$. Since $r(s)$ is bounded in D, $r(s)$ is a constant by Liouville's theorem. On the other hand, in the case $s = 1/2$, we have $f(1/2)^2 = g(1/2)^2 = 1$. This implies the lemma. $\square$

Lemma 2.2. *Let $T > 0$. Let $h(s)$ be a meromorphic function on $\mathbb{C}$ and $r(s) := h(s)/h(1-s)$. Assume that $r(s+T) = r(s)$ holds for all $s \in \mathbb{C}$. If there exist*

$$\lim_{s \to +\infty, s \in \mathbb{R}} h(s)$$

and

$$\lim_{s \to +\infty, s \in \mathbb{R}} h(1-s) \neq 0,$$

then $r(s) = 1$ for all $s \in \mathbb{C}$.

Proof. We assume $s \in \mathbb{R}$ and $k \in \mathbb{N}$. We define

$$c := \lim_{s \to +\infty, s \in \mathbb{R}} h(s).$$

Since we have $r(1/2) = 1$, we obtain

$$c = \lim_{k \to +\infty} h(1/2+kT) = \lim_{k \to +\infty} r(1/2+kT)h(1/2-kT) = \lim_{k \to +\infty} h(1/2-kT).$$

Therefore we obtain $\lim_{s \to +\infty} h(1-s) = c \neq 0$. If $r(s)$ is not a constant, then there exists an x such that $r(x) \neq 1$. Hence, we have

$$c = \lim_{k \to +\infty} h(x+kT) = \lim_{k \to +\infty} r(x+kT)h(1-x-kT) = r(x)c,$$

but this is impossible. $\square$

Note that Lemma 2.1 and Lemma 2.2 correspond to assumptions (a) and (b) in Theorem 2.1, respectively.

Lemma 2.3. *Let $g(s_1, s_2)$ be a meromorphic function on $\mathbb{C}^2$. The solution of the functional equation*

$$g(s_1, s_2) + g(s_2, s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2) \tag{2.12}$$

is

$$g(s_1, s_2) = \zeta_2(s_1, s_2) + \varphi(s_1, s_2),$$

where $\varphi(s_1, s_2)$ is a meromorphic function which satisfies $\varphi(s_2, s_1) = -\varphi(s_1, s_2)$.

Proof. Let g be an arbitrary solution of (2.12). We define

$$F(s_1, s_2) = g(s_1, s_2) - \zeta_2(s_1, s_2).$$

By (2.2) and (2.12) we have

$$F(s_1, s_2) = g(s_1, s_2) - \zeta_2(s_1, s_2) = \zeta_2(s_2, s_1) - g(s_2, s_1).$$

This implies $F(s_2, s_1) = -F(s_1, s_2)$. Therefore we can write

$$g(s_1, s_2) = \zeta_2(s_1, s_2) + \varphi(s_1, s_2), \tag{2.13}$$

where $\varphi(s_1, s_2)$ is a meromorphic function which satisfies $\varphi(s_2, s_1) = -\varphi(s_1, s_2)$. On the other hand, (2.13) actually satisfies (2.12). $\square$

Remark 2.1. Let $f(s)$ be a meromorphic function on $\mathbb{C}$. Assume that $f(s)$ does not have a pole at $s = 0$. If $f(s)$ satisfies the functional equation

$$\zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) = f(s_1)f(s_2) - f(s_1 + s_2), \tag{2.14}$$

then $f(s) = \zeta(s)$.

This claim implies that $\zeta(s)$ can be characterized by the functional equation (2.14). We can prove this claim as follows.

By (2.2) we have

$$f(s_1)f(s_2) - f(s_1 + s_2) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2), \tag{2.15}$$

and by setting $s_1 = 0$ and $s_2 = s$, we obtain

$$f(s)(f(0) - 1) = \zeta(s)(\zeta(0) - 1).$$

Since $\zeta(0) = -1/2$, we obtain

$$f(s) = C\zeta(s), \tag{2.16}$$

where C is a constant. By substituting (2.16) into (2.15), we have

$$(C^2 - 1)\zeta(s_1)\zeta(s_2) = (C - 1)\zeta(s_1 + s_2),$$

and with $s_1 = s_2 = s$,

$$(C^2 - 1)\zeta(s)^2 = (C - 1)\zeta(2s),$$

which is possible if and only if $C = 1$. Hence, we obtain $f(s) = \zeta(s)$.

2.3. **Proof of Theorem 2.1.** Now, we prove our main result.

Proof. We define $C(s_1, s_2) = \Gamma(s_2)/\Gamma(1 - s_1)$. In the case $s_1 + s_2 = 3$ we have

$$C(s_1, s_2) = \frac{\Gamma(3 - s_1)}{\Gamma(1 - s_1)} = (s_1 - 1)(s_1 - 2).$$

By this relation, we see $C(s_2, s_1) = C(s_1, s_2)$ in the case $s_1 + s_2 = 3$. On the other hand, we can easily see $\chi(s)\chi(3-s) = -4\pi^2((s-1)(s-2))^{-1}$ by the definition of $\chi(s)$. Therefore, in the case $s_1 + s_2 = 3$, we obtain $\chi(s_1)\chi(s_2) = -4\pi^2(C(s_1, s_2))^{-1}$. Now, we assume $s_1 + s_2 = 3$. By (2.6) we obtain

$$-\frac{1}{4\pi^2}C(s_1, s_2)(f_2(s_1, s_2) - C(s_1, s_2)^{-1}f(2)) = f_2(1-s_2, 1-s_1) + C(s_1, s_2)\frac{f(3)}{8\pi^2}.$$

By interchanging s_1 and s_2, we obtain

$$-\frac{1}{4\pi^2}C(s_1, s_2)(f_2(s_2, s_1) - C(s_1, s_2)^{-1}f(2)) = f_2(1-s_1, 1-s_2) + C(s_1, s_2)\frac{f(3)}{8\pi^2}.$$

By adding the last two equations and using (2.7), we obtain

$$-\frac{1}{4\pi^2}C(s_1,s_2)(f(s_1)f(s_2)-f(3)-2f(2)C(s_1,s_2)^{-1}) = \\ f(1-s_1)f(1-s_2)-f(-1)+2C(s_1,s_2)\frac{f(3)}{8\pi^2},$$

namely,

$$\begin{aligned} f(s_1)f(3-s_1) &= -4\pi^2(C(s_1,s_2))^{-1}f(1-s_1)f(s_1-2) \\ &= \chi(s_1)\chi(3-s_1)f(1-s_1)f(s_1-2) \end{aligned} \tag{2.17}$$

by $f(2)=-2\pi^2 f(-1)$ and (2.9). If we define $K(s)=f(s)/f(1-s)$, then we have $K(s)K(1-s)=1$ and, by (2.17),

$$\chi(s)\chi(3-s)=\frac{f(s)f(3-s)}{f(1-s)f(s-2)}=K(s)K(3-s). \tag{2.18}$$

On the other hand, if we define $r(s)=K(s)/\chi(s)$ and $h(s)=f(s)/\zeta(s)$, then we have

$$r(s)=\frac{f(s)}{\zeta(s)}\cdot\frac{\zeta(1-s)}{f(1-s)} \tag{2.19}$$

by the definition of $r(s)$,

$$r(s)=\frac{\chi(3-s)}{K(3-s)}=r(s-2) \tag{2.20}$$

by (2.18) and the definition of $r(s)$ and

$$h(s)=r(s)h(1-s) \tag{2.21}$$

by (2.1) and the definition of $K(s)$.

First we assume that (a) holds. Since $\zeta(s)\gg 1$ and $f(s)\ll 1$ in the case $\sigma\geq 2$, $f(s)/\zeta(s)$ is bounded in D. By (a) and (2.9), $f'(-2)\neq 0$ and $f(1-s)=0$ in D if and only if $s=3$. Therefore $\zeta(1-s)/f(1-s)$ is bounded in D, namely, by (2.19), $r(s)$ is also bounded in D. Hence, we obtain $K(s)=\pm\chi(s)$ by setting $f=K$ and $g=\chi$ in Lemma 2.1, and we obtain $K(s)=\chi(s)$ by $K(1/2)=\chi(1/2)=1$. This implies $f=\zeta$ by Hamburger's theorem and (2.9).

Next we assume that (b) holds. Note that

$$h(s)=\sum_{n=1}^{\infty}\frac{\sum_{d|n}a_d\mu(n/d)}{n^s}$$

holds, where μ is the Möbius function. By (b) we have

$$\lim_{s\to+\infty}h(1-s)=\lim_{s\to+\infty}\frac{\chi(s)f(1-s)}{\zeta(s)}=c\neq 0$$

for $s \in \mathbb{R}$. Since (2.20) and (2.21) hold, we obtain $K(s) = \chi(s)$ by Lemma 2.2. This implies $f = \zeta$ by Hamburger's theorem and (2.9).

Hereafter, we assume $s_1, s_2 \in \mathbb{C}$, namely, we do not assume $s_1 + s_2 = 3$. If $f = \zeta$, then, by Lemma 2.3, we can write

$$f_2(s_1, s_2) = \zeta_2(s_1, s_2) + \varphi(s_1, s_2),$$

where φ is a meromorphic function which satisfies $\varphi(s_2, s_1) = -\varphi(s_1, s_2)$. The remaining task is to prove $\varphi = 0$. Note that the pair $f_2 = \zeta_2$ and $f = \zeta$ is a solution of (2.8) by Matsumoto's theorem. By subtracting (2.3) from (2.8) we obtain

$$\frac{\varphi(s_1, s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} = \frac{\varphi(1-s_2, 1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)}.$$

If we assume $\varphi \neq 0$, then we can define

$$G(s_1, s_2) = \frac{\varphi(s_1, s_2)}{\varphi(1-s_2, 1-s_1)} = \frac{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)},$$

and we have

$$G(s_2, s_1) = \frac{-\varphi(s_1, s_2)}{-\varphi(1-s_2, 1-s_1)} = G(s_1, s_2).$$

However, this implies that

$$\frac{\Gamma(1-s_1)}{\Gamma(s_2)} = \frac{\Gamma(1-s_2)}{\Gamma(s_1)}$$

holds, namely, $\sin \pi s_1 = \sin \pi s_2$ holds for all $s_1, s_2 \in \mathbb{C}$. This is impossible. This completes the proof. $\square$

Remark 2.2. We guess that if assumption (b) holds, then the choice of special values of $f(s)$ in Theorem 2.1 can be replaced by other special values, namely, we choose hyperplane $s_1 + s_2 = 2k + 1$ $(0 \neq k \in \mathbb{Z})$ instead of the hyperplane $s_1 + s_2 = 3$ in the proof of Theorem 2.1. However, if assumption (b) does not hold, then assumption (a) must be replaced by a more complicate assumption, because we use (2.9) when we determine the zeros of $f(1-s)$ in D.

3. On the lcm-sum function

The contents in this section are a joint work with Kaneaki Matsuoka (Graduate School of Mathematics, Nagoya University) and based on [11].

3.1. **Introduction.** Pillai [22] first defined the gcd-sum function

$$g(n) = \sum_{j=1}^{n} \gcd(j, n)$$

and studied this function. The function $g(n)$ was defined again by Broughan [5]. Broughan considered

$$G_\alpha(x) = \sum_{n \le x} n^{-\alpha} g(n)$$

for $\alpha \in \mathbb{R}$ and $x \ge 1$, and obtained some asymptotic formulas for $G_\alpha(x)$. The function $G_\alpha(x)$ was studied by some authors (see, for example, [6, 25]). Some generalizations of the function $g(n)$ was considered (see, for example, [4, 27]).

On the other hand, the lcm-sum function

$$l(n) := \sum_{j=1}^{n} \mathrm{lcm}(j, n)$$

was considered by some authors. Alladi [2] studied the sum

$$\sum_{j=1}^{n} (\mathrm{lcm}(j, n))^r \qquad (r \in \mathbb{R},\ r \ge 1)$$

and obtained

$$\sum_{n \le x} \sum_{j=1}^{n} (\mathrm{lcm}(j, n))^r = \frac{\zeta(r+2)}{2(r+1)^2 \zeta(2)} x^{2r+2} + O(x^{2r+1+\epsilon}). \tag{3.1}$$

We define the functions

$$L_a(n) := \sum_{j=1}^{n} (\mathrm{lcm}(j, n))^a$$

$$T_a(x) := \sum_{n \le x} L_a(n)$$

for $a \in \mathbb{Z}$ and $x \ge 1$.

Bordellès studied the sums $T_1(x)$ and $T_{-1}(x)$ and obtained

$$l(n) = \frac{1}{2}((\mathrm{Id}^2 \cdot (\varphi + \tau_0)) * \mathrm{Id})(n),$$

$$\sum_{n \le x} \sum_{j=1}^{n} \mathrm{lcm}(j, n) = \frac{\zeta(3)}{8\zeta(2)} x^4 + O(x^3 (\log x)^{2/3} (\log\log x)^{4/3}) \quad (x > e),$$

$$\sum_{n \le x} \sum_{j=1}^{n} \frac{1}{\mathrm{lcm}(j, n)} = \frac{(\log x)^3}{6\zeta(2)} + \frac{(\log x)^2}{2\zeta(2)} \left(\gamma + \log\left(\frac{\mathcal{A}^{12}}{2\pi} \right) \right) + O(\log x),$$

where $\mathrm{Id}^a(n) = n^a$ $(a \in \mathbb{Z})$,

$$\tau_0(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{otherwise}, \end{cases}$$

$F * G$ is the usual Dirichlet convolution product and $\mathcal{A}$ is the Glaisher-Kinkelin constant [24, p. 25]). Gould and Shonhiwa [9] stated that the log −factors in the error term in the second formula can be removed.

In this paper we study $T_a(x)$ for $a \ge 2$ or $a \le -2$. The following theorems are our main results. These results are proved by the methods similar to those of Bordellès [4, Section 6].

We write $f(x) = O(g(x))$, or equivalently $f(x) \ll g(x)$, where there is a constant $C > 0$ such that $|f(x)| \le Cg(x)$ for all values of x under consideration.

Theorem 3.1. *Let B_n be Bernoulli numbers defined by*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$

If we define

$$\varphi_k(n) := \sum_{d \mid n} \mu(d) \left(\frac{d}{n} \right)^k$$

and

$$M_a(n) := \left(\mathrm{Id}^{2a} \cdot \left(\frac{1}{a+1} \varphi + \frac{1}{2} \tau_0 + \frac{1}{a+1} \sum_{k=1}^{a-1} \binom{a+1}{k+1} B_{k+1} \varphi_k \right) \right)(n),$$

then for $a \in \mathbb{Z}$ we have

$$L_a(n) = (M_a * \mathrm{Id}^a)(n).$$

Theorem 3.2. *Let $x > e$ and $a \in \mathbb{N}$. Then we have*

$$T_a(x) = \frac{\zeta(a+2)}{2(a+1)^2 \zeta(2)} x^{2a+2} + O(x^{2a+1} (\log x)^{2/3} (\log\log x)^{4/3}) \quad (\textit{as } x \to \infty),$$

where the implied constant depends on a.

Theorem 3.3. *Let $x \geq 1$ and $k \in \mathbb{N}$ with $k \geq 2$. Then we have*

$$\sum_{n=1}^{\infty} L_{-k}(n) = \frac{\zeta(k)}{2}\left(1 + \frac{\zeta(k)^2}{\zeta(2k)}\right) \tag{3.2}$$

and

$$T_{-k}(x) = \frac{\zeta(k)}{2}\left(1+\frac{\zeta(k)^2}{\zeta(2k)}\right) - \frac{\zeta(k)x^{-k+1}\log x}{(k-1)\zeta(k+1)} + O(x^{-k+1}) \quad (\textit{as } x \to \infty),$$

where the implied constant depends on k.

We note that the function $L_a(n)$ is not multiplicative for all $a \in \mathbb{Z} \setminus \{0\}$, but we can write $L_a(n)$ $(a \geq 1)$ explicitly by Dirichlet convolution. In the proof of Theorem 3.2 we use this fact. The error estimates in Theorem 3.2 are better than (3.1). Since we have

$$g_r(n) := \sum_{j=1}^{n} (\gcd(j,n))^r > \varphi(n)$$

for all $r \in \mathbb{R}$, the sum

$$\sum_{n=1}^{\infty} g_r(n)$$

is divergent for all r. Therefore the behavior of the sum $T_a(x)$ $(a \in \mathbb{Z}$ and $a \leq -2)$ is completely different from that of the sum

$$\sum_{n \leq x} g_a(n).$$

3.2. **Lemmas for the proof of theorems.** In this section, we collect some auxiliary results and definitions.

Let $B_n(x)$ be Bernoulli polynomials defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$

The following relations are well-known [24, p. 59].

$$B_n(x+1) - B_n(x) = nx^{n-1},$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

$$B_n(0) = B_n(1) = B_n \quad (n > 1).$$

Lemma 3.1. *Let $m, n \in \mathbb{N}$ and*

$$S_n(m) := \sum_{l=1}^{m} l^n.$$

Then we have

$$S_n(m) = \frac{m^{n+1}}{n+1} + \frac{1}{2}m^n + \frac{1}{n+1}\sum_{k=1}^{n-1}\binom{n+1}{k+1}B_{k+1}m^{n-k}.$$

Proof. We have

$$\begin{aligned}
S_n(m) &= \frac{1}{n+1}(B_{n+1}(m+1) - B_{n+1}(1)) \\
&= \frac{1}{n+1}(B_{n+1}(m) + (n+1)m^n - B_{n+1}) \\
&= \frac{1}{n+1}\left(\sum_{k=0}^{n+1}\binom{n+1}{k}B_k m^{n+1-k} + (n+1)m^n - B_{n+1}\right) \\
&= \frac{m^{n+1}}{n+1} + \frac{1}{2}m^n + \frac{1}{n+1}\sum_{k=1}^{n-1}\binom{n+1}{k+1}B_{k+1}m^{n-k}.
\end{aligned}$$

□

We use the following lemmas in the proof of Theorem 3.2.

Lemma 3.2. *Let $r, k \in \mathbb{N}$ with $r > k$ and $x \geq 1$. We have*

$$\sum_{n \leq x} n^r \varphi_k(n) \leq x^{r+1}.$$

Proof. We have

$$\begin{aligned}
\sum_{n \leq x} n^r \varphi_k(n) &= \sum_{n \leq x} n^{r-k} \sum_{d|n} \mu(d) d^k \\
&= \sum_{d \leq x} \mu(d) d^k \big(d^{r-k} + (2d)^{r-k} + \cdots + (d\lfloor x/d \rfloor)^{r-k}\big) \\
&= \sum_{d \leq x} \mu(d) d^r \sum_{j \leq x/d} j^{r-k} \\
&\leq x^{r+1}.
\end{aligned}$$

□

Lemma 3.3. *Let $r \in \mathbb{N}$ and $x > e$. We have*

$$\sum_{n \leq x} n^r \varphi(n) = \frac{x^{r+2}}{(r+2)\zeta(2)} + O(x^{r+1}(\log x)^{2/3}(\log\log x)^{4/3}) \quad (as\ x \to \infty),$$

where the implied constant depends on r.

Proof. We can obtain the lemma by the estimate [29]

$$\sum_{n\le x}\varphi(n)=\frac{x^2}{2\zeta(2)}+O(x(\log x)^{2/3}(\log\log x)^{4/3})$$

and the partial summation formula. □

3.3. **Proof of Theorem 3.1 and Theorem 3.2.**

Proof of Theorem 3.1. We have

$$\begin{aligned}\sum_{j=1}^{n}\left(\frac{j}{\gcd(n,j)}\right)^a&=\sum_{d|n}\frac{1}{d^a}\sum_{\substack{j=1\\ \gcd(j,n)=d}}^{n}j^a\\&=\sum_{d|n}\frac{1}{d^a}\sum_{\substack{k\le n/d\\ \gcd(k,n/d)=1}}(kd)^a=\sum_{d|n}\sum_{\substack{k\le n/d\\ \gcd(k,n/d)=1}}k^a.\end{aligned}$$

By Lemma 3.1 we have

$$\begin{aligned}\sum_{\substack{k\le N\\ \gcd(k,N)=1}}k^a&=\sum_{k\le N}k^a\sum_{d|\gcd(k,N)}\mu(d)=\sum_{d|N}d^a\mu(d)\sum_{m\le N/d}m^a\\&=\sum_{d|N}d^a\mu(d)\Bigg(\frac{1}{a+1}\left(\frac{N}{d}\right)^{a+1}+\frac{1}{2}\left(\frac{N}{d}\right)^{a}+\\&\qquad+\frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\left(\frac{N}{d}\right)^{a-k}\Bigg)\\&=\frac{N^a}{a+1}\sum_{d|N}\mu(d)\frac{N}{d}+\frac{N^a}{2}\sum_{d|N}\mu(d)+\\&\qquad+\frac{N^a}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\sum_{d|N}\mu(d)\left(\frac{d}{N}\right)^k\\&=N^a\left(\frac{1}{a+1}\varphi+\frac{1}{2}\tau_0+\frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\varphi_k\right)(N).\end{aligned}$$

Hence we obtain

$$
\begin{aligned}
L_a(n) &= n^a \sum_{j=1}^{n} \left(\frac{j}{\gcd(n,j)} \right)^a \\
&= \sum_{d|n} \left(\frac{n}{d} \right)^{2a} \cdot \left(\frac{1}{a+1}\varphi + \frac{1}{2}\tau_0 + \frac{1}{a+1} \sum_{k=1}^{a-1} \binom{a+1}{k+1} B_{k+1}\varphi_k \right)(n/d) \cdot d^a \\
&= (M_a * \mathrm{Id}^a)(n).
\end{aligned}
$$

□

Proof of Theorem 3.2. By Lemma 3.2, Lemma 3.3 and Theorem 3.1, we have

$$
\begin{aligned}
\sum_{n\le x} L_a(n) &= \sum_{n\le x} (M_a * \mathrm{Id}^a)(n) = \sum_{d\le x} d^a \sum_{m\le x/d} M_a(m) \\
&= \sum_{d\le x} d^a \sum_{m\le x/d} m^{2a} \left(\frac{1}{a+1}\varphi + \frac{1}{2}\tau_0 + \frac{1}{a+1} \sum_{k=1}^{a-1} \binom{a+1}{k+1} B_{k+1}\varphi_k \right)(m) \\
&= \frac{1}{a+1} \sum_{d\le x} d^a \sum_{m\le x/d} m^{2a}\varphi(m) + O(x^{a+1}) + \\
&\quad + \frac{1}{a+1} \sum_{k=1}^{a-1} \binom{a+1}{k+1} B_{k+1} \sum_{d\le x} d^a \sum_{m\le x/d} m^{2a}\varphi_k(m) \\
&= \frac{1}{a+1} \sum_{d\le x} d^a \left(\frac{1}{(2a+2)\zeta(2)} \left(\frac{x}{d} \right)^{2a+2} + \right. \\
&\quad \left. + O\left(\left(\frac{x}{d} \right)^{2a+1} (\log(x/d))^{2/3} (\log\log(x/d))^{4/3} \right) \right) + \\
&\quad + O(x^{a+1}) + O\left(\sum_{d\le x} d^a (x/d)^{2a+1} \right) \\
&= \frac{x^{2a+2}}{(a+1)(2a+2)\zeta(2)} \sum_{d\le x} \frac{1}{d^{a+2}} + O(x^{2a+1}(\log x)^{2/3}(\log\log x)^{4/3}) + \\
&\quad + O(x^{2a+1}).
\end{aligned}
$$

This implies the theorem. □

3.4. **Proof of Theorem 3.3.**

Proof of Theorem 3.3. Since we have

$$\begin{aligned}
L_{-k}(n) &= \sum_{j=1}^{n} \frac{1}{(\mathrm{lcm}(n,j))^k} = \frac{1}{n^k}\sum_{j=1}^{n} \frac{(\gcd(n,j))^k}{j^k} = \frac{1}{n^k}\sum_{d|n} d^k \sum_{\substack{j=1 \\ \gcd(j,n)=d}}^{n} \frac{1}{j^k} \\
&= \frac{1}{n^k}\sum_{d|n} d^k \sum_{\substack{i\le \frac{n}{d} \\ \gcd(i,\frac{n}{d})=1}} \frac{1}{i^k d^k} \\
&= \frac{1}{n^k}\sum_{d|n} \sum_{\substack{i\le \frac{n}{d} \\ \gcd(i,\frac{n}{d})=1}} \frac{1}{i^k},
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} L_{-k}(n) &= \sum_{n=1}^{\infty} \frac{1}{n^k}\sum_{d|n} \sum_{\substack{i\le \frac{n}{d} \\ \gcd(i,\frac{n}{d})=1}} \frac{1}{i^k} = \sum_{d=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{j^k d^k} \sum_{\substack{i\le j \\ \gcd(i,j)=1}} \frac{1}{i^k} \\
&= \zeta(k)\sum_{j=1}^{\infty} \frac{1}{j^k} \sum_{\substack{i\le j \\ \gcd(i,j)=1}} \frac{1}{i^k} \\
&= \zeta(k)\sum_{n=1}^{\infty} \frac{1}{n^k} \Big(\sum_{\substack{i\le j \\ \gcd(i,j)=1 \\ ij=n}} 1 \Big).
\end{aligned}$$

Also we have

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \Big(\sum_{\substack{i\le j \\ \gcd(i,j)=1 \\ ij=n}} 1 \Big) = 1 + \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n^k} \Big(\sum_{\substack{\gcd(i,j)=1 \\ ij=n}} 1 \Big)$$

and the relation [26, 1.2.8]

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \Big(\sum_{\substack{\gcd(i,j)=1 \\ ij=n}} 1 \Big) = \frac{\zeta(k)^2}{\zeta(2k)}.$$

Therefore we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \Big(\sum_{\substack{i\le j \\ \gcd(i,j)=1 \\ ij=n}} 1 \Big) = 1 + \frac{1}{2}\left(\frac{\zeta(k)^2}{\zeta(2k)} - 1 \right) = \frac{1}{2}\left(1 + \frac{\zeta(k)^2}{\zeta(2k)} \right).$$

This implies (3.2).

By the relation

$$\sum_{n\le x} L_{-k}(n) = \frac{\zeta(k)}{2}\left(1+\frac{\zeta(k)^2}{\zeta(2k)}\right) - \sum_{n>x} L_{-k}(n),$$

the remaining task is to estimate the sum $\sum_{n>x} L_{-k}(n)$. We have

$$\begin{aligned}
\sum_{n>x} L_{-k}(n) &= \sum_{n>x}\frac{1}{n^k}\sum_{d|n}\sum_{\substack{i\le \frac{n}{d}\\ \gcd(i,\frac{n}{d})=1}}\frac{1}{i^k} = \sum_{d=1}^{\infty}\sum_{h>\frac{x}{d}}\frac{1}{(hd)^k}\sum_{\substack{j\le h\\ \gcd(j,h)=1}}\frac{1}{j^k}\\
&= \sum_{d=1}^{\infty}\sum_{h>\frac{x}{d}}\frac{1}{(hd)^k}\sum_{j\le h}\frac{1}{j^k}\sum_{\delta|\gcd(j,h)}\mu(\delta)\\
&= \sum_{d=1}^{\infty}\sum_{h>\frac{x}{d}}\frac{1}{(hd)^k}\sum_{\delta|h}\sum_{m\le\frac{h}{\delta}}\frac{\mu(\delta)}{m^k\delta^k}\\
&= \sum_{d=1}^{\infty}\frac{1}{d^k}\sum_{\delta=1}^{\infty}\sum_{l>\frac{x}{d\delta}}\frac{\mu(\delta)}{l^k\delta^{2k}}\sum_{m\le l}\frac{1}{m^k}\\
&= \sum_{q=1}^{\infty}\frac{1}{q^{2k}}\sum_{d\delta=q}d^k\mu(\delta)\sum_{l>\frac{x}{q}}\frac{1}{l^k}\sum_{m\le l}\frac{1}{m^k}\\
&= \sum_{q=1}^{\infty}\frac{1}{q^{2k}}\sum_{d\delta=q}d^k\mu(\delta)\sum_{l>\frac{x}{q}}\frac{1}{l^k}\left(\zeta(k)-\frac{l^{1-k}}{k-1}+O(l^{-k})\right)\\
&= \sum_{q<x}\frac{1}{q^{2k}}\sum_{d\delta=q}d^k\mu(\delta)\sum_{l>\frac{x}{q}}\frac{1}{l^k}\left(\zeta(k)-\frac{l^{1-k}}{k-1}+O(l^{-k})\right)+\\
&\qquad +\sum_{q\ge x}\frac{1}{q^{2k}}\sum_{d\delta=q}d^k\mu(\delta)\sum_{l>\frac{x}{q}}\frac{1}{l^k}\left(\zeta(k)-\frac{l^{1-k}}{k-1}+O(l^{-k})\right)\\
&=: S_1+S_2,
\end{aligned}$$

say. We have

$$\begin{aligned}
S_1 &= \sum_{q<x}\frac{1}{q^{2k}}\sum_{d\delta=q}d^k\mu(\delta)\sum_{l>\frac{x}{q}}\frac{1}{l^k}\left(\zeta(k)-\frac{l^{1-k}}{k-1}+O(l^{-k})\right)\\
&= \sum_{q<x}\frac{1}{q^{2k}}\sum_{d\delta=q}d^k\mu(\delta)\left(\frac{\zeta(k)}{k-1}(x/q)^{-k+1}+O((x/q)^{-k})-\right.\\
&\qquad\left.-\frac{(\frac{x}{q})^{-2k+2}}{(k-1)(2k-2)}+O((x/q)^{-2k+1})\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q<x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \left(\frac{\zeta(k)}{k-1} (x/q)^{-k+1} + O((x/q)^{-k}) \right) \\
&= \sum_{q<x} \frac{1}{q^{k}} \sum_{d|q} \frac{\mu(d)}{d^k} \left(\frac{\zeta(k)}{k-1} (x/q)^{-k+1} + O((x/q)^{-k}) \right) \\
&= \frac{\zeta(k) x^{-k+1}}{k-1} \sum_{q<x} q^{-1} \sum_{d|q} \frac{\mu(d)}{d^k} + O\left(x^{-k} \sum_{q<x} \sum_{d|q} \frac{|\mu(d)|}{d^k} \right) \\
&= \frac{\zeta(k) x^{-k+1}}{k-1} \sum_{d<x} \sum_{j<\frac{x}{d}} j^{-1} \frac{\mu(d)}{d^{k+1}} + O(x^{-k+1}) \\
&= \frac{\zeta(k) x^{-k+1}}{k-1} \sum_{d<x} \log\left(\frac{x}{d}\right) \frac{\mu(d)}{d^{k+1}} + O(x^{-k+1}) \\
&= \frac{\zeta(k) x^{-k+1} \log x}{(k-1)\zeta(k+1)} + O(x^{-k+1})
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{q\geq x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l>\frac{x}{q}} \frac{1}{l^k} \left(\zeta(k) - \frac{l^{1-k}}{k-1} + O(l^{-k}) \right) \\
&\ll \sum_{q\geq x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l=1}^{\infty} \frac{1}{l^k} \left(\zeta(k) - \frac{l^{1-k}}{k-1} + O(l^{-k}) \right) \\
&\ll \sum_{q\geq x} q^{-k} \sum_{d|q} \frac{|\mu(d)|}{d^k} \\
&\ll x^{-k+1}.
\end{aligned}
$$

Therefore we obtain

$$
\sum_{n>x} L_{-k}(n) = \frac{\zeta(k) x^{-k+1} \log x}{(k-1)\zeta(k+1)} + O(x^{-k+1}).
$$

This completes the proof. $\square$

4. A new construction of the real numbers by alternating series

The contents of this section are based on [10].

4.1. **Introduction.** The purpose of this paper is to present a new method of constructing the complete ordered field of real numbers from the ordered field of rational numbers. Our method is similar to the method which was established by A. Knopfmacher and J. Knopfmacher in [18], but our method is more general. Moreover our result gives infinitely many ways of constructing the complete ordered field of real numbers. As an application of our results, we prove the irrationality of certain series.

A. Knopfmacher and J. Knopfmacher constructed the complete ordered field of real numbers by the Sylvester expansion and the Engel expansion in [17] and by the alternating-Sylvester expansion and the alternating-Engel expansion in [18]. The advantages of these constructions are the fact that those are concrete and do not depend on the notion of equivalence classes. The alternating-Sylvester expansion and the alternating-Engel expansion are generalizations of Oppenheim's expansion (see [21]) and special cases of the alternating Balkema-Oppenheim's expansion (see [13]), which were introduced by A. Knopfmacher and J. Knopfmacher in [18]. The definition of the alternating-Sylvester expansion and the alternating-Engel expansion are the following.

(i) **Alternating-Sylvester expansion**. Let $\alpha \in \mathbb{R}$, $a_0 = [\alpha]$ and $A_1 = \{\alpha\}$, where $\{x\} = x - [x]$. We define, for $n \in \mathbb{N}$ and $A_n > 0$,

$$a_n = \left[\frac{1}{A_n}\right]$$

and

$$A_{n+1} = \frac{1}{a_n} - A_n.$$

Then

$$\alpha = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \dots, \tag{4.1}$$

where $a_1 \geq 1$ and $a_{n+1} \geq a_n(a_n + 1)$ for $n \in \mathbb{N}$.

(ii) **Alternating-Engel expansion**. Let $\alpha \in \mathbb{R}$, $a_0 = [\alpha]$ and $A_1 = \{\alpha\}$. We define, for $n \in \mathbb{N}$ and $A_n > 0$,

$$a_n = \left[\frac{1}{A_n}\right]$$

and

$$A_{n+1} = 1 - a_n A_n.$$

Then

$$\alpha = a_0 + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \dots, \tag{4.2}$$

where $a_1 \geq 1$ and $a_{n+1} \geq a_i + 1$ for $n \in \mathbb{N}$.

The relation

$$\frac{1}{d+1} < \alpha \leq \frac{1}{d} \qquad (\alpha \in (0,1],\ d = [\alpha^{-1}]) \tag{4.3}$$

is used in these expansions. We introduce a new series expansion for every real numbers by using a more general relation

$$\frac{c}{d+1} < \alpha \leq \frac{c}{d} \qquad (\alpha \in (0,1],\ c \in \mathbb{N},\ d = [c\alpha^{-1}]).$$

Definition 4.1 (Generalized alternating-Sylvester expansion)**.** *Let $\alpha \in \mathbb{R}$, $q_0 = [\alpha]$ and $A_1 = \{\alpha\}$. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. We define, for $n \in \mathbb{N}$,*

$$a_n = \left[\frac{c_n}{A_n}\right] \qquad (\textit{for } A_n \neq 0),$$

$$q_n = \begin{cases} \frac{c_n}{a_n} & (A_n \neq 0) \\ 0 & (A_n = 0) \end{cases}$$

and

$$A_{n+1} = q_n - A_n.$$

Then

$$\alpha = q_0 + \sum_{n=1}^{\infty} (-1)^{n-1} q_n. \tag{4.4}$$

If we regard the alternating-Sylvester series (4.1) as an analogue of the regular continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

the generalized alternating-Sylvester series (4.4) is an analogue of the continued fraction

$$a_0 + \cfrac{c_1}{a_1 + \cfrac{c_2}{a_2 + \cfrac{c_3}{a_3 + \cdots}}}.$$

By taking some appropriate $\{c_n\}$, we can get a simple continued fraction representation for some real numbers. For example, we have (see (2.1.22) in [15])

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cdots}}}}.$$

On the other hand, the regular continued fraction representation of π is complicate. Therefore we can expect that if we take some appropriate $\{c_n\}$, then we can get a simple series representation for some real numbers. In fact we can prove irrationality of certain numbers by using such a representation.

The outline of this paper is the following. In Section 4.2 we study some fundamental properties of the generalized alternating-Sylvester series. In Section 4.3 we take an arbitrary sequence of positive integers $\{c_n\}_{n=1}^{\infty}$ such that $c_n \mid c_{n+1}$ for all $n \in \mathbb{N}$, and we prove that the set

$$S(\{c_n\}) = \{\{q_n\}_{n=0}^{\infty} \mid \{q_n\} \text{ appears in (4.4)}\} \tag{4.5}$$

can be identified with the complete ordered field of real numbers $\mathbb{R}$ by introducing the relation $<$ and the operator $+$ and $\cdot$. In other words we prove that $S(\{c_n\})$ becomes an ordered field which is isomorphic to $\mathbb{R}$. Since there exist infinitely many $\{c_n\}$ such that $c_n \mid c_{n+1}$, this implies that there exist infinitely many ways of constructing the complete ordered field of real numbers. Our construction is similar to that in [18]. Therefore our construction is also concrete and does not use the notion of equivalence classes. When we prove that $S(\{c_n\})$ becomes an ordered field, we use a general lemma (see Lemma 4.7). It seems that this lemma can be used in [16], [17] and [18]. In Section 4.4, we prove the irrationality of certain series by Proposition 4.3 and Proposition 4.4.

Remark 4.1. At first glance, it seems that we can define generalized alternating-Engel series as follows :

Let $\alpha \in \mathbb{R}$, $A_1 = \alpha - a_0$ with $0 < A_1 \leq 1$, $a_0 \in \mathbb{Z}$. Let $\{c_n\}$ be a sequence of positive integers. We define, for $n \in \mathbb{N}$ and $A_n \neq 0$,

$$a_n = \left[\frac{c_n}{A_n}\right]$$

and

$$A_{n+1} = c_n - a_n A_n.$$

Then

$$\alpha = a_0 + \frac{c_1}{a_1} - \frac{c_2}{a_1 a_2} + \frac{c_3}{a_1 a_2 a_3} - \dots .$$

However, $a_{n+1} \geq a_n$ does not hold in this series. For example, if we set $A_1 = \alpha = 5/7$, $c_1 = 2$ and $c_2 = 1$, then $a_1 = 2$, $A_2 = 4/7$ and $a_2 = 1$. This is a trouble. In order to simplify the argument we do not argue on this series.

4.2. **Fundamental properties of the generalized alternating Sylvester series.** In this section, we take an arbitrary sequence of positive integers $\{c_n\}_{n=1}^{\infty}$ and fix it.

Proposition 4.1. *The generalized alternating-Sylvester series has the following properties for $n \in \mathbb{N}$.*

(1) *If $A_n \neq 0$, then we have*

$$\frac{c_n}{a_n + 1} < A_n \leq \frac{c_n}{a_n}.$$

(2) *If $A_{n+1} \neq 0$, then we have*

$$a_{n+1} + 1 > \frac{c_{n+1}}{c_n} a_n (a_n + 1).$$

(3) *The inequality $A_n \geq A_{n+1}$ holds. If $A_n \neq 0$, then we even have $A_n > A_{n+1}$.*
(4) *The inequality $q_n \leq 1$ holds.*
(5) *If $A_{n+1} \neq 0$, then we have $a_{n+1} > a_n$.*
(6) *If $A_n \neq 0$, then we have $A_{n+1} < \frac{1}{a_n+1}$.*
(7) *The inequality $q_n \geq q_{n+1}$ holds. If $q_{n+1} \neq 0$, then we even have $q_n > q_{n+1}$.*

Proof. (1) This trivially follows from the definition of the generalized alternating-Sylvester expansion.

(2) From (1) and the definition, we have

$$\begin{aligned} a_{n+1}+1 &> \frac{c_{n+1}}{A_{n+1}} \\ &= \frac{c_{n+1}}{\frac{c_n}{a_n}-A_n} \\ &> \frac{c_{n+1}}{\frac{c_n}{a_n}-\frac{c_n}{a_n+1}} \\ &= \frac{c_{n+1}}{c_n}a_n(a_n+1). \end{aligned}$$

(3) In the case $A_n = 0$, we have $A_n \geq A_{n+1}$. For $A_n \neq 0$, we have

$$A_{n+1} < \frac{c_n}{a_n} - \frac{c_n}{a_n+1} \leq \frac{c_n}{a_n+1} < A_n.$$

(4) By (3), we have $A_n < 1$ for all n. Hence,

$$a_n = \left[\frac{c_n}{A_n}\right] \geq c_n$$

holds. This implies (4).

(5) From (2), we have (5) by using (4).

(6) By (4), we have

$$A_{n+1} < \frac{c_n}{a_n} - \frac{c_n}{a_n+1} = \frac{c_n}{a_n(a_n+1)} \leq \frac{1}{a_n+1}.$$

(7) In the case $q_{n+1} = 0$, we have $q_n \geq q_{n+1}$. For $q_{n+1} \neq 0$, we have

$$q_{n+1} < \frac{c_{n+1}}{c_{n+1}{q_n}^{-1}(a_n+1)-1} \leq \frac{c_{n+1}}{c_{n+1}a_n + c_{n+1} - 1} \leq \frac{1}{a_n} \leq q_n$$

by (2) and (4). □

Remark 4.2. Since we have

$$\sum_{k=1}^{n}(-1)^{k-1}q_k = A_1 + (-1)^{n-1}A_{n+1} \qquad \text{(for all } n \in \mathbb{N}\text{)},$$

the series in (4.4) converges by Proposition 4.1 (5), (6). Hence,

$$(4.6)\quad (-1)^{n-1}\sum_{k=n}^{\infty}(-1)^{k-1}q_k = (-1)^{n-1}\sum_{k=n}^{\infty}(-1)^{k-1}(A_{k+1}+A_k) = A_n$$

holds for all $n \in \mathbb{N}$.

In order to prove Proposition 4.2 we require some lemmas.

We can easily see that the following lemma holds.

Lemma 4.1. *Let* $c, d \in \mathbb{N}$ *and* $\alpha \in (0, 1]$. *Then*

(1) *there does not exist* $d' \in \mathbb{Z}$ *such that*

$$\frac{c}{d+1} < \frac{c}{d'} < \frac{c}{d},$$

(2) $d = [c\alpha^{-1}]$ *is equivalent to*

$$\frac{c}{d+1} < \alpha \leq \frac{c}{d}.$$

Lemma 4.2. *Let* $\alpha, \alpha' \in (0,1]$, $c \in \mathbb{N}$, $d = [c/\alpha]$ *and* $d' = [c/\alpha']$. *If* $c/d \neq c/d'$ *then* $\alpha < \alpha'$ *is equivalent to* $c/d < c/d'$.

Proof. First, we assume $\alpha < \alpha'$. Since $c/(d+1) < \alpha \leq c/d$ and $c/(d'+1) < \alpha' \leq c/d'$ hold by Lemma 4.1 (2), it is sufficient that we consider the following cases.

(1) $\alpha < \alpha' \leq c/d$.
(2) $c/(d'+1) < \alpha \leq c/d < \alpha'$.
(3) $\alpha \leq c/(d'+1) < \alpha'$.

If (1) holds, then we have

$$\frac{c}{d+1} < \alpha < \alpha' \leq \frac{c}{d}.$$

This implies that $c/d = c/d'$ by Lemma 4.1 (2), which is impossible.

If (2) holds, then we have

$$\frac{c}{d'+1} < \frac{c}{d} < \alpha' \leq \frac{c}{d'},$$

which is impossible by Lemma 4.1 (1).

If (3) holds, then we have

$$\alpha \leq \frac{c}{d} \leq \frac{c}{d'+1} < \alpha' < \frac{c}{d'}$$

by Lemma 4.1 (1).

Next, we assume $c/d < c/d'$. Since $c/(d'+1) < c/d$ is impossible by Lemma 4.1 (1), we have

$$\alpha \leq \frac{c}{d} \leq \frac{c}{d'+1} < \alpha'.$$

□

Proposition 4.2. *Let* $\alpha, \alpha' \in \mathbb{R}$ *with* $\alpha \neq \alpha'$. *We define* ${a_n}'$, ${A_n}'$ *and* q'_n *as* a_n, A_n *and* q_n *which appear in the generalized alternating Sylvester expansion of* α', *respectively. Let*

$$i = \min\{j \in \mathbb{N} \cup \{0\} \mid q_j \neq q'_j\}.$$

Then $\alpha < \alpha'$ is equivalent to

$$\begin{cases} q_i < q_i' & (i = 0 \text{ or } 2 \nmid i), \\ q_i > q_i' & (2 \mid i \text{ and } i \geq 2). \end{cases}$$

Proof. First, we consider the case $i = 0$. If $\alpha < \alpha'$, then we have $q_0 = [\alpha] \leq [\alpha'] = q_0'$. Therefore we obtain $q_0 < q_0'$. On the other hand, if $q_0 < q_0'$, then we have $[\alpha] < [\alpha']$. Therefore we obtain $\alpha < \alpha'$.

Next, we assume $i \neq 0$. Then we can write
(4.7)

$$\alpha = q_0 + \sum_{k=1}^{i-1} (-1)^{k-1} q_k + (-1)^{i-1} A_i, \qquad \alpha' = q_0 + \sum_{k=1}^{i-1} (-1)^{k-1} q_k + (-1)^{i-1} A_i'$$

by Remark 4.2. These relations imply that $\alpha < \alpha'$ is equivalent to

$$\begin{cases} A_i < A_i' & (2 \nmid i), \\ -A_i < -A_i' & (2 \mid i \text{ and } i \geq 2). \end{cases}$$

By Proposition 4.1 (1) and Lemma 4.2, this is equivalent to

$$\begin{cases} q_i < q_i' & (2 \nmid i), \\ q_i > q_i' & (2 \mid i \text{ and } i \geq 2). \end{cases}$$

This implies the proposition. □

In order to consider the case $\alpha \in \mathbb{Q}$ we prove the next lemma.

Lemma 4.3. *Let $c \in \mathbb{N}$ and $p/q \in \mathbb{Q} \cap (0, 1]$ with $p, q \in \mathbb{N}$. Let $d = [cq/p]$. Then the numerator of $c/d - p/q$ is less than p. In other words, $cq - dp < p$.*

Proof. We have

$$cq - dp = cq - \left(\frac{cq}{p} - \left\{ \frac{cq}{p} \right\} \right) p \leq \frac{p-1}{p} p = p - 1.$$

□

Proposition 4.3. *The real number α is rational if and only if there exists an $m \in \mathbb{N}$ such that $q_m = 0$.*

Proof. If there exists an $m \in \mathbb{N}$ such that $q_m = 0$, then α is rational. We assume $\alpha = p/q$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. Without loss of generality, we may assume that $q_0 = 0$, $A_1 = p/q$ and $p, q > 0$. By the definition of a_n, A_n and Lemma 4.3, the numerator of A_n is strictly monotonically decreasing. This implies the proposition. □

Propositions 4.1, 4.2 and 4.3 imply that the generalized alternating-Sylvester series is similar to alternating-Sylvester series.

4.3. **Construction of the real numbers.** In this section we take an arbitrary sequence of positive integers $\{c_n\}_{n=1}^{\infty}$ which satisfies the condition $c_n \mid c_{n+1}$ for all $n \in \mathbb{N}$ and fix it. Moreover we identify $\{q_n\}_{n=0}^{\infty} \in S(\{c_n\})$ with $(q_0, q_1, q_2, \dots)$.

Remark 4.3. By the condition $c_n \mid c_{n+1}$ for any $n \in \mathbb{N}$, the inequality in Proposition 4.1 (2) becomes

$$a_{n+1} \geq \frac{c_{n+1}}{c_n} a_n(a_n + 1).$$

If the equality holds in the above and $q_{n+2} = 0$, then we have

$$A_n = q_n - q_{n+1} = \frac{c_n}{a_n + 1}.$$

This contradicts the definition of q_n. Hence, $q_{n+2} \neq 0$ or

$$a_{n+1} > \frac{c_{n+1}}{c_n} a_n(a_n + 1)$$

holds.

In Section 1, we assumed the existence of the real numbers, and we defined $S(\{c_n\})$ in (4.5). In order to use $S(\{c_n\})$ for the construction of the real numbers, here we remove that assumption.

First we will define a set of sequences of rational numbers $T(\{c_n\})$. We will prove $S(\{c_n\}) = T(\{c_n\})$ in Proposition 4.4. For the sake of simplicity, we define a set of sequences of positive integers

$$U(\{c_n\}) := \left\{ \{a_n\} \subset \mathbb{N} \mid \forall n \in \mathbb{N} \left[a_{n+1} \geq \frac{c_{n+1}}{c_n} a_n(a_n + 1) \right] \right\}.$$

Definition 4.2. *Let $\{q_n\}_{n=0}^{\infty}$ be a sequence of rational numbers. We define $\{q_n\} \in T(\{c_n\})$ if and only if*

(1) $q_0 \in \mathbb{Z}$,
(2) $q_n \leq 1$ *for all* $n \in \mathbb{N}$,
(3) *if* $q_1 = 1$, *then* $q_2 \neq 0$,
(4) *if* $q_m = 0$ *for* $m \in \mathbb{N}$, *then* $q_n = 0$ *for all* $n \geq m$,
(5) *there exists a* $\{a_n\} \in U(\{c_n\})$ *such that* $q_n = c_n/a_n$ *for all* $n \in \mathbb{N}$ *if* $q_n \neq 0$, *and*
(6) *if* $q_{n+1} \neq 0$, *then* $q_{n+2} \neq 0$ *or*

$$a_{n+1} > \frac{c_{n+1}}{c_n} a_n(a_n + 1)$$

holds.

We can easily see that the following lemma holds.

Lemma 4.4. *Let* $\{q_n\} \in T(\{c_n\})$ *and* $n \in \mathbb{N}$.

(1) $a_{n+1} > a_n$.
(2) $q_{n+1} \leq \frac{1}{a_n+1}$.
(3) $q_{n+1} \leq q_n$. *If* $q_{n+1} \neq 0$, *then* $q_{n+1} < q_n$.
(4) *The series*

$$\sum_{k=1}^{\infty} (-1)^{k-1} q_k$$

converges.

Proposition 4.4. $S(\{c_n\}) = T(\{c_n\})$.

Proof. $S(\{c_n\}) \subset T(\{c_n\})$ trivially follows by Proposition 4.1 and Remark 4.3. In order to prove $S(\{c_n\}) \supset T(\{c_n\})$, we take $\{q'_n\} \in T(\{c_n\})$ and assume that $q'_0 \in \mathbb{Z}$ and $q'_n = 0$ or $q'_n = c_n/a'_n$ for all $n \in \mathbb{N}$. Since we can set

$$\alpha = q'_0 + \sum_{k=1}^{\infty} (-1)^{k-1} q'_k$$

by Lemma 4.4 (4), we have

$$\alpha = q_0 + \sum_{k=1}^{\infty} (-1)^{k-1} q_k$$

by the generalized alternating-Sylvester expansion. It is sufficient to prove that $q_n = q'_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since the case $q'_1 = 0$ is trivial, we may assume $q'_1 \neq 0$. By considering $[\alpha]$, we have $q_0 = q'_0$ and $\alpha - q_0 = A_1 \leq q'_1 = c_1/a'_1$. If $q'_1 = A_1$, then $q_1 = q'_1$ by Lemma 4.1 (2). If $q'_1 \neq A_1$, then we have $A_1 \geq q'_1 - q'_2 \geq c_1/(a'_1 + 1)$ by $\{a'_n\} \in U(\{c_n\})$ and Definition 4.2 (5). However, $A_1 = q'_1 - q'_2 = c_1/(a'_1+1)$ is impossible because of Definition 4.2 (6). Thus we obtain $c_1/(a'_1+1) < A_1 < c_1/a'_1$. This implies $q_1 = q'_1$ by Lemma 4.1 (2).

Next we suppose that $q_{n-1} = q'_{n-1}$ holds for $n > 1$. Then we have

$$(-1)^{n-1} A_n = \alpha - q_0 - \sum_{k=1}^{n-1} (-1)^{k-1} q_k = \sum_{k=n}^{\infty} (-1)^{k-1} q'_k$$

by Remark 4.2. Hence, we obtain $A_n \leq q'_n = c_n/a'_n$. If $q'_n = A_n$, then $q'_n = q_n$ by Lemma 4.1 (2). If $q'_n \neq A_n$, then we have $A_n \geq q'_n - q'_{n+1} \geq c_n/(a'_n + 1)$ by $\{a'_n\} \in U(\{c_n\})$ and Definition 4.2 (5). By Definition 4.2 (6) we obtain $A_n > c_n/(a'_n + 1)$. Since this implies $q_n = q'_n$, we obtain the assertion of the proposition inductively. □

In the rest of this section, we set $S = S(\{c_n\})$ for simplicity, and we introduce a relation $<$ and operators $+, \cdot$ for S.

First we define the binary relation $<$ on S.

Definition 4.3. *Let $\{p_n\}, \{q_n\} \in S$ with $\{p_n\} \neq \{q_n\}$ and*

$$i = \min\{j \in \mathbb{N} \cup \{0\} \mid p_j \neq q_j\}.$$

We define $\{p_n\} < \{q_n\}$ if and only if

$$\begin{cases} p_i < q_i & (i = 0 \text{ or } 2 \nmid i), \\ p_i > q_i & (2 \mid i \text{ and } i \geq 2). \end{cases}$$

Proposition 4.5. *For any $\{p_n\}, \{q_n\}, \{r_n\} \in S$, we have*

(1) *$\{p_n\} < \{p_n\}$ does not hold (irreflexive law),*
(2) *$\{p_n\} < \{q_n\}$ or $\{p_n\} = \{q_n\}$ or $\{q_n\} < \{p_n\}$ (trichotomy),*
(3) *if $\{p_n\} < \{q_n\}$ and $\{q_n\} < \{r_n\}$ then $\{p_n\} < \{r_n\}$ (transitive law).*

In other words, $<$ is a linear order in the strict sense on S.

Proof. We can easily see that (1) and (2) hold. In order to prove (3), we define

$$i_1 = \min\{j \in \mathbb{N} \cup \{0\} \mid p_j \neq q_j\}, \qquad i_2 = \min\{j \in \mathbb{N} \cup \{0\} \mid q_j \neq r_j\}$$

and $i = \min\{i_1, i_2\}$. Then

$$p_k = q_k = r_k \quad (\text{for any } k \in \{0, 1, \dots, i-1\})$$

and

$$p_i \neq q_i \quad \text{or} \quad q_i \neq r_i$$

hold. If i is odd, then we have

$$\begin{cases} p_i < q_i \text{ and } q_i < r_i & (i = i_1 = i_2), \\ p_i = q_i \text{ and } q_i < r_i & (i = i_2 \neq i_1), \\ p_i < q_i \text{ and } q_i = r_i & (i = i_1 \neq i_2). \end{cases}$$

Therefore we obtain $p_i < r_i$. The other cases can be proved by the same argument. □

If we define

$$Q_S = \{\{q_n\} \in S \mid \text{there exists an } m \in \mathbb{N} \text{ such that } q_m = 0\},$$

we can identify Q_S with $\mathbb{Q}$ by Proposition 4.2 and 4.3. In short, the map

$$\mathbb{Q} \ni \left(q_0 + \sum_{n=1}^{\infty} (-1)^{n-1} q_n\right) \mapsto \{q_n\} \in Q_S$$

is an order-isomorphism. Hence, we may conclude that $\mathbb{Q} \subset S$.

Theorem 4.1. *Let M be a non-empty subset of S. If M is bounded from above (below), then there exists a supremum (an infimum).*

Proof. Since M is bounded from above, there exists a d_0 such that

$$d_0 = \max\{q_0 \in \mathbb{Z} \mid \text{there exists a } (q_0, q_1, \dots) \in M\}.$$

If there does not exist an upper bound for M such that $(d_0, q_1, \dots) \in S$, then $(d_0 + 1, 0, \dots)$ is a supremum for M. We assume that there exists an upper bound for M such that $(d_0, q_1, \dots) \in S$. Since there exists a $(q_0, q_1, \dots) \in M$ such that $q_0 = d_0$, we can define

$$d_1 = \max\{q_1 \in \mathbb{Q} \mid \text{there exists a } (d_0, q_1, \dots) \in M\}$$

from the definition of S and $<$. By the same argument, we can define

$$d_2 = \min\{q_2 \in \mathbb{Q} \mid \text{there exists a } (d_0, d_1, q_2, \dots) \in M\}.$$

In general, if we have defined d_{k-1} for $k > 1$, then we define

$$d_k = \begin{cases} \max\{q_k \in \mathbb{Q} \mid \exists (d_0, d_1, \dots, d_{k-1}, q_k, \dots) \in M\} & (k-1 \text{ is even}), \\ \min\{q_k \in \mathbb{Q} \mid \exists (d_0, d_1, \dots, d_{k-1}, q_k, \dots) \in M\} & (k-1 \text{ is odd}). \end{cases}$$

By the definition of $<$ and $\{d_n\}$, $\{d_n\}$ is the supremum for M. We can prove this as follows. If $\{d_n\}$ is not an upper bound for M, then there exists a $\{q_n\} \in M$ such that $\{d_n\} < \{q_n\}$. By setting $i = \min\{n \in \mathbb{N} \mid d_n \neq q_n\}$, we have $d_i < q_i$ for odd i or $d_i > q_i$ for even i. This contradicts the definition of $\{d_n\}$. On the other hand, if $\{d_n\}$ is not minimum upper bound for M, then there exists an upper bound for M $\{r_n\}$ such that $\{r_n\} < \{d_n\}$. We set $j = \min\{n \in \mathbb{N} \mid d_n \neq r_n\}$. By the definition of $\{d_n\}$, there exists an $X = (x_0, x_1, \dots) \in M$ such that $x_k = d_k$ for $0 \leq k \leq j$. Then we have $\{r_n\} < X \leq \{d_n\}$. This is impossible.

The case of the infimum can be proved by the same argument. □

In order to introduce the algebraic structure for S, we require some preparations.

Definition 4.4. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of rational numbers. We say that $L(a_n)$ holds if and only if, for all $m \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $|a_n| < 1/m$ holds for all $n \geq N$.*

Note that in the usual sense $L(a_n)$ means $\lim_{n \to \infty} a_n = 0$.

The following definition and lemma are the same as in [18, p. 611].

Definition 4.5. *Let $X \in S$ with $X = (x_0, x_1, \dots)$. We define*

$$X_n = (x_0, x_1, \dots, x_n, 0, \dots),$$

where $n \in \mathbb{N}$.

We can easily see that the next lemma holds.

Lemma 4.5. *Let $X \in S$ with $X = (x_0, x_1, \dots)$. Then we have*

(1) $X_{2n} \leq X_{2n+2} \leq X \leq X_{2n+1} \leq X_{2n-1}$,
(2) $L(X_{2n-1} - X_{2n})$,
(3) $\sup X_{2n} = \inf X_{2n-1} = X$.

In order to prove Lemma 4.7, we also require the next lemma.

Lemma 4.6. *Let $\{a_n\}$ be a monotonically increasing sequence of rational numbers which is bounded from above. Let $X = \sup a_n$. Then we have $L(X_{2n-1} - a_n)$.*

Proof. By contradiction. Assume that there exists an m such that

$$\forall N \in \mathbb{N}, \exists n \in \mathbb{N}[n \geq N \text{ and } |X_{2n-1} - a_n| = X_{2n-1} - a_n \geq 1/m]$$

holds. Since we have $X_{2n-1} - a_n \geq X_{2n+1} - a_{n+1}$ by the assumption of the lemma, we have $X_{2n-1} - a_n \geq 1/m$ for all $n \in \mathbb{N}$. On the other hand, by Lemma 4.5, there exists an $N \in \mathbb{N}$ such that

$$X_{2n-1} - X_{2n} < 1/2m$$

holds for all $n \geq N$. Hence, we have

$$\begin{aligned} a_n &\leq X_{2n-1} - \frac{1}{m} \\ &\leq X_{2N-1} - \frac{1}{m} \\ &< X_{2N} - \frac{1}{2m} \end{aligned}$$

for $n \geq N$. This implies that $X_{2N} - (1/2)m$ is an upper bound for $\{a_n\}$. Therefore we obtain

$$\sup a_n \leq X_{2N} - \frac{1}{2m} < X_{2N} \leq \sup X_{2n} = X.$$

This contradicts the definition of X. $\square$

The following lemma is important in the proofs of algebraic properties of S. It seems that this lemma can be used in the work of A. Knopfmacher and J. Knopfmacher [16], [17], [18].

Lemma 4.7. *Let $\{a_n\}, \{b_n\}$ be monotonically increasing sequence of rational numbers which are bounded from above. Then $\sup a_n = \sup b_n$ is equivalent to $L(a_n - b_n)$.*

Proof. First we assume $\sup a_n = \sup b_n$. We set $X = \sup a_n = \sup b_n$. Since

$$|a_n - b_n| \le |a_n - X_{2n-1}| + |X_{2n-1} - b_n|,$$

we have $L(a_n - b_n)$ by Lemma 4.6.

Next we assume $L(a_n - b_n)$. By contradiction. Assume that $\sup a_n \ne \sup b_n$. Without loss of generality, we may assume $\sup a_n < \sup b_n$. We set $X = \sup a_n$. Then there exists an $N \in \mathbb{N}$ such that $X_{2n-1} < b_n$ holds for all $n \ge N$. Since $b_n - X_{2n-1} \le b_{n+1} - X_{2n+1}$ for $n \ge N$, we have

$$|b_n - a_n| = (b_n - X_{2n-1}) + (X_{2n-1} - a_n) \ge b_N - X_{2N-1} > 0$$

for $n \ge N$. This contradicts $L(a_n - b_n)$. □

Now we define the operators on S, and prove that S is an ordered field. (These definitions are the same as in [18].)

Definition 4.6. *Let $X, Y \in S$. We define the following symbols and operators.*

(1) $0 = (0, 0, \dots)(= 0 \in \mathbb{Q})$.
(2) $X + Y = \sup(X_{2n} + Y_{2n})$.
(3) $-X = \sup(-X_{2n-1})$.
(4) $1 = (1, 0, \dots)(= 1 \in \mathbb{Q})$.
(5)

$$X \cdot Y = \begin{cases} \sup(X_{2n} \cdot Y_{2n}) & (X, Y \ge 0), \\ (-X) \cdot (-Y) & (X, Y \le 0), \\ -((-X) \cdot Y) & (X \le 0, Y \ge 0), \\ -(X \cdot (-Y)) & (X \ge 0, Y \le 0). \end{cases}$$

(6)

$$X^{-1} = \begin{cases} \sup((X_{2n-1})^{-1}) & (X > 0), \\ -((-X)^{-1}) & (X < 0). \end{cases}$$

Since $X_{2n} + Y_{2n} \le X_1 + Y_1$, $-X_{2n-1} \le -X_2$, $X_{2n} \cdot Y_{2n} \le X_1 \cdot Y_1 \quad (X, Y \ge 0)$ and $(X_{2n-1})^{-1} \le X_2^{-1} \quad (X > 0)$, these definitions are possible.

Now we prove that $+$ (resp. $\cdot$) shares the same properties with the usual addition (resp. multiplication).

Proposition 4.6. *Let $X, Y, Z \in S$. We have*

(1) $X + Y = Y + X$,
(2) $X + 0 = X$,
(3) $(X + Y) + Z = X + (Y + Z)$,
(4) $X + (-X) = 0$,

(5) *if* $X < Y$*, then* $X + Z < Y + Z$*.*

Proof. (1), (2) These trivially follow from the definition of $+$.

(3) We set $A = X + Y$, which means $L(A_{2n} - (X_{2n} + Y_{2n}))$ by Lemma 4.7. Since

$$|(A_{2n} + Z_{2n}) - (X_{2n} + Y_{2n} + Z_{2n})| = |A_{2n} - (X_{2n} + Y_{2n})|,$$

we have $L((A_{2n} + Z_{2n}) - (X_{2n} + Y_{2n} + Z_{2n}))$. By Lemma 4.7, this implies $\sup(A_{2n} + Z_{2n}) = \sup(X_{2n} + Y_{2n} + Z_{2n})$. Hence, we obtain $(X + Y) + Z = \sup(X_{2n} + Y_{2n} + Z_{2n})$. By the same argument, we can also prove that $X + (Y + Z) = \sup(X_{2n} + Y_{2n} + Z_{2n})$.

(4) We set $A = -X$, which means $L(A_{2n} + X_{2n-1})$ by Lemma 4.7. Since

$$|X_{2n} + A_{2n}| \leq |X_{2n} - X_{2n-1}| + |X_{2n-1} + A_{2n}|,$$

we have $L((X_{2n} + A_{2n}) - 0)$ from Lemma 4.5. This implies $\sup(X_{2n} + A_{2n}) = \sup 0$ by Lemma 4.7. Hence, we obtain (4).

(5) Since $X_{2n} + Z_{2n} < Y_{2n} + Z_{2n}$ holds for sufficiently large n, we have $X + Z \leq Y + Z$. If $X + Z = Y + Z$, then we have $L((X_{2n} + Z_{2n}) - (Y_{2n} + Z_{2n}))$ by Lemma 4.7. In short we have $L(X_{2n} - Y_{2n})$. However, this is impossible by Lemma 4.7. □

From Proposition 4.6 (1), (2), (3) and (4), it follows that S is an abelian group on $+$. Hence, we can use $-(-X) = X$, $-(X + Y) = (-X) + (-Y)$, etc. Moreover we obtain $0 < X \Leftrightarrow 0 + (-X) < X + (-X) \Leftrightarrow -X < 0$, $X < 0 \Leftrightarrow 0 < -X$ and $X < Y \Leftrightarrow -X > -Y$ by Proposition 4.6 (5).

Proposition 4.7. *Let* $X, Y, Z \in S$*. We have*

(1) $X \cdot Y = Y \cdot X$*,*
(2) $X \cdot 1 = X$*,*
(3) $X \cdot Y = -((-X) \cdot Y) = -(X \cdot (-Y))$*,*
(4) $X \cdot X^{-1} = 1 \quad (X \neq 0)$*,*
(5) $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$*,*
(6) *if* $X < Y$ *and* $Z > 0$*, then* $XZ < YZ$*.*

Proof. (1), (2) These trivially follow from the definition of $\cdot$.

(3) In the case $X, Y \geq 0$, by setting $Z = -X$ and $W = -Y$, we have

$$-((-X) \cdot Y) = -(Z \cdot Y) = -(-((-Z) \cdot Y)) = X \cdot Y,$$
$$-(X \cdot (-Y)) = -(X \cdot W) = -(-(X \cdot (-W))) = X \cdot Y.$$

From this case, we can prove the other cases. For example, in the case $X \leq 0$, $Y \geq 0$, we have

$$-(X \cdot (-Y)) = -((-X) \cdot (-(-Y))) = -((-X) \cdot Y) = X \cdot Y.$$

(4) For $X > 0$, we set $A = X^{-1}$, which means $L(A_{2n} - (X_{2n-1})^{-1})$ by Lemma 4.7. Since

$$|X_{2n}A_{2n} - X_{2n}(X_{2n-1})^{-1}| \leq |X_1| \cdot |A_{2n} - (X_{2n-1})^{-1}|,$$

we obtain $L(X_{2n}A_{2n} - X_{2n}(X_{2n-1})^{-1})$. This implies $X \cdot X^{-1} = \sup(X_{2n}(X_{2n-1})^{-1})$ by Lemma 4.7. On the other hand, since

$$|X_{2n}(X_{2n-1})^{-1} - 1| = |(X_{2n-1})^{-1}| \cdot |X_{2n} - X_{2n-1}| \leq |X_2^{-1}| \cdot |X_{2n} - X_{2n-1}|,$$

we obtain $L(X_{2n}(X_{2n-1})^{-1} - 1)$. This implies $X \cdot X^{-1} = 1$. In the case $X < 0$, by (3), we have

$$X \cdot X^{-1} = X \cdot (-((-X)^{-1})) = (-X) \cdot (-X)^{-1} = 1.$$

(5) For $X, Y, Z \geq 0$, we can prove (5) by the same argument as in the proof of Proposition 4.6 (3). By using (3), we can prove the other cases from this case. For example, in the case $X, Z \geq 0$ and $Y \leq 0$, we have

$$\begin{aligned}
(X \cdot Y) \cdot Z &= (-(X \cdot (-Y))) \cdot Z \\
&= -((X \cdot (-Y)) \cdot Z) \\
&= -(X \cdot ((-Y) \cdot Z)) \qquad (-Y > 0) \\
&= X \cdot (-((-Y) \cdot Z)) \\
&= X \cdot (Y \cdot Z).
\end{aligned}$$

(6) For $X, Y \geq 0$, we can prove (6) by the same argument as in the proof of Proposition 4.6 (5). From this case, we can also prove the other cases easily. For example, in the case $X < Y \leq 0$, by (3), we have

$$-(X \cdot Z) = (-X) \cdot Z > (-Y) \cdot Z = -(Y \cdot Z).$$

This implies $X \cdot Z < Y \cdot Z$. $\square$

Proposition 4.8. *Let $X, Y, Z \in S$. We have $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$.*

Proof. First we assume $X, Y, Z \geq 0$. Let $A = Y + Z$, $B = X \cdot Y$ and $C = X \cdot Z$. Then $L(A_{2n} - (Y_{2n} + Z_{2n}))$, $L(B_{2n} - X_{2n}Y_{2n})$ and $L(C_{2n} - X_{2n}Z_{2n})$ holds by Lemma 4.7. Since we have

$$\begin{aligned}
&|X_{2n}A_{2n} - (B_{2n} + C_{2n})| \\
&\qquad = |X_{2n}(A_{2n} - (Y_{2n} + Z_{2n})) + (X_{2n}Y_{2n} - B_{2n}) + (X_{2n}Z_{2n} - C_{2n})| \\
&\qquad \leq |X_1| \cdot |A_{2n} - (Y_{2n} + Z_{2n})| + |X_{2n}Y_{2n} - B_{2n}| + |X_{2n}Z_{2n} - C_{2n}|,
\end{aligned}$$

we obtain $L(X_{2n}A_{2n} - (B_{2n} + C_{2n}))$. This implies $\sup(X_{2n}A_{2n}) = \sup(B_{2n} + C_{2n})$ from Lemma 4.7. This implies $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$.

Next we consider the case $X \geq 0$ and $Y + Z \geq 0$. Since $Y \geq 0$ or $Z \geq 0$ holds by Proposition 4.6 (5), we may assume $Z \leq 0$. Since $-Z \geq 0$, we obtain

$$X \cdot (Y + Z) + X \cdot (-Z) = X \cdot (Y + Z + (-Z)) = X \cdot Y.$$

This is equivalent to $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$ by Proposition 4.7 (3).

By Proposition 4.7 (3), we can easily prove the other cases from these cases. For example, in the case $X \geq 0$ and $Y + Z \leq 0$, we have

$$X \cdot (Y + Z) = -(X \cdot ((-Y) + (-Z))) = -(X \cdot (-Y) + X \cdot (-Z)) = X \cdot Y + X \cdot Z.$$

□

By Propositions 4.5, 4.6, 4.7 and 4.8, S is an ordered field. Since any ordered field which satisfies Theorem 4.1 is isomorphic to $\mathbb{R}$ (see [7]), we obtain the following theorem.

Theorem 4.2. *The set S can be identified with the complete ordered field of real numbers.*

4.4. **An application.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers. For $K \geq 1$, we define a set of sequences of positive integers

$$P(K) := \{\{a_n\} \subset \mathbb{N} \mid \exists N \in \mathbb{N}, \forall n \in \mathbb{N}[n \geq N \Rightarrow a_{n+1} \geq K a_n(a_n+1)]\}.$$

For each $\{a_n\} \in P(K)$ we define

$$f(z; \{a_n\}) = \sum_{n=1}^{\infty} \frac{z^n}{a_n},$$

which is an entire function.

The purpose of this section is to prove the following theorem by using some properties of the generalized alternating-Sylvester expansion.

Theorem 4.3. *Let $\{p_n\} \in P(K)$ and $l \in \{1, 2, 3, \ldots, [K]\}$. Then*

$$f(-l; \{p_n\}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{p_n} l^n$$

is an irrational number.

Proof. We assume that $p_{n+1} \geq Kp_n(p_n+1)$ for all $n \geq N$ $(N \in \mathbb{N})$. We define $a_n = p_{n+2N-1}$ and $c_n = l^{n+2N-1}$. Then we have

$$\begin{aligned} f(-l;\{p_n\}) &= \sum_{n=1}^{2N-1} \frac{(-1)^n}{p_n} l^n + \sum_{n=2N}^{\infty} \frac{(-1)^n}{p_n} l^n \\ &= \sum_{n=1}^{2N-1} \frac{(-1)^n}{p_n} l^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n}{a_n} = A_1 + A_2, \end{aligned}$$

say. Note that $A_1 \in \mathbb{Q}$. Since we have

$$\begin{aligned} a_{n+1} = p_{n+2N} &\geq Kp_{n+2N-1}(p_{n+2N-1}+1) \\ &\geq [K]a_n(a_n+1) \geq \frac{c_{n+1}}{c_n} a_n(a_n+1), \end{aligned}$$

by Definition 4.2 and Proposition 4.4, the series A_2 is the generalized alternating-Sylvester expansion of the number A_2. By Proposition 4.3 we obtain the theorem. □

Remark 4.4. We cannot obtain Theorem 4.3 by using the alternating-Sylvester series. For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{3^{3^n}}$$

is the generalized alternating-Sylvester series, but is not the alternating Sylvester series.

References

[1] S. Akiyama, S. Egami and Y. Tanigawa, Analytic continuation of multiple zeta-functions and their values at non-positive integers, Acta Arith. **98** (2001), 107-116.

[2] K. Alladi, On generalized Euler functions and related totients, in New concepts in Arithmetic Functions, Matscience Report 83, Madras, 1975.

[3] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, Astérisque **61** (1979), 11-13.

[4] O. Bordellès, Mean values of generalized gcd-sum and lcm-sum functions, J. Integer Sequences **10** (2007), Article 07.9.2.

[5] K. A. Broughan, The gcd-sum function, J. Integer Sequences **4** (2001), Article 01.2.2.

[6] K. A. Broughan, The average order of the Dirichlet series of the gcd-sum function, J. Integer Sequences **10** (2007), Article 07.4.2.

[7] L. W. Cohen and G. Ehrlich, The Structure of the Real Number System, D. Van Nostrand Co. 1963.

[8] A.O. Gelfond, Transcendental and algebraic numbers, Dover Publications, Inc. New York. 1960.

[9] H. W. Gould and T. Shonhiwa, Functions of GCD's and LCM's, Indian J. Math. **39** (1997), 11–35.

[10] S. Ikeda, A new construction of the real numbers by alternating series, Nihonkai Math. J. **25**, (2014) 27-43.
[11] S. Ikeda and K. Matsuoka, On the lcm-sum function, J. Integer Sequences, Article 14.1.7.
[12] S. Ikeda and K. Matsuoka, Double analogue of Hamburger's theorem, to appear in Publ. Math. Debrecen.
[13] K.-H. Indlekofer, A. Knopfmacher and J. Knopfmacher, Alternating Balkema-Oppenheim expansions of real numbers, Bull. Soc. Math. Belg. **44** (1992), 1728.
[14] A. Ivić, The Riemann zeta-function. Theory and Applications, Dover Publications, Inc. Mineola, New York. 1985.
[15] W. B. Jones and W. J. Thron, Continued Fractions: Analytic Theory and Applications, Cambridge University Press. 1984.
[16] A. Knopfmacher and J. Knopfmacher, A new construction of the real numbers (via infinite products), Nieuw Arch. Wisk. **5** (1987), 19-31.
[17] A. Knopfmacher and J. Knopfmacher, Two concrete new constructions of the real numbers, Rocky Mountain J. Math. **18** (1988), 813-824.
[18] A. Knopfmacher and J. Knopfmacher, Two constructions of the real numbers via alternating series, Internat. J. Math. Math. Sci. **12** (1989), 603-613.
[19] Y. Komori, K. Matsumoto and H. Tsumura, Functional equations and functional relations for the Euler double zeta-function and its generalization of Eisenstein type, Publ. Math. Debrecen **77** (2010) 15-31.
[20] K. Matsumoto, Functional equations for double zeta-functions, Math. Proc. Cambridge Philos. Soc. **136** (2004) 1-7.
[21] A. Oppenheim, The representation of real numbers by infinite series of rationals, Acta Arith. **21** (1972), 391-398.
[22] S. S. Pillai, On an arithmetic function, J. Annamalai Univ. **2** (1933), 243–248.
[23] T. Rivoal, La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), 267-270.
[24] H. M. Srivastava and J. Choi, Series Associated with Zeta and Related Functions, Kluwer Academic Publishers, 2001.
[25] Y. Tanigawa and W. Zhai, On the gcd-sum function, J. Integer Sequences **11** (2008), Article 08.2.3.
[26] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Second Edition, revised and with a preface by D. R. Heath-Brown, The Clarendon Press, Oxford University Press, New York, 1986.
[27] L. Tóth, A survey of gcd-sum functions, J. Integer Sequences **13** (2010), Article 10.8.1.
[28] A. van der Poorten, A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$. An informal report. Math. Intelligencer **1** (4) (1978/79), 195-203.
[29] A. Walfisz, Weylsche Exponentialsummen in der neueren Zahlentheorie, Leipzig BG Teubner, 1963.
[30] W. V. Zudilin, One of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational, Uspekhi Mat. Nauk **56** (2001), 149-150.