Wiener Amalgam Spaces and Nonlinear Evolution Equations

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Abstract

This thesis comprises of five chapters. Chapter 1, Introduction, describes the background of the study together with the statement of the results. All the notations that will be used in this paper together with the definition of spaces will be established in Chapter 2, Notations. Chapter 3, Fourier Multipliers, presents new boundedness results in the area. Chapter 4, Inclusion Relations, provides embedding between L^p-Sobolev spaces and Wiener amalgam spaces. Finally, Chapter 5–Evolution Equations, is an expository of results relating PDE and amalgam spaces.

Wiener amalgam spaces are a class of spaces of functions or distributions whose norm treats local and global properties simultaneously. By taking advantage of its "nice" properties we were able to prove the boundedness of unimodular Fourier multipliers on Wiener amalgam spaces. For a real-valued homogeneous function μ on $\mathbb{R}^n$ of degree $\alpha \geq 2$, we show the boundedness of the operator $e^{i\mu(D)}$ between the weighted Wiener amalgam space $W_s^{p,q}$ and $W^{p,q}$ for all $1 \leq p, q \leq \infty$ and $s > n(\alpha - 2)|1/p - 1/2| + n|1/p - 1/q|$. This threshold is shown to be optimal for certain (p, q).

We also determined optimal inclusion relations between L^p-Sobolev and Wiener amalgam spaces, which enables us to describe the mapping properties of unimodular Fourier operators $e^{i|D|^\alpha}$ between L^p-Sobolev and Wiener amalgam spaces. Moreover, some Littlewood-Paley type inequalities were derived from the inclusion.

Lastly, we survey some recent progress on Wiener amalgam spaces and modulation spaces and their connection to PDE, i.e., well-posedness results, Strichartz estimates, smoothing estimates, etc.

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Chapter 1

Introduction

1.1 Amalgam spaces

Wiener amalgam spaces consist of functions or distributions whose norm distinguishes between local and global properties. This norm provides a better analysis of local integrability and the decay at infinity of functions, making amalgam spaces advantageous in applications in comparison to the classical L^p spaces.

The norm of a function f in L^p on the real line is given by

$$||f||_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{1/p}.$$

Recall that all rearrangements of a given function have identical L^p norms. This means that it cannot recognize whether a function is, for example, the characteristic function of an interval or the sum of many characteristic functions of small intervals spread widely over $\mathbb{R}$. Also, there are no inclusion properties between any two $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$. On the other hand, subspaces of $L^p(\mathbb{R})$ could only have either "local" or "global" inclusion behaviour. These inherent shortcomings in L^p spaces are non existent in amalgam spaces.

The first appearance of amalgam spaces was due to Norbert Wiener in devising his theory of generalized harmonic analysis. Wiener defined the amalgam spaces $W(L^1, L^2), W(L^2, L^1), W(L^1, L^\infty)$ and $W(L^\infty, L^1)$ in a series of papers [42, 41, 40], where the so-called *standard amalgam* $W(L^p, L^q)$ is defined by the norm

$$||f||_{W(L^p,L^q)} = \left(\sum_{n \in \mathbf{Z}} \left(\int_n^{n+1} |f(t)|^p \, dt \right)^{q/p} \right)^{1/q}, \tag{1.1}$$

with usual modification when p or q is infinity. For a recent development such as the scaling properties of these amalgam spaces, we refer the reader to [7].

In 1980s, H. Feichtinger introduced a generalization of amalgam spaces which enables a vastly wide range of Banach spaces of functions or distributions defined on a locally compact group to be used as local or global components. He used the notation $W(B, C)$ to define a space of functions or distributions which are "locally in B" and "globally in C". Feichtinger called these spaces $W(B, C)$ as *Wiener-type* spaces in recognition to Wiener's prior work. To promote the link between Feichtinger's generalization and the amalgams previously defined, it was suggested by J. Benedetto to call them Wiener amalgam spaces [20]. These are some properties which follow immediately from his theory;

Inclusions. If $B_1 \hookrightarrow B_2$ and $C_1 \hookrightarrow C_2$ then $W(B_1, C_1) \hookrightarrow W(B_2, C_2)$.

Duality. $W(B, C)' = W(B', C')$ whenever a space of test functions is dense in B and C.

Complex interpolation. Complex interpolation can be carried out in each component of $W(B, C)$ separately.

Pointwise multiplications. If $B_1 \cdot B_2 \subset B_3$ and $C_1 \cdot C_2 \subset C_3$ then $W(B_1, C_1) \cdot W(B_2, C_2) \subset W(B_3, C_3)$.

Convolutions. If $B_1 * B_2 \subset B_3$ and $C_1 * C_2 \subset C_3$ then $W(B_1, C_1) * W(B_2, C_2) \subset W(B_3, C_3)$.

As stated earlier, we have a more natural inclusion relations between two amalgam spaces. By choosing weighted L^p spaces, Besov spaces, and Sobolev spaces, etc., as our B or C, we can easily illustrate the properties stated above. For the inclusion, let $p_1 \leq p_2$ and $q_1 \leq q_2$, then

$$W(\mathcal{F}L^{p_1}, L^{q_1}) \hookrightarrow W(\mathcal{F}L^{p_2}, L^{q_2}).$$

The remaining properies will be discussed rigorously in other parts of this thesis.

In the following sections of this chapter, we introduce our results on Fourier multipliers and inclusion relations under Section 1.2 and Section 1.3, respectively. We will compare previous results with our new ones in their respective topics. In section 1.4, we outline the presentation of our survey in Chapter 5 which is the final chapter of this thesis. The survey will focus on the advantages of using modulation and amalgam spaces in solving problems in non-linear dispersive and wave equations.

1.2 Results on Fourier multipliers

A Fourier multiplier $\sigma(D)$ in $\mathbb{R}^n$ is an operator whose action on a test function f is formally defined by

$$\sigma(D)f(x) = \frac{1}{(2n)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} \sigma(\xi) \hat{f}(\xi)\ d\xi.$$

The function σ is called the symbol of the multiplier or simply the multiplier. One can rewrite this operator as a convolution operator

$$\sigma(D)f(x) = \check{\sigma} * f(x),$$

where $\check{\sigma}$ is the (distributional) inverse Fourier transform. These operators are closely related to bounded translation invariant operators [17, 19] and have immense applications to PDEs [1, 6, 22, 30].

In particular, unimodular Fourier multipliers $\sigma(D) = e^{i|D|^\alpha}$ arise naturally as formal solutions for Cauchy problem for dispersive equations given by

$$\begin{cases} \partial_t u - i(\Delta)^{\alpha/2} u = 0 \quad x \in \mathbb{R}^n, t \in \mathbb{R} \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

The cases $\alpha = 1, 2, 3$ are of particular research interest because they correspond to the wave equation, Schrödinger equation and Airy equation, respectively. Boundedness of these multipliers on a particular space S means that the S-properties of the initial condition are preserved by time evolution.

The fundamental problem in the study of Fourier multipliers is to relate the boundedness of $\sigma(D)$ on certain spaces to that of the properties of the symbol σ. In L^p the full resolution of this problem is known as the Hörmander-Mihlin multiplier theorem [19]. Unfortunately, unimodular Fourier multipliers excludes the use of Hörmander-Mihlin due to singularity of the derivatives at the origin and large derivatives at infinity. In fact, the operator $e^{i|D|^\alpha}$ is bounded on L^p if and only if $p = 2$ (see [21]). In view of this unboundedness in L^p, unimodular Fourier multipliers are studied in [1] and [28] in more suitable space, the modulation space $M^{p,q}$, where they proved boundedness. For now, we say modulation spaces are defined by measuring the time-frequency concentration of functions or distributions in the time-frequency plane. Concrete definition of modulation spaces will be given in Chapter 2.

In [1], Bényi, Gröchenig, Okoudjou and Rogers proved the following theorem.

Theorem 1.2.1. *If $\alpha \in [0, 2]$, then the Fourier multiplier $e^{i|D|^\alpha}$ is bounded from $M^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ for all $1 \leq p, q \leq \infty$ and $n \geq 1$.*

The result for $\alpha = 2$ has been already known before from [36]. Now let us define $M_s^{p,q} = \{f \in \mathcal{S}'(\mathbb{R}^n) : (1-\Delta)^{s/2} f \in M^{p,q}\}$ where s represents loss of derivatives. An extention of Theorem 1.2.1 was given by Miyachi, Nicola, Rivetti, Tabacco and Tomita in [28] which is stated as follows:

Theorem 1.2.2. *Let $\alpha > 2$, then the Fourier multiplier operator $e^{i|D|^\alpha}$ is bounded from the weighted modulation space $M_s^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$, if and only if $s \geq (\alpha - 2)n|1/p - 1/2|$.*

These theorems describes the advantages of using modulation spaces because whenever $\alpha \leq 2$, $e^{i|D|^\alpha}$ is bounded, and only have small loss of regularity of order up to $(\alpha - 2)n|1/p - 1/2|$ if $\alpha > 2$. Motivated by these two theorems, we studied in [11] the boundedness of the unimodular Fourier multipliers $e^{i|D|^\alpha}$ with $\alpha > 0$ on Wiener amalgam spaces $W^{p,q}$. Throughout this thesis, we use the notation $W^{p,q}$ for amalgam spaces whose local behaviour is described by Fourier transforms of L^q functions and have a global L^p behaviour. This notation has the same meaning with $W(\mathcal{F}L^q, L^p)$ of Feichtinger's notation. For clarification with our choice of notation we direct the reader to [29] where the authors studied local and global change of variable for modulation, Fourier Lebesgue and Wiener amalgam spaces.

The following theorems are the results we established in [11]

Theorem 1.2.3. *Let $\alpha \geq 2$ and μ be a real-valued homogeneous function on $\mathbb{R}^n$ of degree α which belongs to $C^\infty(\mathbb{R}^n \backslash \{0\})$. Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. Then Fourier multiplier operator $e^{i\mu(D)}$ is bounded from $W_s^{p,q}(\mathbb{R}^n)$ to $W^{p,q}(\mathbb{R}^n)$ whenever*

$$s > n(\alpha - 2)|1/p - 1/2| + n|1/p - 1/q|.$$

We note the analogy of this theorem with the Theorem 1.2.2 for the case on Wiener amalgam spaces. The optimality of the threshold in Theorem 1.2.3 for certain values of p and q is stated as follows.

Theorem 1.2.4. *Let $\alpha \geq 2$ and μ be a real-valued homogeneous function on $\mathbb{R}^n$ of degree α which belongs to $C^\infty(\mathbb{R}^n \backslash \{0\})$. Suppose there exist a point $\xi_0 \neq 0$ at which the Hessian determinant of μ is not zero. Let $max(1/q, 1/2) \leq 1/p$ or $min(1/q, 1/2) \geq 1/p$ and $s \in \mathbb{R}$ and suppose the Fourier multiplier operator $e^{i\mu(D)}$ is bounded from $W_s^{p,q}(\mathbb{R}^n)$ to $W^{p,q}(\mathbb{R}^n)$. Then*

$$s \geq n(\alpha - 2)|1/p - 1/2| + n|1/p - 1/q|.$$

Although we have yet to prove Theorem 1.2.4 for any $1 \leq p, q \leq \infty$, the case $\alpha = 2$ recaptures [8, Proposition 6.1]. It states that if the pointwise multiplier operator $Af(x) = e^{i|x|^2} f(x)$ is bounded from $M_s^{p,q}$ to $M^{p,q}$ then $s \geq n|1/p - 1/q|$. The significance of this proposition is that it showed sharpness of the threshold computed for boundedness of Fourier integral operator(FIO) on modulation spaces $M^{p,q}$ with decay condition on its symbol. One should observe the fact that A is a FIO whose phase $\Phi(x, \eta) = x\eta + \frac{|x|^2}{2}$ and symbol $\sigma \equiv 1$.

1.3 Results on inclusion relations

Research materials that focus on the study of relationships between classical function spaces and modern function spaces help give us a better understanding of the nature of these powerful modern function spaces. In the case of modulation spaces, the works of Toft [35], Sugimoto and Tomita [34] and Wang and Huang [38] gave a full picture on the inclusion relations between Besov spaces $B_s^{p,q}$and modulation spaces $M_s^{p,q}$. We state their work in the following manner.

Theorem 1.3.1. *Let $0 < p, q \leq \infty, s_1, s_2 \in \mathbb{R}$. Then we have*

a. $B_s^{p,q} \subset M^{p,q}$ *if and only if* $s \geq \nu_1(p, q)$, *where*

$$\nu_1(p, q) = max\{0, n(1/q - 1/p), n(1/q + 1/p - 1)\};$$

b. $M^{p,q} \subset B_s^{p,q}$ *if and only if* $s \leq \nu_2(p,q)$, *where*

$$\nu_2(p,q) = min\{0, n(1/q - 1/p), n(1/q + 1/p - 1)\}.$$

This theorem naturally led to establishing the inclusion relations between L^p-Sobolev spaces and modulation spaces $M_s^{p,q}$ [27], in which we write as follows:

Theorem 1.3.2. *Let* $1 \leq p, q \leq \infty$ *and* $s \in \mathbb{R}$. *Then* $L_s^p \hookrightarrow M^{p,q}$ *if and only if one of the following conditions is satisfied.*

a. $q \geq p > 1$ *and* $s > n\nu_1(p,q)$;

b. $p > q$, *and* $s > n\nu_1(p,q)$;

c. $p = 1, q = \infty$ *and* $s \geq n\nu_1(1,\infty)$;

d. $p = 1, q \neq \infty$ *and* $s > n\nu_1(1,q)$.

Conversely, if $L_s^p \hookrightarrow W^{p,q}$, *then* $s \geq n\tau_1(p,q)$.

Theorem 1.3.3. *Let* $1 \leq p, q \leq \infty$ *and* $s \in \mathbb{R}$. *Then* $M^{p,q} \hookrightarrow L_s^p$ *if and only if one of the following conditions is satisfied.*

a. $q \leq p < \infty$ *and* $s < n\nu_2(p,q)$;

b. $p < q$ *and* $s \leq n\nu_2(p,q)$;

c. $p = \infty, q = 1$ *and* $s \leq n\nu_2(\infty,1)$;

d. $p = \infty, q \neq 1$ *and* $s < n\nu_2(\infty,q)$.

Conversely, if $W^{p,q} \hookrightarrow L_s^p$, *then* $s \leq n\tau_2(p,q)$.

Kobayashi, Miyachi and Tomita determined the inclusion relation between modulation spaces $M_s^{p,q}$ and local Hardy spaces h^p for $0 < p \leq 1$, this work is done prior to [27].

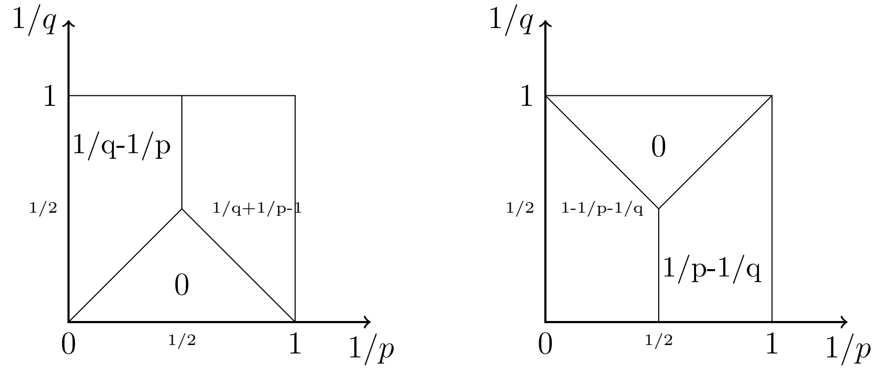

Figure 1.1: The index sets ν_1 (left) and ν_2 (right)

Our work in [10], established the embedding relation between L^p-Sobolev spaces and Wiener amalgam spaces $W^{p,q}$, which also implies the inclusion relations between Besov spaces $B_s^{p,q}$and Wiener amalgam spaces $W^{p,q}$. Although, certain cases of these embedding relations could already be deduced from [27] by using the known embeddings

$$W_s^{p,q} \hookrightarrow M_s^{p,q} \quad \text{for } p \leq q,$$

and

$$M_s^{p,q} \hookrightarrow W_s^{p,q} \quad \text{for } q \leq p,$$

this method is not possible for the remaining cases. Also, our strategy produces sharper estimates even for some cases derived by the method above.

For $(1/p, 1/q) \in [0,1] \times [0,1]$ we define the indices $\tau_1(p,q)$ and $\tau_2(p,q)$ as follows:

$$\tau_1(p,q) = \begin{cases} 0 & \text{if } (1/p, 1/q) \in I_1^* : \min(1/p', 1/2) \geq 1/q \\ 1/p + 1/q - 1 & \text{if } (1/p, 1/q) \in I_2^* : \min(1/q, 1/2) \geq 1/p' \\ 1/q - 1/2 & \text{if } (1/p, 1/q) \in I_3^* : \min(1/p', 1/q) \geq 1/2 \end{cases}$$

$$\tau_2(p,q) = \begin{cases} 0 & \text{if } (1/p,1/q) \in I_1 : \max(1/p',1/2) \le 1/q \\ 1/p+1/q-1 & \text{if } (1/p,1/q) \in I_2 : \max(1/q,1/2) \le 1/p' \\ 1/q-1/2 & \text{if } (1/p,1/q) \in I_3 : \max(1/p',1/q) \le 1/2 \end{cases}$$

where $1/p+1/p' = 1 = 1/q+1/q'$. See Figure 1.2 for a visualization. It is important to compare Figure 1.1 and Figure 1.2 to understand the difference and similarities of index needed for the inclusion relations of $W^{p,q}$ and $M^{p,q}$ to L^p-Sobolev spaces.

The main results in [10] are the following theorems and corollaries.

Theorem 1.3.4. *Let $1 \le p,q \le \infty$ and $s \in \mathbb{R}$. Then $L^p_s \hookrightarrow W^{p,q}$ if one of the following conditions is satisfied.*

a. $p > q, q < 2$ *and* $s > n\tau_1(p,q)$;

b. $p \ne 1$, $max(1/p,1/2) \ge 1/q$ *and* $s \ge n\tau_1(p,q)$;

c. $p = 1, q = \infty$ *and* $s \ge n\tau_1(1,\infty)$;

d. $p = 1, q \ne \infty$ *and* $s > n\tau_1(1,q)$.

Conversely, if $L^p_s \hookrightarrow W^{p,q}$, then $s \ge n\tau_1(p,q)$.

Theorem 1.3.5. *Let $1 \le p,q \le \infty$ and $s \in \mathbb{R}$. Then $W^{p,q} \hookrightarrow L^p_s$ if one of the following conditions is satisfied.*

a. $p < q, q > 2$ *and* $s < n\tau_2(p,q)$;

b. $p \ne \infty$, $min(1/p,1/2) \le 1/q$ *and* $s \le n\tau_2(p,q)$;

c. $p = \infty, q = 1$ *and* $s \le n\tau_2(\infty,1)$;

d. $p = \infty, q \ne 1$ *and* $s < n\tau_2(\infty,q)$.

Conversely, if $W^{p,q} \hookrightarrow L^p_s$, then $s \le n\tau_2(p,q)$.

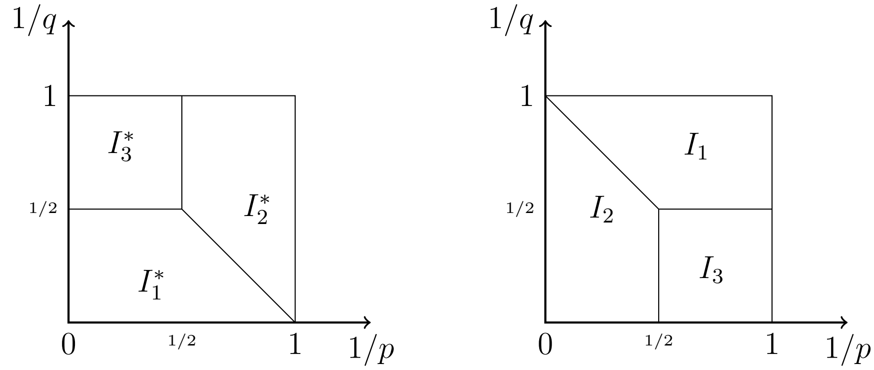

Figure 1.2: The index sets τ_1 (left) and τ_2 (right)

Remark. It is important to point out that our results on Fourier multipliers is fundamental in the proof of the converse of both theorems stated above. This will be in the form of Lemma 4.2.1.

We observe that for both Theorem 1.3.4 and Theorem 1.3.5 conditions (b) and (c) are optimal and in Chapter 4 we also show the optimality of condition (d). Thus, only the sufficiency of $s = n\tau_1(p,q)$ or $s = n\tau_2(p,q)$ in condition (a) of both theorems remains an open problem.

Corollary 1.3.1. *Let $1 \le p \le \infty, 1/p + 1/p' = 1$ and $s \in \mathbb{R}^n$. Then*

$$M^{min(p',2),p} \hookrightarrow \mathcal{F}L^p \hookrightarrow M^{max(p',2),p}.$$

This corollary gives us an understanding of the inclusion relations between modulation spaces and Fourier Lebesgue spaces $\mathcal{F}L^p$ which is an improvement of [35, Prop. 1.7]

The following corollary follows immediately from the inclusion property $L^p_{s+\epsilon} \hookrightarrow B^{p,q}_s \hookrightarrow L^p_{s-\epsilon}$ which can be found in [37, p.97].

Corollary 1.3.2. *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. Then we have*

a. $B^{p,q}_s \hookrightarrow W^{p,q}$ *if* $s > n\tau_1(p,q)$. *Conversely, if* $B^{p,q}_s \hookrightarrow W^{p,q}$, *then* $s \ge n\tau_1(p,q)$;

b. $W^{p,q} \hookrightarrow B^{p,q}_s$ *if* $s < n\tau_2(p,q)$. *Conversely, if* $W^{p,q} \hookrightarrow B^{p,q}_s$, *then* $s \le n\tau_2(p,q)$.

1.4 Evolution equations

The final chapter of this thesis, Chapter 4, is a survey of results on nonlinear dispersive and wave equations under modulation and Wiener amalgam spaces. The presentation of this chapter is partly an adaptation of a previous survey found in [30]. Our aim here is to show the advantages of using modulation spaces or Wiener amalgam spaces in dealing with problems related to nonlinear evolution equations. The topics will include, well-posedness results, Strichartz estimates, and smoothing estimates of certain semigroups, etc. In particular, frequency-uniform decomposition methods will play an important role in our analysis.

Let Q_k be the unit cube centered at k, then $\{Q_k\}_{k\in\mathbb{Z}}$ is a kind of decomposition of $\mathbb{R}^n$ we call Wiener decomposition [41]. Frequency-uniform decomposition operators can be roughly defined as follows:

$$\Box_k \sim \mathcal{F}^{-1}\chi_{Q_k}\mathcal{F}, \quad k \in \mathbb{Z}^n, \tag{1.3}$$

where χ_{Q_k} is the characteristic function of Q_k, and $\mathcal{F}$ is the Fourier transform. In this setting, one can view functions in their localized frequency versions. In Chapter 2, we will use these operators to define discrete versions of modulation and amalgam spaces.

1.4.1 Schrödinger semigroup

The Schrödinger semigroup $S(t) = e^{it\Delta}$ appears in the solutions of (5.3) with $\alpha = 2$. It is known that operator $e^{it\Delta} : L^p \to L^p$ is bounded if and only if $p = 2$. Thus, we can show that

$$\sup_{\varphi\in\mathcal{S}\setminus\{0\}}\frac{||S(t)\varphi||_p}{||\varphi||_p} = \infty, \quad p \neq 2, \tag{1.4}$$

by taking any $\varphi = e^{-(a+ib)|x|^2}$. Hence, one can naturally ask the following question.

Q1. Is there any Banach space X which satisfies

$$\sup_{\varphi\in X\setminus\{0\}}\frac{||S(t)\varphi||_X}{||\varphi||_X}<\infty? \tag{1.5}$$

Moreover, refering to [2, 3], we have the following estimate

$$||S(t)f||_p \leq C|t|^{-n(1/2-1/p)}||f||_{p'}, \tag{1.6}$$

where $2\leq p\leq\infty, 1/p+1/p'=1$. Notice that estimate (1.6) contains singularity at $t=0$, and so we ask another question.

Q2. Is there any Banach space X satisfying the following truncated decay

$$||S(t)f||_X \leq C(1+|t|)^{-\delta(X)}||f||_{X'}, \tag{1.7}$$

where $\delta>0$, X' denotes the dual space of X, with both spaces having same regularity.

Combining (1.6) with some duality arguments we get estimates of the form

$$||S(t)f||_{L_t^qL_x^p}\lesssim ||f||_{L_x^2}. \tag{1.8}$$

Such estimates are called Strichartz estimates, (cf. [18, 23, 33, 43]). Roughly speaking, these estimates expresse a gain in local regularity and decay at infinity both in some L^q-averaged sense. Since amalgam spaces are spaces that control local regularity and decay at infinity separately, it is natural to ask the following.

Q3. Do Strichartz estimates hold in Wiener amalgam spaces?

Now consider the nonlinear Schrödinger equation (NLS)

$$\begin{cases}\partial_t u - i\Delta u = |u|^\kappa u \quad \kappa\in\mathbb{N}\\ u(x,0)=u_0(x),\end{cases} \tag{1.9}$$

If u is a solution of (1.9), then $u_\lambda(t,x) = \lambda^{1/\kappa} u(\lambda^2 t, \lambda x)$ also solves (1.9) with initial datum $\lambda^{1/\kappa} u_0(\lambda x)$. Observing that

$$||u_\lambda(0)||_{\dot{H}^s} = \lambda^{s-n/2+1/\kappa} ||u_0||_{\dot{H}^s} \quad \kappa > 0,$$

we see that whenever $s = s_\kappa := n/2 - 1/\kappa$ the norm $||u_\lambda(0)||_{\dot{H}^s}$ is invariant for all $\kappa > 0$. The Sobolev space $\dot{H}^{s_\kappa}$ is said to be a critical space for NLS [2]. There are several well-posedness results for (1.9) on $\dot{H}^s$ where $s \geq s_\kappa \geq 0$, which corresponds to critical and sub-critical cases in H^s. However, the case $s < s_\kappa$ where H^s is said to be a super-critical space,(1.9) is known to be ill-posed. We pose the question.

Q4. Are there initial data in $u_0 \in H^s$, where $0 \leq s < s_\kappa$, such that NLS has local or global well-posedness results?

All of these questions we pose have positive answers in modulation and Wiener amalgam spaces. Moreover, we note that all these questions also make sense for other equations as well, with only minor variants taken into consideration.

Chapter 2

Notation and Definitions

2.1 Basic symbols

a. We denote the Schwartz class of test functions on $\mathbb{R}^n$ by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and its dual, the space of tempered distributions, by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$.

b. The Fourier transform of $f \in \mathcal{S}$ is given by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx,$$

which is an isomorphism of the Schwartz space $\mathcal{S}$ onto itself that extends to the tempered distributions $\mathcal{S}'$ by duality.

c. The inverse Fourier transform is given by $\mathcal{F}^{-1}f(x) = \check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} f(\xi)\, d\xi$.

d. Let $s \in \mathbb{R}$, we denote $\langle\xi\rangle^s := (1 + |\xi|^2)^{s/2}, \xi \in \mathbb{R}^n$.

e. Given $1 \leq p \leq \infty$, we denote by p' the conjugate exponent of p (i.e. $1/p+1/p' = 1$).

f. We use the notation $u \lesssim v$ to denote $u \leq cv$ for a positive constant c independent of u and v.

g. We use the notation $u \asymp v$ to denote $cu \leq v \leq Cu$ for universal positive constants c, C.

h. The translation and modulation operators are defined by $T_x f(t) = f(t-x)$ and $M_\xi f(t) = e^{it\cdot\xi} f(t)$, respectively.

i. The scaling operator is given by $U_\lambda f(t) = f_\lambda(t) = f(\lambda t)$.

We now recall the definitions of the function spaces to be used in this thesis.

2.2 Function spaces

a. *L^p-Sobolev.* Following the notation of Stein [32], we define the L^p-Sobolev norm by

$$||f||_{L^p_s} = ||(\langle\cdot\rangle^s \widehat{f}(\cdot))^\vee||_{L^p}.$$

We remark that this notation should not be interchanged with the set of all f such that $(1+|x|^2)^{s/2} f$ belongs to L^p.

b. The function space $L^q_{t\in I} L^p_x$ and $L^p_x L^q_{t\in I}$ are defined as follows:

$$||f||_{L^q_{t\in I} L^p_x} = \left(\int_I \left(\int_{\mathbb{R}^n} |f(t,x)|^p \, dx \right)^{q/p} \, dt \right)^{1/q},$$

$$||f||_{L^p_x L^q_{t\in I}} = \left(\int_{\mathbb{R}^n} \left(\int_I |f(t,x)|^q \, dt \right)^{p/q} \, dx \right)^{1/p},$$

and if $I = \mathbb{R}$ we simply write $L^q_t L^p_x := L^q_{t\in I} L^p_x$ and $L^p_x L^q_t := L^p_x L^q_{t\in I}$.

c. *Wiener amalgam spaces.* For $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \neq g \in \mathcal{S}$, the Wiener amalgam space $W^{p,q}_s$ consists of all tempered distributions $f \in \mathcal{S}'$ such that the norm

$$||f||_{W^{p,q}_s} := ||f||_{W(\mathcal{F}L^q_s, L^p)} = \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} |\mathcal{F}(f \cdot T_y g)(\omega)|^q \langle \omega \rangle^{sq} \, d\omega \right)^{p/q} dy \right)^{1/p},$$

is finite, with usual modifications if p or $q = \infty$. If $s = 0$ we simply write $W^{p,q}$ instead of $W^{p,q}_0$. We note that this definition is independent of the choice of window g. Alternatively, we can use the following equivalent discrete norm

$$||f||_{W^{p,q}_s} = || \ ||\{\langle k \rangle^s \varphi(D-k) f\}||_{\ell^q} ||_{L^p},$$

with $\varphi(D-k)f = \mathcal{F}^{-1}(\widehat{f} \cdot T_k \varphi)$, where $\varphi \in \mathcal{S}$ satisfies

$$\operatorname{supp} \varphi \subset (-1,1)^n \qquad \text{and} \qquad \sum_{k \in \mathbb{Z}^n} \varphi(\xi - k) = 1 \qquad \forall \xi \in \mathbb{R}^n. \quad (2.1)$$

Again, this definition is independent of the choice of φ satisfying the conditions above. We denote the closure of the Schwartz class $\mathcal{S}$ in the $W^{p,q}_s$-norm by $\mathcal{W}^{p,q}_s$. If $1 \leq p, q < \infty$ then $\mathcal{W}^{p,q}_s = W^{p,q}_s$.

Remark. Since the χ_{Q_k} in 1.3 cannot be differentiated, it is convenient to substitute by a smooth cut-off function. We now formally define frequency-uniform decomposition operators. Let φ satisfies 2.1 and $\varphi_k(\xi) = \varphi(\xi - k)$ be the translate of φ, then

$$\square_k := \mathcal{F}^{-1} \varphi_k \mathcal{F}, \quad k \in \mathbb{Z}. \quad (2.2)$$

d. *Modulation spaces.* For $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, the modulation space $M_s^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'$ such that the norm

$$||f||_{M_s^{p,q}} = \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} |V_g f(y,\omega)|^p \, dy \right)^{q/p} \langle \omega \rangle^{sq} d\omega \right)^{1/q},$$

is finite, with usual modifications if p or $q = \infty$. Here $V_g f$ denotes the short-time Fourier transform (STFT) of $f \in \mathcal{S}'$ with respect to the window $0 \neq g \in \mathcal{S}$ defined by

$$V_g f(y,\omega) = \int_{\mathbb{R}^n} f(\xi) \overline{g(\xi - y)} e^{-i\xi\cdot\omega} \, d\xi.$$

Notice that $V_g f(y,\omega) = \mathcal{F}(f \cdot T_y \bar{g})(\omega)$, hence $\mathcal{F}M^{q,p} = W^{p,q}$.

The corresponding discrete version of the modulation space norm is given by

$$||f||_{M_s^{p,q}} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} ||\square_k f||_p^q \right)^{1/q}. \tag{2.3}$$

e. *Besov spaces.* Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$. Suppose $\psi_0, \psi \in \mathcal{S}$ are non-negative and satisfy $0 \notin \text{supp } \psi$, and $\sum_{k=0}^{\infty} \psi_k = 1$, where $\psi_k = \psi(\cdot/2^k), k \geq 1$. Then the Besov space $B_s^{p,q}$ consists of $f \in \mathcal{S}$ such that

$$||f||_{B_s^{p,q}} = \left(\sum_{k=0}^{\infty} 2^{ksq} ||\psi_k(D) f||_{L^p}^q \right)^{1/q}$$

is finite, with usual modification if $q = \infty$.

2.3 Basic properties of $W^{p,q}$

Lemma 2.3.1. *Let $p, q, p_i.q_i \in [1, \infty]$ for $i = 1, 2$ and $s_j \in \mathbb{R}$ for $j = 1, 2$. Then*

a. $\mathcal{S} \hookrightarrow W^{p,q} \hookrightarrow \mathcal{S}'$;

b. $\mathcal{S}$ *is dense in* $W^{p,q}$ *if* p *and* $q < \infty$;

c. If $q_1 \le q_2$ *and* $p_1 \le p_2$*, then* $W^{p_1,q_1} \hookrightarrow W^{p_2,q_2}$;

d. If $s_1 \ge s_2$*, then* $W^{p,q}_{s_1} \hookrightarrow W^{p,q}_{s_2}$;

e. $\langle D\rangle^{-s} : W^{p,q} \to W^{p,q}_s$, $f \mapsto (\widehat{f}(\cdot)\langle\cdot\rangle^{-s})^\vee$ *, is an isomorphism;*

f. (Convolution) If $\mathcal{F}L^{q_1} * \mathcal{F}L^{q_2} \hookrightarrow \mathcal{F}L^q$ *and* $L^{p_1} * L^{p_2} \hookrightarrow L^p$*, then*

$$W(\mathcal{F}L^{q_1}, L^{p_1}) * W(\mathcal{F}L^{q_2}, L^{p_2}) \hookrightarrow W(\mathcal{F}L^q, L^p).$$

In particular, $||f * u||_{W(\mathcal{F}L^q,L^p)} \le ||u||_{W(\mathcal{F}L^\infty,L^1)}||f||_{W(\mathcal{F}L^q,L^p)}$;

g. (Complex interpolation) For $0 < \theta < 1$*. Let* $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$ *and* $s = \theta s_1 + (1-\theta)s_2$*. Then*

$$[\mathcal{W}^{p_1,q_1}_{s_1}, \mathcal{W}^{p_2,q_2}_{s_2}]_{[\theta]} = \mathcal{W}^{p,q}_s;$$

h. (Duality) $(\mathcal{W}^{p,q}_s)' = W^{p',q'}_{-s}$*, where* $1/p + 1/p' = 1 = 1/q + 1/q'$, $\quad p, q \ne \infty$.

The proofs of these statements can be found in [13, 14, 15, 16, 36]. Next we give properties of the dilation operator U_λ in Wiener amalgam spaces [11].

We introduce the following indices

$$\mu_1(q,p) = \begin{cases} -1/q & \text{if } (1/p, 1/q) \in I_1^*, \\ 1/p - 1 & \text{if } (1/p, 1/q) \in I_2^*, \\ -2/q + 1/p & \text{if } (1/p, 1/q) \in I_3^*, \end{cases} \tag{2.4}$$

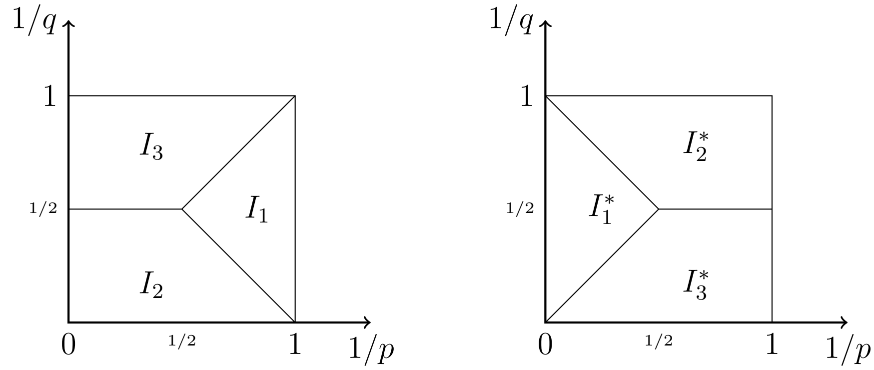

Figure 2.1: The index sets for scaling in $W^{p,q}$

and

$$\mu_2(q,p) = \begin{cases} -1/q & \text{if } (1/p, 1/q) \in I_1, \\ 1/p - 1 & \text{if } (1/p, 1/q) \in I_2, \\ -2/q + 1/p & \text{if } (1/p, 1/q) \in I_3. \end{cases} \tag{2.5}$$

Refer to Figure 2.1 for visualization.

Lemma 2.3.2. *Let $1 \le p, q \le \infty$, and $\lambda \neq 0$. We have the following inequalities.*

a.

$$||U_\lambda f||_{W^{p,q}} \lesssim |\lambda^{-1}|^{n+n\mu_2(q,p)} ||f||_{W^{p,q}} \qquad \forall\ |\lambda| \ge 1, \forall f \in W^{p,q}(\mathbb{R}^n)$$

b.

$$||U_\lambda f||_{W^{p,q}} \lesssim |\lambda^{-1}|^{n+n\mu_1(q,p)} ||f||_{W^{p,q}} \qquad \forall\ 0 < |\lambda| \le 1, \forall f \in W^{p,q}(\mathbb{R}^n)$$

c.

$$||U_\lambda f||_{W^{p,q}} \gtrsim |\lambda^{-1}|^{n+n\mu_1(q,p)} ||f||_{W^{p,q}} \qquad \forall\ |\lambda| \ge 1, \forall f \in W^{p,q}(\mathbb{R}^n)$$

d.

$$||U_\lambda f||_{W^{p,q}} \gtrsim |\lambda^{-1}|^{n+n\mu_2(q,p)} ||f||_{W^{p,q}} \qquad \forall\ 0 < |\lambda| \le 1, \forall f \in W^{p,q}(\mathbb{R}^n)$$

Proof. We use the dilation property of Fourier transforms and the fact that $\mathcal{F}M^{q,p} = W^{p,q}$. We know that $\widehat{U_\lambda f}(\xi) = \lambda^{-n}\hat{f}(\lambda^{-1}\xi)$. Based on [34] we have

$$||U_\lambda f||_{M^{q,p}} \lesssim |\lambda|^{n\mu_1(q,p)}||f||_{M^{q,p}} \qquad |\lambda| \geq 1$$

Equivalently, we have

$$|\lambda|^{-n}||\hat{f}(\lambda^{-1}\xi)||_{W^{p,q}} \lesssim |\lambda|^{n\mu_1(q,p)}||\hat{f}||_{W^{p,q}} \qquad |\lambda| \geq 1$$

Inequality (2) follows from the change of variable $\lambda \mapsto 1/\lambda$. All the remaining estimates follows the same proof by using the appropriate inequality in [34]. □

Chapter 3

Fourier multipliers

This chapter primarily focuses on a particular kind of Fourier multipliers, namely, unimodular Fourier multipliers of the form $e^{i\mu(D)}$. Here, μ is a real-valued homogeneous function on $\mathbb{R}^n$. These types of multipliers are important objects in the study of evolution equations, as we have illustrated in Section 1.2 of Chapter 1. The main goal of this chapter is to expose the behaviour of these multipliers in Wiener amalgam spaces. Although these multipliers are generally unbounded in L^p except when $p = 2$, we will show that we still have boundedness in $W^{p,q}$ at the cost of having zero to a small loss of regularity. We begin by giving sufficient conditions for a Fourier multiplier to be bounded in $W^{p,q}(\mathbb{R}^n)$.

Lemma 3.0.3. *A Fourier multiplier operator $\sigma(D)$ is bounded on all Wiener amalgam spaces $W^{p,q}(\mathbb{R}^n)$ for $n \geq 1$ and $1 \leq p, q \leq \infty$ whenever $\sigma \in M^{\infty,1}$*

Proof. We use the convolution property of Wiener amalgam spaces stated in Lemma 2.0.1. Since $\mathcal{F}L^\infty * \mathcal{F}L^q \hookrightarrow \mathcal{F}L^q$ and $L^1 * L^p \hookrightarrow L^p$, we have $W(\mathcal{F}L^q, L^p) * W(\mathcal{F}L^\infty, L^1) \hookrightarrow W(\mathcal{F}L^q, L^p)$ with

$$||\check{\sigma} * f||_{W^{p,q}} \leq ||\check{\sigma}||_{W(\mathcal{F}L^\infty, L^1)} ||f||_{W^{p,q}}.$$

By the relation $\mathcal{F}M^{q,p} = W(\mathcal{F}L^q, L^p)$ we conclude that if $\sigma \in M^{\infty,1}$, $\sigma(D)f = \check{\sigma} * f$ is bounded on $W^{p,q}(\mathbb{R}^n)$. $\square$

The next corollary follows directly from Lemma 3.0.3 and [1, Cor. 15].

Corollary 3.0.1. *If $\alpha \in [0,1]$, then $e^{i|D|^\alpha}$ is bounded on $W^{p,q}$ for all $1 \le p, q \le \infty$.*

3.1 Sufficient conditions for the boundedness of $e^{i\mu(D)}$

This section contains the proof of Theorem 1.2.3. In addition, we give sufficient conditions for the boundedness of $e^{i\mu(D)}$ on $W^{p,q}$ for $\alpha \in (0,2)$ which is not covered by Theorem 1.2.3.

Throughout this section, $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ will denote a test function such that

$$\chi(\xi) = \begin{cases} 0 & \text{if } |\xi| \ge 2 \\ 1 & \text{if } |\xi| \le 1 \\ 0 \le \chi(\xi) \le 1 & \text{if } 1 \le |\xi| \le 2. \end{cases}$$

Moreover we let $\Phi(\xi) = (1 - \chi(\xi))$.

Lemma 3.1.1. *Let $J = [n/2] + 1$. Suppose that $\partial^\gamma m \in L^2(\mathbb{R}^n)$ for all $\gamma \in N^n$, $|\gamma| \le J$. Then*

$$||\mathcal{F}^{-1}(m)||_{L_1} \le C||m||_{L^2}^{1-n/(2J)} \left(\sum_{|\gamma|=J} ||\partial^\gamma m||_{L^2} \right)^{n/(2J)}. \tag{3.1}$$

The proof of this lemma can be found in [31].

Lemma 3.1.2 (Lemma 3.1 of [28])**.** *Let m be a bounded function on $\mathbb{R}^n$ with compact support. Suppose that m is of class $\mathcal{C}^{[n/2]+1}(\mathbb{R}^n \backslash \{0\})$ and suppose there exists $\epsilon > 0$*

such that

$$|\partial^\gamma \mu(\xi) \leq C_\gamma |\xi|^{\epsilon - |\gamma|}$$

for $|\gamma| \leq [n/2]+1$. *Then* $m \in \mathcal{F}L^1$.

Lemma 3.1.3. *For* $s>0$, *the function* $|\xi|^s/(1+|\xi|^2)^{s/2}$ *is an element of* $M^{\infty,1}$.

Proof. We write

$$\frac{|\xi|^s}{(1+|\xi|^2)^{s/2}} = \frac{|\xi|^s}{(1+|\xi|^2)^{s/2}}\chi(\xi)^2 + \frac{|\xi|^s}{(1+|\xi|^2)^{s/2}}(1-\chi(\xi)^2).$$

The second term in the sum can be distinguished as an element of $\mathcal{C}^k \subset M^{\infty,1}$ for some k. For the first term, we split into $|\xi|^s\chi(\xi) \cdot \frac{\chi(\xi)}{(1+|\xi|^2)^{s/2}}$ where the second factor is again in $\mathcal{C}^k$ and by Lemma 3.1.2 the first factor belongs to $\mathcal{F}L^1$. This ends our proof. □

Lemma 3.1.4. *Let* $\alpha > 0$, $s \in \mathbb{R}$ *and let* μ *be a real-valued function on* $\mathbb{R}^n$ *which belongs to* $\mathcal{C}^{[n/2]+1}$ *supported away from the origin satisfying*

$$|\partial^\gamma \mu(\xi)| \leq C_\gamma |\xi|^{\alpha-|\gamma|} \qquad for\ \gamma \in N^n, |\gamma| \leq [n/2]+1 \tag{3.2}$$

Set $m(\xi) = \Phi(\xi)|\xi|^{-s}e^{i\mu(\xi)}, \quad \xi \in \mathbb{R}^n$. *If* $s > n\alpha/2$, *then* $m \in \mathcal{F}L^{-1}$

Proof. Let $\Phi_0(\xi) = \Phi(\xi) - \Phi(\xi/2)$ and let $\Phi_\nu(\xi) = \Phi_0(2^{-\nu}\xi)$ for $\nu \in \mathbb{N}$ so that $\Phi(\xi) = \sum_{\nu \in \mathbb{N}} \Phi_\nu(\xi)$. Let $m_\nu(\xi) = \Phi_\nu(\xi)|\xi|^{-s}e^{i\mu(\xi)}$. Then from (3.2) and Leibniz rule we have

$$|\partial^\gamma m_\nu(\xi)| \leq C_\gamma 2^{\nu(-s+|\gamma|(\alpha-1))} \quad \text{for} \quad |\gamma| \leq [n/2]+1.$$

Hence,

$$||\partial^\gamma m_\nu(\xi)||_{L_2} \lesssim 2^{n\nu/2-\nu s+\nu|\gamma|(\alpha-1)}$$

and

$$||m_\nu||_{L_2} \lesssim 2^{\nu n/2-\nu s}$$

for $|\gamma| \leq [n/2]+1$. Therefore, by Lemma 3.1.1 we have

$$||\mathcal{F}^{-1}(m_\nu)||_{L_1} \lesssim 2^{\nu n\alpha/2-\nu s}.$$

Finally, we see that

$$||\mathcal{F}^{-1}(m)||_{L_1} \leq \sum_{\nu\in\mathbb{N}} ||\mathcal{F}^{-1}(m_\nu)||_{L_1} \lesssim \sum_{\nu\in\mathbb{N}} 2^{\nu n\alpha/2-\nu s},$$

where the series converges whenever $s > n\alpha/2$. □

Lemma 3.1.5 (Lemma 3.2 of [28]). *Let $\epsilon > 0$. Suppose μ is a real-valued function of class $\mathcal{C}^{[n/2]+1}(\mathbb{R}^n\backslash\{0\})$ satisfying*

$$\partial^\gamma \mu(\xi) \leq C_\gamma |\xi|^{\epsilon-|\gamma|}$$

for $|\gamma| \leq [n/2]+1$. Then $\eta e^{i\mu} \in \mathcal{F}L^1$ for each $\eta \in \mathcal{S}$ with compact support.

Proposition 3.1.1. *Let $\alpha > 0$, $s \in \mathbb{R}$ and μ be a real-valued function on $\mathbb{R}^n$ which belongs to $\mathcal{C}^{[n/2]+1}(\mathbb{R}^n\backslash\{0\})$ satisfying*

$$|\partial^\gamma \mu(\xi)| \leq C_\gamma |\xi|^{\alpha-|\gamma|} \qquad for\ \gamma \in N^n, |\gamma| \leq [n/2]+1. \tag{3.3}$$

If $s > n\alpha/2$, then $e^{i\mu(D)}$ is bounded from $W_s^{p,q}$ to $W^{p,q}$.

Proof. We use property e of Lemma 2.0.1. Hence, by Lemma 3.0.3 we have

$$||e^{i\mu(D)}||_{\mathcal{L}(W_s^{p,q},W^{p,q})} \asymp ||\langle D\rangle^{-s} e^{i\mu(D)}||_{\mathcal{L}(W^{p,q},W^{p,q})} \lesssim ||\langle\xi\rangle^{-s} e^{-i\mu(\xi)}||_{M^{\infty,1}}. \tag{3.4}$$

Now, we only need to show that $\langle\xi\rangle^{-s} e^{i\mu(\xi)} \in M^{\infty,1}$. To do so, we rewrite

$$\langle\xi\rangle^{-s} e^{i\mu(\xi)} = \sigma_1 + \sigma_2 \tag{3.5}$$

where $\sigma_1 = \chi(\xi)\langle\xi\rangle^{-s}e^{-i\mu(\xi)}$ and $\sigma_2 = (1-\chi(\xi))\langle\xi\rangle^{-s}e^{-i\mu(\xi)}$. Then by Lemma 3.1.5, $\sigma_1 \in \mathcal{F}L^1 \subset M^{\infty,1}$. Meanwhile, we write $\sigma_2 = \dfrac{|\xi|^s}{(1+|\xi|^2)^{s/2}} \cdot \Phi(\xi)|\xi|^{-s}e^{i\mu(\xi)}$. By Lemma 3.1.3 and Lemma 3.1.4, both factors of σ_2 are in $M^{\infty,1}$. We conclude that σ_2 is also in $M^{\infty,1}$. by using the multiplication property of modulation spaces. This ends our proof. □

Remark 3.1.1. By interpolation, Proposition 3.1.1 implies Theorem 1.2.3 for $p \leq q \leq p'$ and $p' \leq q' \leq p$. Indeed, take $(p,q) = (1,\infty)$, then interpolating with [28, Theorem 1.1] for $W^{p,p}$ yields the desired threshold for s.

Remark 3.1.2. For $\alpha \in [0,2]$, an improvement of the estimate $s > n\alpha/2$ can be done with the use of [1, Theorem 1], where the boundedness of $e^{i\mu(D)}$ on $M^{p,p} = W^{p,p}$ is known. We interpolate with the case $(p,q) = (1,\infty)$ to get the improved estimate $s > n\alpha/2|1/p - 1/q|$.

The proof of Theorem 1.2.3 relies on the following lemma that gives sufficient condition for the inclusion property of weighted Wiener amalgam spaces.

Lemma 3.1.6. *Let $1 \leq p, q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$. If $q_1 > q_2$ and $s_1 - s_2 > n(1/q_2 - 1/q_1)$, then $W^{p,q_1}_{s_1} \hookrightarrow W^{p,q_2}_{s_2}$.*

Proof. We write $q_2/q_1 + (q_1-q_2)/q_1 = 1$ and $s_2 = s_1 + (s_2 - s_1)$. By Hölder's inequality we have

$$\begin{aligned} ||f||_{W^{p,q_2}_{s_2}} &= ||\ ||\{\langle k\rangle^{s_2}\varphi(D-k)f\}||_{\ell^{q_2}}||_{L^p} \\ &\leq ||\ ||\{\langle k\rangle^{s_1}\varphi(D-k)f\}||_{\ell^{q_1}}||_{L^p}||\{\langle k\rangle^{s_2-s_1}\}||_{\ell^{q_1q_2/(q_1-q_2)}}. \end{aligned}$$

Since $(s_1 - s_2)q_1q_2/(q_1 - q_2) > n$, we have our desired result. □

We are now ready to prove Theorem 1.2.3.

Proof of Theorem 1.2.3. Using the fact that $M^{p,p} = W^{p,p}$ together with the result in [28], we conclude the boundedness of the operator

$$e^{i\mu(D)} : W^{p,p}_{s_2} \to W^{p,p},$$

where $s_2 \geq n(\alpha - 2)|1/p - 1/2|$. By Lemma 2.0.1 $W^{p,p} \hookrightarrow W^{p,q}$ for $p < q$. On the other hand, setting $q_1 = q$, $q_2 = p$ and $s = s_1$ in Lemma 3.1.6 gives us the inclusion $W^{p,q}_s \hookrightarrow W^{p,p}_{s_2}$ when $s - s_2 > n(1/p - 1/q)$ and $p < q$. Thus, the multiplier operator $e^{i\mu(D)} : W^{p,q}_s \to W^{p,q}$ is bounded for $p < q$ whenever $s > n(\alpha - 2)|1/p - 1/2| + n(1/p - 1/q)$. The case for $q < p$ is achieved by duality. This ends our proof.

□

3.2 Necessary condition

Here we give the proof of Theorem 1.2.4 which establishes the optimality of the threshold obtained in Thereom 1.2.3. We need the following lemmas.

Lemma 3.2.1. *Consider the function $M_\xi 1(x) = e^{ix\xi}$. Its short-time Fourier transform is given by*

$$V_g(M_\xi 1)(y, \omega) = e^{iy.(\xi - \omega)} \hat{\bar{g}}(\omega - \xi)).$$

Proof. This follows easily from direct computation. □

Proof of Theorem 1.2.4. Our proof is an adaptation of the arguments used in [28, Section 5]. It suffices to prove Theorem 1.2.4 only for pairs (p, q) satisfying $p \leq q \leq 2$, that is, the shaded region in Figure 3.1. Indeed, suppose, for contradiction, that $e^{i\mu(D)}$ is bounded from W^{p_0,q_0}_s to W^{p_0,q_0} for some $(1/p_0, 1/q_0)$ in $T_1 \backslash T_1'$ such that $s < n(\alpha - 2)(1/p_0 - 1/2) + n(1/p_0 - 1, q_0)$. Then, interpolating with the estimate for point $(1, 0)$ and $s = (\alpha - 2)n/2$ (by Theorem 1.2.3) would yield an improve estimate for all points of the line segment joining $(0,1)$ and $(1/p_0, 1/q_0)$ inside T_1', which is not possible.

From the assumption we have the estimate:

$$||e^{i\mu(D)}\langle D\rangle^{-s} f||_{W^{p,q}} \leq C||f||_{W^{p,q}} \qquad \forall f \in \mathcal{S}(\mathbb{R}^n) \tag{3.6}$$

Let f be a fixed Schwartz function whose Fourier transform is supported in a small neighborhood $\mathcal{U} \subset \mathbb{R}^n \backslash \{0\}$ of the point ξ_0 in the assumption and is equal to 1 in some neighborhood of ξ_0. Using the dilated $U_\lambda f, \lambda \geq 1$, in the above estimate gives us

$$||U_\lambda e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s} f||_{W^{p,q}} \leq C||U_\lambda f||_{W^{p,q}},$$

since $e^{i\mu(D)}\langle D\rangle^{-s} U_\lambda = U_\lambda e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s}$. Using the dilation properties of Wiener amalgam spaces given in Lemma 2.0.2 we have

$$|\lambda|^{n+n\mu_2(q,p)}||e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s} f||_{W^{p,q}} \leq C|\lambda|^{n+n\mu_1(q,p)}||f||_{W^{p,q}}. \tag{3.7}$$

We now compute for a convenient lower bound of (3.7). Applying Lemma 3.2.1 and by interchanging the integrals we see that

$$|V_g(e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s} f)(y,w)| = (2n)^{-n}|\int_{\mathbb{R}^n} e^{iy\xi + i\lambda^\alpha \mu(\xi)}\hat{\bar{g}}(\omega - \xi)\langle \lambda\xi\rangle^{-s}\hat{f}(\xi)d\xi|.$$

Taking the integral over $|\omega| \leq 1$ we have

$$\begin{aligned} ||V_g(e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s} f)(y,\cdot)||_{L^q} &\geq (2n)^{-n}\left(\int_{|\omega|\leq 1} |\int_{\mathbb{R}^n} e^{iy\xi + i\lambda^\alpha \mu(\xi)}\hat{\bar{g}}(\omega - \xi)\langle \lambda\xi\rangle^{-s}\hat{f}(\xi)\ d\xi|^q\ d\omega\right)^{1/q} \\ &\gtrsim |\int_{\mathbb{R}^n} e^{iy\xi + i\lambda^\alpha \mu(\xi)}\hat{\bar{g}}(\xi)\langle \lambda\xi\rangle^{-s}\hat{f}(\xi)\ d\xi|. \end{aligned} \tag{3.8}$$

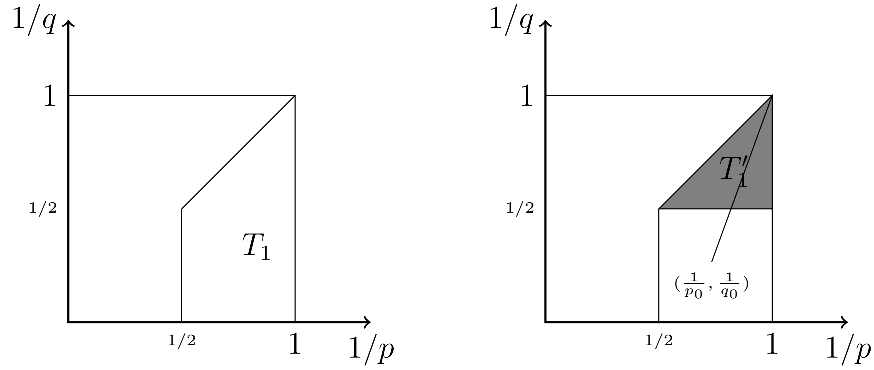

Figure 3.1: Proof by contradiction

Hence by a change of variables $y \mapsto \lambda^\alpha y$ we get

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{iy\xi + i\lambda^\alpha \mu(\xi)} \hat{\bar{g}}(\xi) \langle \lambda\xi \rangle^{-s} \hat{f}(\xi) d\xi|^p \, dy\right)^{1/p} \\ &= \lambda^{\frac{\alpha n}{p}} \left(\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{i\lambda^\alpha (y\xi + \mu(\xi))} \hat{\bar{g}}(\xi) \langle \lambda\xi \rangle^{-s} \hat{f}(\xi) d\xi|^p \, dy\right)^{1/p} \end{aligned} \tag{3.9}$$

Let $y_0 = -\nabla\mu(\xi_0)$. Since the Hessian matrix $d^2\mu(\xi_0)$ of μ at ξ_0 is invertible, it follows from the implicit function theorem that we can find a neighborhood $\mathcal{V}$ of y_0 such that for a small $\mathcal{U}$, the phase $\Phi(\xi) = y\xi + \mu(\xi)$ has a unique non-degenerate critical point $\xi = \xi(y) \in \mathcal{U}$, for every $y \in \mathcal{V}$. Also, choose $g \in \mathcal{S}(\mathbb{R}^n)$, with $\hat{g}(\xi) = 1$ on the support of $\hat{f}$. Now it follows from the stationary phase theorem that for $y \in \mathcal{V}$,

$$\begin{aligned} |\int_{\mathbb{R}^n} e^{i\lambda^\alpha (y\xi + \mu(\xi))} \hat{\bar{g}}(\xi) \langle \lambda\xi \rangle^{-s} \hat{f}(\xi) d\xi| &= \lambda^{-s} |\int_{\mathbb{R}^n} e^{i\lambda^\alpha (y\xi + \mu(\xi))} \hat{\bar{g}}(\xi) \lambda^s \langle \lambda\xi \rangle^{-s} \hat{f}(\xi) d\xi| \\ &\asymp |\det(d^2\mu(\xi(y)))|^{-1/2} \langle \lambda\xi(y) \rangle^{-s} \lambda^{-\alpha n/2} + O(\lambda^{-\alpha(n+2)/2-s}), \end{aligned}$$

where $\det(d^2\mu(\xi(y)))$ is the Hessian matrix of μ at the critical point $\xi(y)$. Indeed, here we use the uniform estimates $|\lambda^s \langle \lambda\xi(y) \rangle^{-s}| \geq C_\lambda$ on the support of $\hat{f}$ and the fact that all derivatives of the phase are uniformly bounded with respect to $y \in \mathcal{V}$. Hence, for some $C > 0$,

$$|\int_{\mathbb{R}^n} e^{i\lambda^\alpha (y\xi + \mu(\xi))} \hat{\bar{g}}(\xi) \langle \lambda\xi \rangle^{-s} \hat{f}(\xi) d\xi| \geq C\lambda^{-\alpha n/2 - s} \qquad y \in \mathcal{V}$$

Combining this estimate with equations (3.8) and (3.9) we arrive to the following estimate

$$||e^{i\mu(\lambda D)}\langle \lambda D\rangle^{-s} f||_{W^{p,q}} \gtrsim \lambda^{\alpha n(1/p-1/2)-s}. \tag{3.10}$$

Using this last estimate with equation (3.7) and letting $\lambda \to +\infty$ gives

$$s \geq \alpha n(1/p - 1/2) + n(\mu_2(q,p) - \mu_1(q,p))$$

Substituting appropriate values of μ_1 and μ_2 for $p \leq q \leq 2$ we have $\mu_2(q,p) - \mu_1(q,p) = 1 - 1/p - 1/q$. Thus, $s \geq (\alpha - 2)n(1/p - 1/2) + n(1/p - 1/q)$ as desired.

□

Chapter 4

Inclusion relations

This chapter is dedicated to the proof of Theorem 1.3.4 and Theorem 1.3.5. We present the sufficient conditions for the inclusion between L^p-Sobolev and Wiener amalgam spaces $W^{p,q}$ in Section 4.1 followed by the necessary conditions in Section 4.2.

4.1 Sufficient condition

Proof of IF part on Theorem 1.3.4. We first prove condition (b). Our strategy is to prove the inclusion $L^p_s \hookrightarrow W^{p,q}$ on four line segments and then interpolate as shown in Figure 4.1. Since $W^{p,p} = M^{p,p}$, we already have the inclusion

$$L^p_{n(2/p-1)} \hookrightarrow W^{p,p}, \quad 1 < p \leq 2 \tag{4.1}$$

due to Theorem 1.3 of [27].

Next we show that $L^p \hookrightarrow W^{p,2}$, $p > 2$ by proving the dual statement $W^{p,2} \hookrightarrow L^p$ where $1 \leq p \leq 2$. By Plancherel's theorem we have the following,

$$W^{p,2} = W(\mathcal{F}L^2, L^p) = W(L^2, L^p) \hookrightarrow W(L^p, L^p) = L^p. \tag{4.2}$$

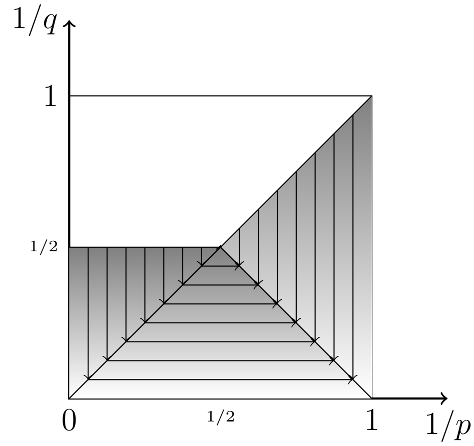

Figure 4.1: Interpolation

Now for $1 \leq p \leq 2$,

$$L^p = W(L^p, L^p) \hookrightarrow W(\mathcal{F}L^{p'}, L^p) = W^{p,p'} \tag{4.3}$$

where the inclusion is due to Hausdorff-Young inequality.

The inclusion $L^p \hookrightarrow W^{p,p}$ for $p \geq 2$ is also done in [27]. We now interpolate these line segments as shown in Figure 4.1, and in view of property (d) of Lemma 2.0.1 this proves condition (b).

Condition (c) follows from (4.3) with $(p, q) = (1, \infty)$ and again property (d) of Lemma 2.0.1.

In proving condition (1), we again use inclusion (4.1) to determine the range of s for the triangular shaded region in Figure 4.2.

By Lemma 3.1.6 we have

$$W^{p,p} \hookrightarrow W^{p,q}_{n(1/p-1/q)-\epsilon}, \qquad p > q.$$

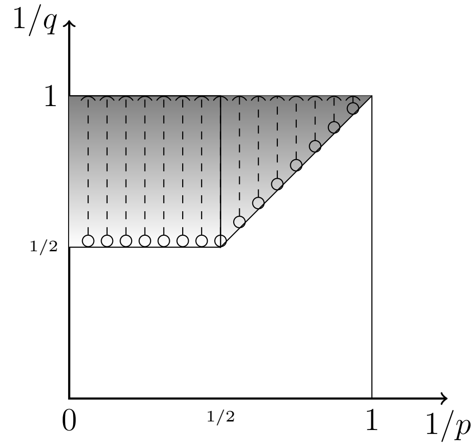

Figure 4.2: Inclusion techniques

Combining this with (4.1) we get

$$L^p_{n(1/p+1/q-1)+\epsilon} \hookrightarrow W^{p,q}, \qquad 1 < q < p \leq 2.$$

To determine the range of s in the square region of Figure 4.2, we again make use of Lemma 3.1.6 for the inclusion

$$W^{p,2} \hookrightarrow W^{p,q}_{n(1/2-1/q)-\epsilon}, \qquad p > 2 \text{ and } q < 2.$$

Together with $L^p \hookrightarrow W^{p,2}$, $p > 2$ we have

$$L^p_{n(1/q-1/2)+\epsilon} \hookrightarrow W^{p,q}.$$

Finally we prove condition (d) with the use of the inclusion $L^1_{n+\epsilon} \hookrightarrow W^{1,1}$, from Theorem 1.3 of [27]. Interpolating this result on the point $(1,1)$ with the point $(1,0)$ yields the desired range of s.

□

4.2 Necessary condition

We begin by introducing a behaviour of the unimodular Fourier operator $e^{i|D|^\alpha}$ with $\alpha \in [0, 1]$. which will be fundamental to our proof.

Lemma 4.2.1. *Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ with $\alpha \in [0, 1]$. Then $e^{i|D|^\alpha}$ is bounded from $W^{p,q}$ to L^p_s if and only if $W^{p,q} \hookrightarrow L^p_s$.*

For the proof we recall that by Corollary 3.0.1, $e^{i|D|^\alpha}$ is bounded on $W^{p,q}$ whenever $\alpha \in [0, 1]$.

Proof. Suppose $e^{i|D|^\alpha}$ is bounded from $W^{p,q}$ to L^p_s :

$$||e^{i|D|^\alpha} f||_{L^p_s} \lesssim ||f||_{W^{p,q}}.$$

Replacing f by $f = e^{-i|D|^\alpha} g$, we have

$$||g||_{L^p_s} \lesssim ||e^{-i|D|^\alpha} g||_{W^{p,q}} \lesssim ||g||_{W^{p,q}},$$

since $e^{-i|D|^\alpha}$ is also bounded from $W^{p,q}$ to L^p_s. This proves that $W^{p,q} \hookrightarrow L^p_s$.

Conversely, assume that $W^{p,q} \hookrightarrow L^p_s$; then we have

$$||e^{i|D|^\alpha} f||_{L^p_s} \lesssim ||e^{i|D|^\alpha} f||_{W^{p,q}} \lesssim ||f||_{W^{p,q}}$$

Hence, $e^{i|D|^\alpha}$ is bounded from $W^{p,q}$ to L^p_s.

□

To show the optimality of condition (d) in Theorem 1.3.4 and Theorem 1.3.5 we need the following lemma [26].

Lemma 4.2.2. *Let $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Then $h^1 \hookrightarrow M^{1,q}_s$ if and only if $s \leq -n/q$. However, in the case $q \neq \infty$, $L^1 \hookrightarrow M^{1,q}_s$ only if $s < -n/q$.*

We recall from [8, Lemma 2.9] the following auxiliary result.

Lemma 4.2.3. *Let φ be the Gauss function $\varphi(t) = e^{-|t|^2}, t \in \mathbb{R}^n$. Then*

$$||\varphi_\lambda||_{W^{p,q}} \asymp (\pi/\lambda^2 + 1)^{n/2(1/q+1/p-1)}(\pi/\lambda^2)^{n/2(1-1/q)}.$$

Lemma 4.2.3 says that $||\varphi_\lambda||_{W^{p,q}} \asymp \lambda^{n(1/q-1)}$ when $\lambda \geq 1$ and $||\varphi_\lambda||_{W^{p,q}} \asymp \lambda^{-n/p}$ in the case of $0 < \lambda \leq 1$.

Lemma 4.2.4. *[34, Lemma 3.8] Let $1 \leq p, q \leq \infty$, $(p, q) \neq (1, \infty), (\infty, 1)$. Define for $\epsilon > 0$*

$$f(t) = \sum_{k \neq 0} |k|^{-n/p-\epsilon} e^{ik\cdot t}\varphi(t) \quad in \quad \mathcal{S}',$$

where φ is the Gauss function. Then $f \in M^{q,p}$ and there exists a constant $C > 0$ such that $||f_\lambda||_{M^{q,p}} \geq C\lambda^{n(1/p-1)+\epsilon}$ for all $0 < \lambda \leq 1$.

The following lemma is a version of Lemma 4.2.4 in Wiener amalgam spaces.

Lemma 4.2.5. *Let $1 \leq p, q \leq \infty$, $(p, q) \neq (1, \infty), (\infty, 1)$. Define for $\epsilon > 0$*

$$\hat{f}(t) = \sum_{k \neq 0} |k|^{-n/p-\epsilon} e^{ik\cdot t}\varphi(t) \quad in \quad \mathcal{S}',$$

where φ is the Gauss function. Then $f \in W^{p,q}$ and there exists a constant $C > 0$ such that $||f_\lambda||_{W^{p,q}} \geq C\lambda^{-n/p-\epsilon}$ for all $\lambda \geq 1$.

Proof. Our proof relies on the fact that $\mathcal{F}M^{q,p} = W^{p,q}$. Using the change of variable $\lambda \to 1/\lambda$ on Lemma 4.2.4 we have

$$\begin{aligned} & ||\hat{f}_{1/\lambda}||_{M^{q,p}} \gtrsim (1/\lambda)^{n(1/p-1)+\epsilon}, \qquad \lambda \geq 1 \\ \iff \quad & \lambda^n ||f_\lambda||_{W^{p,q}} \gtrsim (1/\lambda)^{n(1/p-1)+\epsilon} \qquad \lambda \geq 1, \\ \iff \quad & ||f_\lambda||_{W^{p,q}} \gtrsim \lambda^{-n/p-\epsilon}, \qquad \lambda \geq 1. \end{aligned}$$

□

Proof of the converse of Theorem 1.3.5. We will prove the converse statement in 3 parts based on the division of regions shown in Figure 1.2. First, we deal with region I_3. Let us assume $W^{p,q} \hookrightarrow L^p_s$, then by Lemma 4.2.1 it is equivalent to the statement $e^{i|D|}$ is bounded from $W^{p,q}$ to L^p_s. Then the following estimate holds,

$$||\langle D\rangle^s e^{i|D|} f||_{L^p} \lesssim ||f||_{W^{p,q}} \quad \forall f \in \mathcal{S}. \tag{4.4}$$

Let $g \in \mathcal{S}$ such that

$$\text{supp } \hat{g} \subset \{\xi|2^{-1} < |\xi| < 2\} \text{ and } \hat{g} = 1 \text{ on } \{\xi|2^{-1/2} < |\xi| < 2^{1/2}\} \tag{4.5}$$

We now test this estimate with $f = U_\lambda g, \lambda \geq 1$. Since

$$e^{i|D|}\langle D\rangle^s U_\lambda g = U_\lambda(e^{i|\lambda D|}\langle \lambda D\rangle^s g),$$

it follows from Lemma 2.0.2 that

$$\lambda^{-n/p}||e^{i|\lambda D|}\langle \lambda D\rangle^s g||_{L^p} \lesssim \lambda^{-n-n\mu_2(q,p)}||g||_{W^{p,q}}. \tag{4.6}$$

We now find an appropriate lower bound for estimate (4.6). Using the change of variables $x \mapsto \lambda x$ and the stationary phase theorem we get

$$\begin{aligned}
||e^{i|\lambda D|}\langle \lambda D\rangle^s g||_{L^p} &= ||\int_{\mathbb{R}^n} e^{ix\cdot\xi + i|\lambda\xi|}\langle \lambda\xi\rangle^s \hat{g}(\xi)\ d\xi||_{L^p} \\
&= \lambda^{n/p}||\int_{\mathbb{R}^n} e^{i\lambda(x\cdot\xi + |\xi|)}\langle \lambda\xi\rangle^s \hat{g}(\xi)\ d\xi||_{L^p} \\
&\gtrsim \lambda^{n/p-n/2+s}.
\end{aligned}$$

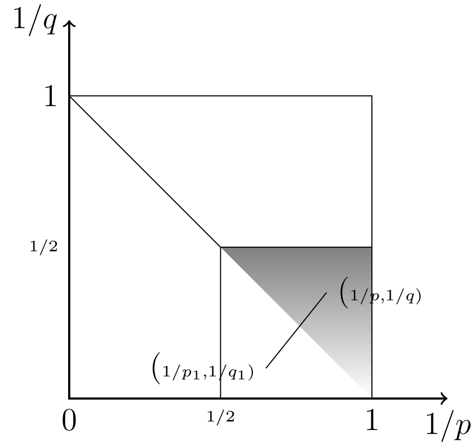

Figure 4.3: Proof by contradiction

Combining this estimate with (4.6) and letting $\lambda \to \infty$ we conclude that the following must hold,

$$s \leq n/2 - n - n\mu_2(q,p).$$

Since $\mu_2(q,p) = -1/q$ on the region shaded in Figure 4.3 we arrive at $s \leq n(1/q - 1/2)$ whenever $W^{p,q} \hookrightarrow L^p_s$. This also informs us of the optimality of our sufficient condition $s \leq \tau(p,q)$ in the shaded region in I_3. We claim that this is true for all points (p,q) in I_3.

Assume otherwise, take any point (p_1, q_1) inside I_3 but outside the shaded region as shown in Figure 4.3, such that $s = n(1/q_1 - 1/2) - \epsilon$. Taking the interpolation with a point (p,q) where $s = n(1/q - 1/2)$ would yield an improvement for the part of the line segment inside the shaded region which is not possible.

Now we prove the result in region I_2, let φ be the Gaussian function so that $\varphi_\lambda(x) = e^{-|\lambda x|^2}$. Let $\lambda \geq 1$, then by the assumption $W^{p,q} \hookrightarrow L^p_s$ and Lemma 4.2.3 we get

$$\lambda^{-n/p+s} \asymp ||\varphi_\lambda||_{L^p_s} \lesssim ||\varphi_\lambda||_{W^{p,q}} \asymp \lambda^{n(1/q-1)}. \tag{4.7}$$

Let $\lambda \to \infty$, then s must satisfy the inequality $s \leq n(1/p + 1/q - 1)$.

Lastly we prove our result in region I_1. By duality we instead prove the converse statement of Theorem 1.3.4 on region I_1^*. Let $\lambda \geq 1$ and assume $L^p_s \hookrightarrow W^{p,q}$, where $s < 0$. Take $s = -\epsilon$, and define

$$\hat{f}(t) = \sum_{k \neq 0} |k|^{-n/p-\epsilon/2} e^{ik\cdot t} \varphi(t),$$

then by Lemma 4.2.5 we have the estimate $||f_\lambda||_{W^{p,q}} \gtrsim \lambda^{-n/p-\epsilon/2}$. From our assumption we can write

$$\lambda^{-n/p-\epsilon/2} \lesssim ||f_\lambda||_{W^{p,q}} \lesssim ||f_\lambda||_{L^p_s} \sim \lambda^{-n/p+s} = \lambda^{-n/p-\epsilon}.$$

However, this is a contradiction which means that $s \geq 0$ whenever $L^p_s \hookrightarrow W^{p,q}$.

To show that condition (d) is in fact optimal, let $1 \leq q < \infty$ and suppose $L^1 \hookrightarrow W^{1,q}_s$. Using Minkowski's inequality we have $W^{1,q}_s \hookrightarrow M^{1,q}_s$. Then we can say that $L^1 \hookrightarrow M^{1,q}_s$. Thus, by Lemma 4.2.2 $s < -n/q$. The case is proved by duality. □

Proof of Corollary 1.3.1. By Theorem 1.3.4 and Theorem 1.3.5 we have

$$W^{p,2} \hookrightarrow L^p \hookrightarrow W^{p,p'}, \qquad 1 \leq p \leq 2$$

and

$$W^{p,p'} \hookrightarrow L^p \hookrightarrow W^{p,2}, \qquad p \geq 2.$$

Thus we combine the above as

$$W^{p,min(p',2)} \hookrightarrow L^p \hookrightarrow W^{p,max(p',2)}.$$

Applying Fourier transform gives us the desired inclusion since $\mathcal{F}W^{p,q} = M^{q,p}$. □

4.3 Applications

4.3.1 Unimodular Fourier multipliers

As one application of our results, we discuss the mapping properties of the unimodular Fourier multiplier operators $e^{i|D|^\alpha}$ between L^p-Sobolev and Wiener amalgam spaces. There have been several works done similarly to this application, some which are presented in Chapter 1. All these works illustrate some advantages in using modulation space or Wiener amalgam spaces. Our work here will tell us what happens if we consider the operator $e^{i|D|^\alpha}$ on Wiener amalgam spaces and L^p-Sobolev spaces.

A straightforward combination of Theorem 1.3.4 and Lemma 4.2.1 yields the following statement. Corollary 4.3.2 is just the dual statement of Corollary 4.3.1.

Corollary 4.3.1. *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $\alpha \in [0,1]$ Then $e^{i|D|^\alpha}$ is bounded from $W^{p,q}$ to L^p_s if one of the following conditions is satisfied.*

a. $p < q, q > 2$ *and* $s < n\tau_2(p,q)$;

b. $p \neq \infty$, $min(1/p, 1/2) \leq 1/q$ *and* $s \leq n\tau_2(p,q)$;

c. $p = \infty, q = 1$ *and* $s \leq n\tau_2(\infty, 1)$;

d. $p = \infty, q \neq 1$ *and* $s < n\tau_2(1,q)$.

Conversely, if $e^{i|D|^\alpha}$ is bounded from $W^{p,q}$ to L^p_s, then $s \leq n\tau_2(p,q)$.

Corollary 4.3.2. *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $\alpha \in [0,1]$ Then $e^{i|D|^\alpha}$ is bounded from L^p_s to $W^{p,q}$ if one of the following conditions is satisfied.*

a. $p > q, q < 2$ *and* $s > n\tau_1(p,q)$;

b. $p \neq 1$, $max(1/p, 1/2) \geq 1/q$ *and* $s \geq n\tau_1(p,q)$;

c. $p = 1, q = \infty$ *and* $s \geq n\tau_1(1, \infty)$;

d. $p = 1, q \neq \infty$ *and* $s > n\tau_1(1,q)$.

Conversely, if $e^{i|D|^\alpha}$ *is bounded from* L^p_s *to* $W^{p,q}$*, then* $s \geq n\tau_1(p,q)$.

4.3.2 Littlewood-Paley type inequalities

Littlewood-Paley theory is a framework that is used to extend certain results in L^2 to L^p. It primarily deals with the expression

$$\left(\sum_{I \in \mathcal{D}} |S_I f|^2 \right)^{1/2}$$

where $\mathcal{D}$ is a collection of dyadic annuli $\{\xi \in \mathbb{R}^n | 2^j < |\xi| < 2^{j+1} : n \in \mathbb{Z}\}$ and S_I is an operator that acts by multiplying the characteristic function of the annulus I in the Fourier transform side, i.e., $\widehat{(S_I f)} = \chi_I \hat{f}$. The classical Littlewood-Paley inequality states that if $1 < p < \infty$ then

$$||f||_{L^p} \lesssim || \left(\sum_{I \in \mathcal{D}} |S_I f|^2 \right)^{1/2} ||_{L^p} \lesssim ||f||_{L^p}, \qquad \forall f \in L^p.$$

Several generalizations of this inequality has been made in literature, in particular, we focus on the following theorem by Rubio de Francia [12].

Theorem 4.3.1. *Let* I *be a bounded* $n-$*dimensional interval, and for each lattice point* $k \in \mathbb{Z}^n$ *consider the translated interval* $k + I$*. Then, for every* $f \in L^p$ *with* $2 \leq p < \infty$*, the following inequality holds:*

$$||(\sum_k |S_{k+I} f|^2)^{1/2}||_{L^p} \leq C_p ||f||_{L^p}.$$

By letting $q = 2$ and $p \geq 2$ in Theorem 1.3.4, we arrived at a Littlewood-Paley type inequality analogous to the work of Rubio de Francia. We write

$$||(\sum_{k\in\mathbb{Z}^n}|\varphi(D-k)f|^2)^{1/2}||_{L^p}\leq C_p||f||_{L^p}.$$

Rubio de Francia also conjectured that for $1 < p < 2$ the modified Littlewood-Paley inequality always holds;

$$||(\sum_k|S_{k+I}f|^{p'})^{1/p'}||_{L^p}\leq C_p||f||_{L^p}.$$

This was later proven to be false by a counterexample given by M. Cowling and T. Tao [9]. Although, if we set $q = p'$ when $1 \leq p \leq 2$ in Theorem 1.3.4 we would have the following estimate which is an analog of the conjecture,

$$||(\sum_{k\in\mathbb{Z}^n}|\varphi(D-k)f|^{p'})^{1/p'}||_{L^p}\leq C_p||f||_{L^p}.$$

We remark that both these results include the critical cases $p = 1$ and $p = \infty$.

Chapter 5

Evolutions equations

In this final chapter, we will survey some recent results concerning evolution equations which is done in the setting of Wiener amalgam spaces and modulation spaces. The results presented here answers the questions we pose in Section 1.4.

Let $S(t) = e^{it\Delta}$ denote the Schrödinger semi-group.

Proposition 5.0.1. *Let $s \in \mathbb{R}, 1 \leq p \leq \infty$ and $0 < q < \infty$. Then we have*

$$||S(t)f||_{M_s^{p,q}} \leq C\,(1+|t|)^{n|1/2-1/p|}\,||f||_{M_s^{p,q}}.$$

This result was done independently in two papers, namely, the works of Wang, Zhao and Guo in [39] and by Bényi, Gröchenig, Okoudjou and Rogers in [1]. In the latter, their result includes a more general semi-group $e^{it\Delta^\alpha}$ with $\alpha \leq 1$. This statement is equivalent to Theorem 1.2.1, while an extension of this can be seen from Theorem 1.2.2. The work of Chen, Fan and Sun [4] obtained estimates for $e^{it(-\Delta)^\alpha}, \alpha > 0$. Looking back at Theorem 1.2.3, we can expect the boundedness of $S(t)$ in Wiener amalgam space $W^{p,q}$ but at the cost of loss of regularity.

Next we consider the truncated decay of $S(t)$.

Proposition 5.0.2. *Let $s \in \mathbb{R}, 2 \leq p \leq \infty, 1/p + 1/p' = 1$ and $0 < q < \infty$. Then we have*

$$||S(t)f||_{M_s^{p,q}} \leq C\,(1+|t|)^{-n|1/2-1/p|}\,||f||_{M_s^{p',q}}$$

The answers to **Q1** and **Q2** are precisely Proposition 5.0.1 and Proposition 5.0.2. These estimates are optimal in the sense that the powers of time variable are sharp [7]. Analogous estimaes are also true for the Klein-Gordon semi-group $G(t) = e^{it\omega^{1/2}}$ where $\omega = I - \Delta$, in modulation space setting.

Proposition 5.0.3. *Let $s \in \mathbb{R}, 1 \leq p \leq \infty$ and $0 < q < \infty$. Then we have*

$$||G(t)f||_{M_s^{p,q}} \leq C\,(1+|t|)^{n|1/2-1/p|}\,||f||_{M_s^{p,q}}$$

$G(t)$ satisfies that following $L^p - L^{p'}$ estimate

$$||G(t)f||_{H_p^{-2\sigma(p)}} \leq C\,(1+|t|)^{-n|1/2-1/p|}\,||f||_{p'} \tag{5.1}$$

where,

$$2 \leq p \leq \infty, 2\sigma(p) = (n+2)(1/2-1/p). \tag{5.2}$$

Proposition 5.0.4. *Let $s \in \mathbb{R}, 2 \leq p \leq \infty, 1/p + 1/p' = 1$ and $0 < q < \infty, \theta \in [0,1]$. Then we have*

$$||G(t)f||_{M_s^{p,q}} \leq C\,(1+|t|)^{-n\theta|1/2-1/p|}\,||f||_{M_{s+2\sigma(p)\theta}^{p',q}}$$

Results for Klein-Gordon semi-group in amalgam spaces are still unknown.

5.1 Strichartz estimates

In the past few decades, there has been several research dedicated to the space-time integrability of the solution of the Cauchy problem for the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 \quad x \in \mathbb{R}^n, t \in \mathbb{R} \\ u(x,0) = u_0(x). \end{cases} \tag{5.3}$$

Let us recall some classical results for the solution $u(t,x) = (e^{it\Delta}u_0)(x)$ of (5.3). First is the L^2 conservation law

$$||e^{it\Delta}u_0||_{L^2_x} = ||u_0||_{L^2_x}, \tag{5.4}$$

then we have the dispersive estimate

$$||e^{it\Delta}u_0||_{L^\infty_x} \lesssim |t|^{-n/2}||u_0||_{L^1_x}. \tag{5.5}$$

By interpolation of (5.3) and (5.5) we get

$$||e^{it\Delta}u_0||_{L^p_x} \lesssim |t|^{-n|1/2-1/p|}||u_0||_{L^{p'}_x}, 2 \le p \le \infty, 1/p + 1/p' = 1 \tag{5.6}$$

We now begin our discussion to give a positive answer to question **Q3**. In [5], Cordero and Nicola proved the following estimate.

Proposition 5.1.1. *We have*

$$||e^{it\Delta}u_0||_{W(\mathcal{F}L^1,L^\infty)} \lesssim \left(\frac{1+|t|}{t^2}\right)^{n/2} ||u_0||_{W(\mathcal{F}L^\infty,L^1)}. \tag{5.7}$$

This estimate is an improvement of (5.5) for any fixed $t \neq 0$ by recalling the following embedding,

$$L^1 = W(L^1,L^1) \hookrightarrow W(\mathcal{F}L^\infty,L^1) \text{ and } W(\mathcal{F}L^1,L^\infty) \hookrightarrow W(L^\infty,L^\infty) = L^\infty.$$

Also, (5.7) recaptures the classical time decay $|t|^{-n/2}$ as $t \to +\infty$. Although, to improve locally in space cost a worsening as $t \to 0$. This is due to the factor $|t|^{-n/2}$ being replace by $|t|^{-n}$ as $t \to 0$. Interpolating (5.7) with (5.4) yields a version of (5.6) that we state in the following theorem. Similarly, this estimate is an improvement of the previously known (5.6).

Theorem 5.1.1 ([5])**.** *We have*

$$||e^{it\Delta}u_0||_{W(\mathcal{F}L^{p'},L^p)} \lesssim \left(\frac{1+|t|}{t^2}\right)^{n(1/2-1/p)} ||u_0||_{W(\mathcal{F}L^p,L^{p'})}. \tag{5.8}$$

Next we move on to space-time estimates. Combining (5.5) with some duality arguments, we get Strichartz estimates. The following theorem collects results from the works of Strichartz [33], Keel and Tao [23], Yajima [43], Ginibre and G. Velo [18].

Theorem 5.1.2. *Let* $2 \leq q, r \leq \infty, 2/q + n/r = n/2, (q, r, n) \neq (2, \infty, 2)(sim.\tilde{q}, \tilde{r})$. *Then we have the Homogeneous Strichartz estimates*

$$||e^{it\Delta}u_0||_{L_t^q L_x^r} \lesssim ||u_0||_{L_x^2}, \tag{5.9}$$

the dual homogeneous Strichartz estimates

$$||\int e^{-is\Delta}F(s)\ ds||_{L^2} \lesssim ||F||_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}, \tag{5.10}$$

and the dual retarded Strichartz estimates

$$||\int_{s<t} e^{-i(t-s)\Delta}F(s)\ ds||_{L_t^q L_x^r} \lesssim ||F||_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}. \tag{5.11}$$

In order to state the next theorem, we need to define the following Wiener amalgam norms:

$$
\begin{aligned}
||F||_{W(L^{q_1},L^{q_2})_t W(\mathcal{F}L^{r_1},L^{r_2})_x} &:= ||||F||_{W(\mathcal{F}L^{r_1},L^{r_2})_x}||_{W(L^{q_1},L^{q_2})_t} \\
&= ||F||_{W(L^{q_1}W(\mathcal{F}L^{r_1},L^{r_2}),L^{q_2})}
\end{aligned}
$$

Theorem 5.1.3 ([5]). *Let $4 \leq q, \tilde{q} \leq \infty, 2 \leq r, \tilde{r} \leq \infty$, such that*

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \tag{5.12}$$

and similarly for $\tilde{q}, \tilde{r}$. Then we have the homogeneous Strichartz estimates

$$||e^{it\Delta}u_0||_{W(\mathcal{F}L^{q/2},L^q)_t W(\mathcal{F}L^{r'},L^r)_x} \lesssim ||u_0||_{L^2_x}, \tag{5.13}$$

the dual homogeneous Strichartz estimates

$$||\int e^{-is\Delta}F(s)\ ds||_{L^2} \lesssim ||F||_{W(\mathcal{F}L^{(\tilde{q}/2)'},L^{\tilde{q}'})_t W(\mathcal{F}L^{\tilde{r}},L^{\tilde{r}'})_x}, \tag{5.14}$$

and the dual retarded Strichartz estimates

$$||\int_{s<t} e^{-i(t-s)\Delta}F(s)\ ds||_{W(\mathcal{F}L^{q/2},L^q)_t W(\mathcal{F}L^{r'},L^r)_x} \lesssim ||F||_{W(\mathcal{F}L^{(\tilde{q}/2)'},L^{\tilde{q}'})_t W(\mathcal{F}L^{\tilde{r}},L^{\tilde{r}'})_x}. \tag{5.15}$$

Notice that for $(q, r) = (\infty, 2)$, estimate (5.13) recaptures (5.9). Moreover, for $q > 4$ we have an improvement for spatial regularity but at the cost of worsening locally, i.e., L^q_t being replaced by $L^{q/2}_t$.

A natural question can be asked regarding Theorem 5.1.3, can it be extended for the case $2 \leq q < 4$? For example, if $q = 2$ we have

$$||e^{it\Delta}u_0||_{W(L^1,L^2)_t W(\mathcal{F}L^{r'},L^r)_x} \lesssim ||u_0||_{L^2_x}, \quad r = 2d/(d-2).$$

The authors of [5] stated that the extension is not possible using the tools used in proving Theorem 5.1.3, namely, Hardy-Littlewood-Sobolev's singular integral theory.

Remark. The Strichartz estimates of Theorem 5.3.1 were applied in [5] to show well-posedness of nonlinear Schrödinger equations or of linear Schrödinger equations with time-dependent potentials. For the sake of completion we state their result in the following section.

5.2 Well-posedness results in Wiener amalgam spaces

In this section, we collect recent results regarding well-posedness of Schrödinger and wave equations in amalgam spaces.

Consider the Cauchy problem for the nonlinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = \lambda |u|^{2k} u \quad t \in \mathbb{R}^n, x \in \mathbb{R}^n, (n \geq 1) \\ u(x,0) = u_0(x) \quad \partial_t u(0,x) = u_1(x), \end{cases} \tag{5.16}$$

where $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$.

Theorem 5.2.1 ([6]). *Let $1 \leq p \leq \infty, s \geq 0, \gamma \geq 0$ and $q' > 2kn$. For every R, there exists $T > 0$ such that for every (u_0, u_1) in the ball B_R center at 0 with radius R in $W(\mathcal{F}L^q_s, L^p_\gamma) \times W(\mathcal{F}L^p_{s-1}, L^q_\gamma)$ there exists a unique solution $u \in C([0,T]; W(\mathcal{F}L^q_s, L^p_\gamma))$*

to (5.16). Furthermore the map $(u_0, u_1) \mapsto u$ *from* B_R *to* $C^0([0,T]; W(\mathcal{F}L^q_s, L^p_\gamma))$ *is Lipschitz continuous.*

Now consider the Cauchy problem for Schrödinger equation with time-dependent potentials

$$\begin{cases} i\partial_t u + \Delta u = V(t,x)u \quad t \in [0,T] = I_T, x \in \mathbb{R}^n, (n \geq 2) \\ u(x,0) = u_0(x) \end{cases} \tag{5.17}$$

with a potential,

$$V \in L^\alpha(I_T; L^p_x), 1/\alpha + n/p \leq 1, 1 \leq \alpha < \infty, n < p \leq \infty. \tag{5.18}$$

Theorem 5.2.2 ([5])**.** *Assume (5.18). Then the Cauchy problem (5.17) has a unique solution* $u \in C(I_T, L^2(\mathbb{R}^n)) \cap L^{q/2}(I_T, W(\mathcal{F}L^{r'}, L^r)) \cap L^2(I_T, W(\mathcal{F}L^{2n/(n+1),2}, L^{2n/(n-1)}))$ *for all* (q,r) *such that* $2/q + n/r = n/2, q > 4, r \geq 2$.

Final remark. In the paper of Wang and Huang [38], they studied the Cauchy problem for generalized Benjamin-Ono (BO), Korteweg-de Vries (KdV) and nonlinear Schrödinger equations, for which the global well-posedness of solutions with small rough data in modulation spaces $M^{2,1}$ is shown. These results are partial answers to **Q4**, since $M^{2,1}$ contains a class of data which are out of control of H^{s_κ} if $s_\kappa < 0$. Their method utilizes smoothing effects in local frequency spaces via frequency-uniform decomposition. As an example, let $S(t) = e^{it\Delta}$ denote the Schrödinger semi-group. Then by Kenig, Ponce and Vega [25, 24] we have the following sharp version of Kato's smoothing effects

$$||S(t)f||_{L^4_x L^\infty_t} \lesssim ||f||_{\dot{H}^{1/4}}. \tag{5.19}$$

Wang and Huang [38] derived a version of (5.19) in localized uniformly partitioned frequency space that could deal with a class of initial data with lower regularity indices. Compare (5.19) with the following estimate [38] ,

$$||\Box_k S(t)f||_{L^p_x L^\infty_t} \lesssim ||\Box_k||_{\dot{H}^{1/p}} \quad k \in \mathbb{Z}, p \geq 4. \tag{5.20}$$

Since $M^{2,1}_s \subset W^{2,1}_s$, working on an analogous result to [38] in $W^{2,1}_s$ means obtaining new local and global well-posedness results for a class of rough data. We are interested to study if estimates (5.20) are applicable to such problem.

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