On a characterization of unbounded homogeneous domains with boundaries of light cone type

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Chapter 1

Introduction

In this thesis we study automorphism groups of some unbounded homogeneous domains in the complex Euclidean space, and give a characterization theorem of a unbounded homogeneous domain by its automorphism group.

1.0.1 Historical background

The study of automorphism groups of domains in the complex Euclidean spaces goes back to Poincaré. Starting with the Riemann mapping theorem, we explain the study of automorphism groups and its connections with the study of homogeneous domains. The Riemann mapping theorem states that any simply connected proper sub-domain in the complex plane is biholomorphic to the unit disk $\mathbb{D}$. This theorem was first stated(shown) by Riemann under some boundary conditions. After some technical development, Carathéodory proved the theorem without boundary conditions, and Koebe generalized the Riemann mapping theorem to the classification of simply connected Riemann surfaces: any simply connected Riemann surface is biholomorphic to one of the three surfaces, the Riemann sphere, the complex plane and the unit disk. As a development of this classification, Koebe and Poincaré showed the uniformization theorem of compact Riemann surfaces, and that are classified by a topological invariant, the genus. On the other hand, one sought a generalization of the Riemann mapping theorem to a higher dimensional case. In the case of dimension two, the unit ball $\mathbb{B}^2$ and the bidisk $\mathbb{D}^2$ are analogous to the unit disk in one dimensional case. However Poincaré proved that these two domains are not biholomorphically equivalent. He described the automorphism groups of these domains and proved that these automorphism groups are not isomorphic. If two domains are biholomorphically equivalent, then the automorphism groups of the domains are isomorphic. Therefore the unit ball and the bidisk in the two dimensional complex Euclidean space are not biholomorphically equivalent. This result shows that it is not sufficient to study higher dimensional complex domains only with topological conditions, in contrast to the one dimensional case. However, the idea of automorphism groups leads us to various studies of complex domains, in particular symmetric domains or more generally homogeneous domains.

A domain is called (holomorphically) homogeneous, if the automorphism group acts transitively, and a domain is called symmetric, if for any point in the domain there exists an involutive automorphism which fixes the point as an isolated point. Every bounded symmetric domain with the Bergman metric is a Hermitian symmetric space, and it is known that every Hermitian symmetric space is homogeneous. The unit ball and the bidisk in the two dimensional complex Euclidean space are both symmetric. Bounded symmetric domains were classified by É. Cartan. He studied Riemannian symmetric spaces extensively, and in consequence of the study he classified Hermitian symmetric spaces. After É. Cartan, Harish-Chandra showed that each Hermitian symmetric space of non-compact type can be realized as a bounded symmetric domain in the complex Euclidean spaces. The study of bounded homogeneous domains in the complex Euclidean spaces was also started by É. Cartan. He showed that the homogeneous bounded domains in dimension at most 3 are all Hermitian symmetric spaces. He did not find a bounded non-symmetric homogeneous domain, so he asked whether all bounded homogeneous domains are symmetric. Piatetskii-Shapiro gave examples of bounded non-symmetric homogeneous domains of dimension 4 and 5 [16]. It was also shown that in dimensions at least 7 there are continuous parameter families of homogeneous bounded domains that are not symmetric [17].

For a general theory of bounded homogeneous domains, the study of automorphism groups is essential. It was shown by H. Cartan that any automorphism group of a bounded domain has a Lie group structure. In particular, for a bounded homogeneous domain, the automorphism group is a Lie group. Thus we have the corresponding Lie algebra, and it is known that the Lie algebra has a structure of j-algebras. The most important class of j-algebras is that of normal j-algebras, which are solvable and split over $\mathbb{R}$. A fundamental theorem due to Piatetskii-Shapiro state that every bounded homogeneous domain has a normal j-algebra in its Lie algebra, and any normal j-algebra corresponds to a bounded homogeneous domain [17]. Although this result does not imply a complete classification of bounded homogeneous domains, it reduces the study of bounded homogeneous domains to that of normal j-algebras. It was shown by Dotti-Miatello that any irreducible homogeneous domain is determined by its automorphism group up to complex conjugates [5]. Therefore for the class of bounded homogeneous domains, a normal j-algebra determines uniquely a bounded homogeneous domain, and automorphism groups characterize bounded homogeneous domains up to complex conjugates in this category.

Not only normal j-algebras, but automorphism groups themselves give important consequence. As the Riemann mapping theorem, let us consider the following problem in a higher dimension: when is a complex manifold biholomorphic to the unit ball $\mathbb{B}^n$? B. Wong gave a geometric characterization theorem of the ball among bounded strongly pseudoconvex domains [19]. Namely, if the domain is bounded strictly pseudoconvex with smooth boundary and homogeneous, then it is biholomorphic to the unit ball $\mathbb{B}^n$. The proof is based on the results of the boundary behavior of the Bergman, Carathéodory and Kobayashi metrics. B. Wong's theorem have a boundary condition and does not consider among all complex manifolds. More general characterization theorem was given among Kobayashi-hyperbolic manifolds [11]: if a complex hyperbolic manifold M of dimension n has the automorphism group isomorphic to $PU(n,1)$, then M is biholomorphic to the unit ball $\mathbb{B}^n$ of dimension n. After these results, Isaev and Kruzhilin determined all complex manifolds with effective action of the unitary group [7], and showed that if, for a complex manifold M of dimension n, the automorphism group is isomorphic to $PU(n,1)$, then M

is biholomorphic to the unit ball $\mathbb{B}^n$ of dimension n or $\mathbb{CP}^n \setminus \overline{\mathbb{B}^n}$, complement of closed ball in projective space of dimension n [8] [9].

Let us consider a general question of a characterization of complex manifolds. If the automorphism groups of two complex manifolds M and N are isomorphic, then is it true that M and N are biholomorphically equivalent? Since there are biholomorphically non-equivalent complex manifolds whose automorphism groups are trivial, this characterization problem does not make sense for such manifolds. Therefore let us restrict our attention to a more reasonable case, homogeneous complex manifolds. In this thesis we only consider homogeneous domains in the complex Euclidean space. For the class of bounded homogeneous domains, a normal j-algebra determines uniquely a bounded homogeneous domain. In contrast to the bounded domains, for unbounded homogeneous domains which possess no bounded realizations, automorphism groups are, in general, not (finite dimensional) Lie groups, and we have not obtained a general theory of the automorphism groups and the characterization theorem. Therefore any unbounded homogeneous domain is of interest. To see difficulty of analysis of unbounded case, consider the Euclidean space $\mathbb{C}^n$. Then the automorphism group is huge, and not a Lie group. Indeed, any holomorphic function and any nowhere vanishing holomorphic function defined on $\mathbb{C}^{n-1}$ present automorphisms by addition and multiplication on one variable, respectively (see [1]). The characterization of $\mathbb{C}^n$ by its automorphism group is given by Isaev [6]. The study of the automorphism group of $\mathbb{C}^n$ is at present progressive. Other important cases of unbounded homogeneous domains are studied by Shimizu and Kodama [12], [13], Byun, Kodama, Shimizu [3], etc.

In this thesis, we proceed with a further example using Kodama and Shimizu's method in [13], and also give a counterexample of the group-theoretic characterization.

1.0.2 Main results

In order to describe our results, let us fix notations here. Let Ω be a complex manifold. An *automorphism* of Ω means a biholomorphic mapping of Ω onto itself. We denote by $\mathrm{Aut}(\Omega)$ the group of all automorphisms of Ω equipped with the compact-open topology. Ω is called *homogeneous* if $\mathrm{Aut}(\Omega)$ acts transitively on Ω. The purpose of our paper is to determine the automorphism group of the unbounded domain

$$D^{n,1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} : -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 > 0\},$$

and give the characterization theorem of $D^{n,1}$ by its automorphism group $\mathrm{Aut}(D^{n,1})$. For these purposes, we also need to consider the domain

$$C^{n,1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} : -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 < 0\},$$

the exterior of $D^{n,1}$ in $\mathbb{C}^{n+1}$. To describe the automorphism groups $\mathrm{Aut}(D^{n,1})$ and $\mathrm{Aut}(C^{n,1})$, we put

$$GU(n,1) = \{A \in GL(n+1, \mathbb{C}) : A^*JA = \nu(A)J, \text{for some}\, \nu(A) \in \mathbb{R}_{>0}\},$$

and the indefinite unitary group

$$U(n,1) = \{A \in GL(n+1, \mathbb{C}) : A^*JA = J\} \subset GU(n,1),$$

where $J = \begin{pmatrix} -1 & 0 \\ 0 & E_n \end{pmatrix}$. Note that $U(1) \times U(n)$ is a maximul compact subgroup of $GU(n,1)$. Consider $\mathbb{C}^*$ as a subgroup of $GU(n,1)$:

$$\mathbb{C}^* \simeq \{ \begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix} : \alpha \in \mathbb{C}^* \} \subset GU(n,1).$$

Note that $\mathbb{C}^*$ is the center of $GU(n,1)$. Since $U(n,1)$ acts transitively on each level sets of $-|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 (\neq 0)$, and $\mathbb{C}^*$ acts on $D^{n,1}$ and $C^{n,1}$, the group $GU(n,1)$ is a subgroup of the automorphism groups of these two domains $D^{n,1}$ and $C^{n,1}$. It can be easily seen that $C^{n,1}$ and $D^{n,1}$ are homogeneous. Now we state our main results.

Theorem 2.3.1. $\mathrm{Aut}(D^{n,1}) = GU(n,1)$ *for* $n > 1$.

We give a group-theoretic characterization theorem of $D^{n,1}$ in the class of complex manifolds contained in Stein manifolds.

Theorem 2.4.3. *Let M be a connected complex manifold of dimension n+1 that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D^{n,1}) = GU(n,1)$ *as topological groups. Then M is biholomorphic to $D^{n,1}$.*

For the domain $C^{n,1}$, the characterization theorem was shown by Byun, Kodama and Shimizu [12](see also remark before Theorem 2.2.2).

A counterexample of the group-theoretic characterization is given:

Theorem 2.5.1. *There exist unbounded homogeneous domains in $\mathbb{C}^n, n \geq 5$ which are not biholomorphically equivalent, while its automorphism groups are isomorphic.*

The domain $D^{n,1}$ is analogous to the de Sitter space

$$\{(x_1, \cdots, x_n) \in \mathbb{R}^n : -x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

The de Sitter space has a well-known property called the Calabi-Markus phenomenon, that is, any isometry subgroups which acts properly discontinuously on the de Sitter space are finite[4]. This phenomenon implies that the de Sitter space has no compact quotient. It is interesting whether similar results occur in other geometry. We will study subgroups of the automorphism group $\mathrm{Aut}(D^{n,1})$ which acts properly discontinuously on $D^{n,1}$ and prove the non-existence of compact quotients of $D^{n,1}$. It is not the precise Calabi-Markus phenomenon, but a rigid phenomenon. This result is given in Appendix.

A sketch of the proof of Theorem 2.4.3

The proof of Theorem 2.3.1 and Theorem 2.5.1 are immediate, so we only give a sketch of the proof of Theorem 2.4.3 here. We have three steps. First we refer Theorem 2.1.17. Then we see that the complex manifold M is biholomorphic to a Reinhardt domain Ω in

$\mathbb{C}^{n+1}$. Furthermore, we can assume that action of $U(1) \times U(n) \subset GU(n,1)$ on Ω induced by the biholomorphism is a linear transformation. Next we consider the $\mathbb{C}^*$-action on Ω induced by the biholomorphism. It is proved in Claim 1 that we have two possibilities. Finally we study the boundary of Ω. Since a orbit of a point in $\mathbb{C}^{n+1}$ by $U(1) \times U(n)$- and $\mathbb{C}^*$-actions together consists generically $2n+1$ real dimensional hypersurfaces, we can determine the boundary $\partial\Omega$ by $U(1) \times U(n)$- and $\mathbb{C}^*$-actions. $\mathbb{C}^*$-action of one of the two possibilities and $U(1) \times U(n)$-action give a orbit of a boundary point which is a light cone type hypersurface, and isomorphic to the boundary of the domains $D^{n,1}$ and $C^{n,1}$. In this case, we see that $\Omega \simeq D^{n,1}$, and $\Omega \not\simeq C^{n,1}$ since $\mathrm{Aut}(C^{n,1})$ is not a Lie group (see Theorem 2.2.1). $\mathbb{C}^*$-action of the other possibility and $U(1) \times U(n)$-action determines an another type boundary and such boundary determines domains whose automorphism groups are not isomorphic to $GU(n,1)$. We must also consider the case that there exist orbits of boundary points by $U(1) \times U(n)$- and $\mathbb{C}^*$-actions together consists real codimension more than one. In this case, however, we can show that the domains determined by these orbits do not have automorphism groups which is isomorphic to $GU(n,1)$. As a consequence, we see that $\Omega \simeq D^{n,1}$ is the only case such that a domain has the automorphism group isomorphic to $GU(n,1)$.

Our paper organizes as follows. In Section 2.1, first we prepare the notion of Reinhardt domains and Kodama-Shimizu's generalized standardization theorem, which is the key lemma for our theorem. We present proofs of some theorems and lemmas for completeness. To determine $\mathrm{Aut}(D^{n,1})$ we need an explicit description of the automorphism group $\mathrm{Aut}(C^{n,1})$, which is considered in Section 2.2. We determine the automorphism groups of $D^{n,1}$ in Section 2.3. In Section 2.4, we prove the characterization theorem of $D^{n,1}$ by its automorphism group $\mathrm{Aut}(D^{n,1})$. In Section 2.5, we construct a counterexample of the characterization of unbounded homogeneous domains.

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Chapter 2

Automorphism groups and the characterization theorem

2.1 Preliminary

2.1.1 Reinhardt domains and Lie group actions

In order to establish terminology and notation, we recall basic facts about Reinhardt domains and Lie group actions, following Kodama and Shimizu [12][13].

Let G be a Lie group and Ω a domain in $\mathbb{C}^n$, and consider a continuous group homomorphism $\rho : G \longrightarrow \mathrm{Aut}(\Omega)$. Then the mapping

$$G \times \Omega \ni (g, x) \longmapsto \rho(g)(x) \in \Omega$$

is continuous, and in fact C^ω. We say that G acts on Ω as a Lie transformation group through ρ. Let $T^n = (U(1))^n$. The n-dimensional torus T^n acts as a group of automorphisms on $\mathbb{C}^n$ by the standard rule

$$T^n \times \mathbb{C}^n \ni (\alpha, z) \longmapsto \alpha \cdot z := (\alpha_1 z_1, \dots, \alpha_n z_n) \in \mathbb{C}^n.$$

By definition, a Reinhardt domain Ω in $\mathbb{C}^n$ is a domain which is stable under this action of T^n. In this way, we have a injective homomorphism $T^n \to \mathrm{Aut}(\Omega)$, whose image is denoted by $T(\Omega)$.

Let f be a holomorphic function on a Reinhardt domain Ω, then f can be expanded uniquely into a Laurent series

$$f(z) = \sum_{\nu \in \mathbb{Z}^n} a_\nu z^\nu,$$

which converges absolutely and uniformly on any compact set in Ω. Here $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$.

We denote by $\Pi(\mathbb{C}^n)$ the group of all automorphisms of $\mathbb{C}^n$ of the form

$$\mathbb{C}^n \ni (z_1, \dots, z_n) \longmapsto (\alpha_1 z_1, \dots, \alpha_n z_n) \in \mathbb{C}^n.$$

where $(\alpha_1, \dots, \alpha_n) \in (\mathbb{C}^*)^n$. For a Reinhardt domain Ω in $\mathbb{C}^n$, we denote by $\Pi(\Omega)$ the subgroup of $\Pi(\mathbb{C}^n)$ consisting of all elements of $\Pi(\mathbb{C}^n)$ leaving Ω invariant.

Lemma 2.1.1 ([12]). *If a holomorphic function f on Ω satisfies the condition that, for some $\nu_0 \in \mathbb{Z}^n$,*

$$f(\alpha \cdot z) = \alpha^{\nu_0} f(z) \text{ for all } \alpha \in T^n \text{ and all } z \in \Omega,$$

then f is of the form $f(z) = a_{\nu_0} z^{\nu_0}$.

Proof. Since we have

$$f(\alpha \cdot z) = \sum_{\nu \in \mathbb{Z}^n} \alpha^\nu a_\nu z^\nu \text{ and } \alpha^{\nu_0} f(z) = \sum_{\nu \in \mathbb{Z}^n} \alpha^{\nu_0} a_\nu z^\nu,$$

it follows from the assumption that, for every $\nu \in \mathbb{Z}^n$, we have

$$\alpha^\nu a_\nu = \alpha^{\nu_0} a_\nu, \text{ for all } \alpha \in T^n$$

This implies that if $a_\nu \neq 0$, then $\nu = \nu_0$. Thus $f(z) = a_{\nu_0} z^{\nu_0}$. □

We need the following lemma to prove the characterization theorem.

Lemma 2.1.2 ([12]). *Let Ω be a Reinhardt domain in $\mathbb{C}^n$. Then $\Pi(\Omega)$ is the centralizer of $T(\Omega)$ in* $\mathrm{Aut}(\Omega)$.

Proof. It is clear that the centralizer of $T(\Omega)$ includes $\Pi(\Omega)$. To prove the inverse inclusion, let $\phi = (\phi_1, \ldots, \phi_n)$ be an element of the centralizer of $T(\Omega)$ in $\mathrm{Aut}(\Omega)$. Then, for every $i = 1, \ldots, n$, we have

$$\phi_i(\alpha \cdot z) = \alpha_i \phi_i(z).$$

By Lemma 2.1.1, it follows that every function ϕ_i is of the form

$$\phi_i(z) = a_i z_i, \quad a_i \in \mathbb{C}.$$

This implies that $\phi \in \Pi(\Omega)$, and the lemma is shown. □

2.1.2 Stein manifolds and the standardization theorem

To state the standardization theorem, first we introduce the notion of Stein manifolds.

Definition 2.1.3. *A complex manifold M is called holomorphically separable if for any $x, y \in M$ with $x \neq y$ there exists a holomorphic function f on M with $f(x) \neq f(y)$.*

Definition 2.1.4. *A complex manifold M is called Stein if the following three conditions are satisfied.*
(i) M is holomorphically separable.
(ii) For any $p \in M$, there exist holomorphic functions $f_1, \ldots, f_n$ on M such that the map $f = (f_1, \ldots, f_n) : M \longrightarrow \mathbb{C}^n$ is an isomorphism of a neighborhood of p onto an open set in $\mathbb{C}^n$.
(iii) M is holomorphically convex, that is, for any compact set $K \subset M$, the set

$$\hat{K} = \{x \in M : |f(x)| \leq \sup_{y \in K} |f(y)| \text{ for all } f \in \mathcal{O}(M)\}$$

is compact.

Definition 2.1.5. *For a complex manifold M, if there exists a Stein manifold $\tilde{M} \supset M$ such that any holomorphic function on M extends holomorphically to $\tilde{M}$, then $\tilde{M}$ is called the smooth envelope of holomorphy of M.*

We state the standardization theorem for torus.

Theorem 2.1.6 (The standardization theorem for torus [2])**.** *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that an injective continuous group homomorphism $\rho : T^n \longrightarrow \mathrm{Aut}(\Omega)$ is given. Then there exists a biholomorphic map F of M onto a Reinhardt domain Ω in $\mathbb{C}^n$ such that*

$$F\rho(T^n)F^{-1} = T^n \subset \mathrm{Aut}(\Omega).$$

We give a proof of the standardization theorem, following Barrett, Bedford and Dadok [2]. Let

$$\exp : \mathbb{R}^n \ni \theta = (\theta_1, \ldots, \theta_n) \longmapsto e^{2\pi i\theta} = (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) \in T^n$$

be the exponential map and let $\mathcal{L}$ be the lattice in $(\mathbb{R}^n)^*$ dual of $\mathbb{Z}^n \subset \mathbb{R}^n$:

$$\mathcal{L} = \{\alpha \in (\mathbb{R}^n)^* : \alpha(\mathbb{Z}^n) \subset \mathbb{Z}\}.$$

Each $\alpha \in \mathcal{L}$ gives the character $\tilde{\alpha}$ on T^n whose value at $e^{2\pi i\theta}$ is $\tilde{\alpha}(e^{2\pi i\theta}) = e^{2\pi i(\alpha\cdot\theta)}$.

We cite three lemmas stated in [2]

Lemma 2.1.7. *Let L be a subset of $\mathcal{L}$. The group generated by L equals $\mathcal{L}$ if and only if the characters in L separates the points of T^n.*

Lemma 2.1.8. *Let $p \in M$. Suppose that $\mathcal{O}(M)$ separates the points of $T^n \cdot p$, the $\rho(T^n)$-orbit of the point p. Then $T^n \cdot p$ is a totally real submanifold of dimension $0 \leq l \leq n$.*

Lemma 2.1.9. *For p in an open dense subset of M the map $e^{2\pi i\theta} \longmapsto e^{2\pi i\theta} \cdot p$ is a diffeomorphism between T^n and $T^n \cdot p$.*

Definition 2.1.10. *For $\alpha \in (T^n)^*$, if there exists $f_\alpha \in \mathcal{O}(M)$ such that $f_\alpha(e^{2\pi i\theta} \cdot z) = e^{2\pi i(\alpha\cdot\theta)} f_\alpha(z)$ for any $z \in M$ and $\theta \in \mathbb{R}^n$, then f_α is called a holomorphic character associated to α.*

For $\alpha \in (T^n)^*$, let f_α and g_α be non-trivial holomorphic characters. Then f_α/g_α is constant on $T^n \cdot p$ and therefore constant on M. Thus there exist at most one character for α up to constant multiple. We put

$$\mathcal{L}_{\mathcal{O}}(M) = \{\alpha \in (T^n)^* : \text{there exists a non zero holomorphic character } f_\alpha\}.$$

Any $f \in \mathcal{O}(M)$ has a Fourier expansion $f = \sum_{\alpha\in\mathcal{L}_{\mathcal{O}}(M)} f_\alpha$, where

$$f_\alpha(z) = \int_{T^n} f(e^{2\pi i\theta} \cdot z)e^{-2\pi i(\alpha\cdot\theta)}d\theta$$

is a holomorphic character associated to α. For a Reinhardt domain $\Omega \subset \mathbb{C}^n$, the above Fourier expansion $f = \sum f_\alpha$ is nothing but the Laurent expansion $f = \sum_{\nu\in\mathbb{Z}^n} a_\nu z^\nu$, and the holomorphic character associated to $\nu \in \mathbb{Z}^n$ is $a_\nu z^\nu$.

Lemma 2.1.11. *$\mathcal{L}_{\mathcal{O}}(M)$ generates $\mathcal{L}$ as a group.*

Proof. Take $p \in M$ as in Lemma 2.1.9. For a holomorphic function f on M, consider its Fourier expansion $f = \sum f_\alpha$. If f_α is restricted to $T^n \cdot p$, then it gives a character on T^n. Since f_α's separate points of $T^n \cdot p$, $\mathcal{L}_{\mathcal{O}}(M)$ generates $\mathcal{L}$ by Lemma 2.1.7. □

For $p \in M$ we define the isotropy subgroup of T^n as

$$T^n_p = \{\theta \in T^n : \theta \cdot p = p\}.$$

Lemma 2.1.12. *For $p \in M$, $T^n_p \neq \{\mathrm{id}\}$ if and only if there exists non-trivial holomorphic character f_α with $f_\alpha(p) = 0$.*

Proof. First assume that $T^n_p = \{\mathrm{id}\}$. Then $T^n \cdot p$ is totally real n-dimensional torus. If $f_\alpha(p) = 0$ for some nontrivial character, then it follows $f_\alpha(T^n \cdot p) = 0$. Thus $f_\alpha \equiv 0$, which is a contradiction.

To prove converse, assume $T^n_p \neq \{\mathrm{id}\}$. Take $\theta(\neq \mathrm{id}) \in T^n_p$. By Lemma 2.1.7 and Lemma 2.1.11, there exists $\alpha \in \mathcal{L}_{\mathcal{O}}(M)$ such that $e^{2\pi i \alpha\cdot\theta} \neq 1$. Then it follows that

$$f_\alpha(p) = f_\alpha(\theta \cdot p) = e^{2\pi i \alpha\cdot\theta} f(p),$$

and thus $f_\alpha(p) = 0$. □

Now we put $S = \{p \in M : T^n_p \neq \{\mathrm{id}\}\}$. Then, by Lemma 2.1.12, S can be written as

$$S = \bigcup_{\alpha \in \mathcal{L}_{\mathcal{O}}(M)} \{f_\alpha = 0\}. \tag{2.1.1}$$

Lemma 2.1.13. *$\mathcal{L}_{\mathcal{O}}(M)$ is given as an intersection of half-spaces, i.e., there exists a subset $\{\beta_j\} \in \mathcal{L}^*$ such that*

$$\mathcal{L}_{\mathcal{O}}(M) = \{\alpha \in \mathcal{L} : \langle \alpha, \beta_j \rangle \geq 0 \text{ for all j}\}.$$

Proof. Write $S = \bigcup S_j$ as a union of irreducible analytic sets S_j. By Lemma 2.1.11, there exist $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_{\mathcal{O}}(M)$ which generate $\mathcal{L}$ as a group. We put $\mu_{j,k}$ the vanishing order of f_α on S_k. For $\alpha \in \mathcal{L}$, it can be written as $\alpha = \nu_1\alpha_1 + \cdots + \nu_n\alpha_n$ for $\nu_i \in \mathbb{Z}$, and therefore

$$f_\alpha = (f_{\alpha_1})^{\nu_1} \cdots (f_{\alpha_n})^{\nu_n}$$

is a meromorphic character associated to α. This character is holomorphic if and only if f_α vanishes to non-negative order on any S_j. It means that $\alpha \in \mathcal{L}_{\mathcal{O}}(M)$ if and only if

$$\nu_1\mu_{1,k} + \cdots + \nu_n\mu_{n,k} \geq 0$$

for each k. If we take

$$\beta_k = \sum_{j=1}^{n} \mu_{j,k}\alpha^*_j,$$

where $\{\alpha^*_j\}$ is the dual basis of $\{\alpha_j\}$, we have shown the lemma.

□

Corollary 2.1.1. *For each irreducible component S_i of S, there exists $\beta_i \in \mathcal{L}^*$ with the following two properties*:

(*i*) *f_α vanishes on S_i if and only if $\langle \alpha, \beta_i \rangle > 0$.*

(*ii*) *$\{\alpha \in \mathcal{L}_{\mathcal{O}}(M) : \langle \alpha, \beta_i \rangle = 0\}$ contains $n-1$ linearly independent elements.*

We prove the local standardization of T^n-action.

Proposition 2.1.14. *For any $p \in M$, there exists a T^n-invariant neighborhood U of p and a biholomorphic map $\phi : U \longrightarrow \phi(U) \subset \mathbb{C}^n$ intertwining the T^n-action on U and an effective linear action of T^n on $\mathbb{C}^n$.*

Proof. Put $k = \dim T^n \cdot p$. Let Λ be the lattice generated by

$$\{\alpha \in \mathcal{L}_{\mathcal{O}}(M) : f_\alpha(p) \neq 0\}$$

and $\beta_1, \ldots, \beta_l$ be a basis of Λ. By Corollary 2.1.1, we have $l \leq k$. Since the corresponding meromorphic characters $f_{\beta_1}, \ldots, f_{\beta_l}$ are nonsingular on $T^n \cdot p$, we can choose a Stein T^n-invariant neighborhood U of $T^n \cdot p$ such that $f_{\beta_i} \in \mathcal{O}(U)$ for $1 \leq i \leq l$. Since $\mathcal{L}_{\mathcal{O}}(M)$ generates $\mathcal{L}$ as a group and $\mathcal{O}(U)$ separates points of U, $f_{\beta_1}, \ldots, f_{\beta_l}$ must separate points on $T^n \cdot p$ and therefore $l \geq k$. Thus $k = l$ and

$$df_{\beta_1} \wedge \cdots \wedge df_{\beta_k}(p) \neq 0.$$

Since U is Stein, there exist $g_1, \ldots, g_n \in \mathcal{O}(U)$ which give local coordinates at p. Expanding g_i into Fourier series we find characters $g_{\alpha_1}, \ldots, g_{\alpha_n}$ such that

$$dg_{\alpha_1} \wedge \cdots \wedge dg_{\alpha_n}(p) \neq 0,$$

and then we can order these characters so that

$$df_{\beta_1} \wedge \cdots \wedge df_{\beta_k} \wedge dg_{\alpha_1} \wedge \cdots \wedge dg_{\alpha_{n-k}}(p) \neq 0$$

holds. The map

$$\phi : U \longrightarrow \phi(U) \subset \mathbb{C}^n$$

given by $\phi = (f_{\beta_1}, \ldots, f_{\beta_k}, g_{\alpha_1}, \ldots, g_{\alpha_{n-k}})$ is locally biholomorphic near $T^n \cdot p$ and one to one on $T^n \cdot p$. Shrinking U if necessary, ϕ is the desired biholomorphic map $\square$

Proposition 2.1.15. *Let $\tilde{M}$ be a Stein manifold with $S \neq 0$, where S is defined as* (2.1.1)*. Then the irreducible components S_j of S satisfy $\bigcap S_j \neq \emptyset$.*

Proof. Let S_j denote any irreducible component of S. Since S is T^n invariant, so is each S_j. We will show $\bigcap_{j=1}^k S_j \neq \emptyset$ for all k. Assume that $\bigcap_{j=1}^k S_j \neq \emptyset$ and $\bigcap_{j=1}^{k+1} S_j = \emptyset$. Since $\tilde{M}$ is Stein, there exists an analytic function f which is 0 on $\bigcap_{j=1}^k S_j$ and 1 on S_{k+1} (see [15]). If we take the average of f on T^n by integration, we obtain a holomorphic function which is constant on orbits of T^n, and thus globally constant. However this contradicts that such the function takes values 0 on $\bigcap_{j=1}^k S_j$ and 1 on S_{k+1}. $\square$

Proof of the standardization theorem for torus. Let $\tilde{M}$ be the Stein manifold that is the envelope of holomorphy of M. and $\tilde{S} = \{z \in \tilde{M} : T^n_z \neq \emptyset\}$. By Proposition 2.1.15, there exists a point $z_0 \in \bigcap \tilde{S}_j$. Let U be a T^n-invariant neighborhood of z_0. Then $\mathcal{L}_{\mathcal{O}}(\tilde{M}) = \mathcal{L}_{\mathcal{O}}(U)$. Indeed, $\alpha \in \mathcal{L}_{\mathcal{O}}(\tilde{M})$ if and only if the set of linear inequalities are satisfied. Since these involve conditions on the $\tilde{S}$, the inequalities hold if and only if they hold in a neighborhood of z_0, and thus $\mathcal{L}_{\mathcal{O}}(\tilde{M}) = \mathcal{L}_{\mathcal{O}}(U)$.

Finally, by Proposition 2.1.14, we may linearize the T^n-action in a neighborhood U of z_0, and the mapping is given as $f = (f_{\alpha_1}, \ldots, f_{\alpha_n})$ with α_j generating $\mathcal{L}_{\mathcal{O}}(U)$. Since $\mathcal{L}_{\mathcal{O}}(\tilde{M}) = \mathcal{L}_{\mathcal{O}}(U)$, it follows that f is holomorphic on $\tilde{M}$. Also, f is one to one, since f_{α_j} generate $\mathcal{O}(\tilde{M})$ and $\tilde{M}$ is holomorphically separable. Thus f is a biholomorphic map onto a Reinhardt domain. $\square$

2.1.3 The generalized standardization theorem

In this section, we recall the generalization of Theorem 2.1.6 given by Kodama and Shimizu.

Proposition 2.1.16 ([12], Proposition1.1)**.** *Let Ω be a bounded Reinhardt domain in $\mathbb{C}^n$. Suppose that*

$$\begin{aligned} \Omega \cap \{z_i = 0\} &\neq \emptyset, \quad 1 \leq i \leq m, \\ \Omega \cap \{z_i = 0\} &= \emptyset, \quad m+1 \leq i \leq n. \end{aligned}$$

If K is a connected compact subgroup of $\mathrm{Aut}(\Omega)$ containing T^n, then there exists a biholomorphic map

$$\phi : \mathbb{C}^m \times (\mathbb{C}^*)^{n-m} \ni (z_1, \ldots, z_n) \longmapsto (w_1, \ldots, w_n) \in \mathbb{C}^m \times (\mathbb{C}^*)^{n-m},$$
$$\begin{cases} w_i = r_i z_{\sigma'(i)} (z'')^{\nu''_i}, & 1 \leq i \leq m, \\ w_i = r_i z_{\sigma''(i)}, & m+1 \leq i \leq n, \end{cases}$$

such that, for $\tilde{\Omega} = \phi(\Omega)$ and $\tilde{K} = \phi K \phi^{-1} \subset \mathrm{Aut}(\tilde{\Omega})$, we have

$$\begin{cases} \tilde{K} = U(k_1) \times \cdots \times U(k_s) \times U(k_{s+1}) \times \cdots \times U(k_l), \\ k_1 + \cdots + k_s + k_{s+1} + \cdots + k_l = n, \\ k_1 + \cdots + k_s = m, \; k_{s+1} = \cdots = k_l = 1, \end{cases}$$

where $r_1, \ldots, r_n \in \mathbb{R}_{>0}$, σ' and σ'' are permutations of $\{1, \ldots, m\}$ and $\{m+1, \ldots, n\}$, respectively, and $(z'')^{\nu''} = z_{m+1}^{\nu_{m+1}} \cdots z_n^{\nu_n}$ for $\nu'' = (\nu_{m+1}, \ldots, \nu_n) \in \mathbb{Z}^{n-m}$.

Using Proposition 2.1.16, we give the proof of the following theorem due to Kodama and Shimizu, which is the key to proving the main theorem in Section 2.4.

Theorem 2.1.17 (The generalized standardization theorem [13])**.** *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy, and let K be a connected compact Lie group of rank n. Assume that an injective continuous group homomorphism ρ of K into $\mathrm{Aut}(M)$ is given. Then there exists a biholomorphic map F of M onto a Reinhardt domain Ω in $\mathbb{C}^n$ such that*

$$F \rho(K) F^{-1} = U(k_1) \times \cdots \times U(k_l) \subset \mathrm{Aut}(\Omega),$$

where $\sum_{j=1}^{l} k_j = n$.

Proof. Since T^n acts effectively on M, by Theorem 2.1.6, there exists a biholomorphic map F_0 of M onto a Reinhardt domain $\Omega_0 \subset \mathbb{C}^n$ such that $F_0\rho(T^n)F_0^{-1} = T^n \subset \mathrm{Aut}(\Omega_0)$. After a suitable permutation of coordinates if we need, we may assume that

$$\begin{aligned} \Omega_0 \cap \{z_i = 0\} &\neq \emptyset, \ 1 \leq i \leq m, \\ \Omega_0 \cap \{z_i = 0\} &= \emptyset, \ m+1 \leq i \leq n. \end{aligned}$$

We put $K_0 = F_0\rho(K)F_0^{-1}$. Take a bounded domain U contained in Ω_0 and satisfying

$$\begin{aligned} U \cap \{z_i = 0\} &\neq \emptyset, \ 1 \leq i \leq m, \\ U \cap \{z_i = 0\} &= \emptyset, \ m+1 \leq i \leq n, \end{aligned}$$

and put

$$\begin{aligned} \Omega_0' &= \{g(z) \in \Omega_0 : g \in K_0, a \in U\} \\ &= \bigcup_{g \in K_0} g(U) = \bigcup_{z \in U} K_0 \cdot z. \end{aligned}$$

Then Ω_0' is preserved by K_0-action, and thus is a bounded Reinhardt domain. Note that Ω_0' also satisfies

$$\begin{aligned} \Omega_0' \cap \{z_i = 0\} &\neq \emptyset, \ 1 \leq i \leq m, \\ \Omega_0' \cap \{z_i = 0\} &= \emptyset, \ m+1 \leq i \leq n, \end{aligned}$$

Then, there exists a biholomorphic map ϕ as in Proposition 2.1.16 such that

$$\begin{cases} \phi\tilde{K}_0\phi^{-1} = U(k_1) \times \cdots \times U(k_s) \times U(k_{s+1}) \times \cdots \times U(k_l), \\ k_1 + \cdots + k_s + k_{s+1} + \cdots + k_l = n, \\ k_1 + \cdots + k_s = m, \ k_{s+1} = \cdots = k_l = 1, \end{cases}$$

We put $F = \phi \circ F_0$ and $\Omega = \phi(\Omega)$. Then Ω is a Reinhardt domain, and F is a biholomorphic map from M onto Ω. We get

$$F\rho(K)F^{-1} = U(k_1) \times \cdots \times U(k_l) \subset \mathrm{Aut}(\Omega)$$

where $\sum k_j = n$. We have shown the theorem. $\square$

2.2 The automorhpsim group of $C^{n,1}$

In this section, we consider the automorphism group $\mathrm{Aut}(C^{n,1})$ of the domain

$$C^{n,1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 < 0\}.$$

Theorem 2.2.1. *For $f = (f_0, f_1, \dots, f_n) \in \mathrm{Aut}(C^{n,1})$, we have*

$$f_0(z_0, z_1, \dots, z_n) = c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0 \text{ or } c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0^{-1},$$

and

$$f_i(z_0, z_1, \dots, z_n) = f_0(z_0, z_1, \dots, z_n)\frac{\sum_{j=0}^n a_{ij}z_j}{\sum_{j=0}^n a_{0j}z_j}, \quad \text{for } i = 1, \dots, n,$$

where c is a nowhere vanishing holomorphic function on $\mathbb{B}^n$, and the matrix $(a_{ij})_{0\le i,j\le n}$ is an element of $PU(n,1)$.

Proof. First we remark that $C^{n,1}$ is biholomorphic to a product domain $\mathbb{C}^* \times \mathbb{B}^n$. In fact, a biholomorphic map is given by

$$\Psi : C^{n,1} \ni (z_0, z_1, \dots, z_n) \mapsto \left(z_0, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) \in \mathbb{C}^* \times \mathbb{B}^n.$$

Therefore, we consider the automorphism group of $\mathbb{C}^* \times \mathbb{B}^n$.

Let $(w_0, w_1, \dots, w_n)$ be a coordinate of $\mathbb{C}^* \times \mathbb{B}^n$, and

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n) \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{B}^n).$$

For fixed $(w_1, \dots, w_n) \in \mathbb{B}^n$, $\gamma_i(\cdot, w_1, \dots, w_n)$ for $i = 1, \dots, n$ are bounded holomorphic functions on $\mathbb{C}^*$. Then, by the Riemann removable singularities theorem and the Liouville theorem, $\gamma_i(\cdot, w_1, \dots, w_n)$ for $i = 1, \dots, n$ are constant. Hence γ_i $(i = 1, \dots, n)$ does not depend on w_0. In the same manner, we see that for the inverse

$$\tau = (\tau_0, \tau_1, \dots, \tau_n) = \gamma^{-1} \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{B}^n)$$

of γ, the functions τ_i for $i = 1, \dots, n$ are independent of w_0. It follows that

$$\overline{\gamma} := (\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathrm{Aut}(\mathbb{B}^n).$$

It is well-known (see[17]) that $\gamma \in \mathrm{Aut}(\mathbb{B}^n)$ is a linear fractional transformation of the form

$$\gamma_i(w_1, w_2, \dots, w_n) = \frac{a_{i0} + \sum_{j=1}^n a_{ij}w_j}{a_{00} + \sum_{j=1}^n a_{0j}w_j}, \quad i = 1, 2, \dots, n,$$

where the matrix $(a_{ij})_{0\le i,j\le n}$ is an element of $PU(n,1)$.

Next we consider γ_0 of γ and τ_0 of τ. By regarding $\overline{\gamma}$ with the standard action of $\mathrm{Aut}(\mathbb{B}^n)$ on $\mathbb{C}^* \times \mathbb{B}^n$, we obtain a biholomorphic map

$$\gamma \circ \overline{\gamma}^{-1}(w_0, w_1, w_2, \dots, w_n) = (\gamma_0(w_0, \overline{\gamma}^{-1}(w_1, w_2, \dots, w_n)), w_1, w_2, \dots, w_n)$$

on $\mathbb{C}^* \times \mathbb{B}^n$. Thus for fixed $(w_1, w_2, \dots, w_n) \in \mathbb{B}^n$, γ_0 is bijective on $\mathbb{C}^*$ with respect to w_0, and $\tau_0(w_0, \overline{\gamma}(w_1, w_2, \dots, w_n))$ is its inverse. Since $\mathrm{Aut}(\mathbb{C}^*) = \{cw, cw^{-1} : c \in \mathbb{C}^*\}$, we have $\gamma_0 = c(w_1, w_2, \dots, w_n)w_0$ or $c(w_1, w_2, \dots, w_n)w_0^{-1}$, where $c(w_1, w_2, \dots, w_n)$ is a nowhere vanishing holomorphic function on $\mathbb{B}^n$.

Since $\Psi^{-1}\mathrm{Aut}(\mathbb{C}^* \times \mathbb{B}^n)\Psi = \mathrm{Aut}(C^{n,1})$, we have shown the theorem.

□

We remark that the group-theoretic characterization of the domain $\mathbb{C}^* \times \mathbb{B}^n$ are proven by Byun, Kodama and Shimizu[3], and in the paper more general domains are treated.

Theorem 2.2.2 (J.Byun, A.Kodama and S.Shimizu [3])**.** *Let M be a connected complex manifold of dimension n+1 that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(C^{n,1})$ *as topological groups. Then M is biholomorphic to $C^{n,1}$.*

2.3 The automorphism group of $D^{n,1}$

In this section, we determine the automorphism group $\mathrm{Aut}(D^{n,1})$ of the domain

$$D^{n,1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 > 0\},$$

which is the exterior of $C^{n,1}$. We assume $n > 1$. We show the following theorem using Theorem 2.2.1 in the previous section.

Theorem 2.3.1. $\mathrm{Aut}(D^{n,1}) = GU(n,1)$ *for* $n > 1$.

Proof. Let $f = (f_0, f_1, \dots, f_n) \in \mathrm{Aut}(D^{n,1})$. If $z_0' \in \mathbb{C}$ is fixed, then the holomorphic functions $f_i(z_0', \cdot, \cdots)$ for $i = 0, \dots, n$, on $D^{n,1} \cap \{z_0 = z_0'\}$ extend holomorphically to $\mathbb{C}^{n+1} \cap \{z_0 = z_0'\}$ by Hartogs theorem. Hence, when z_0 varies, we obtain an extended holomorphic map $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$ such that $f|_{D^{n,1}} \in \mathrm{Aut}(D^{n,1})$. The same consideration for $f^{-1} \in \mathrm{Aut}(D^{n,1})$ shows that there exists a holomorphic map $g : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$, such that $g|_{D^{n,1}} = f^{-1}$. Since $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$ on $D^{n,1}$, the uniqueness of analytic continuation shows that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$ on $\mathbb{C}^{n+1}$. Hence $f \in \mathrm{Aut}(\mathbb{C}^{n+1})$, so that $\mathrm{Aut}(D^{n,1}) \subset \mathrm{Aut}(\mathbb{C}^{n+1})$.

Now we know that $f|_{C^{n,1}} \in \mathrm{Aut}(C^{n,1})$. By Theorem 2.2.1 of the previous section, we have

$$f_0(z_0, z_1, z_2, \dots, z_n) = c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0^{\pm 1},$$

and

$$f_i(z_0, z_1, z_2, \dots, z_n) = f_0(z_0, z_1, z_2, \dots, z_n)\gamma_i\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right)$$

for $i = 1, \dots, n$, where c is a nowhere vanishing holomorphic function on $\mathbb{B}^n$ and

$$\gamma_i(w_1, \dots, w_n) = \frac{a_{i0} + \sum_{j=0}^n a_{ij} w_j}{a_{00} + \sum_{j=0}^n a_{0j} w_j}.$$

If we have

$$f_0(z) = c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0^{-1},$$

considering the Taylor expansion of c at the origin, we see that f_0 is not holomorphic at $z_0 = 0$, which contradicts the fact that f_0 is an entire holomorphic function. Thus we have

$$f_0(z) = c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0.$$

Then the entire functions f_i $(i = 1, \dots, n)$ are expressed as

$$\begin{aligned} f_i(z_0, \dots, z_n) &= \gamma_i\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) c\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0 \\ &= (a_{i0} z_0 + \sum_{j=0}^n a_{ij} z_j) c\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) \Bigg/ \left(a_{00} + \sum_{j=0}^n a_{0j} \frac{z_j}{z_0}\right), \end{aligned}$$

and hence $c(w_1, \ldots, w_n)$ must be divided by $a_{00} + \sum_{j=0}^n a_{0j} w_j$. We now write

$$c(w_1, \ldots, w_n) = (a_{00} + \sum_{j=0}^n a_{0j} w_j) \tilde{c}(w_1, \ldots, w_n),$$

then

$$f_i(z_0, \ldots, z_n) = (a_{i0} z_0 + \sum_{j=0}^n a_{ij} z_j) \tilde{c} \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right).$$

Since $\tilde{c}(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0})$ is holomorphic near $z_0 = 0$, the holomorphic function $\tilde{c}$ must be a non-zero constant C. Consequently, we obtain

$$f(z_0, \ldots, z_n) = \left(C \sum_{j=0}^n a_{0j} z_j, \ldots, C \sum_{j=0}^n a_{nj} z_j \right).$$

Thus we have shown the theorem.

$\square$

2.4 The group-theoretic characterization of $D^{n,1}$ by its automorphism group

We record first some results, which will be used in the proof of the main theorem several times.

Lemma 2.4.1. *Let D and D' be domains in $\mathbb{C}^n$. Then the automorphism groups of domains $\mathbb{C} \times D$, $\mathbb{C}^* \times D'$ and $(\mathbb{C} \times D) \cup (\mathbb{C}^* \times D')$ are not Lie groups.*

Proof. For any nowhere vanishing holomorphic function u on $\mathbb{C}^n$, $f(z) = (u(z_1, \dots, z_n)z_0, z_1, \dots, z_n)$ is an automorphism on each domain. Indeed, the inverse is given by $g(z) = (u(z_1, \dots, z_n)^{-1}z_0, z_1, \dots, z_n)$. Thus the automorphism groups of these domains have no Lie group structures. □

Lemma 2.4.2. *Let p, q, k be non-negative integers and $p + q \geq 2$. For $p + q > k$, any Lie group homomorphism*

$$\rho : SU(p, q) \longrightarrow GL(k, \mathbb{C})$$

is trivial.

Proof. Put $n = p + q$. It is enough to show that the Lie algebra homomorphism

$$d\rho : \mathfrak{su}(p, q) \longrightarrow \mathfrak{gl}(k, \mathbb{C})$$

is trivial. Consider its complex linear extension

$$d\rho_{\mathbb{C}} : \mathfrak{su}(p, q) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathfrak{gl}(k, \mathbb{C}).$$

Since $\mathfrak{su}(p, q) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra, $d\rho_{\mathbb{C}}$ is injective or trivial. On the other hand, $\dim_{\mathbb{C}} \mathfrak{su}(p, q) \otimes_{\mathbb{R}} \mathbb{C} = n^2 - 1 > k^2 = \dim_{\mathbb{C}} \mathfrak{gl}(k, \mathbb{C})$. Thus $d\rho_{\mathbb{C}}$ must be trivial, and so is $d\rho$. □

Now we prove the following main theorem.

Theorem 2.4.3. *Let M be a connected complex manifold of dimension $n+1$ that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D^{n,1}) = GU(n, 1)$ *as topological groups. Then M is biholomorphic to $D^{n,1}$.*

Proof. Denote by $\rho_0 : GU(n, 1) \longrightarrow \mathrm{Aut}(M)$ a topological group isomorphism. Let us consider $U(1) \times U(n)$ as a matrix subgroup of $GU(n, 1)$ in the natural way, and identify $U(n)$ with $\{1\} \times U(n)$. Then, by Theorem 2.1.17, there is a biholomorphic map F from M onto a Reinhardt domain Ω in $\mathbb{C}^{n+1}$ such that

$$F\rho_0(U(1) \times U(n))F^{-1} = U(n_1) \times \cdots \times U(n_s) \subset \mathrm{Aut}(\Omega),$$

where $\sum_{j=1}^{s} n_j = n + 1$. Then, after a permutation of coordinates if we need, we may assume $F\rho_0(U(1) \times U(n))F^{-1} = U(1) \times U(n)$. We define an isomorphism

$$\rho : GU(n, 1) \longrightarrow \mathrm{Aut}(\Omega)$$

by $\rho(g) := F \circ \rho_0(g) \circ F^{-1}$. We will prove that Ω is biholomorphic to $D^{n,1}$.

Put

$$T_{1,n} = \left\{ \begin{pmatrix} u_1 & \\ & u_2 E_n \end{pmatrix} : u_1, u_2 \in U(1) \right\} \subset GU(n,1).$$

Since $T_{1,n}$ is the center of the group $U(1) \times U(n)$, we have $\rho(T_{1,n}) = T_{1,n} \subset \mathrm{Aut}(\Omega)$. Consider $\mathbb{C}^*$ as a subgroup of $GU(n,1)$. So $\mathbb{C}^*$ represents center of $GU(n,1)$. Since $\rho(\mathbb{C}^*)$ is commutative with T^{n+1}, Lemma 2.1.2 tells us that $\rho(\mathbb{C}^*) \subset \Pi(\Omega)$, that is, $\rho(\mathbb{C}^*)$ is represented by diagonal matrices. Furthermore, $\rho(\mathbb{C}^*)$ commutes with $\rho(U(1) \times U(n)) = U(1) \times U(n)$, so that we have

$$\rho\left(e^{2\pi i(s+it)}\right) = \begin{pmatrix} e^{2\pi i\{a_1 s + (b_1 + ic_1)t\}} & \\ & e^{2\pi i\{a_2 s + (b_2 + ic_2)t\}} E_n \end{pmatrix} \in \rho(\mathbb{C}^*),$$

where $s, t \in \mathbb{R}$, $a_1, a_2 \in \mathbb{Z}, b_1, b_2, c_1, c_2 \in \mathbb{R}$. Since ρ is injective, a_1, a_2 are relatively prime and $(c_1, c_2) \neq (0,0)$. To consider the actions of $\rho(\mathbb{C}^*)$ and $U(1) \times U(n)$ on Ω together, put

$$G(U(1) \times U(n)) = \left\{ e^{-2\pi t} \begin{pmatrix} u_0 & \\ & U \end{pmatrix} \in GU(n,1) : t \in \mathbb{R}, u_0 \in U(1), U \in U(n) \right\}.$$

Then we have

$$\begin{aligned} G &:= \rho(G(U(1) \times U(n))) \\ &= \left\{ \begin{pmatrix} e^{-2\pi c_1 t} u_0 & \\ & e^{-2\pi c_2 t} U \end{pmatrix} \in GL(n+1, \mathbb{C}) : t \in \mathbb{R}, u_0 \in U(1), U \in U(n) \right\}. \end{aligned}$$

Note that G is the centralizer of $T_{1,n} = \rho(T_{1,n})$ in $\mathrm{Aut}(\Omega)$.

Let $f = (f_0, f_1, \ldots, f_n) \in \mathrm{Aut}(\Omega) \setminus G$ and consider its Laurent expansions:

$$f_0(z_0, \ldots, z_n) = \sum_{\nu \in \mathbb{Z}^{n+1}} a_\nu^{(0)} z^\nu, \tag{2.4.1}$$

$$f_i(z_0, \ldots, z_n) = \sum_{\nu \in \mathbb{Z}^{n+1}} a_\nu^{(i)} z^\nu, \quad 1 \leq i \leq n. \tag{2.4.2}$$

If f is a linear map of the form

$$\begin{pmatrix} a^{(0)}_{(1,0,\ldots,0)} & 0 & \cdots & 0 \\ 0 & a^{(1)}_{(0,1,0,\ldots,0)} & \cdots & a^{(1)}_{(0,\ldots,0,1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a^{(n)}_{(0,1,0,\ldots,0)} & \cdots & a^{(n)}_{(0,\ldots,0,1)} \end{pmatrix} \in GL(n+1, \mathbb{C}).$$

then f commutes with $\rho(T_{1,n})$, which contradicts $f \notin G$. Thus for any $f \in \mathrm{Aut}(\Omega) \setminus G$, there exists $\nu \in \mathbb{Z}^{n+1} (\neq (1,0,\ldots,0))$ such that $a_\nu^{(0)} \neq 0$ in (2.4.1), or there exists $\nu \in \mathbb{Z}^{n+1} (\neq (0,1,0\ldots,0), \ldots, (0,0\ldots,0,1))$ such that $a_\nu^{(i)} \neq 0$ in (2.4.2) for some $1 \leq i \leq n$.

Remark 2.4.1. *We remark here that, in* (2.4.1) *and* (2.4.2), *there are no negative degree terms of* $z_1, \ldots, z_n$, *since* $\Omega \cup \{z_i = 0\} \neq \emptyset$ *for* $1 \leq i \leq n$ *by the* $U(n)$*-action on* Ω, *and since Laurent expansions are globally defined on* Ω. *Write* $\nu = (\nu_0, \nu') = (\nu_0, \nu_1, \ldots, \nu_n)$ *and* $|\nu'| = \nu_1 + \cdots + \nu_n$. *Let us consider* $\nu' \in \mathbb{Z}^n_{\geq 0}$ *and put*

$$\sum_\nu{}' = \sum_{\nu_0 \in \mathbb{Z}, \nu' \in \mathbb{Z}^n_{\geq 0}}$$

from now on.

Claim 1. $a_1 a_2 c_1 c_2 \neq 0$, *and* $\lambda := c_2/c_1 = a_2/a_1 = \pm 1$.

Proof. To prove the claim, we divide three cases.

Case (i): $c_1 c_2 \neq 0$.
Since $\mathbb{C}^*$ is the center of $GU(n,1)$, it follows that, for $f \in \mathrm{Aut}(\Omega) \setminus G$,

$$f \circ \rho(e^{2\pi i(s+it)}) = \rho(e^{2\pi i(s+it)}) \circ f.$$

By (2.4.1) and (2.4.2), this equation means

$$\begin{aligned} e^{2\pi i\{a_1 s + (b_1 + ic_1)t\}} \sum_\nu{}' a_\nu^{(0)} z^\nu &= \sum_\nu{}' a_\nu^{(0)} (e^{2\pi i\{a_1 s + (b_1+ic_1)t\}} z_0)^{\nu_0^{(0)}} (e^{2\pi i\{a_2 s + (b_2 + ic_2)t\}} z')^{\nu'} \\ &= \sum_\nu{}' a_\nu^{(0)} e^{2\pi i\{a_1 s + (b_1+ic_1)t\}\nu_0^{(0)}} e^{2\pi i\{a_2 s + (b_2+ic_2)t\}|\nu'|} z^\nu \end{aligned}$$

and

$$\begin{aligned} e^{2\pi i\{a_2 s + (b_2 + ic_2)t\}} \sum_\nu{}' a_\nu^{(i)} z^\nu &= \sum_\nu{}' a_\nu^{(i)} (e^{2\pi i\{a_1 s + (b_1+ic_1)t\}} z_0)^{\nu_0^{(i)}} (e^{2\pi i\{a_2 s + (b_2 + ic_2)t\}} z')^{\nu'} \\ &= \sum_\nu{}' a_\nu^{(i)} e^{2\pi i\{a_1 s + (b_1+ic_1)t\}\nu_0^{(i)}} e^{2\pi i\{a_2 s + (b_2+ic_2)t\}|\nu'|} z^\nu, \end{aligned}$$

for $1 \leq i \leq n$. Thus for each $\nu \in \mathbb{Z}^{n+1}$, we have

$$e^{2\pi i\{a_1 s + (b_1+ic_1)t\}} a_\nu^{(0)} = e^{2\pi i\{a_1 s + (b_1+ic_1)t\}\nu_0^{(0)}} e^{2\pi i\{a_2 s + (b_2+ic_2)t\}|\nu'|} a_\nu^{(0)},$$

and

$$e^{2\pi i\{a_2 s + (b_2+ic_2)t\}} a_\nu^{(i)} = e^{2\pi i\{a_1 s + (b_1+ic_1)t\}\nu_0^{(i)}} e^{2\pi i\{a_2 s + (b_2+ic_2)t\}|\nu'|} a_\nu^{(i)},$$

for $1 \leq i \leq n$. Therefore, if $a_\nu^{(0)} \neq 0$ for $\nu = (\nu_0^{(0)}, \nu') = (\nu_0^{(0)}, \nu_1^{(0)}, \ldots, \nu_n^{(0)})$, we have

$$\begin{cases} a_1(\nu_0^{(0)} - 1) + a_2(\nu_1^{(0)} + \cdots + \nu_n^{(0)}) = 0, \\ c_1(\nu_0^{(0)} - 1) + c_2(\nu_1^{(0)} + \cdots + \nu_n^{(0)}) = 0. \end{cases} \tag{2.4.3}$$

Similarly, if $a_\nu^{(i)} \neq 0$ for $\nu = (\nu_0^{(i)}, \nu') = (\nu_0^{(i)}, \nu_1^{(i)}, \ldots, \nu_n^{(i)})$, we have

$$\begin{cases} a_1\nu_0^{(i)} + a_2(\nu_1^{(i)} + \cdots + \nu_n^{(i)} - 1) = 0, \\ c_1\nu_0^{(i)} + c_2(\nu_1^{(i)} + \cdots + \nu_n^{(i)} - 1) = 0, \end{cases} \tag{2.4.4}$$

for $1 \leq i \leq n$.

Suppose $a_\nu^{(0)} \neq 0$ for some $\nu = (\nu_0^{(0)}, \nu_1^{(0)}, \ldots, \nu_n^{(0)}) \neq (1, 0, \ldots, 0)$. Then by (2.4.3) and the assumption $c_1c_2 \neq 0$ it follows that $\nu_0^{(0)} - 1 \neq 0$ and $\nu_1^{(0)} + \cdots + \nu_n^{(0)} \neq 0$. Hence $c_2/c_1 \in \mathbb{Q}$ and $(a_1, a_2) \neq (\pm 1, 0), (0, \pm 1)$ by (2.4.3). On the other hand, if $a_\nu^{(i)} \neq 0$ for some $1 \leq i \leq n$ and $\nu = (\nu_0^{(i)}, \nu_1^{(i)}, \ldots, \nu_n^{(i)}) \neq (0, 1, 0 \ldots, 0), \ldots, (0, 0 \ldots, 0, 1)$, then $\nu_0^{(i)} \neq 0$ and $\nu_1^{(i)} + \cdots + \nu_n^{(i)} - 1 \neq 0$ by (2.4.4) and the assumption $c_1c_2 \neq 0$. In this case, we also obtain $c_2/c_1 \in \mathbb{Q}$ and $(a_1, a_2) \neq (\pm 1, 0), (0, \pm 1)$ by (2.4.4). Consequently, we have

$$\lambda := a_2/a_1 = c_2/c_1 \in \mathbb{Q}$$

by (2.4.3) or (2.4.4).

We now prove that λ is a integer. For the purpose, we assume $\lambda \notin \mathbb{Z}$, that is, $a_1 \neq \pm 1$. First we consider the case $\lambda < 0$. Since $\nu_1^{(i)} + \cdots + \nu_n^{(i)} \geq 0$ for $0 \leq i \leq n$, we have $\nu_0^{(0)} \geq 1$ and $\nu_0^{(i)} \geq 0$ by (2.4.3) and (2.4.4) . Furthermore, the Laurent expansions of the components of $f \in \mathrm{Aut}(\Omega)$ are

$$f_0(z_0, \ldots, z_n) = \sum_{k=0}^{\infty} \sum_{|\nu'|=k|a_1|}{}' a_{\nu'}^{(0)} z_0^{1+k|a_2|} (z')^{\nu'} \tag{2.4.5}$$

and

$$f_i(z_0, \ldots, z_n) = \sum_{k=0}^{\infty} \sum_{|\nu'|=1+k|a_1|}{}' a_{\nu'}^{(i)} z_0^{k|a_2|} (z')^{\nu'} \tag{2.4.6}$$

for $1 \leq i \leq n$. Here we have written $a_{\nu'}^{(0)} = a_{(1+k|a_2|, \nu')}^{(0)}$ and $a_{\nu'}^{(i)} = a_{(k|a_2|, \nu')}^{(i)}$, and so as from now on. Then it follows from (2.4.5) and (2.4.6) that the first degree terms of the Laurent expansions of the composite $f \circ h$ are the composites of the first degree terms of Laurent expansions of f and h, where $h \in \mathrm{Aut}(\Omega)$. We focus on the first degree terms of the Laurent expansions. We put

$$Pf(z) := (a_{(1,0,\ldots,0)}^{(0)} z_0, \sum_{|\nu'|=1}{}' a_{\nu'}^{(1)} (z')^{\nu'}, \ldots, \sum_{|\nu'|=1}{}' a_{\nu'}^{(n)} (z')^{\nu'}). \tag{2.4.7}$$

Then as a matrix we can write

$$Pf = \begin{pmatrix} a_{(1,0,\ldots,0)}^{(0)} & 0 & \cdots & 0 \\ 0 & a_{(0,1,0,\ldots,0)}^{(1)} & \cdots & a_{(0,\ldots,0,1)}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{(0,1,0,\ldots,0)}^{(n)} & \cdots & a_{(0,\ldots,0,1)}^{(n)} \end{pmatrix},$$

which belongs to $GL(n+1,\mathbb{C})$ since f is a biholomorphic map. Hence we have a representation of $GU(n,1)$ given by

$$GU(n,1) \ni g \longmapsto Pf \in GL(n+1,\mathbb{C}),$$

where $f=\rho(g)$. The restriction of this representation to the simple Lie group $SU(n,1)$ is nontrivial since $\rho(U(1)\times U(n)) = U(1)\times U(n)$. However this contradicts Lemma 2.4.2. Thus it does not occur that λ is a negative non-integer.

Next we consider the case $\lambda > 0$ and $\lambda \notin \mathbb{Z}$. Then $\nu_0^{(0)} \leq 1$ and $\nu_0^{(i)} \leq 0$ by (2.4.3) and (2.4.4) since $\nu_1^{(i)} + \cdots + \nu_n^{(i)} \geq 0$ for $0 \leq i \leq n$. Furthermore, the Laurent expansions of f are

$$\begin{aligned} f_0(z_0,\ldots,z_n) &= \sum_{k=0}^{\infty} \sum_{|\nu'|=k|a_1|}{}' a_{\nu'}^{(0)} z_0^{1-k|a_2|}(z')^{\nu'} \\ &= a_{(1,0,\ldots,0)}^{(0)} z_0 + \sum_{|\nu'|=|a_1|}{}' a_{(1-|a_2|,\nu')}^{(0)} z_0^{1-|a_2|}(z')^{\nu'} \\ &\qquad + \sum_{|\nu'|=2|a_1|}{}' a_{(1-2|a_2|,\nu')}^{(0)} z_0^{1-2|a_2|}(z')^{\nu'} + \cdots, \end{aligned}$$

and

$$\begin{aligned} f_i(z_0,\ldots,z_n) &= \sum_{k=0}^{\infty} \sum_{|\nu'|=1+k|a_1|}{}' a_{\nu'}^{(i)} z_0^{-k|a_2|}(z')^{\nu'} \\ &= \sum_{|\nu'|=1}{}' a_{(0,\nu')}^{(i)} (z')^{\nu'} + \sum_{|\nu'|=1+|a_1|}{}' a_{(-|a_2|,\nu')}^{(i)} z_0^{-|a_2|}(z')^{\nu'} \\ &\qquad + \sum_{|\nu'|=1+2|a_1|}{}' a_{(-2|a_2|,\nu')}^{(0)} z_0^{-2|a_2|}(z')^{\nu'} + \cdots \end{aligned}$$

for $1 \leq i \leq n$. We claim that $a_{(1,0,\ldots,0)}^{(0)} \neq 0$. Indeed, if $a_{(1,0,\ldots,0)}^{(0)} = 0$, then $f(z_0,0,\ldots,0) = (0,\ldots,0) \in \mathbb{C}^{n+1}$. This contradicts that f is an automorphism. Take another $h \in \operatorname{Aut}(\Omega)\setminus G$ and put its Laurent expansions

$$\begin{aligned} h_0(z_0,\ldots,z_n) &= \sum_{k=0}^{\infty} \sum_{|\nu'|=k|a_1|}{}' b_{\nu'}^{(0)} z_0^{1-k|a_2|}(z')^{\nu'} \\ h_i(z_0,\ldots,z_n) &= \sum_{k=0}^{\infty} \sum_{|\nu'|=1+k|a_1|}{}' b_{\nu'}^{(i)} z_0^{-k|a_2|}(z')^{\nu'} \end{aligned}$$

for $1 \leq i \leq n$. We have $b_{(1,0,\ldots,0)}^{(0)} \neq 0$ as above. We mention the first degree terms of $f\circ h$. For the first component

$$f_0(h_0,\ldots,h_n) = a_{(1,0,\ldots,0)}^{(0)} h_0 + \sum_{k=1}^{\infty} \sum_{|\nu'|=k|a_1|}{}' a_{\nu'}^{(0)} h_0^{1-k|a_2|}(h')^{\nu'}.$$

Then, for $k > 0$,

$$\begin{aligned} h_0(z)^{1-k|a_2|} &= (\sum_{l=0}^{\infty} \sum_{|\nu'|=l|a_1|}{}' b_{\nu'}^{(0)} z_0^{1-l|a_2|}(z')^{\nu'})^{1-k|a_2|} = z_0^{1-k|a_2|}(\sum_{l=0}^{\infty} \sum_{|\nu'|=l|a_1|}{}' b_{\nu'}^{(0)} z_0^{-l|a_2|}(z')^{\nu'})^{1-k|a_2|} \\ &= (b_{0_n}^{(0)} z_0)^{1-k|a_2|} \left(1 - \frac{1-k|a_2|}{b_{0_n}^{(0)}} z_0^{-|a_2|} \sum_{|\nu'|=|a_1|}{}' b_{\nu'}^{(0)} (z')^{\nu'} + \cdots \right) \end{aligned}$$

Thus $h_0(z)^{1-k|a_2|}$ has the maximum degree of z_0 at most $1-k|a_2| < 1$ and has the minimum degree of z' at least $|a_1| > 1$ in its Laurent expansion. For $|\nu'| = k|a_1|$ and $k > 0$, $(h')^{\nu'}$ has the maximum degree of z_0 at most $-|a_2| < 0$ and the first degree terms of z' are with coefficients of a negative degree z_0 term in its Laurent expansion. Hence the first degree term of Laurent expansion of $f_0(h_0, \ldots, h_n)$ is $a_{(1,0,\ldots,0)}^{(0)} b_{(1,0,\ldots,0)}^{(0)} z_0$.

Similarly, consider

$$f_i(h_0, \ldots, h_n) = \sum_{|\nu'|=1}{}' a_{\nu'}^{(i)} (h')^{\nu'} + \sum_{k=1}^{\infty} \sum_{|\nu'|=1+k|a_1|}{}' a_{\nu'}^{(i)} h_0^{-k|a_2|} (h')^{\nu'}$$

for $1 \le i \le n$. Then, for $k > 0$,

$$h_0^{-k|a_2|} = (b_{0_n}^{(0)} z_0)^{-k|a_2|} \left(1 - \frac{-k|a_2|}{b_{0_n}^{(0)}} z_0^{-|a_2|} \sum_{|\nu'|=|a_1|}{}' b_{\nu'}^{(0)} (z')^{\nu'} + \cdots \right).$$

Thus $h_0^{-k|a_2|}$ has the maximum degree of z_0 at most $-k|a_2| < 0$ and has the minimum degree of z' at least $|a_1| > 1$ in its Laurent expansion. For $|\nu'| = 1 + k|a_1|$ and $k > 0$, $(h')^{\nu'}$ has the maximum degree of z_0 at most $-|a_2| < 0$ and the first degree terms of z' are with coefficients of negative degree z_0 term in its Laurent expansion. Hence the first degree terms of the Laurent expansions of $f_i(h_0, \ldots, h_n)$ is

$$\sum_{j=1}^{n} \sum_{|\nu'|=1}{}' a_{\nu_j}^{(i)} b_{\nu'}^{(j)} (z')^{\nu'},$$

where $\nu_j = (0, \ldots, 0, 1_j, 0, \ldots, 0)$, that is, the j-th component is 1 and the others are 0.

Consequently, the first degree terms of the Laurent expansions of the composite $f \circ h$ are the composites of the first degree terms of Laurent expansions of f and h. Then the same argument as that in previous case shows that this is a contradiction. Indeed, we put Pf as (2.4.7). Then it follows from the above computations that $Pf \in GL(n+1, \mathbb{C})$ since f is an automorphism, and so that we have a representation of $GU(n, 1)$ by

$$GU(n,1) \ni g \longmapsto Pf \in GL(n+1, \mathbb{C}),$$

where $f = \rho(g)$. Therefore this contradicts Lemma 2.4.2, since this representation is nontrivial on $SU(n,1)$ by $\rho(U(1) \times U(n)) = U(1) \times U(n)$. Thus it does not occur that λ is positive non-integer.

Hence we have $\lambda = c_2/c_1 = a_2/a_1 \in \mathbb{Z} \setminus \{0\}$ and $a_1 = \pm 1$. We now prove $\lambda = \pm 1$. By (2.4.3), (2.4.4) and Remark 2.4.1, the Laurent expansions of $f \in \mathrm{Aut}(\Omega)$ are

$$f_0(z_0, \ldots, z_n) = \sum_{k=0}^{\infty} {\sum_{|\nu'|=k}}' a_{\nu'}^{(0)} z_0^{1-k\lambda} (z')^{\nu'},$$

and

$$f_i(z_0, \ldots, z_n) = a_{(\lambda,0,\ldots,0)}^{(i)} z_0^{\lambda} + \sum_{k=0}^{\infty} {\sum_{|\nu'|=1+k}}' a_{\nu'}^{(i)} z_0^{-k\lambda} (z')^{\nu'}$$

for $1 \leq i \leq n$. Consider the actions of $(e^{2\pi i \frac{m}{\lambda}}, 1, \ldots, 1) \in T^{n+1}$ on Ω, for $1 \leq m \leq |\lambda|$. Then

$$\begin{aligned} f_0(e^{2\pi i \frac{m}{\lambda}} z_0, \ldots, z_n) &= \sum_{k=0}^{\infty} {\sum_{|\nu'|=k}}' a_{\nu'}^{(0)} (e^{2\pi i \frac{m}{\lambda}} z_0)^{1-k\lambda} (z')^{\nu'} \\ &= e^{2\pi i \frac{m}{\lambda}} \sum_{k=0}^{\infty} {\sum_{|\nu'|=k}}' a_{\nu'}^{(0)} z_0^{1-k\lambda} (z')^{\nu'} \\ &= e^{2\pi i \frac{m}{\lambda}} f_0(z_0, \ldots, z_n), \end{aligned}$$

and

$$f_i(e^{2\pi i \frac{m}{\lambda}} z_0, \ldots, z_n) = f_i(z_0, \ldots, z_n)$$

for $1 \leq i \leq n$. Thus $(e^{2\pi i \frac{m}{\lambda}}, 1, \ldots, 1) \in T^{n+1}$ for $1 \leq m \leq |\lambda|$ are included in the center $\rho(\mathbb{C}^*)$ of $\rho(GU(n,1))$. Since $a_2 c_2 \neq 0$, we see that the integer λ must be ± 1.

Case (ii): $c_1 \neq 0$, $c_2 = 0$.
In this case, $\Omega \subset \mathbb{C}^{n+1}$ can be written of the form $(\mathbb{C} \times D) \cup (\mathbb{C}^* \times D')$, where D and D' are open sets in $\mathbb{C}^n$. Indeed, $\Omega = (\Omega \cap \{z_0 = 0\}) \cup (\Omega \cap \{z_0 \neq 0\})$. Then $\{0\} \times D := \Omega \cap \{z_0 = 0\} \subset \Omega$ implies $\mathbb{C} \times D \subset \Omega$ by $\rho(\mathbb{C}^*)$- and T^{n+1}-actions on Ω. On the other hand, $\Omega \cap \{z_0 \neq 0\} = \mathbb{C}^* \times D'$ for some open set $D' \subset \mathbb{C}^n$ by $\rho(\mathbb{C}^*)$- and T^{n+1}-actions. Thus $\Omega = (\mathbb{C} \times D) \cup (\mathbb{C}^* \times D')$. Then, by Lemma 2.4.1, $\mathrm{Aut}(\Omega)$ has no Lie group structure, and this contradicts the assumption $\mathrm{Aut}(\Omega) = GU(n,1)$.

Case (iii): $c_1 = 0$ and $c_2 \neq 0$.
As in the previous case, $\Omega \subset \mathbb{C}^{n+1}$ can be written of the form $(D'' \times \mathbb{C}^n) \cup (D''' \times \mathbb{C}^n \setminus \{0\})$ by $\rho(\mathbb{C}^*)$- and T^{n+1}-actions on Ω, where D'' and D''' are open sets in $\mathbb{C}$. Then, by Lemma 2.4.1, $\mathrm{Aut}(\Omega)$ has no Lie group structure, and this contradicts our assumption. $\square$

Since $G = \rho(G(U(1) \times U(n)))$ acts as linear transformations on $\Omega \subset \mathbb{C}^{n+1}$, it preserves the boundary $\partial\Omega$ of Ω. We now study the action of G on $\partial\Omega$. The type of the G-orbits of points in $\mathbb{C}^{n+1}$ consist of four types as follows:

(i) If $p = (p_0, p_1, \ldots, p_n) \in \mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})$, then

$$G \cdot p = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} : -a|z_0|^{2\lambda} + |z_1|^2 + \cdots + |z_n|^2 = 0\}, \tag{2.4.8}$$

where $a := (|p_1|^2 + \cdots + |p_n|^2)/|p_0|^{2\lambda} > 0$ and $\lambda = \pm 1$ by Claim 1.

(ii) If $p' = (0, p'_1, \ldots, p'_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, then

$$G \cdot p' = \{0_1\} \times (\mathbb{C}^n \setminus \{0_n\}). \tag{2.4.9}$$

(iii) If $p'' = (p''_0, 0, \ldots, 0) \in \mathbb{C}^{n+1} \setminus \{0\}$, then

$$G \cdot p'' = \mathbb{C}^* \times \{0_n\}. \tag{2.4.10}$$

(iv) If $p''' = (0, \ldots, 0) \in \mathbb{C}^{n+1}$, then

$$G \cdot p''' = \{0\} \subset \mathbb{C}^{n+1}. \tag{2.4.11}$$

Claim 2. $\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\}))$ *is a proper subset of* $\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})$.

Proof. If $\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})) = \mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})$, then Ω equals one of the following domains by G-actions of type (2.4.9) and (2.4.10) above:

$$\mathbb{C}^{n+1}, \mathbb{C}^{n+1} \setminus \{0\}, \mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\}), \mathbb{C} \times (\mathbb{C}^n \setminus \{0_n\}) \text{ or } \mathbb{C}^* \times \mathbb{C}^n.$$

However these can not occur since all automorphism groups of these domains are not Lie groups, by Lemma 2.4.1. This contradicts that $\mathrm{Aut}(\Omega) = GU(n,1)$. $\square$

By Claim 2, $\partial\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})) \neq \emptyset$. Thus we can take a point

$$p = (p_0, \ldots, p_n) \in \partial\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})).$$

Let

$$\begin{aligned} &a = (|p_1|^2 + \cdots + |p_n|^2)/|p_0|^{2\lambda} > 0, \\ &A_{a,\lambda} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : -a|z_0|^{2\lambda} + |z_1|^2 + \cdots + |z_n|^2 = 0\}. \end{aligned}$$

Note that

$$\partial\Omega \supset A_{a,\lambda}.$$

If $\lambda = 1$, then Ω is included in

$$D_{a,1} = \{|z_1|^2 + \cdots + |z_n|^2 > a|z_0|^2\}$$

or

$$C_{a,1} = \{|z_1|^2 + \cdots + |z_n|^2 < a|z_0|^2\}.$$

If $\lambda = -1$, then Ω is included in

$$D_{a,-1} = \{(|z_1|^2 + \cdots + |z_n|^2)|z_0|^2 > a\}$$

or

$$C_{a,-1} = \{(|z_1|^2 + \cdots + |z_n|^2)|z_0|^2 < a\}.$$

Claim 3. *If $\Omega = D_{a,1}$, then Ω is biholomorphic to $D^{n,1}$.*

Proof. Indeed there exists a biholomorphic map

$$\Phi : D_{a,1} \ni (z_0, z_1, \ldots, z_n) \longmapsto (a^{-1/2} z_0, z_1, \ldots, z_n) \in D^{n,1}.$$

□

We will show that Claim 3 is the only case that a domain has the automorphism group isomorphic to $GU(n,1)$.

Let us first consider the case $\partial\Omega = A_{a,\lambda}$, that is, $\Omega = C_{a,1}, D_{a,-1}$ or $C_{a,-1}$, and we derive contradictions.

Claim 4. *$\mathrm{Aut}(C_{a,1})$ and $\mathrm{Aut}(D_{a,-1})$ are not Lie groups, so $\Omega \neq C_{a,1}, D_{a,-1}$.*

Proof. Indeed, $C_{a,1}$ is biholomorphic to $\mathbb{C}^* \times \mathbb{B}^n$, and $D_{a,-1}$ is biholomorphic to $\mathbb{C}^* \times (\mathbb{C}^n \setminus \mathbb{B}^n)$. The automorphism groups of these domains are not Lie groups, by Lemma 2.4.1. □

Claim 5. $\Omega \neq C_{a,-1}$.

Proof. Suppose $\Omega = C_{a,-1}$. Then, for $f \in \mathrm{Aut}(\Omega) \setminus G$, the Laurent expansions are

$$f_0(z_0, \ldots, z_n) = \sum_{k=0}^{\infty} {\sum_{|\nu'|=k}}' a^{(0)}_{\nu'} z_0^{1+k} (z')^{\nu'},$$

and

$$f_i(z_0, \ldots, z_n) = a^{(i)}_{(\lambda,0,\ldots,0)} z_0^{-1} + \sum_{k=0}^{\infty} {\sum_{|\nu'|=1+k}}' a^{(i)}_{\nu'} z_0^{k} (z')^{\nu'}$$

for $1 \leq i \leq n$. Since $C_{a,-1} \cap \{z_0 = 0\} \neq \emptyset$, negative degree of z_0 does not arise in the Laurent expansions. Therefore

$$\begin{aligned} f_0(z_0, \ldots, z_n) &= \sum_{k=0}^{\infty} {\sum_{|\nu'|=k}}' a^{(0)}_{\nu'} z_0^{1+k} (z')^{\nu'}, \\ f_i(z_0, \ldots, z_n) &= \sum_{k=0}^{\infty} {\sum_{|\nu'|=1+k}}' a^{(i)}_{\nu'} z_0^{k} (z')^{\nu'} \end{aligned}$$

for $1 \leq i \leq n$. Consider

$$Pf(z) = (a^{(0)}_{(1,0,\ldots,0)} z_0, {\sum_{|\nu'|=1}}' a^{(1)}_{(0,\nu')} (z')^{\nu'}, \ldots, {\sum_{|\nu'|=1}}' a^{(n)}_{(0,\nu')} (z')^{\nu'}),$$

as in the proof of Claim 1. Then $Pf \in GL(n+1, \mathbb{C})$, since f is an automorphism. The Laurent expansions of $f \in \mathrm{Aut}(\Omega)$ have no constant terms, so the first degree terms of Laurent expansions of $f \circ h$ are the composites of the first degree terms of Laurent expansions of f and h, where $f, h \in \mathrm{Aut}(\Omega)$. Hence we have a nontrivial representation of $GU(n,1)$, as in the proof of Claim 1, by

$$GU(n,1) \ni g \longmapsto Pf \in GL(n+1, \mathbb{C}),$$

where $f = \rho(g)$, and this contradicts Lemma 2.4.2. Thus $\Omega \neq C_{a,-1}$. □

Let us consider the case $\partial\Omega \neq A_{a,\lambda}$.

Case (I) : $(\partial\Omega \setminus A_{a,\lambda}) \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})) = \emptyset$.
In this case, $\partial\Omega$ is the union of $A_{a,\lambda}$ and some of the following sets

$$\{0_1\} \times (\mathbb{C}^n \setminus \{0_n\}), \mathbb{C}^* \times \{0_n\} \text{ or } \{0\} \subset \mathbb{C}^{n+1}, \tag{2.4.12}$$

by the G-actions on the boundary of type (2.4.9), (2.4.10) and (2.4.11) above. If $\Omega \subset D_{a,-1}$, then sets in (2.4.12) can not be included in the boundary of Ω. Thus we must consider only the case $\Omega \subsetneq D_{a,1}$, $C_{a,1}$ or $C_{a,-1}$.

Case (I-i) : $\Omega \subsetneq D_{a,1}$.
In this case, $\mathbb{C}^* \times \{0_n\}$ can not be a subset of the boundary of Ω, and $\{0\} \in A_{a,1}$. Thus

$$\begin{aligned}\partial\Omega &= A_{a,1} \cup (\{0_1\} \times \mathbb{C}^n),\\ \Omega &= D_{a,1} \setminus (\{0_1\} \times \mathbb{C}^n).\end{aligned}$$

Then, Ω is biholomorphic to $\mathbb{C}^* \times (\mathbb{C}^n \setminus \mathbb{B}^n)$ and $\mathrm{Aut}(\mathbb{C}^* \times (\mathbb{C}^n \setminus \mathbb{B}^n))$ does not have a Lie group structure. This contradicts the assumption that $\mathrm{Aut}(\Omega) = GU(n,1)$. Thus this case does not occur.

Case (I-ii) : $\Omega \subsetneq C_{a,1}$.
In this case, $\{0\} \times (\mathbb{C}^n \setminus \{0_n\})$ can not be a subset of the boundary of Ω, and $\{0\} \in A_{a,1}$. Thus

$$\begin{aligned}\partial\Omega &= A_{a,1} \cup (\mathbb{C} \times \{0_n\}),\\ \Omega &= C_{a,1} \setminus (\mathbb{C} \times \{0_n\}).\end{aligned}$$

Then, Ω is biholomorphic to $\mathbb{C}^* \times (\mathbb{B}^n \setminus \{0_n\})$ and $\mathrm{Aut}(\mathbb{C}^* \times (\mathbb{B}^n \setminus \{0_n\}))$ does not have a Lie group structure. This contradicts the assumption that $\mathrm{Aut}(\Omega) = GU(n,1)$, and this case does not occur.

Case (I-iii) : $\Omega \subsetneq C_{a,-1}$.
In this case, Ω coincides with one of the followings:

$$\begin{aligned}C_1 &= C_{a,-1} \setminus (\{0_1\} \times \mathbb{C}^n) \cup (\mathbb{C} \times \{0_n\}),\\ C_2 &= C_{a,-1} \setminus (\{0_1\} \times \mathbb{C}^n),\\ C_3 &= C_{a,-1} \setminus (\mathbb{C} \times \{0_n\}),\\ C_4 &= C_{a,-1} \setminus \{0_{n+1}\}.\end{aligned}$$

Then C_1 is biholomorphic to $\mathbb{C}^* \times (\mathbb{B}^n \setminus \{0_n\})$, and C_2 is biholomorphic to $\mathbb{C}^* \times \mathbb{B}^n$. The automorphism groups of these domains are not Lie groups. This contradicts the assumption. The proof of Claim 5 also leads that $\Omega \neq C_3, C_4$ since $C_3 \cap \{z_0 = 0\} \neq \emptyset$ and $C_4 \cap \{z_0 = 0\} \neq \emptyset$. Thus this case does not occur.

Case (II) : $(\partial\Omega \setminus A_{a,\lambda}) \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})) \neq \emptyset$.
In this case, we can take a point $p' = (p'_0, \ldots, p'_n) \in (\partial\Omega \setminus A_{a,\lambda}) \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\}))$. Put

$$\begin{aligned}b &= (|p'_1|^2 + \cdots + |p'_n|^2)/|p'_0|^{2\lambda} > 0,\\ B_{b,\lambda} &= \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : -b|z_0|^{2\lambda} + |z_1|^2 + \cdots + |z_n|^2 = 0\}.\end{aligned}$$

We may assume $a > b$ without loss of generality.

Case (II-i) : $\partial\Omega = A_{a,\lambda} \cup B_{b,\lambda}$.
Since Ω is connected, it coincides with

$$C_{a,1} \cap D_{b,1} = \{b|z_0|^2 < |z_1|^2 + \cdots + |z_n|^2 < a|z_0|^2\},$$

or

$$C_{a,-1} \cap D^-_{b,-1} = \{b < (|z_1|^2 + \cdots + |z_n|^2)|z_0|^2 < a\}.$$

These domains are biholomorphic to $\mathbb{C}^* \times \mathbb{B}^n(a,b)$, where

$$\mathbb{B}^n(a,b) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : b < |z_1|^2 + \cdots + |z_n|^2 < a\}.$$

Then $\mathrm{Aut}(\mathbb{C}^* \times \mathbb{B}^n(a,b))$ does not have a Lie group structure by Lemma 2.4.1, and this contradicts our assumption. Thus this case does not occur.

Case (II-ii) : $\partial\Omega \neq A_{a,\lambda} \cup B_{b,\lambda}$.
Suppose $\partial\Omega \cap (\mathbb{C}^* \times \mathbb{C}^n \setminus \{0_n\}) \setminus (A_{a,\lambda} \cup B_{b,\lambda}) \neq \emptyset$, then we can take

$$p'' = (p''_0,\ldots,p''_n) \in \partial\Omega \cap (\mathbb{C}^* \times \mathbb{C}^n \setminus \{0_n\}) \setminus (A_{a,\lambda} \cup B_{b,\lambda}).$$

Then put

$$\begin{aligned} &c = (|p''_1|^2 + \cdots + |p''_n|^2)/|p''_0|^{2\lambda}, \\ &C_{c,\lambda} = \{(z_0,\ldots,z_n) \in \mathbb{C}^{n+1} : -c|z_0|^{2\lambda} + |z_1|^2 + \cdots + |z_n|^2 = 0\}. \end{aligned}$$

We have $A_{a,\lambda} \cup B_{b,\lambda} \cup C_{c,\lambda} \subset \partial\Omega$. However Ω is connected, this is impossible. Therefore this case does not occur. Let us consider the remaining case:

$$\partial\Omega \cap (\mathbb{C}^* \times \mathbb{C}^n \setminus \{0_n\}) \setminus (A_{a,\lambda} \cup B_{b,\lambda}) = \emptyset.$$

However, $\mathbb{C}^* \times \{0_n\}$, $\{0\} \times (\mathbb{C}^n \setminus \{0_n\})$ and $\{0\} \in \mathbb{C}^{n+1}$ can not be subsets of the boundary of Ω since $\Omega \subset C_{a,1} \cap D_{b,1}$ or $\Omega \subset C_{a,-1} \cap D_{b,-1}$. Thus this case does not occur either.

We have shown that $\partial\Omega = A_{a,1}$ and $\Omega = D_{a,1}$ which is biholomorphic to $D^{n,1}$.

$\square$

2.5 A counterexample of the group-theoretic characterization

Theorem 2.5.1. *There exist domains in $\mathbb{C}^n, n \geq 5$ which are not biholomorphically equivalent, while their automorphism groups are isomorphic.*

Proof. Suppose $p, q > 1$ and $p + q = n$. Let

$$D^{p,q} = \{(z_1, \ldots, z_p, w_1, \ldots, w_q) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_p|^2 - |w_1|^2 - \cdots - |w_q|^2 > 0\},$$
$$C^{p,q} = \{(z_1, \ldots, z_p, w_1, \ldots, w_q) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_p|^2 - |w_1|^2 - \cdots - |w_q|^2 < 0\}.$$

If $p \neq q$, then $D^{p,q}$ and $C^{p,q}$ are not biholomorphically equivalent, while $\mathrm{Aut}(D^{p,q}) = \mathrm{Aut}(C^{p,q})$. Indeed, as the proof of Theorem 2.2.1, we take $f \in \mathrm{Aut}(D^{p,q})$. If $(w'_1, \ldots, w'_q) \in \mathbb{C}^q$ is fixed, then the holomorphic functions $f_i(\cdots, w'_1, \ldots, w'_q)$ for $i = 1, \ldots, n$, on $D^{p,q} \cap \{w_1 = w'_1, \ldots, w_q = w'_q\}$ extend holomorphically to $\mathbb{C}^n \cap \{w_1 = w'_1, \ldots, w_q = w'_q\}$ by Hartogs theorem and $p > 1$. Hence, when $w_1, \ldots, w_q$ vary, we obtain a extended holomorphic map $\tilde{f} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $\tilde{f}|_{D^{p,q}} = f \in \mathrm{Aut}(D^{p,q})$. The same consideration for $f^{-1} \in \mathrm{Aut}(D^{p,q})$ shows that there exists a holomorphic map $g : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$, such that $g|_{D^{p,q}} = f^{-1}$. Since $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$ on $D^{p,q}$, the uniqueness of analytic continuation shows that $g \circ \tilde{f} = \mathrm{id}$ and $\tilde{f} \circ g = \mathrm{id}$ on $\mathbb{C}^n$. Hence $\tilde{f} \in \mathrm{Aut}(\mathbb{C}^{\mathrm{n}})$. Now we see that $\tilde{f}|_{C^{p,q}} \in \mathrm{Aut}(C^{p,q})$ and therefore we have a group homomorphism

$$\phi : \mathrm{Aut}(D^{p,q}) \longrightarrow \mathrm{Aut}(C^{p,q}), \quad f \longmapsto \tilde{f}|_{C^{p,q}}.$$

In the same manner, we have

$$\psi : \mathrm{Aut}(C^{p,q}) \longrightarrow \mathrm{Aut}(D^{p,q}), \quad g \longmapsto \tilde{g}|_{C^{p,q}}.$$

by Hartogs theorem and $q > 1$. It is clear that $\phi \circ \psi = \mathrm{id}$ on $\mathrm{Aut}(C^{p,q})$ and $\psi \circ \phi = \mathrm{id}$ on $\mathrm{Aut}(D^{p,q})$. Thus we obtain $\mathrm{Aut}(D^{p,q}) \simeq \mathrm{Aut}(C^{p,q})$. □

We have not yet obtained a explicit description of the automorphism groups $\mathrm{Aut}(D^{p,q})$ for $p, q > 1$. We only expect that $\mathrm{Aut}(D^{p,q}) = GU(p, q)$, where

$$GU(p, q) = \{M \in GL(n, \mathbb{C}) : M^* J M = \nu(M) J, \text{for some}\, \nu(M) \in \mathbb{R}_{>0}\},$$

and $J = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}$.

The difference between $D^{n,1}$ and $D^{p,q}$ for $p, q > 1$ is that the exterior of $D^{n,1}$ is holomorphically convex domain, but that of $D^{p,q}$ is not. It is known that some holomorphically convex homogeneous Reinhardt domains are characterized by its automorphism groups with some additional conditions (see [3], [6] [8], and [12]). We may proceed with the group-theoretic characterization problem for holomorphically convex homogeneous Reinhardt domains, or for homogeneous Reinhardt domains with a holomorphically convex exterior domain.

Chapter 3

Appendix

3.1 The non-existence of compact quotients of $D^{n,1}$

In this section, we show the non-existence of compact quotients of $D^{n,1}$. This theorem is contained in the paper by Mukuno and the author [14]. When we were studying the automorphism group of $D^{n,1}$, Mukuno gave the proof of the theorem. Also he taught the author the Calabi-Markus phenomenon, that was new for the author. Although this theorem is due to Mukuno, we would like to introduce that to the reader.

Theorem 3.1.1. *$D^{n,1}$ has no compact quotients by discrete subgroup of $\mathrm{Aut}(D^{n,1})$ acting properly discontinuously.*

We remark that $C^{n,1}$ has compact quotients since $\mathbb{B}^n$ and $\mathbb{C}^*$ has compact quotients. Recall the following result called the Calabi–Markus phenomenon:

Lemma 3.1.2 (Calabi–Markus[4], Wolf[18])**.** *Let Γ be a subgoup of $O(p, q+1)$ acting properly disontinuously on*

$$\{(x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q+1}) \in \mathbb{R}^{n+1} : -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q+1}^2 = 1\},$$

where $1 < p \leq q$. Then Γ is finite.

Proof. From Theorem 2.3.1, we know that $\mathrm{Aut}(D^{n,1}) = GU(n,1) = \mathbb{R}_{>0} \times U(n,1)$, which acts on the complex Euclidean space as linear transformations. We regard $\mathbb{R}_{>0} \times U(n,1)$ as a subgroup of $\mathbb{R}_{>0} \times O(2n,2)$.

Suppose that there exists a discrete subgroup

$$\Gamma = \{f_m\}_{m=1}^{\infty} \subset \mathbb{R}_{>0} \times O(2n,2)$$

such that Γ acts properly discontinuously on $D^{n,1}$ and that the quotient $D^{n,1}/\Gamma$ is compact. By Selberg's lemma, we may assume without loss of generality that Γ is torsion free. Set $f_m = (r_m, T_m)$, where $r_m \in \mathbb{R}_{>0}$ and $T_m \in O(2n,2)$. It is clear that Γ is not included in $O(2n,2)$ by Lemma 3.1.2. We consider two cases.

First we consider the case where there exists the minimum of the set $\{r_m | 1 < r_m\}$. We denote the minimum by R:

$$R = \min\{r_m | 1 < r_m\}.$$

Then we see that, for any r_m, there exists an integer l such that $r_m = R^l$. Therefore we can write

$$\Gamma = \{f_{l,k} = (R^l, T_{l,k})\}_{l\in\mathbb{Z}, k\in\mathbb{N}}$$

by changing the indexes. Put $\Gamma_0 = \{f_{0,k}\}$, a subgroup of $O(2n, 2)$. By Theorem 3.1.2, it follows that Γ_0 is a finite group. Since Γ_0 is torsion free, $\Gamma_0 = \{\mathrm{id}\}$. Therefore, Γ is the group generated by the element $(R, T) \in \Gamma$. Hence we see that $D^{n,1}/\Gamma$ is not compact.

Next we consider the case where there does not exist the minimum of the set $\{r_m | 1 < r_m\}$. Let R' be the infimum of the set $\{r_m | 1 < r_m\}$:

$$R' = \inf\{r_m | 1 < r_m\}.$$

Then, for any $\epsilon > 0$, by arranging the indexes of the elements of Γ, we can take an infinite distinct sequence

$$R' + \epsilon > r_1 > r_2 > r_3 > \cdots > r_m > \cdots > R'.$$

Let

$$\Pi = \{z_0 = 0\} \subset \mathbb{C}^{n+1}$$

and

$$K = \{z_0 = 0, 1 \leq |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \leq (R' + \epsilon)^2 + 1\} \subset \mathbb{C}^{n+1}.$$

It is clear that K is compact in $D^{n,1}$. Let $\gamma_m = (r_m, T_m)$. We can easily see that $\gamma_m(\Pi)\cap\Pi$ contains a nontrivial linear subspace by the dimension formula of linear map. Then there exist $v_m \in \gamma_m(\Pi) \cap \Pi$ and $w_m \in \Pi$ such that $v_m = \gamma_m(w_m)$ and that $|w_m| = 1$. Note that $w_m \in K$. We see that $|v_m| = r_m|w_m| = r_m \leq R' + \epsilon$, since $v_m \in \Pi$, and thus $v_m \in K$. We obtain that $\gamma_m(K) \cap K \neq \emptyset$ for any $m \geq 1$. However this is a contradiction since Γ acts on properly discontinuously. The proof is complete.

□

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