On a family of Lagrangian submanifolds in bidisks and Lagrangian Hofer metric

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Contents

Chapter 1

Introduction

1.1 Background

In a geometric framework of classical mechanics, symplectic manifolds and the group of Hamiltonian diffeomorphisms naturally appear. In [Ho90], Hofer proved that for the most basic symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ there exists an intrinsic norm on the group of Hamiltonian diffeomorphisms, which is called the *Hofer norm* or *symplectic energy* of Hamiltonian diffeomorphisms. After his remarkable discovery, this new geometry, which is called *Hofer geometry*, has been intensively studied in the framework of modern symplectic geometry (we refer [Po01] and [HZ94] as standard texts on Hofer geometry).

For a Lagrangian submanifold L of a symplectic manifold (M, ω), we denote by $\mathcal{L}(L) = \mathcal{L}(L, M, \omega)$ the set of Lagrangian submanifolds which are Hamiltonian isotopic to L:

$$\mathcal{L}(L) := \{L' \subset M \mid L' = \phi(L) \text{ for some } \phi \in \mathrm{Ham}_c(M, \omega)\}.$$

Here $\mathrm{Ham}_c(M, \omega)$ is the group of compactly supported Hamiltonian diffeomorphisms on (M, ω). Similarly to the case of $(\mathbb{R}^{2n}, \omega_0)$, we have the Hofer norm $\|\phi\|$ on $\mathrm{Ham}_c(M, \omega)$ defined by

$$\|\phi\| := \inf \int_0^1 \left(\max_{p \in M} H(t, p) - \min_{p \in M} H(t, p) \right) dt,$$

where the infimum runs over all compactly supported Hamiltonians $H \in C_c^\infty([0, 1] \times M)$ having time-one map ϕ_H^1 equal to ϕ. Using the Hofer norm, we can define a pseudo metric, which is called the *Lagrangian Hofer pseudo metric* on $\mathcal{L}(L)$, as follows:

$$d(L_0, L_1) := \inf\{\ \|\phi\| \mid \ \phi(L_0) = L_1,\ \phi \in \mathrm{Ham}_c(M, \omega)\}.$$

In other words, $d(L_0, L_1)$ is the minimal Hofer norm which is necessary for transporting L_0 to L_1 by using Hamiltonian diffeomorphisms.

Chekanov showed in [Ch00] that this pseudo-metric d is non-degenerate for any closed and connected Lagrangian submanifolds in tame symplectic manifolds. Although our Lagrangian submanifolds studied in this article are not closed, the same proof as Chekanov's yields that d is also non-degenerate for our cases below (see Section 2.3).

For a given Lagrangian submanifold L in a symplectic manifold (M, ω), it is a fundamental question whether the Lagrangian Hofer metric space $(\mathcal{L}(L), d)$ has an infinite diameter or not. There are some known examples of Lagrangian submanifolds whose Lagrangian Hofer metric space are unbounded (see Section 2.3). In the Hamiltonian case, it is expected that $\mathrm{Ham}(M, \omega)$ is always unbounded with respect to the Hofer norm. In contrast to this, an example of bounded Lagrangian Hofer metric space, which is associated to a *displaceable* Lagrangian submanifold, can be found in [Us13]. As Usher mentioned, this example suggests that there seems to be some relation between intersection rigidity of a Lagrangian submanifold and unboundedness of its Lagrangian Hofer metric space.

1.2 Main Result

Before we state our results, we mention two results which are closely related to ours.

In [Kh09], Khanevsky proved unboundedness of this metric when the ambient space M is an open unit two dimensional disk $B^2 := \{z \in \mathbb{C} \mid |z| < 1\} \subset \mathbb{C}$ and the Lagrangian submanifold L is the real form $Re(B^2) := \{z \in B^2 \mid \mathrm{Im}\ z = 0\}$ of B^2. Seyfaddini generalized Khanevsky's unboundedness result to the case of the real form $Re(B^{2n})$ of higher dimensional open unit ball B^{2n} in [Se14].

In this paper, by adopting Seyfaddini's technique, we prove unboundedness of metric spaces $\mathcal{L}(L)$ for a certain continuous family of non-compact Lagrangian submanifolds in bi-disks, which are mutually non-Hamiltonian isotopic.

Let $B^2(r) \subset \mathbb{C}$ be the open disk of radius $r > 0$ equipped with a symplectic structure $2\omega_0$, where ω_0 is the standard symplectic structure on $\mathbb{C}$ so that $\mathrm{vol}(B^2(r)) = 2\pi r^2$. We simply denote by B^2 the open unit disk $B^2(1)$. We put $(B^2 \times B^2, \bar{\omega}_0) := (B^2(1) \times B^2(1), 2\omega_0 \oplus 2\omega_0)$ and define Lagrangian submanifolds L_δ by

$$L_\delta := T_\delta \times Re(B^2) \subset B^2 \times B^2$$

for each $1/2 < \delta \leq 1$. Here

$$T_\delta := \{|z_1|^2 = 1/(2\delta)\} \subset B^2$$

and $Re(B^2)$ is the real form of B^2.

We study the Lagrangian Hofer metric space $\mathcal{L}(L_\delta, d)$ in this paper. We prove the following.

Theorem 1.1. *For any* $1/2 < \delta \leq 1$*,* $(\mathcal{L}(L_\delta), d)$ *has an infinite diameter.*

In addition to unboundedness, we prove the following inequality for a subfamily of $\{L_\delta\}$.

Theorem 1.2. *For any* $(2+\sqrt{3})/4 < \delta \leq 1$*, there exists a continuous map* $\Phi_\delta : C_c^\infty((0,1)) \to \mathcal{L}(L_\delta)$ *such that*

$$\frac{\|f-g\|_\infty - D_\delta}{C_\delta} \leq d(\Phi_\delta(f), \Phi_\delta(g)) \leq \|f-g\|,$$

where C_δ *and* D_δ *denote positive constants.*

In this statement, $C_c^\infty((0,1))$ denotes the space of compactly supported smooth functions on an open interval $(0,1)$ and the two norms on $C_c^\infty((0,1))$ is defined by

$$\|f\|_\infty := \max_{x\in(0,1)} |f(x)|,$$

and

$$\|f\| := \max_{x\in(0,1)} f(x) - \min_{x\in(0,1)} f(x).$$

These norms are equivalent. We note that $\|f\|_\infty = \|f\|$ for any non-negative functions $f \geq 0$.

Remark 1.1. (1) In [Se14], Seyfaddini proved the same type inequality as in Theorem 1.2 for the case of the real form $Re(B^{2n})$.

(2) As for the condition on δ in Theorem 1.2, see Remark 5.3.

1.3 Comparison with Seyfaddini's result

In this subsection, we compare our result and method with prior research, especially Seyfaddini's work [Se14].

Let $L \subset (M, \omega)$ be a Lagrangian submanifold of (M, ω). In order to obtain unboundedness of $(\mathcal{L}(L), d)$, it is useful to construct a function μ_L : $\mathrm{Ham}_c(M, \omega) \to \mathbb{R}$ with the following properties.

Required properties. *There exist positive constants $C_L, D_L > 0$ such that for any $\phi, \psi \in \mathrm{Ham}_c(M,\omega)$,*

(1) $|\mu_L(\phi)| \leq C_L \|\phi\|$.

(2) $\phi(L) = \psi(L) \Rightarrow |\mu_L(\phi) - \mu_L(\psi)| \leq D_L$.

(3) *There exists a subset $X_{\mu_L} \subset M$ such that*

$$H|_{X_{\mu_L}} \equiv h \ \ (constant) \ \ \Rightarrow \ \ \mu_L(\phi_H^1) = h.$$

The property (2) is, roughly speaking, *"well-definedness"* of μ_L on $\mathcal{L}(L)$. If there exists a function μ_L on $\mathrm{Ham}_c(M,\omega)$ satisfying the properties (1) and (2), we can easily obtain an inequality:

$$\frac{|\mu_L(\phi)| - D_L}{C_L} \leq d(L, \phi(L)).$$

In [EP03], by using a family of conformally symplectic embeddings $\theta_\delta : B^2 \to S^2$, Entov-Polterovich constructed the family of Calabi quasi-morphisms on $\mathrm{Ham}_c(B^2)$ as pullbacks of their single Calabi quasi-morphism on $\mathrm{Ham}_c(S^2)$. Here the parameter δ is taken from some open interval in $\mathbb{R}$.

In [Kh09], Khanevsky slightly modified Entov-Polterovich's Calabi quasi-morphisms on $\mathrm{Ham}_c(B^2)$ and obtained the family of homogeneous quasi-morphisms $\mu^\delta_{Re(B^2)}$ satisfying these properties.

Remark 1.2. The properties (1), (2) and (3) were not listed in [Kh09]. However he proved implicitly that $\mu^\delta_{Re(B^2)}$ has the properties.

Khanevsky found a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}_c(B^2)$ such that $\mu^\delta_{Re(B^2)}(\phi) \neq 0$ for some δ and proved unboundedness of $\mathcal{L}(Re(B^2))$ as follows:

$$d(Re(B^2), \phi^m(Re(B^2))) \geq \frac{m|\mu^\delta_{Re(B^2)}(\phi)| - D_{Re(B^2)}}{C_{Re(B^2)}} \to \infty \ \ (m \to \infty).$$

Thus construction of a non-trivial homogeneous quasi-morphism satisfying the properties is sufficient to obtain the unboundedness.

1.3.1 Seyfaddini's case

In [BEP04], by using a family of conformally symplectic embeddings $\theta_\delta : B^{2n} \to \mathbb{C}P^n$, Biran-Entov-Polterovich constructed the family of Calabi quasi-morphisms on $\mathrm{Ham}_c(B^{2n})$ as pullbacks of the single Calabi quasi-morphism on $\mathrm{Ham_c}(\mathbb{C}P^n)$ constructed in [EP03]. As in Kanevsky's construction, Seyfaddini also obtained the family of non-trivial homogeneous quasi-morphisms $\mu^\delta_{Re(B^{2n})}$ satisfying the properties (1), (2) and (3) by using the symplectic embeddings $\theta_\delta : B^{2n} \to \mathbb{C}P^n$.

Remark 1.3. Khanevsky's proof of the property (2) for his $\mu^\delta_{Re(B^2)}$ depends on the dimension of the ambient space B^2. By a different proof which is applicable in all dimensions, Seyfaddini proved the property (2) for his $\mu^\delta_{Re(B^{2n})}$.

Using this family $\mu^\delta_{Re(B^{2n})}$, Seyfaddini proved unboundedness of $\mathcal{L}(Re(B^{2n}))$. Moreover, he obtained the following theorem.

Theorem 1.3 (Seyfaddini [Se14]). *There exist a map* $\Psi : C^\infty_c((0,1)) \to \mathcal{L}(Re(B^{2n}))$ *and two constants* $C_{Re(B^{2n})}, D_{Re(B^{2n})} \in \mathbb{R}_{>0}$ *such that*

$$\frac{\|f-g\|_\infty - D_{Re(B^{2n})}}{C_{Re(B^{2n})}} \le d(\Psi(f), \Psi(g)) \le \|f-g\|_\infty.$$

In particular, $(\mathcal{L}(Re(B^{2n})), d)$ *has an infinite diameter.*

We explain Seyfaddini's proof. We simply denote by X_δ a subset in the property (3) for each $\mu^\delta_{Re(B^{2n})}$ and regard the parameter δ as an arbitrary element in the open interval $(0,1)$. We note that X_δ can be taken as pairwise disjoint closed subsets.

To define the map $\Psi : C^\infty_c((0,1)) \to \mathcal{L}(Re(B^{2n}))$, Seyfaddini constructed $h : C^\infty_c((0,1)) \to C^\infty_c(Re(B^{2n}))$ satisfying the following property:

$$h(f)|_{X_\delta} \equiv f(\delta) \quad for\ any\ f \in C^\infty_c((0,1)).$$

By using this map h, he defined $\Psi : C^\infty_c((0,1)) \to \mathcal{L}(Re(B^{2n}))$ as follows:

$$\Psi(f) := \phi^1_{h(f)}(Re(B^{2n})).$$

As a result, he obtained the following inequality for any $\delta \in (0,1)$,

$$\frac{|f(\delta)| - D_{Re(B^{2n})}}{C_{Re(B^{2n})}} = \frac{|\mu^\delta_{Re(B^{2n})}(\phi^1_{h(f)})| - D_{Re(B^{2n})}}{C_{Re(B^{2n})}} \le d(Re(B^{2n}), \Psi(f)),$$

where $Re(B^{2n}) = \Psi(0)$. Take a δ' such that $\|f\|_\infty = f(\delta')$. Then one can obtain

$$\frac{\|f\|_\infty - D_{Re(B^{2n})}}{C_{Re(B^{2n})}} \leq d(Re(B^{2n}), \Psi(f)).$$

This is the left hand side inequality of Theorem 1.3 with $g = 0$ for $(M, L) = (B^{2n}, Re(B^{2n}))$. This is the most crucial inequality in the theorem.

1.3.2 Our case

For each Lagrangian submanifold $L_\delta \subset B^2 \times B^2$ with $1/2 < \delta \leq 1$, by using conformally symplectic embeddings $\Theta_{\delta'} : B^2 \times B^2 \to S^2 \times S^2$ with $1/2 < \delta' \leq 1$ (see Section 5.2 for the definition of $\Theta_{\delta'}$), we can also construct a family of homogeneous quasi-morphisms $\mu_{L_\delta}^{\delta'}$ on $\mathrm{Ham_c}(B^2 \times B^2, \bar{\omega}_0)$ from the Entov-Polterovich's Calabi quasi-morphism on $\mathrm{Ham}_c(S^2 \times S^2, \bar{\omega}_{std})$ constructed in [EP03].

When $\delta' = \delta$, the homogeneous quasi-morphisms $\mu_{L_\delta}^{\delta}$ satisfies the properties (1), (2) and (3). By using this single quasi-morphisms $\mu_{L_\delta}^{\delta}$, we can prove Theorem 1.1 as Khanevsky proved unboundedness of $\mathcal{L}(Re(B^2))$ (see Remark 5.4).

However, we have to construct a family of quasi-morphisms on $\mathrm{Ham_c}(B^2 \times B^2, \bar{\omega}_0)$ satisfying the properties (1), (2) and (3) to prove Theorem 1.2. For this purpose, we use the family of Calabi quasi-morphisms on $\mathrm{Ham}_c(S^2 \times S^2, \bar{\omega}_{std})$ constructed by Fukaya-Oh-Ohta-Ono in [FOOO11] instead of Entov-Polterovich's Calabi quasi-morphism (see Section 5.1 and Section 5.2).

1.4 Organization of the thesis

In Chapter 2, we introduce basic notions on symplectic geometry, Hofer geometry and define Lagrangian Hofer metric. In Chapter 3, we recall Calabi quasi-morphisms and symplectic quasi-states which were introduced by Entov-Polterovich in a series of papers [EP03, EP06, EP09]. In Chapter 4, we recall the Lagrangian Floer theory on toric manifolds developed by Fukaya-Oh-Ohta-Ono in [FOOO09-I, FOOO09-II, FOOO11a, FOOO11b], which will be used in the proof of Lemma 5.3. In Chapter 5, we prove Theorem 1.1 and Theorem 1.2.

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Chapter 2

Preliminaries in Hofer geometry

In this chapter, we recall some basic terminologies and define Lagrangian Hofer metric spaces. For comprehensive introductions to symplectic topology and Hofer geometry, refer to standard texts (e.g. [HZ94], [MS95], [Po01]).

2.1 Symplectic manifolds and Lagrangian submanifolds

A pair of a smooth manifold M and a 2-form $\omega \in \Omega^2(M)$ is called a *symplectic manifold* if the 2-form ω is closed and non-degenerate. For a symplectic manifold (M, ω), the 2-form ω is called a *symplectic structure* on M. Non-degeneracy means that if $\omega_p(u, v) = 0$ for all $v \in T_pM$ then $u = 0$ for every tangent space T_pM. From this condition, it turns out that M has even dimension $2n$. Moreover non-degeneracy implies that the top power ω^n does not vanish at any point. Thus M^{2n} is orientable. In this thesis, we denote by $\mathrm{vol}(M)$ the volume of M with respect to the volume form $\omega^n \in \Omega^{2n}(M)$.

The most basic example is the complex vector space $\mathbb{C}^n = \{(z_1, \dots, z_n) \mid z_j = x_j + \sqrt{-1}y_j \in \mathbb{C}, 1 \leq j \leq n\}$ with the standard symplectic structure $\omega_0 := \sum_{j=1}^n dx_j \wedge dy_j$. Of course the open ball $B^{2n}(r) := \{z = (z_1, \dots, z_n) \in \mathbb{C}^{2n} \mid |z| < r\} \subset \mathbb{C}^n$ equipped with the symplectic structure ω_0 is also a symplectic manifold. However, in this thesis, we fix $2\omega_0$ as a symplectic structure on $B^{2n}(r)$.

A *Lagrangian submanifold* $L \subset (M^{2n}, \omega)$ is a submanifold satisfying dim $L = \frac{1}{2}$dim $M = n$ and $\omega \mid_L = 0$. Typical examples of Lagrangian submanifolds in $(B^{2n}(r), 2\omega_0)$ is the real form $Re(B^{2n}(r)) := \{(z_1, \ldots, z_n) \in B^{2n}(r) \mid \mathrm{Im} z_j = 0, 1 \leq j \leq n\}$ and tori $T^n(r_1, \ldots, r_n) := \{z = (z_1, \ldots, z_n) \in B^{2n}(r) \mid |z_j| = r_j, |z| < r\}$.

Given two symplectic manifolds (M_1, ω_1) and (M_2, ω_2), the product $M_1 \times M_2$ is also a symplectic manifold with respect to the symplectic structure $\omega_1 \oplus \omega_2 := \mathrm{pr}_1^* \omega_1 + \mathrm{pr}_2^* \omega_2$, where $\mathrm{pr}_i : M_1 \times M_2 \to M_i$ are i-th projection respectively $(i = 1, 2)$. If $L_1 \subset (M_1, \omega_1)$ and $L_2 \subset (M_2, \omega_2)$ are Lagrangian submanifolds, then it turns out that $L_1 \times L_2$ is also a Lagrangian submanifold in $(M_1 \times M_2, \omega_1 \oplus \omega_2)$.

In this thesis, we deal with the following symplectic manifold and Lagrangian submanifolds. We define $(B^2 \times B^2, \bar{\omega}_0) := (B^2(1) \times B^2(1), 2\omega_0 \oplus 2\omega_0)$ and define Lagrangian submanifolds $L_\delta := T_\delta \times Re(B^2) \subset B^2 \times B^2$ for $1/2 < \delta \leq 1$. Here $T_\delta := T^1(\frac{1}{\sqrt{2\delta}}) \subset B^2$ and $Re(B^2)$ is the real form of B^2.

Let us introduce symplectic diffeomorphisms and Hamiltonian diffeomorphisms.

For two symplectic manifolds (M_1, ω_1) and (M_2, ω_2), a *symplectic diffeomorphism* is a smooth diffeomorphism $f : M_1 \to M_2$ satisfying $f^* \omega_2 = \omega_1$. The group of symplectic diffeomorphisms on (M, ω) is denoted by

$$\mathrm{Symp}(M, \omega) := \{f \in \mathrm{Diff}(M) \mid f^* \omega = \omega\}.$$

Consider a smooth map $F : [0,1] \times M \to M$ such that $F(t, \cdot) \in \mathrm{Symp}(M, \omega)$ for any $t \in [0,1]$. $f_t := F(t, \cdot)$ is called a *symplectic isotopy* of (M, ω). We denote by $\mathrm{Symp}_0(M, \omega)$ the set of all symplectic diffeomorphisms which can be connected with the identity by a symplectic isotopy. We denote by $\mathrm{Symp}^c(M, \omega) \subset \mathrm{Symp}(M, \omega)$ the subset consisting of all compactly supported symplectic diffeomorphisms, and also define $\mathrm{Symp}_0^c(M, \omega) \subset \mathrm{Symp}_0(M, \omega)$ similarly. For any compactly supported time-dependent function $H : [0,1] \times M \to \mathbb{R}$, from non-degeneracy of ω, the time-dependent vector field X_{H_t} is defined by

$$i_{X_{H_t}} \omega = \omega(X_{H_t}, \ \cdot \ ) = dH_t,$$

where $H_t(p) := H(t, p)$. Traditionally, X_{H_t} is called the (time-dependent) *Hamiltonian vector field* and $H \in C_c^\infty([0,1] \times M)$ is called a (time-dependent) *Hamiltonian* on M. Consider the flow ϕ_H^t of X_{H_t} defined by

$$\frac{d}{dt} \phi_H^t = X_{H_t}(\phi_H^t), \quad \phi_H^0 = id.$$

The flow ϕ_H^t is called the *Hamiltonian flow* generated by H. A compactly supported *Hamiltonian diffeomorphism* $\phi : M \to M$ is a one-time map ϕ_H^1

generated by some Hamiltonian H. Denote by $\mathrm{Ham}_c(M,\omega)$ the set of all compactly supported Hamiltonian diffeomorphisms:

$$\mathrm{Ham}_c(M,\omega) := \{\phi \in \mathrm{Diff}(M) \mid \phi = \phi_H^1, \text{ for some } H \in C_c^\infty([0,1]\times M)\}.$$

In case that M is compact, we denote it by $\mathrm{Ham}(M,\omega)$.

We recall some facts about $\mathrm{Ham}_c(M,\omega)$ (see [MS95], [Po01] for more properties and proofs).

Proposition 2.1. $\mathrm{Ham}_c(M,\omega)$ *is a normal subgroup of* $\mathrm{Symp}_0^c(M,\omega)$*.*

In general, $\mathrm{Ham}_c(M,\omega)$ does not coincide with $\mathrm{Symp}_0^c(M,\omega)$. For example, in case of the 2-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ endowed with the area form $\tilde{\omega}_0 := dx \wedge dy$, one can obtain that $\mathrm{Ham}(T^2,\tilde{\omega}_0) \subsetneq \mathrm{Symp}_0(T^2,\tilde{\omega}_0)$ (see e.g. Exercise 10.4 in [MS95]). Under some topological assumption, we can show that $\mathrm{Ham}_c(M,\omega) = \mathrm{Symp}_0^c(M,\omega)$ as follows.

Proposition 2.2. *If* $H_c^1(M,\mathbb{R}) = 0$ *then*

$$\mathrm{Ham}_c(M,\omega) = \mathrm{Symp}_0^c(M,\omega),$$

where $H_c^1(M,\mathbb{R})$ *is the first de Rham cohomology with compact supports.*

Given two time-independent Hamiltonians $F, G \in C^\infty(M)$, we define the *Poisson bracket* $\{F,G\}$ by

$$\{F,G\} := \omega(X_F, X_G) = dF(X_G).$$

We note that two Hamiltonian diffeomorphisms ϕ_F^1 and ϕ_G^1 are commutative if F, G are Poisson commutative (i.e. $\{F,G\} = 0$).

Definition 2.1. When M^{2n} is closed, we define

$$\mathcal{A}(M) := \{F \in C^\infty(M) \mid \int_M F\omega^n = 0\}.$$

When M^{2n} is open, we define $\mathcal{A}(M) := C_c^\infty(M)$.

A Hamiltonian $H \in C_c^\infty([0,1]\times M)$ is called *normalized* Hamiltonian if $H_t \in \mathcal{A}(M)$ for all time $t \in [0,1]$.

Consider a smooth map $F : [0,1]\times M \to M$ such that $F(t,\cdot) \in \mathrm{Ham}_c(M,\omega)$. $f_t := F(t,\cdot)$ is called a *Hamiltonian isotopy* of (M,ω). The next result was established by Banyaga in [Ba78].

Proposition 2.3. *For any Hamiltonian isotopy* $\{f_t\}_{t\in[0,1]}$*, there exists a time-dependent normalized Hamiltonian* $H \in C_c^\infty([0,1] \times M)$ *such that*

$$\frac{d}{dt} f_t = X_{H_t}(f_t) \quad \textit{for all } t \in [0,1]. \tag{2.1.1}$$

Remark 2.1. In this section, the sign convention for Hamiltonian vector fields and the Poisson bracket coincide ones used in [MS95] and [FOOO11].

2.2 Hofer's metric

In this section, we define the Hofer metric on the group of Hamiltonian diffeomorphisms introduced by Hofer in [Ho90].

It is well known that the Lie algebra of $\mathrm{Ham}_c(M,\omega)$ can be identified with $\mathcal{A}(M)$:

$$T_{id}\mathrm{Ham}_c(M,\omega) \ni X_H \leftrightarrow H \in \mathcal{A}(M).$$

Thus, it is natural to define the length of a Hamiltonian isotopy $\{f_t\}_{t\in[0,1]}$ by using a norm $\|\cdot\|$ on $\mathcal{A}(M)$:

$$\mathrm{length}(\{f_t\}) := \int_0^1 \|\frac{d}{dt} f_t\| \; dt = \int_0^1 \|H_t\| \; dt, \tag{2.2.1}$$

where $H \in C_c^\infty([0,1] \times M)$ is a time-dependent normalized Hamiltonian satisfies (2.1.1) in Proposition 2.3.

The pseudo-distance between two Hamiltonian diffeomorphisms $\phi, \psi \in \mathrm{Ham}_c(M,\omega)$ is induced by

$$\rho(\phi,\psi) := \inf \mathrm{length}(\{f_t\}),$$

where the infimum is taken over all Hamiltonian isotopy $\{f_t\}_{t\in[0,1]}$ with $f_0 = \phi$ and $f_1 = \psi$.

The following is obtained immediately.

Properties. *For any* $\phi, \psi, \theta \in \mathrm{Ham}_c(M,\omega)$*,*

- (symmetry) $\rho(\phi,\psi) = \rho(\psi,\phi)$.
- (non-negativity) $0 \le \rho(\phi,\psi)$.
- (triangle inequality) $\rho(\theta,\phi) \le \rho(\theta,\psi) + \rho(\psi,\phi)$.

Moreover, if a norm $\|\cdot\|$ on $\mathcal{A}(M)$ satisfies

$$\|H \circ \psi^{-1}\| = \|H\| \tag{2.2.2}$$

for all $H \in \mathcal{A}(M)$ and $\psi \in \mathrm{Ham}_c(M, \omega)$ then we have

Bi-invariance property. *For any* $\phi, \psi, \theta \in \mathrm{Ham}_c(M, \omega)$,

- $\rho(\psi, \phi) = \rho(\psi\theta, \phi\theta) = \rho(\theta\psi, \theta\phi)$.

We define the normal subgroup $N(\rho)$ of $\mathrm{Ham}_c(M, \omega)$ by

$$N(\rho) := \{\psi \in \mathrm{Ham}_c(M, \omega) \mid \rho(id, \psi) = 0\}.$$

If $N(\rho) = \{id\}$ then ρ is non-degenerate. The following is well known result proved by Banyaga [Ba78].

Theorem 2.1. *For any closed symplectic manifold* (M, ω), $\mathrm{Ham}(M, \omega)$ *is a simple group.*

Consequently, ρ is either non-degenerate or identically zero for closed symplectic manifolds. It is rather non-trivial work to check the non-degeneracy.

In case of $M = \mathbb{R}^{2n}$, Hofer defined the norm $\|\cdot\|_{Hofer}$ on $\mathcal{A}(M)$ by

$$\|H\|_{Hofer} := \max H - \min H,$$

and obtained the non-degeneracy of ρ_H associated to $\|\cdot\|_{Hofer}$ in [Ho90]. In [Po93] Polterovich generalized to some larger class of symplectic manifolds. Finally, the non-degeneracy was proved for all symplectic manifolds by Lalonde and McDuff in [LM95].

Theorem 2.2. *For any symplectic manifolds* (M, ω)*, Hofer's distance function* ρ_H *is non-degenerate.*

This function ρ_H is called the *Hofer metric.* It turn out that $\|\cdot\|_{Hofer}$ satisfies (2.2.2), thus ρ_H is a bi-invariant metric.

The *Hofer norm* $\|\psi\|$ of $\psi \in \mathrm{Ham}_c(M, \omega)$ is defined by

$$\|\psi\| := \rho_H(id, \psi) = \inf \int_0^1 \left(\max_{p \in M} H_t - \min_{p \in M} H_t \right) dt, \tag{2.2.3}$$

where the infimum is taken over all normalized Hamiltonian $H \in C_c^\infty([0,1] \times M)$ with $\phi_H^1 = \psi$. The Hofer norm $\|\psi\|$ is also called the *symplectic energy* of ψ. The next section, we define the distance between two Hamiltonian isotopic Lagrangian submanifolds by using the Hofer norm.

2.3 Lagrangian Hofer's metric

Fix a Lagrangian submanifold $L \subset (M, \omega)$ without boundary. A Lagrangian submanifold $L' \subset (M, \omega)$ is called *Hamiltonian isotopic* to L if there exists a Hamiltonian diffeomorphism $\psi \in \mathrm{Ham}_c(M, \omega)$ such that $\psi(L) = L'$. We denote by $\mathcal{L}(L) = \mathcal{L}(L, M, \omega)$ the set of all Lagrangian submanifolds which are Hamiltonian isotopic to L_0.

$$\mathcal{L}(L) := \{L' \subset (M, \omega) \mid L' = \phi_H^1(L) \text{ for some } H \in C_c^\infty([0,1] \times M)\}.$$

In the previous section, the length of a Hamiltonian isotopy $\{f_t\}_{t \in [0,1]}$ with respect to $\|\cdot\|_{Hofer}$ is defined by

$$\mathrm{length}(\{f_t\}) := \int_0^1 \left(\max_{p \in M} H_t - \min_{p \in M} H_t \right) dt, \tag{2.3.1}$$

where H is a time-dependent Hamiltonian such that X_{H_t} generates $\{f_t\}_{t\in[0,1]}$. As an analogue of this length functional, the length of a path $\{L_t\}_{t\in[0,1]} \subset \mathcal{L}(L)$ is defined by a normalized Hamiltonian $H \in C_c^\infty([0,1] \times M)$ such that $\phi_H^t(L_0) = L_t$:

$$\mathrm{length}(\{L_t\}) := \int_0^1 \left(\max_{p \in L_t} H_t - \min_{p \in L_t} H_t \right) dt. \tag{2.3.2}$$

The following is another expression of $length(\{L_t\})$:

$$\mathrm{length}(\{L_t\}) = \inf_{\phi_F^t(L_0) = L_t} \int_0^1 \left(\max_{p \in M} F_t - \min_{p \in M} F_t \right) dt, \tag{2.3.3}$$

where the infimum is taken over all normalized Hamiltonian $F \in C_c^\infty([0,1] \times M)$ with $\phi_F^t(L_0) = L_t$. For any $L_0, L_1 \in \mathcal{L}(L)$, the *Lagrangian Hofer pseudo-metric* $d(L_0, L_1)$ is defined as infimum of lengths of all paths $\{L_t\}_{t\in[0,1]}$ connecting L_0 and L_1:

$$d(L_0, L_1) := \inf \mathrm{length}(\{L_t\}).$$

More convenient definition of $d(L_0, L_1)$ can be obtained from (2.3.3):

$$d(L_0, L_1) := \inf\{\|\phi\| \mid \phi(L_0) = L_1,\ \phi \in \mathrm{Ham}_c(M, \omega)\}, \tag{2.3.4}$$

where $\|\phi\|$ is the symplectic energy defined by (2.2.3) in the previous section. The following properties are inherited from the Hofer norm.

Properties. *For any* $L_0, L_1, L_2 \in \mathcal{L}(L)$*,* $\phi \in \mathrm{Ham}_c(M, \omega)$*,*

- (symmetry) $d(L_0, L_1) = d(L_1, L_0)$.
- (non-negativity) $0 \leq d(L_0, L_1)$.
- (triangle inequality) $d(L_0, L_2) \leq d(L_0, L_1) + d(L_1, L_2)$.
- (invariance) $d(L_0, L_1) = d(\phi(L_0), \phi(L_1))$.

As in case of Hofer's metric on $\mathrm{Ham}_c(M, \omega)$, the non-degeneracy is rather non-trivial. Chekanov proved the following theorem.

Theorem 2.3 (Theorem 2 in [Ch00])**.** *Let* d' *be a* $\mathrm{Ham}_c(M, \omega)$*-invariant pseudo-metric on* $\mathcal{L}(L)$*. If* d' *is degenerate then* d' *vanishes identically.*

This theorem is originally proved for any closed and connected Lagrangian submanifolds L, moreover, he proved non-triviality of d for any such L in tame symplectic manifolds and obtained non-degeneracy. However, the same proof yields Theorem 2.3 for any connected Lagrangian submanifold which has a Weinstein neighborhood. Therefore, our Theorem 1.1 implies that d is non-degenerate for our Lagrangians L_δ with $1/2 < \delta \leq 1$.

There are some known results about the diameter of the Lagrangian Hofer metric space. For any compact manifold N, Oh and Milinković proved implicitly that the Lagrangian Hofer metric spaces (N, T^*N, ω_{can}) has an infinite diameter. Here ω_{can} is the standard symplectic structure on the cotangent bundle of N (see Theorem Ⅲ in [Oh97] and Theorem 3 in [Mi02]). This is a case which an ambient space M is non-compact and a Lagrangian submanifold L is compact. In [Le08], Leclercq obtained the unboundedness in case of a meridian in a two dimensional torus by using his Lagrangian spectral invariant.

We show two results which state the unboundedness for a continuous family of Lagrangian submanifolds. In [Us13], Usher proved that $\mathcal{L}(L_0 \times L, T^2 \times M)$ is unbounded if the Lagrangian Floer cohomology $HF(L)$ is nonzero. Here $L_0 \subset T^2$ is a meridian in a two dimensional torus and M is any tame symplectic manifold. From this theorem, by taking a continuous family of Lagrangian manifolds with non-vanishing Floer cohomology (see e.g. [FOOO11b]), we may have an example of a family of unbounded Lagrangian Hofer metric spaces associated to these Lagrangian submanifolds. Another result is due to Khanevsky. Let B^2_k be a k times punctured open disk and let $D \subset B^2_k$ be a subset homeomorphic to the closed unit disk. He proved that $\mathcal{L}(\partial D, B^2)$ is unbounded if the area of D is greater than B^2_k (see Section 4.4 in [Kh14]).

An example of bounded Lagrangian Hofer metric space can be found in [Us13]. He proved that $\mathcal{L}(S^1)$ is bounded for the unit circle $S^1 \subset \mathbb{R}^2$. On the other hand, it is obvious that the unit circle is *displaceable* (i.e. there exists a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(\mathbb{R}^2, \omega_0)$ such that $\phi(S^1) \cap S^1 = \emptyset$).

Chapter 3

Calabi quasi-morphisms and symplectic quasi-states

In a series of papers [EP03, EP06, EP09], Entov and Polterovich developed a way to construct *Calabi quasi-morphisms* and *symplectic quasi-states* for some closed symplectic manifold (M, ω). In this chapter, we briefly recall several terminologies and a generalization of their construction.

3.1 Calabi quasi-morphisms

As we mentioned in Section 2.2, Banyaga proved $\mathrm{Ham}(M, \omega)$ is a simple group for any closed symplectic manifold (M, ω). Therefore, there is no non-trivial morphism on $\mathrm{Ham}(M, \omega)$. This leads us to the notion of quasi-morphisms.

A *quasi-morphism* on a group G is a function $\mu : G \to \mathbb{R}$ which satisfies the following property: there exists a constant $D \geq 0$ such that

$$|\mu(g_1 g_2) - \mu(g_1) - \mu(g_2)| \leq D \quad \text{for all } g_1, g_2 \in G.$$

The smallest number of such D is called the *defect* of μ and we denote by D_μ. A quasi-morphism μ is called *homogeneous* if $\mu(g^m) = m\mu(g)$ for all $m \in \mathbb{Z}$.

For any proper open subset $U \subset M$, the subgroup $\mathrm{Ham}_U(M, \omega)$ is defined as the set which consists of all elements $\phi \in \mathrm{Ham}(M, \omega)$ generated by a time-dependent Hamiltonian $H_t \in C^\infty(M)$ supported in U. We denote by $\widetilde{\mathrm{Ham}}_U(M, \omega)$ the universal covering space of $\mathrm{Ham}_U(M, \omega)$. The Calabi morphism $\widetilde{\mathrm{Cal}}_U : \widetilde{\mathrm{Ham}}_U(M^{2n},\omega) \to \mathbb{R}$ is defined by

$$\widetilde{\mathrm{Cal}}_U(\tilde{\phi}_H) := \int_0^1 dt \int_M H_t\omega^n,$$

where $\phi_H^1 \in \mathrm{Ham}_U(M,\omega)$ and $\tilde{\phi}_H$ is the homotopy class of the Hamiltonian path $\{\phi_H^t\}_{t\in[0,1]}$ with fixed endpoints. If ω is exact on U, $\widetilde{\mathrm{Cal}}_U$ descends to $\mathrm{Cal}_U : \mathrm{Ham}_U(M,\omega) \to \mathbb{R}$.

A subset $X \subset M$ is called *displaceable* if there exists a $\phi \in \mathrm{Ham}(M,\omega)$ such that $\phi(X) \cap \overline{X} = \emptyset$.

Definition 3.1 ([EP03])**.** A function $\mu : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ is called a homogeneous Calabi quasi-morphism if μ is a homogeneous quasi-morphism and satisfies

- (Calabi property) If $\tilde{\phi} \in \widetilde{\mathrm{Ham}}_U(M,\omega)$ and U is a displaceable open subset of M, then
$$\mu(\tilde{\phi}) = \widetilde{\mathrm{Cal}}_U(\tilde{\phi}), \tag{3.1.1}$$
where we regard $\tilde{\phi}$ as an element in $\widetilde{\mathrm{Ham}}(M,\omega)$.

For each non-zero element of quantum (co)homology $a \in QH(M)$, the *spectral invariant* $\rho(\ \cdot\ ; a) : C^\infty([0,1]\times M) \to \mathbb{R}$ is defined in terms of Hamiltonian Floer theory (see [Oh97], [Sc00], [Vi92] for the earlier constructions and [Oh05] for the general non-exact case).

In [FOOO11], Fukaya-Oh-Ohta-Ono deformed spectral invariants and obtained $\rho^{\mathfrak{b}}(\ \cdot\ ; a)$ by using an even degree cocycle $\mathfrak{b} \in H^{even}(M,\Lambda_0)$, where a is an element of *bulk-deformed quantum cohomology* $QH_{\mathfrak{b}}(M,\Lambda)$ (see also [Us11] for a similar deformation of spectral invariants). Here coefficient ring Λ_0, which is called *universal Novikov ring*, and its quotient field Λ are defined by

$$\Lambda_0 := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R}_{\geq 0},\ \lim_{i\to\infty} \lambda_i = +\infty \right\},$$

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = +\infty \right\} \cong \Lambda_0[T^{-1}],$$

where T is a formal parameter and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Every element $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$ is generated by some time-dependent Hamiltonian H which is *normalized* in the sense $\int_M H_t\omega^n = 0$ for any $t \in$

$[0,1]$. The spectral invariant $\rho^{\mathfrak{b}}(\,\cdot\,;a)$ has the homotopy invariance property: if F, G are normalized Hamiltonians and $\tilde{\phi}_F = \tilde{\phi}_G$, then $\rho^{\mathfrak{b}}(F;a) = \rho^{\mathfrak{b}}(G;a)$ (see Theorem 7.7 in [FOOO11]). Hence, the spectral invariant descends to $\rho^{\mathfrak{b}}(\,\cdot\,;a) : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ as follows:

$$\rho^{\mathfrak{b}}(\tilde{\phi}_H;a) := \rho^{\mathfrak{b}}(\underline{H};a) \qquad \text{for any } H \in C^\infty([0,1]\times M),$$

where we denote by $\underline{H}$ the normalization of H:

$$\underline{H}_t := H_t - \frac{1}{\mathrm{vol}(M)} \int_{M^{2n}} H_t\ \omega^n, \qquad \mathrm{vol}(M) := \int_{M^{2n}} \omega^n.$$

By using this (bulk-deformed) spectral invariant $\rho^{\mathfrak{b}}(\,\cdot\,;a)$, as in a series of papers [EP03, EP06, EP09], they constructed a function $\mu_e^{\mathfrak{b}} : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ by

$$\mu_e^{\mathfrak{b}}(\tilde{\phi}) := \mathrm{vol}(M) \lim_{m\to+\infty} \frac{\rho^{\mathfrak{b}}(\tilde{\phi}^m;e)}{m},$$

where $e \in QH_{\mathfrak{b}}(M,\Lambda)$ is an idempotent.

The following theorem is the generalization of Theorem 3.1 in [EP03].

Theorem 3.1 (Theorem 16.3 in [FOOO11])**.** *Suppose that there exists a ring isomorphism*

$$QH_{\mathfrak{b}}(M,\Lambda) \cong \Lambda \times Q$$

and $e \in QH_{\mathfrak{b}}(M,\Lambda)$ is the idempotent corresponding to the unit of the first factor of the right hand side. Then the function

$$\mu_e^{\mathfrak{b}} : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$

is a homogeneous Calabi quasi-morphism.

From standard properties of spectral invariants (Theorem 7.8 in [FOOO11]), $\mu_e^{\mathfrak{b}}$ has two additional properties (Theorem 14.1 in [FOOO11]):

1. (Lipschitz continuity) There exists a constant $C \geq 0$ such that for any $\tilde{\psi}, \tilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$,
$$|\mu_e^{\mathfrak{b}}(\tilde{\psi}) - \mu_e^{\mathfrak{b}}(\tilde{\phi})| \leq C\|\tilde{\psi}\tilde{\phi}^{-1}\|.$$
2. (Symplectic invariance) For all $\psi \in \mathrm{Symp}_0(M,\omega)$,
$$\mu_e^{\mathfrak{b}}(\tilde{\phi}) = \mu_e^{\mathfrak{b}}(\psi \circ \tilde{\phi} \circ \psi^{-1}).$$

Here $C \leq \mathrm{vol}(M)$ is easily proved as in Proposition 3.5 of [EP03].

3.2 Symplectic quasi-states

On the other hand, *symplectic quasi-states* are also constructed by using (bulk deformed) spectral invariants. Let $C^0(M)$ be the set of continuous functions on M.

Definition 3.2 (Section 3 in [EP06]). A functional $\zeta : C^0(M) \to \mathbb{R}$ is called symplectic quasi-state if ζ satisfies the following:

1. (Normalization) $\zeta(1) = 1$.
2. (Monotonicity) $\zeta(F_1) \leq \zeta(F_2)$ for any $F_1 \leq F_2$.
3. (Homogeneity) $\zeta(\lambda F) = \lambda\zeta(F)$ for any $\lambda \in \mathbb{R}$.
4. (Strong quasi-additivity) If smooth functions F and G are Poisson commutative: $\{F, G\} = 0$, then $\zeta(F + G) = \zeta(F) + \zeta(G)$.
5. (Vanishing) If supp F is displaceable, then $\zeta(F) = 0$.
6. (Symplectic invariance) $\zeta(F) = \zeta(F \circ \psi)$ for any $\psi \in Symp_0(M, \omega)$.

By using the bulk deformed spectral invariant $\rho^{\mathfrak{b}}(\ \cdot\ ; e)$, a functional $\zeta_e^{\mathfrak{b}} : C^\infty(M) \to \mathbb{R}$ is defined by

$$\zeta_e^{\mathfrak{b}}(H) := -\lim_{m \to +\infty} \frac{\rho^{\mathfrak{b}}(mH; e)}{m}.$$

This functional $\zeta_e^{\mathfrak{b}}$ extends to a functional on $C^0(M)$ as follows. We recall the relation between $\zeta_e^{\mathfrak{b}}$ and $\mu_e^{\mathfrak{b}}$ (see Section 14 [FOOO11]). For any $H \in C^\infty([0,1] \times M)$, by the *shift property* of spectral invariant, we have

$$\rho^{\mathfrak{b}}(\tilde{\phi}_H; e) = \rho^{\mathfrak{b}}(H; e) + \frac{1}{\mathrm{vol}(M)}\mathrm{Cal}_M(H), \tag{3.2.1}$$

where $\mathrm{Cal}_M(H)$ is defined by

$$\mathrm{Cal}_M(H) := \int_0^1 dt \int_{M^{2n}} H_t\ \omega^n.$$

Since $(\tilde{\phi}_H)^m = \tilde{\phi}_{mH}$ for any autonomous Hamiltonian H, the following relation is obtained from (3.2.1)

$$\zeta_e^{\mathfrak{b}}(H) = \frac{1}{\mathrm{vol}(M)} \left(-\mu_e^{\mathfrak{b}}(\tilde{\phi}_H^1) + \mathrm{Cal}_M(H) \right).$$

By the Lipschitz continuity of $\mu_e^{\mathfrak{b}}$, we can extend $\zeta_e^{\mathfrak{b}}$ to a functional on $C^0(M)$. From the same argument in Section 6 in [EP06], this functional $\zeta_e^{\mathfrak{b}}$: $C^0(M) \to \mathbb{R}$ becomes a symplectic quasi-state if one takes an idempotent e from a field factor of $QH_{\mathfrak{b}}(M, \Lambda)$ as in Theorem 3.1.

In this thesis, we define *superheavy subsets* as follows.

Definition 3.3. Let ζ be a symplectic quasi-state on (M, ω). A closed subset X $\subset M$ is called ζ-superheavy if for all $H \in C^0(M)$

$$\min_X H \leq \zeta(H) \leq \max_X H.$$

It is immediately proved that any ζ-superheavy subsets must intersect each other and non-displaceable (see [EP09] for details).

Chapter 4

Brief review of Lagrangian Floer theory

In this chapter, we recall the Lagrangian Floer theory on toric manifolds to prepare notation and terminologies for the proof of Lemma 5.3. The descriptions here are mainly based on Fukaya-Oh-Ohta-Ono's survey [FOOO12a]. See also [FOOO11a], [FOOO11b] for details on toric cases, and [FOOO09-I, FOOO09-II] for more general cases.

4.1 Notation

In this section, we recall the notation used in [FOOO11a], [FOOO11b], [FOOO12a].

Let C be a graded free R module, where R is the coefficient ring. We denote by $C[1]$ the degree shifted module define by $C[1]^d := C^{d+1}$ and define the shifted degree $\deg'$ on $C[1]$ by

$$\deg' x := \deg x - 1.$$

We define B_kC by

$$B_kC := \overbrace{C \otimes \cdots \otimes C}^{k}.$$

There exist an action of the symmetric group $\mathfrak{S}_k$ as follows:

$$\begin{aligned} \sigma \cdot x_1 \otimes \cdots \otimes x_k &:= (-1)^* x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)} \ , \\ * &:= \sum_{i<j;\sigma(i)>\sigma(j)} \deg x_i \cdot \deg x_j \ . \end{aligned}$$

Let $E_kC \subset B_kC$ be the subset of $\mathfrak{S}_k$ invariant elements. We put $B_0C = E_0C = R$ and define

$$BC := \bigotimes_{k=0}^{\infty} B_kC \ , \quad EC := \bigotimes_{k=0}^{\infty} E_kC.$$

There exists a coassociative coalgebra structure $\Delta : BC \to BC \otimes BC$ defined by

$$\Delta(x_1 \otimes \cdots \otimes x_k) := \sum_{i=0}^{k} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k) \ ,$$

where the summand in the case $i = 0$ is $1 \otimes (x_1 \otimes \cdots \otimes x_n)$. By restriction, coassociative coalgebra structure $\Delta : EC \to EC \otimes EC$ is induced. We note that Δ is graded cocommutative on EC.

We define $\Delta^{n-1} : BC \to \overbrace{BC \otimes \cdots \otimes BC}^{n}$ by

$$\Delta^{n-1} := (\Delta \otimes \overbrace{id \otimes \cdots \otimes id}^{n-2}) \circ \cdots \circ (\Delta \otimes id) \circ \Delta.$$

For an element $\mathbf{x} \in BC$, we express $\Delta^{n-1}(\mathbf{x})$ as

$$\Delta^{n-1}(\mathbf{x}) = \sum_{c} \mathbf{x}_c^{n;1} \otimes \mathbf{x}_c^{n;2} \otimes \cdots \otimes \mathbf{x}_c^{n;n}, \tag{4.1.1}$$

where c runs over some index set depending on $\mathbf{x}$.

In Section 3.1, we defined the universal Novikov ring Λ_0 and its quotient field Λ. The universal Novikov ring Λ_0 is a local ring with the maximal ideal Λ_+ defined by

$$\Lambda_+ := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R}_{>0},\ \lim_{i\to\infty} \lambda_i = +\infty \right\}.$$

We define a non-Archimedean valuation $\mathfrak{v}_T$ on Λ as follows

$$\mathfrak{v}_T\bigg(\sum_{i=0}^{\infty} a_i T^{\lambda_i} \bigg) := \inf\{ \lambda_i \mid a_i \neq 0\}, \quad \mathfrak{v}_T(0) := \infty.$$

The above rings Λ, Λ_0 are complete with respect to the valuation $\mathfrak{v}_T$.

4.2 Moduli spaces

We recall the moduli space of the genus zero bordered holomorphic maps. Let (M, ω) be a symplectic manifold and let L be a Lagrangian submanifold. We denote by J a compatible almost complex structure on (M, ω), where *compatible* means that $\omega(\cdot, J\cdot)$ is a Riemannian metric on M.

A *bordered semi-stable curve of genus zero* with $(k+1)$ boundary marked points and ℓ interior marked points $(\Sigma, z_0, \dots, z_k, z_1^+, \dots, z_\ell^+)$ is a connected union of disks D_i^2, $i = 1, \dots, r$ and spheres S_j^2, $j = 1, \dots, s$ with the following properties:

1. Σ is simply connected.
2. D_i^2 and S_j^2 are called irreducible components. The intersection of two different irreducible components is at most one point. A point which belongs to different components is called a *singular point*.
3. For $i \neq i'$, $D_i^2 \cap D_{i'}^2 = \partial D_i^2 \cap D_{i'}^2$ and $D_i^2 \cap S_j^2 = \mathrm{Int}(D_i^2) \cap S_j^2$ for any i, j.
4. The intersection of three different irreducible components is empty.
5. The boundary marked points $z_0, \dots, z_k$ are mutually distinct and none of them coincide with singular points. The order of $z_0, \dots, z_k$ is required to respect the counter-clockwise cyclic order of the boundary of Σ. The interior marked points $z_1^+, \dots, z_\ell^+$ are mutually distinct and none of them coincide with singular points.

A *bordered stable map of genus zero* with $(k+1)$ boundary marked points and ℓ interior marked points is a pair $((\Sigma, z, z^+), u)$ such that (Σ, z, z^+) is a bordered semi-stable curve of genus zero with marked points and $u : (\Sigma, \partial\Sigma) \to (M, L)$ is a continuous map which is J-holomorphic on each of the irreducible components and $((\Sigma, z, z^+), u)$ satisfies the *stability condition*. The stability condition is equivalent to the condition that the automorphism group, which is the set of biholomorphic maps $\psi : (\Sigma, z, z^+) \to (\Sigma, z, z^+)$ satisfying $\psi(z_i) = z_i$, $\psi(z_j^+) = z_j^+$ and $u \circ \psi = \psi$, is finite.

For $\beta \in H_2(M, L; \mathbb{Z})$, we denote by $\mathcal{M}_{k+1;\ell}^{main}(L; \beta)$ the set of all the isomorphism classes of bordered stable maps $((\Sigma, z, z^+), u)$ with $(k+1)$ boundary marked points and ℓ interior marked points satisfy $\beta = [u]$.

We define the evaluation maps $ev_i : \mathcal{M}_{k+1;\ell}^{main}(L; \beta) \to L,\ (i = 0, 1, \dots, k)$ and $ev_j^+ : \mathcal{M}_{k+1;\ell}^{main}(L; \beta) \to L,\ (j = 1, \dots, \ell)$ as follows

$$ev_i([(\Sigma, z, z^+), u]) := u(z_i) \ ,$$

$$ev_j^+([(\Sigma, z, z^+), u]) := u(z_j^+).$$

In [FOOO09-I, FOOO09-II], for any closed relatively spin Lagrangian submanifold L in any closed symplectic manifold, Fukaya-Oh-Ohta-Ono proved $\mathcal{M}^{main}_{k+1;\ell}(L;\beta)$ has a *Kuranishi structure* and they constructed an A_∞ structure on a subcomplex $C^*(L)$ of smooth singular chain complex of L.

To prove our theorem, we need to recall their Lagrangian Floer theory on compact toric manifolds.

4.3 Compact toric manifolds

In this section, after summarizing the constructions of compact toric manifolds, we recall known results on symplectic toric manifolds. The description here is mainly based on [CO06], [FOOO11a], [Au04], [Ca01].

4.3.1 Compact toric manifolds

To obtain a smooth compact toric manifold, we define a *complete fan of regular cones* Σ. We denote by N, $N_{\mathbb{R}}$ the lattice $\mathbb{Z}^n$ and $N \otimes \mathbb{R}$ respectively.

Definition 4.1. A convex set $\sigma \subset N_{\mathbb{R}}$ is called a regular k-dimensional cone if there exist k ($k \geq 1$) linearly independent elements $v_1, \dots, v_k \in N$ such that the set $\{v_i \mid i = 1, \dots, k\}$ is a subset of some $\mathbb{Z}$-basis of N and

$$\sigma = \{a_1 v_1 + \cdots + a_k v_k \mid a_i \in \mathbb{R}_{\geq 0}\}.$$

The set $\{v_i \mid i = 1, \dots, k\}$ determined by σ is called the *integral generators* of σ.

Definition 4.2. A regular cone σ' is called a face of a regular cone σ if the generators of σ' are contained in the set of integral generators of σ. In this case, we write $\sigma' \prec \sigma$.

Definition 4.3. A finite family of regular cones $\Sigma = \{\sigma_1, \dots, \sigma_s \mid \sigma \subset N_{\mathbb{R}}\}$ is called a complete n-dimensional fan of regular cones, if the following conditions are satisfied.

(1) If $\sigma \in \Sigma$ and $\sigma' \prec \sigma$, then $\sigma \in \Sigma$.

(2) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma' \prec \sigma$ and $\sigma \cap \sigma' \prec \sigma'$.

(3) $\sigma_1 \cup \cdots \cup \sigma_s = N_{\mathbb{R}}$.

Let $\Sigma^{(k)}$ be the set of all k-dimensional cones in Σ. For a complete n-dimensional fan of regular cones Σ, we denote by $G(\Sigma) = \{v_1, \dots, v_m\}$ the set of all integral generators of 1-dimensional cones in Σ, where $m = \#\Sigma^{(1)}$.

Definition 4.4. A subset $\mathcal{P} = \{v_{i_1}, \dots, v_{i_p}\} \subset G(\Sigma)$ is called a primitive collection if $\mathcal{P}$ does not generate p-dimensional cones in Σ, while for all k with $1 \leq k < p$ each k-elements subset $\mathcal{P}$ generates a k-dimensional cone in Σ.

Definition 4.5. Let $\mathbb{C}^m$ be an m-dimensional complex vector space with coordinates $z_1, \dots, z_m$ which are in one-to-one correspondence $z_i \leftrightarrow v_i \in G(\Sigma)$, and let $\mathcal{P} = \{v_{i_1}, \dots, v_{i_p}\}$ be a primitive collection in $G(\Sigma)$.

(1) Define the $(m-p)$-dimensional subspace $\mathbb{A}(\mathcal{P}) \subset \mathbb{C}^m$ as follows:

$$\mathbb{A}(\mathcal{P}) := \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_{i_1} = \cdots = z_{i_p} = 0\}.$$

(2) Define the closed subset $Z(\Sigma) \subset \mathbb{C}^m$ as follows:

$$Z(\Sigma) := \bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P}),$$

where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$.

(3) Define the open subset $U(\Sigma) \subset \mathbb{C}^m$ as follows:

$$U(\Sigma) := \mathbb{C}^m \setminus Z(\Sigma).$$

We define a homomorphism $\alpha : \mathbb{Z}^m \to N$ by

$$\alpha(e_i) := v_i,$$

where $\{e_i \mid i = 1, \dots, m\}$ is the basis vectors of $\mathbb{Z}^m$.

When Σ is a complete n-dimensional fan of regular cones, we have an exact sequence:

$$0 \to \mathbb{K} := \mathrm{Ker}(\alpha) \to \mathbb{Z}^m \to N \to 0.$$

We denote by $D(\Sigma)$ the connected commutative subgroup in $(\mathbb{C}^*)^m$ generated by all one-parameter subgroups:

$$\begin{array}{cccc} a_\lambda : \mathbb{C}^* & \longrightarrow & (\mathbb{C}^*)^m \\ \Psi & & \Psi \\ t & \longmapsto & (t^{\lambda_1}, \dots, t^{\lambda_m}), \end{array}$$

where $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{K}$.

Proposition 4.1. *The subgroup $D(\Sigma)$ acts freely on $U(\Sigma) \subset \mathbb{C}^m$.*

The compact toric manifold X_Σ is defined as following:

Definition 4.6. For a complete n-dimensional fan of regular cones Σ, the compact toric manifold associated with Σ is defined by

$$X_\Sigma := U(\Sigma)/D(\Sigma).$$

4.3.2 Symplectic toric manifolds and known results

A $2n$-dimensional *symplectic toric manifold* is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of an n-dimensional torus $\rho : T^n = (S^1)^n \to \mathrm{Symp}(M, \omega)$ and with a choice of a corresponding moment map $\pi = (\pi_1, \ldots, \pi_n) : M \to \mathbb{R}^n$. Two symplectic toric manifolds (M, ω, ρ, π) and $(M', \omega', \rho', \pi')$ are called *isomorphic* if there exists a T^n-equivariant symplectomorphism $\phi : M \to M'$ such that $\pi' \circ \phi = \pi$.

The definition of Hamiltonian action implies that

$$d\pi_i(X) = \omega(X, \tilde{\mathfrak{t}}), \tag{4.3.1}$$

where $\tilde{\mathfrak{t}}$ is the vector field on M introduced by the action of the i-th factor S^1_i of T^n (see [Ca01] for the definition *Hamiltonian action* for general Lie groups).

Remark 4.1. Formula (4.3.1) corresponds with one used in [FOOO11a]. In [FOOO11], they put a factor 2π in the right hand side of (4.3.1).

It is well known that the image $P = \pi(M) \subset \mathbb{R}^n$ is a convex polytope, which is called the *moment polytope* of the symplectic toric manifold (M, ω, ρ, π). Moreover, it turns out that P is a *Delzant polytope* (see [De88]). Hence, in other words, there exist finitely many integral vectors $v_i \in \mathbb{Z}^n$ and constants $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, m)$ such that

- $P = \{u \in \mathbb{R}^n \mid l_i(u) := \langle u, v_i \rangle - \lambda_i \geq 0,\ i = 1, \ldots, m\}$.
- The number of facets $\partial_i P := \{u \in \mathbb{R} \mid l_i(u) = 0\}$ meeting at each vertex p is n. Let $\partial_{i_1}, \ldots, \partial_{i_n}$ be those faces. Then the corresponding integral vectors $v_{i_1}, \ldots, v_{i_n}$ are a basis of $\mathbb{Z}^n$.

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$ and integral vector v_i is a inward-pointing normal vector to the facet $\partial_i P$.

On the other hand, for a Delzant polytope $P \subset \mathbb{R}^n$ defined by affine functions $l_i : \mathbb{R}^n \to \mathbb{R}$ as above, we can obtain a complete n-dimensional fan of regular cones Σ_P whose integral generators of 1-dimensional cones are the set of the integral vectors $v_i \in \mathbb{Z}^n$. We define the compact toric manifold $X_{\Sigma_P} := U(\Sigma_P)/D(\Sigma_P)$ associated to the fan Σ_P. There exist a natural real torus action $\rho_P : T^n \curvearrowright X_{\Sigma_P}$ induced by the torus action $(\mathbb{C}^*)^m/D(\Sigma_P) \curvearrowright X_{\Sigma_P}$. In [De88], Delzant proved the following existence theorem of symplectic toric manifolds (see Section VII.2.a in [Au04]).

Theorem 4.1. *There exists a symplectic structure ω_P on X_{Σ_P} such that the action $\rho_P : T^n \curvearrowright X_{\Sigma_P}$ is Hamiltonian. Moreover, P is the image of a moment map $\pi_P : X_{\Sigma_P} \to \mathbb{R}^n$ associated to the Hamiltonian action.*

In [De88], Delzant also proved that there exists a bijection between isomorphism classes of symplectic toric manifolds and Delzant polytopes.

By Delzant's construction, it turns out that the symplectic structure ω_P is a T^n-invariant Kähler form with respect to the canonical complex structure J on X_{Σ_p}. The following theorem is obtained by Guillemin in [Gu94].

Theorem 4.2. *Define the function $l_\infty : (\mathbb{R}^n)^* \to \mathbb{R}$ by*

$$l_\infty(u) := \langle u, \sum_{i=1}^{m} v_i \rangle.$$

Then we have

$$\omega_P = \sqrt{-1}\partial\bar{\partial}\Big(\pi^*\Big(\sum_{i=1}^{m} \lambda_i(\log l_i) + l_\infty\Big)\Big)$$

over $\mathrm{Int}(P)$.

For each facet $\partial_i P$ $(i = 1, \dots, m)$, we put $D_i := \pi_P^{-1}(\partial_i P)$ (the *irreducible component of toric divisors*). For the principal bundle $\mathrm{pr} : U(\Sigma_P) \to X_{\Sigma_P}$, $\mathrm{pr}^{-1}(D_i)$ is defined by the equation $z_i = 0$ in $U(\Sigma_P) \subset \mathbb{C}^m$.

Let $u \in \mathrm{Int}(P)$. We denote by $L(u) \subset X_{\Sigma_P}$ the inverse image of the moment map $\pi_P : X_{\Sigma_P} \to \mathbb{R}^n$. Then $L(u)$ is a Lagrangian submanifold and an orbit of the T^n-action. We call $L(u)$ a *Lagrangian torus fiber* over $u \in$ *Int*(P). Cho-Oh show the next proposition in [CO06].

Proposition 4.2. *There exist m elements $\beta_i \in H_2(X_{\Sigma_P}, L(u); \mathbb{Z})$ such that β_i is represented by some holomorphic disk $u_i : (D^2, \partial D^2) \to (X_{\Sigma_P}, L(u))$ and*

$$\beta_i \cap [D_j] = \delta_{ij},$$

$$2\pi l_i(u) = \int_{\beta_i} \omega_P,$$

where δ_{ij} *is Kronecker's delta.*

In the end of this section, we state Cho-Oh's theorem (see Section 4 in [CO06]).

Theorem 4.3. (1) (Maslov index formula) *For a symplectic toric manifold* X_{Σ_P}*, and Lagrangian torus fiber* $L(u) \subset X_{\Sigma_P}$*, the Maslov index of any holomorphic disc with boundary lying on* $L(u)$ *is twice the sum of intersection multiplicities of the image of the disc with the codimension 1 submanifolds* D_i *for all* $i = 1, \dots, m$*, where* m *is the number of facets of* P*.*

(2) (Classification theorem) *Any holomorphic map* $u : (D^2, \partial D^2) \to (X_{\Sigma_P}, L(u))$ *can be lifted to a holomorphic map*

$$\tilde{u} : (D^2, \partial D^2) \to (\mathbb{C}^m \setminus Z(\Sigma_P), \mathrm{pr}^{-1}(L(u)))$$

so that each homogeneous coordinates function $z_1(\tilde{u}), \dots, z_m(\tilde{u})$ *is given by Blaschke products with constant factors:*

$$z_i(\tilde{u}) = c_i \prod_{j=1}^{\mu_i} \frac{z - \alpha_{i,j}}{1 - \overline{\alpha}_{i,j} z}$$

for $c_i \in \mathbb{C}^*$ *and* $\mu_i \in \mathbb{Z}_{\geq 0}$ *for each* $i = 1, \dots, m$*.*

(3) (Regularity theorem) *The disks in the classification theorem are Fredholm regular* (*i.e., its linearization map is surjective*).

Remark 4.2. In the above theorem, the each non-negative number μ_i is the intersection multiplicities of the image of the disc with the codimension 1 submanifolds D_i and the each $\alpha_{i,j} \in D^2$ is one of points mapped into D_i.

4.4 Lagrangian Floer theory in toric case

In this section, we briefly recall the A_∞ structure and the Lagrangian Floer theory developed by Fukaya-Oh-Ohta-Ono in toric case (see [FOOO11a], [FOOO11b] for details).

Let C be a graded free Λ_0 module. We define a *G-gapped unital filtered* A_∞ *algebra* by using the notation defined in Section 4.1.

Definition 4.7. A sequence of operators

$$\mathfrak{m}_k : B_k(C[1]) \to C[1]$$

of odd degree for $k \in \mathbb{Z}_{\geq 0}$ is called a G-gapped unital filtered A_∞ algebra on C if $\{\mathfrak{m}_k\}_{k=0}^{\infty}$ satisfies the following.

(1) (A_∞-relation) For any $x_i \in C[1]$,

$$\sum_{k_1+k_2=k+1} \sum_{i=1}^{k_2} (-1)^* \mathfrak{m}_{k_2}(x_1 \otimes \cdots \mathfrak{m}_{k_1}(x_i \otimes \cdots \otimes x_{i+k_1-1}) \cdots \otimes x_k) = 0 \ ,$$

where $* := \sum_{j=1}^{i-1} \deg' x_j$.

(2) $\mathfrak{m}_0(1) \equiv 0 \mod (\Lambda_+)$

(3) (Unitality) There exists an element $\mathbf{e} \in C^0$ such that for any $x, x_i \in C[1]$

- $\mathfrak{m}_{k+1}(x_1 \otimes \cdots \otimes \mathbf{e} \otimes \cdots \otimes x_k) = 0 \quad$ for $k \geq 2,\ k = 0$,
- $\mathfrak{m}_2(\mathbf{e} \otimes x) = (-1)^{\deg x} \mathfrak{m}_2(x \otimes \mathbf{e}) = x \quad$ for $k = 1$. Such $\mathbf{e}$ is called a strict unit.

(4) (G-gappedness) There exists an additive discrete submonoid $G = \{\lambda_i | 0 = \lambda_0 < \lambda_1 < \cdots, i = 0, 1, 2, \cdots\} \subset \mathbb{R}_{\geq 0}$ such that $\mathfrak{m}_k$ is written as

$$\mathfrak{m}_k = \sum_{i=0}^{\infty} \mathfrak{m}_{k,i} T^{\lambda_i},$$

where $\mathfrak{m}_{k,i} : B_k(C[1]) \to C[1]$ is $\mathbb{C}$-linear.

(5) $\mathfrak{m}_{2,0}$ coincides with the product on C up to sign.

After giving the definition of a *bulk* $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$, we give an overview of the construction of gapped unital filtered A_∞ algebra on the de Rham cohomology $H(L; \Lambda_0)$ deformed by a pair of a bulk $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$ and an element $b \in H^1(L; \Lambda_0)$ for a Lagrangian torus fiber L in a symplectic toric manifold (X, ω).

Let a symplectic toric manifold $(X^{2n}, \omega, \pi, \rho) = (X_{\Sigma_P}, \omega_P, \pi_P, \rho_P)$ have the moment polytope P as in Section 4.3.2:

$$P = \{u \in \mathbb{R}^n \mid l_i(u) := \langle u, v_i \rangle - \lambda_i \geq 0,\ i = 1, \ldots, m\} \ .$$

Here m is the number of facets of P. For each facet $\partial_i P$ $(i = 1, \ldots, m)$, we put $D_i := \pi_P^{-1}(\partial_i P)$. We denote by J a subset of $\{1, \ldots, m\}$ and denote $D_J := D_{j_1} \cap \cdots \cap D_{j_k}$. We note that D_J is a real codimension $2k$ submanifold in M and invariant under the T^n-action. If $J = \emptyset$, we put $D_J = X$. We define $\mathcal{A}(\mathbb{Z})$ as the free abelian group generated by D_J and put the degree on it by

$$\deg(D_J) := \text{codim } D_J = 2k.$$

We define

$$\mathcal{A}(\Lambda_+) := \mathcal{A}(\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda_+.$$

In this paper, an element $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$ is called *bulk*.

We note that the homomorphism $\mathcal{A}(\mathbb{Z}) \to H_*(M; \mathbb{Z})$ and the Poincaré duality induce a surjective homomorphism

$$\pi : \mathcal{A}(\Lambda_+) \to H^*(M; \Lambda_+); \quad \mathfrak{b} \mapsto PD[\mathfrak{b}].$$

Remark 4.3. In [FOOO11b] and [FOOO12a], they use $\mathcal{A}(\Lambda_0) = \mathcal{A}(\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda_0$ as the set of bulk (see Section 11 in [FOOO11b]). Since $\mathcal{A}(\Lambda_0)$ is not needed to prove our theorems, we restrict ourselves to the case of $\mathcal{A}(\Lambda_+)$.

We denote by $\mathbf{p}_i$ $(i = 0, 1, \ldots, m, \ldots, B)$ the generator of $\mathcal{A}(\mathbb{Z})$, where $\mathbf{p}_0 = X$ and $\mathbf{p}_i = D_i$ for $i = 1, \ldots, m$. For $I : \{1, \ldots, \ell\} \to \{1, \ldots, B\}$, we define

$$\mathbf{p}_I := \mathbf{p}_{I(1)} \otimes \cdots \otimes \mathbf{p}_{I(\ell)}, \quad [\mathbf{p}_I] := \frac{1}{\ell!} \sum_{\sigma \in \mathfrak{S}_\ell} \mathbf{p}_{I(\sigma(1))} \otimes \cdots \otimes \mathbf{p}_{I(\sigma(\ell))} \in E_\ell \mathcal{A}[2],$$

where $\mathfrak{S}_\ell$ is the symmetric group of order $\ell!$.

For any Lagrangian torus fiber $L(u) \subset X^{2n}$, $u \in \mathrm{P}$, $\beta \in H_2(X, L(u); \mathbb{Z})$ and $I : \{1, \ldots, \ell\} \to \{1, \ldots, B\}$, we define the fiber product in the sense of Kuranishi structure:

$$\mathcal{M}^{main}_{k+1;\ell}(L, \beta; \mathbf{p}_I) := \mathcal{M}^{main}_{k+1;\ell}(L; \beta)_{(ev_1^+, \ldots, ev_\ell^+)} \times_{X^\ell} \mathbf{p}_I.$$

This set consists of all bordered stable maps $[(\Sigma, z, z^+), u] \in \mathcal{M}^{main}_{k+1;\ell}(L; \beta)$ satisfying $u(z_j^+) \in \mathbf{p}_{I(j)}$ for all $j = 1, \ldots, \ell$. Evaluation maps at boundary marked points induce

$$ev = (ev_0, \ldots, ev_k) : \mathcal{M}^{main}_{k+1;\ell}(L, \beta; \mathbf{p}_I) \to L^{k+1}.$$

By using this map, we define an operator

$$\mathfrak{q}_{\ell,k,\beta} : E_\ell \mathcal{A}(\mathbb{Z})[2] \otimes B_k H(L(u); \mathbb{C})[1] \to H(L(u); \mathbb{C})[1].$$

Since $L(u)$ is a torus fiber, there exists a free and transitive T^n-action. We fix a T^n invariant Riemannian metric on $L(u)$. It turns out that harmonic forms with respect to this metric can be identified with T^n invariant forms. Hereafter, we regard the cohomology $H(L(u);\mathbb{C})$ as the set of all T^n invariant forms.

For $h_1,\dots,h_k \in H(L(u);\mathbb{C})$, we define the operator $\mathfrak{q}_{\ell,k,\beta}$ as follows:

$$\mathfrak{q}_{\ell,k,\beta}([\mathbf{p}_I];h_1\otimes\cdots\otimes h_k) := \frac{1}{\ell!}(ev_0)_!(ev_1,\dots,ev_k)^*(h_1\times\cdots\times h_k),$$

where $(ev_0)_!$ denotes the integration along the fiber of the evaluation map $ev_0 : \mathcal{M}^{main}_{k+1;\ell}(L,\beta;\mathbf{p}_I)\to L$.

Remark 4.4. We can define the integration along the fiber when the evaluation map $ev_0 : \mathcal{M}^{main}_{k+1;\ell}(L,\beta;\mathbf{p}_I)\to L$ is a submersion in the sense of Kuranishi structure. By using a T^n *equivariant multisection* $\mathfrak{s}$, we can obtain a perturbed moduli space $\mathcal{M}^{main}_{k+1;\ell}(L,\beta;\mathbf{p}_I)^{\mathfrak{s}}$ and $ev_0 : \mathcal{M}^{main}_{k+1;\ell}(L,\beta;\mathbf{p}_I)^{\mathfrak{s}}\to L$ becomes a submersion and $\mathfrak{q}_{\ell,k,\beta}([\mathbf{p}_I];h_1\otimes\cdots\otimes h_k)$ becomes T^n invariant forms. For more details, see Section 3 and Remark 8.1 of [FOOO12a].

The operator $\mathfrak{q}_{\ell,k,\beta}$ has the following property.

Theorem 4.4 (Theorem 2.1 in [FOOO11b])**.** *For each $\beta\in H_2(L(u);\mathbb{Z})$, $\mathfrak{q}_{\ell,k,\beta}$ satisfies:*

(1) *Let $\mathbf{x}\in B_k(H(L(u);\Lambda_0)[1])$, $\mathbf{y}\in E_\ell(\mathcal{A}(\Lambda_+)[2])$ and let $\Delta^1(\mathbf{y})=\sum_{c_1}\mathbf{y}^{2;1}_{c_1}\otimes \mathbf{y}^{2;2}_{c_1}$, $\Delta^2(\mathbf{x})=\sum_{c_2}\mathbf{x}^{3;1}_{c_2}\otimes\mathbf{x}^{3;2}_{c_2}\otimes\mathbf{x}^{3;3}_{c_2}$. We have*

$$\sum_{\beta=\beta_1+\beta_2}\sum_{c_1,c_2}(-1)^*\mathfrak{q}_{\ell,k,\beta_1}(\mathbf{y}^{2;1}_{c_1};\mathbf{x}^{3;1}_{c_2}\otimes\mathfrak{q}_{\ell,k,\beta_2}(\mathbf{y}^{2;2}_{c_1};\mathbf{x}^{3;2}_{c_2})\otimes\mathbf{x}^{3;3}_{c_2})=0,$$

where $*=\deg'\mathbf{x}^{3;1}_{c_2}+\deg'\mathbf{x}^{3;1}_{c_2}+\deg\mathbf{y}^{2;2}_{c_1}+\deg\mathbf{y}^{2;1}_{c_1}$.

(2) *We put $\mathfrak{m}_{\beta;k}:=\mathfrak{q}_{0,k,\beta}$ and define*

$$\mathfrak{m}_k := \sum_{\beta\in H_2(M,L(u);\mathbb{Z})}\mathfrak{m}_{\beta;k}T^{\beta\cap[\omega]/2\pi}.$$

Then $(H(L(u);\Lambda_0),\{m_k\}_{k=0}^\infty,\mathbf{e}:=PD[L(u)])$ is a gapped unital filtered A_∞ algebra, where $PD[L(u)]$ denotes the Poincaré dual of the fundamental class of $L(u)$.

(3) • *For* $\beta_0 = 0 \in H_2(M, L(u); \mathbb{Z})$ *and for any* $x \in H(L(u); \Lambda_0)[1]$,

$$\mathfrak{m}_{\beta_0,2}(\mathbf{e} \otimes x) = (-1)^{\deg x} \mathfrak{m}_{\beta_0,2}(x \otimes \mathbf{e}).$$

• *For any* $\mathbf{x}_i \in B_{k_i}(H(L(u); \Lambda_0)[1])$ *and any* $\mathbf{y} \in E_\ell(\mathcal{A}(\Lambda_+)[2])$,

$$\mathfrak{q}_{\ell,k_1+1+k_2,\beta}(\mathbf{y}; \mathbf{x}_1 \otimes \mathbf{e} \otimes \mathbf{x}_2) = 0.$$

Next we define

$$\mathfrak{q}_{\ell,k} : E_\ell(\mathcal{A}(\Lambda_+)[2]) \otimes B_k(H(L(u); \Lambda_0)[1]) \to H(L(u); \Lambda_0)[1],$$

$$\mathfrak{q}_{\ell,k} := \sum_{\beta \in H_2(M, L(\mathbf{u}); \mathbb{Z})} \mathfrak{q}_{\ell,k,\beta} T^{\omega \cap \beta / 2\pi}.$$

By using an element $(\mathfrak{b}, b) \in \mathcal{A}(\Lambda_+) \times H^{odd}(L(u); \Lambda_0)$, we deform

$$\mathfrak{m}_k : B_k(H(L(u); \Lambda_0)[1]) \to H(L(u); \Lambda_0)$$

as follows.

Definition 4.8.

$$\mathfrak{m}_k^{\mathfrak{b},b}(x_1 \otimes \cdots \otimes x_k) := \sum_{\substack{l=0, \\ m_0=0,\cdots,m_k=0}}^{\infty} \mathfrak{q}_{l,k+m_0+\cdots+m_k}(\mathfrak{b}^l; \overbrace{b \otimes \cdots \otimes b}^{m_0} \otimes x_1 \otimes \overbrace{b \otimes \cdots \otimes b}^{m_1} \otimes x_2 \otimes \cdots \otimes x_k \otimes \overbrace{b \otimes \cdots \otimes b}^{m_k}). \quad (4.4.1)$$

Fukaya-Oh-Ohta-Ono proved the following.

Theorem 4.5 (Lemma 2.2 in [FOOO11b])**.** *For each* $(\mathfrak{b}, b) \in \mathcal{A}(\Lambda_+) \times H^{odd}(L(u); \Lambda_0)$, $(H(L; \Lambda_0), \mathfrak{m}_k^{\mathfrak{b},b}, \mathbf{e})$ *is a gapped unital filtered* A_∞ *algebra.*

Remark 4.5. In Definition 4.8, we take $b \in H^{odd}(L(u); \Lambda_0)$ instead of taking from $H^{odd}(L(u); \Lambda_+)$. By using a trick due to Cho [Cho08], the convergence problem about right hand side of (4.4.1) is resolved (see Section 12 in [FOOO11a] and Section 8 in [FOOO11b]).

We denote by $\widehat{\mathcal{M}}_{def,weak}(L(u); \Lambda_0)$ the set of all elements $(\mathfrak{b}, b) \in \mathcal{A}(\Lambda_+) \times H^{odd}(L(u); \Lambda_0)$ satisfying $\mathfrak{m}_0^{\mathfrak{b},b}(1) \equiv 0 \bmod \Lambda_+ \mathbf{e}$. If $\widehat{\mathcal{M}}_{def,weak}(L(u); \Lambda_0) \neq \emptyset$, then $\mathfrak{m}_1^{\mathfrak{b},b} \circ \mathfrak{m}_1^{\mathfrak{b},b} = 0$ for any $(\mathfrak{b}, b) \in \widehat{\mathcal{M}}_{def,weak}(L(u); \Lambda_0)$. Thus we can define the *bulk deformed Lagrangian Floer cohomology* by

$$HF((L, \mathfrak{b}, b); \Lambda_0) := \frac{\ker \mathfrak{m}_1^{\mathfrak{b},b}}{\mathrm{Im}\ \mathfrak{m}_1^{\mathfrak{b},b}} .$$

We define the *bulk deformed potential function* as follows:

$$\mathfrak{PO}^u : \widehat{\mathcal{M}}_{def,weak}(L(u); \Lambda_0) \to \Lambda_+,$$

$$\mathfrak{m}_0^{\mathfrak{b},b}(1) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \mathfrak{q}_{\ell,k}(\mathfrak{b}^\ell, b^k) = \mathfrak{PO}^u(\mathfrak{b}, b)\mathbf{e}.$$

Proposition 4.3 (Proposition 2.1 in [FOOO12b]). *For any $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$, there exists a natural inclusion:*

$$\mathcal{A}(\Lambda_+) \times H^1(L; \Lambda_+) \subset \widehat{\mathcal{M}}_{def,weak}, \tag{4.4.2}$$

For any $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$, there exists a natural inclusion:

$$\mathcal{A}(\Lambda_+) \times \frac{H^1(L; \Lambda_0)}{H^1(L; 2\pi\sqrt{-1}\mathbb{Z})} \subset \widehat{\mathcal{M}}_{def,weak} . \tag{4.4.3}$$

In this toric case, there exists a torus action T^n on (X, ω) by the definition. Thus, by identifying $L(u)$ with T^n, we have a canonical basis $\{\mathbf{e}_i\}_{i=1}^n$ of $H^1(L(u); \mathbb{Z})$ represented by dt_i, where t_i is the coordinate of i-th factor of $T^n = (\mathbb{R}/\mathbb{Z})^n$. An element $b \in H^1(L(u); \Lambda_0)$ can be written as $b = \sum_{i=1}^n x_i \mathbf{e}_i$. Put $x_i = x_{i;0} + x_{i;+}$, $x_{i;0} \in \mathbb{C}$, $x_{i;+} \in \Lambda_+$. We define new coordinates y_i^u by

$$y_i^u := \exp(x_{i;0}) \exp(x_{i;+}) \in \Lambda_0 \setminus \Lambda_+,$$

where for $x_{i;+} \in \Lambda_+$

$$\exp(x_{i;+}) := \sum_{k=0}^{\infty} \frac{(x_{i;+})^k}{k!}$$

makes sense in the non-Archimedean sense.

From Proposition 4.3, we can regard the potential function as follows.

$$\mathfrak{PO}^u : \mathcal{A}(\Lambda_+) \times (\Lambda_0 \setminus \Lambda_+)^n \to \Lambda_+.$$

We denote by $\mathfrak{PO}^u_{\mathfrak{b}}$ when we fix a bulk $\mathfrak{b}$.

Fukya-Oh-Ohta-Ono proved the next important relation between the potential function and the Lagrangian Floer cohomology.

Theorem 4.6 (Theorem 3.16 in [FOOO11b]). *If* $(\mathfrak{b}, y = (y_1, \dots, y_n)) \in \mathcal{A}(\Lambda_+) \times (\Lambda_0 \setminus \Lambda_+)^n$ *satisfies*

$$y_i^u \frac{\partial \mathfrak{PO}_{\mathfrak{b}}^u}{\partial y_i^u}(y) = 0, \quad i = 1, \dots, n,$$

then we have

$$HF((L(u), \mathfrak{b}, b); \Lambda_0) \cong H(\mathbb{T}^n; \Lambda_0).$$

4.5 Quantum cohomology and Jacobian ring

In this section, we review the isomorphism between the big quantum cohomology of (X, ω) deformed by a bulk $\mathfrak{b}$ and the Jacobian ring of the deformed potential function by $\mathfrak{b}$ (see [FOOO10] for details).

Let (X^{2n}, ω) be a symplectic manifold. We denote by $\mathcal{M}_\ell(\alpha)$ the moduli space of stable maps in class $\alpha \in H_2(M; \mathbb{Z})$ from genus zero semi-stable curve with ℓ marked points. This moduli space $\mathcal{M}_\ell(\alpha)$ has a virtual fundamental cycle, hence induce a class (see [FO99])

$$ev_*[\mathcal{M}_\ell(\alpha)] \in H_d(X^\ell; \mathbb{Q}),$$

where $d = 2n + 2c_1(X) \cap \alpha + 2\ell - 6$ and ev is an evaluation map:

$$ev = (ev_1, \dots, e_\ell) : \mathcal{M}_\ell(\alpha) \to X^\ell.$$

Let $h_1, \dots, h_\ell$ be closed differential forms on X satisfying

$$\sum_{i=1}^{\ell} \deg h_i = 2n + 2c_1(X) \cap \alpha + 2\ell - 6.$$

The Gromov-Witten invariant is defined by

$$GW_\ell(\alpha; h_1, \dots, h_\ell) := \int_{\mathcal{M}_\ell(\alpha)} ev^*(h_1 \times \cdots \times h_\ell) \in \mathbb{R}.$$

We put $GW_\ell(\alpha; h_1, \dots, h_\ell) = 0$ for differential forms h_i which do not satisfy the degree condition.

A module homomorphism $GW_\ell : H(X; \Lambda_0)^{\otimes \ell} \to \Lambda_0$ is defined by

$$GW_\ell(h_1, \dots, h_\ell) := \sum_{\alpha \in H_2(X;\mathbb{Z})} GW_\ell(\alpha; h_1, \dots, h_\ell) T^{\alpha \cap \omega / 2\pi}.$$

We regard a bulk $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$ as an element in $H(X; \Lambda_+)$ by an obvious surjective homomorphism $\mathcal{A}(\mathbb{Z}) \to H(X; \mathbb{Z})$. The *bulk deformed quantum cup product* $\cup_{\mathfrak{b}}$ is defined as follows.

Definition 4.9. For each $a, b \in H^*(X;\Lambda_0)$, an element $a \cup_{\mathfrak{b}} b \in H^*(X;\Lambda_0)$ is defined by the following formula:

$$\langle a \cup_{\mathfrak{b}} b, c\rangle_{PD} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} GW_{\ell+3}(a, b, c, \mathfrak{b}, \dots, \mathfrak{b}), \tag{4.5.1}$$

where we denote by $\langle \cdot, \cdot \rangle_{PD}$ the Poincaré duality pairing.

The quantum product $\cup_{\mathfrak{b}}$ is graded commutative and associative. We obtain a $\mathbb{Z}_2$-graded commutative ring:

$$QH^*_{\mathfrak{b}}(X;\Lambda_0) := (H^*(X;\Lambda_0), \cup_{\mathfrak{b}}).$$

Remark 4.6. In Definition 4.9, the right hand side converges with respect to the valuation $\mathfrak{v}_T$. If we take $\mathfrak{b} \in H(X;\Lambda_0)$ as a deformation parameter, we need to modify the formula (4.5.1) in order to obtain the convergence (see Remark 5.2 in [FOOO11]).

To define the Jacobian ring of $\mathfrak{PO}^u_{\mathfrak{b}}$, we define a norm on the Laurent polynomial ring $\Lambda[y, y^{-1}] := \Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$. Let a symplectic toric manifold (X^{2n}, ω) has the moment polytope P as in the previous section:

$$P = \{u \in \mathbb{R}^n \mid l_i(u) = \langle u, v_i\rangle - \lambda_i \geq 0,\ i = 1, \dots, m\}\ .$$

For each $u = (u_1, \dots, u_n) \in P$, we put new n variables $y^u_i \in \Lambda[y, y^{-1}]$ by

$$y^u_i := T^{-u_i} y_i. \tag{4.5.2}$$

Any element $f \in \Lambda[y, y^{-1}]$ can be written by y^u_i

$$f = \sum_{(j_1,\dots,j_n)\in\mathbb{Z}^n} f^u_{j_1\cdots j_n} (y^u_1)^{j_1} \cdots (y^u_n)^{j_n},\quad f^u_{j_1\cdots j_n} \in \Lambda.$$

By using this expression, we define a valuation $\mathfrak{v}^u_T$ on $\Lambda[y, y^{-1}]$ as follows:

$$\mathfrak{v}^u_T(f) := \inf\{\mathfrak{v}_T(f^u_{j_1\cdots j_n}) \mid f^u_{j_1\cdots j_n} \neq 0\},\quad \mathfrak{v}^u_T(0) := +\infty.$$

We put

$$\mathfrak{v}^P_T(F) := \inf\{\mathfrak{v}^u_T(f) \mid u \in P\}.$$

This is not a valuation on $\Lambda[y, y^{-1}]$. However, we can define a metric d_P on $\Lambda[y, y^{-1}]$ by

$$d_P(f_1, f_2) := e^{-\mathfrak{v}^P_T(f_1 - f_2)}.$$

We denote by $\Lambda\langle\langle y, y^{-1}\rangle\rangle^P$ the completion of $\Lambda[y, y^{-1}]$ with respect to the norm d_P.

As mentioned in Section 4.4, the potential function $\mathfrak{PO}^u_{\mathfrak{b}}$ is written by u dependent variables y_i^u. We replace the variables y_i^u with new variables y_i defined by the same formula (4.5.2) and denote by $\mathfrak{PO}^u_{\mathfrak{b}}(y_1, \ldots, y_n)$. The following is known.

Theorem 4.7 (Theorem 3.14 in [FOOO11b]). *If we take $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$, then $\mathfrak{PO}^u_{\mathfrak{b}}(y_1, \ldots, y_n)$ converges with respect to d_P. i.e.,*

$$\mathfrak{PO}^u_{\mathfrak{b}} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle^P \quad \textit{for any } u \in \mathrm{Int}(P).$$

Moreover,

$$\mathfrak{PO}^u_{\mathfrak{b}}(y_1, \ldots, y_n) = \mathfrak{PO}^{u'}_{\mathfrak{b}}(y_1, \ldots, y_n)$$

for any $u, u' \in \mathrm{Int}(P)$.

Hence we denote by $\mathfrak{PO}_{\mathfrak{b}}$ the potential function with the variables y_i.

Remark 4.7. In the proof of Lemma 5.3, we use the potential function of the toric manifold $(S^2 \times S^2, \frac{1}{2}\omega_{std} \oplus \frac{1}{2}\omega_{std})$. Since this symplectic manifold is Fano and we use a bulk $\mathfrak{b}$ with $\deg \mathfrak{b} = 2$, the potential function is contained in $\Lambda[y, y^{-1}]$ without the completion.

We next describe the isomorphism, which is called the *Kodaira-Spencer map*, between the quantum cohomology $QH_{\mathfrak{b}}(X; \Lambda)$ and the *Jacobian ring* $\mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}}; \Lambda)$.

The Jacobian ring $\mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}}; \Lambda)$ is defined as follows:

Definition 4.10. *For $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$,*

$$\mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}}; \Lambda) := \frac{\Lambda\langle\langle y, y^{-1}\rangle\rangle^P}{\left(y_i \frac{\partial \mathfrak{P}O_{\mathfrak{b}}}{\partial y_i} : i = 1, \ldots, n\right)},$$

where the denominator is the ideal of $\Lambda\langle\langle y, y^{-1}\rangle\rangle^P$ generated by $y_i \frac{\partial \mathfrak{P}O_{\mathfrak{b}}}{\partial y_i}$.

Remark 4.8. Since we take $\mathfrak{b}$ from not $\mathcal{A}(\Lambda_0)$ but from $\mathcal{A}(\Lambda_+)$, we do not take the closure of the ideal (see Remark 1.2.11 of [FOOO10]).

We now recall the surjective homomorphism

$$\pi : \mathcal{A}(\Lambda_+) \to H^*(M; \Lambda_+); \quad \mathfrak{b} \mapsto PD[\mathfrak{b}].$$

We fix a subset of the basis $\{\mathbf{p}_i\}$ of $\mathcal{A}(\mathbb{Z})$

$$\{\mathbf{p}_{i_j} \mid j = 0, \dots, m', \dots, B'\} \subset \{\mathbf{p}_i \mid i = 1, \dots, m, \dots, B\}$$

so that $\pi(\mathbf{p}_{i_j})$ forms a basis of $H^*(X;\mathbb{Z})$ and $\mathbf{p}_{i_0} = \mathbf{p}_0 = X$, $\deg \mathbf{p}_{i_j} = 2$ for $1 \le j \le m'$. We identify $H^*(X;\Lambda_+)$ with the subspace of $\mathcal{A}(\Lambda_+)$ generated by $\{\mathbf{p}_{i_j}\}$. We put $e_j := \pi(\mathbf{p}_{i_j}) \in H^*(X;\Lambda_+)$. The Kodaira-Spencer map is defined as follows. For any element $\mathfrak{b} \in H(X;\Lambda_+)$, we may write

$$\mathfrak{PO}_{\mathfrak{b}} = \sum a_{k_1\dots k_n}(\mathfrak{b}) y_1^{k_1} \cdots y_n^{k_n},$$

where $\mathfrak{b} = \sum w_j e_j$ and $a_{k_1\dots k_n}(\mathfrak{b})$ is a formal power series of w_j with coefficients in Λ which converges with respect to $\mathfrak{v}_T$. By using this expression, we put

$$\frac{\partial \mathfrak{PO}_{\mathfrak{b}}}{\partial w_j}(\mathfrak{b}) := \sum \frac{\partial a_{k_1\dots k_n}(\mathfrak{b})}{\partial w_j} y_1^{k_1} \cdots y_n^{k_n}.$$

This summation converges in $\Lambda\langle\langle y, y^{-1}\rangle\rangle^P$ for each $\mathfrak{b}$. The Kodaira-Spencer map

$$\mathfrak{ks}_{\mathfrak{b}} : H(X;\Lambda) \to \mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}};\Lambda)$$

is defined by

$$\mathfrak{ks}_{\mathfrak{b}}(e_j) := \left[\frac{\partial \mathfrak{PO}_{\mathfrak{b}}}{\partial w_j}\right].$$

The Kodaira-Spencer map $\mathfrak{ks}_{\mathfrak{b}} : QH_{\mathfrak{b}}(X;\Lambda_0) \to \mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}};\Lambda_0)$ is also defined. In [FOOO10], Fukaya-Oh-Ohta-Ono obtained the following.

Theorem 4.8 (Theorem 1.1.1 in [FOOO10])**.** *The map* $\mathfrak{ks}_{\mathfrak{b}}$ *is a ring isomorphism*

$$QH_{\mathfrak{b}}(X;\Lambda_0) \cong \mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}};\Lambda_0).$$

In particular,

$$QH_{\mathfrak{b}}(X;\Lambda) \cong \mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}};\Lambda).$$

Chapter 5

Proofs of main results

5.1 Brief review of FOOO's results

In [FOOO12b], Fukaya-Oh-Ohta-Ono computed the full potential function of some Lagrangian tori in $S^2 \times S^2$ and they proved superheavyness of these tori in [FOOO11]. In this section, we briefly describe the construction of their superheavy tori.

Let $F_2(0)$ be a symplectic toric orbifold whose moment polytope P is given by

$$P := \{(u_1, u_2) \in \mathbb{R}^2 \mid 0 \leq u_1 \leq 2,\ 0 \leq u_2 \leq 1 - \frac{1}{2}u_1\}.$$

We denote by $\pi : F_2(0) \to P$ the moment map, and denote by $L(u)$ a Lagrangian torus fiber over an interior point $u \in \mathrm{Int}(P)$. Then $F_2(0)$ has one singular point which corresponds to the point $(0, 1)$ in P. They constructed a symplectic manifold $\hat{F}_2(0)$ which is symplectomorphic to $(S^2 \times S^2, \frac{1}{2}\omega_{std} \oplus \frac{1}{2}\omega_{std})$, by replacing a neighborhood of the singularity with a cotangent disk bundle of S^2 (for details, see Section 4 [FOOO12b]). Under the smoothing, Lagrangian torus fiber $L(u)$ is sent to a Lagrangian torus in $S^2 \times S^2$. In particular, we denote by T_τ $(0 < \tau \leq \frac{1}{2})$ this torus corresponding to $L((\tau, 1-\tau)) \subset F_2(0)$.

Remark 5.1. These Lagrangian tori T_τ are not toric fibers with respect to the standard toric structure on $S^2 \times S^2$. Therefore, the full potential function of T_τ can not be determined in terms of the moment polytope data as in Chapter 4.

For these Lagrangian tori $T_\tau \subset S^2 \times S^2$, they obtained the following.

Theorem 5.1 (Fukaya-Oh-Ohta-Ono [FOOO11]). *For any $0 < \tau \leq 1/2$, there exist a bulk $\mathfrak{b}(\tau) \in H^{even}(M, \Lambda_0)$ and idempotents e_τ and e_τ^0, each of which is an idempotent of a field factor of $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$ such that*

(1) *T_τ is $\mu_{e_\tau}^{\mathfrak{b}(\tau)}$-superheavy and $T_{\frac{1}{2}}$ is $\mu_{e_\tau^0}^{\mathfrak{b}(\tau)}$-superheavy.*

(2) *$S^1_{eq} \times S^1_{eq}$ is $\mu_e^{\mathfrak{b}(\tau)}$-superheavy for any idempotent e of a field factor of $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$. In particular,*

$$\psi(T_\tau) \cap (S^1_{eq} \times S^1_{eq}) \neq \emptyset$$

for any symplectic diffeomorphism ψ on $S^2 \times S^2$.

Here $\mu_{e_\tau}^{\mathfrak{b}(\tau)}$ and $\mu_{e_\tau^0}^{\mathfrak{b}(\tau)}$ denote homogeneous Calabi quasi-morphisms associated to the idempotents $e_\tau, e_\tau^0 \in QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$ respectively (see Theorem 3.1).

Remark 5.2. (1) In [FOOO11], (1) is Theorem 23.4 (2), and (2) is Theorem 1.13.

(2) The notion of $\mu_e^{\mathfrak{b}}$-superheavy is defined in Definition 18.5 of [FOOO11] and they remark as Remark 18.6 that $\mu_e^{\mathfrak{b}}$-superheavyness implies $\zeta_e^{\mathfrak{b}}$-superheavyness. In this paper, we need only to use $\zeta_e^{\mathfrak{b}}$-superheavyness.

(3) The quasi-morphisms $\mu_{e_\tau}^{\mathfrak{b}(\tau)}$ and $\mu_{e_\tau^0}^{\mathfrak{b}(\tau)}$ descend to homogeneous Calabi quasi-morphisms on $\mathrm{Ham}(S^2 \times S^2)$ as in [EP03].

Hereafter, we use only above homogeneous Calabi quasi-morphisms

$$\mu_{e_\tau}^{\mathfrak{b}(\tau)} : \mathrm{Ham}(S^2 \times S^2) \to \mathbb{R}$$

with $0 < \tau < 1/2$ and denote them by μ^τ.

5.2 Pullback of the quasi-morphism μ^τ

To obtain quasi-morphisms on $\mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$, we define a conformally symplectic embedding $\Theta_\delta : B^2 \times B^2 \hookrightarrow S^2 \times S^2$ for each Lagrangian submanifold $L_\delta \subset B^2 \times B^2$.

For each $1/2 < \delta \leq 1$, we define a conformally symplectic embedding $\theta_\delta : (B^2, 2\omega_0) \hookrightarrow (S^2, \frac{1}{2}\omega_{std}) \cong (\mathbb{C}P^1, \omega_{FS})$ by

$$\theta_\delta(z) := [\ \sqrt{1 - \delta|z|^2} : \sqrt{\delta}z\ ],$$

where we identify the projective space with a unit sphere by using a stereographic projection with respect to $(1,0,0) \in S^2 \subset \mathbb{R}^3$ after regarding the plane $\{v = (v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_1 = 0\}$ as the complex plane $\mathbb{C}$. We note that $\theta_\delta^*(\frac{1}{2}\omega_{std}) = \delta\omega_0$ and the image of θ_δ is $\{v \in S^2 \mid v_1 < 2\delta - 1\}$. Moreover, by the map θ_δ, the circle $T_\delta \subset B^2$ is mapped onto the equator $S^1_0 := \{v \in S^2 \mid v_1 = 0\}$ and the real form $Re(B^2)$ is mapped into the equator $S^1_{eq} := \{v \in \mathbb{R}^3 \mid v_3 = 0\} \subset S^2$.

Using this conformally symplectic embedding, we define $\Theta_\delta : B^2 \times B^2 \hookrightarrow S^2 \times S^2$ by

$$\Theta_\delta := \theta_\delta \times \theta_\delta : (B^2 \times B^2, \bar{\omega}_0) \hookrightarrow (S^2 \times S^2, \bar{\omega}_{std}), \tag{5.2.1}$$

where $\bar{\omega}_{std}$ denotes the symplectic structure $\frac{1}{2}\omega_{std} \oplus \frac{1}{2}\omega_{std}$ on $S^2 \times S^2$. This is a conformally symplectic embedding for each $1/2 < \delta \leq 1$. Indeed, it is obvious

$$\Theta_\delta{}^*\bar{\omega}_{std} = \delta\bar{\omega}_0.$$

For a time-dependent Hamiltonian F on $B^2 \times B^2$, we define a Hamiltonian $F \circ \Theta_\delta^{-1}$ on $S^2 \times S^2$ by

$$F \circ \Theta_\delta^{-1}(x) := \begin{cases} F(t, \Theta_\delta^{-1}(x)) & (x \in \mathrm{Im}(\Theta_\delta)) \\ 0 & (x \notin \mathrm{Im}(\Theta_\delta)). \end{cases}$$

Since Θ_δ is a conformally symplectic embedding, we obtain

$$\phi^1_{\delta F \circ \Theta_\delta^{-1}} = \Theta_\delta \phi^1_F \Theta_\delta^{-1}.$$

Thus, $\Theta_\delta \phi \Theta_\delta^{-1}$ is a Hamiltonian diffeomorphism on $S^2 \times S^2$ for any $\phi \in \mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$.

We define a family of quasi-morphisms $\mu^\tau_\delta : \mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0) \to \mathbb{R}$ by

$$\mu^\tau_\delta(\phi) := \frac{\delta^{-1}}{\mathrm{vol}(S^2 \times S^2)} \left(-\mu^\tau(\Theta_\delta \phi \Theta_\delta^{-1}) + \mathrm{Cal}_{\Theta_\delta(B^2 \times B^2)}(\Theta_\delta \phi \Theta_\delta^{-1})\right), \tag{5.2.2}$$

where μ^τ are Fukaya-Oh-Ohta-Ono's quasi-morphisms in Section 5.1 and $\mathrm{Cal}_{\Theta_\delta(B^2 \times B^2)}$ is the Calabi morphism on $\mathrm{Ham}_{\Theta_\delta(B^2 \times B^2)}(S^2 \times S^2, \bar{\omega}_{std})$ in Section 3.1. The symplectic structure $\bar{\omega}_{std}$ is exact on $\Theta_\delta(B^2 \times B^2)$, hence the right hand side of (5.2.2) does not depend on the choice of the Hamiltonian generating ϕ. Moreover, by the definition, it turns out that μ^τ_δ are quasi-morphisms. To obtain another expression of μ^τ_δ, we define $\zeta^\tau : C^\infty([0,1] \times S^2 \times S^2) \to \mathbb{R}$ as the following :

$$\zeta^\tau(H) := -\lim_{n \to \infty} \frac{\rho^{\mathfrak{b}(\tau)}(H^{\# n}; e_\tau)}{n},$$

where we denote by $H_1\#H_2$ the concatenation of two Hamiltonian H_1 and H_2 :

$$H_1\#H_2(t,x) := \begin{cases} \chi'(t)H_1(\chi(t),x)) & 0 \le t \le 1/2 \\ \chi'(t-1/2)H_2(\chi(t),x)) & 1/2 \le t \le 1 \end{cases}$$

for a smooth function $\chi : [0,1/2] \to [0,1]$ with $\chi' \ge 0$ and $\chi \equiv 0$ near $t = 0$, $\chi \equiv 1$ near $t = 1/2$. Note that this definition is independent of the function χ since the spectral invariant $\rho^{\mathfrak{b}(\tau)}$ has homotopy invariance property.

By the definition and (3.2.1), one can check that

$$\zeta^\tau(H) = \frac{1}{\mathrm{vol}(S^2 \times S^2)} \left(-\mu^\tau(\phi_H^1) + \mathrm{Cal}_{S^2\times S^2}(H)\right) \tag{5.2.3}$$

for any time-dependent Hamiltonian H and the restriction of ζ^τ to autonomous Hamiltonians corresponds to the bulk-deformed quasi-state $\zeta_{e_\tau}^{\mathfrak{b}(\tau)}$ which is associated to $\mu^\tau = \mu_{e_\tau}^{\mathfrak{b}(\tau)}$.

Therefore, by (5.2.2) and (5.2.3), we obtain the following expression of μ_δ^τ.

Lemma 5.1.

$$\mu_\delta^\tau(\phi_F^1) = \delta^{-1}\zeta^\tau(\delta F \circ \Theta_\delta^{-1}).$$

5.2.1 Properties of quasi-morphisms μ_δ^τ

In this section, we prove some properties of the quasi-morphisms μ_δ^τ by following procedures in [Se14]. Since Proposition 5.1 and Proposition 5.2 are proved by using only standard properties of Calabi quasi-morphisms, two proofs are the same as in [Se14]. However the proof of Proposition 5.3 depends on some properties of Lagrangian submanifolds and ambient spaces, thus we need to modify the proof slightly for our Lagrangian submanifolds $L_\delta \subset B^2 \times B^2$.

Proposition 5.1. *For any $0 < \tau < 1/2$ and $1/2 < \delta \le 1$, we have*

(1) $|\mu_\delta^\tau(\phi)| \le C_\delta \|\phi\|$, *where C_δ is a positive constant.*

(2) *If a time-dependent Hamiltonian H_t on $B^2 \times B^2$ is supported in a displaceable subset for any time $t \in [0,1]$ then we have*

$$\mu_\delta^\tau(\phi_H^1) = 0.$$

Proof. Let ϕ_F^1 be an element in $\mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$. Since the quasi-morphisms μ^τ have Lipschitz continuity property with respect to the Hofer norm on $\mathrm{Ham}(S^2 \times S^2, \bar{\omega}_{std})$ and $\Theta_\delta \phi_F^1 \Theta_\delta^{-1} = \phi^1_{\delta F \circ \Theta_\delta^{-1}}$, we obtain

$$|\mu^\tau(\Theta_\delta \phi_F^1 \Theta_\delta^{-1})| \leq \mathrm{vol}(S^2 \times S^2) \|\phi^1_{\delta F \circ \Theta_\delta^{-1}}\|.$$

By the definition of the Hofer norm, it turns out that

$$\|\phi^1_{\delta F \circ \Theta_\delta^{-1}}\| \leq \delta \|\phi_F^1\|.$$

Hence, we have

$$|\mu^\tau(\Theta_\delta \phi_F^1 \Theta_\delta^{-1})| \leq \delta \mathrm{vol}(S^2 \times S^2) \|\phi_F^1\|.$$

On the other hand, an easily calculation shows that

$$\mathrm{Cal}_{\Theta_\delta(B^2 \times B^2)}(\Theta_\delta \phi_F^1 \Theta_\delta^{-1}) = \delta^3 \int_0^1 dt \int_{B^2 \times B^2} F(t,x)\ \bar{\omega}_0^2.$$

As a result, we can obtain the following:

$$|\mathrm{Cal}_{\Theta_\delta(B^2 \times B^2)}(\Theta_\delta \phi_F^1 \Theta_\delta^{-1})| \leq \delta^3 \mathrm{vol}(B^2 \times B^2) \|\phi_F^1\|.$$

Consequently, it turns out that

$$\begin{aligned} |\mu_\delta^\tau(\phi)| &\leq \frac{\delta^{-1}}{\mathrm{vol}(S^2 \times S^2)} \Big(|\mu^\tau(\Theta_\delta \phi \Theta_\delta^{-1})| + |\mathrm{Cal}_{\Theta_\delta(B^2 \times B^2)}(\Theta_\delta \phi \Theta_\delta^{-1})| \Big) \\ &\leq (1 + \delta^2) \|\phi\|. \end{aligned}$$

Thus (1) is proved.

The property (2) follows immediately from Calabi-property of μ^τ. Indeed, two terms in the definition of μ_δ^τ are canceled each other. □

Let $X \subset S^2 \times S^2$ be a $\zeta_{e_\tau}^{\mathfrak{b}(\tau)}$-superheavy subset. By definition, we have

$$\min_X H \leq \zeta_{e_\tau}^{\mathfrak{b}(\tau)}(H) \leq \max_X H$$

for all autonomous Hamiltonians H on $S^2 \times S^2$. One can obtain the same inequality for $\zeta^\tau : C^\infty([0,1] \times S^2 \times S^2) \to \mathbb{R}$ if a closed subset $X \subset S^2 \times S^2$ is $\zeta_{e_\tau}^{\mathfrak{b}(\tau)}$-superheavy. More precisely, for all time-dependent Hamiltonians H on $S^2 \times S^2$, we have

$$\min_{[0,1] \times X} H \leq \zeta^\tau(H) \leq \max_{[0,1] \times X} H. \tag{5.2.4}$$

This is easily proved as mentioned in [Se14] without the detail. Indeed, we can take two autonomous Hamiltonians $H_{\min}, H_{\max}$ for any time-dependent Hamiltonian H such that $H_{\min} \equiv \min_{[0,1]\times X} H$, $H_{\max} \equiv \max_{[0,1]\times X} H$ on X and $H_{\min} \leq H \leq H_{\max}$ on $S^2 \times S^2$. By applying the anti[1]-monotonicity property of $\rho^{\mathfrak{b}(\tau)}$ (i.e. $H \leq K \Rightarrow \rho^{\mathfrak{b}(\tau)}(H; e_\tau) \geq \rho^{\mathfrak{b}(\tau)}(K; e_\tau)$, see Theorem 9.1 in [FOOO11]) and the fact $H \leq K$ implies $H^{\#n} \leq K^{\#n}$ to above Hamiltonians $H_{\min}, H, H_{\max}$, we can obtain (5.2.4) immediately.

From Lemma 5.1 and this inequality (5.2.4), we obtain the following.

Proposition 5.2. *Suppose a closed subset $X \subset S^2 \times S^2$ is $\zeta_{e_\tau}^{\mathfrak{b}(\tau)}$-superheavy and F is any compactly supported time-dependent Hamiltonian on the bi-disks $B^2 \times B^2$ such that $F \circ \Theta_\delta^{-1} |_X \equiv c$, then*

$$\mu_\delta^\tau(\phi_F^1) = c.$$

Proposition 5.3 is the most important to obtain unboundedness of $(\mathcal{L}(L_\delta), d)$. In [Kh09], Khanevsky proved the similar property and obtained the unboundedness for the case where the ambient space is two-dimensional open ball. In [Se14], by a different proof, Seyfaddini also obtained the similar property for $(\mathcal{L}(Re(B^{2n})), d)$.

Proposition 5.3. *If two Hamiltonian diffeomorphisms $\phi, \psi \in \mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$ satisfy*

$$\phi(L_\delta) = \psi(L_\delta),$$

then we have

$$|\mu_\delta^\tau(\phi) - \mu_\delta^\tau(\psi)| \leq \frac{D_{\mu^\tau}}{\delta \mathrm{vol}(S^2 \times S^2)} \quad \text{for all } \frac{1}{2} < \delta \leq 1,\ 0 < \tau < \frac{1}{2},$$

where D_{μ^τ} denotes the defect of μ^τ.

We prove this proposition by slightly modifying Seyfaddini's proof.

Proof. Throughout the proof, we fix δ, τ with $1/2 < \delta \leq 1$, $0 < \tau < 1/2$, respectively. From the definition of μ_δ^τ and its homogeneity we obtain that

$$\begin{aligned}
&|\mu_\delta^\tau(\phi^{-1}\psi) + \mu_\delta^\tau(\phi) - \mu_\delta^\tau(\psi)| \\
&= |\mu_\delta^\tau(\phi^{-1}\psi) - \mu_\delta^\tau(\phi^{-1}) - \mu_\delta^\tau(\psi)| \\
&= \frac{1}{\delta \mathrm{vol}(S^2 \times S^2)} |\mu^\tau(\Theta_\delta \phi^{-1} \psi \Theta_\delta^{-1}) - \mu^\tau(\Theta_\delta \phi^{-1} \Theta_\delta^{-1}) - \mu^\tau(\Theta_\delta \psi \Theta_\delta^{-1})| \\
&\leq \frac{D_{\mu^\tau}}{\delta \mathrm{vol}(S^2 \times S^2)}.
\end{aligned}$$

[1]Fukaya-Oh-Ohta-Ono used different sign conventions from [EP03, EP06, EP09] (see Remark 4.17 in [FOOO11]).

Consequently, it is sufficient to prove the proposition that $\mu_\delta^\tau(\phi)$ vanishes for Hamiltonian diffeomorphisms ϕ satisfying $\phi(L_\delta) = L_\delta$.

Now we take any Hamiltonian $F \in C_c^\infty([0,1] \times (B^2 \times B^2))$ and assume the Hamiltonian diffeomorphism ϕ_F^1 preserves the Lagrangian submanifold L_δ.

For $0 < s \leq 1$, we define a diffeomorphism $a_s : B^2 \times B^2(s) \to B^2 \times B^2$ by

$$a_s(z_1, z_2) := (z_1, \frac{z_2}{s}).$$

Using this map, we define a compactly supported symplectic diffeomorphism ψ_s for each $0 < s \leq 1$:

$$\psi_s := \begin{cases} a_s^{-1} \phi_F^1 a_s & |z_2| \leq s \\ id & |z_2| \geq s \end{cases}.$$

As compactly supported cohomology group $H_c^1(B^2 \times B^2; \mathbb{R}) = 0$ and $\bar{\omega}_0$ is exact on $B^2 \times B^2$, any isotopy of compactly supported Symplectic diffeomorphisms on $(B^2 \times B^2, \bar{\omega}_0)$ is a compactly supported Hamiltonian isotopy. Thus, for each $0 < s \leq 1$, we can take a time-dependent Hamiltonian $F^s \in C_c^\infty([0,1] \times B^2 \times B^2)$ such that $\psi_s = \phi_{F^s}^1$.

This Hamiltonian diffeomorphisms ψ_s have the following properties:

(1) $\psi_1 = \phi_{F^1}^1 = \phi_F^1$,

(2) ψ_s preserves L_δ for each $0 < s \leq 1$,

(3) There exists a compact subset K_s in B^2 such that F^s is supported in $K_s \times B^2(s) \subset B^2 \times B^2$ for each $0 < s \leq 1$.

Hereafter we fix sufficiently small $\epsilon > 0$ such that $K_\epsilon \times B^2(\epsilon)$ is displaceable inside the bi-disks $B^2 \times B^2$. By Proposition 5.1 (2), it follows that

$$\mu_\delta^\tau(\psi_\epsilon) = 0. \tag{5.2.5}$$

We take a time-dependent Hamiltonian $H \in C_c^\infty([0,1] \times B^2 \times B^2)$ so that $\phi_H^t := \psi_\epsilon^{-1} \psi_{t(1-\epsilon)+\epsilon}$ for $0 \leq t \leq 1$. In particular, we have the time-one map $\phi_H^1 = \psi_\epsilon^{-1} \phi_F^1$ by the above property (1).

We note that Hamiltonian vector field X_{H_t} is tangent to the Lagrangian submanifold L_δ since ϕ_H^t preserves L_δ. Consequently, for each $t \in [0,1]$, $H_t = H(t, \cdot)$ is constant on L_δ. Because of this and non-compactness of L_δ, the restriction of H_t to L_δ is 0 for all $t \in [0,1]$. Since $L_\delta = T_\delta \times Re(B^2)$ is mapped into $S_0^1 \times S_{eq}^1$ by Θ_δ, hence $H \circ \Theta_\delta^{-1}$ vanishes on a torus $S_0^1 \times S_{eq}^1$.

On the other hand $S^1_0 \times S^1_{eq}$ is $\zeta^{\mathfrak{b}(\tau)}_{e_\tau}$-superheavy by Fukaya-Oh-Ohta-Ono's result (Theorem 5.1), therefore we have

$$\mu^\tau_\delta(\phi^1_H) = 0. \tag{5.2.6}$$

Here we used Proposition 5.2.

As a consequence of these two equalities (5.2.5), (5.2.6) and quasi-additivity of μ^τ_δ, it follows that

$$|\mu^\tau_\delta(\phi^1_F)| = |\mu^\tau_\delta(\phi^1_F) - \mu^\tau_\delta(\psi_\epsilon) - \mu^\tau_\delta(\phi^1_H)| \leq \frac{D_{\mu^\tau}}{\delta \mathrm{vol}(S^2 \times S^2)}.$$

Because $(\phi^1_F)^n$ preserves L_δ for any $n \in \mathbb{N}$, we can apply the same argument to $(\phi^1_F)^n$ and obtain $|\mu^\tau_\delta((\phi^1_F)^n)| \leq \delta^{-1}\mathrm{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}$. Since μ^τ_δ is a homogeneous quasi-morphism, we have

$$\mu^\tau_\delta(\phi^1_F) = 0.$$

□

By applying Proposition 5.1 (1) and Proposition 5.3, we obtain the following.

Proposition 5.4. *For any $\phi \in \mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$ and any $\frac{1}{2} < \delta \leq 1,\ 0 < \tau < \frac{1}{2}$, the following inequality holds.*

$$\frac{\mu^\tau_\delta(\phi) - \delta^{-1}\mathrm{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}}{C_\delta} \leq d(L_\delta, \phi(L_\delta)),$$

where D_{μ^τ} is as above.

Proof. We take any $\psi \in \mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$ satisfying $\phi(L_\delta) = \psi(L_\delta)$. From Proposition 5.3, we obtain the following inequality.

$$|\mu^\tau_\delta(\phi) - \mu^\tau_\delta(\psi)| \leq \frac{D_{\mu^\tau}}{\delta \mathrm{vol}(S^2 \times S^2)}.$$

By using Proposition 5.1 (1), we have

$$|\mu^\tau_\delta(\phi)| - \frac{D_{\mu^\tau}}{\delta \mathrm{vol}(S^2 \times S^2)} \leq |\mu^\tau_\delta(\psi)| \leq C_\delta \|\psi\|.$$

Therefore, by definition of the metric d, we obtain the following inequality:

$$|\mu^\tau_\delta(\phi)| - \frac{D_{\mu^\tau}}{\delta \mathrm{vol}(S^2 \times S^2)} \leq C_\delta \cdot d(L_\delta, \psi(L_\delta)).$$

□

5.3 Construction of $\Phi_\delta : C_c^\infty((0,1)) \to \mathcal{L}(L_\delta)$

5.3.1 Locations of FOOO's superheavy tori

To construct a mapping $\Phi_\delta : C_c^\infty((0,1)) \to \mathcal{L}(L_\delta)$ in Theorem 1.2, we describe the locations of Fukaya-Oh-Ohta-Ono's Lagrangian superheavy tori by following Oakley-Usher's result. Let us recall their description. In [OU13], they constructed a symplectic toric orbifold $\mathcal{O}$ which is isomorphic to $F_2(0)$ as symplectic toric orbifolds by gluing $S^2 \times S^2 \setminus \bar{\Delta}$ to $B^4/\{\pm 1\}$. Here $\bar{\Delta}$ denotes anti-diagonal of $S^2 \times S^2$ and B^4 is a four dimensional open ball. The moment map $\pi : \mathcal{O} \to \mathbb{R}^2$, which has the same moment polytope P of $F_2(0)$ in Section 5.1, is expressed on $S^2 \times S^2 \setminus \bar{\Delta}$ by

$$\pi(v,w) = \Big(\frac{1}{2}|v+w| + \frac{1}{2}(v+w)\cdot e_1,\ 1 - \frac{1}{2}|v+w|\Big) \in \mathbb{R}^2$$

for $(v,w) \in S^2 \times S^2 \setminus \bar{\Delta}$ and $e_1 := (1,0,0)$. Therefore one can consider a torus fiber $L(u) \subset F_2(0)$ as $\pi^{-1}(u) \subset S^2 \times S^2 \setminus \bar{\Delta}$ for any interior point u in the moment polytope.

By replacing $B^4/\{\pm 1\}$ by the unit disk cotangent bundle $D_1^*S^2$, they obtained a smoothing $\Pi : \hat{\mathcal{O}} \to \mathcal{O}$ which maps the zero-section of $D_1^*S^2$ to the singularity of $\mathcal{O}$ and whose restriction to $S^2 \times S^2 \setminus \bar{\Delta}$ is the identity mapping. Moreover they gave an explicit symplectic morphism $\hat{\mathcal{O}} \xrightarrow{\sim} S^2 \times S^2$ which is the identity mapping on $S^2 \times S^2 \setminus \bar{\Delta}$. Hence above tori $\pi^{-1}(u)$ are invariant under the smoothing and the symplectic morphism $\hat{\mathcal{O}} \xrightarrow{\sim} S^2 \times S^2$.

Using this construction, Oakley-Usher proved that the Entov-Polterovich's exotic monotone torus in [EP09] is Hamiltonian isotopic to the Fukaya-Oh-Ohta-Ono's torus over $(1/2, 1/2)$ (for details, see the proof of Proposition 2.1 [OU13]).

Proposition 5.5 (Oakley-Usher [OU13])**.** *Fukaya-Oh-Ohta-Ono's superheavy Lagrangian tori T_τ can be expressed as*

$$T_\tau = \left\{(v,w) \in S^2 \times S^2 \mid \frac{1}{2}|v+w| + \frac{1}{2}(v+w)\cdot e_1 = \tau,\ 1 - \frac{1}{2}|v+w| = 1-\tau\right\},$$

where the parameter τ is in $(0, 1/2]$. In particular, the Lagrangian torus $T_{1/2}$ is Entov-Polterovich's exotic monotone torus.

The following corollary is proved by an easily calculation.

Corollary 5.1. *The image of i-th projection* $\mathrm{pr}_i : S^2 \times S^2 \to S^2$ $(i = 1, 2)$ *is*

$$\mathrm{pr}_i(T_\tau) = \left\{v \in S^2 \mid |v \cdot e_1| \le \sqrt{1-\tau^2}\right\}, \qquad (5.3.1)$$

where τ *is* $0 < \tau \leq 1/2$.

By this corollary and the definition of the conformally symplectic embedding $\Theta_\delta : B^2 \times B^2 \hookrightarrow S^2 \times S^2$. We have the following.

Corollary 5.2. *For any* $(2+\sqrt{3})/4 < \delta \leq 1$ *there exists a sufficiently small* $\varepsilon_\delta > 0$ *such that*

$$\bigcup_{\tau \in I_\delta} T_\tau \subset \Theta_\delta(B^2 \times B^2), \qquad I_\delta := [1/2 - \varepsilon_\delta, 1/2].$$

Remark 5.3. The condition $(2+\sqrt{3})/4 < \delta \leq 1$ in Theorem 1.2 guarantees that the image of Θ_δ contains a continuous subfamily of superheavy tori $T_\tau \subset S^2 \times S^2$ as in Corollary 5.2. However, for any $1/2 < \delta \leq 1$, it is likely that there exist $\phi_\delta \in \mathrm{Ham}(S^2 \times S^2)$ such that the image of Θ_δ contains $\cup_{\tau \in I'_\delta} \phi_\delta(T_\tau)$ for some open interval $I'_\delta \subset (0, 1/2]$. In this case, we can show Theorem 1.2 under the weaker assumption $1/2 < \delta \leq 1$.

5.3.2 Construction of Φ_δ

We fix δ with $(2+\sqrt{3})/4 < \delta \leq 1$ and consider the interval $I_\delta = [1/2-\varepsilon_\delta, 1/2]$ in Corollary 5.2. We take a segment J_δ in the moment polytope $P = \pi(\mathcal{O}) \subset \mathbb{R}^2$ defined by

$$J_\delta := \{(\tau, 1-\tau) \mid \tau \in \mathrm{Int}(I_\delta)\} \subset \mathrm{Int}(P).$$

We denote by $B^2(u_0; \sqrt{2}\varepsilon_\delta)$ the open disk of which center is $u_0 := (1/2, 1/2) \in \mathrm{Int}(P)$ and radius is $\sqrt{2}\varepsilon_\delta$. We may take and fix a sufficiently small $\varepsilon_\delta > 0$ so that the open disk $B^2(u_0; \sqrt{2}\varepsilon_\delta)$ is contained in P and moreover the inverse image of $B^2(u_0; \sqrt{2}\varepsilon_\delta)$ under $\tilde{\pi} := \pi \circ \Pi : \hat{O} \to P$ is contained in the image of $\Theta_\delta : B^2 \times B^2 \to S^2 \times S^2$.

We identify J_δ with an open interval $(0, 1)$ and will define a map Φ_δ on $C_c^\infty(J_\delta)$. First, we extend a function $f \in C_c^\infty(J_\delta)$ to the function f_{B^2} on the open disk $B^2(u_0; \sqrt{2}\varepsilon_\delta)$ which is constant along the circle centered at u_0. More explicitly, we define $f_{B^2} : B^2(u_0; \sqrt{2}\varepsilon_\delta) \to \mathbb{R}$ by

$$f_{B^2}(u) := f((\ |u-u_0|/\sqrt{2}, 1 - |u-u_0|/\sqrt{2}\ )), \quad u \in B^2(u_0; \sqrt{2}\varepsilon_\delta) \subset \mathrm{Int}(P).$$

We define $\tilde{f} \in C_c^\infty(B^2 \times B^2)$ for $f \in C_c^\infty(J_\delta)$ as the pull-back:

$$\tilde{f} := \Theta_\delta^* \tilde{\pi}^* f_{B^2}. \tag{5.3.2}$$

By the construction, the restriction of $\tilde{f}$ on $\Theta_\delta^{-1}(T_\tau)$ is constantly equal to $f(\tau)$ for all $1/2 - \varepsilon_\delta < \tau < 1/2$.

Definition 5.1. For any $(2+\sqrt{3})/4 < \delta \leq 1$, we define $\Phi_\delta : C_c^\infty((0,1)) \to \mathcal{L}(L_\delta)$ by the following expression:

$$\Phi_\delta(f) := \phi_{\tilde{f}}^1(L_\delta),$$

where we regard f as an element in $C_c^\infty(J_\delta)$.

For the proof of Theorem 1.2, we prove the next lemma.

Lemma 5.2. *For any $f, g \in C_c^\infty((1/2-\varepsilon_\delta, 1/2))$ there exists a constant $1/2-\varepsilon_\delta < \tau' < 1/2$ such that*

$$|\mu_\delta^{\tau'}(\phi_{\tilde{f}-\tilde{g}}^1)| = \|f-g\|_\infty,$$

where δ is $(2+\sqrt{3})/4 < \delta \leq 1$.

Proof. For any $f, g \in C_c^\infty((1/2-\varepsilon_\delta, 1/2))$, there exists $\tau' \in (1/2-\varepsilon_\delta, 1/2)$ such that

$$\|f-g\|_\infty = \max |f(x)-g(x)| = |f(\tau')-g(\tau')|.$$

Thus $\mu_\delta^{\tau'}(\phi_{\tilde{f}-\tilde{g}}^1)$ is equal to $\|f-g\|_\infty$ because of (5.3.2) and Proposition 5.2. □

5.4 Proof of Theorem 1.1 and Theorem 1.2.

proof of Theorem 1.1. For all $1/2 < \delta \leq 1$, the image of Θ_δ contains the torus $S_0^1 \times S_0^1 \subset (S^2 \times S^2, \bar{\omega}_{std})$. If we take a Hamiltonian $H \in C_c^\infty(B^2 \times B^2)$ for any $h \in \mathbb{R}$ such that $H \equiv h$ on the torus $\Theta_\delta^{-1}(S_0^1 \times S_0^1)$, then we have from Proposition 5.2 and $\zeta_{e_\tau}^{\mathfrak{b}(\tau)}$-superheavyness of $S_0^1 \times S_0^1$

$$\mu_\delta^\tau(\phi_H^1) = h,$$

where we fix any $\tau \in (0, \frac{1}{2})$. By applying Proposition 5.4, we obtain

$$\frac{h - \delta^{-1}\mathrm{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}}{C_\delta} \leq d(L_\delta, \phi(L_\delta)).$$

Since h is an arbitrary constant, Theorem 1.1 is proved. □

Remark 5.4. To prove Theorem 1.1, it is not necessary to use a family of quasi-morphisms on $\mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$. Indeed, since the torus $S_0^1 \times S_0^1$ is superheavy with respect to Entov-Polterovich's symplectic quasi-state ζ_{EP} (see [EP03, EP06, EP09]), we can use the Calabi quasi-morphism μ_{EP} associated to ζ_{EP} instead of Fukaya-Oh-Ohta-Ono's Calabi quasi-morphisms μ^τ.

On the other hand, to prove Theorem 1.2, it is necessary that the image $\Theta_\delta(B^2 \times B^2)$ contains a continuous subfamily of superheavy tori $\phi_\delta(T_\tau) \subset S^2 \times S^2$ for some $\phi_\delta \in \mathrm{Ham}(S^2 \times S^2)$ as mentioned in Remark 5.3.

In this thesis, we consider the case $\phi_\delta = id$. Then we need to use the parameter δ of our Lagrangian submanifolds L_δ with $(2+\sqrt{3})/4 < \delta \leq 1$ as in Corollary 5.2.

proof of Theorem 1.2. First, we will prove the left-hand side inequality. For any $f, g \in C_c^\infty((1/2-\varepsilon_\delta, 1/2))$, we have $\tilde{f}, \tilde{g} \in C_c^\infty(B^2 \times B^2)$ defined by (5.3.2). Then we apply Proposition 5.4 to $\phi_{\tilde{g}}^{-1} \circ \phi_{\tilde{f}}^1 \in \mathrm{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$ to obtain

$$\frac{|\mu_\delta^\tau(\phi_{\tilde{g}}^{-1} \circ \phi_{\tilde{f}}^1)| - \delta^{-1}\mathrm{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}}{C_\delta} \leq d(L_\delta, \phi_{\tilde{g}}^{-1} \circ \phi_{\tilde{f}}^1(L_\delta)), \tag{5.4.1}$$

where $\phi_{\tilde{g}}^{-1}$ is the inverse of $\phi_{\tilde{g}}^1$. By the construction of autonomous Hamiltonians $\tilde{f}$, $\tilde{g}$ in (5.3.2), we find that the Poisson bracket $\{\tilde{f}, \tilde{g}\}_{\bar{\omega}_0}$ vanishes. Thus we have

$$\phi_{\tilde{g}}^{-1} \circ \phi_{\tilde{f}}^1 = \phi_{\tilde{f}-\tilde{g}}^1.$$

Therefore the inequality (5.4.1) becomes

$$\frac{|\mu_\delta^\tau(\phi_{\tilde{f}-\tilde{g}}^1)| - \delta^{-1}\mathrm{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}}{C_\delta} \leq d(\phi_{\tilde{g}}^1(L_\delta), \phi_{\tilde{f}}^1(L_\delta)).$$

By Lemma 5.2, we obtain the following inequality:

$$\frac{\|f-g\|_\infty - \delta^{-1}\mathrm{vol}(S^2 \times S^2)^{-1} D_{\mu^{\tau'}}}{C_\delta} \leq d(\Phi_\delta(f), \Phi_\delta(g)),$$

where the constant τ' depends on f and g. We will prove the following lemma in Section 5.5.

Lemma 5.3. *For any bulk-deformation parameter $\tau \in (0, 1/2)$, the defect D_{μ^τ} of quasi-morphisms μ^τ satisfies*

$$D_{\mu^\tau} \leq 12.$$

Therefore, we obtain the left-hand side inequality by putting $D_\delta := \delta^{-1}\mathrm{vol}(S^2 \times S^2)^{-1} \cdot \sup_\tau D_{\mu^\tau}$.

The right-hand side inequality is proved immediately. Indeed, we can estimate as the following:

$$\begin{aligned} d(\Phi_\delta(f), \Phi_\delta(g)) = d(L_\delta, \phi_{\tilde{g}}^{-1}\phi_{\tilde{f}}^1(L_\delta)) &\leq \|\tilde{f} - \tilde{g}\| \\ &= \|f - g\|. \end{aligned}$$

This completes the proof of Theorem 1.2. □

5.5 Finiteness of D_{μ^τ}

The estimate in Lemma 5.3 can be obtained by almost the same calculation of Proposition 21.7 in [FOOO11]. For this reason, we only sketch the outline of the calculation and use the same notation used in [FOOO11].

proof of Lemma 5.3. From Remark 16.8 in [FOOO11], upper bounds of defects D_{μ^τ} can be taken to be $-12\mathfrak{v}_T(e_\tau)$, where $\mathfrak{v}_T$ is a valuation of bulk-deformed quantum cohomology $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$. The proof of Theorem 5.1 (Theorem 23.4 [FOOO11]) implies that the idempotent $e_\tau \in QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$ can be taken from one of four idempotents in $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$ which decompose quantum cohomology as follows:

$$QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda) = \bigoplus_{(\epsilon_1,\epsilon_2)=(\pm 1,\pm 1)} \Lambda \cdot e^\tau_{\epsilon_1,\epsilon_2}.$$

Here the quantum product in $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2)$ respects this splitting (i.e. it is semi-simple).

Hence, to prove Lemma 5.3, we only have to estimate the maximum valuation of $e^\tau_{\epsilon_1,\epsilon_2}$. For this purpose, we regard $S^2 \times S^2$ as the symplectic toric manifold with the moment polytope:

$$P = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid l_i(u) \geq 0,\ i = 1, \dots, 4\},$$

where

$$l_1 = u_1,\ \ l_2 = u_2,\ \ l_3 = -u_1 + 1,\ \ l_4 = -u_2 + 1.$$

We denote by $\partial_i P := \{l_i(u) = 0\}$ each facets of P and put $D_i := \pi^{-1}(\partial_i P)$, where $\pi : S^2 \times S^2 \to P \subset \mathbb{R}^2$ is the moment map. In the following, we fix

$$e_0 := PD[S^2 \times S^2],\ e_1 := PD[D_1],\ e_2 := PD[D_2],\ e_3 := PD[D_1 \cap D_2]$$

as basis of $H^*(S^2 \times S^2; \mathbb{C})$ and denote by $L(u_0)$ the Lagrangian torus fiber over $(1/2, 1/2) \in P$.

The element $\mathfrak{b}(\tau)$ in Theorem 5.1 is defined by

$$\mathfrak{b}(\tau) := aPD[D_1] + aPD[D_2],\ a := T^{\frac{1}{2}-\tau}. \tag{5.5.1}$$

In our case, since $S^2 \times S^2$ is Fano, the potential function $\mathfrak{PO}_{\mathfrak{b}(\tau)}$ is determined in terms of the moment polytope data. Hence we obtain the following expression as in the proof of Theorem 23.4 [FOOO11]

$$\mathfrak{PO}_{\mathfrak{b}(\tau)} = \mathrm{e}^a y_1 + \mathrm{e}^{-a} y_2 + y_1^{-1}T + y_2^{-1}T,$$

where $y_1, \ldots, y_4$ are formal variables and $\mathrm{e}^a := \sum_{n=0}^{\infty} a^n/n! \in \Lambda_0$ (see Section 3 in [FOOO11b] and Section 20.4 in [FOOO11] for the definition of potential functions for toric fibers).

By Proposition 1.2.16 in [FOOO10], the *Jacobian ring* $\mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}(\tau)};\Lambda)$ of the potential function $\mathfrak{PO}_{\mathfrak{b}(\tau)}$, which is defined as a certain quotient ring of the Laurent polynomial $\Lambda[y_1, \ldots, y_4, y_1^{-1}, \ldots, y_4^{-1}]$ for our case, is decomposed as follows:

$$\mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}(\tau)};\Lambda) = \bigoplus_{(\epsilon_1,\epsilon_2)=(\pm 1,\pm 1)} \Lambda \cdot 1^{\tau}_{\epsilon_1,\epsilon_2},$$

where $1^{\tau}_{\epsilon_1,\epsilon_2}$ is the unit on each component. More explicitly, we have

$$1^{\tau}_{\epsilon_1,\epsilon_2} = \frac{1}{4}[1 + \epsilon_1 \mathrm{e}^{\frac{a}{2}} y_1 T^{-\frac{1}{2}} + \epsilon_2 \mathrm{e}^{-\frac{a}{2}} y_2 T^{-1/2} + \epsilon_1\epsilon_2 y_1 y_2 T^{-1}].$$

We denote by $e^{\tau}_{\epsilon_1,\epsilon_2}$ the idempotent of $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2;\Lambda)$ which corresponds to $1^{\tau}_{\epsilon_1,\epsilon_2}$ under the *Kodaira-Spencer map*:

$$\mathfrak{ks}_{\mathfrak{b}(\tau)} : QH_{\mathfrak{b}(\tau)}(S^2 \times S^2;\Lambda) \to \mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}(\tau)};\Lambda),$$

which is a ring isomorphism (see Theorem 4.8). The same calculation as in Remark 1.3.1 [FOOO10] shows that the Kodaira-Spencer map $\mathfrak{ks}_{\mathfrak{b}(\tau)}$ maps the basis of $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2;\Lambda)$ to the following:

$$\mathfrak{ks}_{\mathfrak{b}(\tau)}(e_0) = [1],\ \mathfrak{ks}_{\mathfrak{b}(\tau)}(e_1) = [\mathrm{e}^a y_1],\ \mathfrak{ks}_{\mathfrak{b}(\tau)}(e_2) = [\mathrm{e}^{-a} y_2],\ \mathfrak{ks}_{\mathfrak{b}(\tau)}(e_3) = [q y_1 y_2].$$

Here $q \in \mathbb{Q}$ is defined as follows (see Definition 6.7 in [FOOO11b]). Let $\beta_1 + \beta_2$ be the element of $H_2(S^2 \times S^2, L(u_0);\mathbb{Z})$ satisfies

$$(\beta_1 + \beta_2) \cap D_i = 1 \ (i = 1, 2)$$

with Maslov index $\mu(\beta_1 + \beta_2) = 4$ and

$$q := ev_{0*}[\mathcal{M}^{main}_{1;1}(L(u_0), \beta_1 + \beta_2; e_3)] \cap L(u_0).$$

The classification theorem of holomorphic disks in [CO06] implies $q = \pm 1$ immediately.

By comparing $e^{\tau}_{\epsilon_1,\epsilon_2}$ with $1^{\tau}_{\epsilon_1,\epsilon_2}$, we can obtain for $(\epsilon_1,\epsilon_2) = (\pm 1, \pm 1)$,

$$e^{\tau}_{\epsilon_1,\epsilon_2} = \frac{1}{4}\big(e_0 + \epsilon_1 \mathrm{e}^{-\frac{a}{2}} T^{-\frac{1}{2}} \cdot e_1 + \epsilon_2 \mathrm{e}^{\frac{a}{2}} T^{-\frac{1}{2}} \cdot e_2 + \epsilon_1\epsilon_2 q^{-1} T^{-1} \cdot e_3\big).$$

Since $a = T^{\frac{1}{2}-\tau}$ and $0 < \tau < 1/2$, we obtain $\mathfrak{v}_T(e^{\tau}_{\epsilon_1,\epsilon_2}) = -1$. This implies Lemma 5.3. □

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