Monopole Walls and Hyperkähler Metrics

Daichi Muranaka

Department of Mathematics, Nagoya University

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Abstract

We present explicit hyperkähler metrics induced from well-separated $SU(2)$ monopole walls which are equivalent to monopoles on $T^2 \times \mathbb{R}$. A hyperkähler manifold is a $4k$-dimensional Riemannian manifold equipped with three good-natured complex structures which are defined in the same manner as pure imaginary quaternion bases. In particular, the metric on a hyperkähler manifold corresponds to a vacuum solution of the self-dual Einstein equations in four-dimensional Euclidean space that is a so-called gravitational instanton. The monopole moduli spaces are hyperkähler manifolds. According to Manton's observation, the equations of geodesic motions on the moduli spaces correspond to the equations of motions induced from the Lagrangian of slowly moving monopoles which start to move with small velocity. Actually, some hyperkähler metrics have been explicitly obtained by calculating the interaction of well-separated monopoles. The metrics are called the Gibbons-Manton metrics. Although the Lagrangian for periodic monopoles diverge in the calculation, the relative Lagrangian can be defined by making the vacuum expectation value of the Higgs field diverge at the expense of the center-of-mass Lagrangian. According to the above methods, we explicitly derive hyperkähler metrics with doubly-periodicity by assuming the asymptotic fields of well-separated monopole-walls and calcurating their interaction. Our metrics are of the ALH type. These are defined on a T^3 fibration over $\mathbb{R}$. The metrics have the modular invariance with respect to the complex structure of the complex torus T^2. We also find a local isometric transformation for the metrics. Moreover, we derive metrics from monopole walls with Dirac-type singularities following methods for periodic monopoles. We specify the maximum number of the singularities for our metrics by using spectral analysis which was used for calculating the boundary conditions and the dimension of the moduli space of such monopole walls. The number is indeed equal to the maximal number of the matter hypermultiplets in the fundamental representation in the corresponding super Yang-Mills theory with eight super charges. In addition to these results, we review fundamental topics such as $SU(2)$ monopoles and the Gibbons-Manton metrics in non-periodic case, monopole walls and spectral analysis.

Contents

Chapter 1

Introduction

In differential geometry, a hyperkähler manifold is a $4k$-dimensional Riemannian manifold with three good-natured complex structures which are defined in the same manner as quaternions. Hyperkähler manifolds can also be defined as $4k$-dimensional Riemannian manifolds whose holonomy group $SO(4k)$ is reduced to the compact symplectic group $Sp(k)$. Hyperkähler manifolds have played important roles in the study of supersymmetric quantum field theories and string theories, especially, in the context of the string compactifications, duality tests and so on.

The explicit metric on a compact hyperkähler manifold is not known except trivial examples. On the other hand, explicit forms of the non-compact hyperkähler metric have been derived in several ways. Among them the most systematic one is the hyperkähler quotient construction [1] (see also [2]). In four-dimensions, the hyperkähler metrics satisfy the self-dual Einstein equations and arise as the gravitational instanton solutions (see e.g. [3]). These can be classified into some categories: the ALE, ALF, ALG and ALH spaces [4] according to their asymptotic volume growth.

In the context of three-dimensional gauge theories, hyperkähler metrics are obtained by considering well-separated monopoles, which is due to Manton's observation [5] that the dynamics of k well-separated BPS monopoles can be approximated as a geodesic motion on the asymptotic moduli space of the BPS k-monopole if the initial velocities of each monopole are substantially small.

Table 1.1: The correspondence of the periodicity of monopole, super Yang-Mills theory and the asymptotic behavior of hyperkähler metric (four-dimensional topology).

Periodicity of monopole	Super Yang-Mills theory	Asymptotic behavior
$\mathbb{R}^3$ (non-periodic)	$\mathcal{N}=4$ SYM on $\mathbb{R}^3$	ALF : S^1 fibration on $\mathbb{R}^3$
$S^1 \times \mathbb{R}^2$ (periodic)	$\mathcal{N}=2$ SYM on $\mathbb{R}^3 \times S^1$	ALG : T^2 fibration on $\mathbb{C}$
$T^2 \times \mathbb{R}$ (doubly-periodic)	$\mathcal{N}=1$ SYM on $\mathbb{R}^3 \times T^2$	ALH : T^3 fibration on $\mathbb{R}$

For a non-periodic BPS k-monopole the moduli space can be written as $\mathcal{M}_k = \mathbb{R}^3 \times (S^1 \times \widetilde{\mathcal{M}}_k^0)/\mathbb{Z}_k$, where the simply-connected part is denoted by $\widetilde{\mathcal{M}}_k^0$, and the degrees of $\mathbb{R}^3$ and S^1 correspond to the center of mass and the gauge degree of global $U(1)$, respectively. The dimensions of the k-monopole moduli $\mathcal{M}_k$ are equal to $4k$. The moduli space $\mathcal{M}_k$ can be identified with the moduli space of a vacuum on the Coulomb branch of the three dimensional $SU(k)$ super Yang-Mills theory with eight supercharges [6]. The relative moduli space of the two-monopole $\widetilde{\mathcal{M}}_2^0$ is known as the Atiyah-Hitchin manifold [7] which is the ALF space with S^1 fibration over $\mathbb{R}^3$. In the case of well-separated BPS monopoles, each monopole carries three moduli of the position and a degree of the $U(1)$ phase modulus. The latter degree corresponds to the electric charge and hence we should include the electrical degree of the dyon. The effective dynamics of the k-dyon system can be described by a sigma model Lagrangian whose target space is the monopole moduli space. Hence the asymptotic metric of the moduli space of the BPS k-monopoles can be obtained by calculating the Lagrangian of interactions of k well-separated BPS monopoles (dyons). The metric is known as the Gibbons-Manton metric [8].

For a periodic BPS k-monopole on $\mathbb{R}^2 \times S^1$, which is called the monopole chain [9, 10, 11], the moduli space is identified with the moduli space of a vacuum on the Coulomb branch of the four dimensional $SU(k)$ super Yang-Mills theory compactified on S^1 with eight supercharges. The relative moduli space of the two-monopole $\widetilde{\mathcal{M}}_2^0$ is the ALG space [12]. Since the periodicity is achieved by a chain of monopoles, the total energy would diverge due to the infinite number of monopoles. However, the Nahm transform can be well-defined and the asymptotic metric of the moduli space of monopole chains is obtained in the same manner as the non-periodic case [13]. The geodesic motion is also discussed [13, 14, 15, 16].

For a doubly-periodic BPS k-monopole on $T^2 \times \mathbb{R}$, which is called the monopole sheet or wall [11, 17] (see also [18]), the moduli space is identified with the moduli space of a vacuum on the Coulomb branch of the five dimensional $SU(k)$ super Yang-Mills theory compactified on T^2 with eight supercharges [19]. One of the examples of the correspondence between the monopole moduli and the vacuum moduli of the five dimensional super Yang-Mills theory is that the number of the Dirac-type singularity corresponds to that of the matter flavor. Asymptotically the relative moduli space of the monopole walls is expected to be the ALH space with T^{3k-3} fibration over $\mathbb{R}^{k-1}$. As far as we know, there are no examples of ALH hyperkähler metrics in the literature except for the classical metric derived from the effective action of the $\mathcal{N} = 1$ super Yang-Mills theory on $\mathbb{R}^3 \times T^2$ by Haghighat and Vandoren [19]. Furthermore, the doubly-periodic monopoles have rich properties on the D-brane interpretation, string duality, and M-theoretic interpretation via the various S,T-duality transformations [20]. Therefore the analysis of the moduli metric would be applied to various situation of the corresponding super Yang-Mills theory, string theory and M-theory.

In this thesis, we derive asymptotic hyperkähler metrics on the moduli space of the monopole walls by calculating the effective sigma model Lagrangian of k well-separated BPS walls following Manton's observation [5]. In our calculation, the BPS wall is assumed to be a doubly-periodic superposition of BPS monopoles. In the non-periodic direction, the walls are assumed to be well-separated to each other compared with the thickness of the monopole wall so that the fields can be well-approximated by superpositions of linearized monopole walls. The metric computed in this thesis is for the case of two identical non-abelian monopole walls, including the Dirac singularities as well. We prove that the induced metrics actually have the modular invariance with respect to a complex structure τ of the complex torus T^2 in addition to the expected periodicity. We also present the metrics of monopole walls with Dirac-type singularities. We see that when we consider k monopole walls the maximum number of singularities is $2k$ by a simple analysis using the Newton polygon. This is consistent with the fact that in the $SU(k)$ super Yang-Mills theory the number of the matter flavor has the upper bound $2k$. This bound is due to the requirement that the super Yang-Mills theory is either conformal or asymptotically free. When the bound is saturated the theory has conformal invariance.

The present metrics would be the most explicit ones of the ALH type derived from the solutions of monopole walls including the case with the Dirac-type singularities. The symmetry and other properties are consistent with the corresponding super Yang-Mills theory [19].

The results are already published in [21].

- M. Hamanaka, H. Kanno and D. Muranaka,
  Hyper-Kähler metrics from monopole walls,
  Phys. Rev. D **89**, 065033 (2014). [arXiv:1311.7143 [hep-th]].

We discuss our main results in the last section of Chapter 4. The remaining sections of the chapter and other chapters are reviews for complete understanding of our results. We refer to some famous books, for example [22], [23] and [7]. Some graphs in this thesis are produced by [24].

This thesis is organized as follows.

In Chapter 2, we review BPS monopoles and the moduli spaces. Bogomolny-Prasad-Sommerfield (BPS) monopoles are topological solitons of $SU(2)$ Yang-Mills-Higgs theory in four-dimensional Minkowski space-time. These lead to non-compact hyperkähler manifolds as the moduli space. Contrary to Dirac monopoles in $U(1)$ gauge theory, $SU(2)$ monopoles are smooth. However, the quantization condition arises from topological characters of the theory. Moreover, each of them can be regarded as N point-like particles if the monopole belongs to a topological sector of topological charge N, at least the cores are well separated. In addition, it is known that the moduli space of $SU(2)$ monopoles has finite dimension $4N$ as long as the Higgs field is massless, which is called the BPS limit. We introduce the $SU(2)$ Yang-Mills-Higgs theory in Section 2.1 and then give an example of the monopoles in Section 2.2. Section 2.3 is devoted to explaining the BPS limit and the moduli space of the monopoles.

In Chapter 3, we review low energy dynamics of the monopoles and metrics on the moduli space of BPS monopoles. If the fields are time-dependent, or if the time component of the gauge field does not vanish, then an $SU(2)$ monopole acquires an electric charge and kinetic energy. Such monopoles are called dyons. The first example of $SU(2)$ dyons is the Julia-Zee dyon [31] obtained soon after the 't Hooft-Polyakov monopole [26]. The dyon is static and the time component of the gauge field is non-zero. The dynamics of BPS monopoles are treated carefully with an additional background gauge condition which constrains deformations of monopoles to be time-dependent gauge transformations and reveals phase moduli of BPS monopoles. The metrics of the moduli spaces of BPS monopoles are given by the kinetic term of the Lagrangian of the theory, and the motion of slowly moving monopoles or dyons which start to move with infinitesimal initial velocities corresponds to the geodesic motion on them. The metrics are hyperkähler as mentioned above. The asymptotic metric of the moduli space is called the Gibbons-Manton metric. We firstly explain the dyons and the dynamics of monopoles in Section 3.1 and 3.2, respectively, and then we review the Gibbons-Manton metric in Section 3.3.

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Chapter 2

BPS Monopoles

In this chapter we mainly review static BPS monopoles. In the first section we firstly introduce $SU(2)$ Yang-Mills-Higgs theory on Minkowski space-time and then explain how the smooth monopoles arise from the theory. The second section is devoted to explaining the 't Hooft-Polyakov monopole which is a typical solution of the field equations with spherical symmetry. In the third section we explain the BPS limit and then define the monopole moduli spaces. The dynamics of monopoles and the moduli space metrics are discussed in the next chapter.

2.1 $SU(2)$ Yang-Mills-Higgs Theory

Firstly, we shall introduce $SU(2)$ Yang-Mills-Higgs theory on four-dimensional Minkowski space-time. The theory is described by pairs of triplet scalar field ϕ^a $(a = 1, 2, 3)$ and triplet vector field A^a_μ $(\mu = 0, 1, 2, 3)$ which are called the Higgs field and the gauge field respectively. A local gauge transformation in the theory is defined by

$$\begin{aligned} \phi^a t_a &=: \phi :\mapsto g\phi g^{-1}\,, \\ A^a_\mu t_a &=: A_\mu :\mapsto gA_\mu g^{-1} + g\partial_\mu g^{-1}\,, \end{aligned} \tag{2.1}$$

where $g(x) \in SU(2)$ is a function defined on Minkowski space-time; the linear combinations ϕ and A_μ are $su(2)$-valued functions; $\{t_a\}$ is a basis of the algebra $su(2)$ with the normalization conditions $\mathrm{Tr}(t_a t_b) = -2\delta_{ab}$ and the commutation relations $[t_a, t_b] = -2\varepsilon_{abc}t_c$.

Here we use the adjoint representation of $SU(2)$; in particular, we take $t_a := \mathrm{i}\sigma_a$, where σ_a are the Pauli matrices. The covariant derivative of the Higgs field and the field strength are defined by, respectively,

$$D_\mu \phi^a := \partial_\mu \phi^a - 2\varepsilon_{abc} A^b_\mu \phi^c \,, \tag{2.2}$$

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - 2\varepsilon_{abc} A^b_\mu A^c_\nu \,. \tag{2.3}$$

Note that under the gauge transformations we have $D_\mu \phi \mapsto g(D_\mu \phi) g^{-1}$ and $F_{\mu\nu} \mapsto gF_{\mu\nu} g^{-1}$. Here the above definitions imply an identity:

$$[D_\mu, D_\nu]\phi^a = -2\varepsilon_{abc} F^b_{\mu\nu} \phi^c \,. \tag{2.4}$$

Namely, in terms of differential geometry, the field strength means curvature, and the gauge field corresponds to a connection. The identity is also denoted by $[D_\mu, D_\nu] = F_{\mu\nu}$. From Jacobi's identity and properties of the anti-symmetric tensors, one finds

$$\varepsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = \frac{1}{3}\varepsilon^{\mu\nu\rho\sigma}\Big([D_\nu,[D_\rho,D_\sigma]] + [D_\rho,[D_\sigma,D_\nu]] + [D_\sigma,[D_\nu,D_\rho]]\Big) = 0 \,,$$

that is, the following Bianchi identity with respect to the dual field strength $\widetilde{F}^{a\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$ holds in general:

$$D_\nu \widetilde{F}^{a\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} D_\nu F^a_{\rho\sigma} = 0 \,. \tag{2.5}$$

For convenience, we define the following $SU(2)$ electric and magnetic fields, respectively,

$$E^a_i := F^a_{i0} \,, \qquad B^a_i := \widetilde{F}^a_{i0} = \frac{1}{2}\varepsilon_{ijk} F^a_{jk} \,. \tag{2.6}$$

Now we introduce the Lorentz invariant Lagrangian density:

$$\mathcal{L} := -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{\lambda}{4}(1 - \phi^a \phi^a)^2 \,, \tag{2.7}$$

where the positive constant λ is a parameter of the model. Note that all the variables in the model are made to be dimensionless. Note also that the model has gauge invariance with respect to $SU(2)$ symmetry. By definition, the Lagrangian

$$L := \int \mathrm{d}^3 x \, \mathcal{L} \,, \tag{2.8}$$

where the integration is over $\mathbb{R}^3$, can be divided as $L = T - V$; the kinetic term is

$$T := \frac{1}{2}\int \mathrm{d}^3x\,(E_i^aE_i^a + D_0\phi^aD_0\phi^a)\,, \tag{2.9}$$

and the potential term is

$$V := \int \mathrm{d}^3x \left\{\frac{1}{4}F_{ij}^aF_{ij}^a + \frac{1}{2}D_i\phi^aD_i\phi^a + \frac{\lambda}{4}(1-\phi^a\phi^a)^2\right\}. \tag{2.10}$$

This implies that the field configurations of the classical vacuum, which minimizes V, should satisfy $F_{ij}^a = 0$, $D_i\phi^a = 0$ and $\phi^a\phi^a = 1$, where the non-zero vacuum expectation value spontaneously breaks the symmetry $SU(2)$ to $U(1)$. Because of $F_{ij}^a = 0$, the gauge field is a pure gauge $A_i^at_a = g\partial_ig^{-1}$, where $g(\boldsymbol{x})$ is an $SU(2)$-valued function on $\mathbb{R}^3$. By a gauge transformation, A_i^a can be made vanishing, and then $D_i\phi^a = \partial_i\phi^a = 0$. Moreover the Higgs field can be fixed to be the standard form $\phi^a = (0,0,1)$ by a certain global gauge transformation. In this situation, $g(\boldsymbol{x})$ is constrained to be $gt_3g^{-1} = t_3$, which implies that $g(\boldsymbol{x})$ is generated by t_3 and has only $U(1)$ symmetry.

The field equations turn out to be the following coupled system of the second order non-linear differential equations:

$$D_\mu D^\mu\phi^a = \lambda\phi^a(1-\phi^b\phi^b)\,, \tag{2.11}$$

$$D_\nu F^{a\mu\nu} = 2\varepsilon_{abc}D^\mu\phi^b\phi^c\,. \tag{2.12}$$

In order to make the energy, $E := T + V$, finite, the solutions must satisfy the conditions of the vacuum at infinity. We further restrict the Higgs field to be $\phi^a(0,0,+\infty) = (0,0,1)$ so that the framing is fixed to be $g(0,0,+\infty) = 1$. The spontaneous breaking of symmetry can again be seen from the equations. Let ϕ_0, W_μ^1, W_μ^2 and a_μ be infinitesimal fields, and put $\phi^a = (0,0,1+\phi_0)$ and $A_\mu^a = (W_\mu^1, W_\mu^2, a_\mu)$. Then, by substituting them into (2.11) and (2.12), one obtains the following wave equations:

$$\begin{aligned}
\partial_\mu\partial^\mu\phi_0 &= -2\lambda\phi_0\,,\\
\partial_\mu(\partial^\mu W^{1\nu} - \partial^\nu W^{1\mu}) &= -4W^{1\nu}\,,\\
\partial_\mu(\partial^\mu W^{2\nu} - \partial^\nu W^{2\mu}) &= -4W^{2\nu}\,,\\
\partial_\mu(\partial^\mu a^\nu - \partial^\nu a^\mu) &= 0\,.
\end{aligned}$$

Namely, the Higgs field obtains mass $\sqrt{2\lambda}$, two of the triplet gauge fields turn out to be weak bosons with mass 2, and the remainder falls into a massless photon.

The possibility of the monopole solutions in the theory can be checked as follows [25]. Let us consider a normalized Higgs field $\hat{\phi}^a := \phi^a/|\phi^a|$ defined on a certain region in $\mathbb{R}^3$ and supposed to be

$$D_\mu \hat{\phi}^a = 0\,. \tag{2.13}$$

Note that if the Higgs field is represented as $\phi^a =: h\hat{\phi}^a$, where h is a positive function, the covariant derivative of the Higgs field can be written as $D_\mu\phi^a = (\partial_\mu h)\hat{\phi}^a$, and one of the field equations is simplified to $\partial_\mu\partial^\mu h = \lambda h(1-h^2)$ with the boundary conditions $h^2 = 1$ and $\partial_\mu h = 0$; the situation always occurs at infinity. Then the gauge field can be obtained from (2.13) as, with an arbitrary smooth function $a_\mu = A^a_\mu\hat{\phi}^a$,

$$A^a_\mu = a_\mu\hat{\phi}^a + \frac{1}{2}\varepsilon_{abc}\hat{\phi}^b\partial_\mu\hat{\phi}^c\,, \tag{2.14}$$

and thus the projection of the field strength toward $\hat{\phi}^a$ can be written as

$$f_{\mu\nu} := F^a_{\mu\nu}\hat{\phi}^a = \partial_\mu a_\nu - \partial_\nu a_\mu + \frac{1}{2}\varepsilon_{abc}\hat{\phi}^a\partial_\mu\hat{\phi}^b\partial_\nu\hat{\phi}^c\,. \tag{2.15}$$

This corresponds to the electromagnetic field tensor in abelian gauge theory because it satisfies parts of $U(1)$ Maxwell's equations with no source, $\partial_\nu f^{\mu\nu} = 0$. Therefore the linearized magnetic field can be defined by $b_i := \frac{1}{2}\varepsilon_{ijk}f_{jk}$, where the surface integral at infinity is identical to the magnetic charge g, *i.e.*,

$$g := \int_{S^2_\infty} \mathrm{d}^2\sigma_i\, b_i = \frac{1}{4}\int_{S^2_\infty} \mathrm{d}^2\sigma_i\, \varepsilon_{ijk}\varepsilon_{abc}\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c\,, \tag{2.16}$$

where the integrations are over the 2-sphere at infinity, S^2_∞, with the area element $\mathrm{d}^2\sigma_i$. Here the region can be replaced with the target space S^2 of $\hat{\phi}^a$, and one finds

$$g = 2\pi N\,, \tag{2.17}$$

where $N \in \mathbb{Z}$ is the winding number of $\hat{\phi}^a$. In other words, continuous maps from S^2_∞ to S^2 belong to the homotopy group $\pi(S^2) = \mathbb{Z}$, and each of them can be classified with the integer N. Note that such maps can not be smoothly deformed to the other topological sectors without any singular transformation; the topological charge N is also called the monopole number. Some configurations of the Higgs field are shown in Figure 2.1. We can see from the configurations that the most fundamental but non-trivial solution is the case, $N = 1$, which obviously has spherical symmetry.

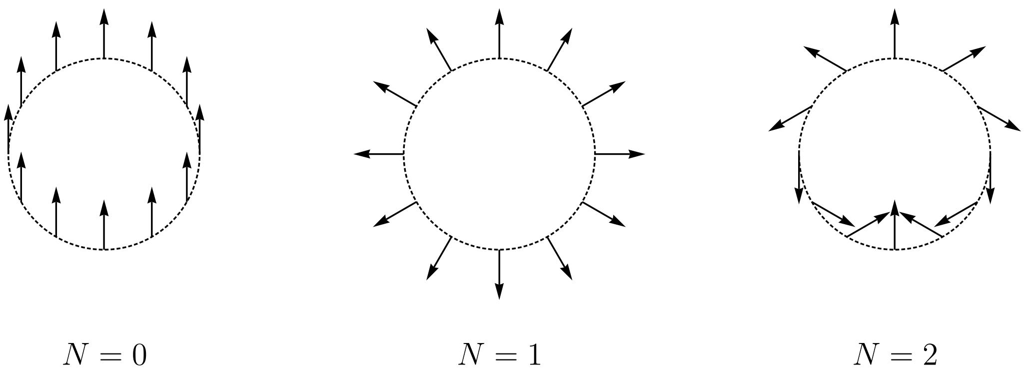

Figure 2.1: Conceptual diagrams of the Higgs field configurations ($N = 0, 1, 2$). In each case, the direction in the internal space of the triplet Higgs field rotates N times when we go around once on the 2-sphere at infinity. Note that all the arrows on the North Pole are directed to the top due to the boundary condition, $\phi^a(0, 0, +\infty) = (0, 0, 1)$.

2.2 The 't Hooft-Polyakov Monopole

The solution of the field equations in the case, $N = 1$, can be obtained as follows. Since the configuration of the Higgs field would be spherically symmetric, the asymptotic form of the Higgs field at infinity can be written as follows:

$$\phi^a = \frac{x^a}{r}\,, \tag{2.18}$$

where $r := (x^i x^i)^{1/2}$. Then, from the boundary conditions, one finds

$$a_i = A_i^a \phi^a = -\frac{1}{2}\varepsilon_{ija}\frac{x^j x^a}{r^3} \tag{2.19}$$

and

$$b_i = B_i^a \phi^a = \frac{x^i}{2r^3} = \frac{g}{4\pi r^2}\frac{x^i}{r}\,. \tag{2.20}$$

The forms imply that the $SU(2)$ monopole may look like a Dirac monopole sitting at the origin when we stay far from the origin; nevertheless the core is smooth. (On the analogy with the Dirac monopoles in $U(1)$ gauge theory, the explicit form of the Higgs field might have a zero at the origin because linearized Bianchi's identity $\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\partial_\nu f_{\alpha\beta} = 0$, which is described by $\hat{\phi}^a := \phi^a/|\phi^a|$ anywhere, would be ill-defined at the origin.)

Based on the asymptotic forms, one may suppose the following ansatz [26] [27]:

$$\phi^a = h(r)\frac{x^a}{r}\,, \qquad A_i^a = -\frac{1}{2}\varepsilon_{ija}(1-k(r))\frac{x^j}{r^2} \tag{2.21}$$

and $A_0^a = 0$, where $h(r)$ and $k(r)$ are unknown functions which should satisfy $h(0) = 0$, $k(0) = 1$, $h(\infty) = 1$, and $k(\infty) = 0$. Note that these fields are static, $\partial_0\phi^a = \partial_0 A_i^a = 0$, and the time component of the gauge field is zero, $A_0^a = 0$. Accordingly, $D_0\phi^a = E_i^a = 0$, and thereby the kinetic energy would vanish at all. In this situation, the field equations (2.11) and (2.12) are simplified to, respectively,

$$D_i D_i \phi^a = -\lambda\phi^a(1-\phi^b\phi^b)\,, \tag{2.22}$$

$$\varepsilon_{ijk} D_j B_k^a = -2\varepsilon_{abc} D_i\phi^b\phi^c\,, \tag{2.23}$$

and the identity (2.4) and the Bianchi identity (2.5) are equivarent to, respectively,

$$\varepsilon_{ijk} D_j D_k \phi^a = -2\varepsilon_{abc} B_i^b \phi^c\,, \tag{2.24}$$

$$D_i B_i^a = 0\,. \tag{2.25}$$

Substituting the ansatz into (2.22) and (2.23) and calculating a little, one obtains the following coupled pair of the second order non-linear differential equations:

$$\frac{\mathrm{d}^2 h}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{2}{r^2}hk^2 - \lambda(1-h^2)h\,, \tag{2.26}$$

$$\frac{\mathrm{d}^2 k}{\mathrm{d}r^2} = \frac{1}{r^2}(k^2-1)k + 4h^2 k\,. \tag{2.27}$$

This system indeed has a solution for each λ, however, it can not be in general obtained without numerical computation; the analytic solution can be found only in the case, $\lambda = 0$. The one-parameter family of solutions is called the 't Hooft-Polyakov monopole. We leave the detailed analysis of the numerical solutions for other books (see *e.g.*, [22]) and only review the case, $\lambda = 0$, in the next section.

2.3 BPS Monopoles and the Moduli Space

By comparing (2.23) and (2.24) in the previous section, one finds that one of the static field equations is automatically satisfied if the following condition holds:

$$B_i^a = D_i\phi^a \,. \tag{2.28}$$

In this case, from the Bianchi identity (2.25), the remaining static field equation (2.22) requires $-\lambda\phi^a(1-\phi^b\phi^b) = 0$, that is, $\lambda = 0$. The condition (2.28) is called the Bogomolny equation, and the case, $\lambda = 0$, is called the Bogomolny-Prasad-Sommerfield (BPS) limit [28]. The Bogomolny equation can also be obtained from the energy. The energy of the static monopoles in the case, $\lambda = 0$, can be written as

$$\begin{aligned} E &= \frac{1}{2}\int \mathrm{d}^3x\,(B_i^aB_i^a + D_i\phi^aD_i\phi^a) \\ &= \frac{1}{2}\int \mathrm{d}^3x\,(B_i^a - D_i\phi^a)^2 + \int \mathrm{d}^3x\,B_i^aD_i\phi^a\,, \end{aligned} \tag{2.29}$$

where the second term of the right-hand side of the second equality can be written in terms of the magnetic charge g, *i.e.*, for Stokes' theorem and the Bianchi identity,

$$g = \int_{S^2_\infty} \mathrm{d}^2\sigma_i\,B_i^a\phi^a = \int \mathrm{d}^3x\,D_i(B_i^a\phi^a) = \int \mathrm{d}^3x\,B_i^aD_i\phi^a\,. \tag{2.30}$$

Thus, one finds the following Bogomolny energy bound:

$$E \geq 2\pi N\,, \tag{2.31}$$

and the Bogomolny equation (2.28) turns out to be an energy minimizing condition for monopoles. (The necessity is not guaranteed.) Note that, in the BPS limit, the mass of the Higgs field vanishes at all, and the norm of the Higgs field at infinity should be tend to a harmonic function, as we have seen in the first section ($h = |\phi^a|$); not only the energy, $E = 2\pi N$, but also the energy density $\mathcal{E}(\boldsymbol{x})$ is determined by purely the configuration of the Higgs field:

$$\mathcal{E}(\boldsymbol{x}) = \frac{1}{2}\nabla^2|\phi|^2\,, \tag{2.32}$$

where $|\phi|^2 := -\frac{1}{2}\mathrm{Tr}(\phi^2) = \phi^a\phi^a$ is the matrix norm of the Higgs field.

In the BPS limit, the pair of the field equations (2.26) and (2.27) with regard to the 't Hooft-Polyakov ansatz is reduced to the following first order differential equations:

$$\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{1}{2r^2}(1-k^2)\,, \qquad \frac{\mathrm{d}k}{\mathrm{d}r} = -2hk\,. \tag{2.33}$$

This coupled system has the following Prasad-Sommerfield solution [29]:

$$h(r) = \coth 2r - \frac{1}{2r}\,, \qquad k(r) = \frac{2r}{\sinh 2r}\,. \tag{2.34}$$

It may be useful to see the properties of the solution. Firstly, from the Taylor expansion of hyperbolic functions, the solution indeed satisfies the boundary conditions at $r \to 0$. Secondary, the asymptotic form of the norm of the Higgs field at infinity is

$$|\phi| = h(r) = 1 - \frac{1}{2r} + O(\mathrm{e}^{-4r})\,, \tag{2.35}$$

which comes from

$$\coth x = \frac{1+\mathrm{e}^{-2x}}{1-\mathrm{e}^{-2x}}\,, \qquad \frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \cdots\,.$$

Thirdly, the energy density can be written in terms of $h(r)$ and $k(r)$ as follows:

$$\mathcal{E}(r) = \frac{1}{4r^4}(1-k^2)^2 + \frac{2}{r^2}h^2k^2\,, \tag{2.36}$$

which indeed has no singularities and is localized at the origin (Figure 2.2). In addition,

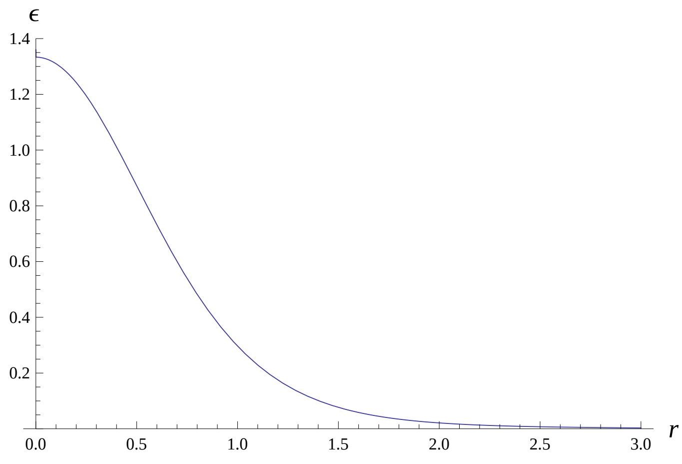

Figure 2.2: The energy density of the Prasad-Sommerfield monopole.

the asymptotic expansion of $\mathcal{E}(r)$ near the origin is

$$\mathcal{E}(r) = \frac{4}{3} - \frac{64}{27}r^2 + \frac{64}{25}r^4 + O(r^6)\,, \tag{2.37}$$

and the energy can also be calculated as follows:

$$E = 4\pi \int_0^\infty \mathrm{d}r\, r^2 \mathcal{E}(r) = 2\pi h(1-k^2)\Big|_0^\infty = 2\pi\,. \tag{2.38}$$

The Prasad-Sommerfield solution is the most basic example of the BPS monopoles, and we use some of the properties in the following chapters.

The moduli space of monopoles has remarkable properties. The N-monopole moduli space is defined as follows. Let $\mathcal{A}_N$ be the space of solutions of the Bogomolny equation (2.28). Here each element is a pair (A_i, ϕ) of the spatial components of the gauge field and the Higgs field which are smooth, satisfy the boundary conditions and are specified with the topological charge N. Let $\mathcal{G}$ also be the space of all non-singular local gauge transformations defined as (2.1). Then the N-monopole moduli space $\mathcal{M}_N$ is defined by the quotient:

$$\mathcal{M}_N := \mathcal{A}_N/\mathcal{G}\,. \tag{2.39}$$

$\mathcal{M}_N$ is a connected and complete Riemannian manifold of dimension $4N$ [30] (the framing parameter is added). In addition, all the moduli space metrics are hyperkähler. Namely, the Bogomolny equation (2.28) can be written in terms of hyperkähler moment maps [1]:

$$\mu_i(Q) := \frac{1}{2}\varepsilon_{ijk}[Q_j, Q_k] - [Q_0, Q_i]\,, \tag{2.40}$$

where $i,j,k = 1,2,3$ and $Q = Q_0 + IQ_1 + JQ_2 + KQ_3$ is a quaternionic skew-adjoint operator. For $D = -\phi + ID_1 + JD_2 + KD_3$, the moment map is a map from the infinite-dimensional quaternionic vector space to the space $\mathfrak{g}^* \otimes \mathbb{R}^3$, where $\mathfrak{g}^*$ is the dual space of $su(2)$-valued functions on $\mathbb{R}^3$; the Bogomolny equation (2.28) is equivarent to $\mu_i(D) = 0$ for all $i = 1,2,3$ due to $F_{ij} = [D_i, D_j]$ and $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$. Note that the moment map is preserved for the action of $\mathcal{G}$ because the Bogomolny equation is gauge invariant. Now the moduli space can be written as the following hyperkähler quotient:

$$\mathcal{M}_N = \bigcap_{i=1}^{3} \mu_i^{-1}(0)/\mathcal{G}\,. \tag{2.41}$$

Although the hyperkähler moment maps are infinite-dimensional, the quotient is a finite-dimensional hyperkähler manifold [32] [7]. The moduli space has the following metric decomposition:

$$\mathcal{M}_N = \mathbb{R}^3 \times \frac{S^1 \times \widetilde{\mathcal{M}}_N^0}{\mathbb{Z}_N}\,, \tag{2.42}$$

where $\mathbb{R}^3 \times S^1$ is flat and decouples from $\widetilde{\mathcal{M}}_N^0$, and $\widetilde{\mathcal{M}}_N^0$ is simply connected and admits an $SO(3)$ isometry group.

The moduli space metrics are well-studied. Especially, the explicit metric for $N = 2$, which can be obtained with some properties of the hyperkähler metrics, is known as the Atiyah-Hitchin metric [7] (see also [22]). On the other hand, the asymptotic metric of the moduli space for general N, which can be derived by considering the force between slowly moving BPS monopoles, is called the Gibbons-Manton metric [5] [8]. We review the dynamics of monopoles and show some moduli space metrics in the next chapter.

Chapter 3

Dynamics of BPS Monopoles

In this chapter we review low energy dynamics of $SU(2)$ monopoles and the metrics of the monopole moduli spaces. We also explain $SU(2)$ dyons which are related to moving monopoles under a time-dependent gauge transformation. In the first section we review the Julia-Zee dyon and the BPS limit of static dyons. The second section is devoted to explaining the monopole dynamics; in particular Manton's observation which indicates how the metrics are obtained from monopole solutions. In the third section we review the Gibbons-Manton metric which is the asymptotic metric of the moduli space and can explicitly be obtained by using formulation of abelian electromagnetism. The derivation in the last section can directly be applied to our calculation in the next chapter.

3.1 The Julia-Zee Dyon

From now on, we suppose that the fields are time-dependent, or the time component of the gauge field is non-vanishing (*i.e.*, either side of (3.7) remains). Then a monopole in general acquires an electric charge and kinetic energy; the electric charge is defined in the same way as the magnetic charge; the non-zero kinetic energy means that the right-hand side of (2.9) does not vanish. Such monopoles are generally called dyons. More precisely, a dyon is a particle or soliton with both magnetic and electric charges, which is not strictly static, although it is stationary in certain gauge. In this situation, it may be useful to write down the equations and the identities in terms of the $SU(2)$ electric and magnetic fields (2.6). The pair of the Bianchi identity (2.5) and a part of the field equation (2.12) is equivarent to the following $SU(2)$ Maxwell equations:

$$D_i B_i^a = 0\,, \tag{3.1}$$

$$D_0 B_i^a + \varepsilon_{ijk} D_j E_k^a = 0\,, \tag{3.2}$$

$$D_i E_i^a = 2\varepsilon_{abc} D_0 \phi^b \phi^c\,, \tag{3.3}$$

$$D_0 E_i^a - \varepsilon_{ijk} D_j B_k^a = 2\varepsilon_{abc} D_i \phi^b \phi^c\,, \tag{3.4}$$

and the identity (2.4) can also be rewritten as

$$D_0 D_i \phi^a - D_i D_0 \phi^a = 2\varepsilon_{abc} E_i^b \phi^c\,, \tag{3.5}$$

$$-\varepsilon_{ijk} D_j D_k \phi^a = 2\varepsilon_{abc} B_i^b \phi^c\,. \tag{3.6}$$

In particular, the Gauss law (3.3) can be expanded as, in terms of $su(2)$ matrices,

$$-D_i D_i A_0 - [\phi, [\phi, A_0]] = D_i \dot{A}_i + [\phi, \dot{\phi}]\,, \tag{3.7}$$

where the dots denote the time derivative. Note that we can always eliminate the time component of the gauge field by the Gauss law at the expense of time dependence.

The 't Hooft-Polyakov ansatz (2.21), which leads static, spherically symmetric solutions, can be extended to the following Julia-Zee ansatz [31]:

$$\phi^a = h(r)\frac{x^a}{r}\,, \qquad A_0^a = j(r)\frac{x^a}{r}\,, \qquad A_i^a = -\frac{1}{2}\varepsilon_{ija}(1-k(r))\frac{x^j}{r^2}\,, \tag{3.8}$$

where $h(r)$, $j(r)$, and $k(r)$ are unknown functions. Note that A_0^a has the same direction as ϕ^a in the internal space, which is called the Julia-Zee correspondence. In this case, the electric field is modified as $E_i^a = D_i A_0^a$, whereas $D_0\phi^a = 0$ again holds due to $\partial_0 \phi^a = \varepsilon_{abc} A_0^b \phi^c = 0$. Hence the solution would acquire the electric charge and non-zero kinetic energy; nevertheless the fields are static. The solution is called the Julia-Zee dyon. Since $D_0\phi^a = 0$, the field equations (2.11) and (2.12) are reduced to as follows:

$$D_i D_i \phi^a = -\lambda \phi^a (1 - \phi^b \phi^b)\,, \tag{3.9}$$

$$D_i E_i^a = 0\,, \tag{3.10}$$

$$D_0 E_i^a - \varepsilon_{ijk} D_j B_k^a = 2\varepsilon_{abc} D_i \phi^b \phi^c \,, \tag{3.11}$$

which can further be simplified by the ansatz to the following equations:

$$\frac{\mathrm{d}^2 h}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{2}{r^2} h k^2 - \lambda(1-h^2)h \,, \tag{3.12}$$

$$\frac{\mathrm{d}^2 j}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}j}{\mathrm{d}r} = \frac{2}{r^2} j k^2 \,, \tag{3.13}$$

$$\frac{\mathrm{d}^2 k}{\mathrm{d}r^2} = \frac{1}{r^2}(k^2-1)k + 4(h^2 - j^2)k \,. \tag{3.14}$$

Although the coupled system can again be solved numerically, the solutions would have an ambiguity; the boundary conditions of $j(r)$ should be $j(0) = 0$ and $j(\infty) = C$, where C is an arbitrary constant. This implies that the electric charge and the kinetic energy are not quantized in classical theory. Note also that the time component of the gauge field vanishes if we take the following time-dependent gauge:

$$g(t, \boldsymbol{x}) = \exp(tA_0(\boldsymbol{x})) \,. \tag{3.15}$$

Then, A_0^a becomes zero, the Higgs field does not change due to the Julia-Zee correspondence, and the spatial components of the gauge field would be still spherically symmetric but have complicated time-dependence. Furthermore, the gauge transformation changes the framing; the framing at t becomes

$$g(t, 0, 0, +\infty) = \exp(tCt_3) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}tC}, \mathrm{e}^{-\mathrm{i}tC}) \,, \tag{3.16}$$

which rotates with the period $2\pi/C$.

The BPS limit in the privious chapter again exists for dyons. We recall that the BPS limit is a condition where the field equations is automatically satisfied as the identities. In the case of dyons, such conditions can be obtained from both sides of (3.4) $\times\, C_2$ $-$ (3.2) $\times\, C_1$ = (3.6), where the constants C_1 and C_2 must be $C_1^2 + C_2^2 = 1$; we set $(C_1, C_2) = (\sin\alpha, \cos\alpha)$ with an arbitrary real constant α. Thus, one obtains the following pair of the Bogomolny equations for dyons:

$$E_i^a = D_i \phi^a \sin\alpha \,, \tag{3.17}$$

$$B_i^a = D_i \phi^a \cos\alpha \,. \tag{3.18}$$

In this situation, for a sequential equation, $E_i^a = B_i^a \tan\alpha$, and the Bianchi identity (3.1), the Gauss law (3.3) requires $2\varepsilon_{abc} D_0\phi^b\phi^c = 0$, however it is always satisfied if the fields are static and the Julia-Zee correspondence holds; the remaining field equation (2.11) again requests $\lambda = 0$. The Bogomolny equations can also be derived from the energy of dyons. If $D_0\phi^a = 0$ and $\lambda = 0$, then the energy can be written as

$$\begin{aligned} E &= \frac{1}{2}\int \mathrm{d}^3x\,(E_i^aE_i^a + B_i^aB_i^a + D_i\phi^aD_i\phi^a) \\ &= \frac{1}{2}\int \mathrm{d}^3x\,(E_i^a - D_i\phi^a\sin\alpha)^2 + \frac{1}{2}\int \mathrm{d}^3x\,(B_i^a - D_i\phi^a\cos\alpha)^2 \\ &\qquad + \sin\alpha\int \mathrm{d}^3x\,E_i^aD_i\phi^a + \cos\alpha\int \mathrm{d}^3x\,B_i^aD_i\phi^a\,, \end{aligned} \tag{3.19}$$

where the last two terms of the right-hand side of the second equality can be written in terms of the $SU(2)$ magnetic and electric charges, *i.e.*, respectively, (2.30) and

$$q = \int_{S^2_\infty} \mathrm{d}^2\sigma_i\,E_i^a\phi^a = \int \mathrm{d}^3x\,D_i(E_i^a\phi^a) = \int \mathrm{d}^3x\,E_i^aD_i\phi^a\,. \tag{3.20}$$

Thus, one finds the following Bogomolny energy bound for dyons:

$$E \geq q\sin\alpha + g\cos\alpha\,, \tag{3.21}$$

and the system of the Bogomolny equations (3.17) and (3.18) turns out to be an energy minimizing condition for dyons.

The Prasad-Sommerfield solution (2.34) can easily be extended for dyons by rescaling each variables, and the analytic solution can be written as follows:

$$h(r) = \coth(2r\cos\alpha) - \frac{1}{2r\cos\alpha}\,, \tag{3.22}$$

$$k(r) = \frac{2r\cos\alpha}{\sinh(2r\cos\alpha)}\,, \tag{3.23}$$

$$j(r) = \coth(2r\cos\alpha)\sin\alpha - \frac{\tan\alpha}{2r}\,. \tag{3.24}$$

Note that the electric charge is given by $q = g\tan\alpha$, while the magnetic charge is again $g = 2\pi$; the energy or rest mass of the dyon is $E = \sqrt{g^2+q^2}$, which follows from

$$\sin\alpha = \frac{|q|}{\sqrt{g^2+q^2}}\,, \qquad \cos\alpha = \frac{|g|}{\sqrt{g^2+q^2}}\,. \tag{3.25}$$

3.2 Dynamics and the Moduli Space Metrics

In order to make the time-dependent fields to be a solution of the Bogomolny equation, we shall introduce an additional condition which would reveal the remaining moduli of BPS monopoles. Let (A_i, ϕ) be a pair of gauge and Higgs fields which are smooth, decay appropriately at infinity, have topological charge N and satisfy the Bogomolny equation (2.28). Note that we do not have to consider A_0 because it obeys the other fields via the Gauss law (3.7). Then we consider a deformation $(A_i + a_i, \phi + \phi_0)$ with infinitesimal fields a_i and ϕ_0. In order to make the deformed fields to be a solution of the Bogomolny equation, the infinitesimal fields should satisfy the following equation:

$$\frac{1}{2}\varepsilon_{ijk}(D^A_j a_k - D^A_k a_j) = D^A_i \phi_0 + [a_i, \phi]\,, \tag{3.26}$$

where we keep the terms up to the linear order. Here the solutions of the equation can be written with an $su(2)$-valued function ω as $(a_i, \phi_0) = (-D^A_i \omega, [\omega, \phi])$, in general, that is, possible deformations must be infinitesimal gauge transformations of this type. Now we shall require the following orthogonality condition to remove non-physical changes of the fields:

$$\int \mathrm{d}^3x \left\{ -\mathrm{Tr}(a_i D^A_i \omega) + \mathrm{Tr}(\phi_0[\omega, \phi]) \right\} = 0\,. \tag{3.27}$$

Here the condition is equivarent to the following background gauge condition:

$$D^A_i a_i + [\phi, \phi_0] = 0\,. \tag{3.28}$$

Note that the condition has the same form as the right-hand side of the Gauss law (3.7); in particular, if we fix the time component of the gauge field to be vanishing, then the condition is satisfied with $(a_i, \phi_0) = (\dot{A}, \dot{\phi})$. Such deformations indeed exist, for example a time-dependent gauge transformation for a static monopole with the following gauge:

$$g(t, \boldsymbol{x}) = \exp(\theta\phi(\boldsymbol{x}))\,, \tag{3.29}$$

where θ is a time-dependent phase parameter, and whose change with time is supposed to be small. Under the gauge transformation with keeping $A_0 = 0$, the fields indeed satisfy $\dot{A}_i = -D_i(\dot{\theta}\phi)$ and $\dot{\phi} = [\dot{\theta}\phi, \phi] = 0$ for an approximation $g(t, \boldsymbol{x}) \simeq 1 + \dot{\theta}\phi\delta t$. In particular, the Higgs field does not change, and thus the magnetic charge is again $g = 2\pi$. Furthermore, from $A_0 = 0$ and the Bogomolny equation, one finds

$$E_i = -\dot{A}_i = \dot{\theta} D_i \phi = \dot{\theta} B_i \,. \tag{3.30}$$

Hence the monopole acquires the electric charge, $q = g\dot{\theta} = 2\pi\dot{\theta}$, and the kinetic energy, $T = (g/2)\dot{\theta}^2 = \pi\dot{\theta}^2$. Namely, in the BPS limit, there are not only position moduli but also phase moduli for each constituent monopole, where the changes of the phases with time generate electric charges and make the monopole to be a dyon. Such an argument may be valid if the cores are well separated, and each of them can be regarded as a dyon. Consequently, a well-separated BPS N-monopole or dyon has $N-1$ relative phase moduli and one total phase modulus, in which the total phase would be conserved, while the relative phases might be exchanged each other through collisions.

Now we consider the metric on the moduli space $\mathcal{M}_N$ defined in Section 2.3 (see [33], [7] and [34]). Since $\mathcal{M}_N$ is a hyperkähler manifold of dimension $4N$, we can use local coordinates $(q_1, \cdots, q_{4N})$ on $\mathcal{M}_N$, and then a point $q := \{q_\alpha\}$ $(\alpha = 1, \cdots, 4N)$ on $\mathcal{M}_N$ denotes a family of gauge equivarent solutions by $(A_i(\boldsymbol{x}; q), \phi(\boldsymbol{x}; q)) \in \mathcal{M}_N$, in which q_α correspond to the parameters of the solutions. Here the Lagrangian for a curve $q(t)$ on $\mathcal{M}_N$ can be written as a sigma model Lagrangian,

$$L(q, \dot{q}) = \frac{1}{2} g_{\alpha\beta}(q)\, \dot{q}_\alpha \dot{q}_\beta - U(q) \,, \tag{3.31}$$

where $g_{\alpha\beta}(q)$ is a Riemannian metric on $\mathcal{M}_N$, and $U(q)$ is a potential. We have seen in the previous chapter that the energy of BPS monopoles is minimized in each topological sector. Therefore, if the initial motion is tangent to $\mathcal{M}_N$, or equivarently the monopole stays static at the initial time and starts to move with small initial velocity, then the resulting motion remains close to $\mathcal{M}_N$, and the Lagrangian might be approximated by the Lagrangian of the slowly moving monopole, in which the forms of $g_{\alpha\beta}$ and U is given by the kinetic and potential terms of the Lagrangian, (2.9) and (2.10), respectively. In particular, if we fix A_0 to be $A_0 = 0$, we have $E_i^a = -\dot{A}_i^a$ and $D_0\phi^a = \dot{\phi}^a$, and the kinetic term might be approximated as the inner product of a tangent vector $(\dot{A}_i, \dot{\phi})$ up to the quadratic order if the fields satisfy the Bogomolny equation at initial time, and $\dot{q}_\alpha$ are sufficiently smal:

$$g_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta \simeq \int \mathrm{d}^3x\,(\dot{A}_i^a\dot{A}_i^a + \dot{\phi}^a\dot{\phi}^a)\,, \tag{3.32}$$

in which the potential for BPS monopoles can be written as

$$U(q) = \int \mathrm{d}^3x \left(\frac{1}{4}F_{ij}^aF_{ij}^a + \frac{1}{2}D_i\phi^aD_i\phi^a\right). \tag{3.33}$$

Although the integration over $\mathbb{R}^3$ of the right-hand side of (3.32) near the initial time would be difficult in general, it can be avoided by replacing the kinetic term with the one of N well separated BPS monopoles if the monopole can be regarded as a so-called well-separated BPS N-monopole that is a BPS N-monopole whose moduli space coordinates can be written as $q = \{\boldsymbol{x}_i, \theta_i\}$ $(i = 1, \cdots, N)$, where $\boldsymbol{x}_i := (x_i, y_i, z_i)$ are positions in $\mathbb{R}^3$, and θ_i are phases; every $|\boldsymbol{x}_i - \boldsymbol{x}_j|$ $(i \neq j)$ are sufficiently large. Namely, the Lagrangian of a BPS N-monopole whose cores look like point particles for other $N-1$ cores has only long-range interaction as abelian electromagnetism. Consequently, the metric on such asymptotic region of $\mathcal{M}_N$ is given by the Lagrangian of interaction of well-separated BPS monopoles or dyons with small velocities and phases (or electric charges).

3.3 The Gibbons-Manton Metric

As we have argued in the previous section, the metric on the asymptotic region of $\mathcal{M}_N$ is given by the Lagrangian of N slowly moviong dyons if the dyons are well separated and each initial velocity is suffitiently small. Here the force between static BPS monopoles or dyons is quite simple; for the existence of static solutions, the force would be cancelled, so that each well separated dyon feels not only the magnetic and electric Coulomb force but also an attractive force produced by the massless Higgs fields. The Lorentz boosted Lagrangian would have the position and electric charge of each dyon, where the electric charges can be replaced with the phases of dyons by using the Legendre transformation. The effective Lagrangian is the one on the asymptotic region of a $4N$-dimensional space which might be the N-monopole moduli space.

The force between well-separated BPS monopoles or dyons can directly be obtained from the field equations [25]. Let us consider a monopole moving in background fields of another monopole or anti-monopole (a monopole with opposite magnetic charge) at rest. If the separation between them is large enough, then the background magnetic and Higgs fields for the moving monopole can be approximated by the following magnetic Coulomb and long-range Higgs fields,

$$b_i \simeq \frac{1}{2r^2}\frac{x_i}{r}\,, \qquad |\phi| \simeq 1 - \frac{1}{2r}\,. \tag{3.34}$$

On the other hand, if the monopole starts to move at $t = 0$ with a constant acceleration $\boldsymbol{a}$ from the origin as a result of the existence of the background fields, then the fields of the monopole at time t can be written as the following Lorentz boosted fields:

$$\phi^a(t, \boldsymbol{x}) = \phi^a\left(\boldsymbol{x} - \frac{1}{2}\boldsymbol{a}t^2\right) \simeq \phi^a(\boldsymbol{x}) - \frac{1}{2}\boldsymbol{a}t^2 \cdot \boldsymbol{\nabla}\,\phi^a(\boldsymbol{x})\,, \tag{3.35}$$

$$A_i^a(t, \boldsymbol{x}) = A_i^a\left(\boldsymbol{x} - \frac{1}{2}\boldsymbol{a}t^2\right) \simeq A_i^a(\boldsymbol{x}) - \frac{1}{2}\boldsymbol{a}t^2 \cdot \boldsymbol{\nabla} A_i^a(\boldsymbol{x})\,, \tag{3.36}$$

$$A_0^a(t, \boldsymbol{x}) = \boldsymbol{a}t \cdot \boldsymbol{A}^a\left(\boldsymbol{x} - \frac{1}{2}\boldsymbol{a}t^2\right) \simeq \boldsymbol{a}t \cdot \boldsymbol{A}^a(\boldsymbol{x})\,, \tag{3.37}$$

where we use $\boldsymbol{a}t$ for the velocity of the moving monopole at time t, and suppose $\boldsymbol{a}t$ to be small compared with the speed of light and estimate the fields up to the order $O(|\boldsymbol{a}|^2)$. Then, the time dependence of the Higgs and gauge fields are, respectively,

$$\partial_0 \phi^a(t, \boldsymbol{x}) \simeq -a_j t \partial_j \phi^a(\boldsymbol{x})\,, \tag{3.38}$$

$$\partial_0 A_i^a(t, \boldsymbol{x}) \simeq -a_j t \partial_j A_i^a(\boldsymbol{x})\,, \tag{3.39}$$

which lead $D_0\phi^a = -a_j t D_j \phi^a$ and $E_i^a = -a_j t F_{ij}^a$, and thereby the static field equations are modified as follows:

$$D_i(D_i + a_i)\phi^a = -\lambda\phi^a(1 - \phi^b\phi^b)\,, \tag{3.40}$$

$$\varepsilon_{ijk}(D_j + a_j)B_k^a = -2\varepsilon_{abc} D_i \phi^b \phi^c\,. \tag{3.41}$$

These are also satisfied if $\lambda = 0$, and the following modified Bogomolny equation holds:

$$B_i^a = D_i\phi^a + a_i\phi^a\,, \tag{3.42}$$

and it can be solved for a_i by using the linearization, $B_i^a = b_i \hat{\phi}^a$ and $D_i \phi^a = \partial_i |\phi| \hat{\phi}$, as

$$a_i = b_i - \partial_i |\phi| \,. \tag{3.43}$$

By substituting the asymptotic background fields (3.34) into (3.43), one finds that the force between a point of monopole and anti-monopole is twice of the expected magnetic force, and there are no force between two well separated BPS monopoles. This is because the massless Higgs field has long-range scalar interaction which is always attractive and whose strength is the same as that of the magnetic force. Here the scalar charge of the interaction can be read off from the coefficient of the $1/r$ term of the asymptotic expansion of $|\phi|$. The result can easily be extended to the case of dyons. The force between two dyons with equal magnetic and electric charges vanishes as follows:

$$\frac{g^2}{4\pi s^2} + \frac{q^2}{4\pi s^2} - \frac{g^2 + q^2}{4\pi s^2} = 0 \,, \tag{3.44}$$

where s is the distance between two dyons and g, q and $(g^2 + q^2)^{1/2}$ are the magnetic, electric and scalar charges of each dyon, respectively.

Now we derive the asymptotic metrics of the moduli spaces of monopoles [5] [8]. Let us consider n well separated BPS dyons. Then the Lagrangian of the n-th dyon can be assumed to be as follows:

$$\begin{aligned} L_n = -(g^2 + q_n^2)^{1/2} \phi \, (1 - \boldsymbol{V}_n^2)^{1/2} + q_n \boldsymbol{V}_n \cdot \boldsymbol{A} - q_n A_0 \\ + g \boldsymbol{V}_n \cdot \widetilde{\boldsymbol{A}} - g \widetilde{A}_0 \,, \end{aligned} \tag{3.45}$$

where g, q_n, $(g^2 + q_n^2)^{1/2}$ and $\boldsymbol{V}_n$ are the magnetic, electric, scalar charges and velocity of the n-th dyon, respectively; ϕ and $(\boldsymbol{A}, A_0)$ are the asymptotic background Higgs and gauge fields, respectively, in which each field is simply a superposition of the remaining fields; $\widetilde{\boldsymbol{A}}$ and $\widetilde{A}_0$ are the dual vector and scalar potentials to $\boldsymbol{A}$ and A_0 defined through the dual electric and magnetic fields $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{B}}$, respectively, so that

$$\boldsymbol{\nabla} \times \widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{B}} = -\boldsymbol{E} = \boldsymbol{\nabla} A_0 + \dot{\boldsymbol{A}} \,, \tag{3.46}$$

$$-\boldsymbol{\nabla} \tilde{A}_0 - \dot{\widetilde{\boldsymbol{A}}} = \widetilde{\boldsymbol{E}} = \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \,. \tag{3.47}$$

Note that the first term of the Lagrangian denotes the long-range scalar interaction, the pair of the second and third terms corresponds to the ordinary Lorentz force in $U(1)$ gauge theory, and the pair of the fourth and fifth terms describes the dual Lorentz force with respect to the magnetic interaction. The explicit form for the Lagrangian of the n-th dyon in the presence of the first dyon can be obtained as follows. Firstly, the asymptotic Higgs field of a monopole in the BPS limit can be written as

$$\phi \simeq v - \frac{g}{4\pi r} + O(\mathrm{e}^{-8\pi v r/g}) \,, \tag{3.48}$$

where v is the vacuum expectation value. (We put $g = 2\pi$ for each dyon as the Julia-Zee dyon with charge $N = 1$ and recover the vacuum expectation value $v = 1$ from the non-dimensionalization of the theory.) Then the corresponding vector potential must satisfy the linearized Bogomolny equation $\boldsymbol{\nabla}\phi = \boldsymbol{\nabla} \times \boldsymbol{A}$; accordingly,

$$\boldsymbol{A} = -\frac{g}{4\pi}\boldsymbol{w} \,, \tag{3.49}$$

where $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{w}(-\boldsymbol{x})$ is a vector function defined so that $\boldsymbol{\nabla} \times \boldsymbol{w} = \boldsymbol{\nabla}(1/r)$. Note that $\boldsymbol{w}$ is the Dirac potential for the harmonic function $1/r$, and the explicit form of $\boldsymbol{w}$ is not concerned in the calculation. Hence the fields of the first dyon can be obtained by simply replacing the fields in the same manner as the Julia-Zee dyon:

$$\phi = v - \frac{(g^2 + q_1^2)^{1/2}}{4\pi r_{n1}} \,, \qquad \boldsymbol{A} = -\frac{g}{4\pi}\boldsymbol{w}_{n1} \,, \qquad A_0 = -\frac{q_1}{4\pi r_{n1}} \,, \tag{3.50}$$

where $\boldsymbol{r}_{ji} := \boldsymbol{x}_j - \boldsymbol{x}_i$, $r_{ji} := |\boldsymbol{r}_{ji}|$, and $\boldsymbol{w}_{ji} := \boldsymbol{w}(\boldsymbol{r}_{ji})$; $\boldsymbol{x}_i$ is the position of the i-th dyon. Furthermore, the dual potentials turn out to be

$$\widetilde{\boldsymbol{A}} = -\frac{q_1}{4\pi}\boldsymbol{w}_{n1} \,, \qquad \widetilde{A}_0 = \frac{g}{4\pi r_{n1}} \,. \tag{3.51}$$

The Higgs field of the moving first dyon can be derived by replacing the static field with the Liénard-Wiechert potential:

$$\phi = v - \frac{(g^2 + q_1^2)^{1/2}}{4\pi s_{n1}}(1 - \boldsymbol{V}_1^2)^{1/2} \simeq v - \frac{g}{4\pi r_{n1}}\left(1 + \frac{q_1^2}{2g^2} - \frac{\boldsymbol{V}_1^2}{2}\right) , \tag{3.52}$$

where $s_{n1} := (r_{n1}^2 - |\boldsymbol{r}_{n1} \times \boldsymbol{V}_1|^2 + O(\boldsymbol{V}_1^2))^{1/2}$, but it can be approximated by r_{n1} because the explicit form of the denominator plays no role in the calculation; we assume that the velocity and the electric charge (or the change of the phase with time) of each dyon are sufficiently small and can be approximated with keeping the terms quadratic in velocities and electric charges as expected in the observation in Section 3.2. The remaining fields can also be obtained by the Lorentz boost with the same approximation:

$$\begin{aligned} \boldsymbol{A} &= -\frac{q_1}{4\pi r_{n1}}\boldsymbol{V}_1 - \frac{g}{4\pi}\boldsymbol{w}_{n1}\,, \qquad & A_0 &= -\frac{q_1}{4\pi r_{n1}} - \frac{g}{4\pi}\boldsymbol{V}_1\cdot\boldsymbol{w}_{n1}\,, \\ \tilde{\boldsymbol{A}} &= \frac{g}{4\pi r_{n1}}\boldsymbol{V}_1 - \frac{q_1}{4\pi}\boldsymbol{w}_{n1}\,, \qquad & \tilde{A}_0 &= \frac{g}{4\pi r_{n1}} - \frac{q_1}{4\pi}\boldsymbol{V}_1\cdot\boldsymbol{w}_{n1}\,. \end{aligned} \tag{3.53}$$

Substituting them into the Lagrangian (3.45) and summarizing terms up to the order of q_n^2, $\boldsymbol{V}_n^2$, q_1^2, $\boldsymbol{V}_1^2$ and so on, one obtains

$$\begin{aligned} L_{n1} = -m_n + \frac{1}{2}m_n\boldsymbol{V}_n^2 + \frac{1}{8\pi r_{n1}}(q_n - q_1)^2 - \frac{g^2}{8\pi r_{n1}}(\boldsymbol{V}_n - \boldsymbol{V}_1)^2 \\ - \frac{g}{4\pi}(q_n - q_1)(\boldsymbol{V}_n - \boldsymbol{V}_1)\cdot\boldsymbol{w}_{n1}\,, \end{aligned} \tag{3.54}$$

where $m_n := v(g^2 + q_n^2)^{1/2}$ is the rest mass of the n-th dyon. Note that the sum of the first and second terms is ordinary kinetic energy, and all the remaining terms are also quadratic if we regard the electric charges as velocities.

It is straightforward to extend the above calculation for the first dyon to all the remaining dyons. The Lagrangian of the n-th dyon in the presence of the other $n-1$ dyons can be written as

$$\begin{aligned} L_n = \frac{1}{2}m_n\boldsymbol{V}_n^2 + \frac{1}{8\pi}\sum_{i=1}^{n-1}\frac{(q_n - q_i)^2}{r_{ni}} - \frac{g^2}{8\pi}\sum_{i=1}^{n-1}\frac{(\boldsymbol{V}_n - \boldsymbol{V}_i)^2}{r_{ni}} \\ - \frac{g}{4\pi}\sum_{i=1}^{n-1}(q_n - q_i)(\boldsymbol{V}_n - \boldsymbol{V}_i)\cdot\boldsymbol{w}_{ni}\,, \end{aligned} \tag{3.55}$$

where we omitted the constant, $-m_n$. Although the Lagrangian is not symmetric for the indices, the total Lagrangian L, which can be obtained by adding the ordinary kinetic terms of all the remaining dyons and the interactions from all the pairs of dyons which do not contain n-th dyon to L_n, can be symmetrized as

$$\begin{aligned} L = \sum_{i=1}^{n}\frac{1}{2}m\boldsymbol{V}_i^2 + \frac{1}{8\pi}\sum_{1\le i<j\le n}\frac{(q_j - q_i)^2}{r_{ji}} - \frac{g^2}{8\pi}\sum_{1\le i<j\le n}\frac{(\boldsymbol{V}_j - \boldsymbol{V}_i)^2}{r_{ji}} \\ - \frac{g}{4\pi}\sum_{1\le i<j\le n}(q_j - q_i)(\boldsymbol{V}_j - \boldsymbol{V}_i)\cdot\boldsymbol{w}_{ji}\,, \end{aligned} \tag{3.56}$$

where $m := vg$, and we use apploximations, $m_i \boldsymbol{V}_i^2 \simeq m\boldsymbol{V}_i^2$. Note that not only the terms after the second term but also the first term can be expressed by the center of mass or relative coordinates; the first term can be expanded as

$$\sum_{i=1}^{n} \frac{1}{2} m \boldsymbol{V}_i^2 = \frac{1}{2n} m \left(\sum_{i=1}^{n} \boldsymbol{V}_i \right)^2 + \sum_{1 \leq i < j \leq n} \frac{1}{2n} m (\boldsymbol{V}_j - \boldsymbol{V}_i)^2 . \tag{3.57}$$

As we have mentioned in Section 3.2, the electric charges q_i is related with the phase moduli θ_i of dyons, so the Lagrangian we have derived can be compaired with the sigma model Lagrangian on the asymptotic region of $\mathcal{M}_N$. The Lagrangian is finally written as the following form:

$$\mathcal{L} = \frac{1}{2} g_{ij} \boldsymbol{V}_i \cdot \boldsymbol{V}_j + \frac{1}{2} h_{ij} (\dot{\theta}_i + \boldsymbol{W}_{ik} \cdot \boldsymbol{V}_k)(\dot{\theta}_j + \boldsymbol{W}_{jl} \cdot \boldsymbol{V}_l) , \tag{3.58}$$

where g_{ij}, h_{ij} and $\boldsymbol{W}_{ij}$ depend only $\boldsymbol{x}^i$, and g_{ij} and h_{ij} are symmetric and invertible. This form is in fact the Gibbons-Hawking type [35]. For the purpose of comparing the Lagrangian (3.58) and (3.56), one considers the following effective Lagrangian:

$$\mathcal{L}_{\text{eff}} := \mathcal{L} - \frac{1}{\kappa} q_i \dot{\theta}_i , \tag{3.59}$$

where κ is a constant which should be fixed by a certain condition. From $\partial \mathcal{L}_{\text{eff}} / \partial \dot{\theta}_i = 0$, one finds that the effective Lagrangian has n conserved quantities which can be regarded as the electric charges q_i, that is,

$$q_i = \kappa h_{ij} (\dot{\theta}_j + \boldsymbol{W}_{jk} \cdot \boldsymbol{V}_k) , \qquad \dot{\theta}_i = \frac{1}{\kappa} h^{ij} q_j - \boldsymbol{W}_{ij} \cdot \boldsymbol{V}_j , \tag{3.60}$$

where h^{ij} is the inverse of h_{ij}. Hence the effective Lagrangian can be rewritten as

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} g_{ij} \boldsymbol{V}_i \cdot \boldsymbol{V}_j - \frac{1}{2\kappa^2} h^{ij} q_i q_j + \frac{1}{\kappa} q_i \boldsymbol{W}_{ij} \cdot \boldsymbol{V}_j . \tag{3.61}$$

By comparing this and the Lagrangian (3.56), one finds

$$\begin{aligned} g_{jj} &= m - \frac{g^2}{4\pi} \sum_{i \neq j} \frac{1}{r_{ij}} , \quad (\text{not summed over } j) , \qquad & g_{ij} &= \frac{g^2}{4\pi} \frac{1}{r_{ij}} , \quad (i \neq j) , \\ \boldsymbol{W}_{jj} &= \quad - \frac{g\kappa}{4\pi} \sum_{i \neq j} \boldsymbol{w}_{ij} , \quad (\text{not summed over } j) , \qquad & \boldsymbol{W}_{ij} &= \frac{g\kappa}{4\pi} \boldsymbol{w}_{ij} , \quad (i \neq j) . \end{aligned}$$

The remaining variables can be also fixed. It is sufficient to set $h^{ij} = (\kappa^2/g^2)g_{ij}$. One also puts $\kappa = 4\pi/g$ in order to avoid the singularities of the Dirac potential. Finally, the asymptotic metric of $\mathcal{M}_N$, the so-called Gibbons-Manton metric, can be written as

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}\boldsymbol{x}_i \cdot \mathrm{d}\boldsymbol{x}_j + g_{ij}^{-1}(\mathrm{d}\theta_i + \boldsymbol{W}_{ik} \cdot \mathrm{d}\boldsymbol{x}_k)(\mathrm{d}\theta_j + \boldsymbol{W}_{jl} \cdot \mathrm{d}\boldsymbol{x}_l)\,, \tag{3.62}$$

where

$$\begin{aligned} g_{jj} &= 2 - \sum_{i\neq j} \frac{1}{r_{ij}}\,, \quad (\text{not summed over } j)\,, \qquad g_{ij} = \frac{1}{r_{ij}}\,, \quad (i \neq j)\,, \\ \boldsymbol{W}_{jj} &= \quad - \sum_{i\neq j} \boldsymbol{w}_{ij}\,, \quad (\text{not summed over } j)\,, \qquad \boldsymbol{W}_{ij} = \boldsymbol{w}_{ij}\,, \quad (i \neq j)\,, \end{aligned}$$

with the conditions that the metric is hyperkähler,

$$\frac{\partial}{\partial x_{ai}} W_{bjk} - \frac{\partial}{\partial x_{bj}} W_{aik} = \varepsilon_{abc} \frac{\partial}{\partial x_{ci}} g_{jk}\,, \qquad \frac{\partial}{\partial x_{ai}} g_{jk} = \frac{\partial}{\partial x_{aj}} g_{ik}\,, \tag{3.63}$$

where $a, b, c = 1, 2, 3$ are space indices. In other words, the set of the above g_{ij} and $\boldsymbol{W}_{ij}$ is a simple but non-trivial solution of the system of the equations (3.63).

The Gibbons-Manton metric is derived by using purely the asymptotic behavior of BPS monopoles, and the procedure would directly be used for the monopoles with periodicities. The asymptotic metrics of the moduli spaces of periodic monopoles are obtained [12], which also include the case that the monopoles have Dirac-type simgularities. In the next chapter we derive the asymptotic metrics of the moduli spaces of doubly-periodic monopoles following this method.

Chapter 4

Monopole Walls and the Metrics

In this chapter we firstly review doubly-periodic monopoles or monopole walls, and we then derive hyperkähler metrics with doubly-periodicity by using Manton's observation reviewed so far. We would like to derive metrics of not only $SU(2)$ monopole walls but also the ones with Dirac-type singularities. Such monopole walls can be treated by using the spectral analysis. In the first section we define monopole walls with some simple examples. In the second section we review the spectral analysis. The third section is devoted to explaining our main results which also include the derivation of the maximum number of Dirac singularities.

4.1 Monopole walls

First of all, we define $U(n)$ monopole walls [20]. Let $x^\alpha := (x, y, z)$ ($\alpha = 1, 2, 3$) denote the coordinates of the three dimensional space $T^2 \times \mathbb{R}$, where x and y are periodic with period one, *i.e.*, $x \sim x + 1$ and $y \sim y + 1$. The Higgs field ϕ and the gauge field A satisfy the Bogomolny equation:

$$*D_A\phi = -F\,, \tag{4.1}$$

where $D_A\phi := \mathrm{d}\phi + [A, \phi]$ and $F := \mathrm{d}A + A \wedge A$. We assume that the gauge group is generally $U(n)$ in which ϕ is an $n \times n$ anti-Hermitian matrix and A denotes a one-form. We frequently express the gauge field by $A = A_x\mathrm{d}x + A_y\mathrm{d}y + A_z\mathrm{d}z$. We treat the fields as real-valued functions in the case of $U(1)$. The energy density is defined by

$$\mathcal{E} := -\frac{1}{2}\mathrm{Tr}(|D_A\phi|^2 + B^2) = \nabla^2|\phi| \,, \tag{4.2}$$

where $|\phi| := -\frac{1}{2}\mathrm{Tr}\,\phi^2$ is the matrix norm of the Higgs field and $B_i := \frac{1}{2}\varepsilon_{ijk}F_{jk}$.

In this situation the simplest $U(1)$ wall is the following constant-energy solution:

$$\phi = 2\pi(Qz + M)\,, \qquad A = 2\pi(Qy\,\mathrm{d}x - p\,\mathrm{d}x - q\,\mathrm{d}y)\,, \tag{4.3}$$

where Q, M and $p, q \in [0, 1)$ are real constants. Here the solution is not doubly-periodic unless we perform appropriate gauge transformations because the gauge field explicitly depends on the periodic coordinates x and y. By contrast, the Higgs field does not depend on any periodic coordinates. This solution has constant energy density $\mathcal{E} = 8\pi^2Q^2$, which is the origin of the name.

Based on the above constant-energy solution, one supposes the asymptotic behavior of $U(n)$ monopole walls to be as follows:

$$\mathrm{EigVal}\,\phi = \left\{2\pi\mathrm{i}(Q_{\pm,\ell}z + M_{\pm,\ell}) + o(1/z) \,|\, \ell = 1, \cdots, n\right\}, \tag{4.4}$$

where $Q_{\pm,\ell}$ and $M_{\pm,\ell}$ are real constants. Here $Q_{\pm,\ell}$, which are called the monopole-wall charges, are indeed rational numbers [20]; if there are $f_\pm$ distinct monopole-wall charges $Q_{\pm j}$ ($j = 1, \cdots, f_\pm$) as $z \to \pm\infty$, the Charn number of the line bundle $E_{\pm j}$ defined at large $|z|$ obey

$$\int_{T_z^2} c_1(E_{\pm j}) = \frac{\mathrm{i}}{2\pi}\int_{T_z^2}\mathrm{Tr}(F_{\pm j}) = -\frac{\mathrm{i}}{2\pi}\int_{T_z^2}\mathrm{Tr}(*D_A\phi_{\pm j}) = r_{\pm j}Q_{\pm j}\,, \tag{4.5}$$

where T_z^2 is the complex torus at z and $r_{\pm j}$ are the multiplicities of $Q_{\pm j}$.

In addition to smooth solutions having the above asymptotic behavior, we consider solutions such as the following $U(1)$ monopole wall with a Dirac-type singularity [20]:

$$\phi = \phi_0 - \frac{1}{2r} - \frac{1}{2}\sum_{j,k\in Z}\left(\frac{1}{r_{jk}} - \frac{1}{e_{jk}}\right) + \pi z\,, \tag{4.6}$$

$$A_+ = \frac{1}{2}\sum_{j,k\in Z}\frac{(y-k)\,\mathrm{d}x - (x-j)\,\mathrm{d}y}{r_{jk}(z + r_{jk})} + \frac{\pi}{2}(3y\,\mathrm{d}x + x\,\mathrm{d}y)\,, \quad (z \geq 0)\,, \tag{4.7}$$

$$A_- = \frac{1}{2}\sum_{j,k\in Z}\frac{(y-k)\,\mathrm{d}x - (x-j)\,\mathrm{d}y}{r_{jk}(z - r_{jk})} + \frac{\pi}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y)\,, \quad (z < 0)\,, \tag{4.8}$$

where the constant ϕ_0 is taken so that the summation would converge; the summations do not include the case, $j = k = 0$; $\boldsymbol{r} := (x, y, z)$, $r := |\boldsymbol{r}|$, $\boldsymbol{e}_{jk} := (j, k, 0)$, $e_{jk} := |\boldsymbol{e}_{jk}|$, and $r_{jk} := |\boldsymbol{r} - \boldsymbol{e}_{jk}|$. Note that the additional constant-energy part shifts the original charges $(Q_-, Q_+) = (-\frac{1}{2}, \frac{1}{2})$, which come from doubly-periodic Green's function [36], to the standard 1-monopole wall charges $(Q_-, Q_+) = (0, 1)$. Such solutions with Dirac-type singularities are called the Dirac monopole walls.

We have shown only some simple examples of $U(1)$ walls because the solutions of the Bogomolny equation with periodicities are more complicated than the non-periodic case and are therefore studied with some specific methods, for example perturbations for the study of an $SU(2)$ monopole wall with four-moduli [11] and numerical computations to obtain a solution of the $SU(2)$ Bogomolny equation without any moduli [17]. However, some crucial features of monopole walls are recently revealed. In particular, the explicit boundary conditions and the dimensions of the moduli spaces are obtaind by using the spectral analysis [20]. The method is sufficient for our computations because Manton's observation needs only the asymptotic behavior of the fields. For this reason, we rely on other papers for details of solutions and review only the spectral analysis in the next section.

4.2 Spectral Analysis

Let us consider the following pair of complex and real equations which can be obtained from the Bogomolny equation (4.1) in the privious section:

$$\begin{cases} [D_z - \mathrm{i}D_y, D_x + \mathrm{i}\phi] = 0\,, \\ [D_z - \mathrm{i}D_y, (D_z - \mathrm{i}D_y)^\dagger] + [D_x + \mathrm{i}\phi, (D_x + \mathrm{i}\phi)^\dagger] = 0\,. \end{cases} \tag{4.9}$$

Here the first equation implies that the characteristic polynomial $F_x := \det[V_x(y, z) - t]$ is a holomorphic function of the complex variable $s := \exp[2\pi(z - \mathrm{i}y)]$, where V_x is the holonomy of $D_x + \mathrm{i}\phi$ around the x-direction. Based on these definitions, we define the x-spectral curve Σ_x by

$$\Sigma_x := \left\{(s, t) \in \mathbb{C}^* \times \mathbb{C}^* \,|\, F_x(s, t) = 0\right\}. \tag{4.10}$$

For later convenience, we also define the x-spectral polynomial $G_x(s,t) := P(s)F_x(s,t)$ with a common denominator $P(s)$ of F_x so as to remove all the negative exponents of each term of $F_x(s,t)$. The y-spectral curve Σ_y is defined in the same way as Σ_x by using $F_y(\tilde{s},\tilde{t}) := \det[V_y(\tilde{s}) - \tilde{t}]$, where $V_y(\tilde{s})$ is the holonomy of $D_y + \mathrm{i}\phi$ around the y-direction. Note that the variables $\tilde{s} := \exp[2\pi(z + ix)]$ and $\tilde{t}$ differ from s and t, respectively. For example, the holonomy with respect to the $U(1)$ constant-energy solution (4.3) around the x-direction is naively

$$\begin{aligned} V_x &= \exp\left[-\mathrm{i}\int_0^1 (A_x + \mathrm{i}\phi)\,\mathrm{d}x\right] \\ &= \exp[2\pi Q(z - \mathrm{i}y) + 2\pi(M + \mathrm{i}p)] = s^Q \mathrm{e}^{2\pi(M+\mathrm{i}p)} \end{aligned} \tag{4.11}$$

which leads a spectral curve, $s = t$, if $Q = 1$ and $M = p = 0$. Note that the holonomy around the y-direction is $V_y = \tilde{s}^Q \mathrm{e}^{2\pi(M+\mathrm{i}q)}$ in some gauge.

The Newton polygon, which is generally derived from a Laurent polynomial, is useful for visualizing the asymptotic behavior of the spectral curve. The Newton polygon $\mathcal{N}_x$ of the x-spectral curve is defined as follows. Firstly, we mark points (a,b) which correspond to the degree of each term $s^a t^b$ of $G_x(s,t)$ on the integer lattice. Then, $\mathcal{N}_x$ is a minimal convex polygon which includes all the marks. In addition, there is another diagram that shows the asymptotic behavior of the spectral curve and relates to the Newton polygon. Namely, the amoeba $\mathcal{A}_x$ of an x-spectral curve $\Sigma_x = \{(s,t) \in \mathbb{C}^* \times \mathbb{C}^* \mid F_x(s,t) = 0\}$ is defined as the image of the logarithmic map:

$$\mathcal{A}_x := \left\{(\log|s|, \log|t|) \in \mathbb{R}^2 \mid F_x(s,t) = 0\right\}. \tag{4.12}$$

The amoeba is in general a connected domain with tentacle-like asymptotes and holes. The relation between the Newton polygon and the amoeba is as follows.

- The number of tentacles is equal to the number of subedges (parts of edges divided by the points) where each asymptote is orthogonal to the corresponding subedge.
- The number of holes of the amoeba is bounded by the number of internal points in the Newton polygon.

- The area Area($\mathcal{N}$) of the Newton polygon $\mathcal{N}$ and the area Area($\mathcal{A}$) of the amoeba $\mathcal{A}$ satisfy an inequality, $\mathrm{Area}(\mathcal{A}) \leq \pi^2 \mathrm{Area}(\mathcal{N})$.

Technically speaking, the amoeba of the x-spectral curve can be visualized as follows. Let $s =: \exp(u + \mathrm{i}\theta)$ so that $(\log|s|, \log|t|) = (u, \log|t(u,\theta)|)$, where (u,θ) is a pair of real parameters; $t(u,\theta)$ is a solution of $F_x(s,t) = 0$ with respect to t. Then the parametric plot of $(u, \log|t(u,\theta)|)$ where θ runs from zero to 2π is a part of the graph of the amoeba. Accordingly, the total graph would be classified by n-color if the gauge group is $U(n)$. The graph can easily be drawn by Mathematica [24].

The characteristic polynomial $F_x(s,t) = \det[V_x(s) - t]$ with respect to a smooth $SU(2)$ monopole wall can be written as follows:

$$F_x(s,t) = t^2 - W_x(s)t + 1\,, \tag{4.13}$$

where $W_x(s) := \mathrm{Tr}\, V_x(s)$, and note that $\det V_x(s) = 1$. Therefore the Newton polygon of an $SU(2)$ monopole wall should be a rhombus with some internal points. For example, the following spectral curves, with a real parameter a, describe a one-parameter family of $SU(2)$ monopole walls with monopole-wall charges $(Q_-, Q_+) = (1,1)$:

$$st^2 - s^2 t - t + s + ast = 0\,. \tag{4.14}$$

The Newton polygon and the amoeba of the spectral curves are shown in ref. [20].

For the spectral curve of the constant-energy solution, the complex variables (s,t) of x-spectral curves asymptotically satisfy $t \sim s^Q \mathrm{e}^{2\pi(M+\mathrm{i}p)}$, that is,

- If $t \to \infty$ or $t \to 0$ while $s \to s_0$, with a constant s_0, the monopole wall has a Dirac-type singularity whose coordinates can be seen from $z - \mathrm{i}y = \frac{1}{2\pi}\log(s_0)$ because the eigenvalue of the Higgs field can not be determined at the point.

- If $s \to \infty$ or $s \to 0$ while $t \to t_0$, with a constant t_0, the real and imaginary parts of $\log(t_0)$ are asymptotic values of eigenvalues of the Higgs field and the holonomy, respectively, because the limits correspond to $z \to \pm\infty$.

- If $s \to \infty$ or $s \to 0$ while $t \sim s^{\alpha/\beta}$, with some relatively prime integers α and positive β, the asymptotic behavior of the eigenvalue of the Higgs field is $t \sim s^{\alpha/\beta}$, and the corresponding charge is $Q = \alpha/\beta$.

The asymptotic behavior of the spectral curve can be read off from the Newton polygon. Moreover, the Newton polygon tells us some crucial features of the monopole wall [20]. The properties can be summarized as follows.

1. The height of the Newton polygon is n if the gauge group is $U(n)$.
2. The number of points on the top or bottom edges of the Newton polygon is equal to $r_{\pm 0}+1$, where $r_{\pm 0}$ is the number of positive or negative Dirac singularities.
3. If an edge of the Newton polygon has a finite tangent $\beta_{\pm j}/\alpha_{\pm j}$ and $r_{\pm j}+1$ points, the monopole wall has charges $Q_{\pm j}=\alpha_{\pm j}/\beta_{\pm j}$ with multiplicities $r_{\pm j}$.
4. The Newton polygon $\mathcal{N}_x$ and $\mathcal{N}_y$ of x- and y-spectral curves, respectively, coincide if the curves come from the same monopole wall.
5. The dimension of the moduli space $\mathcal{M}$ of a monopole wall is given by the number of internal points $\mathrm{Int}\,\mathcal{N}_x$ of the Newton polygon as follows:
$$\dim\mathcal{M}=4\,\mathrm{Int}\,\mathcal{N}_x\,. \tag{4.15}$$
6. The Nahm transform for monopole walls is the operation which exchanges s and t or turns the Newton polygon with respect to the diagonal line.

One can see from the first, second and third properties that the shape of the Newton polygon is strictly restricted by the boundary data. In fact, the boundary conditions of monopole walls can explicitly be written in terms of the monopole-wall charges, asymptotic values of eigenvalues of the Higgs field and the holonomy, and the position of Dirac singularities. On the other hand, the inner points are free from the asymptotic behavior, and hence the dimension of the moduli space relates to the number of these points. The last property implies the necessity of Dirac-type singularities. For example, the Nahm transform for a $U(1)$ Dirac monopole wall with two negative poles leads a smooth $SU(2)$ wall with charges $(Q_-,Q_+)=(0,1)$ [20]. We use the above properties for the derivation of the maximum number of Dirac singularities with respect to $U(2)$ monopole walls with four-moduli in the next section.

4.3 Hyperkähler Metrics from Monopole Walls

Now we derive asymptotic metrics of well-separated monopole walls. In addition to the definition of the monopole walls in the first section, we introduce the metric on $T^2 \times \mathbb{R}$ as follows. Let $\tau := \tau_1 + \mathrm{i}\tau_2$ ($\tau_1, \tau_2 \in \mathbb{R}$) be a standard complex structure on the torus T^2, and then we introduce a holomorphic coordinate $\xi := x + \tau y$. The periodicity is now represented by $\xi \sim \xi + m + \tau n$ ($m, n \in \mathbb{Z}$). By using the vector notation $\boldsymbol{x} := (\xi, z)$, the metric on $T^2 \times \mathbb{R}$ is represented as follows:

$$\begin{aligned}
\mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} &:= \frac{\nu}{\tau_2}(\mathrm{d}x^2 + 2\tau_1 \mathrm{d}x\mathrm{d}y + |\tau|^2 \mathrm{d}y^2) + \mathrm{d}z^2 \\
&= \frac{\nu}{\tau_2}|\mathrm{d}\xi|^2 + \mathrm{d}z^2 =: g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta ,
\end{aligned} \tag{4.16}$$

where the volume of the torus is denoted by $\nu := \sqrt{\det g}$ ($g := (g_{\alpha\beta})$). Note that two dimensional metric has three independent components, and we have traded them with τ_1, τ_2 and ν. One of the crucial features of our construction of doubly-periodic hyperkähler metrics in the following is the invariance under the modular transformation:

$$\xi \mapsto \frac{\xi}{c\tau + d}\,, \qquad \tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad \tau_2 \mapsto \frac{\tau_2}{|c\tau + d|^2}\,, \tag{4.17}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$.

We would like to derive hyperkähler metrics according to Manton's observation, so we firstly find the asymptotic background fields produced by $SU(2)$ monopole walls. Let us consider k well-separated $SU(2)$ monopole walls sitting at the points $\boldsymbol{a}_j := (\xi_j, z_j)$ ($j = 1, \cdots, k$) in which the monopole-wall charges of the j-th wall are supposed to be $(Q_{-j}, Q_{+j}) = (0, 1)$ or $(1, 0)$ so that $k = \sum_j (|Q_{-j}| + |Q_{+j}|)$. (Note that at least the case $(Q_{-j}, Q_{+j}) = (0, 1)$ can be regarded as a smooth $SU(2)$ monopole arranged per unit cell [17]. Note also that it is not clear that multi-monopole walls indeed have the moduli of separation, however, at least the case of $(Q_{-j}, Q_{+j}) = (1, 1)$ has four-moduli [11].) If the separations $|z_j - z_i|$ are large enough compared with the thicknesses of each wall, the fields are well-approximated by superpositions of linearized monopole walls:

$$\phi(\boldsymbol{x}) = v + \sum_{j=1}^{k} \phi^j(\boldsymbol{x} - \boldsymbol{a}_j)\,, \tag{4.18}$$

$$A_\xi(\boldsymbol{x}) = b + \sum_{j=1}^{k} A_\xi^j(\boldsymbol{x} - \boldsymbol{a}_j)\,, \qquad A_z(\boldsymbol{x}) = 0\,, \tag{4.19}$$

where v is the vacuum expectation value of the Higgs field and b is the background gauge field. Furthermore, the asymptotic Higgs field of each monopole wall can be estimated as a superposition of linearized 't Hooft-Polyakov monopoles arranged in a finite $(2M+1)\times(2N+1)$ rhombic lattice:

$$\phi^j(\boldsymbol{x}) = \frac{1}{4\pi}\sum_{m=-M}^{M}\sum_{n=-N}^{N}\frac{-g}{\sqrt{|\xi - m - n\tau|^2 + z^2}}\,, \tag{4.20}$$

where g is the magnetic charge of the 't Hooft-Polyakov monopole. Here, the asymptotic form of the summation for large $|z|$ can be seen from the analysis of the doubly-periodic Green function [36], and we find

$$\phi^j(\boldsymbol{x}) = \frac{g}{2}|z| - gC_{M,N}\,, \tag{4.21}$$

where $C_{M,N}$ is a positive constant diverging linearly in the limit $M, N \to \infty$. Although the summation would diverge, the part of the effective Lagrangian can be defined in the same manner as periodic monopoles [12]. Namely, we set

$$\phi(\boldsymbol{x}) = v_{\mathrm{ren}} + \frac{g}{2}\sum_{j=1}^{k}|z - z_j| \tag{4.22}$$

and keep $v_{\mathrm{ren}} := v - kgC_{M,N}$ finite with v diverging at the same order as $C_{M,N}$. We note that the configuration is not localized in the periodic directions. This implies that the superposition of doubly-periodic monopoles is represented as a constituent monopole wall in the asymptotic region.

The asymptotic gauge field should satisfy the Bogomolny equation, which leads

$$A_\xi^j(\boldsymbol{x}) = \frac{\mathrm{i}\nu g}{8\tau_2}\,\mathrm{sign}(z)\,\bar{\xi}\,, \qquad A_z^j(\boldsymbol{x}) = 0\,, \tag{4.23}$$

where $\mathrm{sign}(z)$ denotes the sign of z, that is, $\mathrm{sign}(z) = 1$ when z is positive and otherwise $\mathrm{sign}(z) = -1$; we employ the following convention of the Hodge star operator:

$$*(\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}) = \frac{\sqrt{|g|}}{(n-p)!}\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_n}\mathrm{d}x^{\nu_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{\nu_n}\,. \tag{4.24}$$

Note that, as seen in the first section, appropriate gauge transformations should be supposed so as to make the gauge field to be doubly-periodic for $\xi \sim \xi + m + \tau n$. This is essential for our calculation because the phase θ of a dyonic monopole wall appears in a zeromode ψ of the Dirac equation used for the Nahm transform [17] in which the $U(1)$ gauge transformations change the fields as $A \mapsto A + \mathrm{d}\theta$ and $\psi \mapsto \psi \mathrm{e}^{\mathrm{i}\theta}$. Namely, our $U(1)$ bundle over the complex torus is non-trivial, and accordingly we have to impose the following twisted boundary condition where the phase of any functions in the fundamental representation of the gauge group shifts as follows:

$$\theta \mapsto \theta + \frac{\nu g}{4} \operatorname{sign}(z)\, y \quad \text{when} \quad \xi \mapsto \xi + 1\,, \tag{4.25}$$

$$\theta \mapsto \theta - \frac{\nu g}{4} \operatorname{sign}(z)\, x \quad \text{when} \quad \xi \mapsto \xi + \tau\,. \tag{4.26}$$

For later convenience, we introduce the following functions:

$$u(z) = \frac{1}{2}|z| - C_{M,N}\,, \qquad w(\boldsymbol{x}) = \frac{\mathrm{i}\nu}{8\tau_2} \operatorname{sign}(z)\, \bar{\xi}\,, \tag{4.27}$$

which satisfy $u(z) = u(-z)$ and $w(\boldsymbol{x}) = w(-\boldsymbol{x})$. Note that $u(z)$ is a harmonic function on $\mathbb{R}$ with δ-function source at the origin.

As seen in the third chapter, the asymptotic metric of the moduli space of monopoles can be obtained by calculating the long-range interactions of dyons, which would directly be applied for our case. Let us suppose the Lagrangian of the ℓ-th monopole wall as

$$\begin{aligned} L_\ell = -(g^2 + q_\ell^2)^{1/2} \phi (1 - \boldsymbol{V}_\ell^2)^{1/2} + q_\ell \boldsymbol{V}_\ell \cdot \boldsymbol{A} - q_\ell A_0 \\ + g \boldsymbol{V}_\ell \cdot \tilde{\boldsymbol{A}} - g \tilde{A}_0\,, \end{aligned} \tag{4.28}$$

where $(g^2 + q_\ell^2)^{1/2}$, q_ℓ and $\boldsymbol{V}_\ell := (\dot{\xi}_\ell, \dot{z}_\ell)$ are the scalar charge, the electric charge and the velocity of the ℓ-th wall respectively; $(\tilde{\boldsymbol{A}}, \tilde{A}_0)$ is the dual potential which satisfy $\tilde{F} = *F$. The background fields ϕ, $\boldsymbol{A}$, A_0, $\tilde{\boldsymbol{A}}$ and $\tilde{A}_0$ are produced by the remaining $k-1$ moving, dyonic monopole walls and can be derived by using the solution previously derived. For $j \neq \ell$, the asymptotic fields of the j-th dyonic monopole wall at rest can be obtained in the same way as the non-periodic monopoles, and we have

$$\phi^j(\boldsymbol{x}) = (g^2 + q_j^2)^{1/2} u(z) \tag{4.29}$$

and

$$\begin{aligned}
&A^j_\xi(\boldsymbol{x}) = gw(\boldsymbol{x})\,, && A^j_z(\boldsymbol{x}) = 0\,, && A^j_0(\boldsymbol{x}) = -q_j u(z)\,, \\
&\tilde{A}^j_\xi(\boldsymbol{x}) = -q_j w(\boldsymbol{x})\,, && \tilde{A}^j_z(\boldsymbol{x}) = 0\,, && \tilde{A}^j_0(\boldsymbol{x}) = -gu(z)\,,
\end{aligned} \tag{4.30}$$

where $u(z)$ and $w(\boldsymbol{x})$ for the monopole wall are given by (4.27). Then, the fields for a moving monopole wall can be obtained by the Lorentz boost. (Note that we can use the ordinary Lorentz boost for the fields because our definition (4.16) of the metric on $T^2 \times \mathbb{R}$ is flat.) Keeping the terms of order q_j^2, $q_j \boldsymbol{V}_j$ and $\boldsymbol{V}_j^2$, we find

$$\begin{aligned}
\phi^j(\boldsymbol{x}) &\simeq (g^2 + q_j^2)^{1/2} u(z)(1 - \boldsymbol{V}_j^2)^{1/2}\,, \\
A^j_\xi(\boldsymbol{x}) &\simeq -q_j u(z) V_{j\xi} + gw(\boldsymbol{x})\,, \\
A^j_z(\boldsymbol{x}) &\simeq -q_j u(z) V_{jz}\,, \\
A^j_0(\boldsymbol{x}) &\simeq -q_j u(z) + g(wV_j^\xi + \bar{w}V_j^{\bar{\xi}})\,, \\
\tilde{A}^j_\xi(\boldsymbol{x}) &\simeq -gu(z) V_{j\xi} - q_j w(\boldsymbol{x})\,, \\
\tilde{A}^j_z(\boldsymbol{x}) &\simeq -gu(z) V_{jz}\,, \\
\tilde{A}^j_0(\boldsymbol{x}) &\simeq -gu(z) - q_j(wV_j^\xi + \bar{w}V_j^{\bar{\xi}})\,,
\end{aligned} \tag{4.31}$$

where the scalar potentials are replaced by the Liénard-Wiechert potentials with the approximation of the distance $(r^2 - |\boldsymbol{r} \times \boldsymbol{V}|^2 + O(\boldsymbol{V}^2))^{1/2}$ by r. Substituting the boosted fields into the Lagrangian for $k = 2$ and keeping terms of the second order in q_1, $\boldsymbol{V}_1$, q_2 and $\boldsymbol{V}_2$, we obtain

$$\begin{aligned}
L_2 = &-m_2 + \frac{1}{2} m_2 \boldsymbol{V}_2^2 + q_2(bV_2^\xi + \bar{b}V_2^{\bar{\xi}}) \\
&+ \frac{g^2}{2} u(z_2 - z_1)(\boldsymbol{V}_2 - \boldsymbol{V}_1)^2 - \frac{1}{2} u(z_2 - z_1)(q_2 - q_1)^2 \\
&+ g(q_2 - q_1)\big\{w_{21}(V_2^\xi - V_1^\xi) + \bar{w}_{21}(V_2^{\bar{\xi}} - V_1^{\bar{\xi}})\big\}\,,
\end{aligned} \tag{4.32}$$

where $m_j := v(g + q_j)^{1/2}$ is the rest mass of the j-th dyonic monopole wall and $w_{ji} := w(\boldsymbol{x}_j - \boldsymbol{x}_i)$. Furthermore, expanding m_j and making symmetrization, we obtain the total Lagrangian L_{21} as follows:

$$L_{21} = \frac{vg}{2}(\boldsymbol{V}_2^2 + \boldsymbol{V}_1^2) + \frac{g^2}{2} u(z_2 - z_1)(\boldsymbol{V}_2 - \boldsymbol{V}_1)^2$$

$$
\begin{aligned}
&- \frac{v}{2g}(q_2^2 + q_1^2) - \frac{1}{2}u(z_2 - z_1)(q_2 - q_1)^2 \\
&+ b(q_2 V_2^{\xi} + q_1 V_1^{\xi}) + g w_{21}(q_2 - q_1)(V_2^{\xi} - V_1^{\xi}) \\
&+ \bar{b}(q_2 V_2^{\bar{\xi}} + q_1 V_1^{\bar{\xi}}) + g \bar{w}_{21}(q_2 - q_1)(V_2^{\bar{\xi}} - V_1^{\bar{\xi}}) \,.
\end{aligned}
\tag{4.33}
$$

The Lagrangian may look ill-defined due to the diverging v, however, it can be replaced by v_{ren} which remains finite. Then, the Lagrangian can be divided into the two parts: $L_{21} = L_{\mathrm{CM}} + L_{\mathrm{rel}}$, where

$$
\begin{aligned}
L_{\mathrm{CM}} = {}& \frac{vg}{4}(\boldsymbol{V}_2 + \boldsymbol{V}_1)^2 - \frac{v}{4g}(q_2 + q_1)^2 \\
&+ \frac{b}{2}(q_2 + q_1)(V_2^{\xi} + V_1^{\xi}) + \frac{\bar{b}}{2}(q_2 + q_1)(V_2^{\bar{\xi}} + V_1^{\bar{\xi}})
\end{aligned}
\tag{4.34}
$$

and

$$
\begin{aligned}
L_{\mathrm{rel}} = {}& \frac{g^2}{2}\left(\frac{v_{\mathrm{ren}}}{2g} + \frac{1}{2}|z_2 - z_1|\right)(\boldsymbol{V}_2 - \boldsymbol{V}_1)^2 - \frac{1}{2}\left(\frac{v_{\mathrm{ren}}}{2g} + \frac{1}{2}|z_2 - z_1|\right)(q_2 - q_1)^2 \\
&+ \left\{\frac{b}{2} + \frac{\mathrm{i}\nu g}{8\tau_2}\,\mathrm{sign}(z_2 - z_1)\,(\bar{\xi}_2 - \bar{\xi}_1)\right\}(q_2 - q_1)(V_2^{\xi} - V_1^{\xi}) \\
&+ \left\{\frac{\bar{b}}{2} - \frac{\mathrm{i}\nu g}{8\tau_2}\,\mathrm{sign}(z_2 - z_1)\,(\xi_2 - \xi_1)\right\}(q_2 - q_1)(V_2^{\bar{\xi}} - V_1^{\bar{\xi}}) \,.
\end{aligned}
\tag{4.35}
$$

The center of mass Lagrangian L_{CM} would diverge while the relative Lagrangian L_{rel} would converge in the limit of $M, N \to \infty$. The asymptotic metric of the moduli space can be read off from the relative Lagrangian. For convenience, we introduce relative variables by $\xi := \xi_2 - \xi_1$, $z := z_2 - z_1$, $\boldsymbol{V} := \boldsymbol{V}_2 - \boldsymbol{V}_1$ and $q := q_2 - q_1$ and further replace the electric charge q in L_{rel} by the relative phase θ via the Legendre transformation,

$$
L'_{\mathrm{rel}} = L_{\mathrm{rel}} + q\dot{\theta} \,. \tag{4.36}
$$

As we will see shortly the coefficient of $q\dot{\theta}$ can be fixed so that the asymptotic metric has the double periodicity. After the Legendre transformation, we obtain the asymptotic metric of the moduli space in the form of the Gibbons-Hawking ansatz [35],

$$
\frac{1}{g}\mathrm{d}s^2 = U \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} + \frac{1}{U}(\mathrm{d}\theta + \boldsymbol{W} \cdot \mathrm{d}\boldsymbol{x})^2 \,, \tag{4.37}
$$

where

$$
U = \frac{v_{\mathrm{ren}}}{2} + \frac{g}{2}|z| \,, \qquad W_{\xi} = \frac{b}{2} + \frac{\mathrm{i}\nu g}{8\tau_2}\,\mathrm{sign}(z)\,\bar{\xi} \,,
$$

$$W_{\bar{\xi}} = \overline{W}_{\xi}\,, \qquad W_z = 0\,. \tag{4.38}$$

At first sight the metric seems to have a constant shift when we go around the closed cycles on T^2, since W_ξ explicitly depends on the coordinate $\bar{\xi}$. However we can confirm the double-periodicity of the metric by observing that the constant shift of W_ξ can be cancelled by the phase shift due to the necessary $U(1)$ gauge transformation in the twisted boundary conditions (4.25) and (4.26), which also determines the coefficient of $q\dot{\theta}$ in (4.36). Furthermore, we can also easily check the invariance of the metric under the modular transformation (4.17). Thus our metric (4.37) is well-defined on $T^3 \times \mathbb{R}$ with local coordinates (θ, ξ, z). Finally the hyperkähler metric (4.37) allows the following local isometries with parameters (α, β, γ);

$$\begin{aligned} &\theta \to \theta + \alpha + \frac{\nu g}{4}\,\mathrm{sign}(z)\,(\beta y - \gamma x)\,, \\ &x \to x + \beta\,, \qquad y \to y + \gamma\,. \end{aligned} \tag{4.39}$$

It is straightforward to extend the above computation for $k = 2$ to the case of general k. The total Lagrangian of the k well-separated dyonic monopole walls can be obtained by generalizing (4.33) as follows

$$\begin{aligned} L_k = {} & \frac{vg}{2}\sum_{j=1}^{k} \boldsymbol{V}_j^2 + \frac{g^2}{2}\sum_{1\le i<j\le k} u(z_j - z_i)(\boldsymbol{V}_j - \boldsymbol{V}_i)^2 \\ & - \frac{v}{2g}\sum_{j=1}^{k} q_j^2 - \frac{1}{2}\sum_{1\le i<j\le k} u(z_j - z_i)(q_j - q_i)^2 \\ & + b\sum_{j=1}^{k} q_j V_j^{\xi} + \sum_{1\le i<j\le k} g w_{ji}(q_j - q_i)(V_j^{\xi} - V_i^{\xi}) \\ & + \bar{b}\sum_{j=1}^{k} q_j V_j^{\bar{\xi}} + \sum_{1\le i<j\le k} g \bar{w}_{ji}(q_j - q_i)(V_j^{\bar{\xi}} - V_i^{\bar{\xi}})\,. \end{aligned} \tag{4.40}$$

This can be decomposed into the two parts $L_k = L_{\mathrm{CM}} + L_{\mathrm{rel}}$, where

$$\begin{aligned} L_{\mathrm{CM}} = {} & \frac{vg}{2k}\left(\sum_{j=1}^{k} \boldsymbol{V}_j\right)^2 - \frac{v}{2kg}\left(\sum_{j=1}^{k} q_j\right)^2 \\ & + \frac{b}{k}\left(\sum_{j=1}^{k} q_j\right)\left(\sum_{j=1}^{k} V_j^{\xi}\right) + \frac{\bar{b}}{k}\left(\sum_{j=1}^{k} q_j\right)\left(\sum_{j=1}^{k} V_j^{\bar{\xi}}\right) \end{aligned} \tag{4.41}$$

and

$$
\begin{aligned}
L_{\rm rel} = & \frac{g^2}{2} \sum_{1 \le i < j \le k} \left(\frac{v_{\rm ren}}{kg} + \frac{1}{2}|z_j - z_i| \right) (\boldsymbol{V}_j - \boldsymbol{V}_i)^2 \\
& - \frac{1}{2} \sum_{1 \le i < j \le k} \left(\frac{v_{\rm ren}}{kg} + \frac{1}{2}|z_j - z_i| \right) (q_j - q_i)^2 \\
& + \sum_{1 \le i < j \le k} \left\{ \frac{b}{k} + \frac{\mathrm{i}\nu g}{8\tau_2} \operatorname{sign}(z_j - z_i) \, (\bar{\xi}_j - \bar{\xi}_i) \right\} (q_j - q_i)(V_j^{\xi} - V_i^{\xi}) \\
& + \sum_{1 \le i < j \le k} \left\{ \frac{\bar{b}}{k} - \frac{\mathrm{i}\nu g}{8\tau_2} \operatorname{sign}(z_j - z_i) \, (\xi_j - \xi_i) \right\} (q_j - q_i)(V_j^{\bar{\xi}} - V_i^{\bar{\xi}}) \, . \qquad (4.42)
\end{aligned}
$$

On the other hand, the Gibbons-Hawking ansatz for general k can be written as

$$
\begin{aligned}
\frac{1}{g} \mathrm{d}s^2 = U_{IJ} \mathrm{d}\boldsymbol{X}_I \cdot \mathrm{d}\boldsymbol{X}_J + U_{IJ}^{-1} (\mathrm{d}\Theta_I + \boldsymbol{W}_{IK} \cdot \mathrm{d}\boldsymbol{X}_K) \\
\cdot (\mathrm{d}\Theta_J + \boldsymbol{W}_{JL} \cdot \mathrm{d}\boldsymbol{X}_L) \, , \qquad (4.43)
\end{aligned}
$$

where $I, J, K, L = 1, \cdots, k-1$; we have introduced the following relative coordinates measured by the position of k-th moving monopole wall:

$$
\Xi_J := \xi_J - \xi_k \, , \qquad Z_J := z_J - z_k \, , \qquad \Theta_J := \theta_J - \theta_k \, , \qquad \boldsymbol{X}_J := (\Xi_J, Z_J) \, .
$$

By comparing the coefficients of (4.42) and the sigma model Lagrangian for the Gibbons-Hawking ansatz, we find, on the diagonal line,

$$
U_{JJ} = (k-1) \frac{v_{\rm ren}}{k} + \frac{g}{2} \sum_{I \neq J} |Z_J - Z_I| \, , \qquad (4.44)
$$

$$
(W_\xi)_{JJ} = (k-1) \frac{b}{k} + \frac{\mathrm{i}\nu g}{8\tau_2} \sum_{I \neq J} \operatorname{sign}(Z_J - Z_I) \, (\bar{\Xi}_J - \bar{\Xi}_I) \, , \qquad (4.45)
$$

while for $I \neq J$,

$$
U_{IJ} = -\frac{v_{\rm ren}}{k} - \frac{g}{2} |Z_J - Z_I| \, , \qquad (4.46)
$$

$$
(W_\xi)_{IJ} = -\frac{b}{k} - \frac{\mathrm{i}\nu g}{8\tau_2} \operatorname{sign}(Z_J - Z_I) \, (\bar{\Xi}_J - \bar{\Xi}_I) \, , \qquad (4.47)
$$

and $(W_{\bar{\xi}})_{IJ} = (\overline{W}_\xi)_{IJ}$ and $(W_z)_{IJ} = 0$ in anywhere.

Finally, we discuss the asymptotic metric of monopole walls with Dirac-type singularities. We note that it is proved that the maximum number of the singularities is four in the case of monopole chains with four-moduli. Accordingly, we shall derive the inequality for the maximum number of Dirac singularities of monopole walls with four-moduli by using the spectral curves and Newton polygons reviewed in the privious section. The key is simply a geometric restriction. For a given number of internal points, the maximum Newton polygon of $U(2)$ monopole walls with singularities must be a trapezoid which has height $n = 2$ and has length of top and bottom edges r_{+0} and r_{-0}, respectively (Figure 4.1). From the shape of the Newton polygon, the maximum number of singularities obvi-

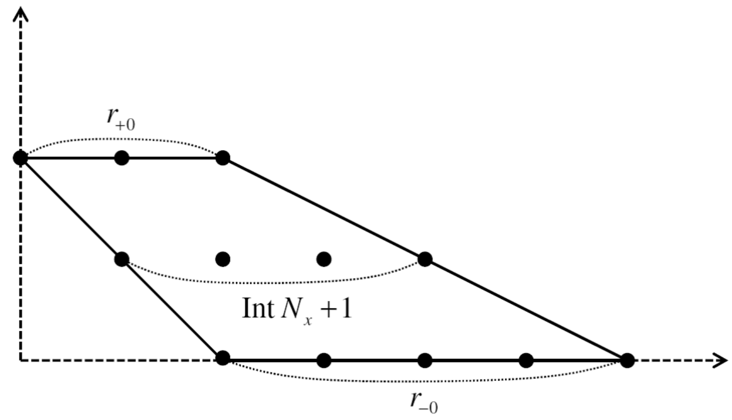

Figure 4.1: The maximum Newton polygon $\mathcal{N}_x$ of a $U(2)$ monopole wall with r_{+0} singularities and r_{-0} singularities.

ously have a relation, $r_{+0} + r_{-0} = 2(\mathrm{Int}\, N_x + 1)$ (which can also be derived by the Pick's formula). Therefore, the total number of singularities $r_0 := r_{+0} + r_{-0}$ is limited by the dimension of the moduli space $\mathcal{M}$ of the monopole walls as

$$r_0 \leq \frac{1}{2} \dim \mathcal{M} + 2 \,. \tag{4.48}$$

Especially the maximum number of singularities of k well-separated monopole walls is $2k$ because the dimension of the relative moduli space is $4(k-1)$. This is consistent with the fact that the maximal number of the matter hypermultiplets in the fundamental representation is $2k$ in the corresponding $SU(k)$ super Yang-Mills theory with eight super charges.

Here we restrict our calculation to the monopole walls with four-moduli, that is, for $k = 2$. Then the maximal number of the Dirac singularities is $r_0 = 4$. Since these singularities are stationary and have no electric charge, the metric can be obtained by simply replacing the vacuum expectation value and the background field by $v+\sum_{\ell=1}^{r_0} g_\ell u(r_{\ell z}-z)$ and $b+\sum_{\ell=1}^{r_0} g_\ell w(\boldsymbol{r}_\ell-\boldsymbol{x})$, respectively, where g_ℓ and $\boldsymbol{r}_\ell := (r_{\ell\xi}, r_{\ell z})$ are the magnetic charges and the positions of each singularity [12]. Substituting them into (4.38), we have

$$\begin{aligned}
U &= \frac{v'_{\mathrm{ren}}}{2} + \frac{g}{2}|z| + \frac{1}{4}\sum_{\ell=1}^{r_0} g_\ell \left| r_{\ell z} - \frac{z}{2} \right| \\
&\qquad\qquad + \frac{1}{4}\sum_{\ell=1}^{r_0} g_\ell \left| r_{\ell z} + \frac{z}{2} \right| , \\
W_\xi &= \frac{b}{2} + \frac{\mathrm{i}\nu g}{8\tau_2}\,\mathrm{sign}(z)\,\bar{\xi} + \frac{\mathrm{i}\nu}{16\tau_2}\sum_{\ell=1}^{r_0} g_\ell\,\mathrm{sign}\Big(r_{\ell z} - \frac{z}{2}\Big)\left(\bar{r}_{\ell\xi} - \frac{\bar{\xi}}{2}\right) \\
&\qquad\qquad\qquad + \frac{\mathrm{i}\nu}{16\tau_2}\sum_{\ell=1}^{r_0} g_\ell\,\mathrm{sign}\Big(r_{\ell z} + \frac{z}{2}\Big)\left(\bar{r}_{\ell\xi} + \frac{\bar{\xi}}{2}\right) , \\
W_{\bar{\xi}} &= \overline{W}_\xi \,, \quad W_z = 0 \,,
\end{aligned} \tag{4.49}$$

where

$$v'_{\mathrm{ren}} := v - \left(2 + \sum_{\ell=1}^{r_0} \frac{g_\ell}{g}\right) g C_{M,N} \tag{4.50}$$

and we assume $\boldsymbol{x}_1 + \boldsymbol{x}_2 = \boldsymbol{0}$.

In the correspondence with $\mathcal{N}=1$ super Yang-Mills theory on $\mathbb{R}^3 \times T^2$, the function $U(z)$ is identified with the low energy effective coupling, or the second derivative of the prepotential on the Coulomb modulus $\mathbb{R}_{>0}$.

Chapter 5

Conclusion

In this thesis, we have explicitly derived hyperkähler metrics whose asymptotic behavior is of ALH type from the low energy dynamics of well-separated monopole walls. The metric in four dimensions is defined on a $T^2 \times S^1$ fibration over $\mathbb{R}$ and enjoys the modular invariance on T^2. We have also derived the maximal number of the Dirac singularities for $U(2)$ monopole walls by using the Newton polygon of the spectral curve. Furthermore, we have reviewed some fundamental topics to give readers complete understanding of the background of our study such as BPS monopoles and the moduli space, dynamics of monopoles, the metrics of the moduli space, monopole walls and the spectral analysis.

One of the next challenges is the low-energy scattering of the monopole walls as a geodesic motion on the moduli space. In the present discussion, the monopoles are assumed to be well-separated and hence the collision process is excluded.

In order to obtain a global metric on the moduli space of monopole walls, we need some ideas such as the one for the Atiyah-Hitchin metric [7] for non-periodic $SU(2)$ two-monopole in BPS limit. On the super Yang-Mills theory side, the region of well-separated monopoles corresponds to the weak coupling region of the moduli space of the Coulomb branch, where the vacuum expectation values of the scalar fields in the vector multiplets are large compared with the dynamical scale of the theory. In order to obtain a global metric which is valid on the whole Coulomb branch, the inclusion of instanton corrections is crucial. A successful example of such computation is the Ooguri-Vafa metric [40]. See also [41] and [42] for recent developments.

In the periodic monopoles, the monopole scattering has been successfully discussed by using the Nahm transform, the spectral curve and the corresponding Hitchin equation [13, 14, 15]. In addition, some recent works would give us ideas for the study of the explicit metrics of the moduli space of monopole walls. More practical classifications of monopole walls by using the Newton polygon are discussed [37]. The numerical analysis of the moduli space metric of monopole walls in terms of spectral curves are performed [38]. The metrics in [39] might also be helpful for our study.

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