On the existence problem of Kähler-Ricci solitons
(ケーラー・リッチソリトンの存在問題について)

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Abstract

In this paper, we consider the existence problem of Kähler-Ricci solitons. Let M be a Fano manifold. We call a Kähler metric $\omega \in c_1(M)$ a Kähler-Ricci soliton if it satisfies the equation $\mathrm{Ric}(\omega) - \omega = L_V \omega$ for some holomorphic vector field V on M. We study the explicit construction of Kähler-Ricci solitons on special projective bundles, called "admissible bundles", which were introduced by Gauduchon and other collaborators to unify previous works on the existence problem of canonical metrics on projective bundles. On admissible bundles, the admissible Kähler-Ricci soliton condition can be written as a simple ODE, and its existence is equivalent to the vanishing of Maschler-Tønnesen invariant. We also study the K-stability for Kähler-Ricci solitons. It is known that a necessary condition for the existence of Kähler-Ricci solitons is the vanishing of the modified Futaki invariant introduced by Tian-Zhu. In a recent work of Berman-Nyström, it was generalized for (singular) Fano varieties and the notion of algebro-geometric stability of the pair (M, V) was introduced. We propose a method of computing the modified Futaki invariant for Fano complete intersections in projective spaces.

Contents

1 Introduction

Let M be a compact complex manifold. We say that a Kähler metric ω is Kähler-Einstein if it satisfies the equation $\mathrm{Ric}(\omega) = c\omega$ for some real constant c. After normalizing constants, we may assume that $c = -1$, 0 or 1. Then $c_1(M)$ is represented by a negative, zero or positive real $(1,1)$-form. These conditions are simply written as $c_1(M) < 0$, $c_1(M) = 0$ and $c_1(M) > 0$ repectively. In the case of compact Riemann surfaces, the existence of a Kähler-Einstein metric follows from the classical uniformization theorem, which states that there exists a unique metric of constant scalar curvature on any compact Riemann surfaces. For higher dimensional manifolds, it is well-known that every compact Kähler manifold with $c_1(M) \leq 0$ admits a unique Kähler-Einstein metric (cf. [Yau78], [Aub76]). On the other hand, in the case of Fano manifolds, i.e., compact complex manifolds with $c_1(M) > 0$, there are examples admitting no Kähler-Einstein metrics. Hence we are especially interested in the case of Fano manifolds.

Now let M be a Fano manifold. In this paper, we study the existence problem of "Kähler-Ricci solitons". A pair (ω, V) of a Kähler form ω and a holomorphic vector field V is called a **Kähler-Ricci soliton** if it satisfies the equation

$$\mathrm{Ric}(\omega) - c\omega = L_V\omega$$

for some constant $c > 0$, where L_V denotes the Lie derivative with respect to V. In particular, if $V \equiv 0$, ω is Kähler-Einstein. After normalizing constants, we may assume that $c = 1$. A Kähler-Ricci soliton gives rise to a self similar solution of the following PDE for a t-dependent Kähler form ω_t, called **Kähler-Ricci flow**:

$$\begin{cases} \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t, \\ \omega_0 \in c_1(M). \end{cases}$$

Generally, Kähler-Ricci flow has good properties that (1) The Kähler condition are preserved under the flow. (2) Any solution ω_t belongs to $c_1(M)$. (3) There is a unique long time solution [Cao85]. Thus we obtain a deformation family ω_t $(0 \leq t < \infty)$ of Kähler forms in $c_1(M)$. If (ω_0, V) is a Kähler-Ricci soliton, a direct computation shows that $\omega_t = (\exp(-\mathrm{Re}(V)t))^*\omega_0$ is a unique solution of Kähler-Ricci flow with initial Kähler form ω_0.

One motivation to study Kähler-Ricci solitons is that they are closely related to the limiting behavior of solutions of Kähler-Ricci flow. Tian-Zhu [TZ07] showed that if M admits a Kähler-Ricci soliton (ω_{KS}, V) and the initial metric ω_0 is invariant under the action of the one-parameter subgroup generated by $\mathrm{Im}(V)$, then any solution of Kähler-Ricci flow converges to the ω_{KS} in the sense of Cheeger-Gromov.

Another motivation is the uniqueness of the vector field V which is the candidate for a Kähler-Ricci soliton (ω_{KS}, V). In particular, the existence of a Kähler-Ricci soliton with respect to a non-zero holomorphic vector field is an obstruction to the existence of Kähler-Einstein metrics. To explain this, we first mention the modified Futaki invariant introduced by Tian-Zhu [TZ02], that is an obstruction to the existence of Kähler-Ricci solitons: let κ be a real valued smooth function on M defined by the equation

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\kappa,$$

here we remark that the function κ (**Ricci potential**) uniquely exists up to an additive constant. Let $\mathfrak{h}$ be the Lie algebra consisting of all holomorphic vector fields on M. Then any $V \in \mathfrak{h}$ can be lifted to the anti-canonical bundle $-K_M$ of M, and naturally acts on the space of Hermitian metrics on $-K_M$. Let h be a Hermitian metric on $-K_M$ such that $\omega = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$ and $\mu_{h,V}$ the holomorphy potential of the pair (h, V) defined by the equation $L_V h = -\mu_{h,V} \cdot h$ (cf. Definition 2.2). Then we can easily check that

$$\begin{cases} i_V\omega = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\mu_{h,V} \\ -\Delta_{\partial}\mu_{h,V} + \mu_{h,V} + V(\kappa) = 0, \end{cases}$$

where $\Delta_{\partial} = -g^{i\bar{j}}\frac{\partial^2}{\partial z^i \partial z^{\bar{j}}}$ denotes the ∂-Laplacian with respect to ω. Then one can easily see that the pair (ω, V) is a Kähler-Ricci soliton if and only if $\kappa = \mu_{h,V}$ holds up to an additive constant. Let $\mathcal{F}$ be a function on $\mathfrak{h}$ defined by

$$\mathcal{F}(V) = -\frac{1}{c_1(M)^n}\int_M e^{\mu_{h,V}}\omega^n,$$

and $\mathrm{Fut}_V(W)$ the modified Futaki invariant defined as the Gâteaux differential of $\mathcal{F}$ at V in the direction W, i.e.,

$$\begin{aligned} \mathrm{Fut}_V(W) &= \frac{d}{dt}\mathcal{F}(V+tW)\Big|_{t=0} = -\frac{1}{c_1(M)^n}\int_M \mu_{h,W}e^{\mu_{h,V}}\omega^n \\ &= \frac{1}{c_1(M)^n}\int_M W(\kappa - \mu_{h,V})e^{\mu_{h,V}}\omega^n. \end{aligned}$$

Hence if there exists a Kähler-Ricci soliton (ω, V), then we have $\kappa = \mu_{h,V}$ (up to an additive constant) and $\mathrm{Fut}_V(W)$ must vanish. They showed that $\mathrm{Fut}_V(W)$ is independent of a choice of $\omega \in c_1(M)$ (In the case when $V \equiv 0$, this function coincides with the original Futaki invariant and its independence was shown by Futaki [Fut83]). Next, we consider $\mathrm{Fut}_V(W)$ from the geometric view point. We denote by $\mathrm{Aut}^0(M)$ the identity component of the group of holomorphic automorphisms of M. Since $\mathrm{Aut}^0(M)$ is a linear algebraic group [Fuj78], the Chevalley decomposition allows us to obtain a semidirect decomposition

$$\mathrm{Aut}^0(M) = \mathrm{Aut}_r(M) \ltimes R_u,$$

where $\mathrm{Aut}_r(M)$ is a reductive subgroup of $\mathrm{Aut}^0(M)$, which is the complexification of a maximal compact subgroup K, and R_u the unipotent radical of $\mathrm{Aut}^0(M)$. We also obtain the corresponding decomposition of $\mathfrak{h}$

$$\mathfrak{h} = \mathfrak{h}_r + \mathfrak{h}_u,$$

where $\mathfrak{h}_r (= \kappa(M)^{\mathbb{C}})$, $\mathfrak{h}_u$ and $\kappa(M)$ denotes the Lie algebras of $\mathrm{Aut}_r(M)$, R_u and K respectively. Tian-Zhu also showed that $\mathcal{F}$ is a real valued proper convex function if restricted to the linear subspace $\mathfrak{h}_{r,\mathbb{R}} := \{W \in \mathfrak{h}_r | \mathrm{Im}(W) \in \kappa(M)\}$. Hence, from the definition of Fut_V, we know that there exists a unique holomorphic vector field $V \in \mathfrak{h}_{r,\mathbb{R}}$ such that $\mathrm{Fut}_V \equiv 0$ on $\mathfrak{h}_r$. Moreover, Saito [Sai14] recently showed that the equation $\mathrm{Fut}_V \equiv 0$ also holds on $\mathfrak{h}_u$ by modifying the earlier Mabuchi's work for classical Futaki invariant [Mab90]. Hence the modified Futaki invariant is, strictly speaking, not an obstruction to the existence of a Kähler-Ricci soliton, but it tells us how to choose the candidate V for a Kähler-Ricci soliton (ω, V).

It is conjectured that the existence of a Kähler-Ricci soliton is equivalent to some stabilities. Let us first see that the case of Kähler-Einstein metrics. Let $\omega_0 \in c_1(M)$ be a Kähler metric on M and $\mathcal{H}$ the space of all Kähler-forms in $c_1(M)$. Mabuchi [Mab86] introduced a functional $\mathrm{Mab}\colon \mathcal{H} \to \mathbb{R}$, called **K-energy map** by integrating Futaki invariant:

$$\mathrm{Mab}(\omega) = -\frac{1}{c_1(M)^n} \int_0^1 \int_M \dot{\varphi}_t (\mathrm{Scal}(\omega_t) - n) \omega_t^n dt, \ \omega \in \mathcal{H},$$

where ω_t $(0 \leq t \leq 1)$ is a path in $\mathcal{H}$ joining ω_0 to ω and φ_t is a smooth function defined by $\omega_t = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$. Then Mab is well-defined, i.e., independent of a choice of such a path ω_t. One can also see that changing a base point $\omega_0 \in \mathcal{H}$ affects only the constant term of Mab. Hence the derivative of the K-energy map is independent of a choice of a base point $\omega_0 \in \mathcal{H}$. In particular, the derivative of the K-energy map along the one-parameter subgroup $\rho_t = \exp(t\mathrm{Re}(V))$ $(t \in \mathbb{R})$ for a holomorphic vector field V is given by the real part of the Futaki invariant of V:

$$\frac{d}{dt}\mathrm{Mab}(\rho_t^* \omega) = \mathrm{Re}(\mathrm{Fut}(V)).$$

By the definition, the critical point of the K-energy map is, if exists, a Kähler-Einstein metric. Hence one might think that the existence of a Kähler-Einstein metric is equivalent to the coercivity[1] of the K-energy map. Tian [Tian97] showed that this statement is true when $\mathrm{Aut}(M)$ is discrete. If this is not the case, it is known that the coercivity of the K-energy map leads to the existence of a Kähler-Einstein metric [CTZ05]. But the converse problem is still open.

Donaldson threw the fresh light on this problem from the view point of geometric invariant theory. He constructed a suitable manifold with a Hamiltonian action of an infinite dimensional group so that its moment map is given by the scalar curvature (cf. [Don97]). He also introduced an algebraic definition of the Futaki invariant and generalized it for any (possibly singular) Fano variety (cf. [Don02]). Then the asymptotic behavior of the K-energy map near the boundary of $\mathcal{H}$ can be controlled by the generalized Futaki invariants of all degenerations of M. Donaldson's formulation of this picture generalizes Ding-Tian's K-stability [DT92] and is called (Donaldson's) **K-stability**: we consider a degeneration of M, called a **test configuration**, which is parametrized by $\mathbb{C}$ and which can be regard as a flat morphism of schemes $\mathcal{M} \to \mathbb{C}$ on which $\mathbb{C}^*$ operates equivariantly, where each fiber of

[1]The word "coercive" is also called "strongly proper", which is defined quantitatively by means of Aubin's functional. See [BN14, Section 3.6] for the precise definition.

$\mathcal{M}$ is (possibly singular) Fano variety. Each test configuration gives rise to a holomorphic vector field on the central fiber (the fiber over $\{0\}$), and we can associate it with a number by means of the generalized Futaki invariant. We say that M is K-stable if this number is non-negative for any test configuration and equals to 0 if and only if this test configuration is trivial. The idea of K-stability comes from Hilbert-Mumford criterion (cf. Section 3.1.1) in geometric invariant theory, which says that we have only to test all degenerations of M parametrized by $\mathbb{C}$ to check the stability. It is known that M is K-stable if and only if M admits a Kähler-Einstein metric. The "if" part was proved by Ding-Tian [DT92], Berman [Ber12] and Stoppa [Stop09]. The "only if" part was recently proved by Chen-Donaldson-Sun [CDS13] and Tian [Tian12].

Berman-Nyström [BN14] generalized Donaldson's K-stability for pairs (M, V) by extending the modified Futaki invariant for normal $\mathbb{Q}$-Fano varieties with log-terminal singularities, and showed that if M admits a Kähler-Ricci soliton with respect to V, then (M, V) is K-polystable. However, it is still an open question whether the K-polystability of (M, V) leads to the existence of a Kähler-Ricci soliton with respect to V.

Contributions

The author's contributions in this paper consist of mainly two parts and are based on [Tak14] and [Tak14-2]:

(1) The author gained the existence result of a Kähler-Ricci soliton on "admissible bundles" which are special projective bundles introduced in [ACGT08]:

Theorem 1.1 ([Tak14])**.** Let M be an admissible bundle and Ω an admissible class on M. We assume that Ω is proportional to $c_1(M)$. Then there exists an admissible Kähler-Ricci soliton in Ω.

It is well-known that there exists a Kähler-Ricci soliton on a toric Fano manifold [WZ04] and a certain $\mathbb{C}P^1$-bundle [TZ02]. Theorem 1.1 generalizes the existence result proved by Tian-Zhu [TZ02] to the case when the dimension of the fiber is greater than 1.

We say that M is an **admissible bundle** if it is a projective bundle of the form $\mathbb{P}(E_0 \oplus E_\infty) \to S$, where E_0 and E_∞ are projectively flat hermitian vector bundles over a compact Kähler manifold S, and the base manifold S is locally a Kähler product $\prod_{a \in \mathcal{A}} S_a$ for some finite subset $\mathcal{A} \subset \mathbb{N}$. Then we assume that each S_a has a constant scalar curvature Kähler metric $(\pm g_a, \pm\omega_a)$ (where $\pm$ is chosen so that ω_a is positive definite) and the condition $c_1(E_\infty)/\mathrm{rank}(E_\infty) - c_1(E_0)/\mathrm{rank}(E_0) = \sum_{a \in \mathcal{A}} [\omega_a/2\pi]$. In short, an **admissible Kähler metric** is a Kähler metric (g, ω) on M parametrized by real constants $\{x_a\}_{a \in \mathcal{A}}$ $(0 < |x_a| < 1)$ and a smooth function $\Theta \colon [-1, 1] \to \mathbb{R}$ satisfying (i) $\Theta > 0$ on $(-1, 1)$, (ii) $\Theta(\pm 1) = 0$, (iii) $\Theta'(\pm 1) = \mp 2$. Then we call $\Omega := [\omega]$ an **admissible Kähler class**. Changing of such a function Θ gives a deformation family of Kähler metrics, but their corresponding Kähler forms (and hence admissible Kähler classes) are the same, which only depends on a choice of real constants $\{x_a\}_{a \in \mathcal{A}}$ (see Definition 2.6 for more detail).

Restricting our attention to admissible Kähler metrics, the equation of admissible Kähler-Ricci soliton can be reduced to a simple ODE for a function $F\colon [-1,1] \to \mathbb{R}$:

$$F'(z) + k \cdot F(z) = P(z),\ F(z) > 0 \text{ on } (-1,1),\ F(\pm 1) = 0,$$

where $k \in \mathbb{R}$, $P(z)$ and $p_c(z)$ are some polynomial functions of z depending only on $\{x_a\}$, and we put $F(z) := \Theta(z) \cdot p_c(z)$. We can solve this equation and get an explicit solution $F(z) = e^{-kz} \int_{-1}^{z} P(t)e^{kt}dt$ under the condition $F(-1) = 0$. Hence the condition $F(1) = 0$ is equivalent to the vanishing of the **Maschler-Tønnesen invariant** $\mathrm{MT}(k) := \int_{-1}^{1} P(t)e^{kt}dt$.

The author showed that if Ω is proportional to $c_1(M)$, $P(t)$ has exactly one root on the interval $(-1,1)$ (cf. Lemma 4.20). Combining with some results proved by Maschler-Tønnesen [MT11], we can show that there exists a unique $k_0 \in \mathbb{R}$ such that $\mathrm{MT}(k_0) = 0$. Then $F(z) = e^{-k_0 z} \int_{-1}^{z} P(t)e^{k_0 t}dt$ satisfies $F(z) > 0$ on $(-1,1)$ and an admissible Kähler-Ricci soliton is naturally constructed from this F. We prove Theorem 1.1 at the end of Section 4.2.2.

In the course of the proof of Theorem 1.1, the author found an explicit relation between the modified Futaki invariant and Maschler-Tønnesen invariant (cf. Lemma 4.20), both of which are obstructions to the existence of admissible Kähler-Ricci solitons.

(2) The author invented the following explicit formula of the function $\mathcal{F}$ (therefore the modified Futaki invariant Fut_V as well) for Fano complete intersections in projective spaces:

Theorem 1.2 ([Tak14-2])**.** Let M be a Fano complete intersection in $\mathbb{C}P^N$, i.e., M is the $(N-s)$-dimensional Fano variety in $\mathbb{C}P^N$ defined by homogeneous polynomials $F_1, \ldots, F_s$ of degree $d_1, \ldots, d_s$ respectively, and $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{i=0}^{N}|z^i|^2\right)$ the Fubini-Study metric of $\mathbb{C}P^N$. We suppose that there exists a constant $m > 0$ such that $m\omega \in c_1(M)$. Let $V \in \mathfrak{sl}(N+1,\mathbb{C})$ be a holomorphic vector field on $\mathbb{C}P^N$ such that $VF_i = \alpha_i F_i$ for some constants α_i $(i = 1, \ldots, s)$. Then we have $m = N + 1 - d_1 - \cdots - d_s$ and the function $\mathcal{F}$ can be written as

$$\mathcal{F}(V) = -\frac{(N-s)!}{d_1 \cdots d_s m^{N-s}} \exp\left(\sum_{i=1}^{s} \alpha_i\right) \int_{\mathbb{C}P^N} \prod_{i=1}^{s}(d_i\omega + d_i\theta_V - \alpha_i)e^{m\theta_V} \cdot e^{m\omega}, \quad (1.1)$$

where $\theta_V := V\log\left(\sum_{i=0}^{N}|z^i|^2\right)$.

From the above theorem, we know that $\mathcal{F}(V)$ can be written as a linear combination of the integrals $I_{0,l} := m^l \int_{\mathbb{C}P^N}(\theta_V)^l e^{m\theta_V}\omega^N$ $(0 \le l \le s)$.

Though we can easily get a method of computing $\mathcal{F}$ using the localization formula for orbifolds in [DT92], our formula (1.1) is still valuable since we need not to assume that M has at worst orbifold singularities. And we also do not require the explicit geometric knowledge of M, V and ω (local coordinates (uniformization), the zero set of V, curvature, etc.). More concretely, in order to apply the localization formula in [DT92] directly to our case, we have to know:

1. The zero set Zero(V) of V, where we assume that Zero(V) consists of disjoint nondegenerate submanifolds $\{Z_i\}$.

2. The values of integrals
$$\int_{Z_i} \frac{e^{m(\omega+\theta_V)}}{\det(L_{i,V}+K_i)},$$
where $L_{i,V}(W) := [V, W]$ denotes an endomorphism and K_i the curvature matrix of the normal bundle of Z_i.

If $s(=\operatorname{codim}(M)) = 1$ and $\dim(Z_i) = 0$, the above integral can be computed by taking local coordinates (or uniformization) around Z_i. However, it is very hard to compute in general.

In the case of Kähler-Einstein metrics, the Futaki invariant of complete intersection was first computed by Lu [Lu99] using the adjunction formula and the Poincare-Lelong formula. Then it was also computed by many mathematicians using different techniques ([PS04], [Hou08] and [AV11]). Our formula (1.1) has in common with Lu's one in that $\mathcal{F}(V)$ is expressed by the degree $d_1, \ldots, d_s$ of defining polynomials of M and the weights $\alpha_1, \ldots, \alpha_s$ of the actions induced by the vector field V. However, we need more knowledge of V to compute the integrals $I_{0,l}$ $(0 \le l \le s)$ (see Section 5.2.3 for more details).

We prove Theorem 1.2 by modifying Lu's approach for Futaki invariant studied in Section 5.1. The author also gave another proof of Theorem 1.2 in quantized settings. First, we define the quantization of the function $\mathcal{F}$ in reference to the quantized modified Futaki invariant introduced by Berman-Nyström [BN14], and then show that this $\mathcal{F}$ coincides with the one defined as an integral invariant if the variety has log-terminal singularities (cf. Proposition 5.12). Thanks to this algebraic formula, we can compute $\mathcal{F}$ by the equivariant Riemann-Roch formula (cf. Lemma 5.16). We study these things in Section 5.2.2. Finally, in Section 5.2.3, the author gave some examples of computing $\mathcal{F}$ for some varieties partly given by [Lu99]. The author gave a new example of a singular cubic surface in $\mathbb{C}P^3$ whose singularity is not log-terminal (cf. Example 5.22).

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2 Preliminaries

Various holomorphic invariants are closely related to holomorphic equivariant cohomology. A famous example is the Futaki invariant. In this section, we first review some basic materials about holomorphic equivariant cohomology.

Then we make a breif review of admissible bundles, which are projective bundles of the form $\mathbb{P}(E_0 \oplus E_\infty) \to S$, where E_0 and E_∞ are projectively flat Hermitian holomorphic vector bundles over a compact Kähler manifold S. Admissible bundles were introduced by Apostolov-Calderbank-Gauduchon-Friedman [ACGT08] to unifiy and generalize previous works on the existence problem of canonical metrics on projective bundles.

2.1 Holomorphic equivariant cohomology

Let M be a complex manifold and G be a Lie group acting holomorphically on M. Denote $\mathfrak{g} := \mathrm{Lie}(G)$ the Lie algebra of G. Then for each $\xi \in \mathfrak{g}$, we denote by $\xi_M^{\mathbb{R}}$, the real holomorphic vector field on M given by

$$\xi_M^{\mathbb{R}}(f)(p) = \frac{d}{dt} f(\exp(-t\xi) \cdot p)\bigg|_{t=0}, f \in C^\infty(M),\ p \in M.$$

and $\xi_M := \frac{1}{2}(\xi_M^{\mathbb{R}} - \sqrt{-1}J\xi_M^{\mathbb{R}})$, the complex holomorphic vector field on M. Let $\mathbb{C}[\mathfrak{g}]$ be the algebra of complex valued polynomial function on $\mathfrak{g}$. We regard each element in $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ as a polynomial function which take values in differential forms. The group G acts on an element $\sigma \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ by

$$(g \cdot \sigma)(\xi) = g \cdot (\sigma(g^{-1} \cdot \xi)), g \in G \text{ and } \xi \in \mathfrak{g}.$$

Let $\mathcal{A}_G(M) = (\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M))^G$ be the space of G-invariant elements in $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$. For $\sigma \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$, we define the bidegree of σ by

$$\mathrm{bideg}(\sigma) = (\deg(\mathrm{P}) + p, \deg(P) + q),$$

where $\sigma = P \otimes \varphi$ ($P \in \mathbb{C}[\mathfrak{g}]$ and $\varphi \in \mathcal{A}^{p,q}(M)$). For instance, $\mathrm{bideg}(\xi) = (1, 1)$. Thus $\mathcal{A}_G(M) = \bigoplus \mathcal{A}_G^{p,q}(M)$ has a structure of a bigraded algebra. We define the equivariant exterior differential $\bar{\partial}_{\mathfrak{g}}$ on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ as

$$(\bar{\partial}_{\mathfrak{g}}\sigma)(\xi) = \bar{\partial}(\sigma(\xi)) + 2\pi\sqrt{-1} i_{\xi_M}(\sigma(\xi)),\ \sigma \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M).$$

Then $\bar{\partial}_{\mathfrak{g}}$ increases by $(0, 1)$ the total bidegree on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$, and preserves $\mathcal{A}_G(M)$. Hence we have a complex $(\mathcal{A}_G(M), \bar{\partial}_{\mathfrak{g}})$.

Definition 2.1. The **holomorphic equivariant cohomology** $H_{\mathfrak{g}}(M)$ of the pair (M, G) is the cohomology of the complex $(\mathcal{A}_G(M), \bar{\partial}_{\mathfrak{g}})$.

Let E be a G-linearized holomorphic vector bundle over M, and $\mathrm{Herm}(E)$ the space of Hermitian metrics on E. The group G acts on $\mathrm{Herm}(E)$ by the formula

$$(g \cdot h)(u, v) = h(g^{-1} \cdot u, g^{-1} \cdot v),\ g \in G \text{ and } u, v \in E.$$

Hence for $\xi \in \mathfrak{g}$, we define the real Lie derivative of $\mathfrak{g}$ on $\mathrm{Herm}(E)$ by

$$L^{\mathbb{R}}_{\xi} h = \frac{d}{dt} \exp(t\xi) \cdot h \Big|_{t=0}$$

and the complex Lie derivative of $\mathfrak{g}$ on $\mathrm{Herm}(M)$ by

$$L_{\xi} h = \frac{1}{2}(L^{\mathbb{R}}_{\xi} h - \sqrt{-1} L^{\mathbb{R}}_{J\xi} h).$$

We can also define the representation of $\mathfrak{g}$ on the space of sections $\Gamma(E)$ in a similar way. Let ∇ be the Chern connection with respect to h, and put

$$\mu_{h,\xi} = L_{\xi} - \nabla_{\xi_M}.$$

Since $\mu_{h,\xi}(fs) = \xi_M f \cdot s + f \cdot L_{\xi} s - \xi_M f \cdot s - f \cdot \nabla_{\xi_M} s = f \cdot \mu_{h,\xi}(s)$ for any $f \in C^{\infty}(M)$ and $s \in \Gamma(E)$, we have $\mu_{h,\xi} \in \Gamma(\mathrm{End}(E))$. Moreover, one can show that

$$L_{\xi} h = -\mu_{h,\xi} \cdot h, \ i_{\xi_M} \theta(h) = -\mu_{h,\xi}, \text{ and } i_{\xi_M} R(h) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \mu_{h,\xi},$$

where $\theta(h) = \partial h \cdot h^{-1}$ is the connection form and $R(h) = \frac{\sqrt{-1}}{2\pi} \bar{\partial}(\partial h \cdot h^{-1})$ is the curvature form with respect to h. Define the equivariant curvature form $R_{\mathfrak{g}}(h)$ by

$$R_{\mathfrak{g}}(h) = R(h) + \mu_{h,\xi}.$$

Then $R_{\mathfrak{g}}(h)$ is $\bar{\partial}_{\mathfrak{g}}$-closed and defines an element in $H^{1,1}_{\mathfrak{g}}(M)$.

Now let us consider the case when $E = L$ is a G-linearized ample line bundle. Let h be a Hermitian metric on L with positive curvature, i.e., $\omega := -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h$ is Kähler.[2] Then $\mu_{h,\xi}$ is a complex valued smooth function on M. Conversely, for a given ω, there exists a Hermitian metric h on L such that $\omega = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h$ up to multiple a constant. Hence the function $\mu_{h,\xi}$ dose not depend on a choice of such a h, which we also denote by $\mu_{\omega,\xi}$.

Definition 2.2. The function $\mu_{h,\xi}$ (resp. $\mu_{\omega,\xi}$) is said to be the **holomorphy potential** of the pair (h, ξ) (resp. (ω, ξ)).

2.2 Admissible bundles

We make a brief review of special projective bundles, called "admissible bundles" (see [ACGT08] for more detail).

Definition 2.3. A projective bundle of the form $M = \mathbb{P}(E_0 \oplus E_{\infty}) \to S$ is called an **admissible bundle** if it satisfies the following conditions:

[2] In this paper, we sometimes say $\omega = -\sqrt{-1}\partial\bar{\partial} \log h \in 2\pi c_1(L)$ is a Kähler form considering the compatibility of notations in other references.

1. S has the universal covering $\tilde{S} = \prod_{a\in\mathcal{A}} S_a$ (for a finite index set $\mathcal{A} \subset \mathbb{N}$) of simply connected Kähler manifolds $(S_a, \pm g_a, \pm\omega_a)$ of complex dimensions d_a with (g_a, ω_a) being pullbacks of tensors on S; here, "$\pm$" means that either $+\omega_a$ or $-\omega_a$ is a Kähler form which defines a Kähler metric denoted by $+g_a$ or $-g_a$ respectively.

2. E_0 and E_∞ are holomorphic projectively-flat Hermitian vector bundles over S of rank d_0+1 and $d_\infty+1$ with $c_1(E_\infty)/\mathrm{rank}E_\infty - c_1(E_0)/\mathrm{rank}E_0 = [\omega_S/2\pi]$ and $\omega_S = \sum_{a\in\mathcal{A}} \omega_a$.

Here the second condition means that we can choose Hermitian metrics on E_0 and E_∞ whose Chern connections have tracelike curvatures $\Omega_0\otimes\mathrm{Id}_{E_0}$ and $\Omega_\infty\otimes\mathrm{Id}_{E_\infty}$ satisfying $\Omega_\infty - \Omega_0 = \sum_{a\in\mathcal{A}} \omega_a$.

Let M be an admissible bundle. We define several notations and give some remarks that we will use later:

- we set the index set $\hat{\mathcal{A}} := \{a \in \mathbb{N}\cup\{0,\infty\} | d_a > 0\}$.
- $e_0 = \mathbb{P}(E_0 \oplus 0)$ (resp. $e_\infty = \mathbb{P}(0\oplus E_\infty)$) denotes a subbundle of M. Then e_0 and e_∞ are disjoint submanifolds of M.
- $\mathbb{P}(E_0) \to S$ (resp. $\mathbb{P}(E_\infty) \to S$) is equipped with the fiberwise Fubini-Study metric with the scalar curvature $d_0(d_0+1)$ (resp. $d_\infty(d_\infty+1)$), which is denoted by (g_0, ω_0) (resp. $(-g_\infty, -\omega_\infty)$).
- Let $\hat{M}$ be the blow-up of M along the set $e_0 \cup e_\infty$, and set $\hat{S} = \mathbb{P}(E_0)\times_S \mathbb{P}(E_\infty) \to S$. Then $\hat{M} \to \hat{S}$ is a $\mathbb{C}P^1$-bundle (cf. Figure 1).
- We define a $U(1)$-action on M by the canonical $U(1)$-action on E_0. Then the Hermitian structures of E_0 and E_∞ induce the (fiberwise) moment map $z : M \to [-1,1]$ of this $U(1)$-action with critical sets $z^{-1}(1) = e_0$ and $z^{-1}(-1) = e_\infty$(we will see the explicit construction of z at the beginning of Section 2.2.2).
- K denotes the infinitesimal generator of the $U(1)$-action on M.
- $\hat{e}_0$ (resp. $\hat{e}_\infty$) denotes the exceptional divisor corresponding to the submanifold e_0 (resp. e_∞).
- Set $M^0 = M\backslash(e_0\cup e_\infty)$. Then $M^0 \to \hat{S}$ is a $\mathbb{C}^*$-bundle, i.e., we have an isomorphism $M^0/\mathbb{C}^* \simeq \hat{S}$, where we remark that $U(1)$ acts on M^0 freely and this action can be extended to the corresponding $\mathbb{C}^*$-action on M^0 (cf. Figure 1).

2.2.1 Admissible Kähler classes and metrics

Now we only deal with special Kähler metrics that have nice properties. Before discussing this, we introduce the Kähler class to which they belong.

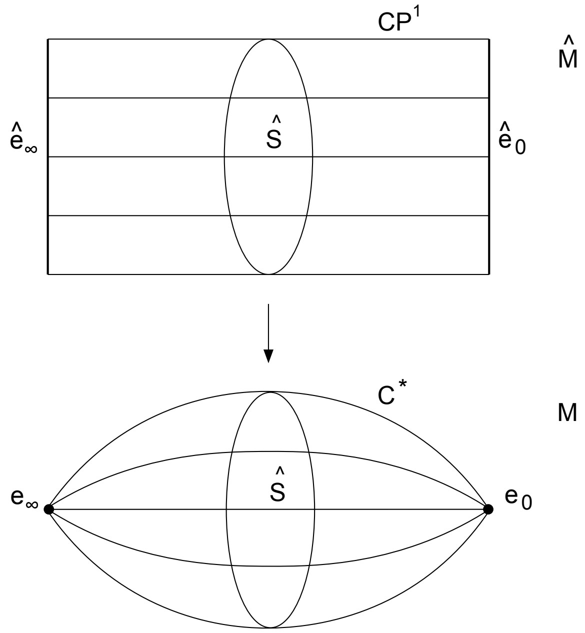

Figure 1: The blow up $\hat{M} \to M$

Definition 2.4. A Kähler class Ω on M is called admissible if there are real constants x_a, with $x_0 = 1$ and $x_\infty = -1$, such that the pullback of Ω to $\hat{M}$ has the form

$$\Omega = \sum_{a \in \hat{\mathcal{A}}} [\omega_a]/x_a + \hat{\Xi}, \tag{2.1}$$

where $\hat{\Xi}$ is the Poincaré dual to $2\pi[\hat{e}_0 + \hat{e}_\infty]$.

We can see that any admissible class Ω has the form

$$\Omega = \sum_{a \in \mathcal{A}} [\omega_a]/x_a + \Xi, \tag{2.2}$$

where the pullback of Ξ to $\hat{M}$ is $[\omega_0] - [\omega_\infty] + \hat{\Xi}$, i.e., the cohomology class $[\omega_0] - [\omega_\infty] + \hat{\Xi}$ vanishes along the fiber $\hat{e}_0 \to e_0$ and $\hat{e}_\infty \to e_\infty$.

Remark 2.5. We call the parameters $\{x_a\}$ the **admissible data** of Ω. Since Ω is Kähler, the admissible data $\{x_a\}$ satisfies the condition:

1. $0 < |x_a| < 1$ for all $a \in \mathcal{A}$
2. x_a has the same sign as g_a.

In this paper, we also assume that $\pm g_a$ has constant scalar curvature $\mathrm{Scal}(\pm g_a) = \pm d_a s_a$, where s_a are constants defined in [ACGT08, Section 1.2].

Definition 2.6. Let Ω be an admissible class with the admissible data $\{x_a\}$. An **admissible Kähler metric** g is the Kähler metric on M which has the form

$$g = \sum_{a\in\hat{\mathcal{A}}} \frac{1+x_a z}{x_a} g_a + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2, \ \omega = \sum_{a\in\hat{\mathcal{A}}} \frac{1+x_a z}{x_a}\omega_a + dz\wedge\theta \tag{2.3}$$

on M^0, where θ is the connection 1-form ($\theta(K) = 1$) with the curvature $d\theta = \sum_{a\in\hat{\mathcal{A}}}\omega_a$, and Θ is a smooth function on $[-1,1]$ satisfying

$$\Theta > 0 \text{ on } (-1,1),\ \Theta(\pm 1) = 0 \ \text{ and } \ \Theta'(\pm 1) = \mp 2. \tag{2.4}$$

The form ω defined in (2.3) is a symplectic form, and the compatible complex structure J of (g,ω) is given by the pullback of the base complex structure and the relation $Jdz = \Theta\theta$.

Remark 2.7. Using the relation $d\theta = \sum_{a\in\hat{\mathcal{A}}}\omega_a$, we can check that ω is closed and $\Omega = [\omega]$. Hence g is a Kähler metric whose Kähler form ω belongs to Ω.

Remark 2.8. The defining equation (2.3) is motivated by the representation of the canonical admissible metric g_c in polar coordinates. In this case, the corresponding function Θ_c is given by $\Theta_c(z) = 1 - z^2$ (cf. Lemma 2.11).

The condition (2.4) is the necessary and sufficient condition for extending a metric g on M^0 which has the form (2.3) to a smooth metric defined on M (cf. Section 2.2.3). In order to compute simply, we sometimes use the function

$$F(z) = \Theta(z)\cdot p_c(z) \tag{2.5}$$

instead of $\Theta(z)$, where $p_c(z) = \prod_{a\in\hat{\mathcal{A}}}(1+x_a z)^{d_a}$ is a polynomial of z. Then the equation (2.4) forces F to fulfill the condition

$$F > 0 \text{ on } (-1,1),\ F(\pm 1) = 0 \ \text{ and } \ F'(\pm 1) = \mp 2p_c(\pm 1). \tag{2.6}$$

Remark 2.9. Generally, this is only necessary condition for F, i.e., we can not restore Θ from F satisfying (2.6). However, it is possible if g is extremal or Generalized Quasi Einstein (GQE)[3] (cf. [ACGT08, Section 2.4] and [MT11, Section 4]).

Remark 2.10. In the original paper [ACGT08, Section 1.3, Section 1.4], admissible classes and admissible metrics are defined by (2.2) and (2.3) "up to scale" respectively because the condition of canonical metrics (cscK, extremal, GQE, etc.) are preserved under the scaling of metrics. In this paper, the argument of scaling metrics sometimes becomes essential. This is why we define them not up to scale.

[3] These are certain kinds of generalizations of constant scalar curvature Kähler metrics.

2.2.2 Constructing canonical admissible Kähler metrics

We start with any (not necessarily Kähler) cohomology class Ω defined by the equation (2.2). Now we can show that the condition mentioned in Remark 2.5 is the necessary and sufficient condition for Ω to be Kähler (and hence, be admissible). We can prove this by constructing the "canonical admissible metric" g_c and its symplectic form ω_c belonging to Ω: Let ρ_0 (resp. ρ_∞) be a $U(1)$-action on E_0 (resp. E_∞) defined by scalar multiplication. Then the moment map of this action is given by $z_0 = \frac{1}{2}r_0^2$ (resp. $z_\infty = \frac{1}{2}r_\infty^2$), where we denote by r_0 (resp. r_∞) the norm function induced by a Hermitian metric on E_0 (resp. E_∞). We also denote by K_0 (resp. K_∞) the infinitesimal generator of ρ_0 (resp. ρ_∞). Let us consider the diagonal $U(1)$-action $\rho := \rho_0 \oplus \rho_\infty$ on $E_0 \oplus E_\infty$ with the moment map $z_0 + z_\infty$. Since ρ is free restricted to the moment level $z_0 + z_\infty = 2$, the restriction of the Hermitian metric on this level set descends to the fiberwise Fubini-Study metric on the quotient manifold M, which we denote by $(g_{M/S}, \omega_{M/S})$. Since ρ_0 commutes with ρ and preserves the Hermitian structure, ρ_0 also descends to a fiberwise Hamiltonian $U(1)$-action $\tilde{\rho}_0$ on M. From general results in symplectic geometry, the moment map of this action is given by $z = z_0 - 1 = 1 - z_\infty$.

We extend $(g_{M/S}, \omega_{M/S})$ to a tensor on M by requiring that the restriction to the horizontal distribution on M is zero. Hence $(g_{M/S}, \omega_{M/S})$ is semi-positive. In order to get a (positive definite) metric on M, we set

$$g_c = \sum_{a \in \mathcal{A}} \frac{1 + x_a z}{x_a} g_a + g_{M/S},\ \omega_c = \sum_{a \in \mathcal{A}} \frac{1 + x_a z}{x_a} \omega_a + \omega_{M/S}.$$

Then (g_c, ω_c) is a Kähler metric with respect to the canonical complex structure J_c on M. Moreover, we have:

Lemma 2.11 ([ACGT08], Lemma 1)**.** For any admissible data $\{x_a\}$, the corresponding canonical Kähler metric on M is of the form (2.3), where

$$\Theta(z) = \Theta_c(z) = 1 - z^2.$$

Proof. The inverse image in $E_0 \oplus E_\infty$ of M^0 may be viewed as an open subset of $\mathcal{O}(-1)_{E_0} \oplus \mathcal{O}(-1)_{E_\infty}$. Then (g_c, ω_c) is the Kähler quotient of the metric defined at the moment level $z_0 + z_\infty = 2$:

$$\sum_{a \in \hat{\mathcal{A}}} \frac{(1 + x_a)z_0 + (1 - x_a)z_\infty}{2x_a} g_a + \frac{dz_0^2}{2z_0} + \frac{dz_\infty^2}{2z_\infty} + 2z_0\theta_0^2 + 2z_\infty\theta_\infty^2, \qquad (2.7)$$

where θ_0, θ_∞ are connection 1-forms for the $U(1)$-line bundles $\mathcal{O}(-1)_{E_0}$, $\mathcal{O}(-1)_{E_\infty}$, with $\theta_0(K_0) = 1$, $\theta_\infty(K_\infty) = 1$, $d\theta_0 = -\omega_0 + \Omega_0$, $d\theta_\infty = \omega_\infty + \Omega_\infty$.[4] Here we take notice of two points: (1) The fiberwise Fubini-Study metric g_0 (resp. g_∞) is normalized to have scalar curvature $d_0(d_0+1)$ (resp. $d_\infty(d_\infty+1)$). (2) We extend

[4]The standard metric of $\mathbb{C}^{n+1}$ can be written as the form $\frac{dz^2}{2z} + 2z\theta^2$, where $2z$ is the square of the norm function and θ a connection of the $U(1)$-line bundle $\mathcal{O}(-1) \to \mathbb{C}P^n$ such that $-d\theta$ is the Fubini-Study metric of $\mathbb{C}P^n$.

$(g_{M/S}, \omega_{M/S})$ to a tensor on M by requiring that the restriction to the horizontal distribution on M is zero. We also remark that Ω_0 (resp. Ω_∞) is degenerate along the fiber $\mathbb{P}(E_0) \to S$ (resp. $\mathbb{P}(E_\infty) \to S$).

Put $\hat{\mathcal{L}} := \mathcal{O}(1)_{E_0} \otimes \mathcal{O}(-1)_{E_\infty}$ and regard M^0 as an open subset of the blow-up $\hat{M} = \mathbb{P}(\mathcal{O} \oplus \hat{\mathcal{L}})$. Since ρ is generated by $K_0 + K_\infty$, the form $\theta_\infty - \theta_0$ is invariant under ρ, which defines the connection θ on $M^0 \to \hat{S}$ and corresponding line bundle $\hat{\mathcal{L}}$ such that $d\theta = d\theta_\infty - d\theta_0 = \omega_{\hat{S}}$ (where, we used $\Omega_\infty - \Omega_0 = \sum_{a \in \mathcal{A}} \omega_a$). Using the relation $z_0 = 1+z$ and $z_\infty = 1-z$ and taking the quotient of (2.7) yields (g_c, ω_c) is an admissible Kähler metric corresponding to the function $\Theta(z) = \Theta_c(z) = 1 - z^2$. $\square$

2.2.3 Symplectic potentials

Let M be an admissible bundle. Since M is a $\mathbb{C}^*$-bundle over $\hat{S}$, we can think M^0 as a family of toric manifolds parametrized by a compact Kähler manifold $\hat{S}$. Thus we can apply some methods in toric geometry to each fiber of M. In particular, we will mention symplectic potentials, which is important because the Kähler potential of admissible metrics are represented by its Legendre transform.

We know that admissible metrics with a fixed symplectic form ω define a deformation family of complex structures. Now we will show that this can be regarded as the same complex structure J_c via $U(1)$-equivariant fiber-preserving diffeomorphisms.

Definition 2.12. A function $u \in C^0([-1,1])$ is called the **symplectic potential** of an admissible Kähler metric $\Theta(z)$ if $u''(z) = 1/\Theta(z)$, $u(\pm 1) = 0$ and $u - u_c$ is smooth on $[-1,1]$, where u_c is the canonical symplectic potential defined by

$$u_c(z) = \frac{1}{2}\left\{(1-z)\log(1-z) + (1+z)\log(1+z) - 2\log 2\right\}. \tag{2.8}$$

The symplectic potential u is uniquely determined by the above condition. If we put

$$y = u'(z) \ \text{ and } \ h(y) = -u(z) + yz, \tag{2.9}$$

then the direct computation shows that $d^c_J y = \theta$ and $dd^c_J h(y) = \omega - \sum_{a \in \hat{\mathcal{A}}} \omega_a / x_a$ on M^0. Let $t : M^0 \to \mathbb{R}/2\pi\mathbb{Z}$ be an angle function (locally defined up to an additive constant) and y_c, h_c the functions corresponding to u_c defined by (2.9). Since $\exp(y + \sqrt{-1}t)$ and $\exp(y_c + \sqrt{-1}t)$ give $\mathbb{C}^*$-coordinates on the fibers, there exists a unique $U(1)$-equivariant fiber-preserving diffeomorphism Ψ of M^0 such that

$$\Psi^* y = y_c, \ \Psi^* t = t \ \text{ and hence } \ \Psi^* J = J_c. \tag{2.10}$$

As J_c and J are integrable complex structures, Ψ extends to a $U(1)$-equivariant diffeomorphism of M leaving fixed any point on $e_0 \cup e_\infty$. Hence $\Psi^*\omega$ is a Kähler form on M with respect to J_c. As $\Psi : (M, J_c) \to (M, J)$ is biholomorphic by the definition of Ψ, we obtain

$$dd^c_{J_c} h(y_c) = dd^c_{J_c} h(\Psi^* y) = \Psi^* dd^c_J h(y) = \Psi^*\omega - \sum_{a \in \hat{\mathcal{A}}} \omega_a / x_a,$$

$$\Psi^*\omega - \omega = dd^c_{J_c}(h(y_c) - h_c(y_c))$$

on M^0.

Lemma 2.13 ([ACGT08], Lemma 3)**.** $h(y_c) - h_c(y_c)$ *is extended smoothly on* M.

Proof. Since Ψ is a diffeomorphism with $\Psi^*y = y_c$, the statement holds if and only if $h(y) - h_c(y)$ is smooth on M. By the definition of symplectic potentials, we know that $h(y) - h_c(y_c) = -(u(z) - u_c(z)) + z(u'(z) - u'_c(z))$ is smooth on M. Hence we have only to show $h_c(y) - h_c(y_c)$ is smooth on M. By (2.8), we can compute $h_c(y) - h_c(y_c)$ as

$$h_c(y) - h_c(y_c) = -\frac{1}{2}\left(\log\left(\frac{1-\tilde{z}}{1-z}\right) + \log\left(\frac{1+\tilde{z}}{1+z}\right)\right),$$

where $\tilde{z} := \Psi^*z$ is the moment map of $\tilde{\omega} := \Psi^*\omega$. Since Ψ is $U(1)$-equivariant and fixes the critical set $e_0 \cup e_\infty$, we have $\tilde{z}(\pm 1) = \pm 1$ regarded as a function of z. Using the formula $d\tilde{z} = z' \cdot dz$, we have

$$\nabla^2\tilde{z} = z'' dz^2 + z' \cdot \nabla^2 z.$$

Since z and $\tilde{z}$ are moment maps of the same $U(1)$-action $\tilde{\rho}_0$ on M, they are Morse-Bott functions with the same critical manifolds e_0 and e_∞ (i.e., the Hessians $\nabla^2 z$ and $\nabla^2\tilde{z}$ are non-degenerate in the normal direction). Hence we have $\tilde{z}'(\pm 1) \neq 0$ and thus $h(y_c) - h_c(y_c)$ is smooth on M. $\square$

Let $\mathcal{K}^{\mathrm{adm}}_\omega$ be the moduli space of admissible metrics with a fixed symplectic form ω. Then we obtain an inclusion map

$$\mathcal{K}^{\mathrm{adm}}_\omega \hookrightarrow \{\text{Kähler form in } (\Omega, J_c)\}$$

defined by $\Theta \mapsto \Psi^*\omega$, where (Ω, J_c) denotes the Dolbeault cohomology class with respect to J_c.

3 Geometric invariant theory

Stability is a condition on a orbit of a group action that should insure that the moduli space of stable orbits be a well-behaved space. Mumford-Fogarty-Kirwan [MFK94] studied an algebraic variety M on which an algebraic group G acts and defined a variety M/G as a projective scheme of the graded ring consisting of all G-invariant functions over M. The main difficulty in this approach is to understand the projection map M to M/G. To define this map, we eliminate certain "bad" orbits and consider only "semistable" ones.

This construction can be also interpreted from a view point of symlectic geometry. Assume that M is a Kähler manifold with a Kähler form ω and a compact group K acts on M by holomorphic isometry. Then Kempf-Ness theorem [KN79] tells us that a $G(:= K^{\mathbb{C}})$-orbit is semistable if and only if its closure contains a zero of the moment map. This picture enables us to generalize this problem for infinite dimensional settings.

In geometry, many important problems can be reduced to a PDE of the form

$$\mu(x) = 0$$

where μ denotes the moment map and x runs over a G-orbit in an infinite dimensional symplectic manifold (see, for instance [DK97]). Donaldson [Don97] showed that the equation of constant scalar curvature Kähler (cscK) metric can be written as this form, which gives us the formal aspects of stability.

3.1 Finite dimensional GIT

3.1.1 GIT stability

Let G be a reductive algebraic group with a finite linear representation V.

Definition 3.1. Let $x \in \mathbb{P}(V)$, then

1. x is semistable if there exists a non-constant G-invariant homogenous polynomial f such that $f(x) \neq 0$.
2. x is polystable if x is semistable and the orbit $G \cdot x$ is closed in $\mathbb{P}(V)_{\mathrm{ss}}$, where we denote by $\mathbb{P}(V)_{\mathrm{ss}}$ the set of all semistable points.
3. x is stable if x is polystable and has discrete stabilizer.

We say that x is unstable if x is not semistable.

In particular, we have the following relations between three notions of stability

$$\text{stable} \Rightarrow \text{polystable} \Rightarrow \text{semistable}.$$

Since the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}(V)$ is just the blow-up at the origin, the G-action defines a G-linearization of the line bundle $\mathcal{O}(-1) \to \mathbb{P}(V)$. We often use the following alternative interpretation of stability:

Proposition 3.2. Let $x \in \mathbb{P}(V)$ and $\hat{x} \in \mathcal{O}(-1)$ a non-zero lift of x, then

1. x is semistable if and only if the closure of the orbit $G \cdot \hat{x}$ dose not intersect the zero section of $\mathcal{O}(-1)$.

2. x is polystable if and only if the orbit $G \cdot \hat{x}$ is closed in $\mathcal{O}(-1)$.

Let A be the graded ring of all polynomials over V and A^G the algebra of all G-invariant elements in A. We define the quotient space $\mathbb{P}(V)//G$ as a projective variety $\mathrm{Proj}A^G$. Then the inclusion $A^G \hookrightarrow A$ induces a rational map $\mathbb{P}(V) \dashrightarrow \mathbb{P}(V)//G$. This map is not defined at unstable points. But restricting attention to the set of all semistable elements $\mathbb{P}(V)_{\mathrm{ss}}$, we have a map $\mathbb{P}(V)_{\mathrm{ss}} \to \mathbb{P}(V)//G$ and thus $\mathbb{P}(V)//G$ can be interpreted as the quotient of the set of all semistable points.

We want to generalize this picture for any polarized manifold (M, L) with a group action, where we assume that G is a reductive algebraic group acting on M homomorphically and can be lifted to a holomorphic action on L.

Definition 3.3. Let $x \in M$ and $\hat{x} \in L_x$ a non-zero lift of x, then

1. x is semistable if the closure of the orbit $G \cdot \hat{x}$ dose not intersect the zero section of L.

2. x is polystable if the orbit $G \cdot \hat{x}$ is closed in L.

We also call a G-orbit is (poly/semi)stable if a point in the orbit is. This definition dose not depend on a choice of point in the orbit.

To see the relation with Proposition 3.2, we should consider the Kodaira embedding $M \hookrightarrow \mathbb{P}(H^0(M, L))^*$. For simplicity, we assume that L is very ample. Then the G-action on L induces an action on $H^0(M, L)$ and L is just the restriction of the hyperplane bundle $\mathcal{O}(1)|_M$.

Let M_{ss} be the set of all semistable elements in M and set $M//G := \mathrm{Proj}A^G$, where $A := \bigoplus_{i=0}^{\infty} H^0(M, L^k)$ is the graded ring of all functions over M. Then we have a map $M_{\mathrm{ss}} \to M//G$.

Let $\lambda : \mathbb{C}^* \to G$ be a nontrivial one-parameter subgroup and $x \in M$. Since M is projective, we can define

$$x_0 = \lim_{t \to 0} \lambda(t) x.$$

Since x_0 is a fixed point of the G-action, we obtain a $\mathbb{C}^*$-action on the fiber L_{x_0}, which has a weight $-w_\lambda(x)$. Or equivalently, $w_\lambda(x)$ is the unique integer such that the limit

$$\lim_{t \to 0} t^{w_\lambda(x)} \lambda(t) \hat{x}$$

exists in L and is not zero, where $\hat{x}$ is a non-zero lift of x (see Figure 2).

Proposition 3.4 (Hilbert-Mumford criterion)**.** Let $x \in M$, then

1. x is semistable if and only if $w_\lambda(x) \geq 0$ for all λ.

2. x is polystable if and only if $w_\lambda(x) \geq 0$ for all λ and equality holds if and only if λ fixes x.

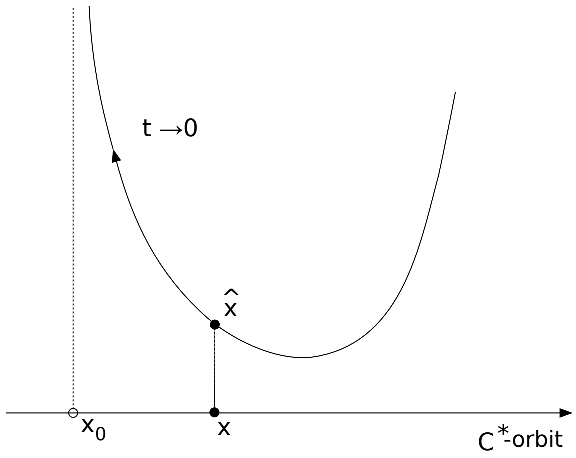

Figure 2: A $\mathbb{C}^*$-orbit of a polystable point x

3. x is stable if and only if $w_\lambda(x) > 0$ for all λ.

By virtue of Hilbert-Mumford criterion, we have only to consider any $\mathbb{C}^*$-orbit to check the stability.

3.1.2 Kempf-Ness theorem

Let (M, L) be an n-dimensional polarized manifold with a Kähler form $\omega \in c_1(L)$ and $\mathfrak{h}$ the Lie algebra consisting of all holomorphic vector fields on M. We assume that a compact group K acts on (M, ω) with a moment map μ by holomorphic isometry.

First, we see that a choice of moment map is equivalent to a choice of linearization of L. Let $\mathfrak{h}_1$ be an ideal of $\mathfrak{h}$ consisting of all holomorphic vector fields on M whose zero set is non-empty.

Lemma 3.5 ([Kob95], Lemma 3, p.109)**.** The space $\mathfrak{h}_1$ coincides with the space of holomorphic vector fields that can be lifted to L.

Proof. Let h be a Hermitian metric on L such that $\omega = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$ and θ its connection form on the associated principal bundle $L^\circ := L - \{\text{zero section}\}$. We fix a non-vanishing holomorphic section σ of L° on a small open set $U \subset M$. Then we have a natural identification $L^\circ|_U \simeq U \times \mathbb{C}^*$ with a coordinate system $(z_1, \ldots, z_n, t)$ defined by σ. We denote by θ_U the pullback of the connection form θ by σ, then

$$\theta_U = \partial(\log ||\sigma||_h^2), \ \ \theta = \theta_U + \frac{1}{t}dt = \partial(\log ||\sigma||_h^2 t\bar{t}), \ \ \omega = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\theta_U,$$

where $\frac{1}{t}dt$ is the Maurer-Cartan form along the fibers. Let V be a holomorphic vector field on M belonging to $\mathfrak{h}_1$. Then by [Kob95, Theorem 4.4], there exists a function f on M such that

$$i_V\omega = \frac{\sqrt{-1}}{2\pi}\bar{\partial}f.$$

We lift V to a vector field $\hat{V}$ on L° of type $(1,0)$ such that

$$-f = \theta(\hat{V}),$$

which determines $\hat{V}$ uniquely. If we write

$$\hat{V} = \sum_i f_{\bar{i}} \frac{\partial}{\partial z^i} + \tau \frac{\partial}{\partial t},$$

then

$$-f = \theta(\hat{V}) = \theta_U(V) + \frac{\tau}{t}.$$

Applying $\bar{\partial}$, we obtain

$$\begin{aligned} -\bar{\partial} f &= \bar{\partial}(i_V \theta_U) + \bar{\partial}\left(\frac{\tau}{t}\right) = -i_V(\bar{\partial}\theta_U) + \bar{\partial}\left(\frac{\tau}{t}\right) \\ &= 2\pi\sqrt{-1} i_V \omega + \bar{\partial}\left(\frac{\tau}{t}\right). \end{aligned}$$

Hence $\bar{\partial}\left(\frac{\tau}{t}\right) = 0$. Since t is holomorphic, so is τ. Thus we have proved $\hat{V}$ is holomorphic. Hence V can be also lifted to a holomorphic vector field on L. This completes the proof of Lemma 3.5. $\square$

Let G be a complexification of K and $\mathfrak{k}$, $\mathfrak{g}$ the Lie algebra of K, G respectively. By Lemma 3.5, the holomorphic vector field ξ_M defined by $\xi \in \mathfrak{k}$ can be lifted to a holomorphic vector field on L by

$$\hat{\xi_M} = \tilde{\xi_M} + \sqrt{-1}\langle \mu, \xi \rangle \underline{t},$$

where $\underline{t} := t\frac{d}{dt}$ denotes the canonical vector field on L. Thus we obtain an infinitesimal action of $\mathfrak{k}$ and its complexified action of $\mathfrak{g}$ on L. We suppose that these infinitesimal action can be integrated to group actions on L.

Theorem 3.6 (Kempf-Ness, [KN79]).

1. A G-orbit is semistable if and only if its closure contains the zero of the moment map. Such a zero is called a "de-stabilizer" of the original G-orbit. The de-stabilizers all lie in the unique polystable orbit in the closure of the original G-orbit.
2. A G-orbit is polystable if and only if it contains a zero of the moment map. The zeros within it form a unique K-orbit.

Outline of the proof. We fix a point $x \in M$, its non-zero lift $\hat{x} \in L_x$ and a K-invariant Hermitian metric on L. The key idea is considering the following function, called **Kempf-Ness function**:

$$\Phi\colon G/K \ni g \mapsto \log ||g \cdot \hat{x}|| \in \mathbb{R}$$

Actually, this map is well-defined since the norm $||\cdot||$ is invariant under the action of K. Hence we have only to consider the $\sqrt{-1}\mathfrak{k}$-direction. For $\xi \in \mathfrak{k}$, we set $\hat{x}_t := \exp(\sqrt{-1}t\xi)\cdot\hat{x}$, $x_t := \exp(\sqrt{-1}t\xi)\cdot x$ and define

$$f(t) := \Phi(\exp(\sqrt{-1}t\xi)) = \log ||\hat{x}_t||, \quad t \in \mathbb{R}.$$

Then

$$f'(t) = \frac{1}{2||\hat{x}_t||^2}\frac{d}{dt}||\hat{x}_t||^2 = -\langle \mu(x_t), \xi \rangle, \tag{3.1}$$

$$f''(t) = d\langle \mu(x_t), \xi \rangle (J\xi_M^{\mathbb{R}}) = 2\pi\omega(\xi_M^{\mathbb{R}}, J\xi_M^{\mathbb{R}}) = ||\xi_M^{\mathbb{R}}(x_t)||^2 \geq 0. \tag{3.2}$$

This yields that Φ is convex along geodesics and g is a critical point of Φ if and only if $\mu(g \cdot x) = 0$. Regarding Φ as a function on the orbit $G \cdot \hat{x}$, we obtain

$$\begin{aligned} x \text{ is polystable} \quad &\Leftrightarrow \quad G \cdot \hat{x} \text{ is closed in } L \\ &\Leftrightarrow \quad \Phi \text{ has a critical point} \\ &\Leftrightarrow \quad G \cdot x \text{ contains the zero of the moment map.} \end{aligned}$$

□

In particular, we have an isomorphism as a set

$$M_{\mathrm{ps}}/G \simeq \mu^{-1}(0)/K, \tag{3.3}$$

where M_{ps} is the set of all polystable points in M, and the RHS is a symplectic quotient at moment level 0. Let us show a simple example.

Example 3.7. This example is a special case of [Szèk06, Example1.2.1]. We define a $U(1)$-action on $\mathbb{C}^2$ by $t \cdot (z_0, z_1) := (tz_0, t^{-1}z_1)$ ($t \in \mathbb{C}; |t| = 1$), which naturally descends to a $U(1)$-action on $\mathbb{C}P^1$. Since $\mathcal{O}(-1)$ is just the blow-up of $\mathbb{C}^2$ at the origin, $U(1)$ also acts on $\mathcal{O}(-1)$ and its dual $\mathcal{O}(1)$. Then the compatible moment map for the action on $\mathcal{O}(1) \to \mathbb{C}P^1$ with respect to the Fubini-Study metric of $\mathbb{C}P^1$ is given by

$$\mu([z_0, z_1])a = \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2}a, \ [z_0, z_1] \in \mathbb{C}P^1, \ a \in \mathbb{R}(\simeq \mathfrak{u}(1)).$$

Hence $\mu^{-1}(0)$ is the "equator" of $\mathbb{C}P^1$. Complexified orbits ($\mathbb{C}^*$-orbits) of this action are $O_1 := [1, 0]$, $O_2 := \{[z_0, z_1] | z_0 \neq 0 \text{ and } z_1 \neq 0\}$ and $O_3 := [0, 1]$. By Kempf-Ness theorem, O_2 is polystable, and O_1, O_3 are unstable. We can also show this by Hilbert-Mumford criterion. For instance, if we set $x_t := [tz_0, t^{-1}z_1]$ for $[z_0, z_1] \in O_2$, we have $\lim_{t \to 0} x_t = [0, 1]$. Then the induced action on $\mathcal{O}(-1)|_{[0,1]}$ is given by $t \cdot (0, 1) = (0, t^{-1})$, which has positive weight, and so is the induced action on $\mathcal{O}(1)|_{[0,1]}$. Similarly, we can show that $x_t := [t^{-1}z_0, tz_1]$ tends to $[1, 0]$ as $t \to 0$ and the weight of the induced action on $\mathcal{O}(1)|_{[1,0]}$ is positive. Hence O_2 is polystable. In this case, the isomorphism (3.3) is

$$O_2/\mathbb{C}^* \simeq (\text{the equator of } \mathbb{C}P^1)/U(1) \simeq \{\text{a point}\}.$$

3.2 Infinite dimensional GIT

Let (M, ω) be a $2n$-dimensional symplectic manifold. We say that an almost complex structure J is ω-compatible if a Riemann metric

$$g_J(u, v) := \omega(u, Jv)$$

is almost Kähler. Let $\mathcal{J}$ be the space of all ω-compatible almost complex structures. Then $\mathcal{J}$ is an infinite dimensional manifold, and for each $J \in \mathcal{J}$, the tangent space at J is given by

$$T_J\mathcal{J} = \{A \in \mathrm{End}(TM) | AJ + JA = 0, \omega(u, Av) = \omega(v, Au) = 0\},$$

where we used that $J^2 = -\mathrm{Id}_{TM}$ and $\omega(Ju, Jv) = \omega(u, v)$. The symplectic form ω gives a natural identification between the tangent bundle and the cotangent bundle, i.e., for $A \in T_J\mathcal{J}$, we define

$$\mu_A(u, v) = \omega(u, Av). \tag{3.4}$$

Then one can check that μ_A is a symmetric anti J-invariant section of $T^*M \otimes T^*M$. Conversely, any symmetric anti J-invariant section of $T^*M \otimes T^*M$ gives rise to an element in $T_J\mathcal{J}$ via the relation (3.4). Since $\mathcal{J}$ is the space of smooth sections of an $Sp(2n)/U(n)$-bundle over M, it carries a natural Kähler structure. Actually, for $\mu \in T_J\mathcal{J}$, we define a complex structure $\hat{J}$ on $\mathcal{J}$ by

$$(\hat{J}\mu)(u, v) := -\mu(Ju, v)$$

and a Kähler metric on $\mathcal{J}$ as an L^2 inner product

$$(\mu, \nu)_J := \int_M g_J(\mu, \nu)\omega^n, \ J \in \mathcal{J}, \ \mu, \nu \in T_J\mathcal{J}.$$

We denote by $\mathcal{J}_{\mathrm{int}}$ a subvariety of $\mathcal{J}$ consisting of all ω-compatible integrable complex structures on M. Let $K := \mathrm{Symp}^0(M, \omega)$ be the identity component of the group of symplectic automorphisms. For simplicity, we assume that $H^1(M) = 0$. Then K is isomorphic to the group of Hamiltonian diffeomorphisms, and the Lie algebra $\mathfrak{k}$ of K consists of all Hamiltonian vector fields on M. We identify $\mathfrak{k}$ with the space of all Hamiltonians with mean value 0, which we denote by $C_0^\infty(M)$ (here, the Lie bracket over $C_0^\infty(M)$ is given by the standard poisson bracket for functions).

For $f, g \in C_0^\infty(M)$, we define a K-invariant inner product on $C_0^\infty(M)$ as an L^2 inner product

$$(f, g) = \int_M fg\omega^n. \tag{3.5}$$

The group K acts on the space $\mathcal{J}$ (as a pullback of a tensor), and this action preserves holomorphic and symplectic structure of $\mathcal{J}$. Donaldson [Don97] showed that

Proposition 3.8. Via the inner product (3.5), the moment map of the action of K on $\mathcal{J}_{\mathrm{int}}$ is given by

$$\mathcal{J}_{\mathrm{int}} \ni J \mapsto \mathrm{Scal}(J) - \overline{\mathrm{Scal}} \in C_0^\infty(M),$$

where $\mathrm{Scal}(J)$ denote the scalar curvature of g_J and $\overline{\mathrm{Scal}}$ the average of scalar curvature, which is independent of a choice of $J \in \mathcal{J}_{\mathrm{int}}$.

Next, we consider what a $G(:= K^{\mathbb{C}})$-orbit is. Since K is infinite dimensional, there may not exist a genuine complexification G. However, we can still define a foliation of $\mathcal{J}$, and we think of the leaves of this foliation as G-orbits. For a fixed $J_0 \in \mathcal{J}_{\mathrm{int}}$, the complexified orbit $G \cdot J_0$ is given by

$$G \cdot J_0 = \{J \in \mathcal{J}_{\mathrm{int}} | {}^\exists \varphi \in \mathrm{Diff}_0(M) \text{ such that } \varphi^* J = J_0\},$$

where we remark that for each $J \in G \cdot J_0$, a choice of φ is unique modulo the action of automorphisms with respect to J_0. Let $\mathcal{H}_0$ be the space of Kähler forms in $([\omega], J_0)$, where $([\omega], J_0)$ denotes the Dolbeault cohomology class with respect to J_0. For simplicity, we assume that the stabilizer of J_0 is discrete. Then there exists a unique $\varphi_J \in \mathrm{Diff}_0(M)$ such that $(\varphi_J)^* J = J_0$, and we can define a map $G \cdot J_0 \to \mathcal{H}_0$ with this φ_J as

$$G \cdot J_0 \ni J \mapsto (\varphi_J)^* \omega \in \mathcal{H}_0.$$

Actually, the infinitesimal action of a Hamiltonian $f \in C_0^\infty(M)$ on ω is

$$L_{-J_0 X_f} \omega = -d(i_{J_0 X_f} \omega) = -dJ_0 df = -2\sqrt{-1}\partial\bar{\partial} f,$$

where X_f is the Hamiltonian vector field corresponding to f. The Kernel of this map is exactly the group of Hamiltonian diffeomorphisms, thus we obtain $G \cdot J_0 / K \simeq \mathcal{H}_0$, where the inverse morphism $\mathcal{H}_0 \to G \cdot J_0/K$ is given by Moser's theorem [Mos65] (i.e., for any $\tilde{\omega} \in \mathcal{H}_0$, there exists an isotopy joining ω to $\tilde{\omega}$). Hence we conclude that each G-orbits can be regarded as the space of Kähler metrics in a fixed Kähler class with respect to a fixed complex structure[5].

[5] In the Calabi-Yau theory, one studies the deformation of Kähler metrics in a fixed Kähler class.

4 Stability of manifolds

4.1 K-stability

Futaki [Fut83] introduced a holomorphic invariant which generalizes the obstruction of Kazan-Warner to prescribe Gauss curvature on $\mathbb{S}^2$. Futaki invariant is defined as an integral invariant, which is a Lie algebraic character from the Lie algebra of holomorphic vector fields into $\mathbb{C}$. The vanishing of this holomorphic invariant is a necessary condition for the existence of a Kähler-Einstein metric. But the problem is that Futaki invariant dose not work when the manifold dose not have a non-trivial holomorphic vector field.

Ding-Tian [DT92] extended Futaki invariant to a new obstruction to the existence of Kähler-Einstein metric on Fano manifolds using the jumping of complex structures. However, this obstruction inherits original analytical definition and hard to use. Besides, we had to assume the normality of varieties.

Finally, Donaldson [Don02] gave a pure algebraic definition of the Futaki invariant, and extending it to singular Fano varieties, which enabled us to define an algebro-geometric stability of manifolds, called K-stability. In the definition of K-stability, we can catch a glimpse of philosophy of Hilbert-Mumford criterion (cf. Proposition 3.4).

4.1.1 Futaki invariant

Let (M, L) be a n-dimensional polarized manifold and h a Hermitian metric on L with positive curvature $\omega = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h \in c_1(L)$ and $\mathfrak{h}$ the Lie algebra consisting of all holomorphic vector field on M. Then there exists a real-valued smooth function κ (called **Ricci potential**) on M such that

$$\mathrm{Ric}(\omega) - \mathbb{H}\mathrm{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\kappa,$$

where $\mathbb{H}\mathrm{Ric}(\omega)$ is the harmonic representative of $\mathrm{Ric}(\omega)$. Then ω is cscK if and only if κ is a constant. In particular, a Kähler form $\omega \in c_1(M)$ is cscK if and only if it is Kähler-Einstein. Calabi [Cal85] extended the original Futaki invariant [Fut83] to an obstruction to the existence of cscK metrics for any polarized manifolds, which is also called **Futaki invariant**:

$$\mathrm{Fut}(W) = \frac{1}{c_1(L)^n}\int_M W(\kappa)\omega^n, \;\; W \in \mathfrak{h}_1,$$

where $\mathfrak{h}_1$ is an ideal of $\mathfrak{h}$ defined in Section 3.1.2. The Futaki invariant Fut is a holomorphic invariant, i.e., independent of a choice of Kähler form $\omega \in c_1(L)$. The vanishing of Fut is a necessary condition to the existence of a cscK metric, but not sufficient. Tian [Tian97] found a Fano manifold whose automorphism group is discrete. Hence Fut is trivial, but dose not admit any Kähler-Einstein metrics.

4.1.2 Ding-Tian's K-stability

Let M be an n-dimensional normal $\mathbb{Q}$-Fano variety[6]. We first introduce a generalization of Futaki invariant. The original definition of (generalized) Futaki invariant (hereafter referred as Futaki invariant) was given by Ding-Tian [DT92]. In this paper, we adopt an alternative characterization shown by Hou [Hou08, Theorem 2.7] as the definition of Futaki invariant. For simplicity, let us make the following assumptions:

1. M is a compact subvariety of a projective manifold N.
2. L is an ample line bundle on N such that on the regular part M_{reg} of M, the isomorphism
$$L|_{M_{\mathrm{reg}}} \simeq -kK_{M_{\mathrm{reg}}} \tag{4.1}$$
holds for some integer k.
3. The Lie group $G := \mathrm{Aut}(M)$ acts on (N, L) such that the isomorphism (4.1) is G-equivariant.

Remark 4.1. In fact, M can be embedded into $\mathbb{C}P^N \simeq H^0(M, -kK_M)$ for a sufficiently large integer k, and $(\mathbb{C}P^N, \mathcal{O}(1))$ satisfies the requirement above.

Definition 4.2. A Hermitian metric h on $-K_{M_{\mathrm{reg}}}$ is said to be admissible if h^k can be extended to a Hermitian metric on L over N under the isomorphisms (4.1).

Let h be an admissible Hermitian metric on $-K_{M_{\mathrm{reg}}}$ with positive curvature $\omega := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$.

Definition 4.3. We define the Futaki invariant by

$$\mathrm{Fut}(W) = -\frac{1}{c_1(M)^n}\int_M \mu_{h,W}\omega^n, \tag{4.2}$$

where $\mu_{h,W}$ is the holomorphy potential of the pair (h, W) (cf. Definition 2.2).

We remark that $\mu_{h,W}$ and ω can be extended continuously to M since h is admissible. Hence the above integral is finite. Futaki invariant is independent of a choice of admissible metric h on $-K_{M_{\mathrm{reg}}}$, and coincides with the Futaki invariant defined in Section 4.1.1 when M is smooth.

Now let M be a smooth Fano manifold and consider the Kodaira embedding $M \hookrightarrow \mathbb{C}P^{N_k}$ ($N_k := \mathrm{dim}H^0(M, -kK_M)$) defined by a basis of $H^0(M, -kK_M)$. Let $V \in \mathfrak{sl}(N_k + 1, \mathbb{C})$ be a holomorphic vector field on $\mathbb{C}P^{N_k}$ and σ_t the one parameter subgroup generated by $\mathrm{Re}(V)$. Set $M_t := \sigma_t(M)$, then it is known that M_t coverges to a subvariety M_∞ in $\mathbb{C}P^{N_k}$ as $t \to \infty$. Then σ_t fixes M_∞. Therefore $\mathrm{Re}V$ is tangent to M_∞. Ding-Tian showed that the following:

[6] We say that a normal projective variety M is $\mathbb{Q}$-Fano if for a large integer k, $-kK_{M_{\mathrm{reg}}}$ can be extended to an ample line bundle over M.

Theorem 4.4 ([DT92], Theorem 0.1 (also refer to [Ber12], [Stop09]))**.** Let M, σ_t, V, M_∞ as above, and assume that the subvariety M_∞ is normal and M admits a Kähler-Einstein metric. Then $\mathrm{Fut}(V)$ is non-negative.

Especially, when V is tangent to the image of M, we have $\mathrm{Fut}(V) \geq 0$, and replacing V with $-V$ forces the converse inequality $\mathrm{Fut}(V) \leq 0$. Thus we obtain:

Corollary 4.5 ([Fut83])**.** If M admits a Kähler-Einstein metric, then $\mathrm{Fut}(V) = 0$ for any holomorphic vector field V on M.

4.1.3 Donaldson's K-stability

Let (M, L) be an n-dimensional polarized manifold.

Definition 4.6. A **test configuration** for (M, L), of exponent r consists of:

- A flat morphism of schemes $\mathcal{M} \to \mathbb{C}$.
- A line bundle $\mathcal{L} \to \mathcal{M}$.
- An equivariant $\mathbb{C}^*$-action ρ on $\mathcal{L} \to \mathcal{M} \to \mathbb{C}$, where $\mathbb{C}^*$-acts on $\mathbb{C}$ by scalar multiplication.

such that the generic fiber (M_1, L_1) is isomorphic to (M, L^r).

In the original definition proposed by Donaldson [Don02], $\mathcal{M}$ is a (not necessarily normal) scheme. However, in the case of anti-canonical polarizations, it is known that we have only to test on the "special" test configurations to check the K-polystability, i.e., test configurations whose central fibers are normal $\mathbb{Q}$-Fano varieties with log-terminal singurarities (cf. Section 4.2.3).

There are a few remarks which immediately follow form the definition. First, all the fibers (M_t, L_t) $(t \in \mathbb{C}^*)$ are isomorphic to the generic fiber (M_1, L_1) via the action ρ. Second, the $\mathbb{C}^*$-action induces an action on the central fiber (M_0, L_0), and on the cohomology $H^0(M_0, kL_0)$ which we denote by ρ_k. We also let $B_k \in \mathrm{End}(H^0(M_0, kL_0))$ be defined by

$$\rho_k(w) = \exp(tB_k) = w^{B_k} \text{ for } w = \exp t \in \mathbb{R}^+ \ (t \in \mathbb{R}).$$

According to [PS07], a test configuration can be embedded into a projective space as in the following:

Lemma 4.7 ([PS07], Lemma 4.1)**.** Let $\mathcal{T}$ be a test configuration of exponent $r = 1$ for the pair (M, L). Let h be a positively curved metric on L. Let k be an integer such that L^k is very ample. Then there is

1. an orthogonal basis $s = (s_0, \ldots, s_{N_k})$ of $H^0(M, kL) \simeq H^0(M_1, kL_1)$.
2. an embedding $I_s : (k\mathcal{L} \to \mathcal{M} \to \mathbb{C}) \hookrightarrow (\mathcal{O}(1) \times \mathbb{C} \to \mathbb{C}P^{N_k - 1} \times \mathbb{C} \to \mathbb{C})$, satisfying the following property: for every $w \in \mathbb{C}^*$ and every $l_t \in kL_t$,

$$I_s(\rho(w) l_t) = (w^{B_k} \cdot I_s(l_t), wt),$$

where w^{B_k} is a diagonal matrix whose eigenvalues are the eigenvalues of $\rho_k(w) \in \mathrm{End}(H^0(M_0, kL_0))$. The matrix B_k is uniquely determined, up to a permutation of the diagonal entries, by k and the test configuration $\mathcal{T}$. Moreover, the basis s is uniquely determined by h and $\mathcal{T}$ up to an element of $U(N_k + 1)$ which commutes with B_k.

In particular, we obtain an embedding $\mathcal{M} \hookrightarrow \mathbb{C}P^{N_k-1} \times \mathbb{C}$ and a family of subschemes of $\mathbb{C}P^{N_k-1}$ parametrized by $t \in \mathbb{C}$. These aspects are very similar to those given by Ding-Tian [DT92] (see Section 4.1.2). But there are some obvious differences between them. Ding-Tain considered only Fano manifolds polarized by the anti-canonical bundle, and restricted their attention to test configurations with normal central fibers. On the other hand, Donaldson [Don02] invented an algebraic definition of Futaki invariant, which can be applied, more widely, to the central fiber of any test configuration.

Let (M, L) be an n-dimensional polarized scheme and W be a holomorphic vector field on M generating a $\mathbb{C}^*$-action which can be lifted to L. We define the quantized Futaki invariant $\mathrm{Fut}_k(W)$ at level k by the weight of the induced $\mathbb{C}^*$-action on the top exterior power $\bigwedge^{N_k} H^0(M, kL)$, i.e.,

$$\mathrm{Fut}_k(W) := \left.\frac{d}{dt}\right|_{t=0} \mathrm{Trace}(e^{tW})|_{H^0(M,kL)} = \sum_{i=1}^{N_k} w_i^{(k)},$$

where $(w_i^{(k)})$ are the joint eigenvalues of the infinitesimal action generated by $\mathrm{Re}(W)$. By the general theory, we know that the integers $\mathrm{Fut}_k(W)$, N_k are given by polynomials of k for sufficiently large k. Moreover,

Lemma 4.8. The quantity $\frac{\mathrm{Fut}_k(W)}{kN_k}$ is bounded, i.e., $\mathrm{Fut}_k(W)$ is at most the degree of N_k plus 1.

Proof. We can prove this in the same way as [PS07, Lemma 3.1]. For simplicity, we may assume that L is very ample. Then M can be embedded to $\mathbb{C}P^{N_1-1}$ and the image of M is a subscheme of $\mathbb{C}P^{N_1-1}$ defined by a homogeneous ideal $I \subset \mathbb{C}[z_1, \ldots, z_{N_1}]$, where $(z_1, \ldots, z_{N_1})$ is the standard coordinates of $\mathbb{C}^{N_1}$. We can write

$$\mathbb{C}[z_1, \ldots, z_{N_1}]/I = \bigoplus_{i \geq 0} S_i/I_i,$$

where $S_i \subset \mathbb{C}[z_1, \ldots, z_{N_1}]$ is the space of polynomials which are homogeneous of degree i and $I_i = S_i \cap I$. Then we have $H^0(M, kL) \simeq S_k/I_k$ for sufficiently large k. This isomorphism is $\mathbb{C}^*$-equivariant if $t \in \mathbb{C}^*$ acts on S_k by the formula

$$t \cdot z_1^{p_1} \cdots z_{N_1}^{p_{N_1}} = t^{p_1 w_1^{(1)} + \cdots + p_{N_1} w_{N_1}^{(1)}} \cdot z_1^{p_1} \cdots z_{N_1}^{p_{N_1}}, \; p_1 + \cdots + p_{N_1} = k.$$

Thus,

$$\{w_1^{(k)}, \ldots, w_{N_k}^{(k)}\} \subset \{p_1 w_1^{(1)} + \cdots + p_{N_1} w_{N_1}^{(1)} | p_1 + \cdots + p_{N_1} = k\}.$$

On the other hand, we clearly have $|p_1 w_1^{(1)} + \cdots + p_{N_1} w_{N_1}^{(1)}| \leq \sup_{i=1,\ldots,N_1} |w_i^{(1)}| \cdot k$, which yields that

$$|w_i^{(k)}| \leq Ck \;\text{ for } i = 1, \ldots, N_k$$

and hence

$$|\mathrm{Fut}_k(W)| \leq CkN_k.$$

This completes the proof of Lemma 4.8. $\square$

By Lemma 4.8, we have an asymptotic expansion as $k \to \infty$:

$$\frac{2\mathrm{Fut}_k(W)}{kN_k} = F_0 + F_1 k^{-1} + O(k^{-2}). \tag{4.3}$$

Definition 4.9. We define the Futaki invariant $\mathrm{Fut}(W)$ of the pair (M, L) with a holomorphic vector field W to be the coefficient F_1.

Proposition 4.10 ([Don02], Proposition 2.2.2)**.** When M is smooth, Donaldson-Futaki invariant is given by

$$F_1 = \frac{1}{c_1(L)^n} \int_M W(\kappa)\omega^n.$$

Proof. By the (equivariant) Riemann-Roch formula, we have

$$\begin{aligned}
N_k &= \int_M \mathrm{ch}(L^k)\mathrm{td}(M) \\
&= \int_M e^{k\omega}\left(1 + \frac{1}{2}\mathrm{Ric}(\omega) + \cdots\right) \\
&= \frac{k^n}{n!}\int_M \omega^n + \frac{k^{n-1}}{2(n-1)!}\int_M \mathrm{Ric}(\omega)\wedge\omega^{n-1} + O(k^{n-2}) \\
&= \frac{c_1(L)^n}{n!}k^n + \frac{c_1(L)^n}{2n!}\overline{\mathrm{Scal}}\cdot k^{n-1} + O(k^{n-2}) \\
&=: Ck^n + Dk^{n-1} + O(k^{n-2}),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Fut}_k(W) &= \frac{d}{dt}\mathrm{Trace}(e^{tW})|_{H^0(M,kL)}\Big|_{t=0} \\
&= \frac{d}{dt}\int_M e^{k(\omega + t\mu_{h,W})}\left(1 + \frac{1}{2}(\mathrm{Ric}(\omega) + t\Delta_\partial\mu_{h,W}) + \cdots\right)\Big|_{t=0} \\
&= \frac{k^n}{2(n-1)!}\int_M \mu_{h,W}\mathrm{Ric}(\omega)\wedge\omega^{n-1} + \frac{k^n}{n!}\int_M\left(\frac{1}{2}\Delta_\partial\mu_{h,W} + k\mu_{h,W}\right)\omega^n + O(k^{n-1}) \\
&= \frac{k^{n+1}}{n!}\int_M \mu_{h,W}\omega^n + \frac{k^n}{2n!}\int_M \mu_{h,V}\mathrm{Scal}(\omega)\omega^n + O(k^{n-1}) \\
&=: Ak^{n+1} + Bk^n + O(k^{n-1}).
\end{aligned}$$

Since

$$\frac{\mathrm{Fut}_k(W)}{kN_k} = \frac{A + Bk^{-1} + O(k^{-2})}{C + Dk^{-1} + O(k^{-2})} = \frac{A}{C} + \frac{AD - BC}{C^2}k^{-1} + O(k^{-2}),$$

we obtain

$$\begin{aligned} F_1 &= 2\cdot\frac{AD-BC}{C^2} \\ &= -\frac{1}{c_1(L)^n}\int_M \mu_{h,W}(\mathrm{Scal}(\omega)-\overline{\mathrm{Scal}})\omega^n \\ &= \frac{1}{c_1(L)^n}\int_M \mu_{h,W}\Delta_\partial\kappa\omega^n \\ &= \frac{1}{c_1(L)^n}\int_M W(\kappa)\omega^n. \end{aligned}$$

□

If we choose another lift of W, then the eigenvalues of the infinitesimal action on $\bigwedge^{N_k} H^0(M,kL)$ are given by $(w_i^{(k)}+k\alpha)$ for some k-independent constant α. Then we have

$$\frac{2\sum_{i=1}^{N_k}(w_i^{(k)}+k\alpha)}{kN_k} = \frac{2\sum_{i=1}^{N_k} w_i^{(k)}}{kN_k} + 2\alpha.$$

Hence the choice of lift of W contributes only the constant term in the RHS of (4.3), and hence $\mathrm{Fut}(W)$ is independent of a choice of lift of W to L.

Let (M,L) be a polarized manifold. A test configuration $\mathcal{T}$ for (M,L) induces the $\mathbb{C}^*$-action on the central fiber (M_0,L_0). We denote by W its infinitesimal generator. We say that a test configuration $\mathcal{T}$ is a **product configuration** if $\mathcal{M}=M\times\mathbb{C}$ and its $\mathbb{C}^*$-action ρ is given by a scalar multiplication to the second factor of $\mathcal{M}$. The analogue of the Hilbert-Mumford criterion in this setting is the following:

Definition 4.11. We say that the pair (M,L) is **K-polystable** if

1. $\mathrm{Fut}(W)\geq 0$ for any test configuration $\mathcal{T}$ for (M,L).
2. M_0 is normal and $\mathrm{Fut}(W)=0$ if and only if the test configuration is a product configuration[7].

Theorem 4.12 ([CDS13] and [Tian12])**.** If a Fano manifold M is K-polystable, then it admits a Kähler-Einstein metric.

In the case of general polarizations, it is conjectured that the existence of a cscK metric is equivalent to more stronger stability, for instance, strong K-stability [Mab09] and uniform K-stability [Szèk06].

4.2 Kähler-Ricci solitons

Tian-Zhu generalized Futaki invariant for the pair (M,V) of a Fano manifold M and a holomorphic vector field V, called the modified Futaki invariant, which is an obstruction to the existence of Kähler-Ricci soliton with respect to V. However, it looks different in character compared to original Futaki invariant. First, we can

[7]There is an example of a non-normal test configuration with $\mathrm{Fut}(W)=0$ (cf. [LX11], [Mab13]).

always choose V so that the modified Futaki invariant vanishes. Second, the modified Futaki invariant is closely related to the canonical lift to the anti-canonical bundle $-K_M$ although Futaki invariant is independent of a choice of lift (see Section 4.1.3).

In the case of anti-canonical polarizations, the author generalized Tian-Zhu's existence result [TZ02] of Kähler-Ricci solitons on a certain $\mathbb{C}P^1$-bundle over special manifold, to an admissible bundle on which the existence of extremal metrics were studied by Gauduchon and other collaborators. We study this construction method in Section 4.2.2.

In [BN14, Theorem 1.3], Berman-Nyström have shown that a normal $\mathbb{Q}$-Fano variety admitting a (singular) Kähler-Ricci soliton necessarily has at worst log-terminal singularities. They studied the existence problem of Kähler-Ricci solitons on this class of varieties. They generalized Donaldson's K-stability to Kähler-Ricci soliton case, and showed that the existence leads to K-polystability for the pair (M, V) (cf. Section 4.2.3).

4.2.1 The modified Futaki invariant

Let M be a n-dimensional Fano manifold and h a Hermitian metric on $-K_M$ with positive curvature $\omega = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h \in c_1(M)$. Let $\mathfrak{h}$ be the Lie algebra consisting of all holomorphic vector fields on M. Let κ be a Ricci potential of ω defined by

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\kappa.$$

Then one can see that ω is a pair (ω, V) is a Kähler-Ricci soliton if and only if $\kappa = \mu_{h,V}$ up to an additive constant. The **modified Futaki invariant** is defined as a functional

$$\mathrm{Fut}_V(W) = \frac{1}{c_1(M)^n}\int_M W(\kappa - \mu_{h,V})e^{\mu_{h,V}}\omega^n, \quad V \in \mathfrak{h},$$

which is independent of a choice of a Hermitian metric h [TZ02, Section 2].

4.2.2 Examples on admissible bundles

We first consider the existence problem of Generalized Quasi Einstein (GQE) metrics, which is a straightfoward extension of Kähler-Ricci solitons for any polarizations defined as follows:

Definition 4.13. Let M be a compact complex manifold with Kähler class Ω. we say that a Kähler form $\omega \in \Omega$ is **Generalized Quasi Einstein** (GQE) if it satisfies the equation

$$\mathrm{Ric}(\omega) - \mathbb{H}\mathrm{Ric}(\omega) = L_V\omega \tag{4.4}$$

for some homolorphic vector field V on M.

Let κ be a Ricci potential of an admissible metric g:

$$\mathrm{Ric}(\omega) - \mathbb{H}\mathrm{Ric}(\omega) = \sqrt{-1}\partial\bar{\partial}\kappa. \tag{4.5}$$

Then a Kähler metric is GQE if and only if its Ricci potential κ is Killing, i.e., $J\mathrm{grad}\kappa$ is a Killing vector field (cf. [Kob95, p.93, 94]).

Examples of GQE metrics on admissible bundles were studied by Maschler and Tønnesen-Friedman [MT11]. They showed that there exists an admissible GQE metric in any admissible class if its admissible data $\{x_a\}$ is sufficiently small, i.e., $|x_a|$ is sufficiently small for all $a \in \mathcal{A}$. In reference to this result, we study the case when Ω is proportional to $c_1(M)$ and prove Theorem 1.1.

Now we start with the review of constructing GQE metrics on admissible bundles. We adopt the notations defined in Section 2.2. Let M be an $n\left(:= \sum_{a\in\hat{\mathcal{A}}} d_a + 1\right)$-dimensional admissible bundle and Ω an admissible class on M. Let $C^\infty([-1,1])$ be the space of smooth functions over the interval $[-1,1]$.

Lemma 4.14 ([MT11], Proposition 3.1)**.** For any admissible metric and $S \in C^\infty([-1,1])$, we have

$$\Delta_\partial S(z) = -\frac{[S'(z)\cdot F(z)]'}{2p_c(z)}. \tag{4.6}$$

Proof.

$$\begin{aligned}
\Delta_\partial S(z) &= -\frac{1}{2}(dd^c S(z),\omega) = -\frac{1}{2}(dJdS(z),\omega) = -\frac{1}{2}(d(S'(z)Jdz),\omega)\\
&= -\frac{1}{2}\left(d\left(S'(z)\frac{F(z)}{p_c(z)}\theta\right),\omega\right) \left(\text{Because } Jdz = \Theta(z)\theta = \frac{F(z)}{p_c(z)}\theta.\right)\\
&= -\frac{1}{2}\left(\left(\frac{[S'(z)F(z)]'}{p_c(z)} - \frac{S'(z)F(z)p_c'(z)}{(p_c(z))^2}\right)dz\wedge\theta,\omega\right)\\
&\quad -\frac{1}{2}\left(S'(z)\frac{F(z)}{p_c(z)}\sum_{a\in\hat{\mathcal{A}}}\omega_a,\omega\right)\left(\text{Because } d\theta = \sum_{a\in\hat{\mathcal{A}}}\omega_a.\right)\\
&= -\frac{1}{2}\frac{[S'(z)F(z)]'}{p_c(z)} + \frac{1}{2}\cdot\frac{S'(z)F(z)}{p_c(z)}\left[\frac{p_c'(z)}{p_c(z)} - \sum_{a\in\hat{\mathcal{A}}}\frac{d_a x_a}{1+x_a z}\right]\\
&= -\frac{1}{2}\frac{[S'(z)F(z)]'}{p_c(z)}\left(\text{Because } (dz\wedge\theta, dz,\wedge\theta) = 1,\ (\omega_a,\omega) = \frac{d_a x_a}{1+x_a z}.\right).
\end{aligned}$$

□

According to [ACG06, (79)], the scalar curvature of an admissible metric is given by

$$\mathrm{Scal}_g(\omega) = \frac{1}{2}\left(\sum_{a\in\hat{\mathcal{A}}}\frac{2d_a s_a x_a}{1+x_a z} - \frac{F''(z)}{p_c(z)}\right). \tag{4.7}$$

Let $C_0^\infty([-1,1])$ be the set of smooth function on $[-1,1]$ normalized so that they integrate to zero when viewed as smooth function on M by compositing with z. Then,

Corollary 4.15 ([MT11], Corollary 3.2)**.** Given an admissible metric g, its Laplacian gives a surjective map from $C_0^\infty([-1,1])$ to itself (considered as a space of smooth functions over M).

Proof. Given $R(z) \in C_0^\infty([-1,1])$, we can obtain an explicit solution of $\Delta_\partial S(z) = R(z)$ outside the critical set of z by virtue of Lemma 4.14. Using de l'Hôpital's rule and (2.6), this solution extends to the critical set of z. $\square$

Taking the trace of (4.5), we obtain

$$\mathrm{Scal}_g(\omega) - \overline{\mathrm{Scal}} = -\Delta_\partial \kappa \tag{4.8}$$

Combining with Corollary 4.15 and (4.7), we have:

Corollary 4.16 ([MT11], Corollary 3.3)**.** The Ricci potential κ of an admissible metric is a function of z.

For any admissible metric, the space of Killing potentials depending only on z is a vector space spanned by 1 and z, i.e., the space of all affine functions of z. Hence,

Lemma 4.17. An admissible metric is GQE if and only if there exists $k \in \mathbb{R}$ such that $\kappa = kz$ up to an additive constant.

Define the polynomial $P(t)$ by

$$P(t) = 2\int_{-1}^{t} \left(\left(\sum_{a \in \hat{\mathcal{A}}} \frac{d_a s_a x_a}{1 + x_a s} \right) \cdot p_c(s) - \frac{\beta_0}{\alpha_0} \cdot p_c(s) \right) ds + 2p_c(-1), \tag{4.9}$$

where α_0 and β_0 are constants defined by

$$\alpha_0 = \int_{-1}^{1} p_c(t)dt \ \text{ and } \ \beta_0 = p_c(1) + p_c(-1) + \int_{-1}^{1} \left(\sum_{a \in \hat{\mathcal{A}}} \frac{d_a s_a x_a}{1 + x_a t} \right) p_c(t)dt. \tag{4.10}$$

Clearly, these quantities are independent of a choice of admissible metric, depend only on M and the admissible class Ω. We added a constant $2p_c(-1)$ to the RHS of (4.9) so that $P(\pm 1) = \mp 2p_c(\pm 1)$ holds. Then $P(t)$ has the following properties:

Lemma 4.18 ([MT11], Lemma 4.3)**.** For any given admissible data, $P(t)$ satisfies: If $d_0 = 0$, then $P(-1) > 0$, otherwise $P(-1) = 0$. If $d_\infty = 0$, then $P(1) < 0$, otherwise $P(1) = 0$. Furthermore, $P(t) > 0$ in some (deleted) right neighborhood of $t = -1$, and $P(t) < 0$ in some (deleted) left neighborhood of $t = 1$. Concretely, we see that if $d_0 > 0$, then $P^{(d_0)}(-1) > 0$ (and the lower order derivatives vanish), while if $d_\infty > 0$, then $P^{(d_\infty)}(1)$ has sign $(-1)^{d_\infty + 1}$ (and the lower order derivatives vanish).

Lemma 4.19. For any admissible metric, we have

$$F'(z) + \kappa'(z) \cdot F(z) = P(z). \tag{4.11}$$

Proof. From (4.6) – (4.10), we get

$$F''(z) + [\kappa'(z) \cdot F(z)]' = P'(z),$$

where we used the relation $\overline{\mathrm{Scal}} = \frac{\beta_0}{\alpha_0}$ (cf. [ACGT08, Section 2.2]). Since $F(z)$ and $P(z)$ have the same boundary condition $F(\pm 1) = P(\pm 1) = \mp 2p_c(\pm 1)$, integrating the above on z yields

$$F'(z) + \kappa'(z) \cdot F(z) = P(z).$$

□

In particular, by Lemma 4.17 and Lemma 4.19, the admissible GQE condition can be written as

$$F'(z) + k \cdot F(z) = P(z)$$

for some constant $k \in \mathbb{R}$. The point is that the admissible GQE condition can be reduced to simple ODE of z. Hence we get an explicit solution

$$F(z) = e^{-kz} \int_{-1}^{z} P(t)e^{kt}dt \tag{4.12}$$

under the boundary condition $F(-1) = 0$. In order to construct a GQE metric from this $F(z)$, we also need $F(z)$ to satisfy $F(1) = 0$. Therefore

$$\mathrm{MT}(k) := \int_{-1}^{1} P(t)e^{kt}dt \tag{4.13}$$

is an obstruction to the existence of admissible GQE metrics with the Ricci potential kz, which we would like to call **Maschler-Tønnesen invariant**. Actually, since $P(t)$ depends only on M and Ω, so MT is.

Next, we assume that Ω is proportional to $c_1(M)$. Then there are two obstructions to the existence of an admissible Kähler-Ricci soliton, namely, the modified Futaki invariant and Maschler-Tønnesen invariant. We study the relation between these two invariants. For any $k \in \mathbb{R}$, let X_J^k be a holomorphic vector field such that $i_{X_J^k}\omega = \sqrt{-1}\bar{\partial}_J kz$, where J is the compatible complex structure induced by an admissible metric. Since K is the infinitesimal generator of the $U(1)$-action on M and the function z is the moment map of this action, we get $i_K\omega = -dz$. Hence $X_J^2 = -JK - \sqrt{-1}K$ and $X_J^k = \frac{k}{2} \cdot X_J^2 = -\frac{k}{2}(JK + \sqrt{-1}K)$.

Lemma 4.20. Let M and Ω are defined as above. Then

1. If we set $\Omega = 2\pi\lambda^{-1}c_1(M)$ for some positive constant λ, then we have $\lambda = \frac{d_0 + d_\infty + 2}{2}$.

2. The modified Futaki invariant and Maschler-Tønnesen invariant have a relation

$$\mathrm{Fut}_{\lambda^{-1}X_J^k}(X_J^2) = \alpha_0^{-1} \exp\left(-\frac{kC}{2\lambda}\right) \mathrm{MT}(k). \tag{4.14}$$

as a function of k, where $C := d_0 - d_\infty$ is a constant.

Proof. In this proof, we consider a fixed admissible metric g whose Kähler form ω belongs to Ω.

(1) Put $g' = \lambda g$ and $\omega' = \lambda\omega$, then (g', ω') defines a Kähler structure and $\omega' \in 2\pi c_1(M)$. Let κ be the Ricci potential of ω. Since the Ricci form is preserved under scaling of ω, κ is also the Ricci potential of ω'. Let $\mu_{\omega',X}$ be the holomorphy potential of the pair (ω', X). Then $\mu_{\omega',X}$ satisfies the equation $-\Delta_{\partial,g'}\mu_{\omega',X}+\mu_{\omega',X}+X(\kappa) = 0$, where $\Delta_{\partial,g'}$ is the ∂-Laplacian with respect to g'. We set $\mu_{\omega',X_J^2} = 2\lambda z - C$ for some constant C, then C is computed by

$$\begin{aligned} C &= -2\Delta_{\partial,g'}\lambda z + 2\lambda z + \kappa'(z)\cdot\Theta(z) \\ &= -2\Delta_{\partial,g} z + 2\lambda z + \kappa'(z)\cdot\Theta(z) \\ &= \frac{F'(z)}{p_c(z)} + 2\lambda z + \kappa'(z)\cdot\Theta(z), \end{aligned} \tag{4.15}$$

where we used (4.6) and $X_J^2(\kappa(z)) = -JK(\kappa(z)) = -d(\kappa(z))(JK) = \kappa'(z)Jdz(K) = \kappa'(z)\cdot\Theta(z)$, and denoted by $\Delta_{\partial,g}$ the ∂-Laplacian with respect to g. In order to find C as above, we take the limit of z to the boundary. Since

$$\frac{F'(z)}{p_c(z)} = \Theta'(z) + \Theta(z)\cdot\frac{p_c'(z)}{p_c(z)} = \Theta'(z) + \Theta(z)\cdot\sum_{a\in\hat{\mathcal{A}}}\frac{x_a d_a}{1+x_a z},$$

using (2.4) and de l'Hôpital's rule, we get

$$\lim_{z\to 1}\frac{F'(z)}{p_c(z)} = -2-2d_\infty.$$

Similarly, we have $\lim_{z\to -1}\frac{F'(z)}{p_c(z)} = 2+2d_0$. Therefore combining with (4.15), we obtain $C = -2-2d_\infty+2\lambda = 2+2d_0-2\lambda$, and hence $C = d_0 - d_\infty$ and $\lambda = \frac{d_0+d_\infty+2}{2}$.

(2) From the argument in (1), we have $\mu_{\omega',X_J^2} = 2\lambda z - C$ and $\mu_{\omega',X_J^k} = k\lambda z - \frac{kC}{2}$. Hence, by (4.11) and (4.15), we have

$$2\lambda z p_c(z) - Cp_c(z) + P(z) = 0.$$

Hence the direct computation shows that

$$\begin{aligned} (2\pi)^n c_1(M)^n &= \lambda^n\int_M \omega^n \\ &= \lambda^n n!\int_M p_c(z)\left(\bigwedge_{a\in\hat{\mathcal{A}}}\frac{(\omega_a/x_a)^{d_a}}{d_a!}\right) dz\wedge\theta \\ &= 2\pi\lambda^n n!\alpha_0 \mathrm{Vol}\left(S, \prod_{a\in\hat{\mathcal{A}}}\frac{\omega_a}{x_a}\right), \end{aligned}$$

$$
\begin{aligned}
&\int_M \mu_{\omega', X_J^2} \exp(\mu_{\omega', \lambda^{-1} X_J^k})(\omega')^n \\
= \;& \lambda^n \int_M (2\lambda z - C) e^{kz - \frac{kC}{2\lambda}} \omega^n \\
= \;& \lambda^n n! \exp\left(-\frac{kC}{2\lambda}\right) \int_M (2\lambda z - C) e^{kz} p_c(z) \left(\bigwedge_{a \in \hat{\mathcal{A}}} \frac{(\omega_a/x_a)^{d_a}}{d_a!} \right) dz \wedge \theta \\
= \;& 2\pi \lambda^n n! \exp\left(-\frac{kC}{2\lambda}\right) \mathrm{Vol}\left(S, \prod_{a \in \hat{\mathcal{A}}} \frac{\omega_a}{x_a}\right) \int_{-1}^{1} (2\lambda z - C) p_c(z) e^{kz} dz \\
= \;& -2\pi \lambda^n n! \exp\left(-\frac{kC}{2\lambda}\right) \mathrm{Vol}\left(S, \prod_{a \in \hat{\mathcal{A}}} \frac{\omega_a}{x_a}\right) \mathrm{MT}(k),
\end{aligned}
$$

where we used the equation $\omega^n/n! = p_c(z) \left(\bigwedge_{a \in \hat{\mathcal{A}}} \frac{(\omega_a/x_a)^{d_a}}{d_a!} \right) dz \wedge \theta$ (cf. [ACGT08, Section 2.2]). Thus we obtain

$$
\mathrm{Fut}_{\lambda^{-1} X_J^k}(X_J^2) = \alpha_0^{-1} \exp\left(-\frac{kC}{2\lambda}\right) \mathrm{MT}(k).
$$

□

Corollary 4.21. We assume the same as above. Then $\Omega = 2\pi c_1(M)$ holds if and only if $d_0 = d_\infty = 0$, i.e., a blow-down occurs. In this case, we have

$$
\mathrm{Fut}_{\lambda^{-1} X_J^k}(X_J^2) = \alpha_0^{-1} \mathrm{MT}(k) \tag{4.16}
$$

for any admissible metrics.

From Lemma 4.18, we know that the polynomial $P(t)$ has at least one root in the interval $(-1, 1)$. If $P(t)$ has exactly one root in the interval $(-1, 1)$, Maschler-Tønnesen invariant has good behavior.

Lemma 4.22 ([MT11], Proposition 4.2)**.** If the function $P(t)$ has exactly one root in the interval $(-1, 1)$, then there exists a unique $k_0 \in \mathbb{R}$ such that $\mathrm{MT}(k_0) = 0$. Moreover, for this k_0, the function $F(z)$ defined by (4.10) satisfies $F > 0$ on $(-1, 1)$, and an admissible GQE metric is naturally constructed from F.

Proof. Since $P(t)$ has exactly one root t_0 in the interval $(-1, 1)$, we may write

$$
P(t) = (t - t_0) p(t).
$$

By Lemma 4.18, $p(t)$ is negative for all $t \in (-1, 1)$. Let us consider the function

$$
G(k) := e^{-kt_0} \int_{-1}^{1} P(t) e^{kt} dt = \int_{-1}^{1} p(t)(t - t_0) e^{k(t - t_0)} dt.
$$

By the direct computation, one can show that $G'(k)$ is negative and $\lim_{k \to \pm\infty} G(k) = \mp\infty$, which yield the existence and uniqueness of k_0 such that $G(k_0) = 0$, or equivalently, $\mathrm{MT}(k_0) = 0$. It is clear that $F(t)$ is positive on $(-1, 1)$ since $P(t)e^{k_0 t}$ changes sign exactly once in $(-1, 1)$. This completes the proof of Lemma 4.22. □

Proof of Theorem 1.1. This is a direct corollary from Lemma 4.20: from the argument in the proof of Lemma 4.20, we know that $P(t) = (C - 2\lambda t)p_c(t)$, where $\lambda = \frac{d_0+d_\infty+2}{2}$ and $C = d_0 - d_\infty$ are constants. Thus $P(t)$ has exactly one root $t = \frac{C}{2\lambda}$ in the interval $(-1, 1)$. Therefore we have the desired result by Lemma 4.22. $\square$

Our theorem includes Tian-Zhu's example on a $\mathbb{C}P^1$-bundle [TZ02, Example 4.1] as a special case:

Example 4.23. We consider an admissible bundle $M := \mathbb{C}P^{l+1}\#\overline{\mathbb{C}P^{l+1}} = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{C}P^l$ for $l \geq 1$. Since the 2nd betti number of $\mathbb{C}P^l$ is 1, every Kähler class on M is admissible up to scale (cf. [ACGT08, Remark 2]), so $c_1(M)$ is admissible up to scale. By Corollary 4.21, there exists an admissible class Ω with the admissible data $x \in (-1, 1)$ $(x \neq 0)$ such that $\Omega = 2\pi c_1(M)$. Then we have

$$\mathrm{MT}(0) = -2\int_{-1}^{1} t(1 + xt)^l dt = -4\sum_{i=1}^{[(l+1)/2]} \binom{l}{2i-1}\frac{x^{2i-1}}{2i+1}.$$

This is a linear combination of monomials of odd degree with negative coefficients, that yields $\mathrm{MT}(0) \neq 0$. Combining with Theorem 1.1, there exists an admissible Kähler-Ricci soliton with respect to a non-trivial holomorphic vector field.

4.2.3 K-stability for Kähler-Ricci solitons

In this section, we study K-stability for Kähler-Ricci solitons introduced by Berman-Nyström [BN14]. Let M be a Fano manifold and a pair (ω, V) a Kähler-Ricci soliton:

$$\mathrm{Ric}(\omega) - \omega = L_V\omega. \tag{4.17}$$

Taking the imaginary part of the above equation yields $L_{\mathrm{Im}(V)}\omega = 0$, hence ω is invariant under the group action generated by $\mathrm{Im}(V)$. We first see that the Ricci soliton vector field V comes from an element in a Lie algebra of a complex torus acting homomorphically on M. More generally, we show that:

Proposition 4.24 ([BN14], Lemma 2.13)**.** Let M be a Fano manifold and V a holomorphic vector field on M. If there exists a Kähler metric ω which is invariant under the action of $\mathrm{Im}(V)$, then there exists a complex torus T_c acting holomorphically on M such that $\mathrm{Im}(V)$ may be identified with an element in the Lie algebra of the corresponding real torus $T \subset T_c$.

Proof. First, we check that the isometry group K of ω is a compact Lie group. This is shown by considering the canonical imbedding $M \hookrightarrow H^0(M, -kK_M)$ and the K-invariant Hilbert norm $||s||^2 := \int_M |s|_k^2\omega^n$ $(s \in H^0(M, -kK_M))$. Actually, K is identified with a subgroup of the group consisting of unitary transformations on $H^0(M, -kK_M)$ with respect to $||\cdot||$, which yields K is compact. Taking the topological closure of the 1-parameter subgroup generated by $\mathrm{Im}(V)$ in K, we get a real torus T as desired. In general, any holomorphic action of a real torus on M can be naturally extended to the corresponding complex torus action on M. $\square$

Definition 4.25. We say that a variety M has log-terminal singularities if we can write $K_{\widetilde{M}} = \pi^* K_M + \sum_i a_i E_i$ with $a_i > -1$ for a log resolution $\pi : \widetilde{M} \to M$, where $a_i \in \mathbb{Q}$ and E_i are exceptional irreducible divisors.

Definition 4.26. A **special test configuration** $\mathcal{T}$ for (M, V) consists of:

- A flat morphism of schemes $\mathcal{M} \to \mathbb{C}$.
- An equivariant $\mathbb{C}^*$-action ρ on $\mathcal{M} \to \mathbb{C}$, where $\mathbb{C}^*$-acts on $\mathbb{C}$ by scalar multiplication.
- A holomorphic vector field $\mathcal{V}$ on $\mathcal{M}$.

such that

1. Each fiber of $\mathcal{M} \to \mathbb{C}$ is a normal $\mathbb{Q}$-Fano variety with log-terminal singularities.
2. The generic fiber (M_1, V_1) is isomorphic to (M, V).
3. The vector field $\mathcal{V}$ preserves the each fiber of $\mathcal{M} \to \mathbb{C}$.
4. The action ρ preserves $\mathcal{V}$.

Let $\mathcal{W}$ be a holomorphic vector field on $\mathcal{M}$ generating the $\mathbb{C}^*$-action ρ, which commutes with $\mathcal{V}$ by the assumption. Then there are two commuting vector fields $V_0 := \mathcal{V}|_{M_0}$, $W_0 := \mathcal{W}|_{M_0}$ tangent to the central fiber M_0.

Remark 4.27. (1) In the original paper [BN14], they studied the existence problem of (singular) Kähler-Ricci solitons on a normal $\mathbb{Q}$-Fano variety M, which is defined as a pair (ω, V) of a Kähler metric ω with the maximal volume (i.e., the volume coincides with the global algebraic top intersection number $c_1(M)^n$) and a holomorphic vector field V on the regular set M_{reg} satisfying the equation (4.17) on M_{reg}. If (M, V) admits a Kähler-Ricci soliton, M have at worst log-terminal singularities [BN14, Theorem 1.3]. In that sense, it seems natural to assume that each fiber of $\mathcal{M}$ has log-terminal singularities.

(2) As the case of cscK is so, we may need to consider more general test configurations, not only special ones. In fact, [WZZ14] introcuced slightly different notion of K-stability, and studied the case of toric degenerations on toric manifolds, on which the existence of Kähler-Ricci solitons has already been shown by Wang-Zhu [WZ04].

Let M be a normal $\mathbb{Q}$-Fano variety with log-terminal singularities. Let V be a holomorphic vector field on M generating a torus action and W a holomorphic vector field on M generating a $\mathbb{C}^*$-action commuting with V. Set

$$N_k := \dim H^0(M, -kK_M).$$

We define the quantized modified Futaki invariant at level k as

$$\mathrm{Fut}_{V,k}(W) := -\sum_{i=1}^{N_k} \exp(v_i^{(k)}) w_i^{(k)}, \tag{4.18}$$

where $(v_i^{(k)}, w_i^{(k)})$ are the joint eigenvalues for commuting action of $\mathrm{Re}(V)$ and $\mathrm{Re}(W)$ on $H^0(M, -kK_M)$ defined by the canonical lift to $-K_M$. In the same way as Lemma 4.8, we know that the limit $\lim_{k\to\infty} \frac{1}{kN_k}\mathrm{Fut}_{V,k}(W)$ exists.

Definition 4.28. We define the modified Futaki invariant for (M, V) as the limit

$$\mathrm{Fut}_V(W) := \lim_{k\to\infty} \frac{1}{kN_k}\mathrm{Fut}_{V,k}(W) \tag{4.19}$$

Remark 4.29. In [BN14], they adopted an alternative definition of the modified Futaki invariant, which is defined as the derivative of the modified K-energy[8].

The analogue of the Hilbert-Mumford criterion in this setting is the following:

Definition 4.30. Let M be a Fano manifold and V a holomorphic vector field on M generating a torus action. We say that the pair (M, V) is **K-polystable** if

1. $\mathrm{Fut}_{V_0}(W_0) \geq 0$ for any special test configuration $\mathcal{T}$ for (M, V).
2. $\mathrm{Fut}_{V_0}(W_0) = 0$ if and only if the special test configuration is a product configuration.

Then Berman-Nyström showed that:

Theorem 4.31 ([BN14], Theorem 1.5)**.** If the pair (M, V) admits a Kähler-Ricci soliton, then (M, V) is K-polystable.

[8] One needs a special test configuration for defining the modified K-energy.

5 Fano complete intersections

In this section, we prove Theorem 1.2. We first recall the preceding results on Futaki invariant shown by Lu [Lu99] and Hou [Hou08]. Then we give two definitions of the function $\mathcal{F}$ (and the modified Futaki invariant). One is a Ding-Tian type integral invariant for normal $\mathbb{Q}$-Fano varieties defined in terms of admissible metrics. The other is a quantized version of $\mathcal{F}$ for normal $\mathbb{Q}$-Fano varieties with log-terminal singularities. These definitions are equivalent when the variety has log-terminal singularities. Correspondingly, we give two proofs of Theorem 1.2. Finally, we give some examples of computing $\mathcal{F}$.

5.1 Preceding results on Futaki invariant

Let M be an n-dimensional normal variety in $\mathbb{C}P^N$ and X a holomorphic vector field on $\mathbb{C}P^N$. Then X can be identified with the linear vector field $\sum_{i,j=0}^N a_{ij} z^i \frac{\partial}{\partial z^j}$ on $\mathbb{C}^{N+1}$ and the traceless matrix $(a_{ij})_{0\le i,j\le N} \in \mathfrak{sl}(N+1,\mathbb{C})$ such that the push-foward of $\sum_{i,j=0}^N a_{ij} z^i \frac{\partial}{\partial z^j}$ with the standard projection $\pi : \mathbb{C}^{N+1} \to \mathbb{C}P^N$ is equal to X.

For a holomorphic vector field X, we define a complex valued smooth function on $\mathbb{C}^{N+1} - 0$ by

$$\theta_X := X\left(\log\left(\sum_{i=0}^N |z^i|^2\right)\right), \tag{5.1}$$

which descends to a smooth function on $\mathbb{C}P^N$. Let $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\sum_{i=1}^N |z^i|^2) \in \pi c_1(\mathcal{O}(1))$ be the Fubini-Study metric of $\mathbb{C}P^N$. Then we have

$$i_X\omega = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\theta_X. \tag{5.2}$$

We say that "X is tangent to M" if and only if $\mathrm{Re}(X)$ leaves M invariant. If M is a hypersurface defined by a homogenous polynomial F of degree d, X is tangent to M if and only if X fixes $[F] \in \mathbb{P}(H^0(M,\mathcal{O}(d)))$, or, equivalently, $XF = \gamma F$ for some constant γ. For any X which is tangent to M, the equation (5.2) can be written as

$$X^i = g^{i\bar{j}}\frac{\partial\theta_X}{\partial x^{\bar{j}}}\ (i = 1,\dots,n),\ \ X = \sum_{i=1}^n X^i\frac{\partial}{\partial x^i}, \tag{5.3}$$

at some smooth point in local holomorphic coordinates $(x^1,\dots,x^n)$ of M, where $(g_{i\bar{j}})$ is the matrix of ω.

Now let M be a normal $\mathbb{Q}$-Fano complete intersection in $\mathbb{C}P^N$ defined by the homogeneous polynomials $F_1,\dots,F_s$ of degree $d_1,\dots,d_s$ respectively and suppose that $m\omega \in c_1(M)$ for some constant $m > 0$. Let W be a holomorphic vector fields on $\mathbb{C}P^N$ such that

$$WF_i = \beta_i F_i$$

for some constants β_i $(i = 1,\dots,s)$. The process to obtain an explicit expression of Futaki invariant is divided to two steps.

Step1. We compute the moment map defined by the canonical lift to $-K_{M_{\rm reg}}$. Let G be the Lie group generated by W. Then G acts on $\mathcal{O}(1)$ and the normal bundle $N_{M_{\rm reg}}$ in a natural manner, which we denote by σ and σ_N respectively. Using the adjunction formula, we know that $m = N+1-d_1-\cdots-d_s$ and

$$-K_{M_{\rm reg}} \simeq \mathcal{O}(m)|_{M_{\rm reg}}, \tag{5.4}$$

where we remark that this isomorphism is not G-equivariant. But the isomorphism

$$-K_{M_{\rm reg}} \simeq \mathcal{O}(N+1)|_{M_{\rm reg}} \otimes (\det N_{M_{\rm reg}})^{-1} \tag{5.5}$$

is G-equivalent if G acts on the RHS by $\sigma^{N+1} \otimes (\det \sigma_N)^{-1}$. Hence we can compute the moment map by studying the G-action on $N_{M_{\rm reg}}$. We first consider the case of hypersurfaces in $\mathbb{C}P^N$.

Lemma 5.1 ([Hou08], Proposition 3.2)**.** Let M be a hypersurface of degree d defined by $F \in H^0(\mathbb{C}P^N, \mathcal{O}(d))$ such that $g \cdot F = \rho(g)F$, where $\rho : G \to \mathbb{C}^*$ is a character of G. Then we have a G-equivariant isomorphism

$$N_{M_{\rm reg}} \simeq \mathbb{C}_{\rho^{-1}}|_{M_{\rm reg}} \otimes \mathcal{O}(d)|_{M_{\rm reg}} \tag{5.6}$$

where $\mathbb{C}_{\rho^{-1}}$ is a trivial bundle on $\mathbb{C}P^N$ with linearization ρ^{-1}.

Proof. Over $M_{\rm reg}$, dF gives a non-vanishing section of $\mathcal{O}(d)|_{M_{\rm reg}} \otimes N^*_{M_{\rm reg}}$, which gives an isomorphism

$$\mathbb{C}_\rho|_{M_{\rm reg}} \simeq \mathcal{O}(d)|_{M_{\rm reg}} \otimes N^*_{M_{\rm reg}}.$$

Since $g \cdot F = \rho(g)F$ for $g \in G$, we have $g \cdot dF = \rho(g)dF$ over $M_{\rm reg}$, which yields that this isomorphism is G-equivariant. □

Lemma 5.2 ([Hou08], Section 3)**.** Let h be the Hermitian metric on $\mathcal{O}(1)$ such that $\omega = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$ is the Fubini-Study metric of $\mathbb{C}P^N$. Then we have

$$\mu_{h^m,W} = \sum_{i=1}^{s} \beta_i + m\theta_W, \tag{5.7}$$

where h^m is the Hermitian metric on $-K_{M_{\rm reg}}$ defined via the isomorphism (5.4).

Proof. Since M is complete, the normal bundle $N_{M_{\rm reg}}$ splits over $M_{\rm reg}$ as

$$N_{M_{\rm reg}} \simeq \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_s).$$

Let ρ_i be the character of the G-action on F_i, then by Lemma 5.1, the G-action on $\det N_{M_{\rm reg}}$ is

$$\rho_1^{-1} \otimes \cdots \otimes \rho_s^{-1} \otimes \sigma^{d_1+\cdots+d_s}.$$

Therefore the G-action on $-K_{M_{\rm reg}}$ is

$$\rho_1 \otimes \cdots \otimes \rho_s \otimes \sigma^m.$$

Thus the moment map $\mu_{h^m,W}$ is given by

$$\mu_{h^m,W} = \sum_{i=1}^{s} \beta_i + m\theta_W.$$

$\square$

Step2. We compute the integral invariant I_k defined as follows: we set $N_i := \{F_i = 0\} \subset \mathbb{C}P^N$ $(i = 1, \dots s)$, and $M_i := N_1 \cap \cdots \cap N_i$ $(i = 1, \dots s)$. Thus we have an increasing sequence of subvariety in $\mathbb{C}P^N$:

$$M = M_s \subset M_{s-1} \subset \cdots \subset M_1 \subset M_0 := \mathbb{C}P^N.$$

For $k = 0, \dots, s$, we define an integral invariant $I_k = I_k(W)$ by

$$I_k := \int_{M_k} (\theta_W + \omega)^{N-k+1}. \tag{5.8}$$

Lemma 5.3 ([Lu99], Lemma 5.1)**.** We have

$$\begin{cases} I_k = d_k I_{k-1} - \beta_k \prod_{i=1}^{k-1} d_i \ \ (1 \leq k \leq s) \\ I_0 = 0, \end{cases} \tag{5.9}$$

where we put $\prod_{i=1}^{0} d_i = 1$ in the case of $k = 1$.

Proof. Define a smooth function ξ_i $(i = 1, \dots, s)$ on $\mathbb{C}P^N$ by

$$\xi_i = \frac{|F_i|^2}{\left(\sum_{i=0}^{N} |z^i|^2\right)^{d_i}}.$$

Using the Poincare-Lelong formula, we obtain

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\xi_k = [N_k] - d_k\omega,$$

where $[N_k]$ is the divisor of the zero locus of F_k. Then we have

$$\begin{aligned} I_k &= (N-k+1)\int_{M_k} \theta_W \omega^{N-k} \\ &= (N-k+1)\int_{M_{k-1}} \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\xi_k + d_k\omega\right) \wedge \theta_W \omega^{N-k} \\ &= (N-k+1)\int_{M_{k-1}} \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\xi_k \wedge \theta_W\omega^{N-k} + \frac{N-k+1}{N-k+2} d_k I_{k-1}. \end{aligned}$$

On the other hand, the direct computation shows that

$$W \log \xi_k = \beta_k - d_k\theta_W.$$

Hence integrating by parts, we obtain

$$\begin{aligned}(N-k+1)\int_{M_{k-1}} \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\xi_k \wedge \theta_W \omega^{N-k} &= -\int_{M_{k-1}} W \log \xi_k \omega^{N-k+1} \\ &= -\beta_k \prod_{i=1}^{k-1} d_i + \frac{1}{N-k+2} d_k I_{k-1}.\end{aligned}$$

Thus we obtain

$$\begin{aligned} I_k &= -\beta_k \prod_{i=1}^{k-1} d_i + \frac{1}{N-k+2} d_k I_{k-1} + \frac{N-k+1}{N-k+2} d_k I_{k-1} \\ &= d_k I_{k-1} - \prod_{i=1}^{k-1} d_i. \end{aligned}$$

The integral $I_0 = (N+1)\int_{\mathbb{C}P^N} \theta_W \omega^N$ is exatly the Futaki invariant $\mathrm{Fut}(W)$ of $\mathbb{C}P^N$ (up to a multiple constant). Since $\mathbb{C}P^N$ admits a Kähler-Einstein metric, the integral I_0 must be zero. $\square$

The equation (5.9) can be written as

$$\frac{I_k}{d_1 \cdots d_k} = \frac{I_{k-1}}{d_1 \cdots d_{k-1}} - \frac{\beta_k}{d_k}.$$

Therefore we have:

Lemma 5.4 ([Lu99], Theorem 5.1)**.** For $k = 1, \ldots, s$, we have

$$I_k = -\prod_{i=1}^{k} d_i \cdot \sum_{i=1}^{k} \frac{\beta_i}{d_i}.$$

Theorem 5.5 ([Lu99], Theorem 1.1)**.** Let M be a normal $\mathbb{Q}$-Fano complete intersection in $\mathbb{C}P^N$ defined by homogeneous polynomials $F_1, \ldots, F_s$ of degree $d_1, \ldots, d_s$ respectively. Let $W \in \mathfrak{sl}(N+1, \mathbb{C})$ be a holomorphic vector field on $\mathbb{C}P^N$ such that

$$W F_i = \beta_i F_i$$

for some constants α_i ($i = 1, \ldots, s$). Then the Futaki invariant $\mathrm{Fut}(W)$ can be written as

$$\mathrm{Fut}(W) = -\sum_{i=1}^{s} \beta_i + \frac{m}{N-s+1} \sum_{i=1}^{s} \frac{\beta_i}{d_i}, \tag{5.10}$$

where $m = N + 1 - d_1 - \cdots - d_s$.

Proof. By Lemma 5.2, we have

$$\begin{aligned}
\mathrm{Fut}(W) &= -\frac{1}{m^{N-s}\prod_{i=1}^{s} d_i}\int_M \left(\sum_{i=1}^{s}\beta_i + m\theta_W\right)(m\omega)^{N-s} \\
&= -\frac{1}{\prod_{i=1}^{s} d_i}\left(\sum_{i=1}^{s}\beta_i\int_M \omega^{N-s} + m\int_M \theta_W\omega^{N-s}\right) \\
&= -\frac{1}{\prod_{i=1}^{s} d_i}\left(\prod_{i=1}^{s} d_i\cdot\sum_{i=1}^{s}\beta_i + \frac{m}{N-s+1}I_s\right).
\end{aligned}$$

Combining with Lemma 5.4, we have the desired result. □

As a corollary, we have:

Corollary 5.6 ([Lu99], Corollary 1.1)**.** Let M be a hypersurface in $\mathbb{C}P^N$ defined by the homogeneous polynomial F of degree d and $W \in \mathfrak{sl}(N+1,\mathbb{C})$ a holomorphic vector field on $\mathbb{C}P^N$ satisfying

$$WF = \beta F$$

for some constant β. Then the Futaki invariant $\mathrm{Fut}(W)$ can be written as

$$\mathrm{Fut}(W) = -\frac{(N+1)(d-1)}{Nd}\beta.$$

In particular, $\mathrm{Re}(\mathrm{Fut}(W))$ and $-\mathrm{Re}(\beta)$ have the same signature.

5.2 The modified Futaki invariant of complete intersections

5.2.1 Calculations of the function $\mathcal{F}$

Let M be an n-dimensional normal $\mathbb{Q}$-Fano variety. We assume the same condition as in Section 4.1.2. Let h be an admissible Hermitian metric on $-K_{M_{\mathrm{reg}}}$ with positive curvature $\omega := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$. We define a function $\mathcal{F}$ on $\mathfrak{h} := \mathrm{Lie}(H)$ by

$$\mathcal{F}(V) = -\frac{1}{c_1(M)^n}\int_M e^{\mu_{h,V}}\omega^n \tag{5.11}$$

and the modified Futaki invariant Fut_V by

$$\mathrm{Fut}_V(W) = \frac{d}{dt}\mathcal{F}(V+tW)\bigg|_{t=0} = -\frac{1}{c_1(M)^n}\int_M \mu_{h,W}e^{\mu_{h,V}}\omega^n, \tag{5.12}$$

Then,

Lemma 5.7. The function $\mathcal{F}$ and Fut_V are independent of the embedding $M \hookrightarrow N$ and a choice of admissible Hermitian metric h on $-K_{M_{\mathrm{reg}}}$.

This was shown in [Hou08, Section 2.3] using the equivariant Chern-Weil theorem. When M has log-terminal singularities, the modified Futaki invariant (5.12) coincides with the one defined in quantized settings (Section 4.2.3). We will show this in Proposition 5.12.

Now let M be a normal $\mathbb{Q}$-Fano complete intersection in $\mathbb{C}P^N$ defined by the homogeneous polynomials $F_1, \dots, F_s$ of degree $d_1, \dots, d_s$ respectively. We adopt the same notations as in Section 5.1. Let V be a holomorphic vector field on $\mathbb{C}P^N$ such that

$$VF_i = \alpha_i F_i$$

for some constants α_i $(i = 1, \dots, s)$. We define the integrals $I_{k,l} = I_{k,l}(V)$ $(k = 0, 1, \dots, s;\ l \geq 0)$ by

$$I_{k,l} = m^l \int_{M_k} (\theta_V)^l e^{m\theta_V} \omega^{N-k}. \tag{5.13}$$

Lemma 5.8. For $k = 1, \dots, s$, $I_{k,0}$ satisfies

$$I_{k,0} = \left(d_k - \frac{m\alpha_k}{N-k+1} \right) I_{k-1,0} + \frac{d_k}{N-k+1} I_{k-1,1}. \tag{5.14}$$

Proof. We can prove (5.14) in the same way as Lemma 5.3. Define a smooth function ξ_i $(i = 1, \dots, s)$ on $\mathbb{C}P^N$ by

$$\xi_i = \frac{|F_i|^2}{\left(\sum_{i=0}^N |z^i|^2 \right)^{d_i}}.$$

Using the Poincare-Lelong formula, we obtain

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \xi_k = [N_k] - d_k \omega,$$

where $[N_k]$ is the divisor of the zero locus of F_k. Then we have

$$\begin{aligned} I_{k,0} &= \int_{M_k} e^{m\theta_V} \omega^{N-k} \\ &= \int_{M_{k-1}} \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \xi_k + d_k \omega \right) \wedge e^{m\theta_V} \omega^{N-k} \\ &= \int_{M_{k-1}} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \xi_k \wedge e^{m\theta_V} \omega^{N-k} + d_k I_{k-1,0}. \end{aligned}$$

On the other hand, using the relation

$$V \log \xi_k = \alpha_k - d_k \theta_V$$

and integrating by parts, we obtain

$$\begin{aligned} &\int_{M_{k-1}} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \xi_k \wedge e^{m\theta_V} \omega^{N-k} \\ &= -\frac{m}{N-k+1} \int_{M_{k-1}} V(\log \xi_k) e^{m\theta_V} \omega^{N-k+1} \\ &= -\frac{m\alpha_k}{N-k+1} I_{k-1,0} + \frac{d_k}{N-k+1} I_{k-1,1}. \end{aligned}$$

Thus we get the desired result. $\square$

Corollary 5.9.

$$c_1(M)^{N-s}\left(=m^{N-s}\int_M \omega^{N-s}\right)=m^{N-s}\prod_{i=1}^s d_i. \tag{5.15}$$

Proof. If we set $V\equiv 0$ in Lemma 5.8, then we have $\alpha_k=0$, $I_{k,1}=0$ and hence $I_{k,0}=d_k I_{k-1,0}$. Hence we have

$$\begin{aligned} c_1(M)^{N-s} &= m^{N-s}\int_M \omega^{N-s}=m^{N-s}I_{s,0} \\ &= m^{N-s}\prod_{i=1}^s d_i\cdot I_{0,0}=m^{N-s}\prod_{i=1}^s d_i. \end{aligned}$$

$\square$

In order to get the explicit expression of $I_{k,0}$, we show the next lemma.

Lemma 5.10. For $k=1,\ldots,s$, the equation

$$\begin{aligned} & \frac{(N-k)!}{m^{N-k}}\int_{\mathbb{C}P^N}\prod_{i=1}^k(d_i\omega+d_i\theta_V-\alpha_i)e^{m\theta_V}\cdot e^{m\omega} \\ + & \frac{(N-k-1)!}{m^{N-k}}\sum_{i=1}^k\int_{\mathbb{C}P^N}(d_i\theta_V-\alpha_i)\cdot\prod_{p\in\{1,\ldots,k\}-\{i\}}(d_p\omega+d_p\theta_V-\alpha_p)e^{m\theta_V}\cdot e^{m\omega} \\ = & \frac{(N-k-1)!}{m^{N-k-1}}\int_{\mathbb{C}P^N}\prod_{i=1}^k(d_i\omega+d_i\theta_V-\alpha_i)\cdot\omega\cdot e^{m\theta_V}\cdot e^{m\omega} \end{aligned} \tag{5.16}$$

holds.

Proof. For $i=0,\ldots,k$, put

$$J_i:=\sum_{1\le p_1<\cdots<p_i\le k}d_{p_1}\cdots d_{p_i}\int_{\mathbb{C}P^N}(d_{q_1}\theta_V-\alpha_{q_1})\cdots(d_{q_{k-i}}\theta_V-\alpha_{q_{k-i}})e^{m\theta_V}\omega^N,$$

where $q_1<\cdots<q_{k-i}$ and $\{q_1,\ldots,q_{k-i}\}=\{1,\ldots,k\}-\{p_1,\ldots,p_i\}$. Then the direct computation shows that

$$\frac{(N-k)!}{m^{N-k}}\int_{\mathbb{C}P^N}\prod_{i=1}^k(d_i\omega+d_i\theta_V-\alpha_i)e^{m\theta_V}\cdot e^{m\omega}=J_0+\sum_{i=1}^k\frac{m^i}{(N-k+1)\cdots(N-k+i)}J_i$$

and

$$\begin{aligned} & \frac{(N-k-1)!}{m^{N-k}}\sum_{i=1}^k\int_{\mathbb{C}P^N}(d_i\theta_V-\alpha_i)\cdot\prod_{p\in\{1,\ldots,k\}-\{i\}}(d_p\omega+d_p\theta_V-\alpha_p)e^{m\theta_V}\cdot e^{m\omega} \\ = & \sum_{i=1}^k\frac{im^i}{(N-k)\cdots(N-k+i)}J_i. \end{aligned}$$

Hence the LHS of (5.16) is

$$J_0 + \sum_{i=1}^{k} \frac{m^i}{(N-k)\cdots(N-k+i-1)} J_i,$$

which is equal to the RHS of (5.16). □

Lemma 5.11. For $k = 1, \dots, s$, $I_{k,0}$ can be written as

$$I_{k,0} = \frac{(N-k)!}{m^{N-k}} \int_{\mathbb{C}P^N} \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}. \tag{5.17}$$

Proof. We will prove (5.17) by induction for k. When $k = 1$, the equation (5.17) coincides exactly with (5.14), so the statement holds.

Next, we assume that (5.17) holds for a fixed k. Then by Lemma 5.8, we have

$$I_{k+1,0} = \left(d_{k+1} - \frac{m\alpha_{k+1}}{N-k} \right) I_{k,0} + \frac{d_{k+1}}{N-k} I_{k,1}.$$

Using the induction hypothesis, we have

$$\frac{m\alpha_{k+1}}{N-k} I_{k,0} = \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} \alpha_{k+1} \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}$$

and

$$\begin{aligned}
& \frac{d_{k+1}}{N-k} I_{k,1} \\
= \ & \frac{d_{k+1}}{N-k} \cdot \frac{d}{dt} I_{k,0}(V + tV) \bigg|_{t=0} \\
= \ & d_{k+1} \frac{(N-k-1)!}{m^{N-k}} \sum_{i=1}^{k} \int_{\mathbb{C}P^N} (d_i\theta_V - \alpha_i) \cdot \prod_{p \in \{1,\dots,k\} - \{i\}} (d_p\omega + d_p\theta_V - \alpha_p) e^{m\theta_V} \cdot e^{m\omega} \\
+ \ & \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} d_{k+1}\theta_V \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}
\end{aligned}$$

Hence combining with Lemma 5.10, we obtain

$$\begin{aligned}
I_{k+1,0} &= d_{k+1}(\text{the LHS of } (5.16)) \\
&+ \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} (d_{k+1}\theta_V - \alpha_{k+1}) \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega} \\
&= \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} \prod_{i=1}^{k+1} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}.
\end{aligned}$$

Therefore the statement holds for $k+1$. □

Proof of Theorem 1.2. By Lemma 5.2 and Corollary 5.9, $\mathcal{F}$ can by written as

$$\begin{aligned}\mathcal{F}(V) &= -\frac{1}{m^{N-s}\prod_{i=1}^{s} d_i}\int_M \exp\left(\sum_{i=1}^{s}\alpha_i + m\theta_V\right)(m\omega)^{N-s} \\ &= -\frac{1}{\prod_{i=1}^{s} d_i}\cdot\exp\left(\sum_{i=1}^{s}\alpha_i\right) I_{s,0}.\end{aligned}$$

Thus, combining with Lemma 5.11, we get the desired formula for $\mathcal{F}$. □

5.2.2 Another proof of Theorem 1.2

Let M be a normal $\mathbb{Q}$-Fano variety with log-terminal singularities. Let V be a holomorphic vector field on M generating a torus action and W a holomorphic vector field on M generating a $\mathbb{C}^*$-action commuting with V. We remark that Theorem 1.2 holds even if the singularities of M are not log-terminal and V is any holomorphic vector field. But we need these assumptions in order to prove Theorem 1.2 from the view point of the quantization of the function $\mathcal{F}$.

Set

$$N_k := \dim H^0(M, -kK_M).$$

We define the quantization of the function $\mathcal{F}$ at level k as

$$\mathcal{F}_k(V) := -k\mathrm{Trace}(e^{V/k})_{H^0(M,-kK_M)} = -k\sum_{i=1}^{N_k}\exp(v_i^{(k)}/k), \tag{5.18}$$

where $(v_i^{(k)})$ are the joint eigenvalues for the action of $\mathrm{Re}(V)$ on $H^0(M, -kK_M)$ defined by the canonical lift of V to $-K_M$. Then the quantized modified Futaki invariant at level k (see Section 4.2.3) is given by the Gâteaux differential of $\mathcal{F}_k$ at V in the direction W:

$$\mathrm{Fut}_{V,k}(W) = \frac{d}{dt}\mathcal{F}_k(V+tW)\bigg|_{t=0} = -\sum_{i=1}^{N_k}\exp(v_i^{(k)}/k)w_i^{(k)}, \tag{5.19}$$

where $(v_i^{(k)}, w_i^{(k)})$ are the joint eigenvalues for the commuting action of $\mathrm{Re}(V)$ and $\mathrm{Re}(W)$. Then we have:

Proposition 5.12. In the case when M is smooth,

(1) We have the asymptotic expansion of $\mathcal{F}_k(V)$ as $k \to \infty$:

$$\mathcal{F}_k(V) = \mathcal{F}^{(0)}(V)\cdot k^{n+1} + \mathcal{F}^{(1)}(V)\cdot k^n + \cdots,$$

where $\mathcal{F}^{(0)}(V)$ is proportional to $\mathcal{F}(V)$.

(2) We have the asymptotic expansion of $\mathrm{Fut}_{V,k}(W)$ as $k \to \infty$:

$$\mathrm{Fut}_{V,k}(W) = \mathrm{Fut}_V^{(0)}(W)\cdot k^{n+1} + \mathrm{Fut}_V^{(1)}(W)\cdot k^n + \cdots,$$

where $\mathrm{Fut}_V^{(i)}(W)$ is the i th order modified Futaki invariant defined in [BN14, Section 4.4], and $\mathrm{Fut}_V^{(0)}(W)$ is proportional to $\mathrm{Fut}_V(W)$.

(3) the i th order modified Futaki invariant $\mathrm{Fut}_V^{(i)}(W)$ is the Gâteaux differential of $\mathcal{F}^{(i)}$ at V in the direction W, i.e.,

$$\frac{d}{dt}\mathcal{F}_k^{(i)}(V+tW)\Big|_{t=0} = \mathrm{Fut}_V^{(i)}(W).$$

In general, when M is a (possibly singular) Fano variety, we have

(4)

$$\mathcal{F}(V) = \lim_{k\to\infty}\frac{1}{kN_k}\mathcal{F}_k(V).$$

(5)

$$\mathrm{Fut}_V(W) = \lim_{k\to\infty}\frac{1}{kN_k}\mathrm{Fut}_{V,k}(W),$$

where the LHS is the modified Futaki invariant defined as an integral invariant (5.12). Thus two definitions of the modified Futaki invariant are equivalent when M has log-terminal singularities.

Proof. The statements (2) and (5) were shown in [BN14, Section 4.4]. (3) is trivial from the definition of $\mathrm{Fut}_{V,k}(W)$.

(1) As with the proof of (2) (cf. [BN14, Section 4.4]) or [WZZ14, Lemma 1.2], $\mathcal{F}_k(V)$ can be calculated by the equivariant Riemann-Roch formula as

$$\begin{aligned}\mathcal{F}_k(V) &= -k\mathrm{Trace}(e^{V/k})_{H^0(M,-kK_M)}\\ &= -k\int_M \mathrm{ch}^{\mathfrak{g}}(-kK_M)\mathrm{td}^{\mathfrak{g}}(M)\\ &= -k\int_M e^{\mu_{h,V}}\cdot e^{k\omega}\mathrm{td}^{\mathfrak{g}}(M)\\ &= -\frac{1}{n!}\int_M e^{\mu_{h,V}}\omega^n\cdot k^{n+1}+O(k^n),\end{aligned}$$

where $\mathrm{ch}^{\mathfrak{g}}$ (resp. $\mathrm{td}^{\mathfrak{g}}$) denotes the equivariant Chern character (resp. the equivariant Todd class). Thus $\mathcal{F}^{(0)}(V) = \frac{c_1(M)^n}{n!}\cdot\mathcal{F}(V)$.

(4) By definition, $\mathcal{F}(V)$ can be written as

$$\mathcal{F}(V) = -\frac{1}{c_1(M)^n}\int_M e^{\mu_{h,V}}\omega^n = -\int_{\mathbb{R}} e^v\nu^V,$$

where ν^V is the push forward measure of the Monge-Ampère measure $\frac{\omega^n}{c_1(M)^n}$ under $\mu_{h,V}$. Let ν_k^V be the spectral measure on $\mathbb{R}$ attached to the infinitesimal action of $\mathrm{Re}(V)$ on $H^0(M,-kK_M)$:

$$\nu_k^V = \frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{v_i^{(k)}/k}.$$

Then by [BN14, Proposition 4.1], ν_k^V converges to ν^V as $k \to \infty$ in a weak topology. Hence we have

$$\frac{1}{kN_k}\mathcal{F}_k(V) = -\frac{1}{N_k}\sum_{i=1}^{N_k} \exp(v_i^{(k)}/k) = -\int_M e^v \nu_k^V \to -\int_{\mathbb{R}} e^v \nu^V = \mathcal{F}(V)$$

as $k \to \infty$. □

Remark 5.13. When M is smooth, by the equivariant Riemann-Roch formula, we have an asymptotic expansion as $k \to \infty$:

$$N_k = \frac{1}{n!}c_1(M)^n \cdot k^n + O(k^{n-1}). \tag{5.20}$$

Combining with Proposition 5.12 (1), we have

$$\frac{1}{kN_k}\mathcal{F}_k(V) = \mathcal{F}(V) + O(k^{-1}) \tag{5.21}$$

as $k \to \infty$. In general, when M is a (possibly singular) Fano variety, we do not know whether we can obtain the expansion (5.21). However, Proposition 5.12 (4) allows us to use the equivariant Riemann-Roch formula formally to compute the leading term of (5.21) (i.e., the limit $\lim_{k\to\infty} \frac{1}{kN_k}\mathcal{F}_k(V)$) even if M has singularities.

Now we give another proof of the main theorem using this algebraic formula for $\mathcal{F}$.

Lemma 5.14 ([AV11], Lemma 5.1)**.** Let B be a holomorphic vector bundle of rank b on a manifold M, then

$$\sum_{i=0}^{b}(-1)^i \mathrm{ch}(\wedge^i B) = c_b(B)\mathrm{td}(B)^{-1}.$$

Proof. Let $r_1, \dots, r_b$ be the Chern roots of B. Since

$$\mathrm{ch}(\wedge^i B^*) = \sum_{1\le p_1 < \cdots < p_i \le b} e^{-(r_{p_1}+\cdots+r_{p_i})},$$

we obtain

$$\begin{aligned}
\sum_{i=0}^{b}(-1)^i \mathrm{ch}(\wedge^i B^*) &= \sum_{i=0}^{b}(-1)^i \sum_{1\le p_1<\cdots<p_i\le b} e^{-(r_{p_1}+\cdots+r_{p_i})} \\
&= \prod_{p=1}^{b}(1-e^{-r_p}) \\
&= \prod_{p=1}^{b} r_p \prod_{p=1}^{b} \frac{1-e^{-r_p}}{r_p} \\
&= c_b(B)\mathrm{td}(B)^{-1}.
\end{aligned}$$

□

Now let M be a Fano complete intersection in $\mathbb{C}P^N$. We will adopt the notation in Section 5.2.1. We further assume that $V \in \mathfrak{sl}(N+1,\mathbb{C})$ is a Hermitian matrix. Then $\mathrm{Im}(V)$ is Killing with respect to the Fubini-Study metric ω.

Lemma 5.15 ([AV11], Lemma 5.2)**.** We have the following asymptotic expansion of N_k as $k \to \infty$:

$$N_k = \frac{d_1 \cdots d_s m^{N-s}}{(N-s)!} \cdot k^{N-s} + O(k^{N-s-1}). \tag{5.22}$$

Lemma 5.16. We have the following asymptotic expansion of $\mathcal{F}_k(V)$ as $k \to \infty$:

$$\mathcal{F}_k(V) = -\exp\left(\sum_{i=1}^{s} \alpha_i\right) \int_{\mathbb{C}P^N} \prod_{i=1}^{s} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega} \cdot k^{N-s+1} + O(k^{N-s}). \tag{5.23}$$

Proof. This proof is essentially based on the argument in [AV11, Lemma 5.3]. The only difference between Lemma 5.16 and [AV11, Lemma 5.3] is the linearization of $-K_M$, to which we only have to pay attention. In order to avoid confusion, let $L(\simeq O(m))$ be a linearized line bundle on $\mathbb{C}P^N$ such that $L|_M$ is isomorphic to $-K_M$ as a linearized line bundle whose linearization is determined by the canonical lift of V to $-K_M$. Let G be the Lie group generated by V and ρ_i be the character of the G-action on F_i. Let $\mathbb{C}_{\rho_i^{-1}}$ be a trivial bundle on $\mathbb{C}P^N$ with linearization ρ_i^{-1}. Set $L_i := \mathcal{O}(d_i) \otimes \mathbb{C}_{\rho_i^{-1}}$ and $B := L_1 \oplus \cdots \oplus L_s$. Then $\mathrm{rank} B = s$ and the section $F := (F_1, \ldots, F_s) \in H^0(\mathbb{C}P^N, B)$ is invariant. Since M is complete, the Koszul complex:

$$0 \to \wedge^s B^* \to \wedge^{s-1} B^* \to \cdots \to B^* \to \mathcal{O}_{\mathbb{C}P^N} \to \mathcal{O}_M \to 0$$

is exact and equivariant, where $\mathcal{O}_M$ denotes the structure sheaf of M. Tensoring by L^k preserves the exactness and equivariance, so we obtain

$$\chi^{\mathfrak{g}}(M, L^k|_M) = \sum_{i=0}^{s} (-1)^i \chi^{\mathfrak{g}}(\mathbb{C}P^N, L^k \otimes \wedge^i B^*),$$

where $\chi^{\mathfrak{g}}$ denotes the Lefschetz number. By the equivariant Riemann-Roch formula and Lemma 5.14, we get

$$\begin{aligned}
\mathcal{F}_k(V) &= -k \sum_{i=0}^{s} (-1)^i \chi^{\mathfrak{g}}(\mathbb{C}P^N, L^k \otimes \wedge^i B^*) \\
&= -k \sum_{i=0}^{s} (-1)^i \int_{\mathbb{C}P^N} \mathrm{ch}^{\mathfrak{g}}(\wedge^i B^*) e^{kc_1^{\mathfrak{g}}(L)} \mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N) \\
&= -k \int_{\mathbb{C}P^N} \left(\sum_{i=0}^{s} (-1)^i \mathrm{ch}^{\mathfrak{g}}(\wedge^i B^*)\right) e^{kc_1^{\mathfrak{g}}(L)} \mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N) \\
&= -k \int_{\mathbb{C}P^N} c_s^{\mathfrak{g}}(B) \mathrm{td}^{\mathfrak{g}}(B)^{-1} e^{kc_1^{\mathfrak{g}}(L)} \mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N) \\
&= -k \int_{\mathbb{C}P^N} \prod_{i=1}^{s} \left(d_i c_1^{\mathfrak{g}}(\mathcal{O}(1)) - \frac{\alpha_i}{k}\right) \cdot \mathrm{td}^{\mathfrak{g}}(B)^{-1} e^{kc_1^{\mathfrak{g}}(L)} \mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N).
\end{aligned}$$

Let h be a Hermitian metric on $\mathcal{O}(1)$ such that $\omega = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$ is the Fubini-Study metric of the $\mathbb{C}P^N$. Then by Lemma 5.2, the equivariant 1st Chern form for $(h, V/k)$ and $(h^m, V/k)$ are written as

$$\omega + \frac{1}{k}\theta_V \in c_1^{\mathfrak{g}}(\mathcal{O}(1)) \text{ and } m\omega + \frac{m}{k}\theta_V + \frac{1}{k}\sum_{i=1}^{s}\alpha_i \in c_1^{\mathfrak{g}}(L)$$

respectively. Both $\mathrm{td}^{\mathfrak{g}}(B)^{-1}$ and $\mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N)$ can be written as the form

$$1 + A + \sum_{i\geq 1}\frac{1}{k^i}B_i,$$

where A (resp. B_i) denotes $2l$-forms ($l \geq 1$ (resp. $l \geq 0$)) not depending on k. Hence we have

$$\begin{aligned}
\mathcal{F}_k(V) &= -k\exp\left(\sum_{i=1}^{s}\alpha_i\right)\int_{\mathbb{C}P^N}\prod_{i=1}^{s}\left(d_i\omega + \frac{1}{k}(d_i\theta_V - \alpha_i)\right)\mathrm{td}^{\mathfrak{g}}(B)^{-1}e^{m\theta_V}\cdot e^{km\omega}\mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N) \\
&= -\exp\left(\sum_{i=1}^{s}\alpha_i\right)\int_{\mathbb{C}P^N}\prod_{i=1}^{s}(d_i\omega + d_i\theta_V - \alpha_i)e^{m\theta_V}\cdot e^{m\omega}\cdot k^{N-s+1} + O(k^{N-s}).
\end{aligned}$$

□

Proof of Theorem 1.2. By Lemma 5.15 and Lemma 5.16, we have an asymptotic expansion as $k \to \infty$:

$$\frac{1}{kN_k}\mathcal{F}_k(V) = -\frac{(N-s)!}{d_1\cdots d_s m^{N-s}}\exp\left(\sum_{i=1}^{s}\alpha_i\right)\int_{\mathbb{C}P^N}\prod_{i=1}^{s}(d_i\omega + d_i\theta_V - \alpha_i)e^{m\theta_V}\cdot e^{m\omega} + O(k^{-1}).$$

On the other hand, by Proposition 5.12 (4), $\frac{1}{kN_k}\mathcal{F}_k(V)$ converges to $\mathcal{F}(V)$ as $k \to \infty$. Hence we have the desired formula. □

5.2.3 Examples of computing $\mathcal{F}$

In this section, we compute $\mathcal{F}$ for several examples (cf. [Lu99, Section 6]). Let M be a Fano complete intersection in $\mathbb{C}P^N$. We will adopt the notation in Section 5.2.1. First, we will mention some results obtained as a corollary of the localization formula in holomorphic equivariant cohomology theory (cf. [Liu95, Theorem 1.6]).

Lemma 5.17. If $V = \mathrm{diag}(\lambda_0, \ldots, \lambda_N)$ is a diagonal matrix with different eigenvalues $\lambda_0, \ldots, \lambda_N$. Then we have

$$I_{0,0} = N!\sum_{i=0}^{N}\frac{e^{m\lambda_i}}{\prod_{p\in\{0,\ldots,N\}-\{i\}}(\lambda_i - \lambda_p)}. \tag{5.24}$$

Since $I_{0,l}$ are given by the derivatives of $I_{0,0}$, we can calculate $I_{0,l}$ for any integer l. On the other hand, by Theorem 1.2, $\mathcal{F}(V)$ can be written as a linear combination of $I_{0,s}$ $(0 \leq l \leq s)$. Hence we can express $\mathcal{F}(V)$ in terms of the eigenvalues of V.

However, we can calculate $\mathcal{F}(V)$ without using Theorem 1.2 in a special case: we assume that M has at worst orbifold singularities and V satisfies the condition

1. V has isolated zero points $\{p_i\}$.

2. V is nondegenerate at each zero point p_i, i.e., for each local uniformization $\pi : U \to U/\Gamma_i \subset M$ with $\pi(U) \cap p_i \neq \emptyset$, $\pi^* V$ vanishes along $\pi^{-1}(p_i)$ and the matrix $B_i = \left(-\frac{\partial v_j^i}{\partial z^k} \right)_{1 \leq j,k \leq N-s}$ is nondegenerate near $\pi^{-1}(p_i)$, where $(z^1, \ldots, z^{N-s})$ is local holomorphic coordinates around $\pi^{-1}(p_i)$ and $V = \sum_{j=1}^{N-s} v_j^i \frac{\partial}{\partial z^j}$.

In the same way as [DT92, Proposition 1.2], we have:

Lemma 5.18. Let M and V be as above. Then we have

$$\mathcal{F}(V) = -\frac{(N-s)!}{d_1 \cdots d_s} \exp\left(\sum_{i=1}^{s} \alpha_i \right) \cdot \sum_i \frac{1}{|\Gamma_i|} \cdot \frac{e^{m\theta_V(p_i)}}{\det B_i}, \tag{5.25}$$

where $|\Gamma_i|$ is the order of the local uniformization group Γ_i at a point p_i.

Remark 5.19. One can extend Lemma 5.17 and Lemma 5.18 to the case when the zero set of V is the sum of nondegenerate submanifolds, where the word "nondegenerate" means that the induced actions of V to the normal bundle of submanifolds are nondegenerate. However, since $I_{0,0}(V)$ and $\mathcal{F}(V)$ are clearly continuous with respect to V, we may think that the equations (5.24) and (5.25) hold in the sense of limit $V_\epsilon \to V$ of any expression. For instance,

Lemma 5.20. Let $m = 1$ and $V = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_2) \in \mathfrak{sl}(4, \mathbb{C})$ a holomorphic vector field on $\mathbb{C}P^3$, where λ_0, λ_1 and λ_2 are different numbers. Then we have

$$\begin{aligned} I_{0,0} &= 6 \left[\frac{e^{\lambda_0}}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)^2} + \frac{e^{\lambda_1}}{(\lambda_1 - \lambda_0)(\lambda_1 - \lambda_2)^2} \right. \\ &+ \left. \frac{\{\lambda_0 + \lambda_1 - 2\lambda_2 + (\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)\} e^{\lambda_2}}{(\lambda_2 - \lambda_0)^2 (\lambda_2 - \lambda_1)^2} \right]. \end{aligned} \tag{5.26}$$

Proof. Let $\epsilon \neq 0$ be a small number. If we set $V_\epsilon := \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2 + \epsilon, \lambda_2 - \epsilon)$, then V_ϵ has different eigenvalues. Hence we can compute $I_{0,0}(V) = \lim_{\epsilon \to 0} I_{0,0}(V_\epsilon)$ directly using (5.24). $\square$

Example 5.21. Let $M \subset \mathbb{C}P^3$ be the zero set of a cubic polynomial $F := z_0 z_1^2 + z_2 z_3 (z_2 - z_3)$, where (z_0, z_1, z_2, z_3) are homogeneous coordinates of $\mathbb{C}P^3$ and $V = \mathrm{diag}(-7t, 5t, t, t)$ $(t \neq 0)$ a holomorphic vector field tangent to M. We compute $\mathcal{F}$ in two methods:

(1) The variety M has a unique quotient singularity at $p_0 := [1, 0, 0, 0]$. If we restricts V to M, V has five zeros $p_0 = [1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$ and

$[0,0,1,1]$. Let $\zeta_i := \frac{z_i}{z_0}$ $(i = 1, 2, 3)$ be Euclidean coordinates defined near p_0. Then we can rewrite F near p_0 in the standard form

$$f = \frac{F}{z_0^3} = \zeta_1^2 - \zeta_3(\zeta_2^2 - 4\zeta_3^2).$$

According to [Liu95, Example 1], we see that there is a uniformization $\phi : \mathbb{C}^2 \to \mathbb{C}^2/\Gamma \subset M$ defined by

$$\phi : \begin{cases} \zeta_1 = uv(u^4 - v^4) \\ \zeta_2 = u^4 + v^4 \\ \zeta_3 = u^2v^2, \end{cases}$$

where Γ is the dihedral subgroup in $SU(2)$ of type D_4. Thus we have $\phi^*(V) = 2tu\frac{\partial}{\partial u} + 2tv\frac{\partial}{\partial v}$. Since the order of the group D_4 is 8, applying Lemma 5.18, we obtain

$$\begin{aligned} \mathcal{F}(V) &= -\frac{2}{3}e^{3t}\left(\frac{1}{8}\cdot\frac{e^{-7t}}{4t^2} + \frac{e^{5t}}{16t^2} + 3\cdot\frac{e^{t}}{-32t^2}\right) \\ &= -\frac{e^{-4t}}{48t^2} - \frac{e^{8t}}{24t^2} + \frac{e^{4t}}{16t^2}. \end{aligned}$$

(2) By Theorem 1.2, we obtain

$$\begin{aligned} \mathcal{F}(V) &= -\frac{2}{3}e^{3t}\int_{\mathbb{C}P^3}(3\omega + 3\theta_V - 3t)e^{\theta_V}e^{\omega} \\ &= -e^{3t}\left\{\left(1 - \frac{t}{3}\right)I_{0,0} + \frac{1}{3}I_{0,1}\right\}. \end{aligned}$$

By Lemma 5.17, we have

$$I_{0,0} = -\frac{e^{-7t}}{128t^3} + \frac{e^{5t}}{32t^3} - \frac{3(1+8t)e^t}{128t^3}$$

and

$$I_{0,1} = \frac{(7t+3)e^{-7t}}{128t^3} + \frac{(5t-3)e^{5t}}{32t^3} - \frac{3(8t^2-15t-3)e^t}{128t^3}.$$

Hence we have

$$\mathcal{F}(V) = -\frac{e^{-4t}}{48t^2} - \frac{e^{8t}}{24t^2} + \frac{e^{4t}}{16t^2}.$$

The next example is a new example of computing $\mathcal{F}$ for a normal $\mathbb{Q}$-Fano variety whose singurarity is not log-terminal.

Example 5.22. Let $M \subset \mathbb{C}P^3$ be the zero locus of the cubic polynomial $F := z_0^3 + z_1^3 + z_2^3 = 0$ and $V = \mathrm{diag}(t, t, t, -3t)$ $(t \neq 0)$ a holomorphic vector field tangent to M. Then M has a unique singularity at $[0,0,0,1]$. Let $\pi : \widetilde{M} \to M$ a resolution of M. By the adjunction formula, we have

$$K_{\widetilde{M}} = \pi^*K_M - E,$$

where E is an exceptional divisor (an elliptic curve). Hence the singularity of M is not log-terminal[9]. By Theorem 1.2, we get

$$\mathcal{F}(V) = -e^{3t}\left\{\left(1-\frac{t}{3}\right) I_{0,0} + \frac{1}{3} I_{0,1}\right\},$$

$$I_{0,0} = \frac{3(8t^2-4t+1)e^t}{32t^3} - \frac{3e^{-3t}}{32t^3},$$

$$I_{0,1} = \frac{3(8t^3-12t^2+9t-3)e^t}{32t^3} + \frac{9(t+1)e^{-3t}}{32t^3}$$

and

$$\mathcal{F}(V) = \frac{(1-4t)e^{4t}}{8t^2} - \frac{1}{8t^2}.$$

Example 5.23. Let $M \subset \mathbb{C}P^4$ be the zero locus defined by

$$\begin{cases} F_1 = z_0 z_1 + z_2^2 \\ F_2 = z_1^2 + z_3 z_4 \end{cases}$$

and $V = \mathrm{diag}(-7t, 3t, -2t, 5t, t)$ $(t \neq 0)$ a holomorphic vector field tangent to M. Then M has a unique quotient singularity at $[1,0,0,0,0]$. By Theorem 1.2, we have

$$\mathcal{F}(V) = -e^{2t}\left\{\left(1-\frac{t}{3}-\frac{t^2}{2}\right) I_{0,0} + \left(\frac{2}{3}-\frac{t}{12}\right) I_{0,1} + \frac{1}{12} I_{0,2}\right\},$$

$$I_{0,0} = \frac{e^{-7t}}{200t^4} - \frac{3e^{3t}}{25t^4} - \frac{24e^{-2t}}{525t^4} + \frac{e^{5t}}{28t^4} + \frac{e^t}{8t^4},$$

$$I_{0,1} = -\frac{(7t+4)e^{-7t}}{200t^4} + \frac{3(4-3t)e^{3t}}{25t^4} + \frac{48(t+2)e^{-2t}}{525t^4} + \frac{(5t-4)e^{5t}}{28t^4} + \frac{(t-4)e^t}{8t^4}$$

and

$$\begin{aligned} I_{0,2} &= \frac{(49t^2+56t+20)e^{-7t}}{200t^4} - \frac{3(9t^2-24t+20)e^{3t}}{25t^4} - \frac{96(t^2+4t+5)e^{-2t}}{525t^4} \\ &+ \frac{5(5t^2-8t+4)e^{5t}}{28t^4} + \frac{(t^2-8t+20)e^t}{8t^4}. \end{aligned}$$

Hence we have

$$\mathcal{F}(V) = -\frac{e^{-5t}}{48t^2} - \frac{e^{7t}}{24t^2} + \frac{e^{3t}}{16t^2}.$$

Here we remark that V has only three zero points $p_1 = [1,0,0,0,0]$, $p_2 = [0,0,0,1,0]$, $p_3 = [0,0,0,0,1]$ in M. Actually, the exponents appeared in the above expression of $\mathcal{F}(V)$ are $-5t = \theta_V(p_1) + 2t$, $7t = \theta_V(p_2) + 2t$, $3t = \theta_V(p_3) + 2t$, so correspond to the three zero points of V.

[9] Generally, a log-terminal singularity is not a quotient singularity (e.g., an ordinary double point of a variety in $\mathbb{C}^n$ $(n \geq 3)$).

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