学位論文

Analysis of effective temperature of non-equilibrium dense matter in holography

ホログラフィーによる非平衡有限密度物質の有効温度の解析

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Contents

Abstract

We study properties of effective temperature of non-equilibrium steady states by using the anti-de Sitter spacetime/conformal field theory (AdS/CFT) correspondence. We consider non-equilibrium systems with a constant flow of current along an electric field, in which the current is carried by both the doped charges and those pair-created by the electric field. We find that the effect of pair-creation raise the effective temperature whereas the current by the doped charges contributes to lower the effective temperature in a wide range of the holographic models. We find that the effective temperature agrees with that of the Langevin systems if we take the limit where the pair creation is negligible.

1 Introduction

Non-equilibrium physics is one of the frontiers of moder physics, and construction of non-equilibrium statistical mechanics is still a great challenge. The difficulty comes from the fact we cannot *a priori* rely on the guiding principle, such as the principle of detailed balance held in equilibrium system. However, we desire to find a fundamental law that governs a wide range of non-equilibrium systems. More precisely, we wish to know if such a fundamental low exists of not. A good place to study for this purpose may be non-equilibrium steady states (NESS). NESS is a system which is driven by a constant external force and out of equilibrium with dissipation while the macroscopic variables do not evolve in time.

Recently, the anti-de Sitter spacetime/conformal field theory (AdS/CFT) correspondence, which is a computational method developed in superstring theory, has been applied to studies of physics of non-equilibrium (see, for example, [1]). In the framework of AdS/CFT[#1], a physical problem in study can be re-formulated into that in gravity theory, and one finds that original problem can be much more easily analyzed in the gravity dual. For example, transport coefficients in NESS have been computed beyond the linear response regime [2, 3, 4]. The typical systems in study are the system of a test particle dragged at a constant velocity in a medium [2, 3] (which we call Langevin systems in this paper) and systems of charged particles with constant flow of current along the external electric field acting on the charge [4] (which we call conductor systems).

It has also been found that the notion of the effective temperature of NESS in these systems naturally appears in the gravity dual picture in terms of the Hawking temperature of analogue black hole [5, 6, 7, 8, 9, 10]. The effective temperature agrees with the ratio between the fluctuation and the dissipation at NESS [7, 10] and it characterizes the correlation functions of fluctuations in NESS[#2]. Therefore, the effective temperature is quite important in the research into non-equilibrium statistical physics.

In this paper, we further study the nature of the effective temperature in holographic models. One of the problems we shall study in this paper is the relationship between the effective temperature of the conductor systems and that in the Langevin systems. Since the conductor systems consist of many charged particles, their effective temperature may be related to that in the Langevin systems where a single (but the same) charged particle is dragged. In general, the effective temperatures of these two systems are

[#1] That is also called holography.

[#2] For the definition of the effective temperature in the literature on non-equilibrium statistical physics, see, for example, [11] for a review.

different from each other. However, as we shall see, if we take the large-mass limit or the large-density limit of the charge carriers in the conductor systems, the effective temperature agrees with that in the Langevin systems. In order to reach the forementioned results, we introduce the mass of the charged particles and the charge density to the analysis of [10], where the zero-mass and zero-density limits have been taken. In [10], it has been found that the effective temperature of NESS can be either higher or lower than the temperature of the heat bath depending on the models and the parameters of the systems. At finite densities, we shall find that the effective temperature can be lower than the temperature of the heat bath even for the models that had the higher effective temperatures at zero density in [10]. Our results imply that the effect of pair creation of charge carriers by the external force is responsible for raising the effective temperature whereas the effect of dragging of the already-existing doped charge carriers is responsible for lowering the effective temperature, at least for our systems.

The organization of the paper is as follows. In Section 2, we briefly review the non-equilibrium steady states and see a example. In Section 3, we overview the AdS/CFT correspondence. In Section 4, we review previous works on computation of an effective temperature of the Langevin system in holographic models. In Section 5, we overview previous works on computation of non-linear conductivity in holographic models. The setup of our model is also explained. One representative of the holographic models of conductor is so-called the D3-D7 model [4]. Therefore, we mainly focus on the D3-D7 model in this paper. In Section 6, the derivation of effective temperature is presented. However, the computations are straightforwardly generalized into other models which we can see in Section 7. Our main results shall be given in Section 6, Section 7.2 [12]. We conclude in Section 8.

2 Non-equilibrium steady states (NESS)

Non-equilibrium steady state (NESS) is a non-equilibrium system where the macroscopic variables are time independent although the energy is dissipated into a heat bath. In order to construct a NESS, we prepare the heat bath made of a large degree of freedom at a temperature T. Then we put a subsystem which has smaller degrees of freedom than that of the heat bath. Turning on a constant external force, it is driven out of equilibrium. After enough time, the subsystem reach a steady state where the in-coming energy and the dissipation are in balance.

In this section, we review the Langevin system within linear-response regime[#3] as an example of NESS. Analysis beyond the linear-response regime is a quite a challenge. Fortunately, however, we can go beyond the linear response theory by using the AdS/CFT correspondence in some cases, which will be discussed in later sections.

2.1 Example of NESS: Brownian motion

We consider the Langevin system of the Brownian motion as an example of NESS, and find possible definition of an effective temperature T_*.

Langevin equation

We consider translational diffusion of a test particle in one dimensional space. The particle is diffused by the heat bath at the temperature T. Since the position of the particle $x(t)$ takes a random vale, its ensemble average vanishes: $\langle x(t)\rangle = 0$. Here we have assumed $x(t=0)=0$. The mean-square displacement, which characterize the amplitude of the fluctuation, is always non-negative: $\langle x^2(t)\rangle \geq 0$. This is a monotonically increasing function of time.

Without an external force, the system is described by the Langevin equation as

$$m\frac{dv}{dt} = -\zeta v + f'(t), \tag{2.1}$$

where m and v are the mass and the velocity of the particle, respectively, ζ is the friction coefficient, and $f'(t)$ stands for the random force. Here we have assumed ζ is a constant value. Then multiplying the position x, we have

$$m\left(\frac{d}{dt}(xv) - v^2\right) = -\zeta vx + xf'(t). \tag{2.2}$$

[#3] For example, see also [13].

Taking the ensemble average, it goes as

$$m\left(\frac{d}{dt}\langle xv\rangle - \langle v^2\rangle\right) = -\zeta\langle vx\rangle + \langle xf'(t)\rangle. \tag{2.3}$$

Then we employ the principle of equipartition of energy $m\langle v^2\rangle = k_BT$, and $\langle xf'\rangle = \langle x\rangle\langle f'\rangle = 0$ since there is no correlation between x and f'. Then we obtain

$$m\frac{d}{dt}\langle xv\rangle = k_BT - \zeta\langle vx\rangle. \tag{2.4}$$

Hence

$$\langle xv\rangle = \frac{1}{2}\frac{d}{dt}\langle x^2\rangle = \frac{k_BT}{\zeta}(1-e^{-t/\tau}), \tag{2.5}$$

where $\tau \equiv m/\zeta$ stands for the relaxation time. The solution under the initial condition $\langle x^2(0)\rangle = 0$ can be given as

$$\langle x^2\rangle = \frac{2k_BT}{\zeta}(t-\tau(1-e^{-t/\tau})). \tag{2.6}$$

In two limits $t \ll \tau$ and $t \gg \tau$, it becomes as

$$\begin{aligned}\sqrt{\langle x^2\rangle} &= v_{th}\ t \propto t \qquad (t \ll \tau),\\ \sqrt{\langle x^2\rangle} &= \sqrt{\frac{2k_BT}{\zeta}t} \propto \sqrt{t} \qquad (t \gg \tau),\end{aligned} \tag{2.7}$$

where $v_{th} \equiv \sqrt{\langle v^2\rangle} = \sqrt{k_BT/m}$ is called thermal velocity, and this implies the particle behaves as the free particle in the short time limit. While, in the large time limit, the behavior implies it does the random walk. In other wards, the particle remembers the the information of the velocity only for the dulation of τ. Thus we can use the relaxation time τ to distinguish the deterministic dynamics and stochastic behavior.

Brownian motion as Random walk

Let us model the Brownian motion. We assume that the particle moves at the constant velocity v_{th} within the mean free time $\tau_f \sim O(\tau)$, then it changes the direction of the velocity suddenly at every interval τ_f. Then the mean free path is given by $\ell_f \equiv v_{th}\tau_f$. This diffusion phenomenon is well-known as the random walk. Then total displacement x in N steps is given as $x = \sum_{i=1}^{N} x_i$, where $|x_i| = \ell_f$. Hence the mean-square displacement is obtained as

$$\left\langle x^2 \right\rangle = \sum_{i=1}^{N} \left\langle x_i^2 \right\rangle + \sum_{i,j \neq i} \langle x_i \cdot x_j \rangle = N\ell_f^2 = \frac{\ell_f^2}{\tau_f} t = v_{th}^2 \tau_f t \propto t, \tag{2.8}$$

where $\langle x_i \cdot x_j \rangle = 0$ since x_i is independent of x_j, and $t = N\tau_f$ has been used. Then, comparing the result to the first equation in (2.7), we have

$$\tau_f = \frac{2k_B T}{v_{th}^2 \zeta} = \frac{2m}{\zeta} = 2\tau. \tag{2.9}$$

This relation is consistent with the fact that the particle forgets the information of the velocity in 2τ.

Derivation from diffusion equation

Here we consider another description of the Brownian motion. We locate the particle at $x = 0$ when $t = 0$. Then the probability $\rho(x,t)$ that the particle is located at x when $t \geq 0$ obeys the following diffusion equation:

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}, \tag{2.10}$$

with the normalization condition

$$\int_{-\infty}^{\infty} \rho(x,t) dx = 1, \tag{2.11}$$

where D stands for the diffusion constant. We impose an initial condition $\rho(x,0) = \delta(x)$. Then the solution of (2.10) is given by a Gaussian distribution:

$$\rho(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right). \tag{2.12}$$

Hence the mean-square displacement is obtained as

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} dx x^2 \rho(x,t) = 2Dt \propto t, \tag{2.13}$$

which agrees with the previous result $\langle x^2 \rangle \propto t$ [#4].

[#4] Another way to get the result (2.13) is as follows. By multiplying x^2 to (2.10) and integrating out, we have

$$\frac{\partial \left\langle x^2 \right\rangle}{\partial t} = 2D,$$

and then $\left\langle x^2 \right\rangle = 2Dt$ by setting $\left\langle x^2(0) \right\rangle = 0$.

Fluctuation-dissipation theorem

Putting (2.7), (2.8) and (2.13) together, we obtain a relation between the diffusion constant D and the friction coefficient ζ as

$$D = \frac{k_B T}{\zeta}. \tag{2.14}$$

This is called the Einstein's relation and is an example of the fluctuation-dissipation theorem. In other words, the friction force $-\zeta v$ is related to the random force f', both of them are the effect of the heat bath.

Furthermore, if the particle has a unit charge, the mobility is defined as $\mu = v/F$, and the relation is written as

$$\frac{D}{\mu} = k_B T. \tag{2.15}$$

Beyond the linear response regime, in general, μ may be defined by the differential mobility as discussed in [14]

$$\frac{D}{\mu} = k_B T_*, \qquad \mu \equiv \frac{\partial v}{\partial F}, \tag{2.16}$$

where F is an external force, and T_* is the effective temperature which can be different from the heat-bath temperature T in general. Since D and μ are observables, we can measure T_*.

In Section 4, 5, 6 and 7, we investigate into T_* in the framework of the AdS/CFT correspondence in detail.

3 AdS/CFT correspondence, gravity and thermodynamics

We overview the AdS/CFT correspondence[#5] and the basics of black hole physics in this section.

3.1 Analogy between the large-N_c gauge theory and the weakly-coupled string tyeory

It has been known that there is an analogy between a large-N_c gauge theory and a string theory, before the discovery of the AdS/CFT correspondence. We go over this old analogy briefly.

Large-N_c gauge theory

Here we consider the SU(N_c) pure Yang-Mills theory at large N_c limit. This theory has only two parameters N_c and the gauge coupling g_{YM}. The Lagrangian density is given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2g_{YM}^2}\mathrm{Tr}\left[F_{\mu\nu}^2\right], \\ F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}A_\mu^b A_\nu^c, \end{aligned} \tag{3.1}$$

where the field strength is defined as $F_{\mu\nu} \equiv F_{\mu\nu}^a t^a$, and the gauge field is defined as $A_\mu \equiv A_\mu^a t^a$ (that is, $A_{\mu,j}^i = A_\mu^a (t^a)^i{}_j$). The generator t^a and the structure constant f^{abc} of the gauge group satisfy $\mathrm{Tr}(t^a t^b) = \delta^{ab}/2$ and $\left[t^a, t^b\right] = i f^{abc} t^c$. Then we introduce the double-line notation, by replacing the twisted line in the Feynman diagram to "strip" with arrow, as shown in Fig 1. The arrow denotes the charge flow (let us call it "color"), and they never hit each other. The propagator is proportional to g_{YM}^2. The Feynman rules for the strip vertices is given by $1/g_{YM}^2$ multiplied by some constants which are independent of g_{YM} or N_c. The color-line loop supplies N_c.

From here we pay attention to bubble diagrams since the partition function is given by them. We introduce the 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c$[#6] which plays a role of effective coupling constant as shown below. The bubble diagrams constructed by V vertices, P propagators and L loops is proportional to

$$\left(\frac{1}{g_{YM}^2}\right)^V \left(g_{YM}^2\right)^P N_c^L = g_{YM}^{2(P-V)} N_c^L = \lambda^{P-V} N_c^{V-P+L}. \tag{3.2}$$

[#5]For example, see also [15, 16].

[#6]We will use another definition, which is twice larger than that, from Section 4.

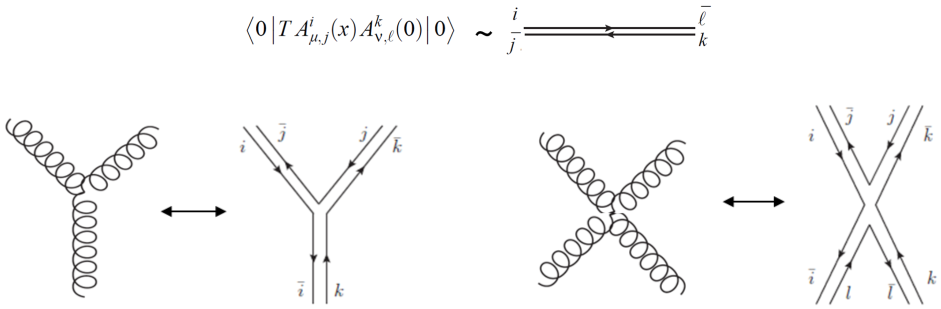

Figure 1: Double line notation

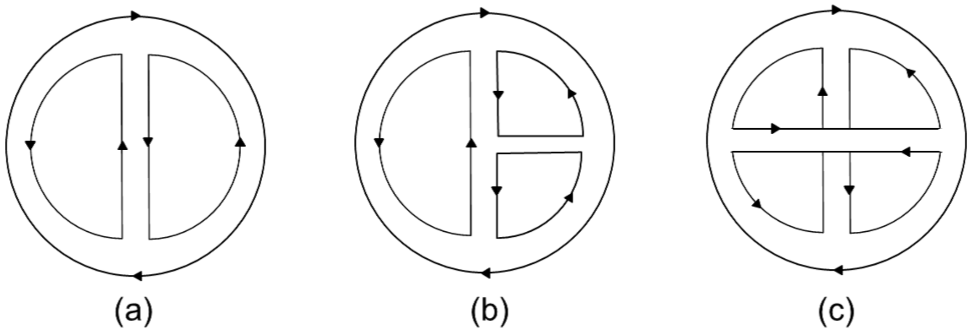

Figure 2: Planar and non-planar diagrams

The diagrams are divided into two types: planar diagrams, which do not include crossing lines as (a) and (b) in Fig. 2, and non-planar diagrams, which include crossing lines as (c) in Fig. 2. In other wards, the planar (non-planar) diagram can (cannot) be drawn on a planar surface. For example, we put a strip into the diagram (a) in Fig. 2, and then (b) or (c) appears. From (a) to (b), an additional loop appears, and then an additional factor λ appears. On the other hand, from (a) to (c), the number of the loops decreases, and then a factor λ/N_c^2 is multiplied. In general, adding a strip increases or decreases a loop with a factor λ or λ/N_c^2, respectively. Thus summation of planar diagrams is written as

$$\sum(\text{planar diagram}) = f_0(\lambda)N_c^2, \qquad f_0(\lambda) \equiv \sum_{n=0}^{\infty} c_n \lambda^n, \tag{3.3}$$

where c_n does not include λ and N_c. By including non-planar diagrams, the summation of the bubble diagrams is given by

$$\sum(\text{bubble}) = f_0(\lambda)N_c^2 + f_2(\lambda)N_c^0 + f_4(\lambda)\frac{1}{N_c^2} + \cdots, \tag{3.4}$$

where $f_i(\lambda)$ is the polynomial function of λ which does not include N_c. Note that, at large N_c, the contribution from the planar diagrams becomes dominant.

Finally we have the partition function as given by

$$\ln Z_{SU(N_c)} = \sum_{h=0}^{\infty} N_c^{\chi} f_{2h}(\lambda), \tag{3.5}$$

where the topological invariant $\chi = V - P + L = 2 - 2h$ is the Euler characteristic, and the genus h is the number of the handle.

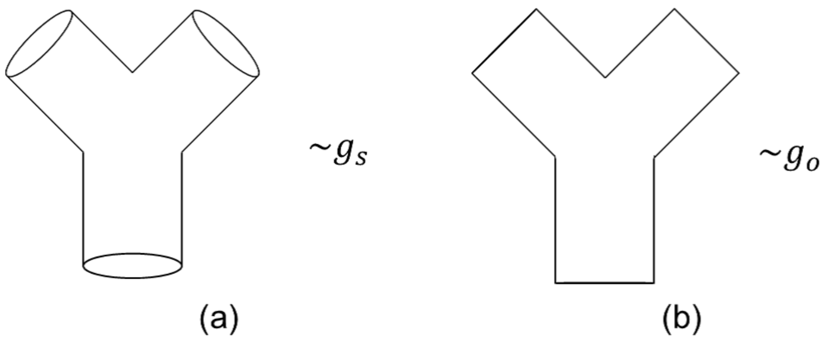

Figure 3: String interactions

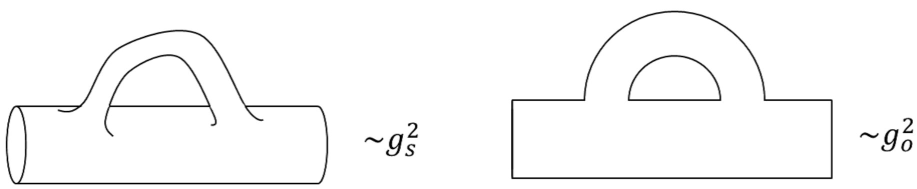

Figure 4: 1-loop diagram of closed string and open string

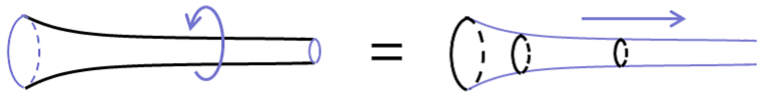

Figure 5: the open-string 1-loop diagram is the same as the closed-string tree diagram.

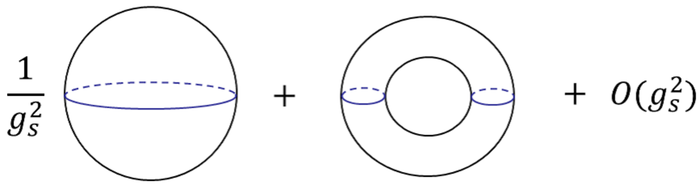

Figure 6: g_s expansion of the partition function

String interaction

Here we overview the interaction of strings briefly. The superstring theory is described in $(9+1)$-dimensional spacetime. There are two types of strings, open string and closed string, and they interact with each other. The closed string includes the graviton, and the open string contains the gauge fields, as their lowest oscillation modes. As the point-like particle sweeps the worldline, the string sweeps the worldsheet.

Typical string interactions are shown in Fig. 3. We define g_s (g_o) as the closed (open) string coupling. When an additional loop appears, the square of the coupling constant is multiplied additionally, as shown in Fig. 4. In other wards, putting a handle corresponds to multiplying g_s^2. Thus the diagrams are characterized by the genus h. For example, a diagram which has h closed-string handles is proportional to $g_s^{2h} = g_s^{2-\chi}$, where $\chi = 2 - 2h$. Then the partition function is given by

$$\ln Z_{string} = \sum_{h=0}^{\infty} \left(\frac{1}{g_s}\right)^{\chi} \tilde{f}_{2h}(\ell_s), \tag{3.6}$$

where ℓ_s is the string length, and $\tilde{f}_{2h}(\ell_s)$ does not contain g_s. The $h = 0$ term gives the classical gravity effect.

Note also that the coupling constants are related as $g_s \sim g_o^2$, since the 1-loop diagram of the open string is topologically same as the diagram of emission of the closed string as shown in Fig. 5. In addition, since the amplitude of emission of the open string is proportional to g_o, we can see $g_{YM} \propto g_o \sim g_s^{1/2}$.[#7]

Comparing (3.5) with (3.6), we find an analogy between them if we identify $g_s \sim 1/N_c$. (3.5) and (3.6) also imply that λ may be related to ℓ_s. We realize that this analogy is promoted to an explicit relationship in the AdS/CFT correspondence, which we describe in the next subsection.

3.2 Example of AdS/CFT correspondence: D3-brane case

The AdS/CFT correspondence is a conjecture proposed by J. Maldacena [17]. Originally, it is a duality between a supergravity on a five-dimensional anti-deSitter spacetime (AdS_5) and an $\mathcal{N} = 4$ SU(N_c) super Yang-Mills (SYM) theory in a four-dimensional spacetime, where $\mathcal{N}$ denotes the number of the supersymmetry. In this case, the SYM theory is a conformal field theory (CFT). However, many other models of the correspondence, including those without conformal invariance, have also been proposed[#8].

D3-brane: $\mathcal{N} = 4$ SYM theory

Here we introduce a D-brane, which is a hypersurface where the open strings can end. The end point carries the quantum number of the gauge group on the brane, and hence N_c overlapping D-branes describe a U(N_c) gauge theory. We call a D-brane which extends into p spacial dimensions a Dp-brane.

We consider the D3-brane in type IIB superstring theory. The open strings are localized on the D-brane. If we put N_c D3-branes on top of each other, then the end point of the string carries the quantum number of the U(N_c) gauge theory. The oscillations are separated into the transverse modes and longitudinal modes to the D3-brane. The transverse and longitudinal modes are identified as the scalar fields and the U(N_c) gauge fields,

[#7]In the viewpoint of the gauge theory in $p+1$ dimensional spacetime, the action is given by

$$S = \frac{1}{g_{YM}^2} \int d^{p+1}x \left(\partial A \partial A + A^2 \partial A + A^4\right),$$

where $p \leq 9$. Then the amplitude of emission of the gauge field is proportional to g_{YM}, which corresponds to the amplitude of emission of the open string. Hence we can see $g_{YM} \propto g_o \sim g_s^{1/2}$. Since only ℓ_s is the dimensionful parameter in the superstring theory, we can see $g_{YM}^2 \propto g_s \ell_s^{p-3}$ by the dimensional analysis $[A] = \left[\ell_s^{-1}\right]$ and $\left[g_{YM}^2\right] = \left[\ell_s^{p-3}\right]$.

[#8]Hence the correspondence is also called holographic theory, gauge/gravity correspondence, and so on.

respectively. The scalar fields are the Nambu-Goldstone bosons associated with the translational symmetry breaking of the six longitudinal directions. These six scalars and the gauge field are a part of the vector multiplet[#9] of $\mathcal{N}=4$ SYM theory. This theory has a global $SO(6)$ symmetry related to interchange of the six scalar fields, and it is called R-symmetry. Furthermore, $\mathcal{N}=4$ SYM theory has conformal symmetry[#10] which is SO(2,4). Thus $\mathcal{N}=4$ SYM theory has the global SO(2,4)$\times$SO(6)$_R$ symmetry.

At $\lambda \gg 1$ limit, we cannot use perturbation. However we can analyze the gauge theory by using another description that is called the AdS/CFT correspondence.

Another description of D3-brane: AdS$_5\times$S^5

The D3-brane in superstring theory has another description in supergravity. Here we consider the theory at the limit $g_s \ll 1$. We consider the low-enrgy effective theory of type IIB superstring theory. When the string length ℓ_s is small enough, the low-energy effective action of the superstring theory is given by

$$S = \frac{1}{16\pi G_{10}} \int dx^{10} \sqrt{-g} \left[R + \cdots + \mathcal{O}(\ell_s^2 R^2) \right], \tag{3.7}$$

where $G_{10} \propto g_s^2 \ell_s^8$ is the Newton constant[#11] in (9+1)-dimensional spacetime, R stands for the scalar curvature, and $(\cdots)$ includes other terms of the supergravity. The terms which include ℓ_s^2 denote the string corrections for the supergravity, and we can neglect them at $\ell_s^2 R \ll 1$.

A solution, which corresponds to N_c D3-branes, is obtained as the black 3-brane geometry:

$$ds_{10}^2 = H^{-1/2} \left(-dt^2 + d\vec{x}^2 \right) + H^{1/2} \left(dr^2 + r^2 d\Omega_5^2 \right), \tag{3.8}$$

[#9]The vector multiplet includes fermion fields on the D3-branes. The fermion fields are the adjoint representation but not the fundamental representation. Hence they do not correspond to quarks.

[#10]See the detail in Appendix A.

[#11]By considering the perturbation of the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, formally the action becomes

$$S = \frac{1}{16\pi G_{10}} \int dx^{10} \left(\partial h \partial h + h \partial h \partial h + h^2 \partial h \partial h + \cdots \right).$$

Then we can read the amplitude of emission of the graviton is proportional to $G_{10}^{1/2}$. In the string picture, the amplitude of closed string emission is proportional to g_s. Hence we conclude $G_{10} \propto g_s^2 \ell_s^8$ by the dimensional analysis $[G_{10}] = \left[\ell_s^8\right]$.

where

$$H = 1 + \frac{L^4}{r^4}, \qquad L^4 \simeq g_s N_c \ell_s^4, \tag{3.9}$$

where the typical length L fixes the curvature of the spacetime and is defined by using ℓ_s and $g_s N_c$.

Then we take the near-horizon limit[#12]:

$$\frac{r}{L} \to 0 \, \left(\frac{r}{\ell_s} \to 0\right), \qquad \frac{r}{\ell_s^2} \equiv u = \text{fixed}, \tag{3.10}$$

where u represents the energy scale of the open string theory. The tension of the open string is proportional to $1/\ell_s^2$, hence the open string which extends from the D-brane to the distance r should have the energy proportional to u. Thus, the operation (3.10) corresponds to that we pay attention to the near horizon $r \ll L$ without changing the physical quantity of the gauge theory.

At the near-horizon limit, we obtain

$$\frac{ds_{10}^2}{\ell_s^2} \xrightarrow[u=\text{fixed}]{r/\ell_s \to 0} \frac{u^2}{u_0^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{u_0^2}{u^2} du^2 + u_0^2 d\Omega_5^2, \tag{3.11}$$

where we define $u_0 \equiv L/\ell_s \simeq g_s N_c = \lambda = \text{fixed}$. The scale ℓ_s defines the unit of the length, while the right-hand side is independent on ℓ_s. Multiplying (3.11) by ℓ_s^2, we obtain

$$ds_{10}^2 \xrightarrow[u=\text{fixed}]{r/L \to 0} \left(\frac{r}{L}\right)^2 \left(-dt^2 + d\vec{x}^2\right) + L^2 \frac{dr^2}{r^2} + L^2 d\Omega_5^2, \tag{3.12}$$

This is $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. The AdS_5 and S^5 have the SO(2,4) and SO(6) symmetry[#13], respectively.

Note that, the symmetry of this geometry coincides with that of $\mathcal{N} = 4$ SYM theory by virtue of the near-horizon limit. Another important point is that we fix $\lambda = g_s N_c$ when we take the near-horizon limit, and hence $g_s \ll 1$ corresponds to the large-N_c limit. Furthermore, we obtain $R \sim L^{-2}$ from (3.12), and hence $\ell_s^2 R \sim u_0^{-2} \sim \lambda^{-2}$. Thus the supergravity limit $\ell_s^2 R \ll 1$ corresponds to the large 't Hooft coupling limit $\lambda \gg 1$.

As a result, we have the correspondence as

$$Z_{\mathcal{N}=4SYM} = Z_{AdS_5 \times S^5}, \tag{3.13}$$

[#12]The decoupling limit is $\ell_s \ll \ell_{obs}$. Here the typical mass scale of the gauge theory is $1/\ell_{obs} \sim r/\ell_s^2$, hence the limit corresponds to $r \ll \ell_s$.

[#13]See also Appendix A.

or more explicitly

$$\left\langle e^{i\int \phi_0 \mathcal{O}} \right\rangle = e^{iS[\phi_0=\phi(r=\infty)]}, \tag{3.14}$$

where ϕ denotes the arbitrary field in the $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry, and $\mathcal{O}$ is the operator conjugate to the source ϕ_0. The relation (3.14) is called the GKP-Witten relation [18, 19]. Furthermore, it is known that the gauge group $\mathrm{U}(N_c)$ is reduced to $\mathrm{SU}(N_c)$ at the near-horizon limit. Hence the classical supergravity theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ corresponds to $\mathcal{N} = 4$ $\mathrm{SU}(N_c)$ SYM theory[#14].

3.3 Finite temperature system

Finite temperature solution at $g_s N_c \gg 1$: AdS_5 black hole $\times$ S^5

Near-extremal D3-branes provide a gravitational representation of $\mathrm{SU}(N_c)$ $\mathcal{N} = 4$ SYM theory at finite temperature T [19], at large N_c, and at strong 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c \gg 1$. The bulk metric described by the near-extremal D3-brane is

$$\begin{aligned} ds_{10}^2 =& H^{-1/2}\left(-hdt^2 + d\vec{x}^2\right) + H^{1/2}\left(h^{-1}dr^2 + r^2 d\Omega_5^2\right) \\ \xrightarrow[u=\text{fixed}]{r/L\to 0}& \left(\frac{r}{L}\right)^2\left(-hdt^2 + d\vec{x}^2\right) + L^2\frac{dr^2}{hr^2} + L^2 d\Omega_5^2, \\ h =& 1 - \left(\frac{r_0}{r}\right)^4, \end{aligned} \tag{3.15}$$

where we have fixed the Hawking temperature when we take the near-horizon limit: $r_0/r =$ fixed. This is the black hole solution which has the horizon, and hence the system corresponds to the gauge theory at the finite temperature. This solution is called AdS_5-Schwarzschild black hole, and this reproduces (3.12) when $r_0/r \to 0$.

Through this thesis, we employ the probe approximation such that the background geometry does not affected by a test string and flavor N_f D-branes. The approximation are realized by setting $N_c \gg N_f$ since the bulk action, the D-brane action, and the string action are $\mathcal{O}(N_c^2)$, $\mathcal{O}(N_f N_c)$, and $\mathcal{O}(1)$, respectively.

In the context of statistical physics, the heat capacity of the bulk sector becomes infinitely large at $N_c \gg 1$. The fact that the bulk sector is not affected by the probe sector means that the bulk remains at thermal equilibrium at temperature T regardless of the probe sector. Hence the bulk plays a role of heat bath.

[#14] See for a review [20].

Hawking temperature of black hole

Here we consider the Hawking temperature of a black hole described by the following metric:

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + \cdots , \tag{3.16}$$

where t and r denote the time and the radial coordinate of the AdS. Here we assume the horizon is located at $r = r_H$. Then near the horizon, the metric approaches to

$$\begin{aligned} ds^2 &\sim -a(r - r_H)dt^2 + \frac{b}{r - r_H}dr^2 + \cdots \\ &= a\rho d\tau^2 + \frac{b}{\rho}d\rho^2 + \cdots , \end{aligned} \tag{3.17}$$

where $\rho = r - r_H$, and we have switched to the Euclidean signature as $t = i\tau$. Furthermore, we change the variable ρ as $dR = d\rho/\sqrt{\rho}$, that is, $R = 2\sqrt{\rho}$. Then the metric is written as

$$\begin{aligned} ds^2 &= a\frac{R^2}{4}d\tau^2 + bdR^2 + \cdots \\ &= b(dR^2 + R^2 d\theta^2) + \cdots , \end{aligned} \tag{3.18}$$

where $\theta = \sqrt{\frac{a}{4b}}\tau$. We find that the θ-direction has to be compactified with the period 2π to avoid the conical singularity:

$$\triangle\theta = 2\pi \quad \Longleftrightarrow \quad \sqrt{\frac{a}{4b}}\triangle\tau = 2\pi,$$

and then we have

$$\triangle\tau = 4\pi\sqrt{\frac{b}{a}} \qquad \left(= 4\pi\frac{1}{\sqrt{-g'_{tt}(g_{rr}^{-1})'}\Big|_{r_H}} \right), \tag{3.19}$$

where the prime stands for ∂_r.

We regard (3.19) as the inverse of the temperature, $\triangle\tau = \beta = 1/T$, and then T is obtained as

$$T = \frac{1}{4\pi}\sqrt{\frac{a}{b}} \qquad \left(= \frac{1}{4\pi}\sqrt{-g'_{tt}(g_{rr}^{-1})'}\Big|_{r_H} \right). \tag{3.20}$$

This T is the Hawking temperature. In the AdS/CFT correspondence, we regard this Hawking temperature as the temperature in the gauge-theory side.

4 Langevin system and effective temperature in holography

In this section, we review [3, 5] where an infinitely massive particle moving in a heat bath is considered in a holographic way. A finite temperature system of $\mathcal{N} = 4$ SYM plasma, which plays a role of a heat bath in the present setup, is dual to the AdS$_5$-Schwarzschild geometry. An infinitely massive quark is dual to a infinitely-long fundamental string inserted from the boundary. When we drive the end point at a constant velocity v, we need to apply a constant external force to the string that is equal to the frictional force [3].

4.1 Setup, classical solution, and drag force

Bulk metric

At the near-horizon limit, The N_c D3-branes provide a gravitational representation of SU(N_c) $\mathcal{N} = 4$ super Yang-Mills (SYM) theory at finite temperature T [19], large N_c, and strong 't Hooft coupling $\lambda \equiv 2g_{YM}^2 N_c \gg 1$.[#15] Here g_{YM} denotes the gauge coupling. The gravitational representation is given by AdS$_5$-Schwarzschild metric $\hat{g}_{\mu\nu}$:

$$ds_5^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu = \frac{r^2}{L^2}\left(-hdt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}\frac{dr^2}{h}, \tag{4.1}$$

times the metric for an S^5 of constant radius L. Here $\vec{x} = (x, y, z)$ are the spatial coordinates along the boundary and $d\Omega_5$ is the volume element of the unit five-sphere S^5. The boundary and horizon are located at $r = \infty$ and $r = r_H$ respectively.

Action and classical solution

A test string, in the back ground metric (4.1), is described by the Nambu-Goto action:

$$\begin{aligned} S &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det g_{\alpha\beta}}, \\ g_{\alpha\beta} &\equiv \hat{g}_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \end{aligned} \tag{4.2}$$

where $g_{\alpha\beta}$ is an induced metric , σ^α are the worldsheet coordinates of the string, and $X^\mu(\sigma)$ is the embedding of the string worldsheet in the spacetime.

[#15] From here to the end of this paper, we use the definition $\lambda \equiv 2g_{YM}^2 N_c$ and the relation $L^4 = 2g_{YM}^2 N_c \alpha'^2$, which are different from the original work [3] by the factor two.

From here, we study this system in the static gauge $\sigma^\alpha = (t, r)$, and define x as $x = X^1(t, r)$. Then the induced metric are written as

$$\begin{aligned} g_{tt} &= \hat{g}_{00} + \hat{g}_{11}\dot{x}^2 = H^{-1/2}\left(\dot{x}^2 - h\right), \\ g_{tr} &= H^{-1/2}\dot{x}x', \\ g_{rr} &= \hat{g}_{11}{x'}^2 + \hat{g}_{rr} = H^{-1/2}\left({x'}^2 + H\right), \end{aligned} \tag{4.3}$$

where $H = L^4/r^4$, $\dot{x} = \partial_t x$ and $x' = \partial_z x$. By substituting them, the string action (4.2) becomes

$$S = \frac{1}{2\pi\alpha'}\int dtdr\mathcal{L}, \qquad \mathcal{L} = -\sqrt{1 + \frac{h}{H}x'^2 - \frac{\dot{x}^2}{h}}. \tag{4.4}$$

Then the equation of motion is given as

$$\partial_\alpha \frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\mu} = -\partial_\alpha\left[\sqrt{-g}g^{\alpha\beta}\partial_\beta X^\mu \hat{g}_{\mu\nu}\right] = 0, \tag{4.5}$$

or

$$\nabla_\alpha P^\alpha_\mu = 0, \qquad P^\alpha_\mu \equiv -\frac{1}{2\pi\alpha'}\hat{g}_{\mu\nu}\partial^\alpha X^\nu, \tag{4.6}$$

where ∇_α is the covariant derivative with respect to $g_{\alpha\beta}$: $\sqrt{-g}\,\nabla_\alpha P^\alpha_\mu = \partial_\alpha\left(\sqrt{-g}P^\alpha_\mu\right)$. P^α_μ is a worldsheet current of spacetime energy-momentum which is carried by the string.

Here we assume that steady state behavior is achieved at late times. Hence we take a suitable ansatz:

$$x(t, r) = vt + \xi(r), \tag{4.7}$$

The justification of the ansatz (4.7), the time-independence of $\xi(r)$, is given as follows. In general, the equation of motion (4.6) under the boundary condition $\dot{x}|_{r=\infty} = v$ can give time-dependent solution $\xi(r, t)$. However, we can show $\dot{\xi} = 0$ if the bulk metric dose not depend on time and we impose $\partial_r\left(\sqrt{-g}P^r_x\right) = 0$. The condition $\partial_r\left(\sqrt{-g}P^r_x\right) = 0$ means that the momentum coming from the external force at the boundary and the momentum deposited to the heat bath at the horizon are in balance. This is requested by the realization of NESS.

Substituting the ansatz into (4.4), we have the following Lagrangian density as

$$L = -\sqrt{1 + \frac{h}{H}\xi'^2 - \frac{v^2}{h}}, \qquad \pi_\xi \equiv \frac{\partial L}{\partial \xi'}. \tag{4.8}$$

Then the equation of motion can be written as $\pi_\xi = \text{const}$. The solution of this equation is obtained as

$$\xi' = \pm \pi_\xi \frac{H}{h} \sqrt{\frac{h - v^2}{h - \pi_\xi^2 H}}. \tag{4.9}$$

We assume that v points in the direction of x, and then a string should expand behind the external quark: ξ' should be positive. If we choose the sign of π_ξ as positive, then the sign in (4.9) should be $+$.

Note that we must require that $\xi(r)$ is everywhere real. Since $h = 0$ at $r = r_H$ and $h = 1$ when $r \to \infty$, $h - v^2$ switches the signature at $h = v^2$. We define $r = r_*$ as the location $h = 1 - r_H^4/r_*^4 = v^2$, that is, $r_* = r_H/\sqrt[4]{1 - v^2}$. In order to make the right-hand side real in (4.9), we require $h - \pi_\xi^2 H$ also switch its sign at $h = v^2$ $(r = r_*)$:

$$\begin{aligned}
0 = h_* - \pi_\xi^2 H_* &= v^2 - \pi_\xi^2 (L^4/r_*^4) \\
\Longleftrightarrow \quad \pi_\xi^2 &= \frac{v^2 r_*^4}{L^4} \\
\Longleftrightarrow \quad \pi_\xi &= \pm \frac{v r_*^2}{L^2} = \pm \frac{v r_H^2}{L^2 \sqrt{1 - v^2}}.
\end{aligned}$$

As we mentioned that before, π_ξ should be $+$ here, hence finally

$$\pi_\xi = \frac{v r_*^2}{L^2} = \frac{v r_H^2}{L^2 \sqrt{1 - v^2}}. \tag{4.10}$$

This is the only way to avoid the appearance of imaginary part in the right-hand side of (4.9).

Then the equation of motion becomes the following form:

$$\xi' = v \frac{r_H^2}{L^2} \frac{H}{h} = v \frac{r_H^2 L^2}{r^2 - r_H^2}, \tag{4.11}$$

hence the solution is

$$\xi = -\frac{L^2}{2 r_H} v \left(\tan^{-1} \frac{r}{r_H} + \log \sqrt{\frac{r + r_H}{r - r_H}} \right), \tag{4.12}$$

where we have put $\xi = 0$ at $v = 0$ as a boundary condition.

Drag force

As is mentioned around (4.6), P_μ^α is the conserved worldsheet current of spacetime energy-momentum. The flow of momentum dp_1/dt goes down

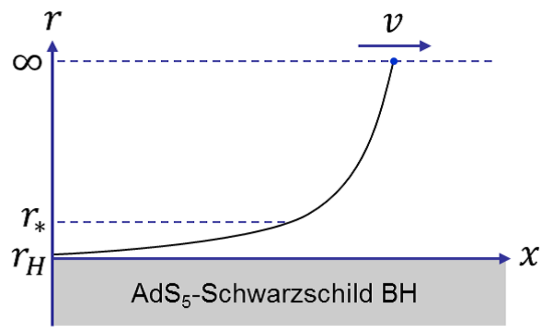

Figure 7: This is a picture of the solution (4.12) at a given time t in which we take the ansatz $x = vt + \xi$ and choose $\mathrm{sign}(\xi') = \mathrm{sign}(\pi_\xi) = +$. r_* is also illustrated which is given by imposing reality of the action.

along the string, which corresponds to the flow from the external quark into the horizon. The momentum flow is obtained from the worldsheet current as

$$\Delta P_1 = \int_{\mathcal{I}} dt \sqrt{-g} P^r_x = \frac{dp_1}{dt} \Delta t, \tag{4.13}$$

where $\mathcal{I}$ is some time interval Δt. Since P^α_μ is conserved, it should not matter to evaluate the integral at any radius r. We choose the direction of p_1 to be negative: it is the drag force, which points opposite the motion. Then the flow is calculated as

$$\begin{aligned} \frac{dp_1}{dt} &= \sqrt{-g} P^r_x = -\frac{\sqrt{1-v^2}}{2\pi\alpha'} \hat{g}_{x\nu} g^{r\alpha} \partial_\alpha X^\nu \\ &= -\frac{r_H^2/L^2}{2\pi\alpha'} \frac{v}{\sqrt{1-v^2}} \\ &= -\frac{\pi\sqrt{2g_{YM}^2 N_c}}{2} T^2 \frac{v}{\sqrt{1-v^2}}, \end{aligned} \tag{4.14}$$

where the Hawking temperature $T = r_H/(\pi L^2)$, which is dual to the temperature of the plasma, and $L^4 = 2g_{YM}^2 N_c \alpha'^2$. This is the final result of the drag force for the infinitely massive particle driven by the constant velocity in the heat bath.

4.2 Fluctuations and effective temperature

If we consider fluctuation modes of the string, then they feel an effective metric, rather than the background metric. A diagonalized effective metric gives the effective temperature, which is observed by the fluctuation modes, as $T_* = \sqrt[4]{1-v^2}T$ [5].

Fluctuation

Here we consider the effective temperature as is discussed in [5]. We expand the equation of motion (4.5) by fluctuation modes of X^μ. In other words, the scalar field X^μ is replaced by $\overline{X}^\mu + \tilde{X}^\mu$, where we regard $\tilde{X}^\mu$ as small fluctuations:

$$\begin{aligned} g_{\alpha\beta} &\equiv \hat{g}_{\mu\nu}\partial_\alpha\left(\overline{X}^\mu + \tilde{X}^\mu\right)\partial_\beta\left(\overline{X}^\nu + \tilde{X}^\nu\right) = \overline{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}, \\ \overline{g}_{\alpha\beta} &\equiv \hat{g}_{\mu\nu}\partial_\alpha\overline{X}^\mu\partial_\beta\overline{X}^\nu, \\ \tilde{g}_{\alpha\beta} &\equiv \hat{g}_{\mu\nu}\left(\partial_\alpha\overline{X}^\mu\partial_\beta\tilde{X}^\nu + \partial_\alpha\tilde{X}^\mu\partial_\beta\overline{X}^\nu\right) + \hat{g}_{\mu\nu}\partial_\alpha\tilde{X}^\mu\partial_\beta\tilde{X}^\nu. \end{aligned} \tag{4.15}$$

where $\tilde{X}^t = \tilde{X}^r = 0$ is imposed by the static gauge. Furthermore, $\hat{g}_{\mu\nu}(X) = \hat{g}_{\mu\nu}(\bar{X})$ in this gauge[#16]. $\bar{g}^{\alpha\beta}$ and $\tilde{g}^{\alpha\beta}$ satisfy

$$\bar{g}^{\alpha\sigma}\bar{g}_{\sigma\beta} = \delta^\alpha_{\ \beta}, \qquad \tilde{g}^{\alpha\beta} = -\bar{g}^{\alpha\gamma}\bar{g}^{\beta\sigma}\tilde{g}_{\gamma\sigma}. \tag{4.16}$$

Then we consider the equation of motion, and (4.5) is expanded perturbatively as follows. An equation of motion at the zeroth order of the fluctuation is merely the same as (4.5):

$$\partial_\alpha\left[\sqrt{-\bar{g}}\bar{g}^{\alpha\beta}\partial_\beta\overline{X}^\mu\hat{g}_{\mu\mu'}(\bar{X})\right] = 0. \tag{4.17}$$

At the first order of the fluctuation, the equation of motion of $\tilde{X}$ is given by

$$\partial_\alpha\left[\sqrt{-\bar{g}}\left(\bar{g}^{\alpha\beta}\partial_\beta\tilde{X}^\mu + \tilde{g}^{\alpha\beta}\partial_\beta\overline{X}^\mu + \frac{1}{2}\bar{g}^{\gamma\sigma}\tilde{g}_{\sigma\gamma}\bar{g}^{\alpha\beta}\partial_\beta\overline{X}^\mu\right)\hat{g}_{\mu\mu'}(\bar{X})\right] = 0. \tag{4.18}$$

Here, we separate (4.18) into a symmetric and anti-symmetric part:

$$\begin{gathered} \partial_\alpha\left[\sqrt{-\bar{g}}\left(\bar{g}^{\alpha\beta}S_{\mu\nu} + A^{\alpha\beta}_{\mu\nu}\right)\partial_\beta\tilde{X}^\nu\right] = 0, \\ S_{\mu\nu} = \hat{g}_{\mu\nu} - \overline{g}_{\rho\sigma}\bar{P}^\rho_{\ \mu}\bar{P}^\sigma_{\ \nu}, \qquad A^{\alpha\beta}_{\mu\nu} = \bar{P}^\alpha_{\ \mu}\bar{P}^\beta_{\ \nu} - \bar{P}^\alpha_{\ \nu}\bar{P}^\beta_{\ \mu}, \end{gathered} \tag{4.19}$$

where $S_{\mu\nu}$ denotes the symmetric part, $S_{\mu\nu} = S_{\nu\mu}$, and A represents the anti-symmetric part, $A^{\alpha\beta}_{\mu\nu} = -A^{\beta\alpha}_{\mu\nu} = -A^{\alpha\beta}_{\nu\mu} = A^{\beta\alpha}_{\nu\mu}$. $\bar{P}^\alpha_{\ \mu'}$ is defined by $\bar{P}^\alpha_{\ \mu'} \equiv \overline{g}^{\alpha\beta}\partial_\beta\overline{X}^\mu\hat{g}_{\mu\mu'}$.

[#16] See also Appendix C.

Effective metric

From here, we consider an effective metric observed by the fluctuation modes $\tilde{X}^{i=1,2,3}$. In the static gauge, $\overline{X}^{\mu}$'s are $\overline{X}^0 = t$, $\overline{X}^r = r$ and $\overline{X}^1 = x(t,r)$, and we imposed $\partial_\alpha \overline{X}^2 = \partial_\alpha \overline{X}^3 = 0$ by the translational invariance. Then $\bar{P}^{\alpha}_{\mu=1,2} = 0$.

The inverse of the induced metric is given as

$$\overline{g}^{\alpha\beta} = \frac{1}{\det \overline{g}_{\alpha\beta}} \begin{pmatrix} \overline{g}_{rr} & -\overline{g}_{rt} \\ -\overline{g}_{tr} & \overline{g}_{tt} \end{pmatrix} = \frac{1}{\overline{g}_{tt}\overline{g}_{rr} - \overline{g}_{tr}^2} \begin{pmatrix} \overline{g}_{rr} & -\overline{g}_{rt} \\ -\overline{g}_{tr} & \overline{g}_{tt} \end{pmatrix}. \tag{4.20}$$

Hence $\bar{P}^t_\mu$, $\bar{P}^r_\mu$ are obtained as follows:

$$\begin{aligned} \bar{P}^r_0 &= -\frac{1}{\det \overline{g}_{\alpha\beta}} \overline{g}_{tr} \hat{g}_{00} = \frac{1}{\det \overline{g}_{\alpha\beta}} \frac{h\dot{x}x'}{H}, \\ \bar{P}^r_1 &= \frac{1}{\det \overline{g}_{\alpha\beta}} \left(-\overline{g}_{tr}\dot{x} + \overline{g}_{tt}x'\right) \hat{g}_{11} = -\frac{1}{\det \overline{g}_{\alpha\beta}} \frac{hx'}{H}, \\ \bar{P}^r_r &= \frac{1}{\det \overline{g}_{\alpha\beta}} \overline{g}_{tt}\hat{g}_{rr} = \frac{1}{\det \overline{g}_{\alpha\beta}} \left(\frac{\dot{x}^2}{h} - 1\right). \end{aligned} \tag{4.21}$$

Here we define $\hat{\mathcal{G}}^{\alpha\beta}_{\mu\nu}$ as follows:

$$\hat{\mathcal{G}}^{\alpha\beta}_{\mu\nu} \equiv \sqrt{-\overline{g}} \left(\overline{g}^{\alpha\beta} S_{\mu\nu} + A^{\alpha\beta}_{\mu\nu}\right). \tag{4.22}$$

For the partial matrix $\hat{\mathcal{G}}^{\alpha\beta}$, we know that the diagonal elements $\hat{\mathcal{G}}^{\alpha\beta}_{\mu\mu}$ ($\mu = \nu$) can be written as

$$\begin{aligned} &\hat{\mathcal{G}}^{\alpha\beta}_{\mu\mu} = \hat{\mathcal{G}}^{\beta\alpha}_{\mu\mu} = \sqrt{-\overline{g}}\overline{g}^{\alpha\beta} S_{\mu\mu} = \sqrt{-\overline{g}}\overline{g}^{\alpha\beta} \left(\hat{g}_{\mu\mu} - \overline{g}_{\rho\sigma}\bar{P}^\rho_\mu \bar{P}^\sigma_\mu\right) \qquad \text{for } \mu = 0, 1, r, \\ &\hat{\mathcal{G}}^{\alpha\beta}_{22} = \hat{\mathcal{G}}^{\alpha\beta}_{33} = \hat{\mathcal{G}}^{\beta\alpha}_{22} = \hat{\mathcal{G}}^{\beta\alpha}_{33} = \sqrt{-\overline{g}}\overline{g}^{\alpha\beta}\hat{g}_{11}. \end{aligned} \tag{4.23}$$

Here, we demonstrate that $A^{\alpha\beta}_{\mu\nu} = 0$ in the static gague. The off-diagonal elements of $\hat{\mathcal{G}}^{\alpha\beta}_{\mu\nu}$ ($\mu \neq \nu$) are

$$\hat{\mathcal{G}}^{\alpha\beta}_{\mu\nu} = \begin{cases} 0 & \text{for } \mu = 2, 3 \text{ or } \nu = 2, 3, \\ \sqrt{-\overline{g}}\overline{g}^{\alpha\beta} \left(-\overline{g}_{\rho\sigma}\bar{P}^\rho_\mu \bar{P}^\sigma_\nu + A^{\alpha\beta}_{\mu\nu}\right) & \text{for others.} \end{cases} \tag{4.24}$$

Hence we need only $\hat{\mathcal{G}}^{\alpha\beta}_{11}$ and $\hat{\mathcal{G}}^{\alpha\beta}_{22} = \hat{\mathcal{G}}^{\alpha\beta}_{33}$: $\hat{\mathcal{G}}^{\alpha\beta}_{0\nu}$ and $\hat{\mathcal{G}}^{\alpha\beta}_{r\nu}$ for any ν are no matter since $\tilde{X}^0 = \tilde{X}^r = 0$, and the off-diagonal components for (μ,ν) satisfy

$\hat{\mathcal{G}}^{\alpha\beta}_{2\nu} = \hat{\mathcal{G}}^{\alpha\beta}_{3\nu} = 0$. That is, $\hat{\mathcal{G}}^{\alpha\beta}_{1\nu}$ $(\nu \neq 1)$ does not affect in our calculation. Thus we define $\hat{\mathcal{G}}^{\alpha\beta}_{\mathrm{L}}$ and $\hat{\mathcal{G}}^{\alpha\beta}_{\mathrm{T}}$ which we need as

$$\begin{aligned} \hat{\mathcal{G}}^{\alpha\beta}_{\mathrm{L}} &\equiv \hat{\mathcal{G}}^{\alpha\beta}_{11} = \frac{1}{2\pi\alpha'}\sqrt{-\bar{g}}\bar{g}^{\alpha\beta}\left(\hat{g}_{11} - \bar{g}_{\rho\sigma}\bar{P}^{\rho}_{1}\bar{P}^{\sigma}_{1}\right), \\ \hat{\mathcal{G}}^{\alpha\beta}_{\mathrm{T}} &\equiv \hat{\mathcal{G}}^{\alpha\beta}_{22} = \hat{\mathcal{G}}^{\alpha\beta}_{33} = \frac{1}{2\pi\alpha'}\sqrt{-\bar{g}}\bar{g}^{\alpha\beta}\hat{g}_{11}. \end{aligned} \tag{4.25}$$

Note that $A^{\alpha\beta}_{\mu\nu}$ does not affect in the gauge, and hence the effective metrics $\hat{\mathcal{G}}_{\mathrm{L}}$ and $\hat{\mathcal{G}}_{\mathrm{T}}$ are symmetric matrices: $\hat{\mathcal{G}}^{\alpha\beta}_{\mathrm{L}} = \hat{\mathcal{G}}^{\beta\alpha}_{\mathrm{L}}$ and $\hat{\mathcal{G}}^{\alpha\beta}_{\mathrm{T}} = \hat{\mathcal{G}}^{\beta\alpha}_{\mathrm{T}}$.

Futhermore, within the ansatz (4.7) based on the time invariance of the Lagrangian, the effective metric for the fluctuation can be written as

$$\hat{\mathcal{G}}^{\alpha\beta}_{T} = (-\bar{g})\,\hat{\mathcal{G}}^{\alpha\beta}_{L} = \frac{1}{2\pi\alpha'}\frac{1}{(-\bar{g})^{1/2}H}\begin{pmatrix} -\xi'^2 - \frac{H}{h} & v\xi' \\ v\xi' & h - v^2 \end{pmatrix}, \tag{4.26}$$

where $\dot{x} = v$, $x' = \xi'$ and hence $\bar{g} = -\left(1 + \frac{h}{H}x'^2 - \frac{\dot{x}^2}{h}\right) = -\left(1 + \frac{h}{H}\xi'^2 - \frac{v^2}{h}\right)$. Substituting the solution of equation of motion (4.12), $\hat{\mathcal{G}}^{\alpha\beta}_{T,L}$ can be written as

$$\hat{\mathcal{G}}^{\alpha\beta}_{\mathrm{T}} = \left(1 - v^2\right)\hat{\mathcal{G}}^{\alpha\beta}_{\mathrm{L}} = \frac{r^4}{2\pi\alpha'\sqrt{1-v^2}}\begin{pmatrix} -\frac{r^4 - \left(1-v^2\right)r_H^4}{\left(r^4 - r_H^4\right)^2} & \frac{v^2 r_H^2}{L^2\left(r^4 - r_H^4\right)} \\ \frac{v^2 r_H^2}{L^2\left(r^4 - r_H^4\right)} & \frac{\left(1-v^2\right)r^4 - r_H^4}{L^4 r^4} \end{pmatrix}, \tag{4.27}$$

where we have used $\sqrt{-\bar{g}} = \sqrt{1-v^2}$. Then we obtain inverse of (4.27) as

$$\hat{\mathcal{G}}^{\mathrm{T}}_{\alpha\beta} = \frac{1}{1-v^2}\hat{\mathcal{G}}^{\mathrm{L}}_{\alpha\beta} = \frac{2\pi\alpha'}{\sqrt{1-v^2}}\begin{pmatrix} \frac{r_H^4}{r^4} - (1-v^2) & \frac{L^2 v^2 r_H^2}{r^4 - r_H^4} \\ \frac{L^2 v^2 r_H^2}{r^4 - r_H^4} & \frac{L^4\left(r^4 - \left(1-v^2\right)r_H^4\right)}{\left(r^4 - r_H^4\right)^2} \end{pmatrix}, \tag{4.28}$$

and then we have

$$\sqrt{-\hat{\mathcal{G}}^{\mathrm{T}}} = \frac{1}{1-v^2}\sqrt{-\hat{\mathcal{G}}^{\mathrm{L}}} = \sqrt{-\left(-\frac{(2\pi\alpha')^2 L^4}{r^4}\right)} = \frac{2\pi\alpha' L^2}{r^2}. \tag{4.29}$$

The effective metrics are diagonalized as follows:

$$\begin{aligned} d\tilde{s}^2 &= \hat{\mathcal{G}}^{T,L}_{tt}dt^2 + \hat{\mathcal{G}}^{T,L}_{tr}dtdr + \hat{\mathcal{G}}^{T,L}_{rt}drdt + \hat{\mathcal{G}}^{T,L}_{rr}dr^2 \\ &= \mathcal{G}^{T,L}_{\tau\tau}d\tau^2 + \mathcal{G}^{T,L}_{rr}dr^2, \end{aligned} \tag{4.30}$$

where $d\tau \equiv dt + \frac{\hat{\mathcal{G}}_{tr}^{T,L}}{\hat{\mathcal{G}}_{tt}^{T,L}} dr$ and hence

$$\mathcal{G}_{\tau\tau}^{T,L} \equiv \hat{\mathcal{G}}_{tt}^{T,L}, \qquad \mathcal{G}_{rr}^{T,L} \equiv \hat{\mathcal{G}}_{rr}^{T,L} - \frac{\hat{\mathcal{G}}_{tr}^{T,L}}{\hat{\mathcal{G}}_{tt}^{T,L}}. \tag{4.31}$$

Then we have the diagonalized effective metric $\mathcal{G}_{\alpha\beta}^{T,L}$ as follows:

$$\begin{aligned} \mathcal{G}_{\alpha\beta}^{T} = \frac{1}{1-v^2}\mathcal{G}_{\alpha\beta}^{L} &= \begin{pmatrix} \hat{\mathcal{G}}_{tt}^{T} & 0 \\ 0 & \hat{\mathcal{G}}_{rr}^{T} - \frac{\left(\hat{\mathcal{G}}_{tr}^{T}\right)^2}{\hat{\mathcal{G}}_{tt}^{T}} \end{pmatrix} \\ &= \frac{2\pi\alpha'}{\sqrt{1-v^2}} \begin{pmatrix} (1-v^2)\left(\frac{r_*^4}{r^4}-1\right) & 0 \\ 0 & \frac{L^4}{r^4-r_*^4} \end{pmatrix}, \end{aligned} \tag{4.32}$$

where $r_* \equiv r_H/\sqrt[4]{1-v^2}$ plays a role of an effective horizon for the fluctuation $\tilde{X}^{\mu=1,2,3}$. Note that the location of the effective horizon is given as $r = r_*$ that has been defined by the reality condition of the action.

To obtain an effective metric at arbitrary r, we should multiply $\mathcal{G}_{\alpha\beta}^{T,L}$ by a constant factor as discussed in [10]. For the 2×2 matrix, unfortunately, we can not do that operation. In order to read the effective temperature, however, we need only $\mathcal{G}_{\alpha\beta}^{T,L}$ at the vicinity of the horizon.

Effective temperature

In the region of $r \sim r_*$, (4.32) is approximated as follows:

$$\mathcal{G}_{\alpha\beta}^{T} = \frac{1}{1-v^2}\mathcal{G}_{\alpha\beta}^{L} \sim \frac{2\pi\alpha'}{\sqrt{1-v^2}} \begin{pmatrix} -\frac{4\left(1-v^2\right)}{r_*}(r-r_*) & 0 \\ 0 & \frac{L^4}{4r_*^3(r-r_*)} \end{pmatrix}. \tag{4.33}$$

Now we can compute the effective temperature as is given in Section 3.3. The effective metric closed to r_* gives

$$\begin{aligned} d\tilde{s}_{T,L}^2 &\sim C_{T,L}\frac{2\pi\alpha'}{\sqrt{1-v^2}}\left(-\frac{4\left(1-v^2\right)}{r_*}(r-r_*)d\tau^2 + \frac{L^4}{4r_*^3(r-r_*)}dr^2\right), \\ C_T &\equiv 1, \qquad C_L \equiv 1-v^2. \end{aligned} \tag{4.34}$$

The effective temperature is given by

$$T_* = \frac{1}{4\pi}\sqrt{\frac{a}{b}}, \tag{4.35}$$

where a and b are given by $ds^2 = -a(r-r_*)+b/(r-r_*)+\cdots$ in the vicinity of r_*. Thus T_* is obtained as

$$T_* = \frac{1}{4\pi}\sqrt{\frac{C_{T,L}\frac{2\pi\alpha'}{\sqrt{1-v^2}}}{C_{T,L}\frac{2\pi\alpha'}{\sqrt{1-v^2}}}\frac{4(1-v^2)/r_*}{L^4/4r_*^3}} = \frac{\sqrt[4]{1-v^2}\,r_H}{\pi L^2}, \tag{4.36}$$

for the fluctuations $\tilde{X}^{1,2,3}$. We stress that the effective temperature T_* can be different from the heat-bath temperature $T = \frac{r_H}{\pi L^2}$ by the factor $\sqrt[4]{1-v^2}$:

$$T_* = \sqrt[4]{1-v^2}\,T. \tag{4.37}$$

As a result, the effective temperature T_* observed by the fluctuations is lower than the temperature of the heat bath, and its dependence on v is highly non-linear. The result (4.37) cannot be explained by using Lorentz boost as discussed in [10]. It reflects non-trivial dynamics of the system.

5 Holographic conductor

In this section we consider the holographic conductor given in [4], which is described by the D3-D7 system.

5.1 Setup and classical solutions

In order to prepare NESS, we need a subsystem coupled to a heat bath. We apply an external force which drives the subsystem into non-equilibrium, and the heat bath absorbs the dissipation. When the work given by the external force and the dissipation into the heat bath are in balance, the subsystem can be realized as NESS. For the conductor systems in this study, the external force is the external electric field E, the subsystem in study is a many-body system of charge carriers, and the heat bath is a system of particles that are neutral in E and are interacting with the subsystem.

One typical realization of the foregoing conductor system in holography is the D3-D7 system with electric field [4]. Since the analysis can be straightforwardly generalized into the cases of other models, we mainly focus on the D3-D7 system in this paper. Let us briefly review the model of [4] to explain our setup and notations.

The field theory realized on the D3-D7 system is a supersymmetric QCD, that consists of a $(3+1)$-dimensional $SU(N_c)$ $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory for adjoint representations (which we call "gluons") and a $\mathcal{N}=2$ hypermultiplet as a sector of fundamental representations (which we call "quarks" or "anti-quarks"). We apply an external force electrically acting on the quark charge (which we call "electric field"), and then the current of the quark charge (which we call "current") appears. The gluon sector plays a role of heat bath since it absorbs the dissipation produced in the quark sector. The picture of the heat bath is established since we take the large-N_c limit where the degrees of freedom of the gluon sector is infinitely large comparing to that of the quark sector. We also take the large 't Hooft coupling limit so that the typical interaction scale of the microscopic process is short enough comparing to that of the macroscopic physics.

In the gravity dual picture, the heat bath of the gluon sector is mapped to the geometry of a direct product of an AdS-Schwarzschild black hole (AdS-BH) and an S^5, whose metric is given by

$$ds^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu = -\frac{1}{z^2}\frac{(1-z^4/z_H^4)^2}{1+z^4/z_H^4}dt^2 + \frac{1}{z^2}\left(1+z^4/z_H^4\right)d\vec{x}^2 + \frac{dz^2}{z^2} + d\Omega_5^2, \tag{5.1}$$

where we have set the AdS radius to be 1. z is the radial coordinate on which the horizon is located at $z = z_H$, and the boundary is at $z = 0$. The Hawking temperature is given by $T = \frac{\sqrt{2}}{\pi z_H}$. The boundary extends along $(t, \vec{x})$ directions, and $d\Omega_5$ stands for the volume element of the S^5. The first equality in (5.1) just shows our notation that $\hat{g}_{\mu\nu}$ denotes the metric of the background geometry. The metric of the S^5 is given by

$$d\Omega_5^2 = d\theta^2 + \sin^2\theta d\psi^2 + \cos^2\theta d\Omega_3^2, \tag{5.2}$$

where θ runs from 0 to $\pi/2$, and ψ varies between 0 and 2π. $d\Omega_3$ denotes the volume element of unit S^3.

The quark sector is mapped to D7-branes on the above geometry. In our study, we consider the case of single flavor, and we introduce only a single D7-brane. We employ the probe approximation since we are taking the large-N_c limit. The D7-brane wraps the S^3 part of the S^5 in such a way that the radius of the S^3 part depends on the radial coordinate z in general. For the original proposal and the details on the D3-D7 system, see [22].

Table 1: The brane configurations in this model

	0	1	2	3	4	5	6	7	8	9
N_c D3	$\surd$	$\surd$	$\surd$	$\surd$						
N_f D7	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$		

Let us exhibit the Dirac-Born-Infeld (DBI) action of the D7-brane for the purpose of defining our notations:

$$\begin{aligned} S_{D7} &= -T_{D7} \int d^8\xi \sqrt{-\det\left(g_{ab} + 2\pi\alpha' F_{ab}\right)}, \\ g_{ab} &= \partial_a X^\mu \partial_b X^\nu \hat{g}_{\mu\nu}, \qquad \partial_a \equiv \frac{\partial}{\partial \xi^a}, \end{aligned} \tag{5.3}$$

where $\alpha'^{-2} = 4\pi g_s N_c = 2g_{YM}^2 N_c = \lambda$, and λ is 't Hooft coupling. $\hat{g}_{\mu\nu}$ is the bulk metric given in (5.1). T_{D7} is the D7-brane tension, ξ^a are the D7-brane's worldvolume coordinates, X^μ represents the location of the D7-brane, g_{ab} is the induced metric and F_{ab} is the field strength of the worldvolume U(1) gauge field (a, b are worldvolume indices). The Wess-Zumino term will not affect in our analysis. In this paper, we employ the static gauge where $\xi^a = (t, \vec{x}, z, \vec{\Omega}_3)$.

We assume the translational invariance along the $\vec{x}$ directions and the rotational invariance on the S^3. Furthermore, we assume the configuration of the D7-brane is time-independent. We introduce the chemical potential for the quark charge, that corresponds to the boundary value of A_t given at (5.9), into this system [23, 24]. We apply the external electric field E along the x^1 direction, which is encoded in the vector potential as $A_x(t,z) = -Et + h(z)$ within our gauge choice and ansatz [4]. We also employ an ansatz $\psi = \text{const.}$ ($\partial_z \psi = 0$) which is consistent with the equation of motion of ψ.

Then the action density, which is divided by $\int d^3\vec{x}$ and we redefine this as S_{D7}, can be written as follows:

$$
\begin{aligned}
S_{D7} &= -\mathcal{N} \int dt dz \cos^3\theta g_{xx} K, \\
K &\equiv \sqrt{|g_{tt}|\, g_{xx} g_{zz} - (2\pi\alpha')^2 \left(g_{xx} {A_t}'(z)^2 + g_{zz} \dot{A}_x(z,t)^2 - |g_{tt}|\, A'_x(z,t)^2 \right)},
\end{aligned}
\tag{5.4}
$$

where x denotes x^1, the dot and the prime stand for ∂_t and ∂_z, respectively. We have integrated over the S^3 directions in the derivation of (5.4), and $\mathcal{N} = 2\pi^2 T_{D7} = \frac{\lambda}{(2\pi)^4} N_c$ in our convention. Within our ansatz and gauge, the induced metric is the same as the bulk metric except for g_{zz} component:

$$
g_{tt} = -\frac{1}{z^2} \frac{(1 - z^4/z_H^4)^2}{1 + z^4/z_H^4}, \quad g_{xx} = \frac{1}{z^2}\left(1 + \frac{z^4}{z_H^4}\right), \quad g_{zz} = \frac{1}{z^2} + \theta'^2. \tag{5.5}
$$

Gauge field

Let us remind ourselves of the analysis given in [4]. The equations of motion for the gauge field are integrated to be

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A'_t(z)} &= -\cos^3\theta g_{xx} \frac{\mathcal{N}(2\pi\alpha')^2 g_{xx} A'_t(z)}{K} \equiv D, \\
\frac{\partial \mathcal{L}}{\partial A'_x(t,z)} &= \cos^3\theta g_{xx} \frac{\mathcal{N}(2\pi\alpha')^2\, |g_{tt}|\, h'(z)}{K} \equiv J,
\end{aligned}
\tag{5.6}
$$

where D and J stand for integral constants. From this result, we can immediately see $D\,|g_{tt}|\, h'(z) = -J g_{xx} A'_t(z)$. From (5.6), we obtain

$$
\begin{aligned}
g_{xx} A'_t(z)^2 &= \frac{1}{(2\pi\alpha')^2} |g_{tt}|\, D^2 \frac{g_{zz}\left(|g_{tt}|\, g_{xx} - (2\pi\alpha')^2 E^2\right)}{\mathcal{N}^2 (2\pi\alpha')^2\, |g_{tt}|\, g_{xx}^3 \cos^6\theta + |g_{tt}|\, D^2 - g_{xx} J^2}, \\
|g_{tt}|\, h'(z)^2 &= \frac{1}{(2\pi\alpha')^2} g_{xx} J^2 \frac{g_{zz}\left(|g_{tt}|\, g_{xx} - (2\pi\alpha')^2 E^2\right)}{\mathcal{N}^2 (2\pi\alpha')^2\, |g_{tt}|\, g_{xx}^3 \cos^6\theta + |g_{tt}|\, D^2 - g_{xx} J^2}.
\end{aligned}
\tag{5.7}
$$

At the vicinity of the horizon $z = z_H$, we can see

$$
\begin{aligned}
g_{xx} {A'_t}^2 &\sim |g_{tt}| \xrightarrow{z \to z_H} 0, \\
g_{zz} \dot{A}_x^2 - |g_{tt}|\, {A'_x}^2 &= g_{zz} E^2 - |g_{tt}|\, h'^2 \xrightarrow{z \to z_H} g_{zz} E^2 - g_{zz} E^2 = 0,
\end{aligned}
\tag{5.8}
$$

then the Lagrangian density becomes zero at $z = z_H$.

At the vicinity of the boundary, the gauge fields can be expanded as

$$\begin{aligned} A_t(z) &= \mu - \frac{1}{2}\frac{D}{\mathcal{N}(2\pi\alpha')^2}z^2 + O(z^4), \\ h(z) &= b + \frac{1}{2}\frac{J}{\mathcal{N}(2\pi\alpha')^2}z^2 + O(z^4). \end{aligned} \tag{5.9}$$

Their leading (non-normalizable) terms give the sources for the dual operators. A_t is dual to the charge density J_t, hence μ is interpreted as the chemical potential. As is discussed in [24] we require $A_t(z_H) = 0$ which determine D as a function of μ. For $h(z)$ we demand simply $b = 0$ since there is no source term corresponding to it at the boundary gauge theory. The sub-leading (normalizable) terms of the asymptotic solution should give expectation values of the dual operators: $\langle J^t \rangle = D, \langle J^x \rangle = J$ by the GKP-Witten relation.

The on-shell DBI action is now given by

$$S_{D7} = -\mathcal{N}\int dzdt\cos^6\theta g_{xx}^{5/2}\left|g_{tt}\right|^{1/2}\sqrt{\frac{g_{zz}\left(\left|g_{tt}\right|g_{xx} - (2\pi\alpha')^2E^2\right)}{\left|g_{tt}\right|g_{xx}^3\cos^6\theta + \frac{\left|g_{tt}\right|D^2 - g_{xx}J^2}{\mathcal{N}^2(2\pi\alpha')^2}}}, \tag{5.10}$$

which can be complex in general. However, we are studying the steady states, and we request the DBI action to be real. This requires the term in the square root to be positive semi-definite for all region of $0 \le z \le z_H$:

$$\frac{g_{zz}\left(\left|g_{tt}\right|g_{xx} - (2\pi\alpha')^2E^2\right)}{\left|g_{tt}\right|g_{xx}^3\cos^6\theta + \frac{\left|g_{tt}\right|D^2 - g_{xx}J^2}{\mathcal{N}^2(2\pi\alpha')^2}} \ge 0, \tag{5.11}$$

which is achieved by setting both the numerator and the denominator flip the signs at the same point, say $z = z_*$, between $z = 0$ and z_H [4]. Then the reality condition is reduced to

$$\left.\left|g_{tt}\right|g_{xx} - (2\pi\alpha')^2E^2\right|_{z=z_*} = 0, \tag{5.12}$$

$$\left.\left|g_{tt}\right|g_{xx}^3\cos^6\theta + \frac{\left|g_{tt}\right|D^2 - g_{xx}J^2}{\mathcal{N}^2(2\pi\alpha')^2}\right|_{z=z_*} = 0. \tag{5.13}$$

From (5.12), we have

$$z_*^2 = \left(\sqrt{e^2+1} - e\right)z_H^2, \qquad e \equiv \frac{|E|}{\frac{\pi}{2}\sqrt{\lambda}T^2}, \tag{5.14}$$

where e is a dimensionless quantity. Then (5.13) gives

$$J^2 = \left(\frac{N_c^2 T^2}{16\pi^2} \sqrt{e^2+1} \cos^6 \theta(z_*) + \frac{d^2}{e^2+1} \right) E^2, \tag{5.15}$$

where the dimensionless quantity d has been defined as

$$d \equiv \frac{D}{\frac{\pi}{2}\sqrt{\lambda} T^2} = \frac{\langle J^t \rangle}{\frac{\pi}{2}\sqrt{\lambda} T^2}. \tag{5.16}$$

From (5.15) we obtain the non-linear conductivity σ as [4]

$$J = \sigma E, \qquad \sigma \equiv \sqrt{\frac{N_c^2 T^2}{16\pi^2} \sqrt{e^2+1} \cos^6 \theta(z_*) + \frac{d^2}{e^2+1}}. \tag{5.17}$$

Scalar field

The Euler-Lagrange equation for θ from (5.4) is coupled to the gauge field. Substituting (5.7) into the equation of motion, it is given by

$$\begin{aligned} &\partial_z \left[\frac{\theta'(z)}{(2\pi\alpha')\sqrt{-g_{11} g_{tt} g_{zz}}} \sqrt{f(z) k(z)} \right] \\ &- 3(2\pi\alpha')\mathcal{N}^2 g_{11}^{5/2} \left(-g_{tt} g_{zz}\right)^{1/2} \sin\theta \cos^5\theta \sqrt{\frac{f(z)}{k(z)}} = 0, \end{aligned} \tag{5.18}$$

and $f(z)$ and $k(z)$ are defined by

$$\begin{aligned} f(z) &\equiv (2\pi\alpha')^2 E^2 + g_{11} g_{tt}, \\ k(z) &\equiv (2\pi\alpha')^2 \mathcal{N}^2 g_{11}^3 g_{tt} \cos^6\theta + J^2 g_{11} + D^2 g_{tt}, \end{aligned} \tag{5.19}$$

for notational simplification. At $z = z_H$, the potential term vanishes. Hence $\partial\mathcal{L}/\partial\partial_z\theta$ is constant at $z = z_H$:

$$\left. \frac{\theta'(z)}{(2\pi\alpha')\sqrt{-g_{11} g_{tt} g_{zz}}} \sqrt{f(z) k(z)} \right|_{z=z_H} = \text{const.} \tag{5.20}$$

The denominator goes to zero at $z = z_H$ since $g_{tt}(z_H) = 0$, and hence the numerator should be zero at $z = z_H$. This implies $\theta'(z_H) = 0$.

The asymptotic solution at the boundary is

$$\theta = \theta_0 z + \theta_2 z^3 + \cdots, \tag{5.21}$$

where θ_0 is related to the current quark mass M_q of the fundamental representation, the mass of the charge carrier, as $M_q = \frac{1}{2}\sqrt{\lambda}T\theta_0$. θ_2 gives the quark condensate as $\langle\bar{\psi}\psi\rangle = -\frac{1}{8}\sqrt{\lambda}N_c T^3\theta_2$ [24].

Here, we proceed further than [4], by investigating into the relationship between $\theta(z_*)$ and M_q in detail for later use. The reason why we are interested in the relationship between M_q and $\theta'(z_*)$ but not $\theta'(z_H)$, is that $z = z_*$ turns out to be the location of "effective horizon" in Section 6.2. The equation of motion for $\theta(z)$ at $z = z_*$ is given by

$$\begin{aligned} \partial_z \frac{\partial \mathcal{L}}{\partial \partial_z \theta(z)} &\sim \partial_z \left[\frac{\theta'(z) f'(z_*)(z - z_*)}{(2\pi\alpha')\sqrt{-g_{11}g_{tt}g_{zz}}} \sqrt{\frac{k'(z_*)}{f'(z_*)}} \right], \\ \frac{\partial \mathcal{L}}{\partial \theta(z)} &\sim 3(2\pi\alpha')\mathcal{N}^2 g_{11}^{5/2} \left(-g_{tt}g_{zz}\right)^{1/2} \sin\theta\cos^5\theta \sqrt{\frac{f'(z_*)}{k'(z_*)}}. \end{aligned} \tag{5.22}$$

Here we have used $f(z_*) = k(z_*) = 0$. Finally we obtain the equation of motion for $\theta(z)$ at $z = z_*$ as

$$\theta'(z)k'(z_*) - 3(2\pi\alpha')^2\mathcal{N}^2 g_{11}^3 \left(-g_{tt}\right) g_{zz} \sin\theta\cos^5\theta = 0. \tag{5.23}$$

We have $\theta'(z_*) = \left(C_2 \mp \sqrt{C_2^2 + C_3^2}\right) / (C_3 z)$ by solving the quadratic equation of θ'_*. Here we consider $\theta'(z_*)$ as a positive value $\theta'(z_*) > 0$. Since $C_2 \geq 0$ and $C_3 \leq 0$, we take the minus sign in the numerator[#17]:

$$\theta'(z_*) = \frac{C_2 - \sqrt{C_2^2 + C_3^2}}{C_3 z_*}, \tag{5.24}$$

where

$$\begin{aligned} C_1 &= -J^2\left(z_H^4 + z_*^4\right)^2 + D^2\left(z_H^8 + 6z_H^4 z_*^4 + z_*^8\right), \\ C_2 &= 4(2\pi\alpha')^2\mathcal{N}^2(z_H^4 + z_*^4)^3\left(z_H^8 + z_*^8\right)\cos^6\theta_* + C_1 z_H^{12} z_*^6 \\ &= (2\pi\alpha')^2\mathcal{N}^2\left(z_H^4 + z_*^4\right)^3\left(3z_H^8 + 2z_H^4 z_*^4 + 3z_*^8\right)\cos^6\theta_* + 8D^2 z_H^{16} z_*^{10}, \\ C_3 &= 3(2\pi\alpha')^2\mathcal{N}^2\left(z_*^4 - z_H^4\right)\left(z_H^4 + z_*^4\right)^4 \sin\theta_*\cos^5\theta_*. \end{aligned} \tag{5.25}$$

One finds $k'(z_*)$ is non-zero at $\theta_* = 0$ and $\pi/2$, $\theta'_* = 0$ is realized at $\theta_* = 0$ and $\pi/2$, that means $0 < \theta_* < \pi/2$ if $\theta'_* \neq 0$.

(5.24) relates $\theta'(z_*)$ to $\theta(z_*)$: the boundary condition at $z = z_*$ is given once we specify $\theta(z_*)$. Then we can solve the equation of motion numerically

[#17] Otherwise, if we choose the plus sign in the numerator, $\theta'(z_*)$ diverges at $\theta(z_*) = 0$ and $\pi/2$.

to find M_q from the boundary value of θ. Fig. 8 demonstrates the behaviors of $\theta(z_*)$ (we may write θ_* as an abbreviation of $\theta(z_*)$) and $\partial_z\theta(z_*)$ as functions of M_q at $T = 0.1$, $E = 0.1$ and $D^2 = 0.1$.#18 We find that $\theta(z_*)$ is a monotonically increasing function of M_q starting from zero at $M_q = 0$ and approaching to $\pi/2$ when $M_q \to \infty$. We also find $\theta'(z_*) = 0$ at $M_q = 0, \infty$. One finds that $\theta'(z_*) = 0$ at $|D| = \infty$ from (5.23) as well.

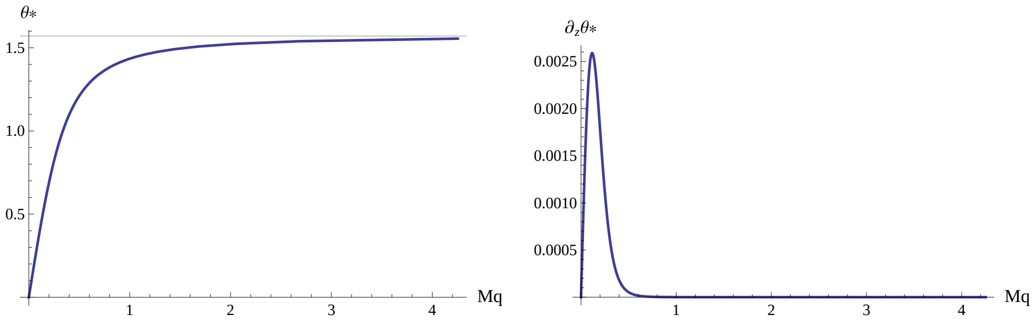

Figure 8: $\theta(z_*)$ and $\partial_z\theta(z_*)$ as functions of M_q. The curves are computed at $T = 0.1$, $E = 0.1$ and $D^2 = 0.1$. The straight line in the left figure indicates the asymptotic value $\theta(z_*) = \pi/2$.

#18 In the numerical computations, we set $2\pi\alpha' = \mathcal{N} = 1$ for simplicity.

6 Fluctuations and effective temperature

This section is the main part of this thesis, which is based on our original work [12].

The main purpose of the present work is to investigate the properties of the effective temperature of NESS. Of course, the notion of temperature in non-equilibrium systems is debatable. In our paper, we define the effective temperature from the relationship between the small fluctuations of physical quantities and the corresponding dissipations [7, 10, 11]. Therefore, analysis of small fluctuations is essential in defining the effective temperature in our study.

The fluctuations of physical quantities correspond to the fluctuations of normalizable modes in the gravity dual. Hence we are most interested in the equations of motion of fluctuations on the probe brane around the background configuration corresponding to NESS.

6.1 Effective metric

Let us consider the fluctuations of X_μ and A_a (which we write $\tilde{X}^\mu$ and $\tilde{A}^a$, respectively) around the solutions obtained in Section 5.1 (which we write $\bar{X}^\mu$ and $\bar{A}^a$, respectively). The equations of motion of $\tilde{X}^\mu$ and $\tilde{A}^a$ are given by perturbing the equations of motion of X^μ and A^a with the replacement $X^\mu \to \bar{X}^\mu + \tilde{X}^\mu$ and $A^a \to \bar{A}^a + \tilde{A}^a$.

It is worth while mentioning for arbitrary setups, and let us begin with the DBI action of a Dp-brane on an arbitrary background geometry whose metric is $\hat{g}_{\mu\nu}$:

$$S = -T_p \int d^{p+1}\xi \, e^{-\Phi} \sqrt{-\det(g_{ab} + 2\pi\alpha' F_{ab})}, \tag{6.1}$$

where T_p is the tension of the Dp-brane, ξ^a are the worldvoulme coordinates, Φ is the dilaton field, $g_{ab} = \partial_a X^\mu \partial_b X^\nu \hat{g}_{\mu\nu}$ is the induced metric, and $F_{ab} = \partial_a A_b - \partial_b A_a$ is the field-strength of the worldvolume $U(1)$ gauge field. The equations of motion of X^μ and A^a are[#19]

$$-\partial_b \left(e^{-\Phi} \omega \sqrt{-G} \hat{g}_{\mu\nu} G^{ab} \partial_a X^\mu \right) + \frac{1}{2} e^{-\Phi} \omega \sqrt{-G} G^{ab} \partial_\nu \hat{g}_{\alpha\beta} \partial_a X^\alpha \partial_b X^\beta = 0, \tag{6.2}$$

$$\partial_a \left(e^{-\Phi} \omega \sqrt{-G} G^{ab} F_{bc} g^{cd} \right) = 0, \tag{6.3}$$

[#19]See Appendix B.

where

$$G_{ab} = g_{ab} - (2\pi\alpha')^2 (Fg^{-1}F)_{ab} \tag{6.4}$$

is the open-string metric [25, 26] and $\omega = (g/G)^{1/4}$. Note that Φ and $\hat{g}_{\mu\nu}$ contain X^μ; g_{ab} contains X^μ and $\partial_a X^\mu$; ω and G_{ab} contain X^μ, $\partial_a X^\mu$ and F_{ab} in general. They may provide non-trivial interactions.

Now, we substitute $X^\mu = \bar{X}^\mu + \tilde{X}^\mu$ and $A^a = \bar{A}^a + \tilde{A}^a$ into (6.2) and (6.3), and consider the equations of motion for $\tilde{X}^\mu$ and $\tilde{A}^a$ to the *linear order* in fluctuations. The equations of motion can be divided into groups of i) the terms with second derivative of fluctuations and ii) the terms with first derivative or without derivative of fluctuations. In Section 6.2, we find that $z = z_*$ plays a role of a horizon of the geometry whose metric is G_{ab}. At the horizon, the terms of i) become dominant because of the redshift and the terms of ii) are negligible. Therefore, if we are interested in the behavior of the fluctuations at the vicinity of $z = z_*$, we need only the terms of i) [27]. The reason why we focus on the vicinity of $z = z_*$ shall be explained shortly.

Of course, the foregoing argument can be justified only when the fluctuations indeed obey the equations of motion on a curved spacetime given by the metric G_{ab}. We show it is indeed the case, at least for some special cases. The terms of i) above in the static gauge can be written as follows[#20]:

$$\begin{aligned} e^{-\Phi}\omega\partial_b\left(\sqrt{-G}G^{ab}\partial_a\tilde{X}^\perp\right) + (\text{terms which contain } \partial_a\bar{X}^\perp) &= 0, & (6.5)\\ e^{-\Phi}\omega g^{cd}\partial_a\left(\sqrt{-G}G^{ab}\tilde{F}_{bc}\right) + (\text{terms which contain } \partial_a\bar{X}^\perp) &= 0, & (6.6)\end{aligned}$$

where $\tilde{F}_{ab} = \partial_a\tilde{A}_b - \partial_b\tilde{A}_a$. The dilaton, the induced metric and the open-string metric contain only the background solutions here. $X^\perp$ denotes X^μ in the directions perpendicular to the worldvolume directions, which are the physical degrees of freedom in the static gauge. Therefore, for the cases with $\partial_a\bar{X}^\perp = 0$,[#21] (6.5) and (6.6) reduce to

$$\begin{aligned} \partial_b\left(\sqrt{-G}G^{ab}\partial_a\tilde{X}^\perp\right) &= 0, & (6.7)\\ \partial_a\left(\sqrt{-G}G^{ab}\tilde{F}_{bc}\right) &= 0, & (6.8)\end{aligned}$$

which are the Klein-Gordon equation and the Maxwell equation, respectively, on a geometry whose metric is G_{ab}.

The reason why we are interested in the equations of motion at the vicinity of $z = z_*$ is that the computations of correlation functions of the fluctuations

[#20] Explicit forms of (6.5) and (6.6) are shown in Appendix C.

[#21] For more general situations, we postpone the analysis in future work [28].

are governed by them in the following sense. Since $z = z_*$ turns out to be a horizon (which we call effective horizon) of the geometry given by the metric G_{ab}, the ingoing-wave boundary condition for fluctuations has to be imposed at $z = z_*$. This means that the correlation functions are parametrized by the Hawking temperature associated with the effective horizon rather than that at the bulk horizon $z = z_H$. Since both the fluctuations and the dissipations are evaluated through the correlation functions, the effective temperature defined by (a generalization of) the fluctuation-dissipation relation at NESS is given by the Hawking temperature of the effective horizon, but not the temperature of the heat bath. Now, (6.7) and (6.8) show that the effective temperature can be read from G_{ab} [5, 6, 7, 8, 9, 10].

6.2 Diagonalization of effective metric

In our setup of D3-D7 model, G_{ab} is given by

$$G_{ab} = g_{ab} + (2\pi\alpha')^2 \begin{pmatrix} \frac{F_{tz}^2}{g_{zz}} + \frac{E^2}{g_{xx}} & \frac{F_{tz}F_{xz}}{g_{zz}} & 0 & 0 & \frac{EF_{xz}}{g_{xx}} \\ \frac{F_{tz}F_{xz}}{g_{zz}} & \frac{F_{xz}^2}{g_{zz}} + \frac{E^2}{g_{tt}} & 0 & 0 & \frac{EF_{tz}}{-g_{tt}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{EF_{xz}}{g_{xx}} & \frac{EF_{tz}}{-g_{tt}} & 0 & 0 & \frac{F_{tz}^2}{g_{tt}} + \frac{F_{xz}^2}{g_{xx}} \end{pmatrix}, \tag{6.9}$$

which has off-diagonal components owing to the non-vanishing field-strength of the worldvoulme gauge field. In order to diagonalize this effective metric, we consider the following transformation for t, x and z:

$$\begin{pmatrix} dt \\ dx \\ dz \end{pmatrix} \longrightarrow \begin{pmatrix} d\tau \\ d\eta \\ d\rho \end{pmatrix} = \begin{pmatrix} dt + \frac{G_{xt}G_{xz} - G_{xx}G_{tz}}{(G_{xt})^2 - G_{xx}G_{tt}} dz \\ dx + \frac{G_{xt}}{G_{xx}} dt + \frac{G_{xz}}{G_{xx}} dz \\ dz \end{pmatrix}, \tag{6.10}$$

and then the diagonalized metric $\mathcal{G}_{ab}$ is

$$\begin{aligned} \mathcal{G}_{\tau\tau} &= -\frac{(2\pi\alpha')^2\mathcal{N}^2 g_{xx}^2 \cos^6\bar{\theta}\,(g_{xx}\,|g_{tt}| - (2\pi\alpha')^2E^2)}{(2\pi\alpha')^2\mathcal{N}^2 g_{xx}^3 \cos^6\bar{\theta} + D^2}, \\ \mathcal{G}_{\rho\rho} &= \frac{(2\pi\alpha')^2\mathcal{N}^2 g_{xx}^3\,|g_{tt}|\,g_{zz}\cos^6\bar{\theta}}{(2\pi\alpha')^2\mathcal{N}^2 g_{xx}^3\,|g_{tt}|\cos^6\bar{\theta} + D^2\,|g_{tt}| - J^2 g_{xx}}, \\ \mathcal{G}_{\eta\eta} &= \frac{\left((2\pi\alpha')^2\mathcal{N}^2 g_{xx}^3\cos^6\bar{\theta} + D^2\right)(g_{xx}\,|g_{tt}| - (2\pi\alpha')^2E^2)}{(2\pi\alpha')^2\mathcal{N}^2 g_{xx}^3\,|g_{tt}|\cos^6\bar{\theta} + D^2\,|g_{tt}| - J^2 g_{xx}}, \\ \mathcal{G}_{22} &= \mathcal{G}_{33} = g_{xx}. \end{aligned} \tag{6.11}$$

Note that the numerator of $\mathcal{G}_{\tau\tau}$ and the denominator of $\mathcal{G}_{\rho\rho}$ contain (5.12) and (5.13), respectively, which go to zero at $z = z_*$. One can check that $\mathcal{G}_{\eta\eta}$

has non-zero and finite value at $z = z_*$. Hence, near z_*, the effective metric behaves as follows:

$$\mathcal{G}_{\tau\tau} \sim -a(z - z_*), \qquad \mathcal{G}_{\rho\rho} \sim b/(z - z_*), \tag{6.12}$$

where

$$\begin{aligned} a &= \left. \frac{(2\pi\alpha')^2 \mathcal{N}^2 g_{xx}^2 \cos^6\bar{\theta}\,(g_{xx}\,|g_{tt}|)'}{(2\pi\alpha')^2 \mathcal{N}^2 g_{xx}^3 \cos^6\bar{\theta} + D^2} \right|_{z=z_*}, \\ b &= \left. \frac{(2\pi\alpha')^2\, \mathcal{N}^2 g_{xx}^3\, |g_{tt}|\, g_{zz} \cos^6\bar{\theta}}{\left(\frac{(2\pi\alpha')^2 \mathcal{N}^2 g_{xx}^3 |g_{tt}| \cos^6\bar{\theta} + D^2 |g_{tt}|}{g_{xx}}\right)' g_{xx}} \right|_{z=z_*}. \end{aligned} \tag{6.13}$$

This means that $z = z_*$ plays the role of the horizon (effective horizon) for the small fluctuations of the normalizable modes on the probe D-brane when (6.7) and (6.8) hold. Then the Hawking temperature associated with the effective horizon, which we call effective temperature T_*, can be read from the ratio of a to b as follows:

$$T_* = \frac{1}{4\pi}\sqrt{\frac{a}{b}}. \tag{6.14}$$

6.3 Results

In this section, we present the results of our analysis on the effective temperature. First, we show two limiting cases where the results are obtained analytically, and then we present numerical results for more general cases.

In our setup, $X^\perp$ corresponds to θ and ψ. We have employed the ansatz (which is consistent with the equation of motion) $\partial_a\bar{\psi} = 0$, and (6.7) always holds for $\tilde{\psi}$. We can also show that $\tilde{\psi}$ decouples from the other modes within the consideration of Section 6.1. Therefore, the notion of the effective temperature for $\tilde{\psi}$ is valid for all the cases presented in this section. For θ, we have found in Section 5.1 that $\partial_a\bar{\theta}$ at $z = z_*$ vanishes when $M_q = 0$, $M_q = \infty$ and when $|D| = \infty$. At these limits, (6.7) for $\tilde{\theta}$ holds, and $\tilde{F}_{ab}$ obeys to (6.8). Therefore, the results for these three limits given in Section 6.3 are valid for all the physical fluctuations $\tilde{\psi}$, $\tilde{\theta}$ and $\tilde{F}_{ab}$.

Infinite-mass limit and high-density limit

T_* depends on $\bar{\theta}(z_*)$ as we can see in (6.13) and (6.14). The relationship between $\bar{\theta}(z_*)$ and physical parameters, such as M_q, is obtained from the nonlinear equation of motion which is solvable only numerically in general.

However, we find that the large mass limit $M_q \to \infty$ corresponds to the limit of $\bar{\theta}(z_*) \to \pi/2$ in Fig. 8, and T_* at this limit can be computed analytically by using this property. We find that the effective temperature behaves as

$$T_* = \frac{1}{4\pi}\sqrt{\frac{64}{z_H^2\sqrt{4+(2\pi\alpha')^2E^2z_H^4}} + \mathcal{O}\left(\theta_* - \pi/2\right)} \longrightarrow \frac{T}{\sqrt[4]{1+e^2}}, \tag{6.15}$$

at the large-mass limit. Note that the effective temperature is *lower* than the heat-bath temperature at finite E.

Let us compare the effective temperature (6.15) to that of the Langevin system given in [5]. In [5], the effective temperature is given as $T_* = \sqrt[4]{1-v^2}\,T$, where v stands for the velocity of the test quark. In our system, the average velocity of the charge carriers is given by the following relationship:

$$v^2 = \left(\frac{J}{D}\right)^2 = \frac{e^2}{1+e^2}. \tag{6.16}$$

The evaluation of the average velocity is justified since the contribution of the pair creation is absent at the large-mass limit. In the presence of the pair creation, the positively charged particles and the negatively charged particles are moving in the opposite directions, and J does not necessarily reflect the average velocity of the charge carriers. Then we obtain from (6.15) and (6.16)

$$T_* = \sqrt[4]{1-v^2}\;T, \tag{6.17}$$

which completely agrees with the result of the Langevin system of infinitely heavy single test particle [5].

One finds that we can also take the high-density limit, $D^2 \to \infty$, in (6.13) and (6.14) analytically. We obtain

$$T_* = \frac{1}{4\pi}\sqrt{\frac{64}{z_H^2\sqrt{4+(2\pi\alpha')^2E^2z_H^4}} + \mathcal{O}\left(\frac{1}{D^2}\right)} \longrightarrow \frac{T}{\sqrt[4]{1+e^2}}, \tag{6.18}$$

which coincides with (6.15). At the large-density limit, the contribution of the doped carriers dominates over that of the pair creation, and we can again justify the estimation (6.16). Although we obtain (6.18) for arbitrary mass, it coincides with (6.17) at the large-density limit.

Numerical results

Numerical computation is necessary for the cases of arbitrary density and arbitrary mass. We show the results from the numerical analysis. We set $2\pi\alpha' = \mathcal{N} = 1$ for simplicity in the numerical computations.

Fig. 9 shows the effective temperature at the massless limit but for various densities. Here we set the heat-bath temperature to be $T = 0.1$ which is indicated by the straight line for reference. We have checked that the result at $D = 0$ agrees with that in [10] where $T_* > T$ at finite E. However, we find that a region of $T_* < T$ appears for $D \neq 0$ when E is small but nonzero. The condition for $T_* < T$ shall be found to be $D^2/T^6 > \lambda N_c^2/32$ at (6.19).

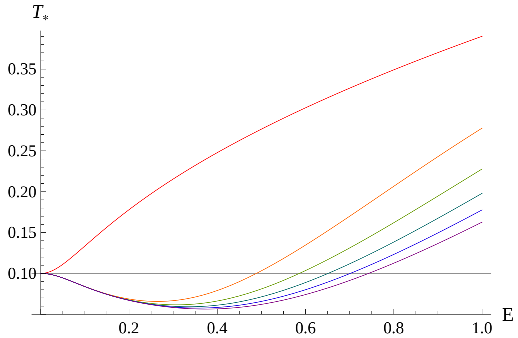

Figure 9: T_* vs. E for massless case at $T = 0.1$. The straight line indicates the temperature of the heat bath. The other curves correspond to different D's from the upper curve ($D^2 = 0$) to the lower curve ($D^2 = 5$) in increments of 1.

The results for finite mass are given in Fig. 10. We present the relationship between T_* and M_q In Fig. 10, we have set $T = 0.1$ and $E = 0.2$. The curves correspond to different D's from the upper line ($D^2 = 0.1$) to the lower one ($D^2 = 1$) in increments of 0.1. One can check the consistency, that T_* at M_q is the same as that at $E = 0.2$ in Fig. 9, and the curves degenerates into the value given by (6.15) in the large M_q region.

Fig. 11 shows T_* vs. D^2 at $T = 0.1$ and $E = 0.8$. The curves correspond to different $\bar{\theta}(z_*)$'s from the upper curve ($\bar{\theta}(z_*) = 0$) to the lower one ($\bar{\theta}(z_*) = 0.9 \times \pi/2$) in increments of $0.1 \times \pi/2$. We present the relationship between T_* and $\bar{\theta}(z_*)$ rather than that for T_* and M_q, but we can read the dependence of T_* on M_q qualitatively since M_q is a monotonically increasing function of $\bar{\theta}(z_*)$ as is demonstrated in Fig. 8[#22]. In this figure, the curves reach the same value given in (6.18) at high densities. Independence of T_* on M_q at high densities can also be seen in Fig. 10 where the dependence of T_* on $\bar{\theta}(z_*)$

[#22]We use $\bar{\theta}(z_*)$ rather than M_q in Fig. 11 and Fig. 13 since $\bar{\theta}(z_*) = \pi/2$ corresponds to $M_q = \infty$, and $\bar{\theta}(z_*)$ is a useful parameter at the large M_q region.

becomes weaker as D goes large.

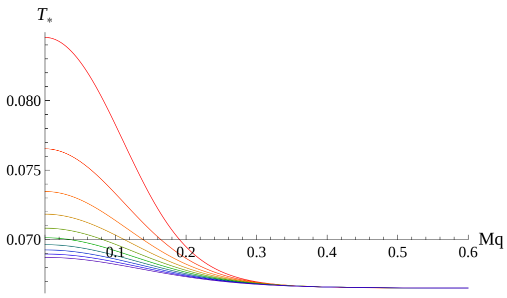

Figure 10: T_* vs. M_q at $T = 0.1$ and $E = 0.2$. The curves correspond to different D's from the upper curve ($D^2 = 0.1$) to the lower curve ($D^2 = 1$) in increments of 0.1.

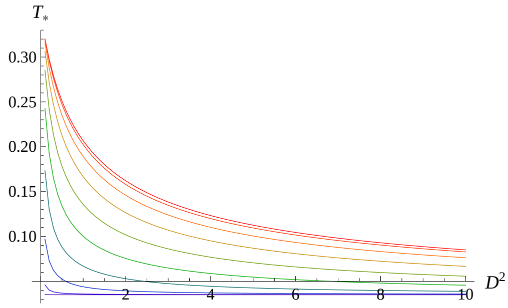

Figure 11: T_* vs. D^2 for various mass at $T = 0.1$ and $E = 0.8$. The curves correspond to different $\bar{\theta}(z_*)$'s from the upper curve ($\bar{\theta}(z_*) = 0$) to the lower curve ($\bar{\theta}(z_*) = 0.9 \times \pi/2$) in increments of $0.1 \times \pi/2$.

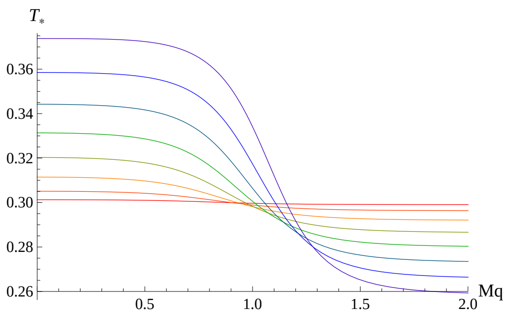

Figure 12: T_* vs. M_q at $T = 0.3$ and $D^2 = 0.2$. The curves correspond to different E's. It varies from $E = 0.1$ to $E = 0.8$ in increments of 0.1 when we follow the intercept on the T_* axis from up to down.

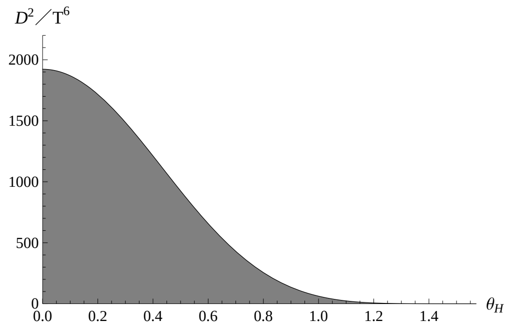

Figure 13: A diagram for behavior of the effective temperature. The boundary line is $D^2/T^6 = I$. The region under the line (filled by gray color) corresponds to the region for $D^2/T^6 < I$ and hence $T_* > T$, whereas the region above the line (the white region) is for $D^2/T^6 > I$ where $T_* < T$.

Fig. 12 shows T_* vs. M_q at $T = 0.3$ and $D^2 = 0.2$. The curves correspond to different E's. It varies from $E = 0.1$ to $E = 0.8$ in increments of 0.1 when we follow the intercept on the T_* axis from up to down. The figure shows that T_* of the system of light carriers increases along E [#23], whereas T_* for the system of heavy carriers decreases along E.

This implies that the pair creation, which is dominant at small M_q, has an effect of raising the effective temperature, whereas the drag effect, which is dominant at large M_q, lowers the effective temperature.

Small E analysis

So far we have found that the effective temperature becomes lower when the density and the mass of the carriers are large. In order to highlight this property, let us examine the effective temperature T_* in the small E region but for arbitrary density and mass. Expanding T_* with respect to E to the order of E^2, we find that $T_* < T$ is realized when the following condition is satisfied:

$$\begin{aligned} \frac{D^2}{T^6} &> I(\bar{\theta}_H), \\ I(\bar{\theta}_H) &\equiv \frac{\lambda N_c^2}{2^9} \cos^{11/2} \bar{\theta}_H \\ &\quad \times \left[4\sqrt{\cos\bar{\theta}_H} + 3\sqrt{2}\sqrt{4 + 7\cos\bar{\theta}_H - 4\cos(2\bar{\theta}_H) + \cos(3\bar{\theta}_H)} \right], \end{aligned} \tag{6.19}$$

where $\bar{\theta}(z_H)$ is abbreviated as $\bar{\theta}_H$.

Fig. 13 shows the behavior of the effective temperature at small but nonzero E for various densities and masses. The region under the line (filled by gray color) corresponds to $D^2/T^6 < I(\bar{\theta}_H)$ and hence $T_* > T$, whereas the region above the line (the white region) is for $D^2/T^6 > I(\bar{\theta}_H)$ where $T_* < T$.

[#23] Precisely speaking, this statement is correct when $|D|$ satisfies (6.19).

7 Effective temperature in general models

In this thesis, this section is also the main part [12].

In this section we present the effective temperature in the large mass and/or large density limit in general models[#24]. We consider a quantum field theory in $(p+1)$-dimensions at temperature T. Then we introduce a probe D$(q+1+n)$-brane, which expands in the $(q+2)$-dimensional spacetime $(t, x^1, \cdots, x^q, r,)$ and wraps the n-dimensional subspace $\vec{\Omega}_n$ of S^{8-p}.

7.1 Setup

To generalize the study in the D3-D7 system, we consider the heat bath to be N_c Dp branes (with $p < 7$) at temperature T. Its holographic dual has the following background metric [29]:

$$ds^2 = \hat{g}_{tt}dt^2 + \hat{g}_{xx}d\vec{x}^2 + \hat{g}_{zz}dz^2 + \hat{g}_{\Omega\Omega}d\Omega^2, \tag{7.1}$$

where $\hat{g}_{\Omega\Omega}$ is the metric on the unit S^{8-p}. The conponents of the bulk metric are given as[#25]

$$\begin{aligned}
\hat{g}_{tt} &= -\left(\frac{L^2}{z^2}\left(1+\frac{z^4}{z_H^4}\right)\right)^{\frac{7-p}{4}}\left(1-\frac{2^{\frac{7-p}{2}}\left(\frac{z}{z_H}\right)^{7-p}}{\left(1+\frac{z^4}{z_H^4}\right)^{\frac{7-p}{2}}}\right)\\
&= -\hat{g}_{xx}\left(1-\frac{(\sqrt{2}L/z_H)^{7-p}}{\hat{g}_{xx}^2}\right),\\
\hat{g}_{xx} &= \left(\frac{L^2}{z^2}\left(1+\frac{z^4}{z_H^4}\right)\right)^{\frac{7-p}{4}}, \qquad \hat{g}_{zz} = \frac{1}{|\hat{g}_{tt}|}\,\frac{L^4}{z^4}\frac{\left(1-\frac{z^4}{z_H^4}\right)^2}{\left(1+\frac{z^4}{z_H^4}\right)},\\
\hat{g}_{\Omega\Omega} &= \frac{1}{\hat{g}_{xx}}\frac{L^4}{z^2}\left(1+\frac{z^4}{z_H^4}\right),
\end{aligned} \tag{7.2}$$

where z_H denotes the location of the horizon. The parameter L, which we set to 1 later, has the dimension of length. Our S^{8-p} metric is

$$\begin{aligned}
d\Omega^2 &= d\theta^2 + \sin^2\theta d\Omega_{7-p-n}^2 + \cos^2\theta d\Omega_n^2,\\
d\Omega_{7-p-n}^2 &= d\psi_1^2 + \sin^2\psi_1 d\psi_2^2 + (\sin\psi_1\sin\psi_2)^2 d\psi_3^2\\
&\quad + \cdots + (\sin\psi_1\sin\psi_2\cdots\sin\psi_{7-p-n})^2 d\psi_{7-p-n}^2,
\end{aligned} \tag{7.3}$$

[#24] See for the massless cases [10].
[#25] See also Appendix D.

where $0 \le \theta \le \pi/2$, $0 \le \psi_j \le \pi$ $(j = 1, 2, \cdots, 6-p-n)$ and $0 \le \psi_{7-p-n} \le 2\pi$[#26]. The Hawking temperature T is obtained as

$$T = \frac{7-p}{2^{\frac{p+3}{4}}\pi} \frac{z_H^{\frac{p-5}{2}}}{L^{\frac{p-3}{2}}}. \tag{7.4}$$

We consider a probe D$(q+1+n)$ brane which fills AdS$_5$ and wraps the $S^n \subset S^{8-p}$. Employing the static gauge $\xi^a = (t, \vec{x}, r, \vec{\Omega}_n)$, the DBI action of the D$(q+1+n)$ brane is

$$\begin{aligned} S_{D(q+1+n)} &= -T_{D(q+1+n)} \int dt d\vec{x} dz d\vec{\Omega}_n e^{-\phi} \sqrt{-\det G_{ab}}, \\ G_{ab} &= \partial_a X^\mu \partial_b X^\nu \hat{g}_{\mu\nu} + 2\pi\alpha' F_{ab}, \\ T_{D(q+1+n)} &= \frac{1}{(2\pi)^{q+1+n} \alpha'^{\frac{(q+1+n)+1}{2}} g_s}, \end{aligned} \tag{7.5}$$

where $T_{D(q+1+n)}$ is the D$(q+1+n)$ brane tension, and the dilaton factor $e^{-\phi}$ is given as the inverse of the following:

$$e^\phi = e^{\phi_0} \left(\frac{1}{z^2} \left(1 + \frac{z^4}{z_H^4} \right) \right)^{(p-3)(7-p)/8}. \tag{7.6}$$

Here we assume the system does not depend on the location on the compact S^n space: X^μ and F_{ab} do not depend on $\vec{\Omega}_n$. In addition, we assume the system is time independent and spatially homogeneous, and hence X^μ and F_{ab} depend only on z. We also assume that $\partial_t \psi_k = \partial_z \psi_k = 0$ which is consistent with the equations of motion. We take $A_z = 0$ gauge, and $A_{x^2} = \cdots = A_{x^q} = 0$ by the rotational invariance. Hence only A_t and $A_x \equiv A_{x^1}$ become nonzero functions. Then we take ansatz $A_x = -Et + h(z)$.

Then the action per unit volume, which we redefine as $S_{D(q+1+n)}$, is now given as follows:

$$\begin{aligned} S_{D(q+1+n)} &= -V_{S^n} T_{D(q+1+n)} \int dt dz e^{-\phi} \hat{g}_{\Omega\Omega}^{n/2} \cos^n \theta \hat{g}_{xx}^{(q-1)/2} K \\ &= -\mathcal{N} \int dt dz w(z) \cos^n \theta \hat{g}_{xx}^{(q-1)/2} K, \\ K &\equiv \sqrt{|g_{tt}|\, g_{xx} g_{zz} - (2\pi\alpha')^2 \left(g_{xx} A_t'(z)^2 + g_{zz} \dot{A}_x(z,t)^2 - |g_{tt}|\, A_x'(z,t)^2 \right)}, \\ \mathcal{N} &= T_{D(q+1+n)} V_{S^n} L^n = \frac{V_{S^n} L^n}{(2\pi)^p \alpha'^{\frac{p+1}{2}} g_s}, \qquad w(z) \equiv e^{-\phi} \left(\hat{g}_{\Omega\Omega}^{1/2} / L \right)^n, \end{aligned} \tag{7.7}$$

[#26] See also Appendix F for more detail of ψ_k.

where we have defined $\lambda = 2g_{YM}^2 N_c$, $g_{YM}^2 = (2\pi)^{p-2} g_s \alpha'^{(p-3)/2}$, and V_{S^n} is the volume of the unit S^n.

Equations of motion of the U(1) gauge fields are given as

$$\begin{aligned} \partial_z \frac{\partial \mathcal{L}}{\partial A_t'(z)} &= \partial_z \left[\frac{-\mathcal{N}(2\pi\alpha')^2 w(z) \cos^n \theta g_{xx}^{(q+1)/2} A_t'(z)}{K} \right] = 0, \\ \partial_z \frac{\partial \mathcal{L}}{\partial A_x'(t,z)} &= \partial_z \left[\frac{\mathcal{N}(2\pi\alpha')^2 w(z) \cos^n \theta g_{xx}^{(q-1)/2} \left| g_{tt} \right| A_x'(t,z)}{K} \right] = 0. \end{aligned} \tag{7.8}$$

By substituting the ansatzes, these equations are written as

$$\begin{aligned} - \frac{\mathcal{N}(2\pi\alpha')^2 w(z) \cos^n \theta g_{xx}^{(q+1)/2} A_t'(z)}{\sqrt{\left| g_{tt} \right| g_{xx} g_{zz} - (2\pi\alpha')^2 \left(g_{xx} A_t'(z)^2 + g_{zz} \dot{A}_x(t,z)^2 - \left| g_{tt} \right| A_x'(t,z)^2 \right)}} &\equiv D, \\ \frac{\mathcal{N}(2\pi\alpha')^2 w(z) \cos^n \theta g_{xx}^{(q-1)/2} \left| g_{tt} \right| h'(z)}{\sqrt{\left| g_{tt} \right| g_{xx} g_{zz} - (2\pi\alpha')^2 \left(g_{xx} A_t'(z)^2 + g_{zz} \dot{A}_x(t,z)^2 - \left| g_{tt} \right| A_x'(t,z)^2 \right)}} &\equiv J, \end{aligned} \tag{7.9}$$

where D and J are charge density and current density as is discussed in the D3-D7 system, and they satisfy $D \left| g_{tt} \right| h'(z) = -J g_{xx} A_t'(z)$. Hence we can rewrite (7.9) as follows:

$$\begin{aligned} g_{xx} A_t'(z)^2 &= \frac{1}{(2\pi\alpha')^2} \left| g_{tt} \right| D^2 \frac{g_{zz} \left(\left| g_{tt} \right| g_{xx} - (2\pi\alpha')^2 E^2 \right)}{\mathcal{N}^2 (2\pi\alpha')^2 w^2 \left| g_{tt} \right| g_{xx}^q \cos^{2n} \theta + \left| g_{tt} \right| D^2 - g_{xx} B^2}, \\ \left| g_{tt} \right| h'(z)^2 &= \frac{1}{(2\pi\alpha')^2} g_{xx} B^2 \frac{g_{zz} \left(\left| g_{tt} \right| g_{xx} - (2\pi\alpha')^2 E^2 \right)}{\mathcal{N}^2 (2\pi\alpha')^2 w^2 \left| g_{tt} \right| g_{xx}^q \cos^{2n} \theta + \left| g_{tt} \right| D^2 - g_{xx} B^2}. \end{aligned} \tag{7.10}$$

Substituting the geuge solutions, the action becomes as follows:

$$\begin{aligned} S_{D(q+1+n)} = -\mathcal{N} \int & dtdz \; w^2 \cos^{2n} \theta g_{xx}^{q-1/2} \\ & \times \sqrt{\frac{(2\pi\alpha')^2 \mathcal{N}^2 \left| \bar{g}_{tt} \right| \bar{g}_{zz} \left(\left| \bar{g}_{tt} \right| \bar{g}_{xx} - (2\pi\alpha')^2 E^2 \right)}{(2\pi\alpha')^2 \mathcal{N}^2 w^2 \left| \bar{g}_{tt} \right| \bar{g}_{xx}^q \cos^{2n} \theta + D^2 \left| \bar{g}_{tt} \right| - B^2 \bar{g}_{xx}}}. \end{aligned} \tag{7.11}$$

We then impose the reality condition on the action as we have did in the case of the D3-D7 system. Hence the following should be satisfied:

$$\begin{aligned} |\bar{g}_{tt}|\, \bar{g}_{xx} - (2\pi\alpha')^2 E^2 \Big|_{z=z_*} &= 0, \\ (2\pi\alpha')^2 \mathcal{N}^2 w^2\, |\bar{g}_{tt}|\, \bar{g}_{xx}^q \cos^{2n}\theta + D^2\, |\bar{g}_{tt}| - B^2 \bar{g}_{xx} \Bigg|_{z=z_*} &= 0, \end{aligned} \tag{7.12}$$

at some point $z = z_*$ which is between $z = 0$ and z_H. From the first relation, we can see z_* as follows:

$$\begin{aligned} z_* &= \sqrt{C - \sqrt{C^2 - 1}}\, z_H, \\ C &\equiv \left(1 + (2\pi\alpha')^2 E^2 \left(\frac{z_H}{\sqrt{2}L}\right)^{7-p}\right)^{\frac{2}{7-p}} = \left(1 + e^2\right)^{\frac{2}{7-p}}, \\ e &\equiv (2\pi\alpha') E \left(\frac{z_H}{\sqrt{2}L}\right)^{\frac{7-p}{2}} = (2\pi\alpha') E \left(\frac{7-p}{4\pi L T}\right)^{-\frac{7-p}{p-5}}. \end{aligned} \tag{7.13}$$

From the second equation in (7.12), we obtain the conductivity σ as

$$B^2 = \sigma^2 E^2, \qquad \sigma^2 \equiv \frac{|\bar{g}_{tt}|}{\bar{g}_{xx} E^2} \left[(2\pi\alpha')^2 \mathcal{N}^2 w^2 \bar{g}_{xx}^q \cos^{2n}\theta + D^2\right] \Bigg|_{z=z_*}. \tag{7.14}$$

Note that only the pair-creation term, the first term in the parenthesis in (7.14), depends on q and n.

For $n \neq 0$ case, the equation of motion of θ is $\partial_z \frac{\partial \mathcal{L}}{\partial \partial_z \theta(z)} - \frac{\partial \mathcal{L}}{\partial \theta(z)} = 0$, where

$$\begin{aligned} \partial_z \frac{\partial \mathcal{L}}{\partial \partial_z \theta(z)} &= \partial_z \left[\frac{-\mathcal{N} w(z) \cos^n \theta\, g_{xx}^{(q-1)/2} \left(|g_{tt}|\, g_{xx} - (2\pi\alpha')^2\, \dot{A}_x(t,z)^2\right) \hat{g}_{\Omega\Omega} \theta'}{K} \right], \\ \frac{\partial \mathcal{L}}{\partial \theta(z)} &= n \mathcal{N} w \tan\theta \cos^n \theta\, g_{xx}^{(q-1)/2} K, \end{aligned} \tag{7.15}$$

where K is defined at (7.7). Thus the equation of motion for $\theta(z)$ near z_* is obtained as follows

$$\begin{aligned} \partial_z \frac{\partial \mathcal{L}}{\partial \partial_z \theta(z)} &= \partial_z \left[\frac{\hat{g}_{\theta\theta} \theta'(z) f'(z_*)(z - z_*)}{(2\pi\alpha') \sqrt{-g_{11} g_{tt} g_{zz}}} \sqrt{\frac{k'(z_*)}{f'(z_*)}} \right], \\ \frac{\partial \mathcal{L}}{\partial \theta(z)} &= n (2\pi\alpha') \mathcal{N}^2 w^2 g_{11}^{q-1/2} \left(-g_{tt} g_{zz}\right)^{1/2} \sin\theta \cos^{2n-1}\theta \sqrt{\frac{f'(z_*)}{k'(z_*)}}, \end{aligned} \tag{7.16}$$

where

$$\begin{aligned} f(z) &\equiv (2\pi\alpha')^2 E^2 + g_{11} g_{tt}, \\ k(z) &\equiv (2\pi\alpha')^2 \mathcal{N}^2 w^2 g_{tt} g_{xx}^q \cos^{2n} + B^2 g_{xx} + D^2 g_{tt}. \end{aligned} \tag{7.17}$$

Then the equation of motion is reduced to

$$\hat{g}_{\Omega\Omega}\theta'(z)k'(z_*) - n(2\pi\alpha')^2 \mathcal{N}^2 w^2 g_{11}^q \,|g_{tt}|\, g_{zz} \sin\theta \cos^{2n-1}\theta = 0. \tag{7.18}$$

Here we can see that $\theta'(z_*) = 0$ when $\theta(z_*) = 0$ or $\pi/2$.

7.2 Effective temperature at large mass limit

The effective temperature of the conductor system is given by

$$\begin{aligned} T_* &= \frac{1}{4\pi}\sqrt{\frac{a}{b}}, \\ a &= \left.\frac{w g_{xx}^{q-1} \cos^{2n}\theta \left(|g_{xx}|\, g_{tt}\right)'}{w g_{xx}^q \cos^{2n}\theta + D^2}\right|_{z_*}, \qquad b = \left.\frac{w\,|g_{tt}|\, g_{zz} g_{xx}^q \cos^{2n}\theta}{\left(\frac{w|g_{tt}|g_{xx}^q \cos^{2n}\theta + D^2|g_{tt}|}{g_{xx}}\right)' g_{xx}}\right|_{z_*}, \end{aligned} \tag{7.19}$$

where w is a model-dependent factor that includes the contributions of the dilaton, the tension of the probe brane and the volume of the compact directions. g_{ab} is the induced metric in the given setup. Note that T_* depends on q and n in general. However, we find that T_* becomes independent of q and n if we take the limit of $D \to \infty$ or $M_q \to \infty$ ($\theta_* = \pi/2$). Furthermore, the effective temperature coincides with each other at both limits, as is the case of the D3-D7 system. The effective temperature at these limits is

$$T_* = \frac{1}{4\pi}\sqrt{\frac{\left(|g_{tt}|\, g_{xx}\right)'}{|g_{tt}|\, g_{zz}}\left(\frac{|g_{tt}|}{g_{xx}}\right)'} = \frac{T}{(1+e^2)^{\frac{1}{7-p}}} = \left(1 - v^2\right)^{\frac{1}{7-p}} T, \tag{7.20}$$

where $e = (2\pi\alpha')E\left(\frac{7-p}{4\pi T}\right)^{\frac{7-p}{5-p}}$, and the average velocity of the charge carrier is given as (6.16). The result, which is the generalization of (6.15) and (6.18), agrees with the effective temperature of the corresponding Langevin system (dragged string system) shown in [10].

8 Summary and discussions

We have analyzed the properties of the effective temperature of NESS in holographic models. Our systems are many-body systems of charge carriers driven by the electric field. We find that at the large-density limit and at the large-mass limit of the charge carriers, the effective temperature agrees with that for the corresponding Langevin system. Let us find possible interpretations of our result.

In the conductor systems, the charge carriers have two origins: those who have doped and those who have pair created. The pair creation is suppressed at the large-mass limit, and the effect of the doped carriers dominates at the large-density limit. This means that the effective temperature of the conductor systems and that of the Langevin systems agree when the role of the doped charge is dominant. This is natural in the sense that the systems of the doped carriers are the many-body systems of the single dragged particle in the same medium. However, our analysis shows more: the effective temperature of the *doped charges* are not affected by the interaction among them at the large-mass limit, since it is independent of the density at this limit. We also found that the effective temperature of the *doped charges* is not affected by the mass at the high-density limit, either. The reason why we have emphasized *doped charges* is that the effect of the pair creation becomes un-important at these limits. For the mutual consistency of these limits, the effective temperatures at these two limits have to agree with each other. We found it is indeed the case. Note that these properties are owing to neither the supersymmetry of the microscopic theory nor the conformal invariance, since we have observed the same properties in general models that do not necessarily have supersymmetry nor conformal invariance[#27].

We have also found that the effective temperature can be lower than the temperature of the heat bath even for the systems which show the higher effective temperature in the neutral case. For example, we found that the effective temperature of the D3-D7 system at finite densities can be lower than the heat-bath temperature in the region of small electric field. These observations lead us to the conclusion that the pair creation of charge carriers has an effect to raise the effective temperature whereas dragging of the doped carriers lowers the effective temperature, in a wide range of holographic models of NESS we have studied.

It has been found that the systems we have studied show non-linear conductivity, and some of them shows even interesting characteristics such

[#27]Note that the supersymmetry is broken in our setup because of the temperature and the density even if the original microscopic theory is supersymmetric.

as negative differential conductivity [30] and non-equilibrium phase transitions [31]. It is interesting to see how the properties of the effective temperature contribute to these phenomena. We leave this for future study.

Acknowledgments

I would like to express my special appreciation and thanks to my adviser Professor Masayasu Harada. You have been a tremendous mentor for me. I would like to thank you for introducing me to research field of QCD and for encouraging my research of hadron physics. You always gave me fruitful suggestions. I would also like to thank my collaborator, Professor Yong-Liang Ma. Your advice about hadron physics was really helpful for me. I appreciate Professor Chiho Nonaka for giving me opportunity to know hot topics in heavy-ion collision physics. I am grateful to everyone of H-lab, E-lab, and QG-lab.

I am thankful to my collaborator, Professor Shin Nakamura. I would like to thank you for introducing me to research field of gauge/gravity correspondence and for discussing my research at large. Without your support, I would not be able to accomplish this thesis.

My work has been supported in part by Global COE Program “Quest for Fundamental Principles in the Universe” of Nagoya University (G07).

A special thanks to my family. Words cannot express how grateful I am to my mother-in-law, father-in-law, my mother, and father for all of the sacrifices that you’ve made on my behalf. Your prayer for me was what sustained me thus far. I would also like to thank all of my friends who supported me in writing, and motivated me to strive towards my goal. At the end I would like express appreciation to my beloved wife Yukiko who spent sleepless nights with me and always supported me.

A Scale invariance in $\mathcal{N}=4$ SU(N_c) SYM and AdS$_5$

Here we comment on the scale-invariance of the $\mathcal{N}=4$ SU(N_c) SYM theory and the AdS$_5$ spacetime.

$\mathcal{N}=4$ supersymmetry (SUSY) has four supercharges. The theory includes gauge fields A_μ, scalar fields ϕ_i $(i=1,\cdots,6)$ and Weyl fermions λ_I $(I=1,\cdots,4)$. The Lagrangian density is given by

$$\mathcal{L}=\frac{1}{g_{YM}^2}\mathrm{Tr}\left[-\frac{1}{2}F_{\mu\nu}^2-(D_\mu\phi_i)^2-\bar{\lambda}_I\gamma^\mu D_\mu\lambda^I+\mathcal{O}(\phi^4)+\mathcal{O}(\lambda\lambda\phi)\right], \quad \text{(A.1)}$$

where $F_{\mu\nu}\equiv F_{\mu\nu}^a t^a$, and the generator of the gauge group t^a satisfies $\mathrm{Tr}(t^a t^b)=\delta^{ab}/2$. γ^μ is the usual gamma matrix, and D denotes the covariant derivative. Note that the $\mathcal{N}=4$ super Yang-Mills theory does not have any parameter having dimensions, and hence that is conformal theory at the classical level.

At the 1-loop level, the β-function is obtained as

$$\beta=-\frac{g_{YM}^3}{48\pi^2}N_c\left(11-2n_f-\frac{1}{2}n_s\right)=0, \quad \text{(A.2)}$$

where $n_f=4$, $n_s=6$ are number of Weyl fermion and real scalar fields, respectively. This theory is scale invariant and is known as a conformal filed theory (CFT). The four-dimensional conformal group is known to be SO(2,4) group.

The metric of AdS$_5$ is given by

$$ds_5^2=\left(\frac{r}{L}\right)^2(-dt^2+d\vec{x}^2)+L^2\frac{dr^2}{r^2}, \quad \text{(A.3)}$$

where the parameter L is called the AdS radius. This spacetime has the SO(2,4) invariance. As a part of the invariance, it has the scale invariance under $x^\mu\to ax^\mu$ and $r\to r/a$. Note that, the energy which is conjugate variable of t has the same transformation property as r. Hence, r will be interpreted as the energy scale of the gauge theory.

B General forms of equations of motion

Here we derive (6.2) and (6.3) from the DBI action (5.3) [26, 32]. In this appendix, we suppress the indices of $\hat{g}_{ab}$, g_{ab}, F_{ab} and G_{ab}. G and its transposition are defined as

$$G=g+F, \qquad G^T=g-F, \quad \text{(B.1)}$$

where g and F are symmetric and anti-symmetric matrix respectively. For simplicity, we set $2\pi\alpha' = 1$ temporarily. Then its inverse $G^{ab} \equiv \left(G^{-1}\right)^{ab}$ is defined as

$$\begin{aligned} G^{-1} &= (g+F)^{-1} = \left[\hat{g}\left(1+g^{-1}F\right)\right]^{-1} = \left(1+g^{-1}F\right)^{-1} g^{-1} \\ &= \Big(1 - g^{-1}F + \left(-g^{-1}F\right)^2 + \cdots\Big)\, g^{-1}, \\ \left(G^{-1}\right)^T &= \Big(1 + g^{-1}F + \left(g^{-1}F\right)^2 + \cdots\Big)\, g^{-1}. \end{aligned} \tag{B.2}$$

We define symmetric part $G^{ab}_{(S)}$ as

$$\begin{aligned} G^{-1}_{(S)} &\equiv \frac{G^{-1} + \left(G^{-1}\right)^T}{2} = \Big(1 + \left(g^{-1}F\right)^2 + \left(g^{-1}F\right)^4 + \cdots\Big)\, g^{-1} \\ &= \Big(1 - \left(g^{-1}F\right)^2\Big)^{-1} g^{-1} = \left(g - Fg^{-1}F\right)^{-1}, \end{aligned} \tag{B.3}$$

and hence its inverse matrix is defined[#28] as $G_{(S)} = g - Fg^{-1}F$. The anti-symmetric part $G^{ab}_{(A)}$ of G^{ab} is

$$\begin{aligned} G^{-1}_{(A)} &\equiv \frac{G^{-1} - \left(G^{-1}\right)^T}{2} = -\Big(g^{-1}F + \left(g^{-1}F\right)^3 + \left(g^{-1}F\right)^5 + \cdots\Big)\, g^{-1} \\ &= -g^{-1}F\Big(1 + \left(g^{-1}F\right)^2 + \left(g^{-1}F\right)^4 + \cdots\Big)\, g^{-1} = -g^{-1}F\left(g - Fg^{-1}F\right)^{-1} \\ &= -g^{-1}FG_{(S)} = -G_{(S)}Fg^{-1}. \end{aligned} \tag{B.4}$$

The determinant of G and $G_{(S)}$ can be written as

$$\begin{aligned} \det G &= \det g \det\left(1 + g^{-1}F\right) = \det g \det\left(1 - g^{-1}F\right), \\ \det G_{(S)} &= \det g \det\left(1 - (g^{-1}F)^2\right) = \det g \left[\det\left(1 + g^{-1}F\right)\right]^2. \end{aligned} \tag{B.5}$$

The lower relation have been obtained by using the upper one. As as result, we have the following relation between $\det G$ and $\det G_{(S)}$ as

$$\det G = \det G_{(S)} \left[\det\left(1 + g^{-1}F\right)\right]^{-1}. \tag{B.6}$$

Then we derive the equation of motion as follows:

$$\delta\sqrt{-\det G} = \frac{1}{2}\sqrt{-\det G}\; G^{ba}\delta G_{ab} = \frac{1}{2}\sqrt{-\det G}\; G^{ba}\left(\delta g_{ab} + \delta F_{ab}\right), \tag{B.7}$$

[#28] Notice that $G^{(S)}$ is not a symmetric part of G: $G_{(S)} \neq \left(G + G^T\right)/2$.

where the variation of g times G^{-1} is

$$\begin{aligned}\delta g_{ab}G^{ba} &= \delta g_{ab}G^{ab}_{(S)} = \delta\left(\hat{g}_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu\right)\cdot G^{ab}_{(S)} \\ &= \left[2\hat{g}_{\mu\nu}\partial_a X^\mu \partial_b \delta X^\nu + \partial_\eta \hat{g}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \delta X^\eta\right] G^{ab}_{(S)},\end{aligned} \tag{B.8}$$

and the variation of F times G^{-1} is

$$\delta F_{ab}G^{ba} = \delta F_{ab}G^{ba}_{(A)} = -2\partial_a \delta A_b G^{ab}_{(A)}. \tag{B.9}$$

Hence $\delta\sqrt{-\det G}$ can be written as

$$\begin{aligned}\delta\sqrt{-\det G} = \frac{1}{2}\omega\sqrt{-\det G_{(S)}} \\ \times \Bigg[&\left[2\hat{g}_{\mu\nu}(X)\partial_a X^\mu \partial_b \delta X^\nu + \partial_\eta \hat{g}_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \delta X^\eta\right] G^{ab}_{(S)} \\ &- 2\partial_a \delta A_b G^{ab}_{(A)}\Bigg],\end{aligned} \tag{B.10}$$

where $\omega \equiv \left[\det\left(1 + g^{-1}F\right)\right]^{-1/2}$ and hence $\det G = \det G_{(S)}\omega^2$ by using (B.6).

Thus the equations of motion are obtained as follows:

$$\begin{aligned}-\partial_b\left(\omega\sqrt{-\det G_{(S)}}G^{ab}_{(S)}\hat{g}_{\mu\nu}(X)\partial_a X^\mu\right) \\ +\frac{1}{2}\omega\sqrt{-\det G_{(S)}}G^{ab}_{(S)}\partial_\nu \hat{g}_{\sigma\eta}(X)\partial_a X^\sigma \partial_b X^\eta = 0, \\ \partial_a\left(\omega\sqrt{-\det G_{(S)}}G^{ab}_{(S)}F_{bc}g^{cd}\right) = 0.\end{aligned} \tag{B.11}$$

These equations are (6.2) and (6.3). Finally we have the following EOMs written by the covariant derivatives:

$$\begin{aligned}&\nabla^2 X^\mu + G^{ab}_{(S)}\Gamma^\mu{}_{\sigma\eta}(\hat{g})\nabla_a X^\sigma \nabla_b X^\eta + G^{ab}_{(S)}\nabla_a X^\mu \nabla_b \ln\omega = 0, \\ &\nabla^a\left(\omega F_{ab}g^{bc}\right) = 0,\end{aligned} \tag{B.12}$$

where ∇_a is the covariant derivative with respect to $G_{(S)}$, and $\Gamma^\sigma{}_{\mu\eta}(\hat{g})$ is written by the back ground metric $\hat{g}$:

$$\begin{aligned}&\partial_a\left(\sqrt{-\det G_{(S)}}G^{ab}_{(S)}\Phi_b\right) = \sqrt{-\det G_{(S)}}\nabla_a \Phi_b \\ &\qquad\qquad (\Phi_b = \partial_b X^\mu, \partial_b\omega, \partial_b \hat{g}_{\mu\nu}(X)), \\ &\Gamma^\mu{}_{\sigma\eta} = \frac{1}{2}\hat{g}^{\mu\nu}(X)\Big[\partial_\sigma \hat{g}_{\eta\nu}(X) + \partial_\eta \hat{g}_{\sigma\nu}(X) - \partial_\nu \hat{g}_{\sigma\eta}(X)\Big].\end{aligned} \tag{B.13}$$

Here note that, the derivative in $\Gamma^\mu_{\sigma\eta}$ is $\partial_\sigma/\partial X^\sigma$ but not $\partial/\partial\xi$.

C Equations of motion for fluctuations

Here, we obtain the explicit forms of (6.5) and (6.6). To achieve them, we expand (B.11) and obtain explicit forms of the equations of motion for fluctuations in the D3-D7 system. The back ground metric is given in (5.1). We employ the static gauge as in the main body of text.

We expand X_μ and A_a as

$$X^\mu \to \bar{X}^\mu + \tilde{X}^\mu, \qquad A^a \to \bar{A}^a + \tilde{A}^a, \tag{C.1}$$

where $\tilde{X}^\mu$ and $\tilde{A}_a$ stand for the fluctuations around the classical solutions $\bar{X}^\mu$ and $\bar{A}_a$. Here we consider expansions to the first order of the fluctuations.

In this section, we need to consider only expansions of $\partial_a \tilde{X}^\mu$ but not $\tilde{X}^\mu$ since the terms vanish. For example, we consider the back ground metric $\hat{g}_{\mu\nu}(X)$. Generally it is expanded as $\hat{g}_{\mu\nu}(X) = \hat{g}_{\mu\nu}(\bar{X}) + \partial_\eta \hat{g}_{\mu\nu}(\bar{X})\tilde{X}^\eta$, however here we consider the case in which the second term vanishes. For example, $\partial_\eta \hat{g}_{\mu\nu}(\bar{X})\tilde{X}^\eta$ does not contribute when $\hat{g}_{\mu\nu}$ depends only on the non dynamical variable r. As the same way, the second term of upper equation in (B.11) vanishes. Then the equations of motion can be written as

$$\begin{aligned} -\partial_b \left(\sqrt{-\det G}\, G^{ab}_{(S)} \hat{g}_{\mu\nu}(\bar{X}) \partial_a X^\mu \right) &= 0, \\ -\partial_a \left(\sqrt{-\det G}\, G^{ab}_{(A)} \right) &= 0. \end{aligned} \tag{C.2}$$

Here we have used $\sqrt{-\det G} = \omega \sqrt{-\det G_{(S)}}$.

Neglecting the second order of fluctuations, the parts of the equations of motion are expanded as follows:

$$\begin{aligned} G_{ab} &= \overline{G}_{ab} + \tilde{G}_{ab}, \\ \overline{G}_{ab} &\equiv \hat{g}_{\mu\nu} \partial_a \overline{X}^\mu \partial_b \overline{X}^\nu + \overline{F}_{ab}, \\ \tilde{G}_{ab} &\equiv \hat{g}_{\mu\nu} \left(\partial_a \overline{X}^\mu \partial_b \tilde{X}^\nu + \partial_a \tilde{X}^\mu \partial_b \overline{X}^\nu \right) + \tilde{F}_{ab}, \end{aligned} \tag{C.3}$$

$$\begin{aligned} \sqrt{-\det G} &= \sqrt{-\det \bar{G}} + \left. \frac{\partial \sqrt{-\det G}}{\partial \partial_a X^\mu} \right|_{\substack{X=\bar{X} \\ A=\bar{A}}} \partial_a \tilde{X}^\mu + \left. \frac{\partial \sqrt{-\det G}}{\partial \partial_a A_b} \right|_{\substack{X=\bar{X} \\ A=\bar{A}}} \partial_a \tilde{A}_b \\ &= \sqrt{-\det \bar{G}} + \sqrt{-\det \bar{G}}\, \bar{G}^{ab}_{(S)} \partial_b \bar{X}^\nu \hat{g}_{\mu\nu}(\bar{X}) \partial_a \tilde{X}^\mu \\ &\quad + \sqrt{-\det \bar{G}}\, \bar{G}^{ab}_{(A)} \partial_a \tilde{A}_b, \end{aligned} \tag{C.4}$$

$$
\begin{aligned}
G_{(S)}^{ab} &= \bar{G}_{(S)}^{ab} + \left.\frac{\partial G_{(S)}^{ab}}{\partial \partial_c X^\mu}\right|_{\substack{X=\bar{X}\\ A=\bar{A}}} \partial_c \tilde{X}^\mu + \left.\frac{\partial G_{(S)}^{ab}}{\partial \partial_c A_d}\right|_{\substack{X=\bar{X}\\ A=\bar{A}}} \partial_c \tilde{A}_d \\
&= \bar{G}_{(S)}^{ab} - \left(\bar{G}_{(S)}^{ca}\bar{G}_{(S)}^{bd} + \bar{G}_{(A)}^{ca}\bar{G}_{(A)}^{bd} + \bar{G}_{(S)}^{cb}\bar{G}_{(S)}^{ad} + \bar{G}_{(A)}^{cb}\bar{G}_{(A)}^{ad} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \partial_d \bar{X}^\sigma \hat{g}_{\mu\sigma}(\bar{X}) \partial_c \tilde{X}^\mu \\
&\quad + \left(\bar{G}_{(S)}^{ac}\bar{G}_{(A)}^{bd} - \bar{G}_{(A)}^{ac}\bar{G}_{(S)}^{bd} + \bar{G}_{(S)}^{bc}\bar{G}_{(A)}^{ad} - \bar{G}_{(A)}^{bc}\bar{G}_{(S)}^{ad} \right) \partial_c \tilde{A}_d,
\end{aligned} \tag{C.5}
$$

$$
\begin{aligned}
G_{(A)}^{ab} &= \bar{G}_{(A)}^{ab} + \left.\frac{\partial G_{(A)}^{ab}}{\partial \partial_c X^\mu}\right|_{\substack{X=\bar{X}\\ A=\bar{A}}} \partial_c \tilde{X}^\mu + \left.\frac{\partial G_{(A)}^{ab}}{\partial \partial_c A_d}\right|_{\substack{X=\bar{X}\\ A=\bar{A}}} \partial_c \tilde{A}_d \\
&= \bar{G}_{(A)}^{ab} + \left(\bar{G}_{(S)}^{ac}\bar{G}_{(A)}^{bd} - \bar{G}_{(A)}^{ac}\bar{G}_{(S)}^{bd} + \bar{G}_{(S)}^{ad}\bar{G}_{(A)}^{bc} - \bar{G}_{(A)}^{ad}\bar{G}_{(S)}^{bc} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \partial_d \bar{X}^\sigma \hat{g}_{\mu\sigma}(\bar{X}) \partial_c \tilde{X}^\mu \\
&\quad - \left(\bar{G}_{(S)}^{ac}\bar{G}_{(S)}^{bd} - \bar{G}_{(A)}^{ac}\bar{G}_{(A)}^{bd} - \bar{G}_{(S)}^{bc}\bar{G}_{(S)}^{ad} + \bar{G}_{(A)}^{bc}\bar{G}_{(A)}^{ad} \right) \partial_c \tilde{A}_d.
\end{aligned} \tag{C.6}
$$

Finally we have the following form of the equations of motion:

$$
\begin{aligned}
\partial_a \left[\sqrt{-\bar{G}} {\gamma_{(XX)}}_{\mu\nu}^{ab} \partial_b \tilde{X}^\nu \right] + \partial_c \left[\sqrt{-\bar{G}} {\gamma_{(XA)}}_{\mu}^{abc} \partial_a \tilde{A}_b \right] &= 0, \\
\partial_a \left[\sqrt{-\bar{G}} {\gamma_{(AA)}}^{abcd} \partial_c \tilde{A}_d \right] + \partial_a \left[\sqrt{-\bar{G}} {\gamma_{(XA)}}_{\mu}^{abc} \partial_c \tilde{X}^\mu \right] &= 0,
\end{aligned} \tag{C.7}
$$

The first terms in (C.7) contain the first terms of (6.5) and (6.6).
where γ's are defined by using $\bar{P}_\mu^a \equiv \bar{G}_{(S)}^{ab} \partial_b \bar{X}^\sigma \hat{g}_{\sigma\mu}(\bar{X})$ as

$$
\begin{aligned}
{\gamma_{(XX)}}_{\mu\nu}^{ab} &\equiv \bar{G}_{(S)}^{ab}\hat{g}_{\mu\nu} + \Big[-\bar{G}_{(S)}^{ab}\bar{G}_{(S)}^{cd} + \bar{G}_{(S)}^{ac}\bar{G}_{(S)}^{bd} - \bar{G}_{(S)}^{ad}\bar{G}_{(S)}^{bc} \\
&\qquad\qquad\qquad + \bar{G}_{(A)}^{ab}\bar{G}_{(A)}^{cd} - \bar{G}_{(A)}^{ad}\bar{G}_{(A)}^{bc} \Big] \bar{P}_{c\mu}\bar{P}_{d\nu}, \\
{\gamma_{(XA)}}_{\mu}^{abc} &\equiv \left(\bar{G}_{(A)}^{ab}\bar{G}_{(S)}^{cd} - \bar{G}_{(A)}^{ac}\bar{G}_{(S)}^{bd} + \bar{G}_{(S)}^{ad}\bar{G}_{(A)}^{bc} + \bar{G}_{(S)}^{ac}\bar{G}_{(A)}^{bd} - \bar{G}_{(A)}^{ad}\bar{G}_{(S)}^{bc} \right) \bar{P}_{d\mu}, \\
{\gamma_{(AA)}}^{abcd} &\equiv \bar{G}_{(A)}^{ab}\bar{G}_{(A)}^{cd} - \left(\bar{G}_{(S)}^{ac}\bar{G}_{(S)}^{bd} - \bar{G}_{(A)}^{ac}\bar{G}_{(A)}^{bd} - \bar{G}_{(S)}^{bc}\bar{G}_{(S)}^{ad} + \bar{G}_{(A)}^{bc}\bar{G}_{(A)}^{ad} \right).
\end{aligned} \tag{C.8}
$$

D Change of coordinate from r to z in general models

Here we consider the AdS$_{p+2}$-Schwarzschild times S^{8-p} where $p < 7$, and change the variable r into z. As discussed in [29], the metric is given by

$$\begin{aligned} ds^2 = \frac{r^{(7-p)/2}}{L^{(7-p)/2}} \left[-\left(1 - \frac{r_H^{7-p}}{r^{7-p}}\right) dt^2 + d\vec{x}^2 \right] \\ + \frac{L^{(7-p)/2}}{r^{(7-p)/2}} \frac{dr^2}{\left(1 - \frac{r_H^{7-p}}{r^{7-p}}\right)} + \frac{L^{(7-p)/2}}{r^{(7-p)/2}} r^2 d\Omega^2, \end{aligned} \tag{D.1}$$

where the superscription of x^i runs from $i = 1$ to p, and r is the radial coordinate. $d\Omega$ denotes the volume element of unit S^{8-p}. r_H is the location of the horizon, and the parameter L has a dimension of length. The Hawking temperature is given by

$$T = c_0^{-1} \frac{r_H^{(5-p)/2}}{L^{(7-p)/2}}, \qquad c_0 = \frac{4\pi}{7-p}. \tag{D.2}$$

First, we change the variable r into $\tilde{r}$ as

$$\tilde{r}^2 = r^2 + \sqrt{r^4 - r_H^4}, \qquad (\tilde{r_H} = r_H), \tag{D.3}$$

and inversely r is written as

$$r = \sqrt{\frac{\tilde{r}^4 + r_H^4}{2\tilde{r}^2}} = \sqrt{\frac{\tilde{r}^4 + \tilde{r}_H^4}{2\tilde{r}^2}}. \tag{D.4}$$

Then we change the variable $\tilde{r}$ into z as

$$\tilde{r} = \sqrt{2}/z, \qquad (\tilde{r}_H = \sqrt{2}/z_H). \tag{D.5}$$

In terms of z, the radial coordinate r can be written as

$$r = \frac{L^2}{z} \sqrt{1 + \frac{z^4}{z_H^4}}, \tag{D.6}$$

and then

$$dr = dr/dz \cdot dz = -\frac{L^2}{z^2} \left(1 - \frac{z^4}{z_H^4}\right) \Bigg/ \sqrt{1 + \frac{z^4}{z_H^4}} dz. \tag{D.7}$$

Hence the metric of r changes as follows:

$$\hat{g}_{rr}dr^2 = \frac{L^{(7-p)/2}}{r^{(7-p)/2}}\frac{dr^2}{\left(1-\frac{r_H^{7-p}}{r^{7-p}}\right)} = \frac{dr^2}{|\hat{g}_{tt}|} = \frac{1}{|\hat{g}_{tt}|}\;\frac{L^4}{z^4}\;\frac{\left(1-\frac{z^4}{z_H^4}\right)^2}{\left(1+\frac{z^4}{z_H^4}\right)}dz^2. \tag{D.8}$$

Finally, we have the metric as

$$ds^2 = \hat{g}_{tt}dt^2 + \hat{g}_{xx}d\vec{x}^2 + \hat{g}_{zz}dz^2 + \hat{g}_{\Omega\Omega}d\Omega^2, \tag{D.9}$$

$$\begin{aligned}
\hat{g}_{tt} &= -\frac{r^{\frac{7-p}{2}}}{L^{\frac{7-p}{2}}}\left(1-\frac{r_H^{7-p}}{r^{7-p}}\right) = -\left(\frac{L^2}{z^2}\left(1+\frac{z^4}{z_H^4}\right)\right)^{\frac{7-p}{4}}\left(1-\frac{2^{\frac{7-p}{2}}\left(\frac{z}{z_H}\right)^{7-p}}{\left(1+\frac{z^4}{z_H^4}\right)^{\frac{7-p}{2}}}\right)\\
&= -\hat{g}_{xx}\left(1-\frac{(\sqrt{2}L/z_H)^{7-p}}{\hat{g}_{xx}^2}\right),\\
\hat{g}_{xx} &= \frac{r^{\frac{7-p}{2}}}{L^{\frac{7-p}{2}}} = \left(\frac{L^2}{z^2}\left(1+\frac{z^4}{z_H^4}\right)\right)^{\frac{7-p}{4}},\\
\hat{g}_{zz} &= \frac{1}{|\hat{g}_{tt}|}\;\frac{L^4}{z^4}\frac{\left(1-\frac{z^4}{z_H^4}\right)^2}{\left(1+\frac{z^4}{z_H^4}\right)},\\
\hat{g}_{\Omega\Omega} &= \frac{L^{(7-p)/2}}{r^{(7-p)/2}}r^2 = L^{(7-p)/2}r^{(p-3)/2} = \frac{r^2}{\hat{g}_{xx}} = \frac{1}{\hat{g}_{xx}}\frac{L^4}{z^2}\left(1+\frac{z^4}{z_H^4}\right),
\end{aligned} \tag{D.10}$$

The Hawking temperature T can be written as

$$T = \frac{7-p}{2^{\frac{p+3}{4}}\pi}\cdot\frac{z_H^{\frac{p-5}{2}}}{L^{\frac{p-3}{2}}} \quad\Longleftrightarrow\quad z_H = 2^{\frac{p+3}{2(p-5)}}L^{\frac{3-p}{5-p}}\left(\frac{7-p}{\pi T}\right)^{-\frac{2}{p-5}}. \tag{D.11}$$

The dilaton factor $e^{-\phi}$ is the inverse of the following:

$$e^{\phi} = e^{\phi_0}r^{(p-3)(7-p)/4} = e^{\phi_0}g_{xx}^{(p-3)/2} = e^{\phi_0}\left(\frac{1}{z^2}\left(1+\frac{z^4}{z_H^4}\right)\right)^{(p-3)(7-p)/8}. \tag{D.12}$$

E Diagonalization of open string metric in general models

Here we show the diagonalization of the effective metric in the general dimension case as in the main body of text. We consider the probe D(q+1+n)

brane, which expands in the $(q+2)$ spacetime $(t, x^1, \cdots, x^q, r,)$ and wraps up the n compacted subspace $\vec{\Omega}_n$ of the S^{8-p}. We assume the open string metric is as follows:

$$\bar{G}^{(S)} = \bar{g} + (2\pi\alpha')^2 \begin{pmatrix} \frac{\bar{F}_{1t}^2}{\bar{g}_{11}} + \frac{\bar{F}_{zt}^2}{\bar{g}_{zz}} & \frac{\bar{F}_{z1}\bar{F}_{zt}}{\bar{g}_{zz}} & 0 & -\frac{\bar{F}_{1t}\bar{F}_{z1}}{\bar{g}_{11}} \\ \frac{\bar{F}_{z1}\bar{F}_{zt}}{\bar{g}_{zz}} & \frac{\bar{F}_{1t}^2}{\bar{g}_{tt}} + \frac{\bar{F}_{z1}^2}{\bar{g}_{zz}} & 0 & \frac{\bar{F}_{1t}\bar{F}_{zt}}{\bar{g}_{tt}} \\ 0 & 0 & D & 0 \\ -\frac{\bar{F}_{1t}\bar{F}_{z1}}{\bar{g}_{11}} & \frac{\bar{F}_{1t}\bar{F}_{zt}}{\bar{g}_{tt}} & 0 & \frac{\bar{F}_{z1}^2}{\bar{g}_{11}} + \frac{\bar{F}_{zt}^2}{\mathrm{gtt}} \end{pmatrix}, \tag{E.1}$$

where $D \equiv \mathrm{diag}(\bar{g}_{22}, \cdots, \bar{g}_{qq})$, and $\bar{g}_{11} = \bar{g}_{22} = \cdots = \bar{g}_{qq}$.

Leaving $G_{xx}^{(S)}$ and dz unchanged, we employ the following diagonalization for t, x^1 and z component as discussed in the case of D3-D7 system in the main text:

$$\begin{pmatrix} dt \\ dx \\ dz \end{pmatrix} \longrightarrow \begin{pmatrix} d\tau \\ d\eta \\ d\rho \end{pmatrix} = \begin{pmatrix} dt + \frac{\bar{G}_{xt}^{(S)}\bar{G}_{xz}^{(S)} - \bar{G}_{xx}^{(S)}\bar{G}_{tz}^{(S)}}{\left(\bar{G}_{xt}^{(S)}\right)^2 - \bar{G}_{xx}^{(S)}\bar{G}_{tt}^{(S)}} dz \\ dx + \frac{\bar{G}_{xt}^{(S)}}{\bar{G}_{xx}^{(S)}} dt + \frac{\bar{G}_{xz}^{(S)}}{\bar{G}_{xx}^{(S)}} dz \\ dz \end{pmatrix}, \tag{E.2}$$

and then the diagonalized metric go as

$$\bar{G}_{ab}^{(S)} \longrightarrow \mathcal{G} = \mathrm{diag}\left(\mathcal{G}_{\tau\tau}, \mathcal{G}_{11}, \mathcal{G}_{22}, \cdots, \mathcal{G}_{qq}, \mathcal{G}_{\rho\rho}\right), \tag{E.3}$$

where

$$\begin{aligned} \mathcal{G}_{\tau\tau} &= \frac{\left((2\pi\alpha')^2\bar{F}_{1t}^2 + \bar{g}_{11}\bar{g}_{tt}\right)\left((2\pi\alpha')^2\left(\bar{F}_{1t}^2\bar{g}_{zz} + \bar{F}_{z1}^2\bar{g}_{tt} + \bar{F}_{zt}^2\bar{g}_{11}\right) + \bar{g}_{11}\bar{g}_{tt}\bar{g}_{zz}\right)}{\bar{g}_{11}\left((2\pi\alpha')^2\left(\bar{F}_{1t}^2\bar{g}_{zz} + \bar{F}_{z1}^2\bar{g}_{tt}\right) + \bar{g}_{11}\bar{g}_{tt}\bar{g}_{zz}\right)} \\ &= -\frac{(2\pi\alpha')^2\mathcal{N}^2 w^2 \bar{g}_{11}^{q-1}\left(|\bar{g}_{11}|\,\bar{g}_{tt} - (2\pi\alpha')^2 E^2\right)\cos^{2n}\theta}{(2\pi\alpha')^2\mathcal{N}^2 w^2 \bar{g}_{11}^q \cos^{2n}\theta + D^2}, \\ \mathcal{G}_{\rho\rho} &= \frac{(2\pi\alpha')^2\left(\bar{F}_{1t}^2\bar{g}_{zz} + \bar{F}_{z1}^2\bar{g}_{tt} + \bar{F}_{zt}^2\bar{g}_{11}\right) + \bar{g}_{11}\bar{g}_{tt}\bar{g}_{zz}}{(2\pi\alpha')^2\bar{F}_{1t}^2 + \bar{g}_{11}\bar{g}_{tt}} \\ &= \frac{(2\pi\alpha')^2\mathcal{N}^2 w^2\,|\bar{g}_{tt}|\,\bar{g}_{zz}\bar{g}_{11}^q\cos^{2n}\theta}{(2\pi\alpha')^2\mathcal{N}^2 w^2\,|\bar{g}_{tt}|\,\bar{g}_{11}^q\cos^{2n}\theta + D^2\,|\bar{g}_{tt}| - B^2\bar{g}_{11}}, \\ \mathcal{G}_{yy} &= (2\pi\alpha')^2\left(\frac{\bar{F}_{1t}^2}{\bar{g}_{tt}} + \frac{\bar{F}_{z1}^2}{\bar{g}_{zz}}\right) + \bar{g}_{11} \\ &= \frac{\left(\bar{g}_{11}\,|\bar{g}_{tt}| - (2\pi\alpha')^2E^2\right)\left((2\pi\alpha')^2\mathcal{N}^2 w^2\bar{g}_{11}^q\cos^{2n}\theta + D^2\right)}{(2\pi\alpha')^2\mathcal{N}^2 w^2\,|\bar{g}_{tt}|\,\bar{g}_{11}^q\cos^{2n}\theta + D^2\,|\bar{g}_{tt}| - B^2\bar{g}_{11}}, \\ \mathcal{G}_{22} &= \cdots = \mathcal{G}_{qq} = \bar{g}_{11}. \end{aligned} \tag{E.4}$$

When we put $p = 3, q = 3, n = 3$ and $w = 1$, then the matrix goes back to the effective metric of the D3-D7 system.

In addition, the effective temperature is obtained as

$$
\begin{aligned}
T_* &= \frac{1}{4\pi}\sqrt{\frac{a}{b}}, \\
a &= \left.\frac{(2\pi\alpha')^2\mathcal{N}^2 w^2 \bar{g}_{11}^{q-1}\cos^{2n}\theta\,(|\bar{g}_{11}|\,\bar{g}_{tt})'}{(2\pi\alpha')^2\mathcal{N}^2 w^2 \bar{g}_{11}^{q}\cos^{2n}\theta + D^2}\right|_{z_*}, \\
b &= \left.\frac{(2\pi\alpha')^2\mathcal{N}^2 w^2\,|\bar{g}_{tt}|\,\bar{g}_{zz}\bar{g}_{11}^{q}\cos^{2n}\theta}{\left(\frac{(2\pi\alpha')^2\mathcal{N}^2 w^2|\bar{g}_{tt}|\bar{g}_{11}^{q}\cos^{2n}\theta + D^2|\bar{g}_{tt}|}{\bar{g}_{11}}\right)'\bar{g}_{11}}\right|_{z_*}.
\end{aligned}
\tag{E.5}
$$

F Example of polar coordinates on S^{d-1}

F.1 Separating S^5 into S^1 and S^3

Here we show an example of the polar coordinate on S^5 (5.2). For our aim, it is enough to consider flat space case. The polar coordinate is represented as follows:

$$
\begin{cases}
x_4 &= r\sin\theta\cos\psi, \\
x_5 &= r\sin\theta\sin\psi, \\
x_6 &= r\cos\theta\cos\phi_1, \\
x_7 &= r\cos\theta\sin\phi_1\cos\phi_2, \\
x_8 &= r\cos\theta\sin\phi_1\sin\phi_2\cos\phi_3, \\
x_9 &= r\cos\theta\sin\phi_1\sin\phi_2\sin\phi_3,
\end{cases}
\qquad
\begin{pmatrix}
0 \le r < \infty \\
0 \le \theta \le \pi/2 \\
0 \le \psi < 2\pi \\
0 \le \phi_{1,2} \le \pi \\
0 \le \phi_3 < 2\pi
\end{pmatrix}
$$

where ψ is an angle of S^1 and ϕ_i $(i = 1, 2, 3)$ denotes angles of S^3, and r satisfies $\sum_{k=4}^{9} x_k^2 = r^2$. The metric is given by

$$
\begin{aligned}
ds^2 &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\psi^2 \\
&\qquad + r^2\cos^2\theta\left[d\phi_1^2 + \sin^2\phi_1 d\phi_2^2 + (\sin\phi_1\sin\phi_2)^2\, d\phi_3^2\right] \\
&= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\psi^2 + r^2\cos^2\theta d\Omega_3^2 = dr^2 + r^2 d\Omega_5^2,
\end{aligned}
\tag{F.1}
$$

where $d\Omega_5^2$ and $d\Omega_3^2$ are defined as

$$
\begin{aligned}
d\Omega_5^2 &= d\theta^2 + \sin^2\theta\,\underbrace{d\psi^2}_{S^1} + \cos^2\theta\,\underbrace{d\Omega_3^2}_{S^3}, \\
d\Omega_3^2 &= d\phi_1^2 + \sin^2\phi_1 d\phi_2^2 + (\sin\phi_1\sin\phi_2)^2\, d\phi_3^2.
\end{aligned}
\tag{F.2}
$$

These are metrics of S^5 and S^3 in this coordinate.

In the holographic case, we should replace the components of the metric in front of dr^2 and $d\Omega_5^2$ as in (5.2), however the definition of $d\Omega_5^2$ is the same as in (F.2).

Then the part of the integral in (5.2) goes as follows:

$$\int d\phi_1 d\phi_2 d\phi_3 \sqrt{\det g_3} = \int d\phi_1 d\phi_2 d\phi_3 \cos^3\theta \sin^2\phi_1 \sin\phi_2 = V_{S^3}\cos^3\theta, \tag{F.3}$$

where the metric of S^3 as $g_3 = \mathrm{diag}(\cos\theta, \cos\theta\sin\phi_1, \cos\theta\sin\phi_1\sin\phi_2)$ by setting $r = 1$, and the volume of S^3 is $V_{S^3} = 2\pi^2$.

F.2 Separating S^{d-1} into S^{k-1} and $S^{(d-1)-k}$

Here we consider more general case. We separate S^{d-1} into S^{k-1} and $S^{(d-1)-k}$ as follows:

$$\begin{cases} x_1 &= r\sin\theta\cos\psi_1, \\ x_2 &= r\sin\theta\sin\psi_1\cos\psi_2, \\ &\vdots \\ x_{k-1} &= r\sin\theta\sin\psi_1\cdots\sin\psi_{k-2}\cos\psi_{k-1}, \\ x_k &= r\sin\theta\sin\psi_1\cdots\sin\psi_{k-2}\sin\psi_{k-1}, \\ x_{k+1} &= r\cos\theta\cos\phi_1, \\ x_{k+2} &= r\cos\theta\sin\phi_1\cos\phi_2, \\ &\vdots \\ x_{d-1} &= r\cos\theta\sin\phi_1\cdots\sin\phi_{d-k-2}\cos\phi_{d-k-1}, \\ x_d &= r\cos\theta\sin\phi_1\cdots\sin\phi_{d-k-2}\sin\phi_{d-k-1}, \end{cases} \quad \begin{pmatrix} 0 \le r < \infty \\ 0 \le \theta \le \pi/2 \\ 0 \le \psi_i \le \pi \\ 0 \le \psi_{k-1} < 2\pi \\ 0 \le \phi_i \le \pi \\ 0 \le \phi_{d-k-1} < 2\pi \end{pmatrix}$$

where ψ_i and ϕ_j are angles of S^{k-1} and $S^{(d-1)-k}$ respectively. r is radius of S^{d-1}. Then the metric is given by

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\Omega_{d-1}^2, \\ d\Omega_{d-1}^2 &= d\theta^2 + \sin^2\theta\, \underbrace{d\Omega_{k-1}^2}_{S^{k-1}} + \cos^2\theta\, \underbrace{d\Omega_{(d-1)-k}^2}_{S^{(d-1)-k}}, \\ d\Omega_{k-1}^2 &= d\psi_1^2 + \sin^2\psi_1 d\psi_2^2 + (\sin\psi_1\sin\psi_2)^2 d\psi_3^2 \\ &\quad + \cdots + (\sin\psi_1\sin\psi_2\cdots\sin\psi_{k-1})^2 d\psi_{(k-1)}^2, \\ d\Omega_{(d-1)-k}^2 &= d\phi_1^2 + \sin^2\phi_1 d\phi_2^2 + (\sin\phi_1\sin\phi_2)^2 d\phi_3^2 \\ &\quad + \cdots + \big(\sin\phi_1\sin\phi_2\cdots\sin\phi_{(d-k-1)}\big)^2 d\phi_{(d-k-1)}^2. \end{aligned} \tag{F.4}$$

Hence the following integral is employed:

$$\int \left(\prod_{j=1}^{d-1-k} d\psi_j \right) \sqrt{\det g_{(d-1-k)}} = V_{S^{d-1-k}} \cos^n \theta, \tag{F.5}$$

where $g_{(d-1)-k}$, given by setting $r = 1$, stands for the metric of $S^{(d-1)-k}$, and $V_{(d-1)-k}$ is the volume of $S^{(d-1)-k}$.

Bibliography

[1] V. E. Hubeny and M. Rangamani, "A Holographic view on physics out of equilibrium," Adv. High Energy Phys. **2010**, 297916 (2010) [arXiv:1006.3675 [hep-th]].

[2] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L. G. Yaffe, "Energy loss of a heavy quark moving through N=4 supersymmetric Yang-Mills plasma," JHEP **0607**, 013 (2006) [hep-th/0605158].

[3] S. S. Gubser, "Drag force in AdS/CFT," Phys. Rev. D **74**, 126005 (2006) [hep-th/0605182].

[4] A. Karch and A. O'Bannon, "Metallic AdS/CFT," JHEP **0709**, 024 (2007) [arXiv:0705.3870 [hep-th]].

[5] S. S. Gubser, "Momentum fluctuations of heavy quarks in the gauge-string duality," Nucl. Phys. B **790**, 175 (2008) [hep-th/0612143].

[6] J. Casalderrey-Solana and D. Teaney, "Transverse Momentum Broadening of a Fast Quark in a N=4 Yang Mills Plasma," JHEP **0704**, 039 (2007) [hep-th/0701123].

[7] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, "Langevin diffusion of heavy quarks in non-conformal holographic backgrounds," JHEP **1012**, 088 (2010) [arXiv:1006.3261 [hep-th]].

[8] K. -Y. Kim, J. P. Shock and J. Tarrio, "The open string membrane paradigm with external electromagnetic fields," JHEP **1106**, 017 (2011) [arXiv:1103.4581 [hep-th]].

[9] J. Sonner and A. G. Green, "Hawking Radiation and Non-equilibrium Quantum Critical Current Noise," Phys. Rev. Lett. **109**, 091601 (2012) [arXiv:1203.4908 [cond-mat.str-el]].

[10] S. Nakamura and H. Ooguri, "Out of Equilibrium Temperature from Holography," Phys. Rev. D **88**, 126003 (2013) [arXiv:1309.4089 [hep-th]].

[11] L. F. Cugliandolo, "The Effective Temperature," J. Phys. A: Math. Theor. 44, 483001 (2011).

[12] H. Hoshino and S. Nakamura, "Effective temperature of nonequilibrium dense matter in holography," Phys. Rev. D **91**, no. 2, 026009 (2015) [arXiv:1412.1319 [hep-th]].

[13] 早川 禮之助, 伊藤 耕三, 木村 康之, 岡野 幸治, 『非平衡系のダイナミクス入門: 動的物性の物理』（培風館, 2006）

[14] K. Sasaki and S. Amari, "Diffusion coefficient and mobility of a brownian particle in a tilted periodic potential," J. Phys. Soc. Jpn. 74, 2226 (2005).

[15] 中村 真, 『ホログラフィック・ゲージ理論入門』, 原子核研究 Vol. 56 Supplement 1, p.3-55 (2011)

[16] 夏梅 誠, 『超弦理論の応用: 物理諸分野での AdS/CFT 双対性の使い方』（サイエンス社, 2012）

[17] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].

[18] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998) [hep-th/9802109].

[19] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2**, 505 (1998) [hep-th/9803131].

[20] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000) [hep-th/9905111].

[21] S. S. Gubser, I. R. Klebanov and A. W. Peet, "Entropy and temperature of black 3-branes," Phys. Rev. D **54**, 3915 (1996) [hep-th/9602135].

[22] A. Karch and E. Katz, "Adding flavor to AdS / CFT," JHEP **0206**, 043 (2002) [hep-th/0205236].

[23] S. Nakamura, Y. Seo, S. J. Sin and K. P. Yogendran, "A New Phase at Finite Quark Density from AdS/CFT," J. Korean Phys. Soc. **52**, 1734 (2008) [hep-th/0611021].

[24] S. Kobayashi, D. Mateos, S. Matsuura, R. C. Myers and R. M. Thomson, "Holographic phase transitions at finite baryon density," JHEP **0702**, 016 (2007) [hep-th/0611099].

[25] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999) [hep-th/9908142].

[26] G. W. Gibbons and C. A. R. Herdeiro, "Born-Infeld theory and stringy causality," Phys. Rev. D **63**, 064006 (2001) [hep-th/0008052].

[27] We have benefited by private communication with S. Kinoshita.

[28] H. Hoshino and S. Nakamura, in progress.

[29] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, "Supergravity and the large N limit of theories with sixteen supercharges," Phys. Rev. D **58**, 046004 (1998) [hep-th/9802042].

[30] S. Nakamura, "Negative Differential Resistivity from Holography," Prog. Theor. Phys. **124**, 1105 (2010) [arXiv:1006.4105 [hep-th]].

[31] S. Nakamura, "Nonequilibrium Phase Transitions and Nonequilibrium Critical Point from AdS/CFT," Phys. Rev. Lett. **109**, 120602 (2012) [arXiv:1204.1971 [hep-th]].

[32] K. Hashimoto, S. Kinoshita, K. Murata and T. Oka, "Electric Field Quench in AdS/CFT," JHEP **1409**, 126 (2014) [arXiv:1407.0798 [hep-th]].