Three-Dimensional Supersymmetric Gauge Theories and Seiberg Duality

（邦題：３次元超対称ゲージ理論とザイバーグ双対性）

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Abstract

We study the three-dimensional supersymmetric gauge theories, their quantum dynamics and their dualities which connect very different-looking theories. These three-dimensional dualities are very similar to the four-dimensional ones. Especially we discuss how the 3d dualities appear from the four dimensional (Seiberg) ones by the dimensional reduction. Although the four-dimensional duality can generically include various matter fields like the adjoint or the tensor matters, there are a few examples in 3d. We especially elaborate on the 3d duality with an adjoint matter. In deriving the 3d duality from 4d we need to carefully treat the monopole operators which appear in the high- and low-energy theories and need to carefully analyze the Coulomb branch of the moduli space of vacua, which is absent in 4d physics. This reduction from 4d is done by the introduction of the effective superpotential induced by the Kaluza-Klein (or being known as the twisted instantons) monopoles. We will give the KK-monopoles and the corresponding potentials with the various matter fields included.

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Chapter 1

Introduction

The standard model which is defined in terms of the gauge theories is very successful throughout the wide range of the energy scale. We can now predict many things at the perturbative level. However there are many areas which are not accessible only by the perturbative approach. For example, Quantum-Chromo-Dynamics (QCD) is strongly coupled in the infrared region, then we need some non-perturbative methods. The lattice gauge theories will give the answer for the non-perturbative regime but it is only numerical. Further, the lattice is not good to deal with the massless fields. We would like to calculate some observables analytically. In some cases, the Schwinger-Dyson equation or the $1/N$ expansion will help us to obtain the non-perturbative information. But these are not complete.

The supersymmetry is a powerful tool to do such exact analysis of the low-energy dynamics of the gauge theories as well as it is believed to be realized at the Tev scale of the real world. If we consider the more and more high supersymmetry, the theory becomes more tractable. For example, if we consider the $\mathcal{N}=4$, the UV divergences are all canceled and the theory is UV-finite [1–4] [5–7] [8] [9–11] [12] [13,14]. In the $\mathcal{N}=2$ theory, the low-energy effective actions are exactly determined [15,16]. In the $\mathcal{N}=1$ theory, the superpotential is protected by the holomorphy from the perturbative renormalization [17,18]. Recently the supersymmetric localization is constructed and it is possible to do the exact calculations of the BPS quantities such as the partition function, the supersymmetric Wilson loops and the superconformal index, which enables us to do the non-perturbative analysis needed for the check of the AdS/CFT conjecture, or some dualities [19,20] [21–23] [24–26] [27]. In this thesis we will focus on the theories with four supercharges.

Some people might say that the dynamics of the highly supersymmetric gauge theories is very less than the usual (non-SUSY) gauge theories. However it is wrong. For example, $\mathcal{N}=4$ super Yang-Mills theory is maximally supersymmetric and there are many non-renormalization theorems. But many theoretical physicist study this theory because it is believe to be dual to the Type IIB superstring theory on $AdS_5 \times S^5$ [28–30] [31] and it gives the playground to test the gauge/string duality. This theory is integrable at least in large N_c but it has the strong relationship with the QCD. The $\mathcal{N}=2$ supersymmetric theory (Seiberg-Witten theory) is one of a few theories which is proven to be confined at low-energy (For example, we can derive the fact that the 3d compact QED also confines [32]). The confinement is one of the most difficult questions in Physics. Then the study of the SUSY gauge theories is necessary for the more and more deep understanding of the quantum aspects of the dynamics. In this thesis we concentrate on the theories with four supercharges, which is a minimal number in four-dimensions.

The Seiberg duality [33] is the conjecture that the 4d $\mathcal{N}=1$ supersymmetric gauge theories with different gauge groups give the identical description in the infrared region, which means that two theories are in the same universality class. We will call one side as the "electric" theory and the other as the "magnetic" theory. In many cases, the one side is a strongly coupled gauge theory, the other side is weakly coupled. Then the Seiberg duality offer the useful description of the strongly coupled system. Although this Seiberg duality is first discovered in four-dimensional gauge theories, this is extended to the diverse dimensions. Although there are many checks of the validity of the Seiberg duality, it is not proven yet.

The relation of the Seiberg dualities between the various dimensions has not been clear since their dynamics have some resemblances but have many and crucial discrepancies. For example, the 3d gauge theories reduced from the 4d one generically have the new phases such as the Coulomb branch which is not always present. Recently, the general procedure to obtain the 3d dualities from the 4d Seiberg dualities was proposed by [34, 35]. In [34, 35], the key steps are to put the theory on $\mathbb{S}^1 \times \mathbb{R}^3$ and to include the non-pertutbative superpotential from the Kaluza-Klein monopoles. They proved that the dynamics of the 4d and 3d are strongly related each other although the 3d and 4d theories have seemingly different dynamics.

Although they showed the generic method to obtain the 3d dualities from 4d, the generalization to the various 4d dualities is not unclear. for example, we have the four-dimensional dualities which include the adjoint matters, symmetric matters and anti-symmetric matters. And there are the dualities between the chiral gauge theories. For these dualities, we don't know how to reduce these 4d dualities to three-dimensions. There are some reasons for these difficulties. This is first because the poor understandings of the quantum dynamics of the 3d gauge theories which come from the various 4d Seiberg dualities by the naive dimensional reduction. The second reason is that including the various matter fields into the duality makes the Coulomb branch of the resulting 3d theories very complicated [36], so the reduction process becomes different-looking and the correct description of the Coulomb branch becomes unclear.

In this thesis, we clarify the relation between the 3d and 4d supersymmetric gauge theories which contain four supercharges and study how to derive the 3d dualities from the 4d dualities. In this thesis we will generalize the procedure to obtain the 3d dualities for more complicated 4d dualities. In particular we will derive the 3d duality with an adjoint matter from the 4d Kutasov-Schwimmer duality concentrating on the unitary gauge groups which was first done in [37]. The most parts of my thesis consist of the analysis of the 3d gauge theories, the 3d dualities and how to derive the 3d dualities from the 4d dualities. These explanations are more and more pedagogical and we will provide discussions from the multifaceted perspective. Then we believe that the review part is also valuable.

The organization of this thesis is as follows. In Chap.2, we will briefly introduce the four-dimensional Seiberg dualities. In Chap.3 we will study the three-dimensional gauge theories and their dynamics. The main attention is to be devoted to the 3d $\mathcal{N} = 2$ supersymmetric gauge theories because it is from the reduction of the 4d $\mathcal{N} = 1$ SUSY. Here we will extensively review what we know about the Coulomb branch. In Chap.4 we will study the three-dimensional dualities. In Chap.5, we will explain how to derive the 3d dualities from the 4d Seiberg duality which only includes the fundamental matters. In Chap.6, we proceed to the 3d duality with an adjoint matter field from the 4d duality [37]. In Chap.7, we summarize what we have done and discuss the future directions. In the appendix, we consider the Callias index theorem, calculate the parity anomaly, and show the perturbative picture of the 3d gauge theories.

Chapter 2

The four-dimensional Seiberg duality

In this chapter we study the four-dimensional Seiberg duality, which was first shown in [33]. The Seiberg duality proposes the another (dual) description, which is easier to analyze in the infrared region, to the strongly coupled SUSY gauge theories which are difficult to study in the IR. It is a powerful tool to analyze the low-energy dynamics. There are tremendous varieties for the Seiberg dualities; various gauge groups $SU(N), SO(N), Sp(2N), \cdots$, various matter contents including adjoint matters and tensor matters, etc. Here we will show the major examples.

2.1 Conventional Seiberg duality

2.1.1 $\mathcal{N} = 1$ SQCD

We first consider the $SU(N_c)$ supersymmetruc gauge theories with four supercharges and with N_f fundamental matters. The exact beta function is as follows.

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N_c - N_f + N_f\gamma(g)}{1 - N_c(g^2/8\pi^2)} \tag{2.1}$$

$$\gamma(g) = -\frac{g^2}{8\pi^2}\frac{N_c^2 - 1}{N_c} + O(g^4) \tag{2.2}$$

where γ is the anomalous dimension of the fundamental matter fields.

Depending on the number of flavors, This theory exhibits the various phases:

$$\begin{aligned}
N_f < N_c :&\ \text{runaway, no stable SUSY vacua} \\
N_f = N_c :&\ \text{IR free non-linear sigma model, complementarity} \\
&\ \text{quantum deformed muduli, chiral symmetry breaking} \\
N_f = N_c + 1 :&\ \text{IR free non-linear sigma model,} \\
&\ \text{confinement without chiral symmetry breaking (s-confining)} \\
N_c + 1 < N_f < \frac{3}{2}N_c :&\ \text{free magnetic} \\
\frac{3}{2}N_c < N_f < 3N_c :&\ \text{interacting non-Abelian Coulomb phase} \\
N_f \geq 3N_c :&\ \text{free electric}
\end{aligned}$$

For $N_f < N_c$, the Affleck-Dine-Seiberg superpotential is generated and there is no SUSY vacua. For $N_f = N_c$, the low energy theory is described by the gauge invariant chiral superfields and the chiral symmetry is always broken at any region of the moduli space. For $N_f = N_c+1$, the theory is again described by the gauge invariant chiral superfields but there is no chiral symmetry breaking at the origin of the moduli space, which is conventionally called "s-confining". For $N_c + 1 < N_f < \frac{3}{2}N_c$, the electric theory is strongly coupled. On the other hand, the magnetic theory is free. For $\frac{3}{2}N_c < N_f < 3N_c$, the theory flows to the interacting superconformal fixed point in the IR. For $N_f \geq 3N_c$, the electric theory is free and the magnetic theory is strongly coupled.

Then the low-energy dynamics becomes very different for each N_f. Seiberg [33] offered the dual description for these low-energy theories, which flows to the same IR fixed point as the original theory. The dual theory contains the different gauge group from the original one and in many cases the dual theory is weakly coupled and then tractable.

2.1.2 Seiberg duality

The electric side

Let us first consider the 4d $\mathcal{N}=1$ $SU(N_c)$ SQCD with N_f flavors. The global charges are as follows. Note that the $U(1)_A$ axial symmetry is anomalous.

Table 2.1: The quantum numbers of the 4d $\mathcal{N}=1$ $SU(N_c)$ with N_f flavors

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_f	1	1	$\frac{N_f-N_c}{N_f}$
$\tilde{Q}$	1	$\bar{N}_f$	-1	$\frac{N_f-N_c}{N_f}$
$M=\tilde{Q}Q$	N_f	$\bar{N}_f$	0	$\frac{2(N_f-N_c)}{N_f}$
$B=Q^{N_c}$	${}_{N_f}\mathrm{C}_{N_c}$	1	N_c	$\frac{N_c(N_f-N_c)}{N_f}$
$\tilde{B}=\tilde{Q}^{N_c}$	1	${}_{N_f}\mathrm{C}_{N_c}$	$-N_c$	$\frac{N_c(N_f-N_c)}{N_f}$

The classical Higgs branch is discussed separately for $N_c > N_f$ and $N_f \geq N_c$. For $N_c > N_f$ the gauge group is generically broken to $SU(N_c - N_f)$ and the N_f^2 number of mass less fields are left at low-energy. For $N_f \geq N_c$, $2N_fN_c - (N_c^2-1)$ mass less fields.

When $N_c = 2$, the slight modification is needed since the $\mathbf{2}$ and $\bar{\mathbf{2}}$ are identical:

Table 2.2: The quantum numbers of the 4d $\mathcal{N}=1$ $SU(2)$ with N_f flavors

	$SU(2N_f)_L$	$U(1)_R$
Q^i	$2N_f$	$\frac{N_f-2}{N_f}$
$V^{ij}=Q^i_aQ^j_b\epsilon^{ab}$	$[\mathbf{ij}]$, ${}_{2N_f}\mathrm{C}_2$	$\frac{2(N_f-2)}{N_f}$

In this case the $SU(2)$ gauge group is higgsed at the generic point of the moduli space. The Higgs branch is $4N_f - 3$ dimensional. For $N_f = 1$, the squark is diagonalised as follows.

$$Q = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad V = \begin{pmatrix} 0 & a^2 \\ -a^2 & 0 \end{pmatrix}. \tag{2.3}$$

For $N_f \geq 2$, V^{ij} are classically constrained to be rank$(V) \leq 2$ which means $\epsilon_{i_1\cdots i_{2N_f}}V^{i_1i_2}V^{i_3i_4} = 0$. This is easily checked by recognizing that the Q is $\mathbf{2}$ representation and that the condition $\epsilon_{i_1\cdots i_{2N_f}}V^{i_1i_2}V^{i_3i_4} = 0$ is equivalent to Pf(A) $= 0$ where A is the 4×4 matrix constructed from V by omitting the $2N_f - 4$ components form V. Pf $A = 0$ means the eigenvalues of A contain 0. So this is equivalent to rank$(A) \leq 2$, which is rank$(M) \leq 2$.

The magnetic side

The magnetic side is the four-dimensional $\mathcal{N}=1$ supersymmetric $SU(N_f - N_c)$ gauge theory with N_f dual flavors $q, \tilde{q}$ and the gauge singlet chiral superfields M which are interacting via the superpotential. The superpotential in magnetic theory takes

$$W = M\tilde{q}q. \tag{2.4}$$

The quantum numbers of these fields are summarized as follows. Note that the $U(1)_A$ axial symmetry has the anomaly, which is the same as the electric side.

Table 2.3: The quantum numbers of the magnetic dual: 4d $\mathcal{N} = 1$ $SU(N_f - N_c)$ with N_f flavors

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\bar{N}_f$	1	$\frac{N_c}{N_f - N_c}$	$\frac{N_c}{N_f}$
$\tilde{q}$	1	N_f	$-\frac{N_c}{N_f - N_c}$	$\frac{N_c}{N_f}$
M	N_f	$\bar{N}_f$	0	$\frac{2(N_f - N_c)}{N_f}$
$N := \tilde{q}q$	$\bar{N}_f$	N_f	0	$\frac{2N_c}{N_f}$

Note that the dynamical scales between the electric and magnetic side should be takes as

$$\Lambda^{3N_c - N_f} \tilde{\Lambda}^{3(N_f - N_c) - N_f} = (-1)^{N_f - N_c} \mu^{N_f}. \tag{2.5}$$

In addition we need to be careful about the identification of the meson operators M_i^j between both sides.

$$M_{\mathsf{electric}} = \mu M_{\mathsf{magnetic}} \tag{2.6}$$

which is the consequence that on the electric side the meson fields are composite and they have the mass dimension 2 and that the mesons on the dual side are the elementary fields so they have the dimension 1. We also have the baryonic operators:

$$B_{\mathsf{ele}} := Q^{N_c}, \qquad \tilde{B}_{\mathsf{ele}} := \tilde{Q}^{N_c} \tag{2.7}$$

$$B_{\mathsf{mag}} := q^{N_f - N_c}, \quad \tilde{B}_{\mathsf{mag}} := \tilde{q}^{N_f - N_c} \tag{2.8}$$

which are naturally identified with each other. Please check that the quantum numbers of the baryon operator indeed match.

We can gauge the $U(1)_B$ baryon symmetry on both sides and obtain the $U(N_c)$ duality. But in 4d the $U(1)$ gauge coupling is IR-free so we do not expect the interesting dynamics for it. So the duality is essencially the same as the $SU(N_c)$ one and the $U(1)_B$ vector multiplet weakly couples to this duality. However the situation becomes completely different if we consider this theory on $\mathbb{R}^3$, where the $U(1)$ gauge coupling is super-renormalizable and IR non-free. Then the 3d (Seiberg) dualities for the $SU(N_c)$ and the $U(N_c)$ would be different.

2.2 Kutasov-Schwimmer duality

Kutasov and Schwimmer constructed the Seiberg duality including an adjoint chiral superfield [38–40]. The theory contains the superpotential for this adjoint field as $W = \mathrm{tr}\, X^{k+1}$. So the chiral ring is truncated and the moduli space which we should care is reduced and the low-energy theory becomes tractable.

Electric side

The "original theory" is the 4d $\mathcal{N} = 1$ supersymmetric $SU(N_c)$ gauge theory with N_f flavors $Q, \tilde{Q}$ and with an adjoint matter X, containing the superpotential $W = \mathrm{tr}\, X^{k+1}$. The global charges are as follows. Again the $U(1)_A$ symmetry is anomalous.

Table 2.4: The quantum numbers of the electric side

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_f	1	1	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
$\tilde{Q}$	1	$\bar{N}_f$	-1	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
X	1	1	0	$\frac{2}{k+1}$
$M_j=\tilde{Q}X^jQ$	N_f	$\bar{N}_f$	0	$2-\frac{4}{k+1}\frac{N_c}{N_f}+\frac{2j}{k+1}$

The superpotential on the electric side is

$$W = \mathrm{Tr}\, X^{k+1}. \tag{2.9}$$

We will extensively consider the weak deformation of this theory by

$$W = \sum_{l=1}^{k} g_l \mathrm{tr}\, X^{l+1} + \lambda \mathrm{tr}\, X \tag{2.10}$$

where the second term is the constraint that X is traceless and λ is a Lagrange multiplier. For the generic values of g_l, the gauge group $SU(N_c)$ breaks as

$$SU(N_c) \to SU(n_1) \times \cdots \times SU(n_k) \times U(1)^{k-1}, \quad \sum_{i=1}^{k} n_i = N_c. \tag{2.11}$$

Each $SU(n_i)$ is the 4d $\mathcal{N}=1$ $SU(n_i)$ SQCD with N_f flavors and with no adjoint field.

Magnetic side

The dual theory is 4d $\mathcal{N}=1$ supersymmetric $SU(kN_f-N_c)$ supersymmetric gauge theory with N_f dual flavors $q, \tilde{q}$, an adjoint matter X and the gauge singlets $M_j,\ j=0,\cdots,k-1$. The superpotential on the magnetic side is

$$W = \mathrm{tr}\, Y^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q. \tag{2.12}$$

The quantum numbers of the field contents on the dual side are summarized as follows. The $U(1)_A$ axial symmetry is again anomalous.

Table 2.5: The quantum numbers of the magnetic side

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\bar{N}_f$	1	$\frac{N_c}{kN_f-N_c}$	$1-\frac{2}{k+1}\frac{kN_f-N_c}{N_f}$
$\tilde{q}$	1	N_f	$-\frac{N_c}{kN_f-N_c}$	$1-\frac{2}{k+1}\frac{kN_f-N_c}{N_f}$
Y	1	1	0	$\frac{2}{k+1}$
M_j	N_f	$\bar{N}_f$	0	$2-\frac{4}{k+1}\frac{N_c}{N_f}+\frac{2j}{k+1}$
$N_j := \tilde{q}Y^jq$	$\bar{N}_f$	N_f	0	$2-\frac{4}{k+1}\frac{kN_f-N_c}{N_f}+\frac{2j}{k+1}$

where N_j's are the dual mesons composed of the dual quarks.

Operator Matching

Mesonic operator

In the original theory side, we define the meson composites which are gauge invariant.

$$M_j := \tilde{Q} X^j Q, \quad j = 0, \cdots, k-1 \tag{2.13}$$

which are naturally identified with the meson chiral superfields in the dual side. On the dual side, these meson fields are the elementary fields which are directly appearing in the UV Lagrangian.

Baryonic operator

We first define the so-called "dressed quarks":

$$Q_{(l)} = X^l Q, \quad l = 0, \cdots, k-1 \tag{2.14}$$

By using these quarks, we can define the baryonic operators:

$$B_{\text{ele}}^{(n_0, \cdots, n_{k-1})} := \left(Q_{(0)}\right)^{n_0} \cdots \left(Q_{(k-1)}\right)^{n_{k-1}}, \quad \sum_{l=0}^{k-1} n_l = N_c \tag{2.15}$$

Similarly we can define the magnetic baryon operators:

$$B_{\text{ele}}^{(m_0, \cdots, m_{k-1})} := \left(q_{(0)}\right)^{m_0} \cdots \left(q_{(k-1)}\right)^{m_{k-1}}, \quad \sum_{l=0}^{k-1} m_l = kN_f - N_c \tag{2.16}$$

The mating of the baryonic operators are following.

$$B_{\text{ele}}^{(n_0, \cdots, n_{k-1})} \sim B_{\text{ele}}^{(m_0, \cdots, m_{k-1})} \tag{2.17}$$

where $m_l = N_f - n_{k-l}, \ l = 0, \cdots, k-1$.

2.3 Duality with an anti-symmetric matter

Next, we consider the theory with tensor matters. In this section we discuss the duality with an anti-symmetric matter. In the next section we will study the duality with a symmetric matter.

Electric side

The original theory consists of the N_f fundamental matters $Q, \tilde{Q}$ and the anti-symmetric matter $X, \tilde{X}$ with the superpotential [41]:

$$W = \mathrm{tr}(X\tilde{X})^{k+1}. \tag{2.18}$$

Table 2.6: The quantum numbers of the electric side

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_X$	$U(1)_B$	$U(1)_R$
Q	N_c	$\square$	1	0	$\frac{1}{N_c}$	$1 - \frac{N_c+2k}{(k+1)N_f}$
$\tilde{Q}$	$\overline{\square}$	1	$\overline{\square}$	0	$\frac{1}{N_c}$	$1 - \frac{N_c+2k}{(k+1)N_f}$
X	anti-symm.	1	1	1	$\frac{2}{N_c}$	$\frac{1}{k+1}$
$\tilde{X}$	$\overline{\text{anti-symm.}}$	1	1	-1	$-\frac{2}{N_c}$	$\frac{1}{k+1}$

The chiral ring elements have following composite operators.

$$M_j := Q(\tilde{X}X)^j \tilde{Q} \quad j = 0, \cdots, k \tag{2.19}$$

$$P_a := Q(\tilde{X}X)^a \tilde{X} Q \quad a = 0, \cdots, k-1 \tag{2.20}$$

$$\tilde{P}_a := \tilde{Q} X (\tilde{X}X)^a \tilde{Q} \tag{2.21}$$

Magnetic side

The dual theory is the $SU(\tilde{N}_c)$ gauge theory with the fundamental dual flavors, anti-symmetric matters $Y, \tilde{Y}$ and singlets $M_j, P_a, \tilde{P}_a$, where $\tilde{N}_c = (2k+1)N_f - 4k - N_c$. The dual superpotential is

$$W_{\mathsf{mag}} = \mathrm{tr}(Y\tilde{Y})^{k+1} + \sum_{j=0}^{k} M_{k-j} q(\tilde{Y}Y)^j \tilde{q} + \sum_{r=0}^{k-1} \left[P_{k-r-1} q(\tilde{Y}Y)^r \tilde{Y} q + \tilde{P}_{k-r-1} \tilde{q} Y (\tilde{Y}Y)^r \tilde{q} \right] \quad (2.22)$$

Table 2.7: The quantum numbers of the magnetic side

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_X$	$U(1)_B$	$U(1)_R$
q	$\tilde{N}_c$	$\bar{N}_f$	1	$\frac{k(N_f-2)}{\tilde{N}_c}$	$\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c+2k}{(k+1)N_f}$
$\tilde{q}$	$\bar{\tilde{N}}_c$	1	$\bar{N}_f$	$-\frac{k(N_f-2)}{\tilde{N}_c}$	$-\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c+2k}{(k+1)N_f}$
Y	anti-symm.	1	1	$\frac{N_c-N_f}{\tilde{N}_c}$	$\frac{2}{\tilde{N}_c}$	$\frac{1}{k+1}$
$\tilde{Y}$	$\overline{\text{anti-symm.}}$	1	1	$-\frac{N_c-N_f}{\tilde{N}_c}$	$-\frac{2}{\tilde{N}_c}$	$\frac{1}{k+1}$
M_j	1	N_f	N_f	0	0	$\frac{\tilde{N}_c-N_c+(2j+1)N_f}{N_f(k+1)}$
P_r	1	anti-symm.	1	-1	0	$\frac{\tilde{N}_c-N_c+2(r+1)N_f}{N_f(k+1)}$
$\tilde{P}_r$	1	1	anti-symm.	1	0	$\frac{\tilde{N}_c-N_c+2(r+1)N_f}{N_f(k+1)}$

2.4 Duality with a symmetric matter

Here we include the symmetric matter into the 4d $\mathcal{N}=1$ supersymmetric $SU(N_c)$ gauge theory with N_f flavors.

Electric side

The theory consists of the N_f fundamental matters $Q, \tilde{Q}$ and the symmetric matters $X, \tilde{X}$ with the superpotential [41]:

$$W = \mathrm{tr}(X\tilde{X})^{k+1}. \quad (2.23)$$

Table 2.8: The quantum numbers of the electric side

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_X$	$U(1)_B$	$U(1)_R$
Q	N_c	N_f	1	0	$\frac{1}{N_c}$	$1 - \frac{N_c-2k}{(k+1)N_f}$
$\tilde{Q}$	$\bar{N}_c$	1	N_f	0	$\frac{1}{N_c}$	$1 - \frac{N_c-2k}{(k+1)N_f}$
X	symm.	1	1	1	$\frac{2}{N_c}$	$\frac{1}{k+1}$
$\tilde{X}$	$\overline{\text{symm.}}$	1	1	-1	$-\frac{2}{N_c}$	$\frac{1}{k+1}$

The composite chiral operators are defined as follows.

$$M_j := Q(\tilde{X}X)^j \tilde{Q}, \quad j = 0, \cdots, k \quad (2.24)$$

$$P_r := Q(\tilde{X}X)^r \tilde{X} Q, \quad r = 0, \cdots, k-1 \quad (2.25)$$

$$\tilde{P}_r := \tilde{Q} X (\tilde{X}X)^r \tilde{Q}, \quad r = 0, \cdots, k-1 \quad (2.26)$$

Magnetic side

The dual theory is the $SU(\tilde{N}_c)$ gauge theory with the fundamental dual flavors, ant-symmetric matters $Y, \tilde{Y}$ and singlets $M_j, P_a, \tilde{P}_a$, where $\tilde{N}_c = (2k+1)N_f + 4k - N_c$. The dual superpotential is

$$W_{\mathsf{mag}} = \mathrm{tr}(Y\tilde{Y})^{k+1} + \sum_{j=0}^{k} M_{k-j} q(\tilde{Y}Y)^j \tilde{q} + \sum_{r=0}^{k-1} \left[P_{k-r-1} q(\tilde{Y}Y)^r \tilde{Y} q + \tilde{P}_{k-r-1} \tilde{q} Y (\tilde{Y}Y)^r \tilde{q} \right] \quad (2.27)$$

Table 2.9: The quantum numbers of the magnetic side

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_X$	$U(1)_B$	$U(1)_R$
q	$\tilde{N}_c$	$\bar{N}_f$	1	$\frac{k(N_f+2)}{\tilde{N}_c}$	$\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c-2k}{(k+1)N_f}$
$\tilde{q}$	$\bar{\tilde{N}}_c$	1	$\bar{N}_f$	$-\frac{k(N_f+2)}{\tilde{N}_c}$	$-\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c-2k}{(k+1)N_f}$
Y	symm.	1	1	$\frac{N_c-\tilde{N}_f}{\tilde{N}_c}$	$\frac{2}{\tilde{N}_c}$	$\frac{1}{k+1}$
$\tilde{Y}$	$\overline{\text{symm.}}$	1	1	$-\frac{N_c-N_f}{\tilde{N}_c}$	$-\frac{2}{\tilde{N}_c}$	$\frac{1}{k+1}$
M_j	1	N_f	N_f	0	0	$\frac{\tilde{N}_c-N_c+(2j+1)N_f}{N_f(k+1)}$
P_r	1	symm.	1	-1	0	$\frac{\tilde{N}_c-N_c+2(r+1)N_f}{N_f(k+1)}$
$\tilde{P}_r$	1	1	symm.	1	0	$\frac{\tilde{N}_c-N_c+2(r+1)N_f}{N_f(k+1)}$

2.5 Duality with anti-symmetric and symmetric matters

Let us consider the duality with an anti-symmetric matter and a symmetric matter [41]. The anomaly cancellation requires that the theory is chiral.

Electric side

The theory consists of the m_f fundamental matters Q, the $\tilde{m}_f$ anti-fundamental matters $\tilde{Q}$, an anti-symmetric matter X, a symmetric matter $\tilde{X}$ with the superpotential:

$$W = \mathrm{tr}(X\tilde{X})^{2(k+1)}. \quad (2.28)$$

The anomaly cancellation is satisfied for $m_f = \tilde{m}_f + 8$.

Table 2.10: The quantum numbers of the electric side

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_X$	$U(1)_B$	$U(1)_R$
Q	N_c	m_f	1	$-(2k+1) + \frac{2(4k+3)}{m_f}$	$\frac{1}{N_c}$	$1 - \frac{N_c+2(4k+3)}{2(k+1)m_f}$
$\tilde{Q}$	$\bar{N}_c$	1	$\tilde{m}_f$	$2k+1+\frac{2(4k+3)}{\tilde{m}_f}$	$\frac{1}{N_c}$	$1 - \frac{N_c-2(4k+3)}{2(k+1)\tilde{m}_f}$
X	anti-symm.	1	1	1	$\frac{2}{N_c}$	$\frac{1}{2(k+1)}$
$\tilde{X}$	$\overline{\text{symm.}}$	1	1	-1	$-\frac{2}{N_c}$	$\frac{1}{2(k+1)}$

The chiral composite operators describing the moduli space are defined as follows.

$$M_j := Q(\tilde{X}X)^j \tilde{Q}, \quad j = 0, \cdots, 2k+1 \quad (2.29)$$

$$P_r := Q(\tilde{X}X)^r \tilde{X} Q, \quad r = 0, \cdots, 2k \quad (2.30)$$

$$\tilde{P}_r := \tilde{Q} X (\tilde{X}X)^r \tilde{Q}, \quad r = 0, \cdots, 2k \quad (2.31)$$

Magnetic side

The dual theory is the $SU(\tilde{N}_c)$ gauge theory with the m_f fundamental dual matters q, the $\tilde{m}_f$ anti-fundamental matters $\tilde{q}$, an anti-symmetric matters Y, asymmetric matter $\tilde{Y}$, and singlets $M_j, P_r, \tilde{P}_r$, where $\tilde{N}_c = \frac{1}{2}(4k+3)(m_f + \tilde{m}_f) - N_c$. The dual superpotential is

$$W_{\mathsf{mag}} = \mathrm{tr}(Y\tilde{Y})^{2(k+1)} + \sum_{j=0}^{k} M_{2k-j+1} q(\tilde{Y}Y)^j \tilde{q} + \sum_{r=0}^{2k} \left[P_{2k-r} q(\tilde{Y}Y)^r \tilde{Y} q + \tilde{P}_{2k-r} \tilde{q} Y (\tilde{Y}Y)^r \tilde{q} \right] \quad (2.32)$$

The quantum numbers of the field contents are below.

Table 2.11: The quantum numbers of the magnetic side

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_X$	$U(1)_B$	$U(1)_R$
q	$\tilde{N}_c$	$\bar{m}_f$	1	$2k+1-\frac{2(4k+3)}{m_f}$	$\frac{1}{\tilde{N}_c}$	$1-\frac{\tilde{N}_c+2(4k+3)}{2(k+1)m_f}$
$\tilde{q}$	$\bar{\tilde{N}}_c$	1	$\bar{\tilde{m}}_f$	$-(2k+1)-\frac{2(4k+3)}{\tilde{m}_f}$	$-\frac{1}{\tilde{N}_c}$	$1-\frac{\tilde{N}_c-2(4k+3)}{2(k+1)\tilde{m}_f}$
Y	anti-symm.	1	1	-1	$\frac{2}{\tilde{N}_c}$	$\frac{1}{2(k+1)}$
$\tilde{Y}$	$\overline{\text{symm.}}$	1	1	1	$-\frac{2}{\tilde{N}_c}$	$\frac{1}{2(k+1)}$

Chapter 3

3d $\mathcal{N}=2$ supersymmetric gauge theories

In this chapter we will extensively review the kinematics and the dynamics of the 3d $\mathcal{N}=2$ supersymmetric gauge theories based on [42,43]. The new feature of the 3d gauge theories, which is absent in the previous 4d theories, is the appearance of the Coulomb branch of the moduli space of vacua. They are parametrized by the so-called "monopole operators" which are the high energy operators and are independent of which branch we are considering. The monopole operator is non-trivially charged under the global symmetries of the theory. So, at the Coulomb phase, some of the global symmetries are broken. We will show the calculation of these charges in many ways, which is worth mentioning here. Please see the appendix for the details of the zero-mode calculus around the monopole background.

The physics on the Coulomb branch is very difficult due to the several reasons. First one is related to the monopole operators. We don't know the explicit form of the monopole operator by the fields of the UV Lagrangian since the monopole operator is one of the disorder operators. The second reason is because the Coulomb moduli generically gets the quantum correction beyond the one-loop perturbation. Then the structure of the Coulomb branch is highly non-trivial. Then, we here show the dynamics of the Coulomb branch so far as known.

One of the most intriguing features is that the Coulomb branch which is generated by the $U(1)$ theory is one-loop exact (+ non-perturbative corrections), which we will show later. On the other hand, the Coulomb branch from the non-abelian gauge sector contains the higher order corrections. However, at the generic point of the Coulomb branch, where the gauge group is broken to $U(1)^{\mathrm{rank}\, G_{\mathrm{gauge}}}$, we have the multiple $U(1)$ dynamics and these sectors are sometimes analyzed only by the one-loop level (+ non-perturbative corrections).

3.1 3d $\mathcal{N}=2$ supersymmetry

3.1.1 SUSY Notation

First we summarize the SUSY notations [44–46]. The 3d $\mathcal{N}=2$ supersymmetry is the dimensional reduction of the 4d $\mathcal{N}=1$ supersymmetry and has the four supercharges:

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \tag{3.1}$$

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \tag{3.2}$$

where Z is a real central charges coming from the P_3 direction. In three space-time dimensions, the central charges are regarded as the background fields of the global conserved currents, which we will discuss later. Note that we have no dotted index unlike the 4d case. The gamma matrix is defined as

$$\sigma^\mu{}_{\alpha\beta} = -1, \sigma^1, \sigma^3. \tag{3.3}$$

Notice that these gamma matrices are symmetric, traceless and real. Then the Lorentz generators are pure imaginary. Thus the corresponding Lie group (and it's representation) is also real. Sometimes another choice is preferable[1]:

$$(\sigma^\mu)_\alpha^{\ \beta} = i\sigma^3, \sigma^2, \sigma^1. \tag{3.4}$$

In this basis the angular momentum $\frac{1}{2}[\gamma_1, \gamma_2]$ is diagonalised.

The conventions for the spinor indices are as follows.

$$(\psi\chi) = \epsilon^{\alpha\beta}\psi_\alpha\chi_\beta \tag{3.5}$$

$$(\psi\chi)^\dagger = -\bar{\chi}\bar{\psi} = -\epsilon^{\alpha\beta}\bar{\chi}_\alpha\bar{\psi}_\beta \tag{3.6}$$

The supercharges, the super-derivatives, their commutation relation and the superspace integrals are as follows:

$$\delta_\epsilon A = i[\epsilon Q - \bar{\epsilon}\bar{Q}, A] = (\epsilon\mathcal{Q} - \bar{\epsilon}\bar{\mathcal{Q}})A \tag{3.7}$$

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\gamma^\mu{}_{\alpha\beta}\bar{\theta}^\beta\partial_\mu \tag{3.8}$$

$$\bar{\mathcal{Q}}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} + i\gamma^\mu{}_{\alpha\beta}\theta^\beta\partial_\mu \tag{3.9}$$

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\gamma^\mu{}_{\alpha\beta}\bar{\theta}^\beta\partial_\mu \tag{3.10}$$

$$\bar{\mathcal{D}}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} - i\gamma^\mu{}_{\alpha\beta}\theta^\beta\partial_\mu \tag{3.11}$$

$$\left\{D_\alpha, \bar{D}_\beta\right\} = i\partial_{\alpha\beta} = -2i\gamma^\mu_{\alpha\beta}\partial_\mu \tag{3.12}$$

$$D\bar{D} = \bar{D}D \tag{3.13}$$

$$\{D_\alpha, D_\alpha\} = 0 \tag{3.14}$$

$$\bar{D}_\alpha(\bar{D}D) = -\frac{1}{2}(\bar{D}\bar{D})D_\alpha \tag{3.15}$$

$$\int d^2\theta\theta^2 = 1, \quad \int d^2\bar{\theta}\bar{\theta}^2 = -1, \quad \int d^4\theta\theta^2\bar{\theta}^2 = -1 \tag{3.16}$$

3.1.2 Gauge multiplet

The massless vector multiplet consists of the gauge field, a real adjoint scalar, two real gauginos. The bosonic d.o.f is 2 on-mass shell. The gauge field strength is defined as

$$W_\alpha = -\frac{1}{4}\bar{D}^2\mathrm{e}^{-V}D_\alpha\mathrm{e}^V, \quad \bar{W}_\alpha = -\frac{1}{4}D^2\mathrm{e}^{-V}\bar{D}_\alpha\mathrm{e}^V. \tag{3.17}$$

$$\mathcal{L} = \frac{1}{g^2}\int d^2\theta\,\mathrm{Tr}\,W_\alpha^2 + \mathsf{h.c.} \tag{3.18}$$

Table 3.1: The R-charges of the vector, real scalar and gaugino fields

	$U(1)_R$
A_μ, σ	0
λ_α	+1
W_α	+1

[1]Other choices are also used: $\gamma^{0,1,2}_{\alpha\beta} = (i\sigma_2, \sigma_3, \sigma_1)$, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+)$

The vector multiplet (or we will call it vector superfield) is expanded in Wess-Zumino gauge as

$$V = -i\theta\bar{\theta}\sigma - \theta\gamma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{3.19}$$

We can alternatively use the linear superfield to express the vector multiplet. The real linear multiplet is defined as follows.

$$D^2\Sigma = \bar{D}^2\Sigma = 0 \tag{3.20}$$

$$\Sigma = 2\pi\mathcal{J}_J = -\frac{i}{2}\bar{D}DV \tag{3.21}$$

$$= \sigma + \theta\bar{\lambda} + \bar{\theta}\lambda + \frac{1}{2}\theta\gamma^\mu\bar{\theta}F^{\nu\rho}\epsilon_{\mu\nu\rho} + i\theta\bar{\theta}D + \frac{i}{2}\bar{\theta}^2\theta\gamma^\mu\partial_\mu\bar{\lambda} + \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\sigma \tag{3.22}$$

Note that the conserved current $J^\mu := \epsilon^{\mu\nu\rho}F_{\mu\rho}$ is contained in the $\theta\gamma^\mu\bar{\theta}$ component. This is generalized (or reversed): If we have any conserved currents, these currents can be regarded as the components of certain linear superfields. The central charge which appears in the SUSY algebra is identified with the background field of this linear superfield. For example,

$$\int d^4\theta\Phi^\dagger e^{m_{\mathsf{real}}\theta\bar{\theta}}\Phi \ni m_{\mathsf{real}}^2|\varphi|^2 + im_{\mathsf{real}}\bar{\psi}\psi \tag{3.23}$$

gives $Z = m_{\mathsf{real}}$ and this chiral superfield Φ saturates the BPS bound.

The kinetic term of the linear superfield is written as

$$\mathcal{L} = \int d^4\theta\left(-\frac{1}{e^2}\Sigma^2 - \frac{k}{4\pi}\Sigma V - \frac{\zeta}{2\pi}V\right), \tag{3.24}$$

where k is the Chern-Simons level and ζ is the Fayet-Iliopoulos term.

3.1.3 Matter multiplet

The chiral superfield is defined just like the 4d SUSY.

$$\Phi(y,\theta) = Q(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{3.25}$$

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \tag{3.26}$$

whose field contents are a complex scalar, two (real) Majorana fermions, and one complex auxiliary field. The matter multiplet has the following kinetic term

$$\mathcal{L}_{\mathsf{matter}} = \int d^4\theta\, Q^\dagger \mathrm{e}^{nV + im\theta\bar{\theta}}Q \tag{3.27}$$

where we consider the $U(1)$ gauge theory. The scalar potential becomes

$$V_{\mathsf{classical}} = \frac{e^2}{32\pi^2}\left(2\pi n|Q|^2 - \zeta - k\sigma\right)^2 + (m + n\sigma)^2|Q|^2 \tag{3.28}$$

where k is the Chern-Simons level and ζ is the Fayet-Iliopoulos term. It is obvious that the Fayet-Iliopoulos term also affect the central charge.

The quantum corrections for the matter and linear multiplets are described as follows.

$$\mathcal{L}_{\mathsf{matter,eff}} = \int d^4\theta K(\Phi,\bar{\Phi}) + \left(\int d^2\theta W(\Phi) + \mathsf{h.c.}\right) \tag{3.29}$$

$$\mathcal{L}_{\mathsf{gauge,eff}} = -\int d^4\theta f(\Sigma) + \frac{1}{4\pi}(k_{eff}\Sigma V + 2\zeta_{eff}V). \tag{3.30}$$

The Wess-Zumino model

Let us consider the Wess-Zumino model with the superpotential $W = \Phi^3$. Since the R-charges of the superpotential should be 2, the $U(1)_R$ charge of the chiral superfield Φ is

$$R[\Phi] = \frac{2}{3}. \tag{3.31}$$

Since all the operators should satisfy the inequality:

$$D \geq |R|, \tag{3.32}$$

the quantum dimension of Φ is

$$D[\Phi] = \frac{2}{3}. \tag{3.33}$$

Since the canonical dimension of ϕ is 1/2, the WZ model with the superpotential $W = \Phi^3$ is a interacting superconformal fields theory in the IR. Note that in 4d we lead to the Gaussian fixed point in the IR.

The Seiberg dual magnetic theory

In this thesis we discuss the Seiberg duality in 3d, so it would be helpful to mention the component action of the magnetic theory. The theory always contains the gauge singlet chiral superfield M with the superpotential:

$$W = M\tilde{q}q \tag{3.34}$$

where q and $\tilde{q}$ are the (dual) quark chiral superfield which are not gauge singlets. We omitted the all the indices, the gauge indices and flavor indices.

The component form of this matter fields are as follows.

$$\mathcal{L}_M = \int d^4\theta\, M^\dagger M + \left(\int d^2\theta M\tilde{q}q + \text{c.c.}\right) \tag{3.35}$$

$$= -\partial_\mu M \partial^\mu M + i\psi_M \partial\!\!\!/ \bar{\psi}_M + F_M^\dagger F_M + \Big(F_M q\tilde{q} + F_q M\tilde{q} + F_{\tilde{q}} M q - \frac{1}{2}\psi_q\psi_M\tilde{q} - \frac{1}{2}q\psi_M\psi_{\tilde{q}} - \frac{1}{2}\psi_q M\psi_{\tilde{q}} + \text{c.c.}\Big) \tag{3.36}$$

The corresponding scalar potential with the D-terms for the dual quarks becomes

$$V = g^2\left(q^\dagger T^a q - \tilde{q}^\dagger T^a \tilde{q}\right)^2 + |\phi q|^2 + |\phi\tilde{q}|^2 + |M\tilde{q}|^2 + |qM|^2 + |q\tilde{q}|^2. \tag{3.37}$$

If we include the Chern-Simons term, this is modified into

$$V = g^2\left(-\frac{k}{4\pi}\phi^a + q^\dagger T^a q - \tilde{q}^\dagger T^a \tilde{q}\right)^2 + |\phi q|^2 + |\phi\tilde{q}|^2 + |M\tilde{q}|^2 + |qM|^2 + |q\tilde{q}|^2. \tag{3.38}$$

3.1.4 The non-abelian extension

We only show the component action for the non-abelian SUSY gauge theory.

$$\begin{aligned}
\mathcal{L}_{\mathsf{matter}} &= \int d^4\theta\, \Phi^\dagger e^{2V-2m_r}\Phi && (3.39)\\
&= -\mathcal{D}_\mu Q^\dagger \mathcal{D}_\mu Q - |(\phi+m_r)Q|^2 - \frac{i}{2}\psi\sigma^\mu\mathcal{D}_\mu\bar{\psi} + \frac{i}{2}\mathcal{D}_\mu\psi\sigma^\mu\bar{\psi} - \psi(\phi+m_r)\bar{\psi} \\
&\quad + F^\dagger F + i\sqrt{2}(Q^\dagger\lambda\psi - Q\bar{\lambda}\bar{\psi}) + Q^\dagger DQ && (3.40)\\
\mathcal{L}_{\mathsf{gauge}} &= \frac{1}{32\pi}\mathrm{Im}\int d^2\theta\, \mathrm{Tr}\, W_\alpha W^\alpha && (3.41)\\
&= \mathrm{Tr}\left[-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}\mathcal{D}^\mu\phi\mathcal{D}_\mu\phi - \frac{1}{g^2}i\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda} + \frac{1}{2g^2}DD\right] && (3.42)\\
\mathcal{L}_{\mathsf{CS}} &= \frac{k}{4\pi}\mathrm{Tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A - \lambda\bar{\lambda} + 2\phi D\right] && (3.43)\\
\mathcal{L}_{\mathsf{FI}} &= \xi\int d^4\theta\, \frac{2}{\sqrt{2N}}\mathrm{Tr}V = \frac{1}{2}\xi D_{\mathsf{abelian}} && (3.44)
\end{aligned}$$

where the FI term can be introduced only for the $U(N_c)$ gauge group.

3.2 Coulomb branch

Here we mention what we know about the Coulomb branch [46]. The most important fact is that the $U(1)$ Coulomb branch is one-loop (semi-classical) exact. So the Coulomb branch only have the non-perturbative corrections [47].

Effectiely the Coulomb branch of the $U(1)^r$ is described by

$$\mathcal{L}_{eff} = -\int d^4\theta f(\Sigma_i) \tag{3.45}$$

where $\Sigma = -\frac{i}{2}\bar{D}DV$ is a real linear superfield. The kinetic terms are

$$ds^2 = f^{rs}d\sigma_r d\sigma_s \quad f^{rs} = \partial^r\partial^s f(\sigma) \tag{3.46}$$

where $\sigma_r = \Sigma_r|_\theta = \bar{\theta} = 0$.

We can dualize the linear multiplets in the following way [43,46,48].

$$\mathcal{L} = \int d^4\theta\left(-f(\Sigma) + (\Phi^r + \bar{\Phi}^r)\frac{\Sigma_r}{2\pi}\right) \tag{3.47}$$

where Φ_r are the chiral superfields and Σ_r are the "un-constrained" superfields. First we consider the equation of motions for Φ_r. The Lagrangian can be rewritten up to total derivatives as

$$\mathcal{L} \sim \int d^2\bar{\theta}\left(D^2(\Phi^r\frac{\Sigma_r}{2\pi}) + \bar{\Phi}^r\frac{D^2\Sigma_r}{2\pi}\right) \tag{3.48}$$

which implies

$$D^2\Sigma_r = 0. \tag{3.49}$$

In a similar way, we obtain $\bar{D}^2\Sigma_r = 0$. These equations of motion says that the Σ_r is the real linear superfield.

Next, let us consider the equations of motion for Σ_r. This gives

$$\frac{1}{2\pi}\left(\Phi^r + \bar{\Phi}^r\right) = \frac{\partial f(\Sigma)}{\partial\Sigma_r}. \tag{3.50}$$

Inverting this relation and solve for Σ_r, we obtain the following dual action:

$$S_{eff} = \int d^3x d^4\theta K(\Phi^r + \bar{\Phi}^r) \tag{3.51}$$

where $K(\Phi_r + \bar{\Phi}) = -f(\Sigma) + (\Phi^r + \bar{\Phi}^r)\frac{\Sigma_r}{2\pi}$.

The inverse Legendre transform is

$$\partial_{\Phi^r} K = \partial_{\bar{\Phi}^r} K = \frac{1}{2\pi}\Sigma_r. \tag{3.52}$$

Thus the Kahler metric becomes

$$ds^2 = 2K_{rs} d\Phi^r d\bar{\Phi}^s. \tag{3.53}$$

The corresponding Kähler form is

$$\Omega = \frac{i}{2} K_{r\bar{r}} d\Phi^r \wedge d\bar{\Phi}^{\bar{r}} = K_{rs} d\phi^r \wedge da^s = \frac{1}{4\pi} d\sigma_r \wedge da^r \tag{3.54}$$

which is exact and there is no corrections.

Note that in the component expression, we have

$$\Sigma_r|_{\theta=\bar{\theta}=0} = \sigma_r \tag{3.55}$$

$$\left(\Phi^r + \bar{\Phi}^r\right)\big|_{\theta=\bar{\theta}=0} = 2\phi^r \tag{3.56}$$

$$\left(\Phi^r - \bar{\Phi}^r\right)\big|_{\theta=\bar{\theta}=0} = 2ia^r. \tag{3.57}$$

3.3 Real masses

We consider the 3d theories which arises from the dimensional reduction of the 4d $\mathcal{N} = 1$ SUSY gauge theories. Thus we can formulate the 3d theories with the superfields. In 3d, the vector multiplet contains the real scalar fields coming from the A_3 direction of the 4d gauge fields. In addition we can gauge the global symmetries and turn on the real masses by intorducing the vavs for the background fields $A_3^{\mathsf{global}} = \phi^{\mathsf{global}}$. So the generic kinetic term contains

$$\int d^4\theta \Phi^\dagger e^{2V - 2m\bar{\theta}\theta} \Phi = \cdots - \bar{\psi}_a^i (\phi_b^a \delta_i^j + m_i^j \delta_b^a) \psi_j^b + \cdots . \tag{3.58}$$

where we only showed the fermion mass terms.

We first note that the masses introduced here are real because these masses are introduced through the Kähler and they are the $\theta\bar{\theta} = (\theta\bar{\theta})^\dagger$ components. Furthermore it is not possible to change the phase of the real masses by changing the pass of the fermion fields.

Second we should notice that the real masses can be introduce for the single chiral superfields. Generically we need two chiral superfields in order to turn on the complex masses. For example, recall the SQCD case:

$$W_{\mathsf{complex\,mass}} = m_{\mathsf{complex}} \tilde{Q} Q. \tag{3.59}$$

However the real masses require the single chiral superfield. Then the massive multiplets are generically able to be in the short (BPS) multiplets.

We note that the sign of the real masses are flipped under the parity transformation. Then the operators which break the parity should appears in the effective action when we introduce the real masses.

3.4 Parity Anomalies

In three space-time dimensions, the chirality is absent and thus there is no chiral anomaly. However there should exist the parity anomaly [49, 50] [51]. The anomaly is always related to the existence of the gauge fields which couple to the some currents. So if we weakly gauge the global symmetry and turn on the corresponding gauge field, we have the effective Chern-Simons terms between the gauge fields which are originally present in the theory or are the newcomers by gauging the global symmetry. Their couplings break the parity symmetry, thus this is called a "parity anomaly".

Their effective couplings can be written as

$$(k_{ij})_{eff} = k_{ij} + \sum_f \frac{1}{2}(q_f)_i (q_f)_j \mathrm{sign} M_f \tag{3.60}$$

where the summation are done for all the fermions which can couple to the currents and M_f is the effective masses for the fermions. The effective mass M_f schematically takes the following form.

$$\begin{aligned} M_f &= \text{(real and complex masses)} + \text{(masses from the Yukawa terms)} &(3.61)\\ &= m_f + \sum_i (q_f)_i \phi_i &(3.62) \end{aligned}$$

The parity anomaly can make the on-point function of the current non-zero:

$$\langle J^\mu \rangle_A = \mathrm{sign}(m) \frac{e^2}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho}. \tag{3.63}$$

Please see the appendix for the derivation.

3.5 Effective Chern-Simons level and current mixiing

In this thesis we don't consider the Chern-Simons term at all. However the interaction between the fermions and scalars induces the CS level shift because the Yukawa term act as the effective mass terms. The effective Lagrangian should take the following form [46].

$$\begin{aligned} \mathcal{L}_{\mathsf{gauge}} = &-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4e'^2} F^{\mathsf{global}}_{\mu\nu} F^{\mu\nu}_{\mathsf{global}} + \frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu_{\mathsf{matter}} \\ &+ A^{\mathsf{global}}_\mu J^\mu_{\mathsf{global}} + \frac{k_{\mathsf{global,gauge}}}{4\pi} \epsilon^{\mu\nu\rho} A^{\mathsf{global}}_\mu \partial_\nu A_\rho \end{aligned} \tag{3.64}$$

where we consider the $U(1)$ gauge theory with CS level k and some global symmetries. We here weakly gauging the global symmetry. $k_{\mathsf{global,gauge}}$ is generated effectively due to the loops of the massive fermions.

$$k_{\mathsf{global,gauge}} = \frac{1}{2} \sum_{f \in \text{fermions}} q_{U(1)_{\mathsf{gauge}}, f} \, q_{U(1)_{\mathsf{global}}, f} \, \mathrm{sign} M_f \tag{3.65}$$

where M_f is the effective mass terms for the fermions.

Then the equations of motion of A^{global}_μ gives

$$-\frac{1}{e'^2} \partial_\nu F^{\nu\mu}_{\mathsf{global}} = J^\mu_{\mathsf{global}} + \frac{k_{\mathsf{global,gauge}}}{4\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho = J^\mu_{\mathsf{global}} + k_{\mathsf{global,gauge}} J^\mu_J$$

where $J^\mu_J = \frac{1}{4pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$ is the topological current Hodge-dual to the $U(1)$ gauge group. Let us consider the Gauss law constraint for E^{global}_i:

$$\begin{aligned} -\int d^2x \frac{1}{e'^2} \partial_\nu F^{\nu\mu}_{\mathsf{global}} = \frac{1}{e'^2} \oint_{\mathbf{S}^1_\infty} E_i dx^i \equiv Q_{\text{Gauss}} &= J^0_{\mathsf{global}} + k_{\mathsf{global,gauge}} J^0_J \\ &= Q_{\mathsf{global}} + k_{\mathsf{global,gauge}} Q_J. \end{aligned} \tag{3.66}$$

Then for the states with $Q_{\text{Gauss}} = 0$, we have the mixing between the global and topological current. In this way, the topologically charged objects are also charged under the global $U(1)$ symmetry.

In any cases which contain the non-abelian gauge symmetry, we can apply the above technique to compute the mixing of the currents. But in practice it is easy to use the fermion zero-mode counting and the index theorems around the topological solitons, which is discussed in the Appendix.

The Coulomb branch is lifted if the k_{eff} is non-zero.

$$k_{\mathsf{eff}} = k + \frac{1}{2}\sum_f T_2(r_f)\mathrm{sign}(M_f(\phi)) \tag{3.67}$$

where M_f is the effective real mass which consists of

$$M_f(\phi) = m_f^{\mathsf{real}} + n_f\phi, \tag{3.68}$$

where n_f is the charge of the matter fields coupling to the gauge field.

3.6 Coulomb branch coordinates and Monopole operators

In 3d $\mathcal{N} = 2$ supersymmetric gauge theories, the vector superfield contains the adjoint scalar field coming from the A_3 of the 4d gauge field and the dual photon. So we have the Coulomb branch in which the gauge group is generically broken to their maximum torus. We need to introduce the notions of Coulomb branch coordinates and monopole operators [52, 53] [46].

3.6.1 The Coulomb branch coordinates

Let us first consider the abelian gauge theory (3d QED). The photon can be dualized into the compact real scalar field.

$$\partial_\mu a = \frac{\pi}{g_3^2}\epsilon_{\mu\nu\rho}F^{\nu\rho}. \tag{3.69}$$

In this definition, the dual photon a is dimensionless and the quantization of the magnetic field makes the dual photon compact $a \sim a + 2\pi$. Thus the natural coordinate of the dual photon is like $\exp(ia)$. If we consider the 3d $\mathcal{N} = 2$ SUSY which is from the dimensional reduction of the 4d $\mathcal{N} = 1$ SUSY, we have the additional adjoint scalars coming from the $A_3 = \phi$ direction. Then the Coulomb branch in $\mathcal{N} = 2$ is naturally described by

$$X_+ = \exp\left(\frac{2\pi\phi}{g_3^2} + ia\right) \tag{3.70}$$

$$X_- = \exp\left(\frac{-2\pi\phi}{g_3^2} - ia\right) \tag{3.71}$$

For $U(N_c)$ case in which we have $U(N_c) \to U(1)^{N_c}$, the Coulomb branch is parametrized by

$$V_i \sim \exp\left(\frac{\phi_i}{g_3^2} + ia_i\right) \quad i = 1, \cdots, N_c. \tag{3.72}$$

For $SU(N_c)$,

$$Y_i \sim \exp\left(\frac{\phi_i - \phi_{i+1}}{g_3^2} + i(a_i - a_{i+1})\right) \quad i = 1, \cdots, N_c - 1. \tag{3.73}$$

3.6.2 The monopole operator

The Coulomb branch coordinates introduced above are not satisfactory for describing the low-energy dynamics of the Coulomb phase since these are not gauge invariant and they are only effective description of the Coulomb phase and need the UV completion of such operators. It is not easy to solve these problems but we can say that the monopole operator which is naively a monopole-creating (or vortex-creating) operator indeed flows to the Coulomb branch coordinate at low-energy.

The monopole operator is a disorder operator and its 4d counterpart is a 'tHooft loop. The 'tHooft loop is the phase which is acquired by the magnetic point particle, monopole. If we dimensionally reduce the theory to 3d, the 'tHooft loop becomes the local operator which we call monopole operator. Since the monopole operator is not the order operator but the disorder operator, considering the monopole operator corresponds to the singular behavior of the gauge fields.

The monopole operator is a UV operator, so is independent of which phases we are in. The monopole operator is very complicated from the perspective of the elementary Lagrangian language. Let us more characterize the monopole operators. The insertion of the monopole operator at $x_\mu = 0$ make the behavior of the gauge field singular.

$$D^2\Sigma = 0, \quad \bar{D}^2\Sigma = 2\pi q_J \delta^{(3)}(x - x_0)\theta^2 \tag{3.74}$$

which means

$$\Sigma|_{\theta=\bar{\theta}=0} := \sigma \to \frac{q_J}{2r}, \qquad r \to 0. \tag{3.75}$$

Then the monopole operator is related to the Coulomb branch coordinate: $V_\pm \sim \exp(2\pi\sigma/g^2 + \cdots)$.

3.7 3d $\mathcal{N} = 2$ SQED

In this section we study the 3d $\mathcal{N} = 2$ SQED. The four-dimensional QED is an IR free theory and can be studied by the perturbative method. In 3d, on the other hand, the dynamics becomes very non-trivial even if the theory only contains the $U(1)$ gauge group. The gauge coupling in 3d always has the mass dimension 1:

$$\left[g^2\right] = M \tag{3.76}$$

Then the dimensionless coupling is defined as

$$\hat{g}^2 = \frac{g^2}{E}. \tag{3.77}$$

Therefore at low-energy the gauge coupling becomes large and the theory is strongly coupled.

In the supersymmetric extension of the 3d QED whose field contents are the photon, the gaugino, the electrons, we have the two branches of the moduli space of vacua: Higgs and Coulomb. The Coulomb branch is described by $V = \mathrm{e}^{\frac{\Phi}{g^2}}$, $\Phi = \phi + i\gamma$, where γ is a dual photon and ϕ is the scalar field of the vector multiplet. The Higgs branch is described by the "mesons" $M := Q^i\tilde{Q}_j$ which is classically constrained by the condition $\mathrm{rank}(M) \le 1$. This constraint is equivalent to the condition $\epsilon_{jki_1\cdots i_{N_f-2}} M^j_a M^k_b = 0$.

The quantum numbers are summarized as follows. The axial symmetry is not anomalous in 3d.

Table 3.2: The quantum numbers of the 3d $\mathcal{N} = 2$ SQED

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	1	1	0	0
$\tilde{Q}$	1	$\bar{N}_f$	1	0	0
M	N_f	$\bar{N}_f$	2	0	0
$V_\pm$	1	1	$-N_f$	±1	N_f

Note that the $U(1)_J$ symmetry shifts the dual photon: $\gamma \to \gamma + \mathrm{const}$. Then at the Coulomb branch, the $U(1)_J$ symmetry is spontaneously broken. However, on the Higgs branch, all the massless fields are not charged under the $U(1)_J$ symmetry. Since the Coulomb branch meets Higgs branch at the origin of the moduli space, $\phi = 0$, the size of the circle of the dual photon γ should be zero at the origin of the moduli space, $\phi = 0$. Then the Coulomb branch is pinched at the origin and is separated into the two region which we label these two region by

$$V_+ \sim e^{\frac{\Phi}{g^2}}, \quad V_- \sim e^{-\frac{\Phi}{g^2}}. \tag{3.78}$$

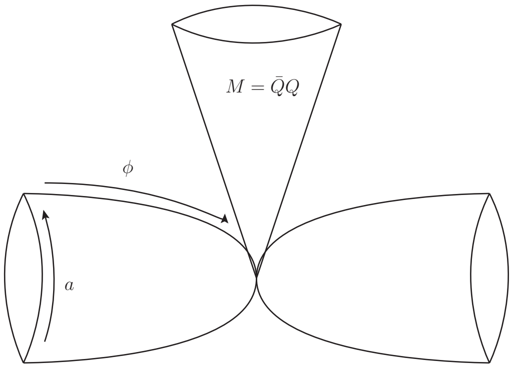

Figure 3.1: The quantum structure of the moduli space of vacua for 3d SQED

The above picture can be also investigated by the perturbative analysis, please see the appendix.

Let us calculate the parity anomaly for the UV theory and the theory whose field contents are only the chiral superfields $M, V_\pm$. Then we now gauge the global symmetries and introduce the vector superfields to couple them to the global currents.

For the UV theory(3d $\mathcal{N} = 2$ SQED)

$$k_{RR} = \frac{1}{2}[(-1)(-1)N_f \text{sign}M_Q + (-1)(-1)N_f \text{sign}M_{\tilde{Q}} + (+1)(+1)\text{sign}M_\lambda] \in \mathbb{Z} + \frac{1}{2} \tag{3.79}$$
$$k_{JJ} = 0 \in \mathbb{Z} \tag{3.80}$$
$$k_{AA} = \frac{1}{2}[(+1)(+1)N_f \text{sign}M_Q + (+1)(+1)N_f \text{sign}M_{\tilde{Q}}] = 0 \in \mathbb{Z} \tag{3.81}$$
$$k_{RJ} = 0 \in \mathbb{Z} \tag{3.82}$$
$$k_{RA} = \frac{1}{2}[N_f(-1)(+1)\text{sign}M_Q + N_f(-1)(+1)\text{sign}M_{\tilde{Q}}] = 0 \in \mathbb{Z} \tag{3.83}$$
$$K_{AJ} = 0 \in \mathbb{Z} \tag{3.84}$$

For the theory with $M, V_\pm$:

$$k_{RR} = \frac{1}{2}[(-1)(-1)N_f^2 \text{sign}M_M + (N_f - 1)(N_f - 1)(\text{sign}M_{V_+} + \text{sign}M_{V_-})] \tag{3.85}$$
$$\overset{N_f\text{: odd}}{\longrightarrow} \mathbb{Z} + \frac{1}{2} \tag{3.86}$$
$$\overset{N_f\text{: even}}{\longrightarrow} \mathbb{Z} \tag{3.87}$$
$$k_{JJ} = \frac{1}{2}[(+1)(+1)\text{sign}M_{V_+} + (-1)(-1)\text{sign}M_{V_-}] \tag{3.88}$$
$$= \frac{1}{2}[\text{sign}((+1)\phi_{U(1)_J}) + \text{sign}((-1)\phi_{U(1)_J})] = 0 \in \mathbb{Z} \tag{3.89}$$
$$k_{AA} = \frac{1}{2}[N_f^2(+2)(+2)\text{sign}M_M + (-N_f)(-N_f)(\text{sign}M_{V_+} + \text{sign}M_{V_-})] \in \mathbb{Z} \tag{3.90}$$
$$k_{RJ} = \frac{1}{2}[(+N_f)(+1)\text{sign}M_{V_+} + (+N_f)(-1)\text{sign}M_{V_-}] \in \mathbb{Z} \tag{3.91}$$
$$k_{RA} = \frac{1}{2}[N_f^2(-1)(2)\text{sign}M_M + (N_f)(-N_f)(\text{sign}M_{V_+} + \text{sign}M_{V_-})] \in \mathbb{Z} \tag{3.92}$$
$$k_{AJ} = \frac{1}{2}[(-N_f)(+1)\text{sign}M_{V_+} + (-N_f)(-1)\text{sign}M_{V_-}] \in \mathbb{Z} \tag{3.93}$$

This suggests that the SQED with $N_f = 1$ is effectively described by the theory which contains only the chiral superfields $M, V_{\pm}$. This is one example of the mirror symmetry which will be discussed next chapter.

3.7.1 The effective description

We can investigate the above (quantum) moduli space by using the theory which only contains the chiral superfields (not a gauge theory).

$N_f = 1$

The moduli space is described by the three chiral superfields $M, V_{\pm}$ which parametrize the one dimensional Higgs branch and the two cones of the Coulomb branch, with the following superpotential.

$$W = -MV_+V_- \tag{3.94}$$

This is one well-known example of the 3d mirror symmetries.

$N_f > 1$

We find that the following superpotential is consistent with the global symmetries.

$$W = -N_f \left(V_+V_- \det M\right)^{\frac{1}{N_f}} \tag{3.95}$$

This effective superpotential is only valid for the moduli space far away from the origin since at the origin this is singular. At the origin of the moduli space of vacua there would be new massless degrees of freedom and we should add some new d.o.f to this superpotential. We can find in the next chapter that the dual theory is given by the mirror symmetry.

3.8 3d $\mathcal{N} = 2$ $SU(2)$ SQCD

Here we will review the quantum structure of the 3d $\mathcal{N} = 2$ $SU(2)$ SQCD. But we will shortly consider the classical moduli space. The Coulomb branch is parametrized by the two scalar fields. One of them is the adjoint scalar coming from the A_3 direction:

$$\phi = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}, \tag{3.96}$$

where we can restrict ϕ to be $\phi \geq 0$ because the Weyl symmetry flips the sign of ϕ, that is to say, ϕ and $-\phi$ are gauge equivalent and are related each other. The second real scalar is the dual photon which is compact due to the quantization of the magnetic flux. These two scalars are combined into one complex scalar:

$$V \sim \exp\left(\frac{\phi}{g_3^2} + ia\right). \tag{3.97}$$

The quantum numbers are as follows.

Table 3.3: Global charges of the 3d $\mathcal{N} = 2$ $SU(2)$ SQCD

	$SU(2N_f)_L$	$U(1)_A$	$U(1)_R$
Q	$\mathbf{2N_f}$	1	0
M	$[\mathbf{ij}]$	2	0
V	$\mathbf{1}$	$-2N_f$	$2N_f - 2$

where the mesonic operator is defined as

$$M_{ij} := Q_i^a Q_j^b \epsilon_{ab} \tag{3.98}$$

$$= \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_{N_f} \\ & & & -\lambda_{N_f} & 0 \end{pmatrix} \tag{3.99}$$

and this is classically constrained by

$$\mathrm{rank} M \leq 2 \Leftrightarrow \epsilon^{i_1 \cdots i_{2N_f}} M_{i_1 i_2} M_{i_3 i_4} = 0 \quad \text{for} \quad N_f > 2 \tag{3.100}$$

$$\Leftrightarrow \mathrm{Pf} M = 0 \qquad \text{for} \quad N_f = 2. \tag{3.101}$$

Notice that for $N_f = 2$, the constraint $\epsilon^{ijkl} M_{ij} M_{kl} = 0$ means

$$\epsilon^{ijkl} M_{ij} M_{kl} \propto \lambda_1 \lambda_2 = 0, \tag{3.102}$$

$$\det M = \lambda_1^2 \lambda^2, \tag{3.103}$$

$$\mathrm{Pf} M = \lambda_1 \lambda_2 = 0. \tag{3.104}$$

For $N_f = 1$, the quark fields can be parametrized

$$Q_i^a = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad M_{ij} = \begin{pmatrix} 0 & c^2 \\ -c^2 & 0 \end{pmatrix}, \tag{3.105}$$

where we used the gauge, global symmetry and the D-flatness condition. Thus the Higgs branch for $N_f = 1$ is one complex-dimensional and parametrized by

$$M = M_{12} = Q_1^a Q_2^b \epsilon_{ab} = 2c^2 \tag{3.106}$$

For $N_f \geq 3$, the condition for M is equivalent to the condition that the any 4×4 minor matrices A which are constructed from M satisfy

$$\mathrm{Pf} A = 0. \tag{3.107}$$

In what follows, we will study the quantum modifications of the above (classical) pictures.

3d $\mathcal{N} = 2$ $SU(2)$ pure SYM: $N_f = 0$

Since there is no fundamental quark, the monopole only has two gaugino zero-modes. Thus it may appear in the superpotential. The monopole-instanton in 3d generates the following non-perturbative superpotential:

$$W = \frac{1}{V} \tag{3.108}$$

which is a type of the runaway and lifts all of the Coulomb branch. Then there is no SUSY vacua. (Actually we should multiply the dimensionfull parameter to make the superpotential mass-dimension 2.)

$N_f = 1$

The superpotential is not generated since all the fields are not charged under the $U(1)_R$ symmetry, but the quantum effect deforms the moduli space. The quantum moduli space is constrained by

$$MV = 1. \tag{3.109}$$

This condition is realized in the superpotential by using the Lagrange multiplier:

$$W = \lambda(MV - 1) \tag{3.110}$$

We can introduce the complex mass to the quark and we obtain the $N_f = 0$ theory:

$$W = \lambda(MV - 1) + mM. \tag{3.111}$$

The equations of motion are as follows.

$$MV = 1 \tag{3.112}$$
$$\lambda V + m = 0 \tag{3.113}$$

If we integrate out the massive fields, we obtain

$$W_{\text{low energy}} = \frac{m}{V} \tag{3.114}$$

which implies the high- and low-energy monopole operators are related by

$$V_{low} = V_{high}/m. \tag{3.115}$$

$N_f = 2$

For $N_f = 2$, the holomorphy argument produces the following superpotential.

$$W = -V\mathrm{Pf}M \tag{3.116}$$

Again, this is consistent with the results of the small flavors. For example we introduce the complex masses for M_{34}. Then the superpotential becomes

$$W = -V\mathrm{Pf}M + mM_{34} \tag{3.117}$$

The equation of motion for M_{34} is

$$-VM_{12} + m = 0. \tag{3.118}$$

Then the low-energy superpotential where the massive quarks are all integrated out is

$$W = \lambda(-VM_{12} + m). \tag{3.119}$$

Again we can find the relation between the monopole operators at high and low energy:

$$V_{low} = V_{high}/m. \tag{3.120}$$

In this way we can find that the superpotential generated by the holomorphy argument indeed produces the correct quantum picture of the moduli space of vacua.

The theory with $N_f = 2$ flavors is described by the theory containing only the chiral superfields with the above superpotential. We can check this by the parity anomaly matching:

$$k_{RR,eff} \overset{\text{UV}}{=} \frac{1}{2}\left[2N_f N_c(-1)^2 \mathrm{sign} M_Q + (N_c^2 - 1)(+1)^2 \mathrm{sign} M_\lambda\right] \in \mathbb{Z} + \frac{1}{2} \tag{3.121}$$
$$\overset{\text{IR}}{=} \frac{1}{2}\left[N_f(2N_f - 1)(-1)^2 \mathrm{sign} M_M + (2N_f - 3)^2 \mathrm{sign} M_V\right] \in \mathbb{Z} + \frac{1}{2} \tag{3.122}$$
$$k_{AA,eff} \overset{\text{UV}}{=} \frac{1}{2}\left[2N_f N_c(+1)^2 \mathrm{sign} M_Q\right] \in \mathbb{Z} \tag{3.123}$$
$$\overset{\text{IR}}{=} \frac{1}{2}\left[N_f(2N_f - 1)(+2)^2 \mathrm{sign} M_M + (-2N_f)^2 \mathrm{sign} M_V\right] \in \mathbb{Z} \tag{3.124}$$
$$k_{RA,eff} \overset{\text{UV}}{=} \frac{1}{2}[2N_f N_c(-1)(+1)\mathrm{sign} M_Q] \in \mathbb{Z} \tag{3.125}$$
$$\overset{\text{IR}}{=} \frac{1}{2}[N_f(2N_f - 1)(-1)(+2)\mathrm{sign} M_M + (2N_f - 3)(-2N_f)\mathrm{sign} M_V] \in \mathbb{Z} \tag{3.126}$$
$$k_{SU(2N_f),eff} \overset{\text{UV}}{=} \frac{1}{2}\underbrace{N_c \cdot 1}_{Q} \in \mathbb{Z} \tag{3.127}$$
$$\overset{\text{IR}}{=} \frac{1}{2} 2N_f(2N_f - 4) \in \mathbb{Z} \tag{3.128}$$

$N_f > 2$

The symmetry argument gives the following superpotential.

$$W = -(N_f - 1)\,(V\mathrm{Pf}M)^{\frac{1}{N_f-1}} \tag{3.129}$$

which is well describing the moduli space far away from the origin of the moduli space of vacua. At the origin we need the additional massless degrees of freedom.

3.9 3d $\mathcal{N} = 2$ $SU(3)$ SQCD

Here we will investigate the moduli space of vacua for the case of $SU(3)$ gauge group. In this case, the Coulomb branch is parametrized by two region:

$$\phi_1 \geq \phi_2 \geq 0 \geq \phi_3 \tag{3.130}$$

and

$$\phi_1 \geq 0 \geq \phi_2 \geq \phi_3. \tag{3.131}$$

We need to introduce the different Coulomb branch coordinates (operators) for each region. But we should impose the constraints in order that these operators are defined continuously throughout the different regions of the moduli space.

The region $\phi_1 \geq \phi_2 \geq 0 \geq \phi_3$ is parametrized by the two operators Y_1 and Y_2. On the other hand, the region $\phi_1 \geq 0 \geq \phi_2 \geq \phi_3$ is described by $\tilde{Y}_1$ and $\tilde{Y}_2$. The Weyl-invariant combination $Y = \prod Y_i$ is globally defined, hence this operator imposes one constraint for the four Coulomb branch coordinates $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$.

$$Y = Y_1 Y_2 = \tilde{Y}_1 \tilde{Y}_2 \tag{3.132}$$

The global charges of these operators are calculated as follow (please see the appendix).

Table 3.4: The quantum numbers of the 3d $\mathcal{N} = 2$ $SU(3)$ SQCD

	$U(1)_R$	$U(1)_A$
$\det M$	0	$2N_f$
Y	$2N_f - 4$	$-2N_f$
Y_1	-2	0
Y_2	$2N_f - 2$	$-2N_f$
$\tilde{Y}_1$	$2N_f - 2$	$-2N_f$
$\tilde{Y}_2$	-2	0

Thus the monopole-instanton generates the superpotential:

$$W = \frac{1}{Y_1} + \frac{1}{\tilde{Y}_2} \tag{3.133}$$

and lifts the Coulomb branch coordinates Y_1 and $\tilde{Y}_2$. To recap, the two operators are lifted and we have one constraint. So we have the $4 - 2 - 1 = 1$ independent operator which is precisely $Y = Y_1 Y_2 = \tilde{Y}_1 \tilde{Y}_2$. This argument to obtain the independent monopole operators is quite powerful and it can be applied to any gauge theory with various matter fields [36].

3.10 3d $\mathcal{N} = 2$ $SU(N_c)$ SQCD

Next, we proceed to the 3d $\mathcal{N} = 2$ $SU(N_c)$ SQCD. Let's us consider the specific region of the Coulomb branch: $\phi_1 \geq \phi_2 \geq \cdots \geq \phi_k > 0 > \phi_{k+1} \geq \cdots \geq \phi_{N_c-1} \geq \phi_{N_c} = -\phi_1$. The zero-modes of the fundamental quarks around the monopole background are counted as follows [36] (This argument is equivalent to the index theorem but this is more physical explanation.).

The size of the instanton ρ_i is characterized by

$$\frac{1}{\rho_i} = \frac{1}{2}|\phi_i - \phi_{i+1}| \tag{3.134}$$

On the other hand, the fundamental quarks have the coupling with ϕ_i, ϕ_{i+1}. Thus the effective mass for the quark is

$$m^i_{eff} = \frac{1}{2}|\phi_i + \phi_{i+1}|. \tag{3.135}$$

The effective mass for the fundamental quarks are calculated as follows. First we should divide the vacuum expectation values for ϕ as

$$\phi = \begin{pmatrix} \frac{1}{2}(\phi_1+\phi_2) & & & \\ & \frac{1}{2}(\phi_1+\phi_2) & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\phi_1-\phi_2) & & & \\ & -\frac{1}{2}(\phi_1-\phi_2) & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}. \tag{3.136}$$

The first term is invariant under the $SU(2)$ rotation but the second term is not invariant. The effective mass is identical to this invariant part.

If the quark mass is lighter than the mass of the instanton, the quark has the zero-modes. For $m_{eff} > 1/\rho$, there is no zero-mode. Then, for $\phi_i > \phi_{i+1} > 0$, we have only the gluino zero-modes around the monopole-instanton background. For $\phi_i > 0 > \phi_{i+1}$, we also have the quark zero-modes in addition to the gaugino zero-modes.

The alternative derivation of the global charges of the monopole operator is given in the appendix, where the index theorem is used. the quantum numbers of the fields are summarized as follows.

Table 3.5: The quantum numbers of the 3d $\mathcal{N}=2$ $SU(N_c)$ SQCD

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
Q	N_f	1	1	1	0
$\tilde{Q}$	1	$\bar{N_f}$	-1	1	0
M	N_f	$\bar{N_f}$	0	2	0
$V_{j\neq k}$	1	1	0	0	-2
V_k	1	1	0	$-2N_f$	$2(N_f-1)$
$V := \prod V_j$	1	1	0	$-2N_f$	$2(N_f-N_c+1)$

The Coulomb branch of this theory is classically N_c-1-dimensional:

$$Y_j \sim \exp\left(\frac{(\phi_j - \phi_{j+1})}{g_3^2} + i(a_j - a_{j+1})\right) \tag{3.137}$$

But almost all the Coulomb branch is lifted by the AHW-type superpotential which is generated by the monopoles. On the Coulomb branch the gauge symmetry breaks as

$$SU(N_c) \to SU(N_c-2) \times U(1)^2. \tag{3.138}$$

In this region other moduli can take

$$\mathrm{rank} M \leq N_c - 2,\ B = \tilde{B} = 0. \tag{3.139}$$

3d $\mathcal{N} = 2$ $SU(N_c)$ pure SYM: $N_f = 0$

Let us consider the pure SYM theory. The symmetry argument gives the following superpotential for the Coulomb branch coordinates:

$$W = \sum_{i=1}^{N_c-1} \frac{1}{V_i}, \tag{3.140}$$

which lifts all of the Coulomb branch, leading to the absence of the stable SUSY vacua. This should be compared with the 4d $\mathcal{N} = 1$ pure SYM where there is some discrete SUSY vacua.

N_f flavors

Next we include the fundamental matters. The monopole generates

$$W = \sum_{j \neq k} \frac{1}{V_j}. \tag{3.141}$$

This lifts almost all the Coulomb branch and there is complex one-dimensional branch un-lifted.

$$\phi_1 > \phi_2 = \cdots = \phi_{N_c-1} = 0 > \phi_{N_c} = -\phi_1, \tag{3.142}$$

which is parametrized by

$$V = \prod_{j=1}^{N_c-1} \sim \exp\left(\frac{\phi_1 - \phi_{N_c}}{g_3^2} + i(a_1 - a_2)\right) \tag{3.143}$$

$N_f < N_c - 1$

The similar argument of the Affleck-Dine-Seiberg superpotential [54, 55] in 4d is applied to this case:

$$W = (N_f - N_c - 1)(V \det M)^{\frac{1}{N_f - N_c - 1}} \tag{3.144}$$

$N_f = N_c - 1$

In this case we have the quantum constraints between the Higgs and Coulomb branch coordinates.

$$V \det M = 1, \tag{3.145}$$

where the scale factor is omitted. Then the large vev region of M corresponds to the small vev region of V.

$N_f = N_c$

The low-energy effective theory is describe by the following effective superpotential.

$$W = -V(\det M - B\tilde{B}) \tag{3.146}$$

Let us check the parity anomaly matching between the UV theory and the effective theory.

$$k_{RR,eff} \overset{\text{UV}}{=} \frac{1}{2}\left[N_c N_f(\text{sign}M_Q + \text{sign}M_{\tilde{Q}}) + (N_c^2-1)\text{sign}M_\lambda\right] \in \mathbb{Z} + \frac{1}{2}(N_c^2+1) \tag{3.147}$$

$$\overset{\text{IR}}{=} \frac{1}{2}\left[N_f^2\text{sign}M_M + 2\text{sign}M_{B,\tilde{B}} + (2(N_f-N_c+1)-1)\text{sign}M_Y\right] \in \mathbb{Z} + \frac{1}{2}(N_f^2+1) \tag{3.148}$$

$$k_{BB,eff} \overset{\text{UV}}{=} \frac{1}{2}\left[N_f N_c(+1)^2\text{sign}M_Q + N_f N_c(-1)^2\text{sign}M_{\tilde{Q}}\right] \in \mathbb{Z} \tag{3.149}$$

$$\overset{\text{IR}}{=} \frac{1}{2}\left[(+N_c)^2\text{sign}M_B + (-N_c)^2\text{sign}M_{\tilde{B}}\right] \in \mathbb{Z} \tag{3.150}$$

$$k_{AA,eff} \overset{\text{UV}}{=} \frac{1}{2}\left[N_f N_c(+1)^2\text{sign}M_Q + N_f N_c(+1)^2\text{sign}M_{\tilde{Q}}\right] \in \mathbb{Z} \tag{3.151}$$

$$\overset{\text{IR}}{=} \frac{1}{2}\left[N_f^2(+2)^2\text{sign}M_M + (+N_c)^2\text{sign}M_B + (+N_c)^2\text{sign}M_{\tilde{B}} + (-2N_f)^2\text{sign}M_Y\right] \in \mathbb{Z} \tag{3.152}$$

$$k_{SU(N_f)_L,eff} \overset{\text{UV}}{=} \frac{1}{2}\underbrace{\overbrace{N_c}^{N_c\text{ colors}} \cdot 1}_{Q} \in \frac{1}{2}N_c + \mathbb{Z} \tag{3.153}$$

$$\overset{\text{IR}}{=} \frac{1}{2}\underbrace{N_f \cdot 1}_{M^i_j} \in \frac{1}{2}N_f + \mathbb{Z} \tag{3.154}$$

$$k_{SU(N_f)_R,eff} \overset{\text{UV}}{=} \frac{1}{2}\underbrace{\overbrace{N_c}^{\bar{N}_c\text{ color}} \cdot 1}_{\tilde{Q}} \in \frac{1}{2}N_c + \mathbb{Z} \tag{3.155}$$

$$\overset{\text{IR}}{=} \frac{1}{2}\underbrace{N_f \cdot 1}_{M^i_j} \in \frac{1}{2}N_f + \mathbb{Z} \tag{3.156}$$

Although the parity anomaly matching is weaker check of the duality, the parity anomalies of the UV and IR theories are indeed identical.

$N_f > N_c$

We have no effective description for $N_f > N_c$.

3.11 3d $\mathcal{N}=2$ $U(N_c)$ SQCD

We can obtain the $U(N_c)$ theory by gauging the $U(1)_B$ global (baryon) symmetry. The quantum numbers of the fields are as follows.

Table 3.6: The quantum numbers of the 3d $\mathcal{N}=2$ $U(N_c)$ SQCD

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_R$
Q	N_f	1	1	0
$\tilde{Q}$	1	$\bar{N}_f$	1	0
M	N_f	$\bar{N}_f$	2	0
$V_\pm$	1	1	$-N_f$	N_f-N_c-1

The monopole operators listed above are at least semi-classically defined as follows.

$$V_+ \sim \exp\left(\frac{\phi_1}{g_3^2} + ia_1\right), \quad V_- \sim \exp\left(-\frac{\phi_{N_c}}{g_3^2} - ia_{N_c}\right). \tag{3.157}$$

$N_f = N_c$

The un-gauged theory is the $SU(N_c)$ with $N_f = N_c$ flavors which is effectively described by the gauge singlets $V, M, B, \tilde{B}$ with the superpotential $W = -V(\det M - B\tilde{B})$. By gauging the $U(1)_B$ symmetry, only the baryon operators are charged, so the $U(1)_B^{gauged}$ sector is the $\mathcal{N} = 2$ SQED with one (vector-like) flavor. Thus the dual description to it is the XYZ model:

$$W = -V_+^{U(1)_B} V_-^{U(1)_B} (B\tilde{B}) \tag{3.158}$$

Combining all the superpotential we arrive at

$$W = -V(\det M - B\tilde{B}) - V_+^{U(1)_B} V_-^{U(1)_B} (B\tilde{B}). \tag{3.159}$$

Integrating out the massive fields V and $(B\tilde{B})$ we have

$$W = -V_+^{U(1)_B} V_-^{U(1)_B} \det M. \tag{3.160}$$

The lower values of N_f is obtained by turning on the masses for the quarks.

$N_f > N_c$

The holomorphy tells us that the effective superpoential is

$$W \sim (V_+ V_- \det M)^{\frac{1}{N_f - N_c + 1}} \tag{3.161}$$

This is singular at the origin of the moduli space of vacua analogous to the 4d $\mathcal{N} = 1$ SQCD with $N_f > N_c$, which implies that some new massless degrees of freedom are needed at the origin. However this superpotential are correctly describing the moduli space away from the origin.

Notice that since the scalar potential in this case becomes

$$V = g^2 \left(Q^\dagger T^a Q - \tilde{Q} T^a \tilde{Q}^\dagger \right)^2 + |\phi Q|^2 + |\phi \tilde{Q}|^2, \tag{3.162}$$

the meson composite $M := \tilde{Q}Q$ generically can take the vev as $\operatorname{rank} M \leq N_c$ for $V_+ = V_- = 0$, $\operatorname{rank} M \leq N_c - 1$ if the one of the Coulomb moduli is non-zero and $\operatorname{rank} M \leq N_c - 2$ for $V_\pm \neq 0$.

3.12 The relation between 3d and 4d SUSY gauge theories

The $\mathcal{N} = 2$ SUSY gauge theories discussed here are the dimensional reduction of the corresponding 4d SUSY gauge theories with four supercharges. So the field contents are the same as the 4d ones except for the adjoint scalars coming from the A_3 direction. However, the dynamics of these theories are very different. If we want to connect the dynamics of the 3d and 4d, we need to put the 4d theory on the $\mathbb{S}^1 \times \mathbb{R}^3$ and take into account the dynamics of the theory on the compact manifold. In order to relate the physics between 3d and 4d, we should consider the so-called Kaluza-Klein monopoles which generate the effective superpotential for the Coulomb branch.

3.12.1 The relation of the gauge couplings

The 3d and 4d gauge coupling is related by the dimensional reduction as

$$2\pi r g_3^2 = g_4^2 \tag{3.163}$$

where r is the radius of the $\mathbb{S}^1$. The dynamical scale in 4d becomes

$$\Lambda^b = \exp\left(\frac{-8\pi^2}{g_4^2}\right) = \exp\left(\frac{-4\pi}{r g_3^2}\right) \tag{3.164}$$

where b is the coefficient of the one-loop beta-function: $b = 3N_c - N_f$. The Coulomb moduli is compact if we put the theory on the $\mathbb{S}^1$:

$$\phi \sim \phi + \frac{1}{r} \tag{3.165}$$

The 4d limit is

$$g_4^2 : \text{ fixed}, \tag{3.166}$$
$$r \to \infty, \tag{3.167}$$

which means $g_3^2 \to 0$ and the Coulomb branch becomes very small. So in the 4d limit we should integrate over the Coulomb branch coordinates.

On the other hand, the 3d limit corresponds to

$$g_3^2 : \text{ fixed}, \tag{3.168}$$
$$r \to 0, \tag{3.169}$$

which means $g_4^2 \to 0$ and $\Lambda^b \to 0$.

3.12.2 $SU(2)$ with N_f flavors

First we show the global charges of the fields.

Table 3.7: The quantum numbers of the 4d $\mathcal{N} = 1$ $SU(2)$ SQCD

	$SU(2N_f)_L$	$U(1)_A$	$U(1)_R$
Q	$\mathbf{2N_f}$	1	0
M	$\mathbf{N_f(2N_f-1)}$	2	0
V	1	$-2N_f$	$2N_f - 2$
$\eta = \Lambda^b$	1	$2N_f$	$-2N_f + 4$

The Kaluza-Klein monopole (or we often call it the twisted instanton because this is in fact the instanton of the $\mathbb{S}^1 \times \mathbb{R}^3$.) generates the effective potential for the Coulomb branch, which is proportional to $\eta = \Lambda^b \sim \mathrm{e}^{-\frac{1}{g_4^2}}$:

$$W_{KK} = \eta V. \tag{3.170}$$

This superpotential is always generated for any N_f since the Kaluza-Klein monopole has no fermonic zero-modes from the fundamental quark and only has two zero-modes from the gaugino. This is intriguing feature with contrast to the fact that the monopole has the quark zero-modes.

$N_f = 0$

In this case we have two contributions from the monopole and the KK-monopole. Then the superpotential is

$$W = \frac{1}{V} + \eta V. \tag{3.171}$$

The equation of motion of V gives

$$V = \pm \frac{1}{\sqrt{\eta}}. \tag{3.172}$$

This observation tells us two important facts. The first one is that in the 3d limit where the second term in the superpotential vanishes the Coulomb branch is lifted and there is no stable

SUSY vacua. The second one is that the effect of the KK-monopole stabilizes the potential and there is two SUSY vacua now. This is consistent with that the 4d $\mathcal{N}=1$ pure SYM has the two SUSY vacua with the mass gap which means that there is no massless degree of freedom.

Thus if we would like to connect the dynamics between 3d and 4d, we should take the KK-monopoles into account. This is a crucial step for deriving the 3d Seiberg duality from the 4d Seiberg duality.

$N_f = 1$

The effective superpotential becomes

$$W = \lambda(MV - 1) + \eta V. \tag{3.173}$$

By integrating out V, we arrive at

$$MV = 1 \tag{3.174}$$
$$\lambda M + \eta = 0 \tag{3.175}$$

$$W_{4d,eff} = \frac{\eta}{M}. \tag{3.176}$$

$N_f = 2$

The effective superpotential becomes

$$W = -V \mathrm{Pf} M + \eta V. \tag{3.177}$$

By integrating out V, we obtain the quantum constraint in 4d:

$$\mathrm{Pf} M \sim \eta. \tag{3.178}$$

$N_f > 2$

In a similar way we have

$$W = -(N_f - 1)\,(V \mathrm{Pf} M)^{\frac{1}{N_f - 1}} + \eta V \tag{3.179}$$
$$\rightarrow \left(\frac{\mathrm{Pf} M}{\eta}\right)^{\frac{1}{N_f - 2}}. \tag{3.180}$$

3.12.3 $SU(N_c)$ with N_f flavors

First we show the quantum numbers of the fields and the dynamical scale. Note that since in 4d there is a chiral anomaly, so the global symmetry generically becomes anomalous and spurious.

Table 3.8: The quantum numbers of the 4d $\mathcal{N}=1$ $SU(N_c)$ SQCD

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
Q	N_f	1	1	1	0
$\tilde{Q}$	1	$\bar{N}_f$	-1	1	0
M	N_f	$\bar{N}_f$	0	2	0
$V_{j\neq k}$	1	1	0	0	-2
V_k	1	1	0	$-2N_f$	$2(N_f-1)$
$V := \prod V_j$	1	1	0	$-2N_f$	$2(N_f - N_c + 1)$
$\eta = \Lambda^b$	1	1	0	$2N_f$	$2(N_c - N_f)$

$N_f = 0$: pure SYM

The effective superpotential including the effect of the twisted instantons is as follows [56].

$$W = \sum_{j=1}^{N_c-1} \frac{1}{Y_j} + \eta Y_1 \cdots Y_{N_c-1} \tag{3.181}$$

The equations of motion for Y_j are

$$\eta Y = \frac{1}{Y_j}, \tag{3.182}$$

which means $Y_c^N = \frac{1}{\eta^{N_c-1}}$. Then the values of the field Y can vary in N_c ways. So we have the N_c vacua which is consistent with the corresponding 4d physics:

$$W \sim \Lambda_4^3 \tag{3.183}$$

$0 < N_f < N_c - 1$ flavors

The effective superpotential is determined by the holomorphy.

$$W = (N_c - N_f - 1)\left(Y \det M\right)^{\frac{1}{N_f - N_c + 1}} + \eta Y \tag{3.184}$$

The equation of motion for Y is

$$\eta = Y^{\frac{N_c - N_f}{N_f - N_c + 1}} (\det M)^{\frac{1}{N_f - N_c + 1}} \tag{3.185}$$

If we substitute this into the superpotential, we have

$$W \sim \left(\frac{\Lambda_4^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}} \tag{3.186}$$

which is known as the Affleck-Dine-Seiberg superpotential [55, 57].

$N_f = N_c - 1$ flavors

The superpotential which describes the quantum moduli and the KK-monopole effect is

$$W = \lambda(Y \det M - 1) + \eta Y \tag{3.187}$$

The F-term equations are

$$\lambda \det M + \eta = 0 \tag{3.188}$$

$$\lambda Y \,\mathrm{cof} M = 0 \tag{3.189}$$

$$Y \det M = 1. \tag{3.190}$$

These three equations have no solution. This can be regarded in two ways. One way to understand this is that the theory is 3d theory deformed with the KK-monopole effects and there is no stable SUSY vacua. The other way is to consider the theory on $\mathbb{S}^1 \times \mathbb{R}^3$ and it also has no stable SUSY vacua. The second interpretation comes back to the 4d theory when the size of the compactification r is infinity. Remember that the 4d $\mathcal{N} = 1$ $SU(N_c)$ SUSY SQCD with $N_f = N_c - 1$ has no stable SUSY vacua due to the Affleck-Dine-Seiberg superpotential.

$N_f = N_c$ flavors

The effective superpotential is

$$W = Y(B\tilde{B} - \det M) + \eta Y \tag{3.191}$$

The Coulomb branch is lifted and the Higgs branch is deformed. This is the same as the quantum moduli space of the 4d $\mathcal{N} = 1$ $SU(N_c)$ SUSY SQCD with $N_f = N_c$. If we regard this as the 3d theory with deformed by the KK-monopoles, the moduli space of the deformed 3d theory is identical to the 4d one.

$N_f = N_c + 1$ **flavors**

Generally, higher values of N_f is more and more difficult to find the effective description in terms of the gauge invariant chiral superfields. However in the case with $N_f = N_c + 1$ we can find the effective description:

$$W = \left(Y(\det M + BM\tilde{B})\right)^{\frac{1}{2}} + \eta Y \tag{3.192}$$

The equation of motion for Y is

$$Y^{\frac{1}{2}} = -\frac{1}{2\eta}\left(\det M + BM\tilde{B}\right)^{\frac{1}{2}} \tag{3.193}$$

Integrating out Y, the 4d effective superpotential becomes

$$W = -\frac{1}{4\eta}\left(\det M + BM\tilde{B}\right). \tag{3.194}$$

$N_f > N_c + 1$ **flavors**

We have no effective description by the gauge invariant variables. One of the difficulties is the appearance of the new massless degrees of freedom at the origin of the moduli space of vacua. If we want to describe this theory in different manners, we need to search for another gauge theories which usually have the different gauge group with the original theory but flow to the same IR fixed point as the original one.

3.13 $SU(4)$ with an anti-symmetric matter

Now we add one anti-symmetric matter to the $SU(4)$ SQCD [36]. Let us consider the Coulomb branch parametrized by

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4 = -\sigma_1 - \sigma_2 - \sigma_3. \tag{3.195}$$

In this theory we have the various flat directions. The first one is the branch parametrized by the quark fields $Q, \tilde{Q}$. These fields couple to the adjoint scalars σ_α. Then the flat directions are determined by

$$\sigma_\alpha Q^\alpha = -\sigma_\alpha \tilde{Q}^\alpha = 0. \tag{3.196}$$

The another moduli comes from the anti-symmetric matter A. This field also couples to the adjoint scalar σ_α. Then the flatness condition is

$$(\sigma_\alpha + \sigma_\beta)A^{\alpha\beta} = 0. \tag{3.197}$$

When some of the components of σ_α become zero, at that point of the Coulomb moduli the Higgs branch emerges. For example, at $\sigma_2 = 0$, the Coulomb branch is pinched (or shrinks) and is divided into two regions. The Higgs branch which is parametrized by $\tilde{Q}^i_2 Q^2_j$ sticks to this point $\sigma_2 = 0$, where i, j are the flavor indices. At $\sigma_3 = 0$, the Higgs branch parametrized by $\tilde{Q}^i_3 Q^3_j$ merges with the Coulomb branch. At $\sigma_2 + \sigma_3 = 0$, the additional Higgs branch arises due to the presence of the anti-symmetric matter $A^{\alpha\beta}$. In conclusion, the Coulomb branch are divided into two parts at $\sigma_2 = 0$, $\sigma_3 = 0$ and $\sigma_2 + \sigma_3 = 0$. Then the Coulomb branch is divided into four regions. For each region, we need the original coordinates which are eventually related each other. So the question is how many independent operators we do need as the coordinate of the Coulomb branch. In [36], the authors gave the very clear answer to this. So we will here review it.

The Coulomb branch coordinates

Let us first calculate the fermion zero-modes around the monopole-instantons. If there is a monopole-instanton labeling by the left-upper $SU(2)$, the corresponding vevs for the adjoint scalar could be seen as

$$\sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \sigma_4 \end{pmatrix} \tag{3.198}$$

$$= \begin{pmatrix} \frac{1}{2}(\sigma_1 - \sigma_2) & & & \\ & -\frac{1}{2}(\sigma_1 - \sigma_2) & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\sigma_1 + \sigma_2) & & & \\ & \frac{1}{2}(\sigma_1 + \sigma_2) & & \\ & & \sigma_3 & \\ & & & \sigma_4 \end{pmatrix} \tag{3.199}$$

The first term is related to the scale of the breaking of the corresponding $SU(2)$ gauge group and is related to the instanton size. On the other hand the second term is related to the effective masses for the quark fields.

$$m_{\text{eff}} = \frac{1}{2}(\sigma_1 + \sigma_2) + \sigma_{3,4} \tag{3.200}$$

Comparing the inverse instanton size with the effective masses, we can calculate the fermion zero-modes.

Region I : $\sigma_1 > 0 > \sigma_2 > \sigma_3 > \sigma_4$

The Coulomb branch is labeled by $Y_{1,2,3}$. We have the additional zero-modes from the matter fields to Y_1. So the effective superpotential is

$$W = \frac{1}{Y_2} + \frac{1}{Y_3}. \tag{3.201}$$

Region II : $\sigma_1 > |\sigma_3| > \sigma_2 > 0 > \sigma_3 > \sigma_4$

The Coulomb branch is labeled by $\tilde{Y}_{1,2}$, Y_3. We have the additional zero-modes from the matter fields to $\tilde{Y}_{1,2}$. So the effective superpotential is

$$W = \frac{1}{Y_3}. \tag{3.202}$$

Region III : $\sigma_1 > \sigma_2 > |\sigma_3| > 0 > \sigma_3 > \sigma_4$

The Coulomb branch is labeled by Y_1', $\tilde{Y}_{2,3}$. Note that we need the new variable $\tilde{Y}_3$ because the Higgs branch parametrized by the anti-symmetric matter $A^{\alpha\beta}$ pinches the Coulomb branch and the Coulomb branch is divided into two region at the point where we across from the region II to the region III.

We have the additional zero-modes from the matter fields to $\tilde{Y}_{2,3}$. So the effective superpotential is

$$W = \frac{1}{Y_1'}. \tag{3.203}$$

Region IV : $\sigma_1 > \sigma_2 > \sigma_3 > 0 > \sigma_4$

The Coulomb branch is labeled by $Y_{1,2,3}'$. We have the additional zero-modes from the matter fields to Y_3'. So the effective superpotential is

$$W = \frac{1}{Y_1'} + \frac{1}{Y_2'}. \tag{3.204}$$

Totaly we have the nine operators in order to parametrize the regions of the Coulomb branch. But the four operators are lifted by the non-perturbative AHW-type super potentials. The operator $Y := \prod_{i=1,2,3} Y_i$ is globally defined throughout the four region I,II,III and IV. So the continuity conditions imposes the three constraints for these operators. Then, we need the

$$\underbrace{9}_{\substack{\text{Coulomb branch}\\ \text{coordinates}}} - \underbrace{4}_{\text{AHW}} - \underbrace{3}_{\text{Continuity}} = 2 \tag{3.205}$$

types of the monopole operators for describing the Coulomb branch. The first one is

$$Y = \prod_{i=1}^{3} Y_i. \tag{3.206}$$

Indeed this is Weyl-invariant and preferable one. The second operator is proposed in [36] as follows:

$$\tilde{Y} := \sqrt{Y\tilde{Y}_2}. \tag{3.207}$$

This can be seen as follows. The non-perturbative superpotentials lift almost all the Coulomb branch as we considered above. Finally we have two dimensional Coulomb branch which is

$$\sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & -\sigma_2 & \\ & & & -\sigma_1 \end{pmatrix}. \tag{3.208}$$

This is checked as follows. For example, if we modify the third component as $\sigma_3 = -\sigma_2 \to \sigma_3' = -\sigma_2 + \delta\sigma$, $\sigma_4 = -\sigma_1 \to \sigma_4' = -\sigma_1 - \delta\sigma$, where $\delta\sigma > 0$. This shifted vacua correspond to the region III discussed above. Then the instanton produces the repulsive force between σ_1 and σ_2. This is equivalent to moving the σ_2 to $|\sigma_3| = |-\sigma_2 + \delta\sigma|$, which means that this repulsive force makes $\delta\sigma \to 0$. This observation can be applied for $\delta\sigma < 0$ and we conclude that the un-lifted Coulomb branch is parametrized by

$$\begin{aligned} \sigma &= \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & -\sigma_2 & \\ & & & -\sigma_1 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_1 - \sigma_2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -(\sigma_1 - \sigma_2) \end{pmatrix} + \begin{pmatrix} \sigma_2 & & & \\ & \sigma_2 & & \\ & & -\sigma_2 & \\ & & & -\sigma_2 \end{pmatrix} && (3.209) \\ &=: Y + \tilde{Y} && (3.210) \end{aligned}$$

Note that in region I and IV, we have the one-dimensional Coulomb branch, so we need only Y.

Dirac quantization condition

Let us consider the minimal magnetic charge and the allowed forms of the corresponding monopole operators [36,58]. Let us first consider the operator $Y :\ \mathrm{diag}(\sigma, 0, 0, -\sigma)$ which corresponds to the breaking of the gauge group as

$$SU(4) \to SU(2) \times U(1) \times U(1) \tag{3.211}$$

whose $U(1)$ generators are specified as

$$t_{U(1)_1} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}, \quad t_{U(1)_2} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \tag{3.212}$$

Next, $\tilde{Y}$: $\text{diag}(\sigma, \sigma, -\sigma, -\sigma)$ is related to the breaking:

$$SU(4) \to SU(2) \times SU(2) \times U(1) \tag{3.213}$$

whose $U(1)$ generator is defined as follows.

$$t_{U(1)_3} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{3.214}$$

Let us consider the Dirac's quantization condition for $SU(4) \to SU(2) \times SU(2) \times U(1)$. The $U(1)$ rotation contains the following group element

$$\exp\left(i\pi\, t_{U(1)_3}\right) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \equiv -\mathbf{1}, \tag{3.215}$$

which is also contained in the center of $SU(2) \times SU(2)$:

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \mathbb{Z}_2 \times \mathbb{Z}_2 \in \text{ center of } SU(2) \times SU(2). \tag{3.216}$$

So when we go around the non-trivial loop, the group element such as $\exp(-4\pi i e Q_3)$ need not to be identity, where e is the electric charge and Q_3 is the monopole charge corresponding to $t_{U(1)_3}$. Namely we have

$$\exp(-4\pi i e Q_3) = -\mathbf{1}. \tag{3.217}$$

So we have the half of the monopole charge, which explains why the monopole operator $\tilde{Y} := \sqrt{Y\tilde{Y}_2}$ contains the square root. The full magnetic charge is defined as

$$M = \frac{1}{2}Q_3 + T_3^{(1)} + T_3^{(2)} \tag{3.218}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{3.219}$$

$$= \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}. \tag{3.220}$$

We can again see that the magnetic charge corresponding $Q_3 = t_{U(1)_3}$ is indeed half integer.

We have alternative explanation for the difference of the minimal magnetic charges. Let us first consider the breaking $SU(4) \to SU(2) \times U(1) \times U(1)$ corresponding to giving the vev to Y. In this case, the fundamental matters in $SU(4)$ decomposes as

$$\square \to (\square, 0, -1) \oplus (\bullet, 1, 1) \oplus (\bullet, -1, 1) \tag{3.221}$$

Since the $SU(2)$ singlets have the minimal $U(1)$ charges, the minimal monopole charges are allowed. On the other hand, the story of the breaking $SU(4) \to SU(2) \times SU(2) \times U(1)$ is different. Indeed under this breaking we have the $SU(2)$ doublet fields which have the minimal $U(1)$ charges. But there remain the $SU(2) \times SU(2)$ dynamics which are strongly coupled then confined. Thus the low-energy effective description should be written in the gauge singlets composite chiral superfields. So the minimal electric charges are doubled. This again leads to the above minimal magnetic charges.

3.14 3d $\mathcal{N}=2$ $SO(N_c)$ or $O(N_c)$ SQCD

Let us consider the orthogonal gauge groups. The theory consists of the $\mathcal{N}=2$ vector multiplet with $O(N_c)$ or $SO(N_c)$ gauge groups and the N_f fundamental (vector representation) matters [59]. The quantum numbers are summarized as follows.

Table 3.9: The quantum numbers of the 3d $\mathcal{N}=2$ $O(N_c)$ SQCD

	$SU(N_f)$	$U(1)_A$	$U(1)_R$
W_α	$\mathbf{1}$	0	1
Q	N_f	1	0
$M^{ij} := Q^i_a Q^j_a$	$\mathbf{\frac{N_f(N_f+1)}{2}}$	2	0

The Higgs branch is parametrized by the gauge singer meson fields:

$$M^{ij} := Q^i_a Q^j_a. \tag{3.222}$$

For $N_f < N_c$, the gauge group is partly broken to $O(N_c - N_f)$. For $N_f \geq N_c$, the gauge group is completely broken. For $N_f \geq N_c$ and $SO(N_c)$ gauge group, we have the additional gauge singlets, "baryons":

$$B^{i_1 \cdots i_{N_c}} := \frac{1}{N_c!} Q^{i_1}_{a_1} \cdots Q^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}. \tag{3.223}$$

The Coulomb branch, on the other hand, is described by the adjoint scalars which are from the vector multiplet. For $(S)O(N_c = 2n)$, the adjoint scalar can be diagonalized. Arranging the eigenvalues, we obtain

$$\phi = \mathrm{diag}(\phi_1, \cdots, \phi_n, -\phi_1, \cdots, -\phi_n), \tag{3.224}$$

where $\phi_1 \geq \phi_2 \geq \cdots \geq \phi_{n-1} \geq |\phi_n|$. For $O(2n)$, we can additionally impose $\phi_n \geq 0$. For $SO(N_c = 2n+1)$, we have

$$\phi = \mathrm{diag}(\phi_1, \cdots, \phi_n, -\phi_1, \cdots, -\phi_n, 0), \tag{3.225}$$

where $\phi_1 \geq \phi_2 \geq \cdots \geq \phi_{n-1} \geq \phi_n \geq 0$. For the generic values of the ϕ_i's, the gauge group is broken to the maximal torus $U(1)^n$.

We define the Coulomb branch coordinates as follows. For $O(2n)$,

$$Y_1 \sim e^{\Phi_1 - \Phi_2}, \ \cdots, \ Y_{n-1} \sim e^{\Phi_{n-1} - \Phi_n}, \ Y_n \sim e^{\Phi_{n-1} + \Phi_n}. \tag{3.226}$$

For $O(2n+1)$,

$$Y_1 \sim e^{\Phi_1 - \Phi_2}, \ \cdots, \ Y_{n-1} \sim e^{\Phi_{n-1} - \Phi_n}, \ Y_n \sim e^{2\Phi_n}. \tag{3.227}$$

The monopole background mixes the $U(1)_J$ symmetries dual to the $U(1)^n$ with the other $U(1)$ symmetry, $U(1)_A$ and $U(1)_R$. Then these operators have the non-trivial charges under the global $U(1)$ symmetries. For $O(2n)$ and $\phi_n > 0$, the quantum numbers of these monopole operators are as follows.

Table 3.10: The quantum numbers with $O(2n)$, $\phi_n > 0$

	$SU(N_f)$	$U(1)_A$	$U(1)_R$
Y_i $(i = 1, \cdots, n-1)$	$\mathbf{1}$	0	-2
Y_n	$\mathbf{1}$	$-2N_f$	$2N_f - 2$

For $O(2n)$ and $\phi_n < 0$,

Table 3.11: The quantum numbers with $O(2n)$, $\phi_n < 0$

	$SU(N_f)$	$U(1)_A$	$U(1)_R$
Y_i $(i = 1, \cdots, n-2, n)$	$\mathbf{1}$	0	-2
Y_{n-1}	$\mathbf{1}$	$-2N_f$	$2N_f - 2$

For $O(2n+1)$,

Table 3.12: The quantum numbers with $O(2n+1)$

	$SU(N_f)$	$U(1)_A$	$U(1)_R$
Y_i $(i = 1, \cdots, n-1)$	$\mathbf{1}$	0	-2
Y_{n-1}	$\mathbf{1}$	$-2N_f$	$2N_f - 2$

Chapter 4

Three-dimensional dualities

We will here study the several three-dimensional dualities: Mirror symmetries, Aharony duality, Kim-Park duality and Giveon-Kutasov duality with attention to the $SU(N_c)$ or $U(N_c)$, $O(N_c)$ gauge groups. Some of these dualities are derived from the 4d dualities in the next chapter. We will also explain the un-gauging technique to obtain the $SU(N_c)$ duality from the $U(N_c)$ duality. Notice again that the $U(1)$ part becomes non-trivial in 3d since the gauge coupling inevitably becomes strong in the IR even if $U(1)$.

4.1 The mirror symmetry

Although we are interested in the 3d Seiberg duality which connects the Higgs branch of the original theory to the Higgs branch of the dual theory, we first mention the three-dimensional mirror symmetry [42, 60, 61] with $\mathcal{N}=2$ supersymmetry because it serves as the good example of studying the quantum dynamics of the 3d SUSY gauge theories and it will be extensively used for deriving the 3d duality from the 4d duality.

4.1.1 The mirror of the XYZ model

We first consider the most simplest mirror symmetry. In this case the mirror theory does not contain the gauge degrees of freedom at all.

XYZ model

The XYZ model is a theory with three chiral superfields with the following superpotential and with no gauge field.

$$W = -XYZ, \tag{4.1}$$

which is consistent with the global symmetries:

Table 4.1: The global charges of the XYZ model

	$U(1)_A$	$U(1)_J$	$U(1)_{R'}$
X	2	0	2/3
Y	-1	$+1$	2/3
Z	-1	-1	2/3

The R-charges of the fields X, Y, Z are identical due to the $\mathbb{Z}_3$-symmetry which exchange these fields each other. Other $U(1)$ transformations rotate the chiral superfields X, Y, Z respectively. Then if there is no superpotential, we have three Abelian global symmetries $U(1)_{X,Y,Z}$ aside from the $U(1)_R$ symmetry. However the presence of the superpotential explicitly breaks the one of these three global symmetries. Then we have only two global Abelian symmetries $U(1)_{A,J}$.

The F-flatness condition reads

$$\frac{\partial W}{\partial X} = \frac{\partial W}{\partial Y} = \frac{\partial W}{\partial Z} = 0 \tag{4.2}$$

$$\longleftrightarrow yz = xz = xy = 0 \tag{4.3}$$

$$\longleftrightarrow y = z = 0 \quad \text{or} \quad x = z = 0 \quad \text{or} \quad x = y = 0, \tag{4.4}$$

where the lower-case letters x, y, z denote the scalar components of the chiral superfields X, Y, Z. The moduli space becomes as follows.

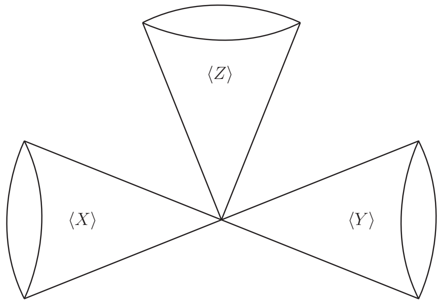

Figure 4.1: The (quantum) moduli space of the XYZ model

At the origin of the moduli space the theory flows to the IR fixed point where the theory would be a superconformal field theory.

3d $\mathcal{N} = 2$ SQED with $N_f = 1$ flavors

The mirror pair of the 3d $\mathcal{N} = 2$ XYZ model is the 3d $\mathcal{N} = 2$ SQED with $N_f = 1$ flavors. The field contents are as follows.

- $U(1)$ vector multiplet which consists of a vector field, a real scalar, Weyl fermion (two Majorana (real) fermions).
- The chiral superfields $Q, \tilde{Q}$. These are charged under the $U(1)$ gauge symmetry as ± 1 respectively.

Table 4.2: The quantum numbers of the 3d $\mathcal{N} = 2$ SQED with one flavor

	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	1	0	0
$\tilde{Q}$	1	0	0
M	2	0	0
$V_\pm$	-1	± 1	1

The Higgs branch is complex one-dimensional, which is labeled by $M = Q\tilde{Q}$. On the other hand the Coulomb branch is parametrized by the $V_\pm \sim \exp(\pm\Phi/g^2)$, $\Phi = \phi + ia$. These three moduli $M, V_\pm$ meet at the origin of the moduli spaces, which is effectively described by the superpotential

$$W = -MV_+V_-. \tag{4.5}$$

which is precisely the XYZ model. We can see the parity anomaly matching with the XYZ model. Please see the previous chapter.

The $U(1)_R$ charge here is different from the $U(1)_{R'}$ charge in the XYZ model. They are related each other by

$$R' = R + \frac{1}{3}A \tag{4.6}$$

where $R,\ R'$ are the R-charges and A is the $U(1)_A$ charges. In this charge assignment, the operator matching is

$$X \sim M, \quad V_+ \sim Y, \quad V_- \sim Z \tag{4.7}$$

although this assignment is arbitrary because the XYZ model has the $\mathbb{Z}_3$-symmetry.

4.1.2 The mirror of the 3d $\mathcal{N}=2$ SQED with N_f flavors

3d $\mathcal{N}=2$ SQED with N_f flavors

This theory was deeply studied in [42, 60]. It consists of the N_f flavors of the vector-like chiral matters $Q, \tilde{Q}$ with charge ± 1 and a photon, a gaugino. The global charges are following.

Table 4.3: The global charges of the 3d $\mathcal{N}=2$ SQED with N_f flavors

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	1	1	0	0
$\tilde{Q}$	1	$\bar{N_f}$	1	0	0
$M := Q^i\tilde{Q}_j$	N_f	$\bar{N_f}$	2	0	0
$V_\pm$	1	1	$-N_f$	± 1	N_f

Through the non-perturbative effect, the superpotentials are generated depending on the number of the flavors. For $N_f = 1$,

$$W = -MV_+V_- \tag{4.8}$$

and for $N_f > 1$

$$W = -N_f(V_+V_- \det M)^{\frac{1}{N_f}}. \tag{4.9}$$

These are consistent with the above assignment of the quantum numbers.

The mirror theory

The mirror description of the 3d $\mathcal{N}=2$ SQED with N_f flavors is given as follows.

- 3d $\mathcal{N}=2$ SUSY and $U(1)^{N_f}/U(1)$ gauge groups.
- N_f chiral superfields q_i $(i = 1, \cdots, N_f)$. q_i is charged as $+1$ under the $U(1)_i$ and -1 under the $U(1)_{i+1}$. Then the one of the $U(1)$s is not coupled to the matter fields.
- singlets: S_i $(i = 1, \cdots, N_f)$
- superpotential: $W = \sum_i S_i q_i \tilde{q}^i$

Table 4.4: The local $U(1)^{N_f}$ charges

	$U(1)_1$	$U(1)_2$	$U(1)_3$	$\cdots$	$U(1)_{N_f-1}$	$U(1)_{N_f}$	$\sum_{i=1}^{N_f} U(1)_i$
q_1	1	-1	0	$\cdots$	0	0	0
q_2	0	1	-1	$\cdots$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
q_{N_f}	-1	0	0	$\cdots$	0	1	0

In the original side, the moduli space is spanned by

$$\begin{aligned} \text{Higgs branch}:\ & M_i^j \qquad \text{dim.} = 2N_f - 1 \\ \text{Coulomb branch}:\ & V_\pm. \end{aligned}$$

On the mirror dual, the moduli space is spanned by

$$\begin{aligned} \text{Higgs branch}:\ & N_\pm \\ \text{Coulomb branch}:\ & W_{i,\pm} \qquad \text{dim.} = N_f - 1 \\ \text{Additional branch}:\ & S_i \qquad \text{dim.} = N_f. \end{aligned}$$

The operator matching is as follows.

$$V_\pm = N_\pm \tag{4.10}$$

$$M_i^{i-1} \sim W_{i,+}, \quad M_{i-1}^i \sim W_{i,-} \tag{4.11}$$

where

$$N_- := q_1 q_2 \cdots q_{N_f} \tag{4.12}$$

$$N_+ := \tilde{q}^1 \tilde{q}^2 \cdots \tilde{q}^{N_f}. \tag{4.13}$$

and $W_{j,\pm}$ is the Coulomb branch coordinate of the $U(1)^{N_f}/U(1)$.

4.2 Karch duality

This is the first Seiberg-like duality discussed in 3d [62] without Chern-Simons term. In this duality the Higgs and Coulomb branch structure are not interchanged. In this sense, the Karch duality is not a mirror symmetry. This duality has the $Sp(2N_c)$ gauge group. So we first review the quantum dynamics of the $3d$ $\mathcal{N} = 2$ supersymmetric $Sp(2N_c)$ gauge theories with N_f flavors.

The $Sp(2N_c)$ gauge theories also have the Higgs and Coulomb branches. The Higgs branch is parametrized by the mesons:

$$M_{ij} = \omega_{ab} Q_i^a Q_j^b, \tag{4.14}$$

where ω_{ab} is the invariant tensor of the $Sp(2N_c)$ group. This anti-symmetric tensor satisfies

$$g^T \omega g = \omega, \quad \omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \quad g \in Sp(2N_c). \tag{4.15}$$

We need not to take into account the baryonic operators which are expressed by the mesons. The Coulomb branch is parametrized by the adjoint scalar ϕ and the dual photons. The adjoint scalar can be diagonalized and arranged as

$$\phi = \text{diag.}(\phi_1, \cdots, \phi_{N_c}, -\phi_1, \cdots, -\phi_{N_c}), \quad \phi_1 \geq \phi_2 \geq \cdots \geq \phi_{N_c} \geq 0 \tag{4.16}$$

The adjoint scalar and dual photons are usually combined into the complex scalars:

$$\Phi_i = \phi_i + i\gamma_i \tag{4.17}$$

The instanton factors are as follows.

$$Y_J \sim e^{\Phi_j - \Phi_{j+1}}, \quad j = 1, \cdots, N_c - 1 \tag{4.18}$$

$$Y_{N_c} \sim e^{2\Phi_{N_c}} \tag{4.19}$$

The quantum numbers of the fields are as follows.

Table 4.5: Th quantum numbers of the 3d $\mathcal{N} = 2$ $Sp(2N_c)$ gauge theory with N_f flavors

	$SU(2N_f)$	$U(1)_A$	$U(1)_R$
Q	$\mathbf{2N_f}$	1	0
M	$\mathbf{N_f(2N_f-1)}$	2	0
$Y := \prod Y_j$	1	$-2N_f$	$2(N_f - N_c)$

Next, we will study the quantum moduli structure.

$N_f = 0$

For the case of the pure SYM, two gaugino zero modes are always present. So the classical Coulomb branch is all lifted.

$$W = \sum_{j=1}^{N_c} \frac{1}{Y_j} \tag{4.20}$$

Then there is no stable vacua.

$0 < N_f < N_c$

The holomorphy argument enable us to produce the following superpotential:

$$W = (N_c - Nf)(Y\mathrm{Pf}M)^{\frac{1}{N_f - N_c}} \tag{4.21}$$

$N_f = N_c$

The classical moduli space is deformed and there is a merging of the Coulomb and Higgs branches.

$$Y\mathrm{Pf}M = 1 \tag{4.22}$$

So the large Y region corresponds to the small M region.

$N_f = N_c + 1$

The holomorphy argument gives the following superpotential:

$$W = -Y\mathrm{Pf}M. \tag{4.23}$$

The theory is described by the chiral superfields, M and Y even at the origin of the moduli space. Then the theory is (s-)confining-like.

$N_f \geq N_c + 2$

At the origin of the moduli space of vacua, where the Higgs and Coulomb branches meet together, we have the interacting superconformal field theory. Then we need some new degrees of freedom at the origin. Karch offered the dual theory which flows to the same IR fixed point [62].

The dual theory is 3d $\mathcal{N} = 2$ supersymmetric $Sp(2(N_f - N_c - 1))$ gauge theory. The field contents are summarized as follows. In the Karch duality there is an internal $SU(2)$ symmetry.

Table 4.6: The quantum numbers of the magnetic side

	$Sp(2(N_f-N_c-1))$	$SU(2N_f)$	$U(1)_A$	$U(1)_R$	$SU(2)$
q	$\square$	$\mathbf{2N_f}$	-1	1	1
t	$\square$	$\mathbf{1}$	N_f	N_c-N_f+1	$\mathbf{2}$
M	1	$\mathbf{N_f(2N_f-1)}$	2	0	1
Y	1	$\mathbf{1}$	$-2N_f$	$2(N_f-N_c)$	1
$\tilde{Y}$	1	$\mathbf{1}$	0	2	1

The dual theory contains the superpotential:

$$W = Mqq + Ytt + \tilde{Y} \tag{4.24}$$

which is needed for lifting the unnecessary moduli on the dual side.

4.3 Aharony duality

We here study about the Aharony duality [59, 63] which is the duality between the 3d $\mathcal{N}=2$ supersymmetric gauge theories with $USp(2N_c)$, $U(N_c)$ and $O(N_c)$ gauge groups. Unlike the Karch duality, the magnetic side of the Aharony duality does not contain the internal $SU(2)$ symmetry whose dynamics makes the theory involved.

4.3.1 $USp(2N_c)$ Aharony duality

We will first discuss the duality with the $USp(2N_c)$ gauge symmetry because in the $USp(2N_c)$ theory we need not to care about the baryonic operators and hence it is simple.

Electric theory

The original theory is the 3d $\mathcal{N}=2$ supersymmetric $USp(2N_c)$ gauge theory with $2N_f$ fundamental flavors. This is the same theory as the electric side of the Karch duality.

Table 4.7: The quantum numbers of the electric side

	$SU(2N_f)$	$U(1)_A$	$U(1)_R$
Q	$\mathbf{2N_f}$	1	0
M	$\mathbf{N_f(2N_f-1)}$	2	0
$Y := \prod Y_j$	1	$-2N_f$	$2(N_f-N_c)$

Let us study the moduli space of vacua on the electric side. For $Y=0$, the rank of the meson composite $M := Q_iQ_j$ generically satisfies

$$\mathrm{rank}\, M \le 2N_c. \tag{4.25}$$

For $Y \neq 0$,

$$\mathrm{rank}\, M \le 2(N_c-1). \tag{4.26}$$

Magnetic theory

The dual theory is the 3d $\mathcal{N}=2$ supersymmetric $USp(2(N_f-N_c-1))$ gauge theory with $2N_f$ fundamental dual flavors and the singlets M, Y which are naturally identified with the mesons and the Coulomb branch coordinate on the electric side. The dual theory contains the superpotential.

$$W = Mqq + \tilde{Y}Y \tag{4.27}$$

where $\tilde{Y}$ is the monopole operator of the dual gauge group. The quantum numbers are as follows.

Table 4.8: The quantum numbers of the magnetic side

	$SU(2N_f)$	$U(1)_A$	$U(1)_R$
q	$\mathbf{2N_f}$	-1	0
M	$\mathbf{N_f(2N_f-1)}$	2	0
Y	1	$-2N_f$	$2(N_f-N_c)$
$\tilde{Y}$	1	$2N_f$	$-2(N_f-N_c-1)$

Naively the equation of motion for $\tilde{Y}$ gives $Y=0$. However it is not correct since $\tilde{Y}$ becomes a fundamental field at the origin of the moduli space: $\tilde{Y}=0$ and since we have no good description of the moduli space at the origin [63]. At the region of the moduli space far away from the origin, we have the effective superpotential

$$W_{eff} = \left(\tilde{Y}\mathrm{Pf}(q_i q_j)\right)^{\frac{1}{N_f-N_c-1}}, \tag{4.28}$$

which is not good at the origin of the moduli space because this superpotential is singular at the origin and we need some new massless degrees of freedom at the origin.

Let us check the moduli space of vacua on the dual side:

$$W_{\mathsf{total}} = Mqq + \tilde{Y}Y + \left(\tilde{Y}\mathrm{Pf}(q_i q_j)\right)^{\frac{1}{N_f-N_c-1}}. \tag{4.29}$$

First the equation of motion for Y gives $\tilde{Y}=0$ which means that the Coulomb branch on the dual side is lifted. On the other hand, the equation of motion for $\tilde{Y}$ gives a vev for Y. This is a consistent picture with the electric side.

For rank $M=2N_c$, the low-energy theory is the $USp(2(N_f-N_c-1))$ gauge theory with $2(N_f-N_c)$ flavors. Then the low-energy effective superpotential becomes

$$W = Mqq + \tilde{Y}Y - \tilde{Y}\mathrm{Pf}(q_i q_j). \tag{4.30}$$

The corresponding equations of motion are

$$Y:\ \tilde{Y}=0, \tag{4.31}$$

$$\tilde{Y}:\ Y=\mathrm{Pf}(q_i q_j), \tag{4.32}$$

$$M_{ij}:\ q_i q_j = 0. \tag{4.33}$$

Then these equations result in $Y=0$ which is consistent with the electric side.

For rank $M=2(N_c+1)$, the low-energy theory is the $USp(2(N_f-N_c-1))$ gauge theory with $2(N_f-N_c-1)$ flavors. Then the effective superpotential reduces to

$$W = Mqq + \tilde{Y}Y + \lambda(\tilde{Y}\mathrm{Pf}(q_i q_j) - 1) \tag{4.34}$$

whose equations of motion are

$$Y:\ \tilde{Y}=0, \tag{4.35}$$

$$\tilde{Y}:\ Y+\lambda\mathrm{Pf}(q_i q_j)=0, \tag{4.36}$$

$$\lambda:\ \tilde{Y}\mathrm{Pf}(q_i q_j) - 1 = 0. \tag{4.37}$$

$$\tag{4.38}$$

Then there is no solution and rank $M=2(N_c+1)$ is not a vacuum, which is again identical to the electric side.

4.3.2 $U(N_c)$ Aharony duality

Next, we will consider the duality with the $U(N_c)$ gauge symmetry.

Electric theory

The original theory consists of

- 3d $\mathcal{N} = 2$ SUSY and $U(N_c)$ gauge groups.
- N_f chiral multiplets $Q, \tilde{Q}$ which are the vector-like matters.

The Higgs branch is parameterized by $M = Q\tilde{Q}$ and the Coulomb branch is given by two chiral superfields (so-called monopole operators) $V_\pm$ which is un-lifted by the monopole-instanton corrections.

Table 4.9: The quantum numbers of the electric side

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	1	1	0	0
$\tilde{Q}$	1	$\bar{N}_f$	1	0	0
M	N_f	$\bar{N}_f$	2	0	0
$V_\pm$	1	1	$-N_f$	± 1	$N_f - N_c + 1$

Magnetic theory

The dual theory consists of

- 3d $\mathcal{N} = 2$ SUSY and $U(N_f - N_c)$ gauge groups.
- N_f chiral multiplets $q, \tilde{q}$ which are the vector-like matters.
- singlets: $M, V_\pm$
- superpotential: $W = Mq\tilde{q} + V_+\tilde{V}_- + V_-\tilde{V}_+$

where the fields appearing in the superpotential are the monopole operators of the dual gauge group. So this duality is not UV-completed and it is only IR expression of the duality. But it is sufficient because the Seiberg-type duality is always IR duality. Then it is okay that the various types of UV completions exist.

Table 4.10: The quantum numbers of the magnetic side

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	N_f	1	-1	0	0
$\tilde{q}$	1	$\bar{N}_f$	-1	0	0
M	N_f	$\bar{N}_f$	2	0	0
$V_\pm$	1	1	$-N_f$	± 1	$N_f - N_c + 1$
$\tilde{V}_\pm$	1	1	N_f	± 1	$N_c - N_f + 1$

If the rank of the singlets $\langle M \rangle$ is N_c, we have at low-energy $U(N_f - N_c)$ gauge theory with $N_f - N_c$ flavors and the superpotential:

$$W = Mq\tilde{q} + V_+\tilde{V}_- + V_-\tilde{V}_+ + \tilde{V}_+\tilde{V}_- \det \tilde{q}q, \tag{4.39}$$

where the last term comes due to the dynamics of $N_f = N_c$.

The equations of motion are

$$V_+ = \tilde{V}_+ \det \tilde{q}q = 0 \tag{4.40}$$
$$V_- = \tilde{V}_- \det \tilde{q}q = 0. \tag{4.41}$$

For rank $M = N_c - 1$, the low-energy theory is $U(N_f - N_c)$ with $N_f - N_c + 1$ flavors. Then the effective description of the moduli is given by

$$W = Mq\tilde{q} + V_+\tilde{V}_- + V_-\tilde{V}_+ + \left(\tilde{V}_+\tilde{V}_- \det \tilde{q}q\right)^{\frac{1}{(N_f-N_c+1)-(N_f-N_c)+1}}. \tag{4.42}$$

The corresponding equations of motion are

$$V_+ + \frac{1}{2}\tilde{V}_-^{-\frac{1}{2}}\left(\tilde{V}_+ \det \tilde{q}q\right)^{\frac{1}{2}} = 0 \tag{4.43}$$
$$V_- + \frac{1}{2}\tilde{V}_+^{-\frac{1}{2}}\left(\tilde{V}_- \det \tilde{q}q\right)^{\frac{1}{2}} = 0 \tag{4.44}$$

which mean

$$V_+V_- = \frac{1}{4}\det \tilde{q}q = 0 \tag{4.45}$$

The last equality comes from the equation of motion for M. Then one of $V_\pm$ can be non-zero, which is consistent with the electric side.

For rank $\langle M \rangle > N_c$, the instanton correction lifts all the moduli space and there is no stable SUSY vacua.

The twisted instanton

When we consider these theories on $\mathbb{S}^1 \times \mathbb{R}^3$, we have the additional superpotential from the KK-monopoles.

$$W = \eta V_+ V_- \tag{4.46}$$

where $\eta = \mathrm{e}^{-\frac{1}{g_4^2}} \sim \mathrm{e}^{-\frac{1}{rg_3^2}}$ is the dynamical scale of the corresponding 4d theory.

The examples for small colors and flavors

Let us consider the case with $(N_c, N_f) = (1, 2)$. The magnetic theory is also the $U(1)$ with the singlets $V_\pm, M_i^j$ with the following superpotential

$$W = M_i^j \tilde{q}_j q^i + V_+\tilde{V}_- + V_-\tilde{V}_+. \tag{4.47}$$

The electric theory also has the mirror description by the 3d $\mathcal{N} = 2$ $U(1)$ gauge theory with two flavors and two singlets S_1, S_2 and with the superpotential:

$$W = \sum_{i=1,2} S_i \tilde{q}_i q^i. \tag{4.48}$$

4.3.3 $O(N_c)$ Aharony duality

Let us consider the Aharony duality with the $O(N_c)$ gauge symmetry.

Electric theory

The original theory consists of

- 3d $\mathcal{N} = 2$ SUSY and $O(N_c)$ gauge groups.
- N_f chiral multiplets Q_a^i which are vector representations.

The Higgs branch is parameterized by $M^{ij} := Q^i_a Q^j_a$ and there is no baryonic operator such as $B^{i_1 \cdots i_{N_c}} := Q^{i_1}_{a_1} \cdots Q^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}$. The Coulomb branch is given by one chiral superfield Y, which is un-lifted by the monopole-instanton correction.

Table 4.11: The quantum numbers of the electric side

	$SU(N_f)_L$	$U(1)_A$	$U(1)_R$
Q	$\mathbf{N_f}$	1	0
M	$\frac{1}{2}\mathbf{N_f}(\mathbf{N_f}+\mathbf{1})$	2	0
Y	$\mathbf{1}$	$-N_f$	$N_f - N_c + 2$

Magnetic theory

The dual theory consists of

- 3d $\mathcal{N}=2$ SUSY and $O(N_f - N_c + 2)$ gauge groups.
- N_f chiral multiplets q which are the vector (fundamental) representations.
- elementary gauge-singlet chiral superfields: M, Y
- superpotential: $W = \frac{1}{2} M^{ij} q_i q_j + \tilde{Y} Y$

where the fields appearing in the superpotential are the monopole operators of the dual gauge group. So this duality is not UV-completed and it is only the IR expression of the duality. The quantum numbers for these fields summarized below.

Table 4.12: The quantum numbers of the magnetic side

	$SU(N_f)_L$	$U(1)_A$	$U(1)_R$
q	$\bar{\mathbf{N}}_\mathbf{f}$	-1	0
M	$\frac{1}{2}\mathbf{N_f}(\mathbf{N_f}+\mathbf{1})$	2	0
Y	$\mathbf{1}$	$-N_f$	$N_f - N_c + 2$
$\tilde{Y}$	$\mathbf{1}$	N_f	$N_c - N_f$

4.4 Kim-Park duality

We will here include an adjoint matter field to the above Aharony duality, which is called Kim-Park duality [64]. In this case, the monopole operators are a bit intricate [64, 65]. We denote the bare monopole corresponding to the magnetic flux $|\pm 1, 0, \cdots, 0\rangle$ as $v_{0,\pm}$. We in addition define the dressed monopole operators.

$$v_{i,\pm} := v_{0,\pm} (X_{11})^i \tag{4.49}$$

In the state formalism, the adjoint scalar excitations are

$$v_{1,\pm} = X_{11} | \pm 1, 0, \cdots, 0\rangle \tag{4.50}$$

$$\mathrm{Tr} X' | \pm 1, 0, \cdots, 0\rangle \tag{4.51}$$

where the adjoint scalar field is

$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & X' \end{pmatrix}. \tag{4.52}$$

The operator product $\mathrm{Tr}X' v_{0,\pm}$ is generated by $v_{1,\pm}$ and $v_{0,\pm}\mathrm{Tr}X$. This means that the independent monopole operators are

$$v_{i,\pm} = (X_{11})^i v_{0,\pm}. \tag{4.53}$$

The duality was conjectured and guessed by the resemblance of the 4d Kutasov-Schwimmer duality and the 3d $U(N_c)$ Aharony duality.

Electric theory

The original theory consists of

- 3d $\mathcal{N}=2$ SUSY and $U(N_c)$ gauge groups.
- N_f chiral multiplets $Q, \tilde{Q}$ which are the vector-like matters.
- an adjoint chiral superfield: X
- the superpotential: $W = \mathrm{Tr}X^{k+1}$

The superpotential for the adjoint matter truncates the chiral ring at $O(X^{k-1})$. The global charges are as follows, where we keep the R-charges of the quarks as the generic values since we do not know the true values which are realized in the IR.

Table 4.13: The quantum numbers of the electric side

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	1	1	0	r
$\tilde{Q}$	1	$\bar{N}_f$	1	0	r
X	1	1	0	0	$\frac{2}{k+1}$
$M_j := \tilde{Q}X^jQ$	N_f	$\bar{N}_f$	2	0	0
$V_{j,\pm}$	1	1	$-N_f$	±1	$N_f(1-r) - \frac{2}{k+1}(N_c-1) + \frac{2j}{k+1}$

Magnetic theory

The dual theory consists of

- 3d $\mathcal{N}=2$ SUSY and $U(kN_f - N_c)$ gauge groups.
- N_f chiral multiplets $q, \tilde{q}$ which are the vector-like matters.
- an adjoint matter: Y
- singlets: $M_j, V_{j,\pm}$ $(j = 0, \cdots, k-1)$
- The superpotential: $W = \mathrm{Tr}\, Y^{k+1} + \sum_{j=0}^{k-1} \left(M_j \tilde{q} Y^{k-1-j} q + V_{j,+}\tilde{V}_{k-1-j,-} + V_{j,-}\tilde{V}_{k-1-j,+} \right)$

where $\tilde{V}_{j,\pm}$ are the monopole operators on the dual side. The singlets M_j are identified with the mesonic operator $\tilde{Q}X^jQ$ on the original side and the singlets $V_{j,\pm}$ are the monopole operators on the original side. The Aharony duality type superpotential on the magnetic side lifts the Coulomb branch of the magnetic side.

Table 4.14: The quantum numbers of the magnetic side

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	1	$\bar{N}_f$	-1	0	$-r+\frac{2}{k+1}$
$\tilde{q}$	N_f	1	-1	0	$-r+\frac{2}{k+1}$
Y	1	1	0	0	$\frac{2}{k+1}$
M_j	N_f	$\bar{N}_f$	2	0	0
$V_{j,\pm}$	1	1	$-N_f$	± 1	$N_f(1-r)-\frac{2}{k+1}(N_c-1)+\frac{2j}{k+1}$
$N_j := \tilde{q}V^j q$	N_f	$\bar{N}_f$	-2	0	$-2r+\frac{4}{k+1}+\frac{2j}{k+1}$
$\tilde{V}_{j,\pm}$	1	1	N_f	± 1	$N_f(r-1)+\frac{2}{k+1}(N_c+1)+\frac{2j}{k+1}$

Matching of the chiral rings

The chiral ring structures of the both theories are quite complicated compared with the duality with the fundamental quarks. Actually we need to compute the superconformal indices on both sides. But naively we can match the operators on both sides as follows:

$$\mathrm{Tr}\, X^j \leftrightarrow \mathrm{Tr}\, Y^j \tag{4.54}$$

$$\tilde{Q} X^j Q \leftrightarrow M_j \tag{4.55}$$

$$\mathrm{Tr}\left(V_{0,\pm}^{U(N_c)} X^j\right) \leftrightarrow V_{j,\pm} \tag{4.56}$$

We should note that the $\mathrm{Tr}\, X^j$ are not all independent each other because they are related by the characteristic equation and they are constrained by the superpotential $W = \mathrm{Tr}\, X^{k+1}$. So the independent set is defined as $\mathrm{Tr}\, X^j,\ j = 1, \cdots, \min(k-1, N_c)$.

4.5 Un-gauging

Here we will introduce the un-gauging technique [34, 66] to obtain the $SU(N_c)$ gauge theory from the $U(N_c)$ one. If we consider the theory with the $U(1)$ gauge groups as the subgroups, let's say $U(N_c)$, the theory have the global topological $U(1)_J$ symmetries which is Hodge-dual to the $U(1)$ part of the gauge groups: $F_{\mu\nu} = \mathrm{tr}\, F^a_{\mu\nu} T^a$. We can gauge this $U(1)_J$ symmetry which introduces the following term.

$$A^J_\mu J^{J,\mu} \sim A^J_\mu \epsilon^{\mu\nu\rho} F^{U(1)}_{\nu\rho} \tag{4.57}$$

which is precisely the mixed Chern-Simons term for the $U(1) \subset U(N_c)$ and $U(1)_J^{gauged}$. This term lift the Coulomb branch at the low-energy and the resulting theory is a $U(N_c) \times U(1)_J \to SU(N_c)$ gauge theory. This technique can be used for deriving the $SU(N_c)$ dualities from the $U(N_c)$ dualities.

4.6 Giveon-Kutasov duality

The Giveon-Kutasov duality is the duality with the Chern-Simons term [67]. We here consider the CS gauge theories with only the fundamental matters or with both the fundamental and adjoint matters.

4.6.1 Giveon-Kutasov duality

Electric side

The original theory consists of

- 3d $\mathcal{N} = 2$ SUSY and $U(N_c)_k$ gauge groups, where the index means the CS level.

- N_f flavors $Q, \tilde{Q}$ which are the vector-like matters.

The theory is supersymmetric if and only if $N_f + k - N_c \geq 0$ [67, 68]. The quantum numbers for the electric side are as follows. We should notice that since the theory contains the $U(N_c)$ gauge symmetry, so we have the topological $U(1)_J$ global symmetry. But all the fields in the UV Lagrangian are not charged under the $U(1)_J$ symmetry.

Table 4.15: The quantum numbers of the electric side

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_R$
Q	N_f	1	1	0
$\tilde{Q}$	1	$\bar{N}_f$	1	0

For $N_f < N_c$, the gauge group is generically broken from $U(N_c)$ to $U(N_c - N_f)$. Then the low-energy theory contains the

$$\underbrace{2N_fN_c}_{Q,\tilde{Q}} - \underbrace{\left[\overbrace{N_c^2}^{U(N_c)} - \overbrace{(N_c - N_f)^2}^{U(N_c-N_f)}\right]}_{\#\text{ of massive gauge fields}} = N_f^2 \tag{4.58}$$

chiral superfields which are given by mesons M. For $N_f \geq N_c$, the gauge group is completely broken and the

$$2N_fN_c - N_c^2 \tag{4.59}$$

mass less chiral superfields remain.

Magnetic side

The dual theory is

- 3d $\mathcal{N} = 2$ SUSY and $U(N_f - N_c + |k|)_{-k}$ gauge groups.
- N_f flavors $q, \tilde{q}$.
- singlets M
- superpotential $W = M\tilde{q}q$.

The quantum numbers for the magnetic side are as follows.

Table 4.16: The quantum numbers of the magnetic side

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_R$
q	$\bar{N}_f$	1	-1	1
$\tilde{q}$	1	N_f	-1	1
M	N_f	$\bar{N}_f$	2	0

For $N_f < N_c$, the magnetic theory has the moduli labeled by M, where M can take the arbitrary vevs. On the other hand, the dual quarks are generically all massive due to the superpotential. Then there is no flat direction for $q, \tilde{q}$.

For $N_f \geq N_c$, the argument becomes a bit complicated. Let us consider the region such as $\mathrm{rank}\, M > N_c$. The theory flows to the $U(N_f - N_c + |k|)_{-k}$ gauge theory with $N_f - \mathrm{rank}\, M$ flavors. Then the magnetic theory is supersymmetric if and only if

$$|k| + \underbrace{(N_f - \mathrm{rank}\, M)}_{\#\,\text{of flavors}} - \underbrace{(N_f - N_c + |k|)}_{\#\,\text{of colors}} = N_c - \mathrm{rank}\, M \geq 0. \tag{4.60}$$

Then we obtain the constraint $\mathrm{rank}\, M \leq N_c$ on the dual side.

4.6.2 Giveon-Kutasov duality with adjoint matter

We can include one adjoint matter into the Giveon-Kutasov duality [69, 70].

Electric side

The original theory consists of

- 3d $\mathcal{N} = 2$ SUSY and $U(N_c)_k$ gauge groups, where the index means the CS level.
- N_f flavors $Q, \tilde{Q}$ which are the vector-like matters.
- an adjoint chiral superfield X
- superpotential $W = \mathrm{tr} X^{n+1}$

Magnetic side

The dual theory is

- 3d $\mathcal{N} = 2$ SUSY and $U(nN_f - N_c + n|k|)_{-k}$ gauge groups.
- N_f flavors $q, \tilde{q}$.
- an adjoint matter Y
- singlets M_j $(j = 0, \cdots, n-1)$
- superpotential $W = \mathrm{tr} Y^{n+1} + \sum_{j=0}^{n-1} M_j \tilde{q} Y^{n-1-j} q$.

4.7 Flow between Aharony and Giveon-Kutasov dualities

Note that the Aharony duality and the Giveon-Kutasov duality are related to each other by the RG flow [46]. This gives the UV completion of the Aharony duality and explains the mysterious superpotential of the Aharony magnetic theory.

4.7.1 GK from Aharony

Let us start with the Aharony duality with $N_f + |k|$ flavors. We can add the real masses for the $|k|$ flavors in equal sign, say $m, \tilde{m} > 0$. On the electric side, we have the $U(N_c)$ with $N_f + |k|$ flavors which flows to the 3d $\mathcal{N} = 2$ $U(N_c)_k$ SUSY Chern-Simons-Matter theory with N_f flavors. The introduction of the Chern-Simons level is due to the fact that the signs of the real masses for the fundamental quarks and the anti-fundamental quarks are identical. Then the integration of the massive quarks introduces the Chern-Simons level.

On the dual side, all the signs of the real masses flip because these real masses are the background gauging of the $SU(N_f + |k|) \times SU(N_f + |k|) \times U(1)_A$. Then the low energy theory on the dual side is 3d $\mathcal{N} = 2$ $U(N_f + |k| - N_c)_{-k}$ SUSY Chern-Simons-Matter theory with the N_f dual flavors and with the superpotential

$$W = M \tilde{q} q. \tag{4.61}$$

where the monopole operators $\tilde{V}_{\pm}$ describing the Coulomb branch of the magnetic side and the gauge singlet chiral superfields $V_{\pm}$ are all charged under the $U(1)_A$ axial symmetry. So these fields are massive under the background gauging of the $U(1)_A$ symmetry and they are decoupled from the low-energy dynamics.

4.7.2 Aharony from GK

Let us alternatively start with the Giveon-Kutasov duality with the gauge group $U(N_c)_{k=-1}$ with the N_f+1 flavors. The dual theory is the $U(N_f+2-N_c)_{k'=1}$ with the N_f+1 flavors and the meson singlets. If we turn on the real masses for the last flavor in the same sign for the fundamental and anti-fundamental quarks, the electric side flows to the $U(N_c)_{k=0}$ with N_f flavors. In this process, the low-energy theory contains the Coulomb branch which is absent at high-energy theory. In this deformation, the low-energy limit of the magnetic theory should be taken at $\phi_{N_f-N_c+1} = - - \phi_{N_f-N_c+2} = m$. Then the dual gauge group $U(N_f-N_c+2)$ is broken to

$$U(N_f-N_c+2)_{k=-1} \to U(N_f-N_c)_{k=0} \times U(1)_{-\frac{1}{2}} \times U(1)_{-\frac{1}{2}}. \tag{4.62}$$

The two $U(1)$ theories are dual to the theories with one chiral superfield which we denote as $X_{\pm}$. The Affleck-Harvey-Witten type superpotential gives the following superpotential:

$$W_{\mathsf{AHW}} = X_+ V_-^{U(N_f-N_c)} + X_- V_+^{U(N_f-N_c)} \tag{4.63}$$

which derives the Aharony dual theory.

Chapter 5

3d Seiberg duality from the conventional 4d Seiberg duality

In this chapter we will explain how to derive the 3d Seiberg duality from the conventional 4d Seiberg duality which includes only the fundamental matters. This was first proposed by Aharony et al. [34], where the gauge groups $SU(N_c), U(N_c), Sp(2N_c)$ are considered. This was immediately generalized to the $SO(N_c)$ gauge group in [35].

5.1 The general procedure of deriving the 3d dualities

In [34], their authors claimed that the following steps are important for deriving the 3d dualities from the 4d dualities by the dimensional reduction. These three steps are generic for the diverse 4d dualities, but the specific procedures differ spending on the matter contents and the gauge groups.

- We need to put the theory on $\mathbb{S}^1$ and include the non-perturbative effect of the $\mathbb{S}^1$ which is generated by the KK-monopoles.
- We need to take the low-energy limit ad $E \ll \Lambda, \tilde{\Lambda}, 1/r$. The theory effectively looks like 3d one although the Coulomb branch is now compact.
- The Coulomb branch of the $\mathbb{S}^1 \times \mathbb{R}^3$ is compact. If we obtain the genuine 3d theories, we should take some deformation or some procedure which omit the global structure of the Coulomb branch. In many cases, the effective potential from the KK-monopoles lifts the this compact Coulomb branch. So we need not to care this compactness. However if we would like to obtain the normal 3d gauge theories which have the non-compact Coulomb branch, we need to add some deformations to the theory.

For example, when we consider the pure SYM theory in 3d and 4d, where there is no fundamental matter, we have no SUSY vacua in 3d while in 4d there is some discrete SUSY vacua, which are captured by the calculation of the Witten indices. Then the naive dimensional reduction of the 4d pure SYM does not give the 3d pure SYM at all! As we explained before, we need to include the superpotential from the KK-monopoles in order to connect the 3d and 4d pure SYMs.

5.2 3d Seiberg duality for $SU(N_c)$

Here we show the three-dimensional Seiberg duality which only contains the fundamental matters. We first introduce the duality without the derivation. In the proceeding section we will derive it from 4d. This was first done by [34].

The electric theory

The electric side is the three-dimensional $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ gauge theory with N_f fundamental flavors $Q, \widetilde{Q}$ with no superpotential. The quantum charges are as follows. We have the three $U(1)$ global symmetries. These are the rotations of Q and $\tilde{Q}$, and the R-symmetry.

Table 5.1: The quantum numbers on the electric side

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
Q	N_c	N_f	1	1	1	0
$\widetilde{Q}$	$\bar{N}_c$	1	$\bar{N}_f$	-1	1	0
M	1	N_f	$\bar{N}_f$	0	2	0
Y	1	1	1	0	$-2N_f$	$2(N_f - N_c + 1)$

Here we denoted the Coulomb branch of the $SU(N_c)$ as Y which are un-lifted by the superpotential from the monopole-instantons.

The magnetic theory

The magnetic side is not a $SU(\tilde{N}_c)$ theory but a $U(\tilde{N}_c)$, where $\tilde{N}_c = N_f - N_c$. It is a three dimensional $\mathcal{N} = 2$ supersymmetric $U(N_f - N_c)$ gauge theory with N_f flavors $(\mathbf{N_f} - \mathbf{N_c})_1 : q, \tilde{q}$, single flavor (electron) $\mathbf{1}_{-(N_f - N_c)} : b, \tilde{b}$ and singlets M, Y with the superpotential $W = M\tilde{q}q + Y\tilde{b}b + \tilde{X}_- + \tilde{X}_+$

Table 5.2: The quantum numbers on the magnetic side

	$U(N_f - N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
q	$\square$	$\bar{N}_f$	1	0	-1	1
$\tilde{q}$	$\overline{\square}$	1	$\bar{N}_f$	0	-1	1
b	$\mathbf{1}_{N_c - N_f}$	1	1	N_c	N_f	$N_c - N_f$
$\tilde{b}$	$\mathbf{1}_{N_f - N_c}$	1	1	$-N_c$	N_f	$N_c - N_f$
M	1	N_f	$\bar{N}_f$	0	2	0
Y	1	1	1	0	$-2N_f$	$2(N_f - N_c + 1)$
$\tilde{X}_\pm$	1	1	1	0	0	2

Notice again that just like the Aharony duality, the magnetic side includes the low-energy Coulomb branch coordinates in the superpotential, which should be UV completed. Then the magnetic side is meaningful as the low-energy effective theory. But it is sufficient because the Seiberg duality is only the IR duality.

5.2.1 The operator matching

First, the meson operator of the dual side is as usual identified with the composite operator on the electric side:

$$M \sim \tilde{Q}Q \tag{5.1}$$

where their mass dimensions should be balanced on each side by multiplying the scale factor but we omitted here for simplicity. The matching of the baryonic operator is not usual one:

$$B := Q^{N_c} \sim q^{N_f - N_c} b \tag{5.2}$$

$$\tilde{B} := \tilde{Q}^{N_c} \sim \tilde{q}^{N_f - N_c} \tilde{b}. \tag{5.3}$$

This is because the anti-symmetric combinations of q are not gauge invariant under the $U(1) \subset U(N_f - N_c)$ and we should combine Q with b. Please check the correspondence of the global charges on both sides.

The Coulomb branch coordinate on the electric side is identified with the singlet Y on the magnetic side.

$$V^{SU(N_c)} \sim Y, \tag{5.4}$$

where we denote the electric Coulomb branch as $V^{SU(N_c)}$. This means that the Coulomb moduli on the electric side is given as the elementary field in the magnetic theory. The quantum picture of the Coulomb branch on the dual side is very difficult if we have no superpotential for the Coulomb moduli. But the existence of the superpotential on the magnetic side simplify the structure of the moduli space on the magnetic side.

5.2.2 The check of the 3d duality

We will here test the above duality. The full test of the moduli spaces between the two sides is difficult. So we can test the moduli spaces by deforming the both theories by including the additional superpotential or by turning on the vevs.

The deformation by $W = \eta Y$

In the electric side the deformation $W = \eta Y$ lifts the Coulomb branch. On the dual side, the superpotential is deformed as

$$W_{\mathsf{magnetic}} = M\tilde{q}q + Y\tilde{b}b + \tilde{X}_+ + \tilde{X}_- + \eta Y. \tag{5.5}$$

The equation of motion for Y sets

$$\langle \tilde{b}b \rangle = -\eta, \tag{5.6}$$

which breaks the gauge symmetry $U(N_f - N_c) \to SU(N_f - N_c)$. The monopole operators are mapped as follows.

$$\tilde{X}_+ = \tilde{X}_- = \tilde{Y} / \langle b\tilde{b} \rangle \tag{5.7}$$

Then the superpotential becomes

$$W_{\mathsf{magnetic}} = M\tilde{q}q + \tilde{\eta}\tilde{Y} \tag{5.8}$$

The small flavors

The duality is derived for $N_f > N_c + 1$. Let us consider the smaller flavors.

For $N_f = N_c + 1$, the dual theory is the $U(1)$ gauge theory with $N_f + 1$ flavors with the superpotential

$$W = M\tilde{q}q + Y\tilde{b}b + \tilde{X}_+^{U(1)} + \tilde{X}_-^{U(1)} \tag{5.9}$$

For flowing the $N_f = N_c$ flavors we need to turn on the mass term for the quarks of the last flavor in the electric theory:

$$W = m\tilde{Q}_{N_f} Q^{N_f} = mM_{N_f}^{N_f}. \tag{5.10}$$

And we turn on the vevs for Y and M such that $\operatorname{rank} M = N_c = N_f - 1$. Thus we obtain at the low-enrgy the following superpotential.

$$W = M_{N_f}^{N_f} \tilde{q}_{N_f} q^{N_f} + \sqrt{Y \det \langle M \rangle}(\tilde{X}_+ + \tilde{X}_-) + mM_{N_f}^{N_f} \tag{5.11}$$

The $U(1)$ theory is now one-flavor theory, so it has the dual description by the XYZ model.

$$W_{XYZ} = -\tilde{X}_+\tilde{X}_- N, \quad N := \tilde{q}^{N_f} q_{N_f} \tag{5.12}$$

The total superpotential is

$$W_{\text{total}} = M^{N_f}_{N_f} N + \sqrt{Y \det \langle M \rangle}(\tilde{X}_+ + \tilde{X}_-) + m M^{N_f}_{N_f} - \tilde{X}_+ \tilde{X}_- N. \tag{5.13}$$

The corresponding equations of motion (F-term equations) are as follows.

$$M^{N_f}_{N_f} : \; N = -m \tag{5.14}$$

$$N : \; M^{N_f}_{N_f} - \tilde{X}_+ \tilde{X}_- = 0 \tag{5.15}$$

$$\tilde{X}_+ : \; \tilde{X}_- N = -m \tilde{X}_- = \sqrt{Y \det \langle M \rangle} \tag{5.16}$$

$$\tilde{X}_- : \; \tilde{X}_+ N = -m \tilde{X}_+ = \sqrt{Y \det \langle M \rangle} \tag{5.17}$$

Integrating out the massive fields $M^{N_f}_{N_f}, \; N, \; \tilde{X}_\pm$, we have

$$W_{\text{low energy}} = -Y \det \langle M \rangle / m \tag{5.18}$$

Including the superpotential for $b, \tilde{b}$: $W = Y \tilde{b} b$, the total low energy superpotential becomes

$$W = \frac{Y}{m} \left(m \tilde{b} b - \det M \right) \tag{5.19}$$

where Y/m is understood as the low-energy monopole operator. This is the correct superpotential describing the 3d $\mathcal{N} = 2$ SQCD with $N_f = N_c$ flavors.

The matching of the moduli space

The meson operator M can classically take the vevs of rank $M \leq N_c$ in the electric side. However on the dual side the vev of the operator M seems to be arbitrary. In order to show that rank $M \leq N_c$ on the dual side, we need to incorporate the quantum dynamics of the dual theory.

Let us consider the case with $N_f = N_c + 1$ and turn on the vev such as rank $\langle M \rangle = N_c$. The low-energy theory becomes as follows.

$$W_{\text{mag}} = M^{N_f}_{N_f} q^{N_f} \tilde{q}_{N_f} + Y b \tilde{b} + \sqrt{\det \langle M \rangle}(\tilde{X}_+ + \tilde{X}_-) \tag{5.20}$$

$$\overset{\text{mirror}}{=} Y b \tilde{b} + \sqrt{\det \langle M \rangle}(\tilde{X}_+ + \tilde{X}_-) \tag{5.21}$$

where since the first and second term are the XYZ models which are dual to the 3d $\mathcal{N} = 2$ SQEDs with one flavor, the two sectors are changed to the mirror theories. $\tilde{X}_\pm$ are mapped to the meson operator of the mirror theory and the superpotential acts as the mass terms. Then we cannot turn on the additional vevs for $M^{N_f+1}_{N_f+1}$ and Y. This is consistent with the electric side.

Next we consider the following vevs:

$$\langle Y \rangle \neq 0 \tag{5.22}$$

$$\text{rank} \, \langle M \rangle = N_c - 1 \tag{5.23}$$

At the low-energy limit the superpotential becomes as follows.

$$W_{\text{mag}} = \overbrace{M}^{2\times 2\,\text{matrix}} q \tilde{q} + \sqrt{\langle Y \rangle \det \langle M \rangle}(\tilde{X}_+ + \tilde{X}_-) \tag{5.24}$$

$$\overset{\text{Aharony dual}}{=} M q \tilde{q} + \overbrace{N}^{N := q\tilde{q}} d \tilde{d} + V_- \tilde{X}_+ + V_+ \tilde{X}_- + \sqrt{\langle Y \rangle \det \langle M \rangle}(\tilde{X}_+ + \tilde{X}_-) \tag{5.25}$$

where in the second line we switched to the Aharony dual with $N_c = 1, N_f = 2$ and the $d, \tilde{d}$ are the dual quarks. The equations of motions for M and N give

$$M = -d \tilde{d}, \quad N = 0. \tag{5.26}$$

Then the low-energy superpotential becomes

$$W_{\text{mag, low-energy}} = V_- \tilde{X}_+ + V_+ \tilde{X}_- + \sqrt{\langle Y \rangle \det \langle M \rangle}(\tilde{X}_+ + \tilde{X}_-) \tag{5.27}$$

The equations of motion for $\tilde{X}_\pm$ mean $V_\pm \neq 0$. This is not allowed because the Coulomb branch in $U(1)$ gauge theory always satisfies $V_+ V_- = 0$. Thus there is no supersymmetric vacuum for $\langle Y \rangle \neq 0, \;$ rank $\langle M \rangle = N_c - 1$. This is again consistent with the electric side.

The $U(1)_B$ gauging

We can gauge the $U(1)_B$ flavor symmetry and obtain the $U(N_c)$ duality, which is known as the Aharony duality. On the electric side the gauge group becomes $SU(N_c) \times U(1)_B$. If we correctly normalize the $U(1)_B$ charge, which is always possible, we can think of this as $U(N_c) \simeq \frac{SU(N_c)\times U(1)_B}{\mathbb{Z}_{N_c}}$. Then the electric theory is simply the 3d $\mathcal{N}=2$ supersymmetric $U(N_c)$ gauge theory with N_f flavors.

On the magnetic side, the gauge group is $U(N_f - N_c) \times U(1)_B$. Under the $U(1)_B$, only the electron $b, \tilde{b}$ is charged. Then the $U(1)_B$ sector is the 3d $\mathcal{N}=2$ SQED with one flavor. Then the dual theory is the XYZ model with the superpotential:

$$W = -V_+^{U(1)_B} V_-^{U(1)_B} N \tag{5.28}$$

where $N = b\tilde{b}$. Thus the total superpotential is

$$W_{\mathsf{mag,total}} = Mq\tilde{q} + Yb\tilde{b} + \tilde{X}_+ + \tilde{X}_- - V_+^{U(1)_B} V_-^{U(1)_B} N \tag{5.29}$$

The equations of motion are as follows.

$$N:\ Y = V_+^{U(1)_B} V_-^{U(1)_B} \tag{5.30}$$

After integrating out the massive superfields and flowing to the low-energy, the monopole operators $\tilde{X}_\pm$ becomes

$$\tilde{X}_+ = X_+^{U(N_f-N_c)} V_-^{U(1)} \tag{5.31}$$

$$\tilde{X}_- = X_-^{U(N_f-N_c)} V_+^{U(1)}, \tag{5.32}$$

where $\tilde{X}_\pm$ are the monopole operators of the high energy $U(N_f - N_c) \times U(1)_B$ theory while $X_\pm^{U(N_f-N_c)}$ are the monopole operators of the low-energy $U(N_f - N_c)$ theory. Note that the $U(1)_B$ sector is dualized into the theory with only the chiral superfields and with no gauge groups. Finally we obtain the following superpotential.

$$W = M\tilde{q}q + X_+^{U(N_f-N_c)} V_-^{U(1)} + X_-^{U(N_f-N_c)} V_+^{U(1)} \tag{5.33}$$

This is precisely the magnetic theory of the Aharony duality.

5.3 The duality on $\mathbb{S}^1 \times \mathbb{R}^3$

We next consider the duality on $\mathbb{S}^1 \times \mathbb{R}^3$ in order to derive the duality discussed above. This duality almost the same as the conventional four-dimensional Seiberg duality. The monopole-instanton contribution to the superpotential however should be taken into account on both sides. Then the duality is summarized as follows.

The electric theory

The supersymmetric $SU(N_c)$ gauge theory with N_f fundamental flavors $Q, \widetilde{Q}$ with $W = \eta Y$, where Y is the unlifted Coulomb branch coordinate.

The magnetic theory

The supersymmetric $SU(N_f - N_c)$ gauge theory with N_f fundamental flavors $q, \tilde{q}$ and gauge singlets(meson) M with superpotential $W = M\tilde{q}q + \tilde{\eta}\tilde{Y}$, where $\tilde{Y}$ is the Coulomb branch coordinate in the dual theory.

We can think of this duality as the 3d duality with the compact Coulomb branches or as the duality on the $\mathbb{S}^1 \times \mathbb{R}^3$ manifold. However since in both cases the Coulomb branch parametrized by A_3 is all lifted by the η terms, the duality does not concern about the compactness of the Coulomb branch.

The global symmetry

We denote the instanton action(vertex) as η like $\eta \sim \exp\left(-\frac{1}{Rg_3^2}\right) \sim \exp\left(\frac{1}{g_4^3}\right) \sim \Lambda^b$, where b is the one-loop beta function coefficient. The naive 3d limit is taking $R, \eta \to 0$ with g_3^2 fixed. On the other hand the four dimensional limit is taken as $R \to \infty$ with g_4^2 fixed. In the 4d limit the monopole operators $Y, \widetilde{Y}$ are should be integrated out. The limit where we get the 3d duality is $E \ll \eta^{1/b}, \tilde{\eta}^{1/\tilde{b}}, 1/r$.

Table 5.3: The quantum numbers of the theories on $\mathbb{S}^1 \times \mathbb{R}^3$

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
Q	N_f	1	1	1	0
$\widetilde{Q}$	1	$\bar{N}_f$	-1	1	0
M	N_f	$\bar{N}_f$	0	2	0
Y	1	1	0	$-2N_f$	$2(N_f - N_c + 1)$
η	1	1	0	$2N_f$	$2(N_c - N_f)$
q	$\bar{N}_f$	1	$\frac{N_c}{N_f - N_c}$	-1	1
$\tilde{q}$	1	N_f	$-\frac{N_c}{N_f - N_c}$	-1	1
$\tilde{Y}$	1	1	0	$2N_f$	$2(N_c - N_f + 1)$
$\tilde{\eta}$	1	1	0	$-2N_f$	$2(N_f - N_c)$

Flowing to the no η theory

In order to obtain the duality without η term, which is equal to obtaining the duality with the un-lifted Coulomb branch, we will start with the $N_f + 1$ flavors theory and turn on the real mass for the last flavor. This is just the background field gauging of the $SU(N_f+1) \times SU(N_f+1) \times U(1)_B$ flavor symmetry. On the electric theory side, we have real masses for the $Q, \widetilde{Q}$:

$$m_{\text{real}}^{Q} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & m \end{pmatrix}, \ \tilde{m}_{\text{real}}^{\widetilde{Q}} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & -m \end{pmatrix}. \tag{5.34}$$

Thus at the low energy limit we obtain the N_f flavor theory. The high and low energy monopole operators are related as

$$Y^{\text{high}} = m_{\text{complex}} Y^{\text{low}}. \tag{5.35}$$

Then since we deform the theory by $m_{\text{complex}} = 0$, the superpotential becomes

$$W = \eta Y^{\text{high}} = \eta \, m_{\text{complex}} Y^{\text{low}} \tag{5.36}$$

We can map these real masses into the dual theory side easily as follows.

$$m_{\text{real}}^{Q} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & m \end{pmatrix} = a_0 T_0 + \sum_{i=1}^{N_f} a_i T_i^{SU(N_f)_R} \tag{5.37}$$

where T_0 and T_i are the generator of the flavor gauge group. We use the following notation.

$$T_0 = \frac{\sqrt{2}}{\sqrt{N_f+1}} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \tag{5.38}$$

$$T_i = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} \tag{5.39}$$

$$\mathsf{Tr}(T^a T^b) = 2\delta^{ab} \tag{5.40}$$

In this normalization we find

$$a_0 = \frac{m}{\sqrt{2(N_f+1)}} \tag{5.41}$$

$$a_{1,\cdots,N_f-1} = \frac{m}{N_f+1} \tag{5.42}$$

$$a_{N_f} = -\frac{N_f}{N_f+1} m. \tag{5.43}$$

These real masses are mapped to

$$\tilde{m}^q_{\mathsf{real}} = B[q]\, a_0 T_0 - \sum_{i=1}^{N_f} a_i T_i \tag{5.44}$$

$$= \frac{N_c}{N_f+1-N_c} a_0 T_0 - \sum_{i=1}^{N_f} a_i T_i \tag{5.45}$$

$$= \begin{pmatrix} m_1 & & & \\ & \ddots & & \\ & & m_1 & \\ & & & m_2 \end{pmatrix} \tag{5.46}$$

$$m_1 = \frac{m}{N_f+1-N_c}, \; m_2 = \frac{N_c - N_f}{N_f+1-N_c} m \tag{5.47}$$

In the dual theory side all dual quarks are massive at $\tilde{\sigma} = 0$. This means that $\tilde{\sigma} = 0$ is not a vacuum any more. The correct vacuum which describes the electric theory is

$$\tilde{\sigma} = \begin{pmatrix} -m_1 & & & \\ & \ddots & & \\ & & -m_1 & \\ & & & -m_2 \end{pmatrix}, \tag{5.48}$$

which means the gauge symmetry breaking as $SU(N_f+1-N_c) \to SU(N_f-N_c) \times U(1)$. The gauge group $SU(N_f-N_c) \times U(1)$ now can be written as $U(N_f-N_c)$. This is not always okay because the $U(N)$ group is the semi-direct product of $SU(N)$ and $U(1)$: $SU(N) \rtimes U(1)$. But now the field contents do not care about the difference between $U(N)$ and $SU(N) \times U(1)$. Then we can think of $SU(N_f-N_c) \times U(1)$ as the $U(N_f-N_c) = SU(N_f-N_c) \rtimes U(1) \simeq \frac{SU(N_f-N_c)\times U(1)}{\mathbb{Z}_{N_f-N_c}}$.

The low-energy field contents are summarized as follows.

- The N_f dual quarks $q, \tilde{q}$ which are from the $N_f - N_c$ components of the original dual quarks.
- The electron $b, \tilde{b}$ which are originally $q^{N_f+1}_{N_f+1-N_c}, \tilde{q}^{N_f+1-N_c}_{N_f+1}$.
- The $N_f \times N_f$ meson singlets M: They come from the left-upper components of the high-energy meson fields M

- The singlet Y: This comes from the last component of the meson $M^{N_f+1}_{N_f+1}$

The other fields in the high-energy theory are all massive and integrated out.

We summarize the quantum numbers of the electric and magnetic theory. The $U(1)_B$ charge is ambiguous because the other $U(1)$ symmetries including the $U(1)$ part of the gauge symmetry can mix with $U(1)_B$. Then we need to choose the convenient one. We found that the following choice for the $U(1)_B$ is convenient for the later purpose of obtaining the $U(N_c)$ duality.

Table 5.4: The global charges of the 3d $\mathcal{N}=2$ $SU(N_c)$ theory with N_f flavors

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
Q	N_c	N_f	1	1	1	0
$\widetilde{Q}$	$\bar{N}_c$	1	$\bar{N}_f$	-1	1	0
M	1	N_f	$\bar{N}_f$	0	2	0
Y	1	1	1	0	$-2N_f$	$2(N_f-N_c+1)$

Table 5.5: The global charges of the dual theory

	$U(N_f-N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
q	$\square$	$\bar{N}_f$	1	0	-1	1
$\tilde{q}$	$\bar{\square}$	1	N_f	0	-1	1
b	$\mathbf{1}_{N_c-N_f}$	1	1	N_c	N_f	N_c-N_f
$\tilde{b}$	$\mathbf{1}_{N_f-N_c}$	1	1	$-N_c$	N_f	N_c-N_f
M	1	N_f	$\bar{N}_f$	0	2	0
Y	1	1	1	0	$-2N_f$	$2(N_f-N_c+1)$
$\tilde{X}_\pm$	1	1	1	0	0	2

The superpotential on the magnetic side becomes

$$W^{\text{high}}_{\text{magnetic}} = M\tilde{q}q + \tilde{\eta}\tilde{Y} \tag{5.49}$$

$$\rightarrow M\tilde{q}q + M^{N_f+1}_{N_f+1}\tilde{q}^{N_f+1-N_c}_{N_f+1}q^{N_f+1}_{N_f+1-N_c} + \tilde{\eta}\tilde{Y} \tag{5.50}$$

$$= M\tilde{q}q + Y\tilde{b}b + \tilde{\eta}\tilde{Y}. \tag{5.51}$$

The high-energy monopole operator is identified as

$$\tilde{Y} \sim \tilde{X}_-^{U(N_f-N_c)}. \tag{5.52}$$

We will derive the above relation. First, we notice that the vacuum of the dual theory was taken at

$$\tilde{\sigma} = \begin{pmatrix} -m_1 & & & \\ & \ddots & & \\ & & -m_1 & \\ & & & -m_2 \end{pmatrix}. \tag{5.53}$$

But in our notation we should rearrange the eigenvalues as follows.

$$\tilde{\sigma} = \begin{pmatrix} -m_2 & & & \\ & -m_1 & & \\ & & \ddots & \\ & & & -m_1 \end{pmatrix}. \tag{5.54}$$

This is because the eigenvalues should be $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \cdots \geq \tilde{\sigma}_{N_f+1-N_c}$ in the Coulomb branch which we are looking at and we assumed $m > 0$, then $m_1 > 0,\ m_2 < 0$. So the fluctuation around this vacuum becomes

$$\tilde{\sigma} = \begin{pmatrix} -m_2 + \delta\tilde{\sigma}_1 & & & \\ & -m_1 + \delta\tilde{\sigma}_2 & & \\ & & \ddots & \\ & & & -m_1 + \delta\tilde{\sigma}_{N_f-N_c+1} \end{pmatrix}. \tag{5.55}$$

So the Coulomb branch of the low-energy $U(N_f - N_c)$ is parametrized by

$$\tilde{X}_+^{U(N_f-N_c)} \sim \exp\left(\frac{\delta\tilde{\sigma}_2}{g_3^2}\right) \tag{5.56}$$

$$\tilde{X}_-^{U(N_f-N_c)} \sim \exp\left(-\frac{\delta\tilde{\sigma}_{N_f-N_c+1}}{g_3^2}\right) \tag{5.57}$$

where we have not written the dependence on the dual photons for the simplicity but its form is manifest. Here we should care about that the $\delta\tilde{\sigma}_j (j = 1, \cdots, N_f - N_c + 1)$ must be traceless because they originally arise from the adjoint representation of the $SU(N_f - N_c + 1)$. Thus the small deformation

$$\delta\tilde{\sigma}_1 \to \delta\tilde{\sigma}_1 + \epsilon,\ \epsilon > 0 \tag{5.58}$$

corresponds to

$$\delta\tilde{\sigma}_2 \to \delta\tilde{\sigma}_2 - \#\epsilon \tag{5.59}$$

$$\delta\tilde{\sigma}_{N_f-N_c+1} \to \delta\tilde{\sigma}_{N_f-N_c+1} - \#\epsilon \tag{5.60}$$

Then the high-energy monopole operator

$$Y \sim \exp\left(\frac{\tilde{\sigma}_1 - \tilde{\sigma}_{N_f-N_c+1}}{g_3^2}\right) \tag{5.61}$$

is identified with

$$\tilde{X}_-^{U(N_f-N_c)} \sim \exp\left(-\frac{\delta\tilde{\sigma}_{N_f-N_c+1}}{g_3^2}\right). \tag{5.62}$$

Since we have the gauge symmetry breaking $SU(N_f+1-N_c) \to SU(N_f-N_c) \times U(1)$, we have the AHW-type superpotential from the $U(1)$ part:

$$W_{\mathsf{AHW}} = \tilde{X}_+^{U(N_f-N_c)} \tag{5.63}$$

Including all the superpotential explained above, we obtain the superpotential on the magnetic side which explains the moduli on the electric side.

$$W_{\mathsf{magnetic}}^{\mathsf{low}} = M\tilde{q}q + Y\tilde{b}b + \tilde{X}_- + \tilde{X}_+. \tag{5.64}$$

5.4 The $U(1)_B$ gauging: obtaining the $U(N_c)$ duality

The three dimensional $\mathcal{N} = 2$ supersymmetric $U(N_c)$ gauge theory is obtained by gauging the $U(1)_B$ global baryon symmetry. We assign the $1/N_c$ charge to the fundamental quarks under the $U(1)_B$ symmetry. The gauge group becomes $SU(N_c) \times U(1)_B$. However we consider this as $U(N_c) = U(1) \ltimes SU(N_c) \simeq (SU(N_c) \times U(1))/\mathbb{Z}_{N_c}$ gauge group. The superpotential becomes

$$W_{\mathsf{electric}} = \eta X_+ X_- \tag{5.65}$$

$$W_{\mathsf{magnetic}} = \tilde{\eta}\tilde{X}_+\tilde{X}_- + M\tilde{q}q \tag{5.66}$$

We can apply the $U(1)_B$ gauging to the duality with no η term. The electric theory becomes the $U(N_c)$ gauge theory. On the other hand, at the magnetic side, the only $b, \tilde{b}$ fields are charged under the $U(1)_B$. Then the low energy effective theory is equivalent to the 3d $\mathcal{N} = 2$ XYZ model:

$$W = -V_+^{U(1)_B} V_-^{U(1)_B} N_{b\tilde{b}} \tag{5.67}$$

Thus the superpotential totally becomes

$$W = Mq\tilde{q} + YN_{b\tilde{b}} + \tilde{\eta}\tilde{X}_- + \tilde{X}_+ - V_+^{U(1)_B} V_-^{U(1)_B} N_{b\tilde{b}} \tag{5.68}$$

Integrating out $N_{q\tilde{q}}$, we obtain the $U(N_f - N_c)$ gauge theory with the monopole operator $\hat{X}_\pm$. The matching of the monopole operators between the low and high energy are as follows.

$$\tilde{X}_+ = \hat{X}_+ V_-^{U(1)_B}, \quad \tilde{X}_- = \hat{X}_- V_+^{U(1)_B} \tag{5.69}$$

where $V_\pm^{U(1)_B}$ are identified with the monopole operator of the electric theory. The resulting superpotential is as follows.

$$W = Mq\tilde{q} + \hat{X}_- V_+^{U(1)_B} + \hat{X}_+ V_-^{U(1)_B} \tag{5.70}$$

which is the Aharony dual.

5.5 Seiberg duality for $U(N_c)$

We can derive the Aharony duality directly from the 4d Seiberg duality. We will start with the conventional Seiberg duality with $U(1)_B$ gauged.

5.5.1 The dimensional reduction from $\mathbb{R}^3 \times \mathbb{S}^1$

We can easily construct the $U(N)$ duality on $\mathbb{R}^3 \times \mathbb{S}^1$ or the 3d $U(N)$ duality with compact Coulomb branch by adding the non-perturbative superpotential from the KK-monopoles. These duality should contain the superpotentials which lift the Coulomb branch, so the duality connects the Higgs branch between the electric and magnetic sides. Then the duality is simple to analyze. But we would like to obtain the duality which also contains the un-lifted non-compact Coulomb branch. In order to do so, we need to deform the theories by the real masses and need to flow to the IR.

The electric theory

The theory is a 3d $\mathcal{N} = 2$ supersymmetric $U(N_c)$ gauge theory with $N_f + 2$ flavors $Q, \widetilde{Q}$ with $W = \eta\, X_+ X_-$, where $X_\pm$ are the monopole operators of the electric Coulomb branch. We consider the following real mass deformation:

$$m_Q = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & m & \\ & & & & -m \end{pmatrix}, \quad \tilde{m}_{\widetilde{Q}} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & -m & \\ & & & & m \end{pmatrix} \tag{5.71}$$

The last two falvor components of the quarks are massive and integrated out. Since the signs of the masses for the fundamental and anti-fundamental quarks are flipped, there is no introduction of the Chern-Simons term. The low-energy theory is a 3d $\mathcal{N} = 2$ supersymmetric $U(N_c)$ with N_f flavors with no superpotential.

The dual theory

The dual theory is a 3d $\mathcal{N}=2$ supersymmetric $U(N_f+2-N_c)$ gauge theory with N_f+2 flavors $q, \tilde{q}$ and singlets M, and with the superpotential $W = M\tilde{q}q + \tilde{\eta}\tilde{X}_+\tilde{X}_-$, where $\tilde{X}_\pm$ are the monopole operators of the magnetic Coulomb branch.
The real mass deformation in the electric theory side is mapped as follows.

$$m_q = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & -m & \\ & & & & m \end{pmatrix}, \quad \tilde{m}_{\tilde{q}} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & m & \\ & & & & -m \end{pmatrix} \tag{5.72}$$

Furthermore the meson fields also obtain the real masses and almost all the off-diagonal components decouple from the low-energy effective theory, remaining as follows.

$$M = \begin{pmatrix} & & & 0 & 0 \\ & \hat{M} & & \vdots & \vdots \\ & & & 0 & \vdots \\ 0 & \cdots & 0 & M^{N_f+1}_{N_f+1} & 0 \\ 0 & \cdots & \cdots & 0 & M^{N_f+2}_{N_f+2} \end{pmatrix}, \tag{5.73}$$

where $\hat{M}$ is the $N_f \times N_f$ matrix.

The low-energy theory should be taken at

$$\widetilde{\sigma} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & -m & \\ & & & & m \end{pmatrix}. \tag{5.74}$$

This breaks the gauge symmetry as follows.

$$U(N_f+2-N_c) \to U(N_f-N_c) \times U(1) \times U(1) \tag{5.75}$$

Notice that this breaking dose not contain the introduction of the Chern-Simons terms(levels) for these gauge groups.

The monopole operator matching

At high-energy the monopole operators which break the gauge symmetry and create the magnetic flux like $U(N_f+2-N_c) \to U(N_f+1-N_c) \times U(1)$ are

$$\tilde{X}_i \sim \exp\left(\frac{\sigma_i}{g_3^2} + ia_i\right), \quad i = 1, \cdots, N_c \tag{5.76}$$

Generically Affleck-Harvey-Witten type superpotential lifts the most of Coulomb branch. The remaining branch is labeled by

$$\tilde{X}_+ \sim \exp\left(\frac{\sigma_1}{g_3^2} + ia_1\right), \quad \tilde{X}_- \sim \exp\left(-\frac{\sigma_{N_c}}{g_3^2} - ia_{N_c}\right) \tag{5.77}$$

If we consider alternatively $SU(\tilde{N}_c)$ gauge theory, the monopole operator becomes

$$\tilde{Y} = \tilde{X}_+\tilde{X}_-. \tag{5.78}$$

When the gauge group is broken like $U(N_f+2-N_c) \to U(N_f-N_c) \times U(1) \times U(1)$, the monopole operator flows to

$$\tilde{X}_+ = \tilde{V}_{1+}, \quad \tilde{X}_- = \tilde{V}_{2-}, \tag{5.79}$$

where $\tilde{V}_{1\pm}$ are describing the Coulomb branch of the first $U(1)$ and $\tilde{V}_{2\pm}$ correspond to the second $U(1)$. The low-energy theory has additional two types of monopole operators corresponding to the remaining $U(1)^2$ part in addition to the $U(N_f-N_c)$ part:

$$U(1): \ \tilde{V}_{1\pm} \tag{5.80}$$

$$U(1): \ \tilde{V}_{2\pm} \tag{5.81}$$

$$U(N_f-N_c): \hat{\tilde{X}}_{\pm} \tag{5.82}$$

This relation is checked in the semiclassical region but we need do rearrangement of the σ matrix;

$$\widetilde{\sigma} = \begin{pmatrix} m+\delta\sigma_1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -m+\delta\sigma_{N_f+2-N_c} \end{pmatrix} \tag{5.83}$$

Low-energy limit The superpotential of the low energy theory becomes

$$W = \hat{M}\tilde{q}q + M_1N_1 + M_2N_2 - N_1\tilde{V}_{1+}\tilde{V}_{1-} - N_2\tilde{V}_{2+}\tilde{V}_{2-} + \tilde{\eta}\tilde{V}_{1+}\tilde{V}_{2-} + \hat{\tilde{X}}_+\tilde{V}_{1-} + \hat{\tilde{X}}_-\tilde{V}_{2+} \tag{5.84}$$

The equations of motion for this superpotential are following.

$$M_1 = \tilde{V}_{1+}\tilde{V}_{1-} \tag{5.85}$$

$$M_2 = \tilde{V}_{2+}\tilde{V}_{2-} \tag{5.86}$$

$$\tilde{\eta}\tilde{V}_{2-} = N_1\tilde{V}_{1-} = 0 \tag{5.87}$$

$$\tilde{\eta}\tilde{V}_{1+} = N_2\tilde{V}_{2+} = 0 \tag{5.88}$$

Then,

$$W_{\text{low energy}} = \hat{M}\tilde{q}q + \hat{\tilde{X}}_+\tilde{V}_{1-} + \hat{\tilde{X}}_-\tilde{V}_{2+} \tag{5.89}$$

This is precisely the Aharony duality. We can identify the monopole operator $\tilde{V}_{1-}, \tilde{V}_{2+}$ as the Coulomb branch coordinates of the low energy effective theory of the "electric" one.

The Global Symmetries

We here list the quantum numbers of the high- and low-energy field contents.

Table 5.6: The quantum numbers of the high energy theories

	$SU(N_f+2)_L$	$SU(N_f+2)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f+2	1	1	0	0
$\tilde{Q}$	1	$\overline{N_f+2}$	1	0	0
M	N_f+2	$\overline{N_f+2}$	2	0	0
$X_\pm$	1	1	$-(N_f+2)$	±1	$(N_f+2)-N_c+1$
q	$\overline{N_f+2}$	1	-1	0	1
$\tilde{q}$	1	N_f+2	-1	0	1
$\tilde{X}_\pm$	1	1	N_f+2	±1	$N_c-(N_f+2)+1$
Λ^b	1	1	$2(N_f+2)$	0	$2(N_c-(N_f+2))$
$\tilde{\Lambda}^{\tilde{b}}$	1	1	$-2(N_f+2)$	0	$2((N_f+2)-N_c)$

Table 5.7: The quantum numbers of the low-energy Coulomb branch coordinates

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
$\hat{\tilde{X}}_\pm$	1	1	N_f	± 1	$N_c - N_f + 1$
$\tilde{V}_{1+}$	1	1	$N_f + 2$	1	$N_c - N_f - 1$
$\tilde{V}_{1-}$	1	1	$-N_f$	-1	$N_f - N_c + 1$
$\tilde{V}_{2+}$	1	1	$-N_f$	1	$N_f - N_c + 1$
$\tilde{V}_{2-}$	1	1	$N_f + 2$	-1	$N_c - N_f - 1$

5.6 The Ungauging: $SU(N)$ duality from $U(N)$ duality

In this subsection we will discuss the un-gauging technique, which is the inverse process of the gauging of the global symmetries. If we consider the $U(1)$ or $U(N_c)$ gauge theories, there is the topological $U(1)_J$ global symmetry according to the presence of the $U(1)$ factors of the gauge groups. We can gauge the $U(1)_J$ symmetry.

Let us start with the $U(N_c)$ Aharony duality and gauge the $U(1)_J$ topological symmetry for concreteness. This means introducing the Chern-Simons term between the original $U(1) \subset U(N_c)$ and $U(1)_J$ gauge fields. The Chern-Simons term is a topological mass for the corresponding gauge fields. In this case $U(1) \subset U(N_c)$ and $U(1)_J$ gauge fields are massive and at low-energy these gauge fields are integrated out.

The original theory

On the electric theory side, the 3d $\mathcal{N} = 2$ supersymmetric $U(N_c)$ gauge theory with N_f flavors has the global $U(1)_J$ topological symmetry. We can gauge this $U(1)_J$ symmetry, which introduces the mixed Chern-Simons term like

$$\mathcal{L} \ni A_\mu^{J,new} J^{J\mu} \sim A_\mu^{J,new} \epsilon^{\mu\rho\sigma} F_{\rho\sigma}^{U(1),old}. \tag{5.90}$$

Then in the low-energy limit we obtain the 3d $\mathcal{N} = 2$ supersymmetric $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ gauge theory with N_f flavors.

Table 5.8: The quantum numbers of the $SU(N_c)$ gauge theory

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
Q	$\square$	$\square$	1	1	1	0
$\tilde{Q}$	$\overline{\square}$	1	$\overline{\square}$	-1	1	0
M	1	$\square$	$\overline{\square}$	0	2	0
$Y^{SU(N_c)}$	1	1	1	0	$-2N_f$	$2(N_f - N_c + 1)$

The magnetic dual

The dual theory, on the other hand, is the 3d $\mathcal{N} = 2$ supersymmetric $U(N_f - N_c)$ gauge theory with N_f flavors and singlets $M, X_\pm$ with the superpotential

$$W = M\tilde{q}q + X_+\tilde{X}_- + X_-\tilde{X}_+. \tag{5.91}$$

The $U(1)_J$ gauging gives us the 3d $\mathcal{N} = 2$ supersymmetric $U(N_f - N_c) \times U(1)_J^{\text{gauged}}$ gauge theory with the mixed Chern-Simons term. In this case, however, we cannot forget about the $U(1) \times U(1)$ dynamics since the singlets $X_\pm$ are charged under the topological $U(1)_J$ symmetry in contrast to the electric side where there is no elementary field charged under the $U(1)_J$ symmetry.

The magnetic dual of the $SU(N_c)$ with N_f flavors is summarized as follows. The magnetic theory is the 3d $\mathcal{N}=2$ supersymmetric $U(N_f-N_c)\times U(1)_J^{\text{gauged}}$ gauge theory with the mixed Chern-Simons term of the CS level 1. The field contents are summarized as follows.

Table 5.9: The quantum numbers of the $U(N_f-N_c)\times U(1)_J^{\text{gauged}}$ gauge theory

	$U(N_f-N_c)\times U(1)_J$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
q	$(\mathbf{N_f-N_c})_{1,0}$	$\overline{N_f}$	1	$\frac{N_c}{N_f-N_c}$	-1	1
$\tilde{q}$	$(\overline{\mathbf{N_f-N_c}})_{-1,0}$	1	N_f	$-\frac{N_c}{N_f-N_c}$	-1	1
M	$\mathbf{1}_{0,0}$	N_f	$\overline{N_f}$	0	2	0
$X_\pm$	$\mathbf{1}_{0,\pm1}$	1	1	0	$-N_f$	N_f-N_c+1
$\tilde{X}_\pm^{U(N_f-N_c)}$	$\mathbf{1}_{0,\pm1}$	1	1	0	N_f	N_c-N_f+1
$V_\pm^{U(1)_J}$	$\mathbf{1}_{\pm(N_f-N_c),0}$	1	1	0	N_f	N_c-N_f

The superpotential on the dual side is as follows.

$$W_{\text{mag}}^{U(N_f-N_c)\times U(1)_J}=M\tilde{q}q+X_+\tilde{X}_-^{U(N_f-N_c)}+X_-\tilde{X}_+^{U(N_f-N_c)} \tag{5.92}$$

We know the dual description of the 3d $SU(N_c)$ theory with N_f flavors, which is $U(N_f-N_c)$ gauge theory with N_f flavors and $b,\tilde{b}$ and singlets M,Y with the superpotential

$$W_{\text{another magnetic}}^{U(N_f-N_c)}=M\tilde{q}q+Y\tilde{b}b+\tilde{X}_++\tilde{X}_-. \tag{5.93}$$

We would like to understand how this superpotential $W=M\tilde{q}q+Y\tilde{b}b+\tilde{X}_++\tilde{X}_-$ emerges from the dual description and the dynamics of the $U(N_f-N_c)\times U(1)_J^{gauge}$ gauge theory. Then we need to match the operators on both sides.

The operator matching

By the symmetry argument, we obtain the following operator matching:

Table 5.10: The operator matching

$U(N_f-N_c)$ gauge theory	$U(N_f-N_c)\times U(1)_J$ gauge theory
Y: singlet which describes the Coulomb branch of $SU(N_c)$	X_+X_-: charge less under $U(1)_J$
$b,\tilde{b}$: single flavor electron	$\tilde{v}_+^{U(1)},\tilde{v}_-^{U(1)}$: $U(1)_J$ Coulomb branch
$\tilde{X}_\pm$: $U(N_f-N_c)$ Coulomb branch	$\tilde{X}_+^{U(N_f-Nc)}X_-$, $\tilde{X}_+^{U(N_f-N_c)}X_-$

The superpotential of the $U(N_f-N_c)$ gauge theory becomes

$$W_{\text{another magnetic}}^{U(N_f-N_c)}=M\tilde{q}q+Y\tilde{b}b+\tilde{X}_++\tilde{X}_- \tag{5.94}$$

$$=\text{tr}MN_{\tilde{q}q}+X_+X_-\,\tilde{v}_+^{U(1)},\tilde{v}_-^{U(1)}+\tilde{X}_++\tilde{X}_- \tag{5.95}$$

$$=\text{tr}MN_{\tilde{q}q}+N_{X_+X_-\text{meson}}\,\tilde{v}_+^{U(1)}\tilde{v}_-^{U(1)}+\tilde{X}_+^{U(N_f-Nc)}X_-+\tilde{X}_-^{U(N_f-Nc)}X_+ \tag{5.96}$$

In this way the correct superpotential of the $U(N_f-N_c)\times U(1)_J$ gauge theory is described by the $U(N_f-N_c)$ another dual. Notice that the second term is generated by the dynamics of the $U(1)_J$ gauge symmetry, which is equivalent to the $\mathcal{N}=2$ supersymmetric XYZ model in the IR.

The global charges

First, we list the global charges of the $U(N_f-N_c)$ gauge theory dual to the $SU(N_f-N_c)$ gauge theory.

Table 5.11: The quantum numbers of the $U(N_f - N_c)$ dual theory

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
M	N_f	$\bar{N}_f$	0	2	0
$N_{\tilde{q}q}$	$\bar{N}_f$	N_f	0	-2	2
b	1	1	N_c	N_f	$N_c - N_f$
$\tilde{b}$	1	1	$-N_c$	N_f	$N_c - N_f$
Y	1	1	0	$-2N_f$	$2(N_f - N_c + 1)$
$\tilde{X}_\pm$	1	1	0	0	2

On the other hand, the $U(N_f - N_c) \times U(1)_J$ theory has $U(1)_{J'} \times U(1)_A \times U(1)_R$ global abelian symmetries, where $U(1)_{J'}$ symmetry is a topological global symmetry Hodge-dual to the $U(1)_J$ gauge symmetry. The $U(1)_{J'}$ symmetry is recognized as the $U(1)_B$ baryon symmetry. The global charges of the $U(1)_J$ monopole operators are determined only from the contribution of the $X_\pm$ fields.

Table 5.12: The quantum numbers of the $U(N_f - N_c) \times U(1)_J$ dual theory

	$U(1)_J$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_{J'}$	$U(1)_A$	$U(1)_R$
M	0	N_f	$\bar{N}_f$	0	2	0
$N_{\tilde{q}q}$	0	$\bar{N}_f$	N_f	0	-2	2
$X_\pm$	± 1	1	1	0	$-N_f$	$N_f - N_c + 1$
$\tilde{X}_\pm^{U(N_f-N_c)}$	± 1	1	1	0	N_f	$N_c - N_f + 1$
$\tilde{v}_\pm^{U(1)}$	0	1	1	± 1	N_f	$N_c - N_f$

Chapter 6

3d Seiberg duality with the adjoint matter from 4d

We will here derive the Kim-Park duality [64] and the $SU(N_c)$ version [66] of it, which are the 3d duality with an adjoint matter, from the corresponding 4d dualities. This was first done in [37]. The $SU(N_c)$ duality is derived also from the un-gauging of the Kim-Park duality. Just like the derivation of the 3d Seiberg duality in the previous chapter, the key step to obtain the 3d duality is to derive the effective superpotential by the KK-monopoles. Including the adjoint matter makes the numbers of the fermion zero-modes large. Then the naive argument leads to the absence of such a superpotential. However the superpotential for the adjoint matter renders the number of the fermion zero-modes. Thus we can have the superpotential from the KK-monopoles.

6.1 Kim-Park duality

Let us include the adjoint matter field to the conventional Aharony duality, which is known as the Kim-Park duality. The "electric" theory is the three-dimensional $\mathcal{N}=2$ supersymmetric $U(N_c)$ gauge theory with N_f fundamental vector-like matters $Q, \widetilde{Q}$ and an adjoint matter X and the superpotential $W = \mathsf{Tr}X^{k+1}$. This theory is simply the dimensional reduction of the four-dimensional $\mathcal{N}=1$ supersymmetric $U(N_c)$ gauge theory with N_f flavors and one adjoint matter. Then the 3d electric theory is the naive dimensional reduction of the electric theory of the 4d Kutasov-Schwimmer duality.

The "magnetic" theory is the three-dimensional $\mathcal{N}=2$ supersymmetric $U(kN_f - N_c)$ gauge theory with N_f fundamental vector-like matters $q, \tilde{q}$, an adjoint matter Y, the singlet $M_j,\ j = 0, \cdots, k-1$ and $v_{0,\pm}, \cdots, v_{k-1,\pm}$ with the superpotential $W = \mathsf{Tr}Y^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q + \sum_{j=0}^{k-1}(v_{j,+}\tilde{v}_{k-1-j,-} + v_{j,-}\tilde{v}_{k-1-j,+})$, where $v_{0,\pm}$ and $\tilde{v}_{0,\pm}$ are the minimal bare monopoles of electric side and magnetic side respecttively. $v_{j\neq 0,\pm}$ and $\tilde{v}_{j\neq 0,\pm}$ are the monopole operators dressed by the adjoint matter. The Coulomb branches of the electric side are described by the elementary chiral superfields $v_{0,\pm}, \cdots, v_{k-1,\pm}$ in the magnetic side.

The global charges are as follows. Note that M_j are identified with the operator in the electric side as $M_j \sim \tilde{Q} X^j Q$ and we listed the four-dimensional instanton factors for the later purpose. Of course these instantons only appear in the 4d theories. In 4d, the $U(1)_A$ and $U(1)_R$ which we used are anomalous, so the instanton factors are charged under these symmetries.

The global symmetries

Table 6.1: The quantum numbers of the two theories

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	1	1	0	r
$\tilde{Q}$	1	$\bar{N}_f$	1	0	r
X, Y	1	1	0	0	$\frac{2}{k+1}$
M_j	N_f	$\bar{N}_f$	2	0	$2r + \frac{2j}{k+1}$
$v_{j,\pm}$	1	1	$-N_f$	± 1	$-N_f r + N_f - \frac{2}{k+1}(N_c - 1) + \frac{2j}{k+1}$
q	$\bar{N}_f$	1	-1	0	$-r + \frac{2}{k+1}$
$\tilde{q}$	1	N_f	-1	0	$-r + \frac{2}{k+1}$
$\tilde{v}_{j,\pm}$	1	1	N_f	± 1	$N_f r - N_f + \frac{2}{k+1}(N_c + 1) + \frac{2j}{k+1}$
Λ^b	1	1	$2N_f$	0	$2N_f r - 2N_f + \frac{4}{k+1}N_c$
$\tilde{\Lambda}^{\tilde{b}}$	1	1	$-2N_f$	0	$-2N_f r + 2N_f - \frac{4}{k+1}N_c$

Table 6.2: The quantum numbers

Here we calculated the (spurionic) charges of the monopole operators and instanton factors as follows.

$$\begin{aligned} R[\mathrm{e}^{\Phi_1}] &= -R[\psi_Q]N_f - R[\lambda](N_c - 1) - R[\psi_X](N_c - 1) \\ &= -(r-1)N_f - (+1)(N_c - 1) - \left(\frac{2}{k+1} - 1\right)(N_c - 1) \\ &= -N_f r + N_f - \frac{2}{k+1}(N_c - 1) \end{aligned} \tag{6.1}$$

$$A[\mathrm{e}^{\Phi_1}] = -A[\psi_Q]N_f = -N_f \tag{6.2}$$

$$U(1)_R: \ \theta \to \theta + \underbrace{(r-1)\cdot 2N_f}_{Q,\tilde{Q}} + \underbrace{(+1)\cdot 2N_c}_{\lambda} + \underbrace{\left(\frac{2}{k+1} - 1\right)\cdot 2N_c}_{X} \tag{6.3}$$

$$U(1)_A: \theta \to \theta + \underbrace{(+1)\cdot 2N_f}_{Q,\tilde{Q}} \tag{6.4}$$

Here we denoted $b, \tilde{b}$ as the one-loop beta function coefficients on the electric and magnetic sides respectively and Λ and $\tilde{\Lambda}$ are the dynamical scales on both sides.

The superpotential induced by the Kaluza-Klein monopole

Next we derive the effective superpotential induced by the KK-monopoles. If we put the theory on the $\mathbb{S}^1 \times \mathbb{R}^3$, we should include the additional superpotential in the low-energy dynamics. The symmetry argument and counting the fermion zero-modes imply that the following superpotentials are induced by the "twisted instanton" which is usually called as a KK monopole.

$$W_{\mathsf{electric}} = \sum_{i+j=k-1} \eta\, v_{i,+} v_{j,-} \tag{6.5}$$

$$W_{\mathsf{magnetic}} = \sum_{i+j=k-1} \tilde{\eta}\, \tilde{v}_{i,+} \tilde{v}_{j,-} \tag{6.6}$$

The Kaluza-Klein monopole has the too many fermion zero modes to appear in the superpotential but does not have any fundamental quark zero-mode. This is a crucial point. In this theory, the adjoint fermion zero-modes are coming from the gaugino and the adjoint chiral superfield X. Then the Kaluza-Klein monopole has the ψ_X fermion zero-modes which should be contracted by scalar-fermion interactions coming from the superpotential $W = \mathsf{Tr}X^{k+1}$. Finally the KK monopole can have the correct fermion zero modes to appear in the superpotential, that is to say, only the two gaugino zero mode, and appears in the superpotential.

Here we will give a detail about its derivation. First we count the fermionic zero-modes around the Kaluza-Klein monopole. The Callias index theorem and $\mathbb{R}^3 \times \mathbb{S}^1$ index theorem [43, 71–77] say that the fundamental quark zero mode is absent and only the adjoint fermonic zero modes contribute. The each adjoint field contributes two fermonic modes. The theory contains the gaugino and adjoint fermion coming from the adjoint chiral superfield X. So we have the four fermionic zero modes. So the KK-monopole vertex take the following form.

$$\mathcal{M}_{\mathsf{KK}} = \mathrm{e}^{-S_0}\mathrm{e}^{\sigma+ia}\lambda^2\psi_X^2 \tag{6.7}$$

This is too many fermonic zero mode to appear in the superpotential since the superpotential usually generate the interactions containing the two fermion fields. However we have the superpotential for the adjoint field X like $W = \mathrm{Tr}\, X^{k+1}$, which generates the potential:

$$V \ni (X_{11})^{k-1}\psi_{X,1a}\psi_{X,a1}. \tag{6.8}$$

This vertex is used to contract the fermionic zero-modes arising from the adjoint field X. In other words, the superpotential $W = \mathrm{Tr}X^{k+1}$ generate the mass term for ψ_X and there in no zero-mode for ψ_X. In the end the KK monopole vertex has only two fermonic zero-modes from the gaugino, which can contribute to the superpotential like (6.5).

$$\mathcal{M'}_{\mathsf{KK}} \sim \mathrm{e}^{-S_0}\mathrm{e}^{\sigma+ia}\lambda^2(X_{11})^{k-1} \tag{6.9}$$

The weak perturbation

In deriving the 3d duality, it is helpful to consider the weak perturbation of the both theories by the potentials for the adjoint matter X, Y, which break the $U(1)_R$ symmetry explicitly, generating a new $U(1)_R$ symmetry at low-energy. We consider the following deformation of the superpotential [39]:

$$W = \sum_{j=0}^{k} \frac{s_j}{k+1-j} \mathsf{Tr}\, X^{k+1-j}. \tag{6.10}$$

The minima of the superpotential are following.

$$W'(x) = \sum_{j=0}^{k} s_j x^{k-j} = s_0 \prod_{j=1}^{k} (x - a_j) \tag{6.11}$$

When the $\{a_j\}$ are distinct, the adjoint field becomes massive (This is easily checked by shifting the field as $X \to \langle X \rangle + \delta X$) and the gauge symmetry is broken as

$$U(N_c) \to U(i_1) \times \cdots \times U(i_k), \quad \sum_{a=1}^{k} i_a = N_c. \tag{6.12}$$

The low-energy effective theory is the sum of the $\mathcal{N} = 2$ $U(i_a)$ theory with N_f fundamental quarks with no adjoint matter and with no superpotential. The monopole operators in the electric side at high-energy $v_{j,\pm}$ $(j = 0, \cdots, k-1)$ are correctly describing the Coulomb branches of the $U(i_1) \times \cdots \times U(i_k)$ gauge groups at low-energy. Then we can identify them as

$$\begin{aligned} v_{j,\pm} &\sim \text{linear combination of } v_{U(i_{j+1})}^{low\ energy} \\ v_{U(i_{j+1})}^{low\ energy} &: \text{ Coulomb branch coordinates of the } U(i_{j+1}) \end{aligned} \tag{6.13}$$

This weak perturbation sometimes helps us to study the duality especially on the dual side. We can use this perturbation for the derivations of both $U(N_c)$ and $SU(N_c)$ dualities.

6.1.1 The Kim-Park dual from 4d

We will here derive the Kim-Park duality using the four-dimensional Seiberg duality. We will start with the four-dimensional $SU(N_c)$ Kutasov-Schwimmer duality [38, 39]. In order to derive the 3d $U(N_c)$ duality, we first need to gauge the $U(1)_B$ global symmetry of the $SU(N_c)$ Kutasov-Schwimmer. The $U(1)$ dynamics is the IR free in 4d. There is no interesting strong coupling phenomenon at least in 4d.

The "electric" theory is the four-dimensional $\mathcal{N}=1$ supersymmetric $U(N_c)$ gauge theory with the N_f fundamental matters Q, $\widetilde{Q}$, one adjoint field X with the superpotential $W = \mathsf{Tr} X^{k+1}$.

The "magnetic" theory is the four-dimensional $\mathcal{N}=1$ supersymmetric $U(kN_f - N_c)$ gauge theory with the N_f fundamental matters q, $\tilde{q}$, one adjoint field Y and the singlets M_j, $j = 0, \cdots, k-1$ with the superpotential $W = \mathsf{Tr} Y^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q$.

We straightforwardly construct the duality on $\mathbb{R}^3 \times \mathbb{S}^1$ by adding the superpotential generated by the KK-monopoles to the 4d duality. The obtained duality can be effectively seen as the 3d duality with the compact Coulomb branch. However since in this case the superpotential from the KK-monopoles lifts the all of the Coulomb branch, then the compactness of the Coulomb branch does not matter.

Next we do the mass deformation to the theories in order to obtain the duality without η terms. We show how to do it in both electric and magnetic sides respectively. In this process, the electric and magnetic theories get the non-compact Coulomb branches.

The electric theory with η term

First we put the electric theory on a circle and compactify it. The theory is $U(N_c)$ gauge theory with the N_f fundamental matters Q, $\widetilde{Q}$, one adjoint field X with the superpotential $W = \mathsf{Tr} X^{k+1} + \sum_{i+j=k-1} \eta\, v_{i,+} v_{j,-}$, which contain the non-perturbative superpotential from the KK monopoles. We would like to obtain the theory without η term. Then we will start with $N_f + 2$ flavors and turn on the real masses for the $SU(N_f+2) \times SU(N_f+2)$ flavor symmetries:

$$m = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & m & \\ & & & & -m \end{pmatrix}, \quad \tilde{m} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & -m & \\ & & & & m \end{pmatrix} \tag{6.14}$$

Notice that this is traceless as it should be. Integrating out the massive flavors, we obtain at the low-energy the three dimensional $\mathcal{N}=2$ $U(N_c)$ gauge theories with N_f flavors and one adjoint matter with no introduction of the Chern-Simons terms. The global charges are precisely the one of the Kim-Park duality Table 6.2.

If we turn on the potential (6.10), then we obtain the sum of the $U(i_j)$ gauge theory with N_f+2 flavors and no adjoint matter. In the presence of the real mass deformations (6.14) we can take the low energy limit at $Q^{U(i_j)}_{N_f+1} = Q^{U(i_j)}_{N_f+2} = \tilde{Q}^{U(i_j)}_{N_f+1} = \tilde{Q}^{U(i_j)}_{N_f+2} = 0$ and obtain the sum of the $U(i_j)$ gauge theory with N_f flavors and no adjoint matter, where $Q^{U(i_j)}$s mean the $U(i_j)$ fundamental quarks. In the limit of $s_j = 0$ $(j \neq 0)$, we expect there may be the $U(N_c)$ gauge theory with N_f flavors and one adjoint field with the superpotential $W = \mathrm{tr}\, X^{k+1}$.

The magnetic theory with η term

Next we will put the dual theory on a circle and do the same deformation as the electric side. On the dual side we have the $U(k(N_f+2) - N_c)$ gauge theory with the $N_f + 2$ fundamental matters q, $\tilde{q}$, one adjoint field Y and the singlets M_j $(j = 0, \cdots, k-1)$ with the superpotential $W = \mathsf{Tr} Y^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q + \sum_{j=0}^{k-1} \tilde{\eta}\, \tilde{v}_{j,+} \tilde{v}_{k-1-j,-}$.

The real mass deformation in the electric side (6.14) is the $SU(N_f+2) \times SU(N_f+2)$ background gauging. Then it is easily mapped to the real mass of the dual side. The real masses in the magnetic theory are

$$m_{\mathsf{dual}} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & -m & \\ & & & & m \end{pmatrix}, \quad \tilde{m}_{\mathsf{dual}} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & m & \\ & & & & -m \end{pmatrix}. \tag{6.15}$$

We need to find the correct vacuum which corresponds to the vacuum of the electric side and take a low-energy limit. It helps us to use the weak deformation (6.10). With the introduction of the weak deformation, the dual gauge group is broken in two steps. At first the gauge group is broken by the deformation of the theory by $W = \sum_{j=0}^{k} \frac{s_j}{k+1-j} \mathsf{Tr}\, Y^{k+1-j}$;

$$U(k(N_f+2)-N_c) \to U(r_1) \times \cdots \times U(r_k), \quad \sum_{i=1}^{k} r_i = kN_f - N_c + 2k \tag{6.16}$$

In this breaking all the components of the adjoint matter become massive and then cannot have vevs. At the second step, the gauge group is broken by the vev of the $\tilde{A}_3 = \tilde{\sigma}$ adjoint scalar field in the vector superfield.

$$\widetilde{\sigma}_{U(r_i)\,\mathsf{part}} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & -m & \\ & & & & m \end{pmatrix} \tag{6.17}$$

$$\begin{aligned} U(k(N_f+2)-N_c) &\to U(r_1) \times \cdots \times U(r_k), \quad \sum_{i=1}^{k} r_i = kN_f - N_c + 2k \\ &\to (U(r_1-2) \times U(1)^2) \times \cdots \times (U(r_k-2) \times U(1)^2) \end{aligned} \tag{6.18}$$

At the low-energy the adjoint chiral superfield Y is massive and the low-energy effective theory is the sum of the $U(r_i) \times U(1)^2$ theory with fundamental matter with meson singlets. The matter fields decompose as

- $\hat{q}_f^{U(r_i-2)}, \hat{\tilde{q}}_{\tilde{f}}^{U(r_i-2)}$ $(f, \tilde{f} = 1, \cdots, N_f.\ i = 1, \cdots, k)$:
  $U(r_i-2)$ fundamental quarks which consist of N_f flavors. The $f, \tilde{f}$ are the flavor indices.
- $q_i^a, \tilde{q}_i^a$ $(a = 1, 2\ \ i = 1, \cdots, k)$:
  $\mathbf{1}_{\pm\delta_{a1}, \pm\delta_{a2}}$ representation under the gauge group $U(r_i) \times U(1)^2$. The lower indices indicate the corresponding $U(1)$ charges.
- $\hat{M}_j$ $(j = 0, \cdots, k-1)$:
  $N_f \times N_f$ meson singlets which come from the left-upper components of the mesons.
- $M_{aj} (a = 1, 2; j = 0, \cdots, k-1)$:
  Singlets coming from the N_f-th and (N_f+1)-th components of the mesons M_j. $M_{1j} \equiv M_{N_f+1, N_f+1}$, $M_{2j} \equiv M_{N_f+2, N_f+2}$

We define the composite operators;

$$\hat{N}_{i,j} := \hat{\tilde{q}}^{U(r_i-2)} Y^j \hat{q}^{U(r_i-2)} \tag{6.19}$$

$$N_{ai}^j := \tilde{q}_i^a Y^j q_i^a. \tag{6.20}$$

Strictly speaking, since we are in the broken phase (6.16) where we have no adjoint matter Y, we should write Y field as the vacuum expectation value $\langle Y \rangle$. However since we finally take the limit where the weak deformations are switched off and we expect the adjoint fields to recover as the massless degree of freedom, we write the adjoint matter Y naively as the dynamical field. In the following, since we deal with Y as the vevs, the upper indixes of $\hat{N}_i^j, N_{ai}^j$ does not distinguish the independent chiral operators. The upper indises are only important in turning off the weak deformations.

In addition, we define the monopole operators which describe the Coulomb branches of $(U(r_1 - 2) \times U(1)^2) \times \cdots \times (U(r_k - 2) \times U(1)^2)$:

$$\hat{\tilde{V}}_{j,\pm} : \text{ Coulomb branch of the } U(r_{j+1} - 2). \ (j = 0, \cdots, k-1) \tag{6.21}$$

$$\tilde{v}^{i,a}_{U(1)} : \text{ Coulomb branch of the } U(1)^2. \ (i = 0, \cdots, k-1. \ a = 1, 2) \tag{6.22}$$

In this notation we have the following superpotential at low-energy,

$$W = \sum_{i=1}^{k}\sum_{j=0}^{k-1} \hat{M}_j \hat{N}_{i,k-1-j} + \sum_{i=1}^{k}\sum_{j=0}^{k-1}\left(M_{1j}N_{1,i}^{k-1-j} + M_{2j}N_{2,i}^{k-1-j}\right) + \sum_{j=0}^{k-1} \tilde{\eta}\, \tilde{v}_{j,+}\tilde{v}_{k-1-j,-}$$
$$+ \sum_{\substack{a=1,2 \\ i=1,\cdots,k}} N^0_{a,i}\tilde{v}^{i,a}_{U(1),+}\tilde{v}^{i,a}_{U(1),-} + \sum_{j=1}^{k}\left(\hat{\tilde{V}}_{j,+}\tilde{v}^{j,1}_{U(1),-} + \hat{\tilde{V}}_{j,-}\tilde{v}^{j,2}_{U(1),+}\right) \tag{6.23}$$

where the fifth terms are due to the $U(1)^2$ dynamics. The $U(1)$ part is the $\mathcal{N} = 1$ supersymmetric QED with one flavor. Then we can use the dual description of the $\mathcal{N} = 2$ XYZ model with the above superpotential. The sixth and seventh terms come from the effect of the Affleck-Harvey-Witten type superpotential corresponding to the breaking $U(r_i) \to U(r_i - 2) \times U(1)^2$. $\hat{\tilde{V}}_{j,\pm}$ should be identified with the monopole operators of the $U(kN_f - N_c)$ in the limit $s_j \ (j \neq 0) \to 0$ and the monopole operators $\tilde{v}_{j,\pm}$ in the high energy should be identified with the linear combinations of the monopole operators $\tilde{v}^1_{j,+}, \ \tilde{v}^2_{j,-}$ of the $U(1)^2$ part.

The equations of motion drop the second, third, fourth, and fifth terms off, we finish with the Kim-Park magnetic dual in the limit of $s_1, \cdots, s_k = 0$. The magnetic theory eventually becomes the $U(kN_f - N_c)$ gauge theory with N_f flavors, one adjoint matter and singlets $M_j, \ v_{i,\pm}$ with the superpotential

$$W = \mathsf{Tr} Y^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q + \sum_{j=0}^{k-1}(v_{j,+}\tilde{v}_{k-1-j,-} + v_{j,-}\tilde{v}_{k-1-j,+}) \tag{6.24}$$

where the $U(1)^2$ Coulomb branch coordinates $\tilde{v}^{j,1}_{U(1),-}, \ \tilde{v}^{j,2}_{U(1),+}$ are identified with the chiral superfields $v_{j,\pm}$ in the Kim-Park magnetic theory and the Coulomb branch coordinates of the $U(kN_f - N_c)$ are denoted as $\tilde{v}_{j,\pm}$. Notice that the $U(1)^2$ part was dualized to the theory which only contains the chiral superfields with no gauge symmetry. In the limit of $s_j \ (j \neq 0) \to 0$, the part with $U(1)^2 \times \cdots \times U(1)^2$ gauge symetry becomes the $U(k) \times U(k)$ gauge theory. Each $U(k)$ theory contains one fundamental matter and an adjoint field. The vacuum of the $N_f = 1$ theory usually has a runaway behavior [42]. In this case, however, the various terms in the superpotential may stabilize the vacuum. In deriving the above duality we assumed that the enhancement of the gauge symmetry to $U(k)$ does not change the duality and the $U(1)$ physics correctly produces the duality.

6.1.2 The ungauging of the Kim-Park dual

The Kim-Park duality has the global topological $U(1)_J$ symmetry which we can gauge it. In the electric side we obtain the $U(N_c) \times U(1)_J$ gauge theory with N_f flavors. In the electric side there are no elementary fields which are charged under the $U(1)_J$ symmetry. Then the gauging of the $U(1)_J$ symmetry only introduces the mixed Chern-Simons term (we can think it as the topological mass.) between the $U(1)_J$ and $U(1) \subset U(N_c)$, flowing at the low-energy limit to the $SU(N_c)$ gauge theory which we want.

In the magnetic side, on the other hand, we have the elementary fields charged under the $U(1)_J$ global symmetry. These fields $X_\pm$ are the chiral superfields which describe the Coulomb branch of the electric theory. Then we cannot forget the $U(1)_J \times U(1) \subset U(kN_f - N_c)$ gauge dynamics.

The gauging of the $U(kN_f - N_c)$ magnetic theory is the $U(kN_f - N_c) \times U(1)_J$ gauge theory with the N_f fundamental matter, one adjoint field X and the singlets $M, v_{i,\pm}$ with the superpotential $W = \mathsf{Tr} Y^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q + \sum_{j=0}^{k-1} (v_{j,+} \tilde{v}_{k-1-j,-} + v_{j,-} \tilde{v}_{k-1-j,+})$. The singlets $v_{i,\pm}$ are only charged under the new $U(1)_J$ gauge symmetry. Now we have the following Coulomb branch; the Coulomb branch for the $U(kN_f - N_c)$ and the one for the gauged $U(1)_J$. Notice that the adjoint field X is coupled to the $U(kN_f - N_c)$ gauge theory but is not to the $U(1)_J$ gauge theory. Then the $U(1)_J$ gauge theory is the three dimensional $\mathcal{N} = 2, N_f = k$ supersymmetric QED with the above superpotential.

6.2 $SU(N)$ with adjoint matter

In this section we will derive the three-dimensional duality for $SU(N_c)$ with one adjoint matter from the reduction of the 4d duality. We will finally obtain the 3d duality proposed by [66] who used the un-gauging techique in deriving it. The 4d Kutasov-Schwimmer duality which we employ is the same as the previous section but we do not gauge the $U(1)_B$ flavor symmetry.

6.2.1 4d Kutasov-Schwimmer duality

We first recall the 4d duality which we will use. This is well known as the Kutasov-Schwimmer duality [39]. The electric and magnetic theories are as follows.

The electric theory

4d $\mathcal{N} = 1$ $SU(N_c)$ SUSY gauge theory with N_f flavors $Q, \tilde{Q}$ and one adjoint matter X with the superpotential $W = \mathrm{Tr}\, X^{k+1}$.

The magnetic theory

4d $\mathcal{N} = 1$ $SU(kN_f - N_c)$ SUSY gauge theory with N_f flavors $q, \tilde{q}$ and one adjoint matter Y and singlets M_j $(j = 0, \cdots, k-1)$ with the superpotential $W = \mathrm{Tr}\, Y^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q$.

Using this duality and compactifying the both theories on $\mathbb{S}^1$ we obtain the three-dimensional duality. In such a process we need to include the non-perturbative superpotential which arises from the twisted instantons.

6.2.2 3d Kutasov-Schwimmer duality (Park-Park duality)

We will put these four-dimensional theories on a circle. In the three-dimensional limit which is the case for the infinitesimally small circle, we have no axial anomaly, then we show the additional $U(1)_A$ symmetry in the following table. But we should keep the size of the circle finite in order to derive the 3d dualities. After the deformation of turning off the η terms, we can think of these theories as the genuine 3d theories which contain the un-lifted Coulomb branch.

The field contents are the same as the four-dimensional ones, but in three-dimensions there is a Coulomb branch which should be naturally identified with the chiral superfield. We will use the symbol $V, \tilde{V}$ for the Coulomb branch coordinates. Due to the presence of the adjoint matters, the monopole operators are dressed. Then we distinguish the dressed ones V_i, V_{ij} from the bare monopoles V by the indices.

We show the quantum numbers of the field contents in 4d theories and $\mathbb{R}^3 \times \mathbb{S}^1$ theories. The choices for the $U(1)_A$ and $U(1)_R$ symmetries are the anomalous combinations in 4d, but it is convenient for the 3d perspective.

Table 6.3: Quntum numbers of the electric side

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
Q	N_c	N_f	1	1	1	0
$\tilde{Q}$	$\bar{N_c}$	1	$\bar{N_f}$	-1	1	0
X	adj.	1	1	0	0	$\frac{2}{k+1}$
M_j	1	N_f	$\bar{N_f}$	0	2	$\frac{2j}{k+1}$
V_j	1	1	1	0	$-2N_f$	$2N_f - \frac{4}{k+1}(N_c-1) + \frac{2j}{k+1}$
V_{ij}	1	1	1	0	$-2N_f$	$2N_f - \frac{4}{k+1}(N_c-1) + \frac{2(i+j)}{k+1}$
Λ^b	1	1	1	0	$2N_f$	$-2N_f + \frac{4N_c}{k+1}$

Table 6.4: Quantum numbers of the magnetic side

	$SU(kN_f-N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
q	$\square$	$\bar{N_f}$	1	$\frac{N_c}{kN_f-N_c}$	-1	$\frac{2}{k+1}$
$\tilde{q}$	$\bar{\square}$	1	N_f	$-\frac{N_c}{kN_f-N_c}$	-1	$\frac{2}{k+1}$
Y	adj.	1	1	0	0	$\frac{2}{k+1}$
M_j	1	N_f	$\bar{N_f}$	0	2	$\frac{2j}{k+1}$
$\tilde{V}_j$	1	1	1	0	$2N_f$	$-2N_f + \frac{4}{k+1}(N_c+1) + \frac{2j}{k+1}$
$\tilde{V}_{ij}$	1	1	1	0	$2N_f$	$-2N_f + \frac{4}{k+1}(N_c+1) + \frac{2(i+j)}{k+1}$
$\tilde{\Lambda}^{\tilde{b}}$	1	1	1	0	$-2N_f$	$2N_f - \frac{4N_c}{k+1}$

The global charges of the (twisted) instanton factor Λ^b, $\tilde{\Lambda}^{\tilde{b}}$ are calculated as follows, which is of course the spurionic symmetries for the four dimensional gauge theories on $\mathbb{R}^3 \times \mathbb{S}^1$

$$U(1)_R: \quad \theta \to \theta + \underbrace{(+1)\cdot 2N_c}_{\lambda} + \underbrace{\left(\frac{2}{k+1}-1\right)\cdot 2N_c}_{X} + \underbrace{2N_f(-1)}_{Q,\tilde{Q}} \tag{6.25}$$

$$U(1)_A: \quad \theta \to \theta + \underbrace{2N_f(+1)}_{Q,\tilde{Q}} \tag{6.26}$$

$$R[\Lambda^b] = -2N_f + \frac{4Nc}{k+1} \tag{6.27}$$

$$A[\Lambda^b] = 2N_f \tag{6.28}$$

where b is the one loop coefficient of the beta function.

$$b = \overbrace{\frac{11}{3}T_{\text{adj.}}}^{\text{vector}} - \overbrace{\sum \frac{2}{3}T_f}^{\text{fermion}} - \overbrace{\sum \frac{1}{3}T_s}^{\text{scalar}} \tag{6.29}$$

$$= \frac{11}{3}N_c - \frac{2}{3}(\underbrace{N_c}_{\lambda} + \underbrace{N_c}_{X} + \underbrace{2N_f\cdot\frac{1}{2}}_{Q,\tilde{Q}}) - \frac{1}{3}(\underbrace{2N_f\cdot\frac{1}{2}}_{Q,\tilde{Q}} + \underbrace{N_c}_{X}) \tag{6.30}$$

$$= 2N_c - N_f \tag{6.31}$$

The monopole operators

We consider the $SU(N_c)$ gauge theories and the monopole background. We restrict ourself to the part of Coulomb branch which is labeled by $\phi_1 > \phi_2 > \cdots > \phi_K > 0 > \phi_{K+1} > \cdots > \phi_{N_c}$ without loss of generality, where ϕ_i is the adjoint scalar fields of the vector superfields. In this case

Callias index theorem tells us that the numbers of the fermonic zero-modes around the monopole-instantons are

$$\text{quark zero-mode: } N_{\Box} = \begin{cases} 1 & (\phi_K > 0 > \phi_{K+1}) \\ 0 & (\phi_i > \phi_{i+1} > 0 \text{ or } 0 > \phi_i > \phi_{i+1}) \end{cases} \tag{6.32}$$

$$\text{adj. fermion zero-mode: } N_{\mathsf{adj.}} = 2. \tag{6.33}$$

Then the global charges of the monopole operators are as follows,

Table 6.5: Quntum numbers of the monopole operators

	$U(1)_A$	$U(1)_R$
Y_1	0	$-2-2(\frac{2}{k+1}-1) = -\frac{4}{k+1}$
$\vdots$	$\vdots$	$\vdots$
Y_{K-1}	0	$-\frac{4}{k+1}$
Y_K	$-2N_f$	$-2-2(\frac{2}{k+1}-1)-2N_f(-1) = 2N_f - \frac{4}{k+1}$
$\vdots$	$\vdots$	$\vdots$
Y_{N_c-1}	0	$-\frac{4}{k+1}$
$V = \prod Y_i$	$-2N_f$	$2N_f - \frac{4}{k+1}(N_c-1)$

where we use the following notations for the bare and dressed monopole operators.

$$V = \prod_{i=1}^{N_c-1} Y_i \tag{6.34}$$

$$Y_i \sim \exp\left(\frac{\Phi_i - \Phi_{i+1}}{g^2}\right) \tag{6.35}$$

$$V_i \sim (X_{11})^i V \tag{6.36}$$

$$V_{ij} \sim (X_{11})^i (X_{N_cN_c})^j V \tag{6.37}$$

The chiral superfield Φ_i consists of the adjoint scalar ϕ_i and the dual photon a_i, which is defined as $\Phi_i = \frac{\phi_i}{g_3^2} + ia_i$. The tilde in the above monopole operators means that the r.h.s of (6.35) is only legitimate at the semi-classical domain with large ϕ_i. At the small ϕ_i, the 3d gauge coupling has a non-trivial loop correction and the metric on the Coulomb branch becomes intricate, so the definition of the monopole operator is involved. At the large ϕ_i region we can think that Y_is are the effective variables of the Coulomb branch except for the fact that the Y_is are not gauge-invariant. At the origin of the Coulomb branch, Y_is are not good description of the moduli because Y_is are fundamental (elementary) variables rather than the effective ones. Then the use of the equations of motion for Y_i is dangerous near the origin of the moduli space. If we want to make the monopole operators gauge-invariant, we need to combine the monopole operators in a Weyl-invariant way:

$$V = \prod_{i=1}^{N_c-1} Y_i \tag{6.38}$$

This combination is globally defined throughout all the region of the moduli space (all the chamber) and is expected to be un-lifted even if we include the non-perturbative effects.

The powers of X_{11} and $X_{N_cN_c}$ are truncated at $O(X^{k-1})$ due to the superpotential $W = \operatorname{Tr} X^{k+1}$ whose F-term equation gives $X^k \sim 0$.

The relation of the monopole operators between the $SU(N_c)$ and $U(N_c)$ gauge theories with an adjoint matter is as follows.

$$V = V_+ V_- \tag{6.39}$$

$$V_{ij} = V_{+i} V_{-j} \tag{6.40}$$

KK monopole induced superpotential

By the symmetry argument we assume the following superpotential is generated by the 1-KK monopole (twisted instanton) configuration.

$$W_{\mathsf{ele}} = \sum_{i+j=k-1} \Lambda^b V_{ij} = \sum_{i+j=k-1} \eta V_{ij} \tag{6.41}$$

$$W_{\mathsf{mag}} = \sum_{i+j=k-1} \tilde{\Lambda}^{\tilde{b}} \tilde{V}_{ij} = \sum_{i+j=k-1} \tilde{\eta} \tilde{V}_{ij} \tag{6.42}$$

The index theorem says that the fermionic zero modes around the KK monopole solution are too many for the KK monopoly to contribute to the superpotential. However we now have the adjoint matter which interact via the superpotential as

$$W = \mathsf{tr} X^{k+1} \tag{6.43}$$

This superpotential produce the following interaction.

$$\mathcal{L} \ni (X_{11})^j \, \psi_{X,1N_c} \, (X_{N_c N_c})^{k-1-j} \, \psi_{X,N_c 1} \tag{6.44}$$

Then the KK monopole vertex

$$\mathcal{M}_{\mathsf{KK}} \sim \mathrm{e}^{-S_0} \mathrm{e}^{\phi + i\sigma} \lambda^2 \psi_X^2 \tag{6.45}$$

is modified to

$$\mathcal{M}'_{\mathsf{KK}} \sim \mathrm{e}^{-S_0} \mathrm{e}^{\phi + i\sigma} \lambda^2 \, (X_{11})^j \, (X_{N_c N_c})^{k-1-j} \tag{6.46}$$

which is precisely coming from the above non-perturbative superpotential (6.41).

The dimensional reduction of the 4d Kutasov-Schwimmer duality with η terms.

Just like in the $U(N_c)$ duality, we first construct the duality on $\mathbb{R}^3 \times \mathbb{S}^1$. This can be easily done by including the KK monopole induced superpotential.

The electric side is the three dimensional $\mathcal{N}=2$ supersymmetric $SU(N_c)$ gauge theory with N_f fundamental flavors $Qs, \tilde{Q}s$ and one adjoint field X with the superpotential $W = \mathsf{tr} X^{k+1} + \sum_{i+j=k-1} \eta V_{ij}$. The magnetic side is the three dimensional $\mathcal{N}=2$ supersymmetric $SU(kN_f - N_c)$ gauge theory with N_f fundamental flavors $qs, \tilde{q}s$ and one adjoint field Y with the superpotential $W = \mathsf{tr} Y^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q + \sum_{i+j=k-1} \tilde{\eta} \tilde{V}_{ij}$.

In order to obtain the duality without the η terms we start with the N_f+1 flavors as the $U(N_c)$ case. We turn on the real mass for the last flavor in the electric theory side, which is the background gauging of the $SU(N_f+1) \times SU(N_f+1) \times U(1)_B$ flavor symmetries. In the dual side the corresponding real mass is mapped as follows.

$$m_r^{\mathsf{dual}} = \begin{pmatrix} m_1 & & & \\ & \ddots & & \\ & & m_1 & \\ & & & m_2 \end{pmatrix}, \quad \widetilde{m}_r^{\mathsf{dual}} = \begin{pmatrix} -m_1 & & & \\ & \ddots & & \\ & & -m_1 & \\ & & & -m_2 \end{pmatrix} \tag{6.47}$$

$$m_1 = \frac{k}{kN_f + k - N_c} m, \quad m_2 = \frac{N_c - kN_f}{kN_f + k - N_c} m \tag{6.48}$$

The electric theory flows to the $SU(N_c)$ gauge theory with N_f flavors with no η terms since the high energy monopole operators vanish due to the absence of the complex masses as discussed in [34]. The quantum numbers of the electric theory are following,

Table 6.6: Quantum numbers of the 3d $SU(N_c)$ theory on the electric side

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
Q	N_c	N_f	1	1	1	r
$\tilde{Q}$	$\bar{N}_c$	1	$\bar{N}_f$	-1	1	r
X	adj.	1	1	0	0	$\frac{2}{k+1}$
M_j	1	N_f	$\bar{N}_f$	0	2	$2r+\frac{2j}{k+1}$
V_{ij}	1	1	1	0	$-2N_f$	$2N_f(1-r)-\frac{4}{k+1}(N_c-1)+\frac{2(i+j)}{k+1}$

where we keep the R-charges of the quarks as generic values.

The flow to the low-energy in the magnetic side is complicated and it is helpful to consider the weak deformation of the theory. In the next subsubsection we will introduce the potential for the adjoint field which breaks the $U(1)_R$ symmetry and we will find the dual description.

Before doing so, we need to care about the following fact. Without the weak deformation presented below, the magnetic theory will flow as follows. The adjoint scalar field which comes from the A_3 part takes the vev as

$$\tilde{\sigma} = \begin{pmatrix} -m_1 & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & -m_1 & & & \\ & & & & -m_2 & & \\ & & & & & \ddots & \\ & & & & & & -m_2 \end{pmatrix} \tag{6.49}$$

Then the gauge symmetry breaks as follows.

$$SU(k(N_f+1)-N_c) \to SU(kN_f-N_c)\times SU(k)\times U(1)/(\mathbb{Z}_{kN_f-N_c}\times\mathbb{Z}_k) \tag{6.50}$$

or

$$SU(k(N_f+1)-N_c) \to S\left(U(kN_f-N_c)\times U(k)\right). \tag{6.51}$$

It might be more rigorous to consider the symmetry breaking as the second one (6.51) than the first one. But it is easy to consider the first one (6.50) practically. To obtain the dual theory, we need to change the $U(1)$ and the $U(k)$ dynamics to the dual description. But this is very difficult at present. The one difficulty would be related to the fact that the minimal monopole charge is smaller than the usual one since $\mathbb{Z}_{kN_f-N_c}\times\mathbb{Z}_k$ is factored out from the broken gauge symmetry. This fact makes the relation between the high- and low-energy monopole operators quite complicated. Then the superpotential from the KK-monopoles takes the complicated form in terms of the low-energy monopole operators, leaving the some supersymmetric vacua.

The second difficulty is due to the theory with $SU(k)$ gauge symmetry. This part of the full theory contains one fundamental flavor and one adjoint matter. This theory seems to have no SUSY vacua since the Affleck-Harvey-Witten type superpotential lift the all the SUSY vacua. However, this theory contains the complicated superpotential and this may give the stable SUSY vacua. Again the complicated relation between the high- and low-energy monopole operators make it difficult to analyze the stable SUSY vacua.

Then we will adopt the weak deformation in the next subsection, which successfully produces the 3d Kutasov-Schwimmer duality. We should remember that turning off the weak deformation for the adjoint matter gives the enhanced gauge symmetry $U(k)$ or $SU(k)$. However we expect this enhancement would not destroy the duality obtained by the weak deformation.

Weak deformation of the theory

The flow in the magnetic side is complicated and it is helpful to consider the weak deformation of the theory. We consider the weak perturbation of the theory by

$$W_{\mathsf{ele}} = \sum_{i=0}^{k} g_i \mathsf{Tr} X^{i+1} \tag{6.52}$$

$$W_{\mathsf{mag}} = \sum_{i=0}^{k} g_i \mathsf{Tr} Y^{i+1} \tag{6.53}$$

where g_0 is the Lagrange multiplier imposing the constraint such that $\mathsf{Tr}X = 0$. This weakly perturbed theory has generically the k vacua for the vev of X. Then the gauge symmetry is generically broken by the vev of the adjoint field X like $SU(N_c) \to SU(i_1) \times SU(i_2) \times \cdots \times SU(i_k) \times U(1)^{k-1}$, where $\sum_{j=1}^{k} i_j = N_c$.

Again we should note the following fact: It might be better to consider that this breaking should be actually written as $S(U(i_1) \times \cdots \times U(i_k))$. This might be important when considering the matching of the monopole operators between at the high and low energies. In this case the monopole operators which are required to describe the Coulomb branch become quite difficult. And we do not know how to find such set of the monopole operator. In this thesis, so we don't consider this possibility and concentrate on the breaking $SU(N_c) \to SU(i_1) \times SU(i_2) \times \cdots \times SU(i_k) \times U(1)^{k-1}$, where $\sum_{j=1}^{k} i_j = N_c$. This possibility is left as the future work.

In the electric side, we start with $N_f + 1$ flavors and take $A_3 = \sigma = 0$ and flow to the low energy limit with Q^{N_f+1} and $\tilde{Q}^{N_f+1}$ integrated out. By tuning off the g_i $(i \neq k)$ we finally get the $SU(N_c)$ gauge theory with N_f flavors and an adjoint field X recovered at the low energy. This is a three-dimensional version of the electric theory in the Kutasov-Schwimmer duality.

In the dual side, $W = \sum_{i=0}^{k} g_i \mathsf{Tr}\, Y^{i+1}$ breaks the gauge group as $SU(kN_f - N_c) \to SU(n_1) \times \cdots \times SU(n_k) \times U(1)^{k-1}$, $\sum_{i=1}^{k} n_i = kN_f - N_c$. When we start with $N_f + 1$ flavors and turn on the real masses corresponding to the electric side, we have the additional breaking of the gauge symmetry by the vevs of $\tilde{A}_3 \equiv \tilde{\sigma}$ as follows.

$$\begin{aligned} SU(k(N_f+1) - N_c) &\to SU(N_1) \times \cdots \times SU(N_k) \times U(1)^{k-1} \\ &\to SU(n_1) \times \cdots \times SU(n_k) \times SU(m_1) \times \cdots \times SU(m_k) \times U(1)^k \times U(1)^{k-1} \end{aligned} \tag{6.54}$$

where

$$n_i = N_i - m_i \tag{6.55}$$

$$\sum_{i=i}^{k} N_i = kN_f - N_c + k \tag{6.56}$$

$$\sum_{i=1}^{k} m_i = k, \quad \sum_{i=1}^{k} n_i = kN_f - N_c \tag{6.57}$$

$$\tilde{\sigma}_{N_1 \times N_1} = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{1N_1} \end{pmatrix}, \cdots, \tilde{\sigma}_{N_k \times N_k} = \begin{pmatrix} a_{k1} & & \\ & \ddots & \\ & & a_{kN_k} \end{pmatrix} \tag{6.58}$$

where a_{ij} take the value of $-m_1$ or $-m_2$ and satisfy the condition that $\sum_{i,j} a_{ij} = 0$ and (the number of $-m_1$) $= kN_f - N_c$ due two the traceless condition of the adjoint field $\tilde{\sigma}$.

We need to choose the correct vacuum which corresponds to the electric theory which is

$$\tilde{\sigma}_{N_1 \times N_1} = \begin{pmatrix} -m_1 & & & \\ & \ddots & & \\ & & -m_1 & \\ & & & -m_2 \end{pmatrix}, \cdots, \tilde{\sigma}_{N_k \times N_k} = \begin{pmatrix} -m_1 & & & \\ & \ddots & & \\ & & -m_1 & \\ & & & -m_2 \end{pmatrix} \tag{6.59}$$

Then the gauge group is broken as

$$SU(k(N_f+1)-N_c) \to SU(N_1-1)\times\cdots\times SU(N_k-1)\times U(1)^{k-1}\times U(1)\times U(1)^{k-1}. \quad (6.60)$$

Here we will clarify the $U(1)^{2k-1}$ part. The first $U(1)^{k-1}$ corresponds to the generator coming from the left upper $(kN_f-N_c)\times(kN_f-N_c)$ part. The second $U(1)$ is following.

$$T = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{pmatrix}, \quad \mathrm{Tr}\, T = 0 \quad (6.61)$$

where we show the case with $kN_f - N_c = 3,\ k=2$ for simplicity. The last $U(1)^{k-1}$ is the right lower part:

$$T' = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}. \quad (6.62)$$

which is also the case with $kN_f - N_c = 3,\ k=2$.

In order to obtain the original $SU(N_c)$ duality with an adjoint matter and with no η term we need to tune the potential as $g_i(i=0,\cdots,k-1)\to 0$ in the electric and magnetic side.Thus the gauge group $SU(N_1-1)\times\cdots\times SU(N_k-1)\times U(1)^{k-1}$ is expected to recover the dual $SU(kN_f-N_c)$ gauge group with the adjoint matter in the limit of $g_i(i=0,\cdots,k-1)\to 0$. Before doing so, we will change the dynamics of the second $U(1)^{k-1}$ to the dual description with no gauge group. Here we do not expect that the recovery of the $SU(k)$ gauge group from the right-lower $U(1)^{k-1}$ in the limit of $g_i(i=0,\cdots,k-1)\to 0$ modify the low-energy duality. This is only an assumption and is not justified because the non-diagonal components of the $k\times k$ adjoint fields become massless and these should be taken into account to the low-energy dynamics in the limit with $g_i \to 0$. Although this is a working assumotion, it will be found this nicely works. The theory can be seen as $SU(kN_f-N_c)\times U(1)\times U(1)^{k-1} \cong U(kN_f-N_c)\times U(1)^{k-1}$ gauge theory at this stage.

We may alternatively have the $U(kN_f-N_c)\times SU(k)$ gauge theory in the limit of $g_i(i=0,\cdots,k-1)\to 0$. If this is correct, we have the various fields from the adjoint fields Y and $q,\tilde{q}$. In this thesis we will not consider this possibility.

Taking the limit as $g_i\ (i=k)\to 0$ we obtain the $U(kN_f-N_c)\times U(1)^{k-1}$ theory whose matter contents are as follows.

- $U(kN_f-N_c)$ fundamental matters: N_f flavors $q,\tilde{q}$
- $U(kN_f-N_c)$ adjoint matter: One adjoint field Y which comes from the $(kN_f-N_c)\times(kN_f-N_c)$ left-upper components of the original adjoint fields Y. With the weak deformation Y should be replaced with the vacuum expectation value.
- The singlets $M_j, \tilde{M}_j\ (j=0,\cdots,k-1)$: M_j fields also have the $N_f\times N_f$ flavor indices. On the other hand $\tilde{M}_j$ are the flavor singlets because they are originally from the last component of M_j.
- The singlets $Y_i\ (i=1,\cdots,k)$, which comes from the diagonal parts of the right-bottom $k\times k$ components of the adjoint field Y. These fields should be recognized as the vacuum expectation values $\langle Y_i\rangle$.
- $U(1)^{k-1}$ charged fields: $q_i,\ \tilde{q}_i\ \ (i=1,\cdots,k)$. We consider the gauge group $U(1)^{k-1}$ as $U(1)^k/U(1)$. In this perspective the charge assignment of $q_i,\tilde{q}_i$ are precisely the same as the 3d $\mathcal{N}=2$ mirror symmetry [42, 61]. They are also charged under the $U(1)\subset U(kN_f-N_c)$.

The superpotential becomes as follows.

$$W = \mathrm{Tr}Y^{k+1} + \sum_{i=1}^{k} Y_i^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q + \sum_{j=0}^{k-1}\sum_{i=1}^{k} \tilde{M}_j \tilde{q}_i Y_i^{k-1-j} q_i + \sum_{j=0}^{k-1} \tilde{\eta} \tilde{V}_{j,k-1-j} \tag{6.63}$$

where Y in the superpotential should be constant vevs and Y fields are all massive. However we loosely write the superpotential as presented above.

We define a new singlet fields $S_i := \sum_{j=0}^{k-1} \tilde{M}_j Y_i^{k-1-j}$ $(i = 1, \cdots, k)$, yielding the following superpotential.

$$W = \mathrm{Tr}Y^{k+1} + \sum_{i=1}^{k} Y_i^{k+1} + \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q + \sum_{i=1}^{k} S_i \tilde{q}_i q_i + \sum_{j=0}^{k-1} \tilde{\eta} \tilde{V}_{j,k-1-j} \tag{6.64}$$

The $U(1)^{k-1}$ theory has the dual description by the mirror symmetry due to the appearance of the superpotential $W = \sum_{i=1}^{k} S_i \tilde{q}_i q_i$. The mirror theory is the three-dimensional $\mathcal{N} = 2$ supersymmetric $U(kN_f - N_c) \times U(1)^{\mathsf{mirror}}$ gauge theory. The field contents are summarized as follows.

- $U(kN_f - N_c)$ fundamentals: N_f flavors $q, \tilde{q}$.
- $U(kN_f - N_c)$ adjoint: one adjoint field Y
- The $N_f \times N_f$ gauge singlets: M_j $(j = 0, \cdots, k-1)$
- $U(1)$ $\mathcal{N} = 2$ SQED with k flavors $b_i, \tilde{b}_i$ $(i = 1, \cdots, k)$

The quantum numbers of the fields are calculated by identifying the baryonic operators betwen the electric and magnetic theories.

$$B_{\mathsf{electric}}^{n_1, n_2, \cdots, n_k} \equiv Q_{(0)}^{n_1} \cdots Q_{(k-1)}^{n_k}, \quad \sum_{j=1}^{k} n_j = N_c \tag{6.65}$$

$$B_{\mathsf{magnetic}}^{m_1, \cdots, m_k} \equiv V_{-}^{U(1)^{\mathsf{mirror}}} q_{(0)}^{m_1} \cdots q_{(k-1)}^{m_k}, \quad m_j = N_f - n_{k+1-j} \tag{6.66}$$

where the magnetic baryon operators contain the Coulomb branch coordinate of the mirror $U(1)^{\mathsf{mirror}}$ because in the $U(kN_f - N_c) \times U(1)^{k-1}$ theory, $V_{-}^{U(1)^{\mathsf{mirror}}}$ is identified with $N_- := q_1 q_2 \cdots q_k$ and the baryonic operators become $B_{\mathsf{magnetic}}^{m_1, \cdots, m_k} = q_{(0)}^{m_1} \cdots q_{(k-1)}^{m_k} \prod_{i=1}^{k} q_i$, which are the natural baryonic operators to be identified with the electric side.

The quantum numbers are summarized as follows. Actually we have an ambiguity about the choice of the $U(1)_B$ charges to mix it with the other $U(1)$ symmetries. The $U(1)_R$ charges of the $b_i, \tilde{b}_i$ fields are different from the conventional assignment of the mirror symmetry. This is due to the presence of the superpotential $W = \sum_{j=0}^{k-1} b_{j+1} \tilde{V}_{k-1-j,-}^{U(kN_f - N_c)}$, which breaks the usual $U(1)_R$ symmetry and generate a new $U(1)_R$ symmetry at the low energy. Notice that the $U(1)^{\mathsf{mirror}}$ symmetry is the gauging of the topological $U(1)$ symmetry corresponding to $U(1) \subset U(kN_f - N_c)$ gauge group and the $U(1) \subset U(kN_f - N_c)$ symmetry is the gauging of the topological symmetry coresponding to the $U(1)^{\mathsf{mirror}}$ gauge group. Thus we have the Chern-Simons coupling between $U(1)^{\mathsf{mirror}}$ and $U(1) \subset U(kN_f - N_c)$.

Table 6.7: Quntum numbers of the mirror theory

	$U(kN_f-N_c)\times U(1)^{\text{mirror}}$	$U(1)_B$	$U(1)_A$	$U(1)_R$
q	$\left(\square_{\frac{1}{kN_f-N_c}},0\right)$	0	-1	$-r+\frac{2}{k+1}$
$\tilde{q}$	$\left(\overline{\square}_{-\frac{1}{kN_f-N_c}},0\right)$	0	-1	$-r+\frac{2}{k+1}$
Y	$(\text{adj.},0)$	0	0	$\frac{2}{k+1}$
b_i	$(1_0,1)$	0	$-N_f$	$-(r-1)N_f-\frac{2N_c}{k+1}+\frac{2i}{k+1}$
$\tilde{b}_i$	$(1_0,-1)$	0	$-N_f$	$-(r-1)N_f-\frac{2N_c}{k+1}+\frac{2i}{k+1}$
M_j	$(1_0,0)$	0	2	$2r+\frac{2j}{k+1}$
$V_{\pm}^{U(1)^{\text{mirror}}}$	$(1_{\pm1},0)$	$\mp N_c$	kN_f	$(r-1)kN_f+\frac{2kN_c}{k+1}$
$\tilde{V}_{j,\pm}^{U(kN_f-N_c)}$	$(1_0,\pm1)$	0	N_f	$(r-1)N_f+\frac{2}{k+1}(N_c+1)+\frac{2j}{k+1}$

The superpotential is

$$W=\operatorname{Tr}Y^{k+1}+\sum_{j=0}^{k-1}M_j\tilde{q}Y^{k-1-j}q+\sum_{j=0}^{k-1}\tilde{\eta}\tilde{V}_{j,k-1-j} \tag{6.67}$$

The monopole operators at high energy should be identified as follows.

$$\tilde{V}_{j,k-1-j}=b_{j+1}\tilde{V}_{k-1-j,-}^{U(kN_f-N_c)},\quad (j=0,\cdots,k-1). \tag{6.68}$$

In addition to the above superpotential we have the Affleck-Harvey-Witten type superpotential [78] which is generated by the gauge symmetry breaking $SU(k(N_f+1)-N_c)\to U(kN_f-N_c)\times U(1)^{k-1}\underset{\text{mirror}}{\approx}U(kN_f-N_c)\times U(1)^{\text{mirror}}$:

$$W_{\text{AHW}}=\sum_{j=0}^{k-1}\tilde{b}_{j+1}\tilde{V}_{k-1-j,+}^{U(kN_f-N_c)} \tag{6.69}$$

Putting these into the superpotential we obtain the dual (mirror) theory,

$$W=\operatorname{Tr}Y^{k+1}+\sum_{j=0}^{k-1}M_j\tilde{q}Y^{k-1-j}q+\sum_{j=0}^{k-1}\left(b_{j+1}\tilde{V}_{k-1-j,-}^{U(kN_f-N_c)}+\tilde{b}_{j+1}\tilde{V}_{k-1-j,+}^{U(kN_f-N_c)}\right), \tag{6.70}$$

where $\tilde{\eta}$ is absorbed by the field rescaling. This is precisely in accord with with the result obtained by the un-gauging technique [66].

6.2.3 Summary of the 3d $SU(N)$ Kutasov-Schwimmer-Kim-Kim-Park duality

We summarize the 3d $SU(N_c)$ duality with an adjoint matter which we found here.

The electric side

The electric side is simple. $SU(N_c)$ gauge theory with N_f fundamental flavors $Q,\widetilde{Q}$s and an adjoint matter X with the superpotential $W=\operatorname{tr}X^{k+1}$. The charge assignment is summarized in the above table.

The magnetic side

The $\mathcal{N}=2$ supersymmetric $U(kN_f-N_c)\times U(1)$ gauge theory with N_f flavors $q,\tilde{q}$, one adjoint field Y, k charged fields b_i, $\tilde{b}_i$ under the $U(1)$ and the gauge singlets M_j. The superpotential is following.

$$
\begin{aligned}
W = \mathrm{Tr} Y^{k+1} &+ \sum_{j=0}^{k-1} M_j \tilde{q} Y^{k-1-j} q \\
&+ \sum_{j=0}^{k-1} \left(b_{k-1-j} V_{j,+}^{U(kN_f-N_c)} + \tilde{b}_{k-1-j} V_{j,-}^{U(kN_f-N_c)} \right)
\end{aligned}
\tag{6.71}
$$

This theory which was obtained from the 4d duality is identical to the one which was considered by [66] in which the authors used the un-gauging technique to obtain the $SU(N_c)$ duality from the $U(N_c)$ Kim-Park duality [64].

Chapter 7

Summary

In this thesis we discussed the three-dimensional $\mathcal{N}=2$ supersymmetric gauge theories, their quantum dynamics, the non-perturbative effects coming from the semi-classical analysis, the relation between the three- and four-dimensional Seiberg duality. We extensively reviewed what we know to date about the Coulomb branch and its dynamics. We also revealed how the duality of the three-dimensional $\mathcal{N}=2$ supersymmetric gauge theory with N_f fundamental matters and one adjoint matter with a superpotential $W = \mathrm{Tr}\, X^{k+1}$ comes from the four-dimensional Kutasov-Schwimmer duality.

In the 3d SUSY gauge theories we inevitably have the adjoint scalar which in some cases makes the analysis easier and in some cases makes more difficult than the 4d SUSY theories. Especially the appearance of the Coulomb branch due to the adjoint scalar makes the 3d physics more interesting and more complicated. The description of the Coulomb branch is always effective and the loop corrections should be considered at higher order. However the flat direction of this adjoint scalar makes the theory more tractable. At the large vev region of the adjoint scalar we can rely on the semi-classical analysis. Then the one loop plus the instanton analysis gives the correct quantum picture of the gauge theories. The monopole-instanton generates the non-perturbative superpotential and lifts almost all the Coulomb branch which are flat direction at the classical and one-loop level. In this thesis we derived the various superpotentials. One of these is the superpotential generated by the monopoles. The second one is the superpotential generated by the KK-monopoles, which is important when considering the $\mathbb{S}^1 \times \mathbb{R}^3$ theories and the reduction of the 4d physics to 3d. We found that when we include the adjoint matter the different form of the potential is generated by the KK monopoles. This superpotential was crucial for deriving the 3d duality with an adjoint matter from the 4d duality.

We investigated the relation between the three- and four-dimensional SUSY gauge theories and their 3d, 4d Seiberg dualities. The 3d Seiberg duality is derived from the 4d Seiberg duality if we include the non-perturbative superpotential frm the KK-monopoles. Especially we derived the duality of the three-dimensional $\mathcal{N}=2$ supersymmetric gauge theory with N_f fundamental matters and one adjoint matter with a superpotential $W = \mathrm{Tr}\, X^{k+1}$ from the four-dimensional Kutasov-Schwimmer duality. We especially concentrated on $U(N_c)$, $SU(N_c)$ gauge groups and the dualities obtained here were the ones which are given by [64,66]. While in [64,66] the dualities were guessed by the generalization of the Aharony duality and by the un-gauging technique respectively, we offered the derivation of these dualities in this paper. The duality for the $U(N_c)$ with an adjoint matter was derived in a similar way of the derivation of the Aharony duality from the 4d $U(N_c)$ Seiberg duality [63]. Hence, the superpotential from the KK-monopoles, Affleck-Harvey-Witten and the XYZ model from the $U(1)^2$ part play a crucial role to modify the Coulomb branch on the magnetic side. The $SU(N_c)$ duality with an adjoint matter was derived in a same way as the $U(N_c)$ case, where we used the $\mathcal{N}=2$ mirror symmetry which is the generalization of the duality between the SQED with $N_f = 1$ and the XYZ model, which makes the Coulomb branch of the dual gauge theory lifted together with the Affleck-Harvey-Witten type superpotential.

It would be worth analyzing the derivation in this paper without the weak perturbation for the adjoint matters X and Y, in which we will have to include the dynamics of the non-abelian gauge group $SU(k)$ for the $SU(N_c)$ duality and $U(k) \times U(k)$ for the $U(N_c)$ duality. In deriving the duality we assumed that the $U(1)^{k-1}$ part does not enhance to $SU(k)$ in the limit of $s_i \to 0$ for $SU(N_c)$ and this enhancement does not modify the duality we found. In the $U(N_c)$ cases we relied on the similar assumption. This was only the working assumption that discarding the off-diagonal part of the adjoint fields would not be important for deriving dualities, hence it should be investigated more carefully. It would be also worth considering other gauge groups, for example $O(N_c)$, $SO(N_c)$ and $Sp(2N_c)$ with an adjoint matter.

It is important to extend the procedure of deriving the 3d dualities from 4d to the dualities with various matter fields as [36] (See [41] for examples of the 4d dualities). In [41] the symmetric tensor matter or anti-symmetric tensor matter are added to the 4d dualities and the chiral theories are also considered. It is interesting to study the corresponding 3d dualities. Although the procedure to obtain the 3d dualities is the same as ones in [34–36] and in this paper, it would be more and more subtle because the structure of the Coulomb branch becomes complicated. To obtain the 3d dualities, it is necessary to match the monopole operators at high- and low-energy theory when we turn on the real masses and take the low energy limit and it is very subtle task. This extension is left as a future work.

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I am forever grateful to my mother and this thesis is devoted to my mother.

Appendix A

The fermion zero-modes on the inatanton-monopole background

The number of the fermion zero-modes around the monopole background is calculated by the Callias index theorem [71]:

$$N = \frac{1}{2}\sum_{w} \mathrm{sign}\,(\mathrm{Tr}\, w\phi)\,\mathrm{Tr}(wg) \tag{A.1}$$

where w is the weight of the fermion considered and the sum is taken over all the weights. In this appendix we resterict ourself to the vanishing real masses. g is the dual simple root which represents the particular monopole we consider. Please see the references [43] for the physical derivation of the index theorem and the notation of the Lie algebras for [79].

A.1 $U(n)$ case

Cartan

In this case the dual simple roots take the same form as the simple root. Let's us define the Cartan subalgebra.

$$h_1 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix},\ h_2 = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & -1 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}, \cdots,\ h_{n-1} = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -1 \end{pmatrix},$$

$$h_n = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \tag{A.2}$$

$$\begin{cases} h_j = E_1 - E_{j+1} \quad (j = 1, \cdots, n-1) \\ h_n = I \end{cases} \tag{A.3}$$

The corresponding monopole operators become as follows.

$$V_1 = \mathrm{e}^{\phi_1 - \phi_2},\ V_2 = \mathrm{e}^{\phi_1 - \phi_3}, \cdots, V_{n-1} = \mathrm{e}^{\phi_1 - \phi_n},\ V_n = \mathrm{e}^{\phi_1 + \cdots + \phi_n} \tag{A.4}$$

We here assume that we are in the region of the Coulomb branch of $\phi_1 > \cdots > \phi_k > 0 > \phi_{k+1} > \cdots > \phi_n$.

The fundamental representation

The weights of the fundamental representation are labeled by

$$w_1 = \overbrace{\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}}^{n\times n}, \quad w_2 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \cdots, \quad w_n = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \tag{A.5}$$

which are of course n-dimensional.

For $g = h_j \quad (j \neq n)$, the zero-modes are counted as

$$\begin{aligned} N &= \frac{1}{2}\sum_{i=1}^{n} \text{sign}(w_i\phi)\text{Tr}(w_i g) && \text{(A.6)} \\ &= \frac{1}{2}\sum_{i} \phi_i(\delta_{i1} - \delta_{i,j+1}) && \text{(A.7)} \\ &= \frac{1}{2}(\text{sign}\phi_1 - \text{sign}\phi_{j+1}) && \text{(A.8)} \\ &= \begin{cases} 0 & (\phi_1 > \phi_{j+1} > 0) \\ 1 & (\phi_1 > 0 > \phi_{j+1}). \end{cases} && \text{(A.9)} \end{aligned}$$

For $g = h_n$, zero-modes become

$$\begin{aligned} N &= \frac{1}{2}\sum_{i=1}^{n} \text{sign}(\phi_i) && \text{(A.10)} \\ &= \frac{1}{2}(k - (n-k)) = k - \frac{n}{2}. && \text{(A.11)} \end{aligned}$$

Thus the fundamental fermion zero-modes are summarized for each instanton factor as follows:

$$\begin{aligned} V_1 = e^{\phi_1 - \phi_2} &: \#\text{ of quark zero-modes} = 0 && \text{(A.12)} \\ V_2 = e^{\phi_1 - \phi_3} &: \#\text{ of quark zero-modes} = 0 && \text{(A.13)} \\ &\vdots \\ V_{k-1} = e^{\phi_1 - \phi_k} &: \#\text{ of quark zero-modes} = 0 && \text{(A.14)} \\ V_k = e^{\phi_1 - \phi_{k+1}} &: \#\text{ of quark zero-modes} = 1 && \text{(A.15)} \\ &\vdots \\ V_{n-1} = e^{\phi_1 - \phi_n} &: \#\text{ of quark zero-modes} = 1 && \text{(A.16)} \\ V_n = e^{\phi_1 + \phi_2 + \cdots + \phi_n} &: \#\text{ of quark zero-modes} = k - \frac{n}{2} && \text{(A.17)} \end{aligned}$$

If we alternatively consider the following instanton factor

$$t = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \tag{A.18}$$

which is represented by $V_+ = e^{\phi_1}$, then the zero-modes around this instanton are calculated as

$$V_+ = (V_1 V_2 \cdots V_n)^{\frac{1}{n}} : \#\text{ of quark zero-modes} = \frac{1}{n}\left(1\cdot(n-k) + k - \frac{n}{2}\right) = \frac{1}{2}. \tag{A.19}$$

Notice that the number of the fermion zero-modes around each instanton factor depends on where the Coulomb branch we are in. Then it is important to consider the branch parametrized by $\phi_1 > \cdots > \phi_k > 0 > \phi_{k+1} > \cdots > \phi_n$. However the combination V_+ are not dependent of k.

Actually in this thesis we are considering the 3d $\mathcal{N}=2$ SQCD which is a vector-like theory. Then we have double the number of quarks which come from Q and $\tilde{Q}$. And we consider the N_f flavors. So the fundamental sermonic zero-modes becomes:

$$V_+ = e^{\phi_1} : \# \text{ of quark zero-modes} = \frac{1}{2} \cdot \underbrace{2}_{Q,\tilde{Q}} \cdot \underbrace{N_f}_{N_f \text{ flavors}} \tag{A.20}$$

Note that the above calculation of the fermonic zero-modes around the monopole-instanton are independent of the choices of the matrices h_i. We can use the following matrices:

$$h_1 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix},\ h_2 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}, \cdots,\ h_{n-1} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix},$$

$$h_n = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \tag{A.21}$$

which are natural for the simple roots.

The adjoint representation

In the supersymmetric gauge theories we inevitably have the gauginos which are the adjoint fermions. So, next we will consider the adjoint fermion zero-modes. The roots (weights for the adjoint rep.) is given by

$$\alpha_{ij} = w_i - w_j \quad (i \neq j) \tag{A.22}$$

For $g = h_K \quad (K \neq n)$, the adjoint fermion zero-modes are counted by the Callias index theorem:

$$\begin{aligned} N &= \frac{1}{2} \sum_{\substack{i \neq j \\ i,j=1}}^{n} \text{sign}\left(\text{Tr}\alpha_{ij}\phi\right) \text{Tr}(\alpha_{ij} h_K) && \text{(A.23)} \\ &= \frac{1}{2} \sum_{i,j} \text{sign}(\alpha_{ij}\phi)(\delta_{i1} - \delta_{iK+1} - \delta_{j1} + \delta_{jK+1}) && \text{(A.24)} \\ &= \sum_{1 \le i < j \le n} (\delta_{i1} - \delta_{iK+1} - \delta_{j1} + \delta_{jK+1}) && \text{(A.25)} \\ &= (n-1) - (n-K-1) + K && \text{(A.26)} \\ &= 2K && \text{(A.27)} \end{aligned}$$

where we again assumed that we are in the Coulomb branch: $\phi_1 > \cdots > \phi_k > 0 > \phi_{k+1} > \cdots > \phi_n$.

For $g = h_n$

$$\begin{aligned} N &= \frac{1}{2} \sum_{\substack{i \neq j \\ i,j=1}}^{n} \text{sign}\left(\text{Tr}\alpha_{ij}\phi\right) \text{Tr}(\alpha_{ij} h_n) && \text{(A.28)} \\ &= \frac{1}{2} \sum_{\substack{i \neq j \\ i,j=1}}^{n} \text{sign}\left(\text{Tr}\alpha_{ij}\phi\right) \text{Tr}(\alpha_{ij}) && \text{(A.29)} \\ &= 0 && \text{(A.30)} \end{aligned}$$

Thus the adjoint fermion zero-modes for each instanton factor becomes:

$$V_1 = e^{\phi_1-\phi_2} : \# \text{ of gaugino zero-modes} = 2K|_{K=1} = 2 \tag{A.31}$$
$$V_2 = e^{\phi_1-\phi_3} : \# \text{ of gaugino zero-modes} = 2K|_{K=2} = 4 \tag{A.32}$$
$$\vdots$$
$$V_{k-1} = e^{\phi_1-\phi_k} : \# \text{ of gaugino zero-modes} = 2K|_{K=k-1} = 2(k-1) \tag{A.33}$$
$$V_k = e^{\phi_1-\phi_{k+1}} : \# \text{ of gaugino zero-modes} = 2K|_{K=k} = 2k \tag{A.34}$$
$$\vdots$$
$$V_{n-1} = e^{\phi_1-\phi_n} : \# \text{ of gaugino zero-modes} = 2K|_{K=n-1} = 2(n-1) \tag{A.35}$$
$$V_n = e^{\phi_1+\phi_2+\cdots+\phi_n} : \# \text{ of gaugino zero-modes} = 0. \tag{A.36}$$

If we alternatively consider the following instanton factor

$$t = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \tag{A.37}$$

which is represented by $V_+ = e^{\phi_1}$, then the zero-modes around this instanton are calculated as

$$V_+ = (V_1 V_2 \cdots V_n)^{\frac{1}{n}} : \# \text{ of gaugino zero-modes} \tag{A.38}$$
$$= \frac{1}{n}(2 + 4 + \cdots + 2(n-1) + 0) \tag{A.39}$$
$$= \frac{1}{n}\sum_{j=1}^{n-1} 2k \tag{A.40}$$
$$= n - 1. \tag{A.41}$$

The calculation of the global charges of the monopole operators

Let us calculate the quantum numbers of the monopole operators for 3d $\mathcal{N} = 2$ supersymmetric $U(n)$ gauge theories with the N_f fundamental vector-like flavors. The monopole operator $V_+ \sim e^{\phi_1}, \ \ \phi_1 > 0$ has the non-trivial charges under the $U(1)_R$ and $U(1)_A$ global symmetries:

$$R[V_+] = -\underbrace{(-1)}_{R[Q]=R[\tilde{Q}]} \cdot \underbrace{\frac{1}{2}}_{\substack{\text{fundamental}\\ \text{fermion}\\ \text{zero-mode}}} \cdot \underbrace{2N_f}_{\substack{Q,\tilde{Q}:\\ N_f \text{ flavors}}} - \underbrace{(+1)}_{R[\lambda]} \underbrace{(n-1)}_{\substack{\text{adjoint}\\ \text{fermion}\\ \text{zero-mode}}} = N_f - n + 1 \tag{A.42}$$

$$A[V_+] = -\underbrace{(+1)}_{A[Q]=A[\tilde{Q}]} \cdot \underbrace{\frac{1}{2}}_{\substack{\text{fundamental}\\ \text{fermion}\\ \text{zero-mode}}} \cdot \underbrace{2N_f}_{\substack{Q,\tilde{Q}:\\ N_f \text{ flavors}}} = -N_f \tag{A.43}$$

The result is summarized below.

	$U(1)_R$	$U(1)_A$
V_+	$N_f - n + 1$	$-N_f$

The quantum numbers for $V_- \sim e^{-\sigma_n}, \ \ \phi_n < 0$ are calculated in the same way.

A.2 $SU(n)$ case

For $SU(n)$ group and the corresponding Lie algebra, their properties are summarized as follows.

$$\text{dimensions}: \ n^2-1 \tag{A.44}$$
$$\text{rank}: \ n-1 \tag{A.45}$$
$$= \text{dim. of the Cartan sub algebra} \tag{A.46}$$
$$= \# \text{ of simple roots} \tag{A.47}$$
$$\text{roots}: \ (n^2-1)-(n-1) = n^2-n \tag{A.48}$$

The dual simple roots are identical to the roots.

$$h_i = E_i - E_{i+1} = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} \quad (i = 1, \cdots, n-1) \tag{A.49}$$

Cartan sub-algebra

The Cartan is the diagonalized matrices and we need to impose the traceless condition.
Cartan:

$$t = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \middle| \sum_{i=1}^{n} a_1 = 0 \right\} \tag{A.50}$$

Fundamental representation

The fundamental representation is n-dimensional. The eigenvectors are parametrized by the following bases. The number of the weight vectors is n and the number of the components of the each weight vector is $n-1$.
The bases:

$$\Box : \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} : \text{n-dimensional} \right\} \tag{A.51}$$

The action of the Cartan for the bases:

$$t \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathrm{Tr}(w_1 t) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{A.52}$$

The weights for the fundamentals:

$$w_1 = \overbrace{\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}}^{n\times n}, \cdots, w_n = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{A.53}$$

The zero-modes for the fundamental fermion are calculated as follows.

$$
\begin{aligned}
N &= \frac{1}{2}\sum_{i=1}^{n} \mathrm{sign}\,(\mathrm{Tr}\, w_i \phi)\, \mathrm{Tr}\, w_i h_k && \text{(A.54)}\\
&= \frac{1}{2}\sum_{i=1}^{n} \mathrm{sign}\phi_i \mathrm{Tr}\, w_i h_k && \text{(A.55)}\\
&= \frac{1}{2}(\mathrm{sign}\phi_k + (-1)\mathrm{sign}\phi_{k+1}) && \text{(A.56)}\\
&= \begin{cases} 1 & (\phi_k > 0 > \phi_{k+1}) \\ 0 & (\phi_k > \phi_{k+1} > 0 \;\text{ or }\; 0 > \phi_k > \phi_{k+1}) \end{cases} && \text{(A.57)}
\end{aligned}
$$

Adjoint representation

The adjoint representation is $(n^2 - 1)$-dimensional.

The bases for the adj. representation:

$$
\mathsf{Adj.} = \left\{ \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \cdots \right\} \tag{A.58}
$$

In the calculation of the zero-modes we should sum over all the bases. But we need only to sum over the diagonal bases which are called roots since the off-diagonal matrices have no overlap with the simple roots. Then automatically $\mathrm{tr}\, w_{\mathsf{adj}} h_j = 0$. So we can concentrate on the roots.

Weights for the adjoint rep. (roots):

$$
\begin{aligned}
W_{\text{adj.}} &= \{\alpha_{ij} = w_i - w_j\} = \{\pm(w_i - w_j) | i < j\} && \text{(A.59)}\\
\dim W_{\text{adj.}} &= 2 \times {}_nC_2 = n(n-1) && \text{(A.60)}
\end{aligned}
$$

The simple roots:

$$
w_i - w_j \;\; (i < j) \tag{A.61}
$$

The dual simple roots are identical to the simple roots in $SU(N)$. Notice that the number of the weights is less than the dimensions of the representation. But it is okay because the bases which are not the weights don't contribute to the zero mode calculation.

The zero modes for the adjoint fermion:

$$
\begin{aligned}
N &= \frac{1}{2}\sum_{i,j=1}^{n} \mathrm{sign}\,(\mathrm{Tr}\, \alpha_{ij} \phi)\, \mathrm{Tr}\, \alpha_{ij} h_k && \text{(A.62)}\\
&= \frac{1}{2}\sum \mathrm{sign}(\phi_i - \phi_j)(\delta_{ik} - \delta_{ik+1} - \delta_{jk} + \delta_{jk+1}) && \text{(A.63)}\\
&= \sum_{i<j} (\delta_{ik} - \delta_{ik+1} - \delta_{jk} + \delta_{jk+1}) && \text{(A.64)}\\
&= 2 && \text{(A.65)}
\end{aligned}
$$

So the adjoint fermions always contribute to the monopole-instanon with two zero-modes.

The symmetric representation

The symmetric representation has the dimensions: $\frac{n(n+1)}{2}$. This state is represented by the tensor product of the fundamental representations with symmetrized combinations:

$$
\mathsf{symm.} = \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i) \tag{A.66}
$$

The action of the Cartan to this state is

$$(t \otimes 1 + 1 \otimes t)\,\mathsf{symm.} = (a_i + a_j)\mathsf{symm.} \tag{A.67}$$

The weights for the symmetric representation become

$$W = \left\{ \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 2 & \\ & & & 0 \end{pmatrix}, \cdots \right\} \tag{A.68}$$

$$\dim W = {}_nC_2 + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} \tag{A.69}$$

The anti-symmetric representation

The antisymmetric representation is $\frac{n(n-1)}{2}$-dimensional. The corresponding state is expressed as

$$\mathsf{anti\ symm.} = \frac{1}{2}(v_i \otimes v_j - v_j \otimes v_i). \tag{A.70}$$

$$(t \otimes 1 + 1 \otimes t)\,\mathsf{anti\ symm.} = \frac{1}{2}(a_i + a_j)v_i \otimes v_j - \frac{1}{2}(a_j + a_i)v_j \otimes v_i \tag{A.71}$$

$$= (a_i + a_j)\,\mathsf{anti\ symm.} \tag{A.72}$$

The weights for the anti-symmetric representation are

$$W = \left\{ \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}, \cdots \right\} \tag{A.73}$$

$$\dim W = {}_nC_2 = \frac{n(n-1)}{2} \tag{A.74}$$

rank 3 symmetric tensor

The dimensions of the rank 3 symmetric tensor is $\frac{n(n+1)(n+2)}{6}$. The corresponding state is

$$\frac{1}{3!}(v_{(i} \otimes v_j \otimes v_{k)}). \tag{A.75}$$

A.3 $SO(n)$

Next, we proceed to the orthogonal groups [59]. Again the adjoint fermion zero-modes are always 2. The arguments are different depending on whether n is even or odd.

A.3.1 $SO(2n)$

For $SO(2n)$ group and the corresponding Lie algebra, their properties are summarized as follows.

$$\text{dimensions}:\ n(2n-1) \tag{A.76}$$

$$\text{rank}:\ n \tag{A.77}$$

$$= \text{dim. of the Cartan sub algebra} \tag{A.78}$$

$$= \#\ \text{of simple roots} \tag{A.79}$$

$$\text{roots}:\ n(2n-1) - n = 2n^2 - 2n \tag{A.80}$$

Cartan

The Cartan subalgebra can be taken as

$$t_{2n} = \mathrm{diag}\,\overbrace{(\theta_1, \cdots, \theta_n, -\theta_1, \cdots, -\theta_n)}^{2n}. \tag{A.81}$$

Roots

The root system of the $SO(2n)$ is called D_n in the Dynkin classification. The set of roots are

$$\Delta = \{\pm(\epsilon_i \pm \epsilon_j)|1 \leq i \neq j \leq n\}, \quad \# = {}_nC_2 \times 4 + 2n = 2n^2 \tag{A.82}$$
$$\Delta_{\mathsf{positive}} = \{\epsilon_i \pm \epsilon_j|1 \leq i < j \leq n\} \tag{A.83}$$
$$\Delta_{\mathsf{simple}} = \{\epsilon_j - \epsilon_{j+1}, \epsilon_{n-1} + \epsilon_n|j = 1, \cdots, n-1\}, \quad \# = n \tag{A.84}$$

where the action of ϵ^j to the Cartan is

$$\epsilon_i \cdot \mathrm{diag}(z_1 \cdots, z_n, -z_1, \cdots, -z_n) = z_i. \tag{A.85}$$

Dual simple roots

The dual root system to D_n is itself, that is to say, a self-dual. Then the dual simple roots are the same as the simple roots above. However in actual calculation it is better to consider the Lie algebra $so(E_{2n})$ isomorphic to $so(2n)$. In this case the (dual) simple root becomes as follows.

$$h_i = (e_i - e_{i+1}) - (e_{i+n} - e_{i+n+1}), \quad i = 1, \cdots, n-1 \tag{A.86}$$
$$h_n = (e_{n-1} + e_n) - (e_{2n-1} + e_{2n}) \tag{A.87}$$

where

$$e_1 = \mathrm{diag}\,\overbrace{(1, 0, \cdots, 0)}^{2n} \tag{A.88}$$
$$\vdots$$
$$e_{2n} = \mathrm{diag}(0, \cdots, 0, 1). \tag{A.89}$$

Let us consider the following branch:

$$\phi = \mathrm{diag.}(\phi_1, \cdots, \phi_n, -\phi_1, \cdots, -\phi_n), \quad \phi_i \geq \phi_{i+1}, \ \phi_{n-1} \geq |\phi_n| \tag{A.90}$$

which is always realized by the operation of Weyl group. Notice that the sign of ϕ_n cannot be arbitralily tuned.

Instanton factors

The monopole-instantons are characterized by the following monopole operators which are represented by the effective coordinates of the Coulomb branch at the semi-classical region of the moduli space.

$$Y_j \sim \exp\left[\frac{1}{2}\mathrm{Tr}(h_j\Phi)\right] \tag{A.91}$$
$$= \exp\left[(\text{dual simple roots} \cdot \phi)\right] \tag{A.92}$$
$$= \exp\left[(\text{simple roots} \cdot \phi)\right] \tag{A.93}$$
$$= \exp\left[(\epsilon_j - \epsilon_{j+1}) \cdot \Phi\right], \quad \exp\left[(\epsilon_{n-1} + \epsilon_n) \cdot \Phi\right] \tag{A.94}$$

$$Y_1 \sim e^{\Phi_1 - \Phi_2}, \quad \cdots, \quad Y_{n-1} \sim e^{\Phi_{n-1} - \Phi_n}, \quad Y_n \sim e^{\Phi_{n-1} + \Phi_n}. \tag{A.95}$$

Fundamental representation

The weights of the fundamental rep. are as follows.

$$w_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \cdots, w_{2n} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{A.96}$$

We can think of this as the bases of the fundamental representation.

For $g = h_j,\ (j = 1, \cdots, n-2)$, the Callias index theorem says that

$$N = \frac{1}{2}\sum_w \text{sign}\,(\text{Tr}\, w\phi)\,\text{Tr}(wg) \tag{A.97}$$

$$= \frac{1}{2}\sum_{i=1}^{2n} \text{sign}(\phi_i)(\delta_{ij} - \delta_{ij+1} - \delta_{ij+n} + \delta_{ij+n+1}) \tag{A.98}$$

$$= \frac{1}{2}(\text{sign}\phi_j - \text{sign}\phi_{j+1} - \text{sign}\phi_{j+n} + \text{sign}\phi_{j+n+1}) \tag{A.99}$$

$$= \frac{1}{2}\left((+1) - (+1) - (-1) + (-1)\right) = 0. \tag{A.100}$$

For $g = h_{n-1}$, we have

$$N = \frac{1}{2}\sum_w \text{sign}\,(\text{Tr}\, w\phi)\,\text{Tr}(wg) \tag{A.101}$$

$$= \frac{1}{2}\sum_{i=1}^{2n} \text{sign}(\phi_i)(\delta_{in-1} - \delta_{in} - \delta_{i2n-1} + \delta_{i2n}) \tag{A.102}$$

$$= \frac{1}{2}(\text{sign}\phi_{n-1} - \text{sign}\phi_n - \text{sign}\phi_{2n-1} + \text{sign}\phi_{2n}) \tag{A.103}$$

$$= \frac{1}{2}\left((+1) - \text{sign}\phi_n - (-1) + \text{sign}\phi_{2n}\right) \tag{A.104}$$

$$= \frac{1}{2}\left(2 - 2\,\text{sign}\phi_n\right) \tag{A.105}$$

$$= \begin{cases} 0 & (\phi_n > 0) \\ 2 & (\phi_n < 0) \end{cases}. \tag{A.106}$$

For $g = h_n$, we have

$$N = \frac{1}{2}\sum_w \text{sign}\,(\text{Tr}\, w\phi)\,\text{Tr}(w((e_{n-1} + e_n) - (e_{2n-1} + e_{2n}))) \tag{A.107}$$

$$= \frac{1}{2}\sum_{i=1}^{2n} \text{sign}(\phi_i)(\delta_{in-1} + \delta_{in} - \delta_{i2n-1} - \delta_{i2n}) \tag{A.108}$$

$$= \frac{1}{2}(\text{sign}\phi_{n-1} + \text{sign}\phi_n - \text{sign}\phi_{2n-1} - \text{sign}\phi_{2n}) \tag{A.109}$$

$$= \frac{1}{2}\left((+1) + \text{sign}\phi_n - (-1) - \text{sign}\phi_{2n}\right) \tag{A.110}$$

$$= \frac{1}{2}\left(2 + 2\,\text{sign}\phi_n\right) \tag{A.111}$$

$$= \begin{cases} 2 & (\phi_n > 0) \\ 0 & (\phi_n < 0) \end{cases}. \tag{A.112}$$

A.3.2 $SO(2n+1)$

For $SO(2n+1)$ group and the corresponding Lie algebra, their properties are summarized as follows:

$$\text{dimensions}: \ n(2n+1) \tag{A.113}$$
$$\text{rank}: \ n \tag{A.114}$$
$$= \text{dim. of the Cartan sub algebra} \tag{A.115}$$
$$= \# \ \text{of simple roots} \tag{A.116}$$
$$\text{roots}: \ n(2n+1) - n = 2n^2 \tag{A.117}$$

Cartan

The Cartan subalgebra can be taken as

$$t_{2n+1} = \text{diag} \overbrace{(\theta_1, \cdots, \theta_n, -\theta_1, \cdots, -\theta_n, 0)}^{2n+1}. \tag{A.118}$$

Roots

The root system of the $SO(2n+1)$ is called B_n in the Dynkin classification. The set of roots are

$$\Delta = \{\pm(\epsilon_i \pm \epsilon_j), \pm\epsilon_j | 1 \le i \ne j \le n\}, \quad \# = {}_nC_2 \times 4 = 2n(n-1) \tag{A.119}$$
$$\Delta_{\textsf{positive}} = \{\epsilon_i \pm \epsilon_j, \epsilon_j | 1 \le i < j \le n\} \tag{A.120}$$
$$\Delta_{\textsf{simple}} = \{\epsilon_j - \epsilon_{j+1}, \epsilon_n | j = 1, \cdots, n-1\}, \quad \# = n \tag{A.121}$$

where the action of ϵ^j to the Cartan is

$$\epsilon_i \cdot \text{diag}(z_1 \cdots, z_n, -z_1, \cdots, -z_n, 0) = z_i. \tag{A.122}$$

We can alternatively use the following representation for the roots.

$$\frac{1}{2} h_i = \frac{1}{2} \left[(e_i - e_{i+1}) - (e_{i+n} - e_{i+n+1})\right], \quad i = 1, \cdots, n-1 \tag{A.123}$$
$$\frac{1}{4} h_n = \frac{1}{4} \left[2e_n - 2e_{2n}\right] \tag{A.124}$$

where

$$e_1 = \text{diag} \ \overbrace{(1, 0, \cdots, 0)}^{2n+1\text{-components}} \tag{A.125}$$
$$\vdots$$
$$e_{2n+1} = \text{diag}(0, \cdots, 0, 1). \tag{A.126}$$

Dual simple roots

The dual simple roots are not themselves. The dual root system of the $B_n : so(2n+1)$ is the $C_n : sp(2n)$. We will check this statement below. Rememer that the dual roots are defined by

$$\boldsymbol{\alpha}^* = \frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}. \tag{A.127}$$

Then the dual simple roots of the $so(2n+1)$ are

$$\frac{1}{2} h_i^* := \frac{\frac{1}{2} h_i}{\text{Tr} \left(\frac{1}{2} h_i\right)^2} = \frac{1}{2} h_i$$
$$= \frac{1}{2} \left[(e_i - e_{i+1}) - (e_{i+n} - e_{i+n+1})\right] \tag{A.128}$$
$$h_n^* := \frac{\frac{1}{4} h_n}{\text{Tr} \left(\frac{1}{4} h_n\right)^2} = \frac{1}{2} h_n$$
$$= e_n - e_{2n} \tag{A.129}$$

which are precisely the simple roots of the $sp(2n)$.

Instanton factors

The monopole-instantons are characterized by the following monopole operators which are represented by the effective coordinates of the Coulomb branch at the semi-classical region of the moduli space.

$$
\begin{aligned}
Y_j &\sim \exp\left[\frac{1}{2}\mathrm{Tr}(h_j^* \Phi)\right], \quad \exp\left[\mathrm{Tr}(h_n^*\Phi)\right] && \text{(A.130)}\\
&= \exp\left[(\text{dual simple roots}\cdot\phi)\right] && \text{(A.131)}\\
&= \exp\left[(\text{simple roots of } sp(2n)\cdot\phi)\right] && \text{(A.132)}\\
&= \exp\left[(\epsilon_j - \epsilon_{j+1})\cdot\Phi\right], \quad \exp\left[2\epsilon_n\cdot\Phi\right] && \text{(A.133)}\\
&= \exp\left[\frac{1}{2}\mathrm{Tr}(h_j^{sp(2n)}\Phi)\right], \quad \exp\left[\mathrm{Tr}(h_n^{sp(2n)}\Phi)\right]. && \text{(A.134)}
\end{aligned}
$$

$$
Y_1 \sim e^{\Phi_1-\Phi_2}, \quad \cdots, \quad Y_{n-1}\sim e^{\Phi_{n-1}-\Phi_n}, \quad Y_n \sim e^{2\Phi_n}. \qquad \text{(A.135)}
$$

Fundamental representation

The weights are given by e_j. Let us consider the Coulomb branch parametrized by

$$
\phi = \mathrm{diag}(\phi_1,\phi_2,\cdots,\phi_n,-\phi_1,\cdots,-\phi_n,0), \quad \phi_1\geq\phi_2\geq\cdots\geq\phi_n\geq 0 \qquad \text{(A.136)}
$$

For $g = h_k^* = h_k$ $(k=1,\cdots,n-1)$, the zero-modes are calculated as follows.

$$
\begin{aligned}
N &= \frac{1}{2}\sum_{i=1}^{2n}\mathrm{sign}\left(\mathrm{Tr}\, e_i\phi\right)\mathrm{Tr}\, e_i g && \text{(A.137)}\\
&= \frac{1}{2}\sum_{i=1}^{2n}\mathrm{sign}(\phi_i)(\delta_{ik}-\delta_{ik+1}-(\delta_{ik+n}-\delta_{ik+n+1})) && \text{(A.138)}\\
&= \frac{1}{2}\left(\mathrm{sign}(\phi_k)-\mathrm{sign}(\phi_{k+1})-\mathrm{sign}(\phi_{k+n})+\mathrm{sign}(\phi_{k+n+1})\right) && \text{(A.139)}\\
&= \frac{1}{2}\left[(+1)-(+1)-(-1)+(-1)\right]=0 && \text{(A.140)}
\end{aligned}
$$

For $g = 2h_n^* = h_n = 2e_n - 2e_{2n}$, the zero-modes become

$$
\begin{aligned}
N &= \frac{1}{2}\sum_{i=1}^{2n}\mathrm{sign}\left(\mathrm{Tr}\, e_i\phi\right)\mathrm{Tr}\,\left(e_i h_n\right) && \text{(A.141)}\\
&= \sum_{i=1}^{2n}\mathrm{sign}(\phi_i)(\delta_{in}-\delta_{i2n}) && \text{(A.142)}\\
&= \mathrm{sign}(\phi_n)-\mathrm{sign}(\phi_{2n}) && \text{(A.143)}\\
&= (+1)-(-1) = 2 && \text{(A.144)}
\end{aligned}
$$

Note that the magnetic charge is two times lager than the simple roots. This is because the gauge group is now $SO(2n+1)$ and the Dirac's quantization condition differs from the one of the $SU(n)$.

A.4 $O(n)$

A.4.1 $O(2n)$

For $O(2n)$ group and the corresponding Lie algebra, their properties are summarized as follows:

$$\begin{aligned} \text{dimensions}: &\ n(2n-1) && \text{(A.145)}\\ \text{rank}: &\ n && \text{(A.146)}\\ &= \text{dim. of the Cartan sub algebra} && \text{(A.147)}\\ &= \#\ \text{of simple roots} && \text{(A.148)}\\ \text{roots}: &\ n(2n-1)-n = 2n^2-2n && \text{(A.149)} \end{aligned}$$

which are the same as the $SO(2n)$ but in $O(2n)$ we have the additional reflection symmetry.

Cartan

The Cartan subalgebra can be taken as

$$t_{2n} = \text{diag}\overbrace{(\theta_1, \cdots, \theta_n, -\theta_1, \cdots, -\theta_n)}^{2n}. \tag{A.150}$$

Dual simple roots

The dual simple roots are as follows.

$$h_i = (e_i - e_{i+1}) - (e_{i+n} - e_{i+n+1}), \quad i = 1, \cdots, n-1 \tag{A.151}$$
$$h_n = (e_{n-1} + e_n) - (e_{2n-1} + e_{2n}) \tag{A.152}$$

Let us consider the following branch:

$$\phi = \text{diag.}(\phi_1, \cdots, \phi_n, -\phi_1, \cdots, -\phi_n), \quad \phi_i \geq \phi_{i+1},\ \phi_{n-1} \geq \phi_n \geq 0 \tag{A.153}$$

which is always possible by the Weyl group.

The fundamental (vector) representation: $2n$-dimensional

The weights of the fundamental representation are as follows.

$$w_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \cdots, w_{2n} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{A.154}$$

These are the bases for the vector representation.

For $g = h_j,\ (j = 1, \cdots, n-1)$, the Callias index theorem says that

$$\begin{aligned} N &= \frac{1}{2}\sum_w \text{sign}\left(\text{Tr}\, w\phi\right) \text{Tr}(wg) && \text{(A.155)}\\ &= \frac{1}{2}\sum_{i=1}^{2n} \text{sign}(\phi_i)(\delta_{ij} - \delta_{ij+1} - \delta_{ij+n} + \delta_{ij+n+1}) && \text{(A.156)}\\ &= \frac{1}{2}(\text{sign}\phi_j - \text{sign}\phi_{j+1} - \text{sign}\phi_{j+n} + \text{sign}\phi_{j+n+1}) && \text{(A.157)}\\ &= \frac{1}{2}(+1 - (+1) - (-1) + (-1)) && \text{(A.158)}\\ &= 0. && \text{(A.159)} \end{aligned}$$

For $g = h_n$, we have

$$N = \frac{1}{2}\sum_w \mathrm{sign}\,(\mathrm{Tr}\,w\phi)\,\mathrm{Tr}(wh_n) \tag{A.160}$$
$$= \frac{1}{2}\sum_{i=1}^{2n} \mathrm{sign}(\phi_i)(\delta_{in-1} + \delta_{in} - \delta_{i2n-1} - \delta_{i2n}) \tag{A.161}$$
$$= \frac{1}{2}(\mathrm{sign}\phi_{n-1} + \mathrm{sign}\phi_n - \mathrm{sign}\phi_{2n-1} - \mathrm{sign}\phi_{2n}) \tag{A.162}$$
$$= \frac{1}{2}(+1 + 1 - (-1) - (-1)) \tag{A.163}$$
$$= 2. \tag{A.164}$$

The adjoint representation

The weights for the adjoint representation, which is called roots, are defined as follows.

$$\alpha_{ij} = \pm e_i \pm e_j \quad (i, j = 1, \cdots, 2n;\ i \neq j) \tag{A.165}$$

The dimension is ${}_{2n}C_n = n(2n-1)$-dimensional.

A.5 $Sp(2n)$

The group $Sp(2n)$ means that the defining representation is $2n$-dimensional. The properties of the correspondingg Lie algebra are summarized as follows.

$$\text{dimensions}:\ n(2n+1) \tag{A.166}$$
$$\text{rank}:\ n \tag{A.167}$$
$$= \text{dim. of the Cartan sub algebra} \tag{A.168}$$
$$= \#\ \text{of simple roots} \tag{A.169}$$
$$\text{roots}:\ n(2n+1) - n = 2n^2 \tag{A.170}$$

Cartan

$$t = \{\mathrm{diag.}(i\theta_1, \cdots, i\theta_n, -i\theta_1, \cdots, -i\theta_n) |\, \theta_i \in \mathbb{R}\}\,. \tag{A.171}$$

Roots

The Lie algebra of the $Sp(2n)$ is a C_n root system.

$$\Delta_{\text{all roots}} = \{\pm(\epsilon_i \pm \epsilon_j), \pm 2\epsilon_i | 1 \leq i \neq j \leq n\}\,, \quad \# = {}_nC_2 \times 4 + 2n = 2n^2 \tag{A.172}$$
$$\Delta_{\text{simple}} = \{\epsilon_j - \epsilon_{j+1}, 2\epsilon_n | j = 1, \cdots, n-1\}\,, \quad \# = n \tag{A.173}$$

where the action of the ϵ_j is

$$\epsilon_i \cdot \mathrm{diag}(z_1, \cdots, z_n, -z_1, \cdots, -z_n) = z_i \tag{A.174}$$

We can represent the root system as follows.

$$h_{i-j} = (e_i - e_j) - (e_{i+n} - e_{j+n}) \tag{A.175}$$
$$h_{i+j} = (e_i + e_j) - (e_{i+n} + e_{j+n}) \tag{A.176}$$
$$h_{2j} = e_j - e_{j+n} \tag{A.177}$$

The simple roots are as follows.

$$\frac{1}{2}h_i = \frac{1}{2}\left[(e_i - e_{i+1}) - (e_{i+n} - e_{i+1+n})\right], \quad i = 1, \cdots, n-1 \tag{A.178}$$
$$h_n = e_n - e_{2n} \tag{A.179}$$

Dual roots

The dual of the $sp(2n)$ is the $so(2n+1)$. Then the dual simple roots are as follows.

$$\frac{1}{2}h_i^{so(2n+1)\text{simple}} = \frac{1}{2}\left[(e_i - e_{i+1}) - (e_{i+n} - e_{i+n+1})\right], \quad i = 1, \cdots, n-1 \tag{A.180}$$

$$h_n^{so(2n+1)\text{simple}} = 2e_n - 2e_{2n} \tag{A.181}$$

Instanton factors

The monopole-instantons are characterized by the following monopole operators which are represented by the effective coordinates of the Coulomb branch at the semi-classical region of the moduli space.

$$Y_j \sim \exp\left[\frac{1}{2}\mathrm{Tr}(h_j \Phi)\right] \tag{A.182}$$

$$= \exp\left[(\text{dual simple roots} \cdot \phi)\right] \tag{A.183}$$

$$= \exp\left[(\text{simple roots of } so(2n+1) \cdot \phi)\right] \tag{A.184}$$

$$= \exp\left[(\epsilon_j - \epsilon_{j+1}) \cdot \Phi\right], \quad \exp\left[\epsilon_n \cdot \Phi\right] \tag{A.185}$$

$$Y_1 \sim e^{\Phi_1 - \Phi_2}, \quad \cdots, \quad Y_{n-1} \sim e^{\Phi_{n-1} - \Phi_n}, \quad Y_n \sim e^{\Phi_n} \tag{A.186}$$

where the h_j represent the dual simple root of the $sp(2n)$, which is the simple root of the $so(2n+1)$[1].

Fundamental representation

The weights of the fundamental representation are denoted as e_i $(i = 1, \cdots, 2n)$. Let us consider the instanton factor $e^{\epsilon_1 \cdot \phi} = e^{\phi_1}$. The fundamental fermion zero-modes are calculated as follows.

$$N_{\mathsf{fund.}} = \frac{1}{2}\sum_{i=1}^{2n} \mathrm{sign}(\mathrm{tr}\, e_i \phi)\mathrm{tr}(e_i(e_1 - e_{n+1})) \tag{A.187}$$

$$= \frac{1}{2}\sum_{i=1}^{2n} \mathrm{sign}(\phi_i)\left(\delta_{i1} - \delta_{in+1}\right) \tag{A.188}$$

$$= \frac{1}{2}\left(\mathrm{sign}\phi_1 - \mathrm{sign}\phi_{n+1}\right) \tag{A.189}$$

$$= \frac{1}{2}\left((+1) - (-1)\right) = 1. \tag{A.190}$$

In this thesis we consider the $Sp(2n)$ gauge theories with $2N_f$ fundamental flavors. Then the fundamental zero-modes are $2N_f$.

[1] In the paper by Karch [62], the instanton factor is defined as $Y_n \sim e^{2\phi_n}$. But I think it would be better to denote Y_n as $Y_n \sim e^{\Phi_n}$.

Adjoint representation

The roots are given by $\pm(\epsilon_i \pm \epsilon_j)$.

$$\begin{aligned}
N_{\mathsf{adj.}} &= \frac{1}{2}\sum_{1\le i\le j\le n} \mathrm{sign}(\pm(\epsilon_i+\epsilon_j)\cdot\phi)\,(\pm(\epsilon_i+\epsilon_j)\cdot\epsilon_1) \\
&\quad + \frac{1}{2}\sum_{1\le i<j\le n} \mathrm{sign}(\pm(\epsilon_i-\epsilon_j)\cdot\phi)\,(\pm(\epsilon_i-\epsilon_j)\cdot\epsilon_1) && \text{(A.191)} \\
&= \left[\sum_{1\le i\le j\le n} \mathrm{sign}(\phi_i+\phi_j)\,(\delta_{1i}+\delta_{1j})\right] + \left[\sum_{1\le i<j\le n} \mathrm{sign}(\phi_i-\phi_j)\,(\delta_{1i}-\delta_{1j})\right] && \text{(A.192)} \\
&= \left[\underbrace{(n-1)}_{i=1;j=2,\cdots,n} + \underbrace{2}_{i=j=1}\right] + \left[\underbrace{(n-1)}_{i=1;j=2,\cdots,n}\right] && \text{(A.193)} \\
&= 2n. && \text{(A.194)}
\end{aligned}$$

Appendix B

Parity Anomaly

In this appendix, we calculate the parity anomaly by the one-loop calculation. First we consider the current one-point function such as

$$\langle J^{\mu}(x) \rangle \,, \tag{B.1}$$

which is obviously zero for the theories with Poincare invariance. However we consider the some background field of the gauge fields, then this one point function would be non-zero. Let us consider the following graph which indeed contributes to this one-point function, where the current is inserted in the left vertex and the wavy line is the gauge field coupling to the fermion. the dashed line should be recognized as follows.

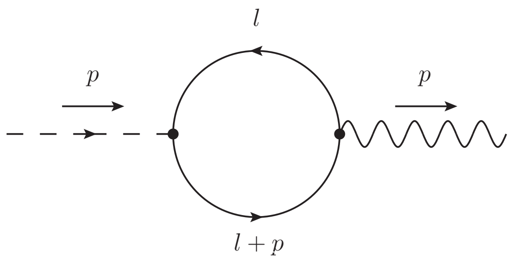

Figure B.1: Current 1-point function

If the current is related to the global symmetry, the dashed line indicates the gauge field which couples to this global symmetry. So we need to weakly gauge this global symmetry. On the other hand if the current corresponds to the gauge symmetry, the dashed line should be replaced by the wavy line or some new gauge fields. Then if this one-point function is non-zero, the mixed Chern-Simons terms between the global-global, global-gauge, ore gauge-gauge vector fields. Please note that the coupling of the gauge field at the right vertex is necessary for the existence of the anomaly, but it is not required that the left vertex couples to the external gauge field. This is the same as the case with the well-known four-dimensional chiral anomaly.

This graph gives

$$(-1)(-ie) \int \frac{d^D l}{(2\pi)^D} \mathrm{Tr} \left[\gamma^{\mu} \frac{\not{l} + \not{p} + m}{(l+p)^2 - m^2} \gamma^{\rho} \frac{\not{l} + m}{l^2 - m^2} \right] \tag{B.2}$$

where the factor (-1) is due to the fermion loop. We should note that the almost all terms vanish but the terms proportional to $\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho})$ remain since in 3d the trace formulas of the gamma matrices is

$$\mathrm{Tr}\left[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\right] = -2i\epsilon^{\mu\nu\rho}. \tag{B.3}$$

We also have

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu} \tag{B.4}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{B.5}$$

By the introduction of the Feynman parameter the denominator becomes

$$\begin{aligned}
\text{denominator} &= \int_0^1 dx \, \frac{1}{\left[x(l+p)^2 - xm^2 + (1-x)l^2 - (1-x)m^2\right]^2} & \text{(B.6)}\\
&= \int_0^1 dx \, \frac{1}{\left[(l+xp)^2 - \Delta\right]^2} & \text{(B.7)}\\
&= \int_0^1 dx \, \frac{1}{[l'^2 - \Delta]^2} & \text{(B.8)}
\end{aligned}$$

where $\Delta = m^2 + p^2 x(x-1)$.

The numerators has the following structures.

$$\mathrm{Tr}\left[m^2\gamma^\mu\gamma^\rho\right] = 2m^2 g^{\mu\rho} \tag{B.9}$$

$$\begin{aligned}
\mathrm{Tr}\left[m\gamma^\mu(\not{l} + \not{p})\gamma^\rho + m\gamma^\mu\gamma^\rho\not{l}\right] &= m\mathrm{Tr}\left[\gamma^\mu\not{p}\gamma^\rho\right] & \text{(B.10)}\\
&= 2im\,\epsilon^{\mu\rho\sigma}p_\sigma & \text{(B.11)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^\mu\not{l}\gamma^\rho\not{l}\right] &= \left(l'^2\left(\frac{2}{D} - 1\right) - x^2p^2\right)\mathrm{Tr}\left[\gamma^\mu\gamma^\rho\right] + 4x^2p^\mu p^\rho & \text{(B.12)}\\
\mathrm{Tr}\left[\gamma^\mu\not{p}\gamma^\rho\not{l}\right] &= -4xp^\mu p^\rho + 2xg^{\mu\rho}p^2 & \text{(B.13)}
\end{aligned}$$

where we used the fact that the $\mathrm{Tr}(\gamma^\rho) = 0$ and we omitted the linear terms in l'. Let us first calculate the l'^2-proportional term:

$$\begin{aligned}
&\left(\frac{2}{D} - 1\right)\mathrm{Tr}\left[\gamma^\mu\gamma^\rho\right]\int_0^1 dx \int \frac{d^Dl}{(2\pi)^D}\frac{l'^2}{(l'^2 - \Delta)^2}\\
&= \left(\frac{2}{D} - 1\right)\mathrm{Tr}\left[\gamma^\mu\gamma^\rho\right]\int_0^1 dx\,\frac{-i}{(4\pi)^{3/2}}\frac{D}{2}\Gamma\left(1 - \frac{D}{2}\right)\left(\frac{1}{\Delta}\right)^{1-\frac{D}{2}} & \text{(B.14)}\\
&= \left(\frac{2}{D} - 1\right)\mathrm{Tr}\left[\gamma^\mu\gamma^\rho\right]\int_0^1 dx\,\frac{3i}{8\pi}\Delta^{1/2} & \text{(B.15)}
\end{aligned}$$

The other therms are calculated as follows.

$$\begin{aligned}
&\left(2m^2g^{\mu\rho} - 4xp^\mu p^\rho + 2xg^{\mu\rho}p^2 + 2im\epsilon^{\mu\rho\sigma}p_\sigma - 2x^2p^2 + 4x^2p^\mu p^\rho\right)\int_0^1 dx \int \frac{d^Dl}{(2\pi)^D}\,\frac{1}{[l'^2 - \Delta]^2}\\
&= \left(2m^2g^{\mu\rho} - 4xp^\mu p^\rho + 2xg^{\mu\rho}p^2 + 2im\epsilon^{\mu\rho\sigma}p_\sigma - 2x^2p^2 + 4x^2p^\mu p^\rho\right)\int_0^1 dx\frac{i\sqrt{\pi}}{(4\pi)^{3/2}}\left(\frac{1}{m^2 + x(x-1)p^2}\right)^{1/2}\\
& & \text{(B.16)}
\end{aligned}$$

where the first term cancels the contribution from (B.15). Combining all the contribution, we have

$$\frac{i}{8\pi}\int_0^1 dx\left[4x(1-x)(p^2g^{\mu\rho} - p^\mu p^\rho) + 2im\epsilon^{\mu\rho\sigma}p_\sigma\right]\Delta^{-1/2}. \tag{B.17}$$

The first term is the one-loop vacuum polarization contribution and the second term is the anomalous contribution. In three space-time dimensions we encounter the non-locality in the wave-function renormalization since at one-loop level we only have the following dimensionless combination:

$$\frac{e^2}{\sqrt{p^2}} \tag{B.18}$$

where e^2 is the gauge coupling and it has the mass dimension 1. Then the renormalization procedure in 3d becomes highly complicated. In order to derive the parity anomaly, it is simpler to consider the heavy particle limit where

$$m \to \infty. \tag{B.19}$$

Multiplying the photon polarization vector ϵ we thus obtain

$$(\text{B.2}) \cdot \epsilon = (2im\, \epsilon^{\mu\rho\sigma} \epsilon_\rho p_\sigma) \int_0^1 dx \frac{i}{(4\pi)^{3/2}} \sqrt{\pi} \left(\frac{1}{m^2 + p^2 x(x-1)} \right)^{1/2} \tag{B.20}$$

$$= \frac{ie}{4\pi} \frac{m}{|m|} \epsilon^{\mu\sigma\rho} p_\sigma \epsilon_\rho. \tag{B.21}$$

Then the one-point function under the background field becomes

$$\langle J^\mu(x) \rangle_A = \frac{e}{4\pi} \text{sign}(m)\, \epsilon^{\mu\sigma\rho} \partial_\sigma A_\rho(x) \tag{B.22}$$

$$= \frac{e}{8\pi} \text{sign}(m)\, \epsilon^{\mu\nu\rho} F_{\nu\rho}. \tag{B.23}$$

where we switched to the configuration space. Here we derived the parity anomaly by using the dimensional regularization. This is not problematic because the parity anomaly comes from the finite loop graphs. If you be careful about the regularization dependence, you can adopt the other regularization scheme, for example the cut-off regularization. We should notice that the parity anomaly is only generated by the massive fermion loops.

Appendix C

Parturbative analysis of the 3d gauge theories

In this appendix we discuss the perturbative behavior of the 3d gauge theories with much attention to the structure of the Coulomb branch [43]. We restrict ourself to the one-loop calculation, but it suffices to the non-trivial information of the dynamics on the Coulomb branch. Of course we will need the higher order corrections for the understanding of the dynamics of the Higgs branch.

C.1 3d SQED

Let us consider the 3d $\mathcal{N}=2$ supersymmetric $U(1)$ gauge theory with $N_f=1$ flavor. In 3d we have the two types of scalar fields coming from the vector multiplet, $\Sigma|_{\theta=\bar{\theta}=0}=\phi$ which is the fourth component of the 4d gauge field and a dual photon $\mathrm{Im}\Phi|_{\theta=\bar{\theta}=0}=a$. The moduli space of vacua is labeled by these two coordinates. The one-loop metric on the moduli space is calculated as

$$ds^2=\frac{1}{4}\left(\frac{1}{e^2}+\frac{1}{|\phi|}\right)d\phi^2+\left(\frac{1}{e^2}+\frac{1}{|\phi|}\right)^{-1}da^2. \tag{C.1}$$

Then the size of the circle of the dual photon becomes small when the vev of the anther scalar $|\phi|$ goes to zero. The shape of the moduli space of vacua is as follows, where we consider the vanishing real masses.

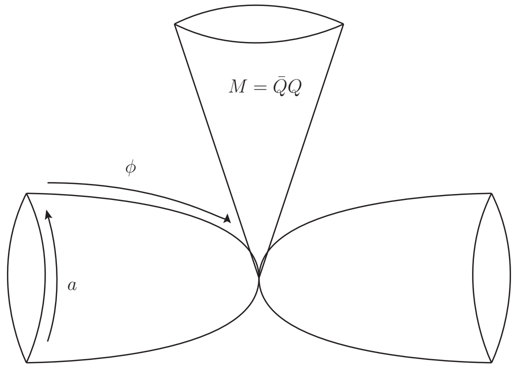

Figure C.1: The quantum structure of the moduli space of vacua for 3d SQED with $N_f=1$

This metric is somewhat confusing because the two scalar fields comes from the different superfields which are related the duality transformation presented in this thesis. Now we derive this metric from the linear superfield perspective. The effective action for the linear superfield becomes at one-loop

$$f(\Sigma) = \frac{1}{2e^2}\Sigma^2 + \Sigma \log\left(\frac{\Sigma}{e^2}\right). \tag{C.2}$$

The Legendre transformation becomes

$$\begin{aligned}\Phi + \bar{\Phi} &= \frac{\partial f(\Sigma)}{\partial \Sigma} \\ &= \frac{1}{e^2}\Sigma + \log\left(\frac{\Sigma}{e^2}\right) + 1\end{aligned} \tag{C.3}$$

$$\Phi = \frac{1}{2e^2}\Sigma + \frac{1}{2}\log\left(\frac{\Sigma}{e^2}\right) + \frac{1}{2} + i\sigma \tag{C.4}$$

$$d\Phi = \frac{1}{2}\left(\frac{1}{e^2} + \frac{1}{r}\right) dr + id\sigma \tag{C.5}$$

We should notice that at $\phi = 0$, the real part of Φ diverges and the coordinate Φ is not good parametrization everywhere. The Kahler potential is also defined by the Legendre transformation,

$$K(\Phi + \bar{\Phi}) = f(\Sigma) - \Sigma(\Phi + \bar{\Phi}). \tag{C.6}$$

Then, the inverse Legendre transform is

$$K'(\Phi + \bar{\Phi}) = -\Sigma \overset{\theta=\bar{\theta}=0}{\longrightarrow} -\phi. \tag{C.7}$$

$$K''(\Phi + \bar{\Phi}) = \left(\frac{\partial(\Phi + \bar{\Phi})}{\partial \Sigma}\right)^{-1} \tag{C.8}$$

$$= \left[\frac{\partial}{\partial \Sigma}\left(\frac{1}{e^2}\Sigma + \log\left(\frac{\Sigma}{e^2}\right) + 1\right)\right]^{-1} \tag{C.9}$$

$$= \left(\frac{1}{e^2} + \frac{1}{\Sigma}\right)^{-1} \to \left(\frac{1}{e^2} + \frac{1}{\phi}\right)^{-1} \tag{C.10}$$

Finally the metric becomes

$$ds^2 = K''(\Phi + \bar{\Phi})\, d\Phi d\bar{\Phi} \tag{C.11}$$

$$= \left(\frac{1}{e^2} + \frac{1}{\Sigma}\right)^{-1}\left[\frac{1}{2}\left(\frac{1}{e^2} + \frac{1}{r}\right) dr + id\sigma\right]\left[\frac{1}{2}\left(\frac{1}{e^2} + \frac{1}{r}\right) dr - id\sigma\right] \tag{C.12}$$

$$= \frac{1}{4}\left(\frac{1}{e^2} + \frac{1}{|\phi|}\right) dr^2 + \left(\frac{1}{e^2} + \frac{1}{|\phi|}\right)^{-1} d\sigma^2. \tag{C.13}$$

Again we recovered the one-loop metric.

C.2 3d $SU(2)$ SQCD

The Coulomb branch of the $SU(2)$ SQCD is labeled again by the two scalar fields: ϕ, a. The dual photon a is periodic:

$$a \sim a + 2\pi. \tag{C.14}$$

Furthermore we have the Weyl reflection symmetry:

$$(\phi, a) \to (-\phi, -a). \tag{C.15}$$

The classical metric for these moduli is as follows.

$$ds^2 = \frac{1}{4e^2} d\phi^2 + e^2 da^2 \tag{C.16}$$

The one-loop quantum corrections for pure SYM modifies this as

$$ds^2 = \frac{1}{4}\left(\frac{1}{e^2} - \frac{2}{|\phi|}\right) d\phi^2 + \left(\frac{1}{e^2} - \frac{2}{|\phi|}\right)^{-1} da^2. \tag{C.17}$$

From this one-loop metric, we see that at the point $|\phi| = 2e^2$ there is a singularity. This suggests us that we need the more higher order corrections or non-perturbative corrections. As the rsults, the metric would become non-singular.

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