— Dissertation —

Chiral Four-Dimensional Heterotic String Vacua From Covariant Lattices

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Abstract

The covariant lattice formalism provides a consistent method for the construction of chiral four-dimensional heterotic string vacua. In this work, we seek to develop a systematic understanding of this corner of the string landscape, and also attempt to clarify the relationship with asymmetric orbifolds. Chiral covariant lattice models are classified using the theory of lattice genera, and by means of the Smith-Minkowski-Siegel mass formula a lower bound of $O(10^{10})$ on the number of $\mathcal{N} = 1$ supersymmetric models is calculated. We also perform an exhaustive enumeration of models for two genera corresponding to certain supersymmetric Z_3 and Z_6 asymmetric orbifolds. In the Z_3 case there exist precisely 2030 models, and for these we carry out a general analysis of discrete flavor and R-symmetries. The Z_6 case produced in total $O(10^7)$ models, but computational resources were insufficient for the elimination of duplicates among them. Finally, we discuss three-generation models from both genera in detail.

Contents

Frequently Used Symbols

$\mathbb{Z}$	The ring of integers, or the lattice of integers.
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$	The fields of rational, real and complex numbers, respectively.
Z_N	The cyclic groups of order N.
S_N	The group of permutations of N objects.
A_n, D_n, E_n	The root lattices/systems or the Lie algebras of ADE type.
$\mathbb{Z}_{(n)}$	The one-dimensional lattice with vectors of minimal norm n.
$\mathcal{G}$	A genus of lattices.
$\Gamma_{22,14}$	A covariant lattice.
$(\Gamma_{22})_{\mathrm{L}}$	A left-mover lattice of a covariant lattice.
$(\Gamma_{14})_{\mathrm{R}}$	A right-mover lattice of a covariant lattice.
$\mathscr{G}$	The graph of containing all right-mover lattices.
$\mathscr{G}_i$	The connected components of $\mathscr{G}$.

Chapter 1

Introduction

String theory may eventually provide a consistent quantum-mechanical unification of elementary particle physics with gravity. Although there is numerous evidence that superstring theory at its core is a unique theory, it possesses a vast landscape of vacuum states with diverse physical properties. Nevertheless, the prospect of having a unified theory of all forces and matter in nature has spawned countless efforts on string model building. Yet, this search for a four-dimensional string vacuum that matches the physical properties of our universe may appear like that for a needle in a haystack. However, the standard model of particle physics (SM), which successfully describes particles and interactions at energy scales below about one hundred GeV, has several attributes that greatly constrain the search area of the string model building programme. First and foremost, there is the gauge symmetry group,

$$G_{\mathrm{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y \,,$$

together with three chiral generations of quarks and leptons. String models which reproduce these properties exist in a considerable number, such as in symmetric orbifold compactifications of the heterotic string [1–12]. Nonetheless, there are still many open problems in these models, such as moduli stabilization. It is also difficult to obtain entirely realistic fermion mass hierarchies and mixing angles [13], yet interesting flavor symmetries appear to be abundant in orbifold compactifications [14–16]. Other issues such as proton decay have seen positive developments recently [17, 18].

A feature that has driven the success of symmetric orbifolds is that they are to a large extent comprehensible in a purely geometric manner: one employs a six-dimensional compact space having certain singularities, possibly with a nontrivial configuration of background fields. However, there also exist intrinsically non-geometric compactifications such as the more general asymmetric orbifolds [19–21], which have also been considered for model building [22–25].

The covariant lattice formalism provides another such non-geometric construction of four-dimensional heterotic string vacua [26–28]. These vacua have some characteristic properties. For example, all chiral covariant lattice theories possess a gauge symmetry of rank 22. However, since G_{SM} is of rank 4, some of these extra gauge symmetries must either be broken at a high energy scale, or survive as a completely hidden symmetry. Another feature of covariant lattice theories is that only rather small representations of the gauge group may appear at the massless level. In particular, adjoint Higgs fields which are indispensable for grand unified gauge theories are excluded. Hence, the search for realistic models restricts to gauge groups which contain G_{SM} as is, or in semi-simple extensions such as Pati-Salam. Also, many covariant lattice models possess an anomalous $U(1)_A$, which may be useful for supersymmetry breaking. Moreover, an advantage of covariant lattice models is that there is only one modulus, the dilaton, and hence the problem of moduli stabilization is simplified.

Even though the heterotic covariant lattice construction lacks an intuitive geometrical picture, there is still a chance to find realistic models. Yet, hitherto only few models have been constructed explicitly using this formalism [29]. Moreover, a systematic classification of models has only been achieved in ten spacetime dimensions [30], while in eight dimensions 444 chiral models were found to exist [31]. The phenomenologically interesting case of four dimensions was considered to some extent in [32]. Some covariant lattice models can also be obtained from other constructions. For example, it is known that by bosonization certain asymmetric orbifolds are equivalent to a covariant lattice theory [33–35]. Also, there are some overlaps with free fermionic constructions and Gepner models [36].

In this thesis, we attempt to develop a systematic understanding of chiral heterotic covariant lattice models and clarify the connection to asymmetric orbifolds (cf. [37]). We also investigate some phenomenological aspects such as discrete flavor and R-symmetries, and finally provide models with three chiral generations of standard model matter.

An Overview

Any string vacuum is described by a conformal field theory on the world-sheet, which is the two-dimensional analogue of the world-line of a point particle. All degrees of freedom on closed string world-sheets split into a left-mover and a right-mover part. In the covariant lattice formalism, the internal degrees of freedom, i.e. those unrelated to the four-dimensional spacetime, are expressed in terms of free bosons with certain periodic boundary conditions. These boundary conditions force the momenta to lie on a lattice $\Gamma_{22,14}$. This lattice — called covariant lattice — has to obey certain consistency conditions: it must be even and self-dual due to the requirement of modular invariance, and also allow for supersymmetry on the world-sheet. A special feature of *chiral* four-dimensional covariant lattice models is that $\Gamma_{22,14}$ contains a sublattice of the form $(\Gamma_{22})_{\mathrm{L}} \oplus \overline{(\Gamma_{14})}_{\mathrm{R}}$.

In this work we provide a (computer aided) classification of right-mover lattices $(\Gamma_{14})_{\mathrm{R}}$ which solve the constraints imposed by world-sheet supersymmetry and chiralness, the result being that there are in total 99 such lattices. Among them, 19 lattices lead to $\mathcal{N} = 1$ spacetime supersymmetry and we relate some of them to asymmetric orbifolds. In order to construct a complete model, one has to combine a right-mover lattice with an appropriate left-mover lattice $(\Gamma_{22})_{\mathrm{L}}$. In fact, due to modular invariance, all lattices $(\Gamma_{22})_{\mathrm{L}}$ that can be combined with a chosen $(\Gamma_{14})_{\mathrm{R}}$ constitute a genus, so the very well-developed theory of lattice genera can be used to study them (an introduction on the subject can be found e.g. in [38]). Most importantly, a genus $\mathcal{G}$ contains only finitely many lattices, and a lower bound on their number $|\mathcal{G}|$ can be evaluated by means of the Smith-Minkowski-Siegel mass formula (refer to [39] for details). This also means that the total number of chiral models is finite. For some relevant genera $\mathcal{G}_{\mathrm{L}}$ of left-mover lattices we calculate a lower bound on $|\mathcal{G}_{\mathrm{L}}|$. In particular, these lower bounds suggest that there are at least $O(10^{10})$ four-dimensional covariant lattice models realizing $\mathcal{N}=1$ spacetime supersymmetry. Furthermore, there exist computational methods which allow the explicit construction of all lattices in a genus, but these are only practicable for reasonably small genera. In this work, we conduct an exhaustive enumeration of some genera $\mathcal{G}_{\mathrm{L}}$ of left-mover lattices, two of which correspond to certain classes of Z_3 and Z_6 asymmetric orbifold models.

In the Z_3 case there exist precisely 2030 models, and for these we perform a general analysis of discrete flavor and R-symmetries. In particular, a flavor symmetry group related to $\Sigma(216 \times 3)$ [40] and a Z_3 R-symmetry was found. We also perform a search for realistic models, and present a three-generation model in detail. However, all these asymmetric Z_3 models possess exotic matter fields (especially with color charge) that are rather difficult to decouple. Also, one can show quite generally that the flavor structure of these models is not very realistic. We expect some of these problems to be cured in the Z_6 case. There, the exhaustive enumeration resulted in a total of $O(10^7)$ models, but it was not possible to eliminate duplicates due to insufficient computational resources. We also do not carry out a general search for realistic models in this case, and instead just present a particular three-generation model. This model possesses only color singlet exotic matter and potentially has realistic mass hierarchies: at three-point level there is a Yukawa term only for the top quark, while the other quarks and leptons must obtain their masses from higher order terms which are naturally suppressed.

This work is organized as follows. In Chapter 2, we review the required aspects of the covariant lattice construction. In particular, we introduce the constraints from modular invariance, world-sheet supersymmetry and the requirement of a chiral four-dimensional spectrum. These are central to the classification of chiral models which we describe in Chapter 3. In Chapter 4, we consider phenomenological aspects of some of these models, and also present three-generation spectra. Finally, in Chapter 5 we summarize and comment on possible future directions this research could take.

Chapter 2

Covariant Lattices and Orbifolds

2.1 The Covariant Lattice Formalism

In this section we briefly introduce the covariant lattice (or bosonic supercurrent) formalism. We also derive constraints from the requirement of a chiral four-dimensional spectrum. For a more detailed introduction, please refer to the review [28].

2.1.1 Four-Dimensional Heterotic Strings

Any string theory vacuum is described by a two-dimensional world-sheet conformal field theory (CFT). Here, we consider such CFTs in their Euclidean form, and adopt the common convention where the world-sheet coordinate is represented by a complex number z. The algebra of conformal transformations in two dimensions is generated by the energy-momentum tensor, which splits into a right-mover (chiral) part $T(z)$ and a left-mover (anti-chiral) part $\overline{T}(\overline{z})$ due to current conservation. The chiral energy-momentum tensor obeys an operator product expansion (OPE) of the form

$$T(z)T(w) \sim \frac{c_{\mathrm{R}}/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,, \tag{2.1}$$

and its Laurent modes L_n span a copy of the Virasoro algebra. In above equation, "$\sim$" denotes the omission of non-singular terms. The anti-chiral component $\overline{T}(\overline{z})$ obeys a similar OPE with c_{R} replaced by c_{L}. The OPE of $T(z)$ ($\overline{T}(\overline{z})$) with a conformal field $\phi(z,\overline{z})$ in particular determines its conformal weights h_{R} (h_{L}), which are also given by the L_0 ($\overline{L}_0$) eigenvalue of the state in the Hilbert space obtained from the operator-state correspondence.

Note, that the conformal anomalies are proportional to the respective central charges c_{L} and c_{R}, which must vanish in order for a CFT to describe a sensible string vacuum. World-sheet CFTs usually possess additional symmetries. In particular, local symmetries are relevant because upon quantization they introduce Fadeev-Popov ghost fields which contribute to the central charges. Common to all string theories is a local diffeomorphism (or reparametrization) symmetry, which after gauge fixing gives rise to a chiral bc

and an anti-chiral $\overline{bc}$-ghost system with respective central charges $c_{\mathrm{L}} = c_{\mathrm{R}} = -26$. The world-sheet CFT of a string vacuum with spacetime fermions must further be locally supersymmetric. The number of local supersymmetries $(N_{\mathrm{L}}, N_{\mathrm{R}})$ may differ for left-movers and right-movers, and string theories where $N_{\mathrm{L}} \neq N_{\mathrm{R}}$ are called heterotic. The phenomenologically interesting case is $(N_{\mathrm{L}}, N_{\mathrm{R}}) = (0, 1)$ (or equivalently $(1, 0)$), which is what is usually referred to as "the" heterotic string. Due to the single local right-mover supersymmetry, a chiral $\beta\gamma$-ghost system with central charge $c_{\mathrm{R}} = 11$ must be included, hence resulting in a ghost sector with total central charges $(c_{\mathrm{L}}, c_{\mathrm{R}}) = (-26, -15)$. Heterotic strings also have a superconformal symmetry generated by a supercurrent $G(z)$ which satisfies the following OPEs:

$$T(z)G(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{(z-w)}\,, \tag{2.2}$$

$$G(z)G(w) \sim \frac{2c_{\mathrm{R}}/3}{(z-w)^3} + \frac{2T(w)}{(z-w)}\,. \tag{2.3}$$

These, together with the OPE (2.1) form what is known as a $N = 1$ super-Virasoro algebra.

Here, we want to consider $(0, 1)$-heterotic strings propagating in a four-dimensional flat Minkowski spacetime. The world-sheet CFT must then contain at least the four bosons $X^\mu(z, \overline{z})$ and their superpartners, the chiral Neveu-Schwarz-Ramond fermions $\psi^\mu(z)$. Note, that each boson contributes with $c_{\mathrm{L}} = c_{\mathrm{R}} = 1$, and each chiral fermion with $c_{\mathrm{R}} = 1/2$ to the total central charge. Hence, in order to cancel the conformal anomalies we require an internal unitary CFT with central charges $(22, 9)$.

Vertex operators. String scattering amplitudes are calculated from certain correlation functions of the world-sheet CFT in which each incoming and outgoing string is represented by a vertex operator. When constructing vertex operators for physical states, it turns out to be convenient to bosonize the ψ^μ and the $\beta\gamma$-ghost CFTs (for the ghosts this is in fact a "rebosonization"). Let us begin with the ψ^μ, which satisfy the OPEs

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{(z-w)}\,. \tag{2.4}$$

Here, we use the sign convention $\eta^{\mu\mu} = (-1, 1, 1, 1)$. By introducing two chiral bosons, H^1 and H^2 with the standard OPEs

$$H^\alpha(z)H^\beta(w) \sim -\delta^{\alpha\beta}\log(z-w) \tag{2.5}$$

one expresses the fermion fields as follows:

$$\tfrac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1) = {:}e^{\pm iH^1}{:}\,, \tag{2.6}$$

$$\tfrac{1}{\sqrt{2}}(\psi^2 \pm i\psi^3) = {:}e^{\pm iH^2}{:}\,, \tag{2.7}$$

where the colons denote the usual free field normal ordering. Note, that in above bosonization we ignore a slight complication: the exponentials of different bosons commute, whereas the corresponding expressions of the ψ^μ anticommute. This can be resolved by introducing cocycle operators, which we will discuss in a related context in Subsection 2.1.3. Further, note that the fields $\psi^\mu(z)$ may either all have periodic (Neveu-Schwarz) or anti-periodic (Ramond) boundary conditions (w.r.t. the coordinate σ in $z = e^{\tau+i\sigma}$). Vertex operators then contain the exponential $:e^{i\lambda\cdot H}:$. The OPEs of the vertex operators with the ψ^μ must be local in the NS-sector, and hence the components of λ have to be integral. In the R-sector they must be half-integral in order to produce the necessary square-root branch cuts. One also sees from the contribution of λ_2 to the overall four-dimensional helicity of a state that spacetime bosons (fermions) belong to the NS(R)-sector, as in the heterotic string there is no other way for half-integer helicities to arise.

Regarding the $\beta\gamma$-ghost system we only note that the ghost number current $:\beta\gamma:$ can be expressed as

$$:\beta\gamma: = i\partial\phi\,. \tag{2.8}$$

Here, ϕ is a boson with opposite sign in the OPE,

$$\phi(z)\phi(w) \sim \log(z-w)\,, \tag{2.9}$$

and with energy momentum-tensor

$$T_\phi(z) = \frac{1}{2}:\partial\phi\partial\phi:(z) + \frac{i}{2}(1-2\lambda)\partial^2\phi(z)\,. \tag{2.10}$$

The exponential $:e^{iq\phi}:$ produces states with definite ghost charge q, and these states have an unusual conformal weight $h_{\mathrm{R}} = -q(q+2)/2$. This is unbounded from below, and one must ask which values of q may appear in vertex operators. In fact, it turns out that vertex operators come in various equivalent "pictures", for which the ghost charge q differs by integers. The form the vertex operators is particularly simple in the canonical ghost picture, where Neveu-Schwarz states carry ghost charge -1, while Ramond states carry ghost charge $-1/2$. Vertex operators in other pictures are then obtained by an operation called picture-changing: let $V_q(z)$ be a vertex operator in the q-picture. The corresponding vertex operator in the $q+1$-picture is given by

$$V_{q+1}(z) = :\mathscr{P}_{+1}V_q:(z)\,, \tag{2.11}$$

where $\mathscr{P}_{+1}(z) = :G(z)e^{i\phi(z)}:$ denotes the picture-changing operator. Here, the supercurrent $G(z)$ is of the form

$$G(z) = i\psi^\mu\partial X_\mu(z) + G_{\mathrm{int}}(z) + G_{\mathrm{ghost}}(z)\,, \tag{2.12}$$

where G_{int} denotes the supercurrent of the internal $c_{\mathrm{R}} = 9$ SCFT and G_{ghost} denotes the contribution from the ghosts. Note, that picture-changing is crucial for the calculation of scattering amplitudes: due to an anomaly of the ghost number current, the ghost numbers of all vertex operators in a correlation function on the sphere must add up to -2. Although this is trivially fulfilled e.g. for boson-fermion-fermion scattering amplitudes in the canonical pictures, other correlators necessitate picture changing.

Summarizing, a general vertex operator contains (among other excitations)

$$V_{\lambda,q}(z) = :e^{i\lambda\cdot H+iq\phi}:(z)\,, \tag{2.13}$$

where

$$(\lambda_1,\lambda_2,q)\in\mathbb{Z}^3\,, \qquad \text{(NS)} \tag{2.14}$$
$$(\lambda_1,\lambda_2,q)\in(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})+\mathbb{Z}^3\,. \qquad \text{(R)} \tag{2.15}$$

It is customary to treat the λ and q on the same footing by realizing that the set of vectors (2.14) and (2.15) together forms a lattice. In general, by a lattice we mean a discrete subset of a real vector space with a bilinear inner product, that is closed under addition and subtraction. For any lattice Λ one also defines a dual lattice Λ^*, which is given by the vectors in the $\mathbb{R}$-span of Λ that have integral inner product with all vectors in Λ. Note, that the set of vectors (2.14) and (2.15) together is closed under addition and subtraction. Furthermore, the operator product

$$:e^{i\lambda\cdot H+iq\phi}:(z)\ :e^{i\lambda'\cdot H+iq'\phi}:(w)\ \sim\ \frac{:e^{i(\lambda+\lambda')\cdot H+i(q+q')\phi}:(w)}{(z-w)^{-\lambda\cdot\lambda'+qq'}}+\ldots \tag{2.16}$$

provides us with a natural inner product $\lambda\cdot\lambda'-qq'$ of signature $(2,1)$. It turns out that the lattice given by the vectors (2.14) and (2.15) together with this inner product is the dual lattice of $D_{2,1}$ (the Lorentzian lattices $D_{n,m}$ are described in detail in Appendix A.3). In the following, we shall denote it by $D_{2,1}^{\text{st}*}$, where the superscript "st" indicates that it encodes spacetime quantum numbers. Note, that in the language of conjugacy classes, the NS-sector constitutes the union $(0)\cup(v)$, and the R-sector the union $(s)\cup(c)$.

Physical states. In the quantization of the string one has to restrict to a subset of all vertex operators (2.13) for unitarity reasons. There are several equivalent ways to treat this, the most systematic one being the BRST formalism. Here we adopt the light-cone formalism, which is adequate if one is only interested in the spectrum. In this formalism, physical states belong to a transverse Hilbert space $\mathscr{H}^\perp$ in which there are no oscillator excitations in the X^0, X^1, ψ^0, ψ^1, and ϕ directions (there are also no bc and $\overline{bc}$-ghost excitations). In the canonical ghost picture, the light-cone gauge then fixes (λ_1, q) as

$$(\lambda_1,q)=(0,-1)\,, \qquad \text{(NS)} \tag{2.17}$$
$$(\lambda_1,q)=(-\tfrac{1}{2},-\tfrac{1}{2})\,. \qquad \text{(R)} \tag{2.18}$$

This can also be expressed using the decomposition

$$D_{2,1}^{\text{st}}\supset D_1^{\text{hel}}\oplus D_{1,1}^{\text{ghost}}\,, \tag{2.19}$$

where $\lambda_2\in D_1^{\text{hel}*}$ and $(\lambda_1,q)\in D_{1,1}^{\text{ghost}*}$. Light-cone states are then associated either with the vector conjugacy class (v) of $D_{1,1}^{\text{ghost}}$ in the Neveu-Schwarz case, or with the spinor conjugacy class (s) in the Ramond case. A physical state must further have conformal weights $h_{\mathrm{R}} = h_{\mathrm{L}} = 1$. These weights are given by

$$h_{\mathrm{L}} = -\tfrac{1}{2}M^2 + \overline{N}_X + h_{\mathrm{L}}^{\mathrm{int}}\,, \tag{2.20}$$
$$h_{\mathrm{R}} = -\tfrac{1}{2}M^2 + N_X + h_{\mathrm{R}}^{\mathrm{int}} + N_{H^2} + \tfrac{1}{2}\lambda_1^2 + \tfrac{1}{2}\lambda_2^2 - \tfrac{1}{2}q^2 - q\,, \tag{2.21}$$

where one has to insert the appropriate values of λ_1 and q from (2.17) or (2.18), respectively. Also in above equations, M^2 denotes the four-dimensional mass of the state and N_X ($\overline{N}_X$) denotes the chiral (anti-chiral) oscillator number of the transversal $X^{2,3}(z,\overline{z})$ CFT. Furthermore, $(h_{\mathrm{L}}^{\mathrm{int}}, h_{\mathrm{R}}^{\mathrm{int}})$ denotes the conformal weights of the excitations of the internal CFT, and N_{H^2} denotes the oscillator number of the boson H^2 that was introduced in the bosonization of the transversal fermions ψ^2 and ψ^3. The condition $h_{\mathrm{L}} + h_{\mathrm{R}} = 2$ is a mass-shell condition, whereas the condition $h_{\mathrm{L}} - h_{\mathrm{R}} = 0$ is known as "level matching". Also, note that in the NS(R)-sector there is a contribution of at least $h_{\mathrm{R}} = 1/2$ ($h_{\mathrm{R}} = 5/8$) from the ψ^μ–$\beta\gamma$ CFTs.

Modular invariance. Not all choices for the internal CFT give rise to consistent string vacua. Let us introduce the one-loop partition function

$$\mathcal{Z}(\tau) = \mathrm{Tr}_{\mathscr{H}^\perp,\,k_0=k_1=0}\left[(-1)^F q^{(L_0-1)}\overline{q}^{(\overline{L}_0-1)}\right]. \tag{2.22}$$

Here, $q = e^{2i\pi\tau}$ and the trace is to be taken over all states in $\mathscr{H}^\perp$ without momenta in the $X^{0,1}$ directions. The eigenvalues of L_0 and $\overline{L}_0$ is given by equations (2.21) and (2.20), respectively. Furthermore, F denotes the spacetime fermion number. In a consistent string theory, the partition function $\mathcal{Z}(\tau)$ must be invariant under the modular group $PSL(2,\mathbb{Z})$ generated by the transformations

$$T : \tau \mapsto \tau + 1\,, \tag{2.23}$$
$$S : \tau \mapsto -\frac{1}{\tau}\,. \tag{2.24}$$

This modular invariance condition allows the unambiguous definition of closed string torus amplitudes (note that there are only closed strings in heterotic theories). There are also similar conditions from higher-loop amplitudes, but in the relevant cases these are implied from one-loop modular invariance. Modular invariance at all loops also ensures anomaly freedom of the effective four-dimensional theory (that is, a Green-Schwarz like cancellation).

2.1.2 Covariant Lattices

As a consequence of anomaly cancellation, any four-dimensional heterotic string theory requires an internal unitary CFT with central charges $(c_{\mathrm{L}}, c_{\mathrm{R}}) = (22, 9)$. Here, we follow the approach of the bosonic supercurrent formalism [26, 27] and consider only internal

CFTs that are realized entirely in terms of free bosons $X^I_{\rm L}(\overline{z})$, $I \in \{1,\dots,22\}$ and $X^i_{\rm R}(z)$, $i \in \{1,\dots,9\}$ that are compactified on a lattice $\Gamma^{\rm int*}_{22,9}$. These bosons obey the OPEs

$$X^I_{\rm L}(\overline{z})X^J_{\rm L}(\overline{w}) \sim -\delta^{IJ}\log(\overline{z}-\overline{w})\,, \tag{2.25}$$

$$X^i_{\rm R}(z)X^j_{\rm R}(w) \sim -\delta^{ij}\log(z-w)\,, \tag{2.26}$$

and the energy-momentum tensors are given by

$$T(z) = -\frac{1}{2}:\!\partial X_{\rm R}\cdot\partial X_{\rm R}\!:(z)\,, \tag{2.27}$$

$$T(\overline{z}) = -\frac{1}{2}:\!\overline{\partial} X_{\rm L}\cdot\overline{\partial} X_{\rm L}\!:(\overline{z})\,. \tag{2.28}$$

The vertex operator that inserts a state with momentum eigenvalues $(k_{\rm L},k_{\rm R}) \in \Gamma^{\rm int*}_{22,9}$ has the form

$$V_{k_{\rm L},k_{\rm R}}(z,\overline{z}) = \,:\!e^{ik_{\rm L}\cdot X_{\rm L}(\overline{z})+ik_{\rm R}\cdot X_{\rm R}(z)}\!:\,, \tag{2.29}$$

where we again ignore cocycle operators at the moment. The OPE of above vertex operators,

$$V_{k_{\rm L},k_{\rm R}}(z,\overline{z})V_{k'_{\rm L},k'_{\rm R}}(w,\overline{w}) \;\sim\; \frac{V_{k_{\rm L}+k'_{\rm L},k_{\rm R}+k'_{\rm R}}(w,\overline{w})}{(z-w)^{-k_{\rm R}\cdot k'_{\rm R}}(\overline{z}-\overline{w})^{-k_{\rm L}\cdot k'_{\rm L}}}+\dots\,, \tag{2.30}$$

again provides us with a natural inner product $k_{\rm L}\cdot k'_{\rm L} - k_{\rm R}\cdot k'_{\rm R}$ of signature $(22,9)$ on $\Gamma^{\rm int*}_{22,9}$. By adjoining the $D^{\rm st*}_{2,1}$ part from the bosonization of the ψ^μ–$\beta\gamma$ CFT one constructs vertex operators $V_{k_{\rm L},k_{\rm R},\lambda,q} = V_{k_{\rm L},k_{\rm R}}V_{\lambda,q}$, and checks that $(k_{\rm L},k_{\rm R},\lambda,q)$ naturally belongs to the orthogonal sum,

$$\Gamma^{\rm int*}_{22,9}\oplus\overline{D^{\rm st*}_{2,1}}\,, \tag{2.31}$$

which has signature $(23,11)$. Here and in the following, $\overline{\Lambda}$ denotes the lattice which is identical to Λ, except that its inner product is amended by an additional minus sign.

Modular invariance. The OPEs of the vertex operators $V_{k_{\rm L},k_{\rm R},\lambda,q}(z,\overline{z})$ is non-local because not all inner products are integers. Also, the partition function obtained from (2.22) is not modular invariant. In order to construct a local and modular invariant theory, one must restrict to a subset of vectors in $\Gamma^{\rm int*}_{22,9}\oplus\overline{D^{\rm st*}_{2,1}}$ and also make additional assumptions on $\Gamma^{\rm int}_{22,9}$.

Let us in the following assume that $\Gamma^{\rm int}_{22,9}$ is an even lattice (i.e. all vectors contained in this lattice are supposed to have even norm) and that $\Gamma^{\rm int*}_{22,9}/\Gamma^{\rm int}_{22,9}$ consists of four cosets which we call (0), (s), (v) and (c). Furthermore, these cosets shall form a Z_4 group and the vectors $k=(k_{\rm L},k_{\rm R})$ shall have norms

$$k^2 \in 2\mathbb{Z}\,,\ \text{ for } k\in(0)\,, \tag{2.32}$$

$$k^2 \in \tfrac{5}{4}+2\mathbb{Z}\,,\ \text{ for } k\in(s)\,, \tag{2.33}$$

$$k^2 \in 1+2\mathbb{Z}\,,\ \text{ for } k\in(v)\,, \tag{2.34}$$

$$k^2 \in \tfrac{5}{4}+2\mathbb{Z}\,,\ \text{ for } k\in(c)\,, \tag{2.35}$$

very much in analogy to the $D_{22,9}$ lattice. Then, we can construct an even lattice

$$\Gamma_{22,10} = (0,0) \cup (v,v) \supset \Gamma^{\text{int}}_{22,9} \oplus \overline{D^{\text{hel}}_1}\,, \tag{2.36}$$

similar to the decomposition $D_{22,10} \supset D_{22,9} \oplus \overline{D_1}$. Then, the cosets $\Gamma^*_{22,10}/\Gamma_{22,10}$ have the same properties as those of $D_{22,10}$, i.e. they form a $Z_2 \times Z_2$ group and their norms are given by

$$u^2 \in 2\mathbb{Z}\,, \text{ for } u \in (0)\,, \tag{2.37}$$
$$u^2 \in 1 + 2\mathbb{Z}\,, \text{ for } u \in (s)\,, \tag{2.38}$$
$$u^2 \in 1 + 2\mathbb{Z}\,, \text{ for } u \in (v)\,, \tag{2.39}$$
$$u^2 \in 1 + 2\mathbb{Z}\,, \text{ for } u \in (c)\,, \tag{2.40}$$

where $u = (k_{\rm L}, k_{\rm R}, \lambda_2)$. Here, we use the same naming as for the corresponding $D_{22,10}$ conjugacy classes. One can go a step further and construct a lattice

$$\Gamma_{22,9;2,1} \supset \Gamma_{22,10} \oplus \overline{D^{\text{ghost}}_{1,1}} \tag{2.41}$$

as the following union of conjugacy classes:

$$\Gamma_{22,9;2,1} = (0,0) \cup (s,s) \cup (v,v) \cup (c,c)\,. \tag{2.42}$$

This lattice is called odd covariant lattice. It is a sublattice of $\Gamma^{\text{int}*}_{22,9} \oplus \overline{D^{\text{st}*}_{2,1}}$, and the coset decomposition (2.42) also holds in the same form for

$$\Gamma_{22,9;2,1} \supset \Gamma^{\text{int}}_{22,9} \oplus \overline{D^{\text{st}}_{2,1}}\,. \tag{2.43}$$

One now verifies that this lattice is odd integral (all vectors have integral inner product and there are vectors of odd norm) and self-dual, i.e. $\Gamma_{22,9;2,1} = \Gamma^*_{22,9;2,1}$.

We now check that restricting to vertex operators $V_{k_{\rm L},k_{\rm R},\lambda,q}$ with $(k_{\rm L}, k_{\rm R}, \lambda, q) \in \Gamma_{22,9;2,1}$ results in a local and modular invariant theory. First, let us note that the contribution of the transversal $X^{2,3}(z,\bar z)$ CFT to the partition function (2.22) is modular invariant by itself. Hence, we must show modular invariance of the partition function corresponding to the internal CFT and the bosonization of the $\psi^{2,3}$ CFT, which is given by

$$\mathcal{Z}_{22,10}(\tau) = \frac{1}{\overline{\eta(\tau)}^{22}\eta(\tau)^{10}} \left(\sum_{(k_{\rm L},k_{\rm R},\lambda_2)\in(v)} - \sum_{(k_{\rm L},k_{\rm R},\lambda_2)\in(s)} \right) \bar q^{\frac{1}{2}k_{\rm L}^2} q^{\frac{1}{2}(k_{\rm R}^2+\lambda_2^2)}\,. \tag{2.44}$$

Here one sums only over the vectors of the conjugacy classes (s) and (v) of $\Gamma_{22,10}$ due to the truncation conditions (2.17) and (2.18), and the relative sign comes from the insertion of $(-1)^F$ in (2.22). The function $\mathcal{Z}_{22,10}(\tau)$ is obviously invariant under the T-transformation (2.23) because we sum only over odd lattice vectors (cf. (2.38) and (2.39)), and due to the following property of the Dedekind η-function:

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau)\,. \tag{2.45}$$

In order to show invariance of $\mathcal{Z}_{22,10}(\tau)$ under the S-transformation (2.24), let us first introduce the general partition function of a CFT of n left-moving and m right-moving bosons compactified on an integral lattice $\Lambda_{n,m}$ of signature (n,m):

$$\mathcal{Z}_{\Lambda_{n,m}}(\tau) = \frac{1}{\overline{\eta(\tau)}^n \eta(\tau)^m} \sum_{(x_\mathrm{L},x_\mathrm{R})\in\Lambda_{n,m}} \bar{q}^{\frac{1}{2}x_\mathrm{L}^2} q^{\frac{1}{2}x_\mathrm{R}^2} \,. \tag{2.46}$$

This function is S-modular invariant if and only if $\Lambda_{n,m}$ is self-dual (this is shown by Poisson resummation). Now, from the conjugacy classes of $\Gamma_{22,10}$ we can construct two self-dual lattices $(0)\cup(v)$ and $(0)\cup(s)$. Hence, the partition functions $\mathcal{Z}_{(0)\cup(v)}(\tau)$ and $\mathcal{Z}_{(0)\cup(s)}(\tau)$ are S-modular invariant. However, then also

$$\mathcal{Z}_{22,10}(\tau) = \mathcal{Z}_{(0)\cup(v)}(\tau) - \mathcal{Z}_{(0)\cup(s)}(\tau) \tag{2.47}$$

is modular invariant, as we claimed. Finally, locality follows from the fact that the inner product on $\Gamma_{22,9;2,1}$ is integral. Showing that such a covariant lattice theory is also multi-loop modular invariant is more tricky [28].

The bosonic string map. Let us now make the observation that the conjugacy classes $(0,0)$ and (v,v) in equation (2.42) have even norm, whereas (s,s) and (c,c) have odd norm. Then, by replacing the $D_{1,1}^{\text{ghost}}$ factor in the decomposition $\Gamma_{22,10}\oplus D_{1,1}^{\text{ghost}}$ with a D_4^{ghost} factor we can again construct a lattice

$$\Gamma_{22,14} = (0,0)\cup(s,s)\cup(v,v)\cup(c,c) \supset \Gamma_{22,10}\oplus D_4^{\text{ghost}} \,, \tag{2.48}$$

which is again self-dual. It is also even since the conjugacy classes (s), (v) and (c) of D_4 have odd norm. The lattice $\Gamma_{22,14}$ is called even covariant lattice (or just covariant lattice) and has signature $(22,14)$. Above replacement is also known as the bosonic string map [41, 42], because by adjoining a right-moving E_8 factor to $\Gamma_{22,14}$ we obtain a lattice that serves as a bosonic string compactification. This map also plays a role in the proof of multi-loop modular invariance.

Exchanging $D_{1,1}^{\text{ghost}}$ with D_4^{ghost} is also equivalent to replacing $D_{2,1}^{\text{st}}$ by D_5^{st}, thus giving a similar decomposition of the form

$$\Gamma_{22,14} \supset \Gamma_{22,9}^{\text{int}} \oplus \overline{D_5^{\text{st}}} \,. \tag{2.49}$$

Recall that we required that the cosets in $\Gamma_{22,9}^{\text{int}*}/\Gamma_{22,9}^{\text{int}}$ satisfy properties similar to those in $D_{22,9}^*/D_{22,9}$. These conditions are sufficient for the evenness and self-duality of $\Gamma_{22,14}$. It is also possible to show their necessity. Hence, we can construct a modular invariant theory from $\Gamma_{22,9}^{\text{int}}$ if and only if there exists an even and self-dual lattice $\Gamma_{22,14}$ that decomposes as in (2.49).

2.1.3 Bosonic Realizations of Supersymmetry

The covariant lattice must obey additional consistency conditions from superconformal invariance of the internal $c_\mathrm{R} = 9$ CFT: there has to exist a supercurrent $G_{\text{int}}(z)$ that obeys the OPEs (2.2) and (2.3). Obviously, we need to express the supercurrent in terms of the $X^i_{\mathrm{R}}(z)$. The OPE (2.2) states that $G_{\mathrm{int}}(z)$ is a primary field of conformal weight 3/2. This condition is solved by the following expression:

$$G_{\mathrm{int}}(z) = \sum_{s^2=3} A(s) :e^{is\cdot X_{\mathrm{R}}(z)}: \varepsilon(s,\hat{p}_{\mathrm{R}}) + \sum_{r^2=1} iB(r)\cdot\partial X_{\mathrm{R}}(z) :e^{ir\cdot X_{\mathrm{R}}(z)}: \varepsilon(r,\hat{p}_{\mathrm{R}})\,. \tag{2.50}$$

Here, one sums over two sets of vectors r and s of norm 1 and 3, respectively. Also, the coefficients $B^i(r)$ are subject to a transversality condition,

$$r\cdot B(r) = 0\,. \tag{2.51}$$

Moreover, in equation (2.50),

$$\hat{p}^i_{\mathrm{R}} = \frac{1}{2\pi}\oint dz\, \partial X^i_{\mathrm{R}}(z) \tag{2.52}$$

denotes the zero-mode (momentum operator) and $\varepsilon(s,t)$ is a cocycle that has to be introduced to ensure correct statistics. The constraints from the OPE (2.3) on the coefficients $A(s)$ and $B^i(r)$ shall not be considered in full generality here. Instead, we will discuss a special case in Subsection 2.1.5, where we additionally require a chiral four-dimensional spectrum.

Constraint vectors. Due to an additional consistency condition, for each vector r and s that appears in (2.50) with non-zero coefficient, the covariant lattice $\Gamma_{22,14}$ must contain all vectors $(k_{\mathrm{L}}, k_{\mathrm{R}}, x)$ of the form

$$(0,r,v) \text{ and } (0,s,v)\,, \tag{2.53}$$

called constraint vectors of the supercurrent [28]. In above notation, the third entry corresponds to the subspace spanned by $\overline{D_5^{\mathrm{st}}}$, and v denotes a weight in the vector conjugacy class of D_5. This condition ensures that the picture-changing operator acts on the states of the theory in a well defined manner. It also arranges that the states of the theory appear in representations of the four-dimensional Lorentz algebra.

Since the covariant lattice is an even lattice and v^2 is an odd integer, an immediate consequence is that the vectors r and s span an odd integral lattice. In the following, we denote this "supercurrent lattice" by Ξ, and sums in expressions such as (2.50) are to be interpreted as sums over vectors in Ξ.

The cocycle. We now need to discuss some properties of the 2-cocycle, $\varepsilon : \Xi\times\Xi \mapsto \mathbb{C}^\times$ (here, $\mathbb{C}^\times$ shall denote the non-zero complex numbers). It is subject to the conditions

$$\varepsilon(s,t)\,\varepsilon(s+t,u) = \varepsilon(s,t+u)\,\varepsilon(t,u)\,, \tag{2.54}$$

$$\varepsilon(s,t) = (-1)^{s\cdot t + s^2 t^2}\,\varepsilon(t,s)\,, \tag{2.55}$$

for all $s, t, u \in \Xi$. These conditions are preserved if the cocycle is multiplied by a coboundary $\delta\eta(s,t)$, i.e.

$$\varepsilon(s,t)' = \varepsilon(s,t)\,\eta(s)\,\eta(t)\,\eta(s+t)^{-1} \tag{2.56}$$

is also a valid cocycle. Then, by choosing η appropriately, the cocycle can be gauged to satisfy

$$\varepsilon(0,s) = 1\,, \tag{2.57}$$
$$\varepsilon(s,-s) = 1\,, \tag{2.58}$$
$$\varepsilon(s,t) = \varepsilon(t,-s-t) = \varepsilon(-s-t,s)\,, \tag{2.59}$$
$$\varepsilon(s,t)^* = \varepsilon(-t,-s) = \varepsilon(s,t)^{-1}\,, \tag{2.60}$$

for all $s, t \in \Xi$ (cf. also [43]). These properties are assumed in the rest of this work.

A particularly useful way to obtain an explicit realization is as follows. First, define $\varepsilon(b_i, b_j)$ for a basis $\{b_1, \ldots, b_9\}$ of Ξ in such a way that the conditions

$$\varepsilon(b_i,b_i) = 1\,, \tag{2.61}$$
$$\varepsilon(b_i,b_j)\,\varepsilon(b_j,b_i) = 1\,, \tag{2.62}$$
$$\varepsilon(b_i,b_j) = (-1)^{b_i\cdot b_j + b_i^2 b_j^2}\,\varepsilon(b_j,b_i)\,, \tag{2.63}$$

are satisfied, and then consider the linear continuation to $\Xi \times \Xi$. One can show that this implies the other properties.

2.1.4 Massless Spectra and Symmetries

Let us now discuss the spectra of covariant lattice theories. Physical states must satisfy (2.17)–(2.18) and (2.20)–(2.21). This results in the conditions:

$$\tfrac{1}{2}M^2 = \tfrac{1}{2}k_{\mathrm{L}}^2 + \overline{N}_X + N_{\mathrm{L}} - 1\,, \tag{2.64}$$
$$\tfrac{1}{2}M^2 = \tfrac{1}{2}k_{\mathrm{R}}^2 + N_X + \tfrac{1}{2}\lambda_2^2 + N_{\mathrm{R}} - \tfrac{1}{2}\,. \tag{2.65}$$

Here, N_{L} (N_{R}) denotes the oscillator number of the 22 left-mover (10 right-mover) bosons corresponding to $\Gamma_{22,10}$, and N_X ($\overline{N}_X$) is the oscillator number of the right-moving (left-moving) part of $X^{2,3}$. Also, $(k_{\mathrm{L}}, k_{\mathrm{R}}, \lambda_2)$ belongs either to the (v) conjugacy class of $\Gamma_{22,10}$ for the NS sector, or to (s) conjugacy class for the R sector. Note, that the internal CFT given by the lattice $\Gamma_{22,9}^{\mathrm{int}}$ always gives rise to a left-mover and a right-mover Kac-Moody algebra $\mathfrak{g}_{\mathrm{L}}$ and $\mathfrak{g}_{\mathrm{R}}$ of level one. The algebra $\mathfrak{g}_{\mathrm{L}}$ is spanned by the currents $i\overline{\partial}X_{\mathrm{L}}^I$ and $:e^{ik_{\mathrm{L}}\cdot X_{\mathrm{L}}}:$, where $k_{\mathrm{L}}^2 = 2$ and $(k_{\mathrm{L}}, 0) \in \Gamma_{22,9}^{\mathrm{int}}$ (and similarly for $\mathfrak{g}_{\mathrm{R}}$). These Kac-Moody algebras contain zero-mode Lie algebras $\mathfrak{g}_{\mathrm{L}}^0$ and $\mathfrak{g}_{\mathrm{R}}^0$, which are important because the states at a given mass level always form a representation of $\mathfrak{g}_{\mathrm{L}}^0 \oplus \mathfrak{g}_{\mathrm{R}}^0$.

There is another method to formally determine the spectrum which makes use of the bosonic string map, and it is especially convenient if one is just interested in the massless states. In that case, we solve

$$0 = \tfrac{1}{2}x_{\mathrm{L}}^2 + \overline{N}_X + N_{\mathrm{L}} - 1\,, \tag{2.66}$$
$$0 = \tfrac{1}{2}x_{\mathrm{R}}^2 + N_X + N_{\mathrm{R}} - 1\,, \tag{2.67}$$

where N_X, $\overline{N}_X$ and N_{L} are as above, but now $(x_{\mathrm{L}}, x_{\mathrm{R}}) \in \Gamma_{22,14}$ and N_{R} is supposed to include oscillations in the D_4^{ghost} directions (note, that physically there are no oscillator excitations in the D_4^{ghost} direction). Then, one formally obtains complete representations of the zero-mode $SO(10)$ Lie-algebra associated with D_5^{st}. Recall though that there is no one-to-one relationship between physical states and D_5^{st} lattice vectors. Rather, by the truncation rules (2.17) and (2.18) the $SO(10)$ representation allows to read off the spacetime Lorentz representation:

$$\mathbf{45} \text{ of } D_5^{\mathrm{st}} \longrightarrow \text{(left-mover) gauge vector boson,} \tag{2.68}$$
$$\mathbf{10} \text{ of } D_5^{\mathrm{st}} \longrightarrow \text{complex scalar,} \tag{2.69}$$
$$\mathbf{16} \text{ of } D_5^{\mathrm{st}} \longrightarrow \text{Weyl fermion,} \tag{2.70}$$
$$\overline{\mathbf{16}} \text{ of } D_5^{\mathrm{st}} \longrightarrow \text{Weyl fermion (CPT conjugate state).} \tag{2.71}$$

for $\overline{N}_X = 0$ and $x_{\mathrm{L}}^2/2 + N_{\mathrm{L}} = 1$, as well as

$$\mathbf{45} \text{ of } D_5^{\mathrm{st}} \longrightarrow \text{graviton + complex dilaton,} \tag{2.72}$$
$$\mathbf{10} \text{ of } D_5^{\mathrm{st}} \longrightarrow \text{(right-mover) gauge vector boson,} \tag{2.73}$$
$$\mathbf{16} \text{ of } D_5^{\mathrm{st}} \longrightarrow \text{gravitino + dilatino,} \tag{2.74}$$
$$\overline{\mathbf{16}} \text{ of } D_5^{\mathrm{st}} \longrightarrow \text{gravitino + dilatino (CPT conjugate states).} \tag{2.75}$$

for $\overline{N}_X = 1$ and $x_{\mathrm{L}}^2 = 0$. In above massless case also x_{R}^2 is determined as $x_{\mathrm{R}}^2 = 0$ for the **45**, $x_{\mathrm{R}}^2 = 1$ for the **10**, and $x_{\mathrm{R}}^2 = 3/4$ for the **16** and $\overline{\mathbf{16}}$ of D_5^{st}. Further, note that above mass shell equations (2.66) and (2.67) resemble those of the bosonic string. Let us now discuss some generic properties of the massless spectrum.

Gauge and discrete symmetries. The effective theory of a covariant lattice model always has a compact gauge symmetry group G_{L} with Lie-algebra $\mathfrak{g}_{\mathrm{L}}^0$, as for each current in $\mathfrak{g}_{\mathrm{L}}$ there exists a corresponding gauge boson in the spectrum. Due to the $U(1)^{22}$ subalgebra spanned by the zero-modes $\hat{p}_{\mathrm{L}}^I$ of the 22 currents $i\bar{\partial}X_{\mathrm{L}}^I$, one observes that the rank of this left-mover gauge-group is always 22.

Under special circumstances there are also right-moving gauge bosons in the spectrum [44]. One sees that this happens exactly when D_5^{st} enhances to some $D_{5+k}^{\mathrm{st}} \supset D_5^{\mathrm{st}}$. However, in these cases the spectrum is always non-chiral, because the spinor representation of D_{5+k}^{st} always decompose into non-chiral pairs of spinors of the D_5^{st} subalgebra. In any case the rank nine Lie algebra $\mathfrak{g}_{\mathrm{R}}^0$ gives rise to a continuous symmetry group G_{R} of the CFT correlation functions. Yet this symmetry does not result in a global symmetry of the effective theory. In fact, couplings obtained from amplitudes which do not require picture changing are G_{R} invariants. An example would be the three-point coupling in the effective superpotential of a supersymmetric theory. This coupling is calculated from the amplitude

$$\langle V_{-1}^{(1)}(z_1,\bar{z}_1)V_{-1/2}^{(2)}(z_2,\bar{z}_2)V_{-1/2}^{(3)}(z_3,\bar{z}_3)\rangle, \tag{2.76}$$

which does not require picture-changing (here, V_q denotes an operator in the q-picture). However, for example the four-point coupling that is obtained from the amplitude

$$\langle :\mathscr{P}_{+1}V_{-1}^{(1)}(z_1,\bar{z}_1): V_{-1}^{(2)}(z_2,\bar{z}_2)V_{-1/2}^{(3)}(z_3,\bar{z}_3)V_{-1/2}^{(4)}(z_4,\bar{z}_4)\rangle, \tag{2.77}$$

is not G_{R} invariant because the internal supercurrent $G_{\mathrm{int}}(z)$ which enters $\mathscr{P}_{+1}$ transforms nontrivially under G_{R}. Nonetheless, there may be a nontrivial subgroup $H_{\mathrm{R}} \subset G_{\mathrm{R}}$ which stabilizes $G_{\mathrm{int}}(z)$, and this subgroup gives a valid symmetry of the effective theory. Note, that for chiral theories H_{R} is necessarily discrete. For non-chiral theories H_{R} may contain continuous symmetries, which are necessarily gauged and lead to above described right-moving gauge bosons.

Spacetime supersymmetry. Spacetime supersymmetry is attained whenever the covariant lattice contains vectors that extend D_5^{st} to one of the exceptional root lattices E_6^{st} ($\mathcal{N}=1$), E_7^{st} ($\mathcal{N}=2$), and E_8^{st} ($\mathcal{N}=4$). Supersymmetric spectra are tachyon-free and always contain the corresponding gravitinos.

The case $\mathcal{N}=1$ shall be of special importance to us. The states solving (2.67) then necessarily come in representations of E_6^{st}, the relevant ones for the massless case being $\mathbf{78}$, $\mathbf{27}$ and $\overline{\mathbf{27}}$. These branch as

$$\begin{aligned}
\mathbf{78} &\to \mathbf{45}_0 \oplus \mathbf{16}_{1/4} \oplus \overline{\mathbf{16}}_{-1/4} \oplus 1_0\,,\\
\mathbf{27} &\to \mathbf{16}_{-1/12} \oplus \mathbf{10}_{1/6} \oplus 1_{-1/3}\,,\\
\overline{\mathbf{27}} &\to \overline{\mathbf{16}}_{1/12} \oplus \mathbf{10}_{-1/6} \oplus 1_{1/3}\,,
\end{aligned} \tag{2.78}$$

under $E_6 \to SO(10)\times U(1)$. Hence, we identify $\mathbf{78}$ with a vector multiplet and $\mathbf{27}$ with a chiral multiplet ($\overline{\mathbf{27}}$ contains CPT conjugate states).

The $\mathcal{N}=4$ case also deserves some extra mentioning. Then, $\Gamma_{22,14}$ contains an $\overline{E_8^{\mathrm{st}}}$-sublattice, which is self-dual by itself. Hence, the covariant lattice can be written as an orthogonal sum,

$$\Gamma_{22,14} = \Gamma_{22,6} \oplus \overline{E_8^{\mathrm{st}}}\,, \tag{2.79}$$

where $\Gamma_{22,6}$ is self-dual and known as Narain lattice. This scenario describes the most general toroidal compactification of ten-dimensional heterotic strings including constant background fields [45, 46]. The enhancement of D_5^{st} to E_8^{st} then arises due to the bosonization of the compactified right-mover NSR-fermions.

2.1.5 Covariant Lattices for Chiral Models

Now, we require that the four-dimensional effective theory has a chiral spectrum. Then, it can be shown that chiralness is spoiled if the supercurrent lattice Ξ contains vectors r

of norm 1 [28]. In that case, D_5^{st} is enhanced to some $D_{5+k}^{\mathrm{st}} \supset D_5^{\mathrm{st}}$, which causes massless fermion matter to appear in vector-like pairs. Hence, in the following we assume the absence of norm 1 vectors and use the ansatz

$$G_{\mathrm{int}}(z) = \sum_{s^2=3} A(s) :e^{is\cdot X_{\mathrm{R}}(z)}: \varepsilon(s, \hat{p}_{\mathrm{R}}) \tag{2.80}$$

for the internal supercurrent. Note that the absence of norm 1 vectors is only a necessary condition for chiralness and it is nonetheless possible to obtain non-chiral models this way.

The supercurrent equations. Until now, we did not consider equation (2.3). So, the next step is to calculate the GG-OPE using the ansatz in equation (2.80) and compare it with the r.h.s. of equation (2.3). Then, one obtains the following system of quadratic equations in the coefficients $A(s)$:

$$\sum_{s^2=3} |A(s)|^2 s^i s^j = 2\delta^{ij}\,, \tag{2.81}$$

$$\sum_{\substack{s^2=t^2=3\\ s+t=u}} A(s)A(t)\varepsilon(s,t) = 0 \ \text{ for all } u^2 = 4\,, \tag{2.82}$$

$$\sum_{\substack{s^2=t^2=3\\ s+t=u}} A(s)A(t)\varepsilon(s,t)\,(s^i - t^i) = 0 \ \text{ for all } u^2 = 2\,. \tag{2.83}$$

Here, we also imposed a hermiticity condition,

$$G_{\mathrm{int}}(z)^\dagger = (z^*)^{-3} G_{\mathrm{int}}(1/z^*)\,, \tag{2.84}$$

which implies $A(s)^* = A(-s)$. A generalized version of equations (2.81)–(2.83) which includes also non-vanishing $B(r)$ can be found in [31].

Left-right decomposition of the covariant lattice. As an immediate consequence of equation (2.81), the supercurrent lattice Ξ must completely span the nine-dimensional space it resides in. Then, the constraint vectors $(0, s, v)$ generate a negative-definite 14-dimensional sublattice $\overline{\Gamma}_{14}$ of $\Gamma_{22,14}$. We also define $(\Gamma_{22})_{\mathrm{L}}$ as the 22-dimensional lattice of vectors belonging to the orthogonal complement of $\overline{\Gamma}_{14}$ in $\Gamma_{22,14}$, and similarly $\overline{(\Gamma_{14})}_{\mathrm{R}}$ as the lattice of vectors in the orthogonal complement of $(\Gamma_{22})_{\mathrm{L}}$ in $\Gamma_{22,14}$. Clearly, $\Gamma_{14} \subseteq (\Gamma_{14})_{\mathrm{R}}$, and we obtain the following decomposition:

$$\Gamma_{22,14} \supset (\Gamma_{22})_{\mathrm{L}} \oplus \overline{(\Gamma_{14})}_{\mathrm{R}}\,. \tag{2.85}$$

This decomposition does not contain self-glue, i.e. all elements of $\Gamma_{22,14}$ that belong to the $\mathbb{R}$-span of $\overline{(\Gamma_{14})}_{\mathrm{R}}$ also lie in $\overline{(\Gamma_{14})}_{\mathrm{R}}$, and those belonging to the $\mathbb{R}$-span of $(\Gamma_{22})_{\mathrm{L}}$ also lie in $(\Gamma_{22})_{\mathrm{L}}$. In the following, we always assume that decompositions such as in (2.85) are self-glue free, and, that the lattice dimensions on the l.h.s. and on the r.h.s. of the inclusion relation match.

Symmetries of the supercurrent equations. Equations (2.81)–(2.83) possess several symmetries (also cf. [28]). First, note that the internal $c_R = 9$ right-mover CFT always contains the Kac-Moody algebra $\mathfrak{g}_R$ spanned by the currents $i\partial X_R^i(z)$. To these currents corresponds the $U(1)^9$ symmetry group which is infinitesimally generated by the $\hat{p}_R^i$. Then, if $G_{\text{int}}(z)$ as in (2.80) satisfies the super-Virasoro algebra, so does the conjugate

$$U(\xi)G_{\text{int}}(z)U(\xi)^\dagger = \sum_{s^2=3} A(s)e^{is\cdot\xi} :e^{is\cdot X_R(z)}: \varepsilon(s, \hat{p}_R) \,, \tag{2.86}$$

where $U(\xi) = e^{i\xi\cdot\hat{p}_R}$ denotes an element of the symmetry group. One also verifies that if $A(s)$ solves (2.81)–(2.83), then so does $A(s)e^{is\cdot\xi}$.

Whenever there exist norm 2 vectors in Ξ, the $U(1)^9$ symmetry is enlarged to a non-Abelian group by additional Frenkel-Kac currents. Then, the norm 3 vectors correspond to weights of a (in general reducible) representation of this non-Abelian group, and group transformations can be used to set some of the $A(s)$ to zero. In this case it may happen that the $A(s)$ become non-zero only on a sublattice $\Xi' \subset \Xi$, which can then be considered more fundamental than Ξ. Besides these continuous symmetries, there may also be additional discrete symmetries (e.g. those induced by lattice automorphisms) that transform one solution into another.

In any case, above symmetries only correspond to a change of basis, and as such produce physically equivalent supercurrents. However, one can not rule out the possibility that equations (2.81)–(2.83) allow for distinct solutions not related by a physical symmetry. Then, one obtains inherently different string vacua. These vacua share the same spectrum, but string amplitudes which necessitate picture changing may differ.

2.2 Asymmetric Orbifolds

Orbifolding [1, 2] is an elegant method that allows the construction of new string vacua from existing ones by dividing out a finite group G_{orb} of symmetries. The spectrum of the new theory then contains an untwisted sector and (possibly several) twisted sectors. The untwisted sector is obtained simply by projecting on G_{orb}-invariant states of the original theory, but is not modular invariant by itself. Modular invariance is only restored by including the twisted sectors, which contain strings with an additional set of boundary conditions. With orbifold constructions it is possible to reduce the number of supersymmetries. It is in particular possible to obtain chiral $\mathcal{N} = 1$ theories from a $\mathcal{N} = 4$ toroidally compactified string.

2.2.1 The Asymmetric Orbifold Construction

Let us now introduce a Z_N orbifold construction that starts from a Narain compactified heterotic string [19, 20]. The internal $(c_L, c_R) = (22, 9)$ CFT of Narain compactification contains 22 left-mover bosons $X_L^I(\bar{z})$ and 6 right-mover bosons $X_R^i(z)$ compactified on a Narain lattice $\Gamma_{22,6}$, as well as six right-mover NSR-fermions $\psi_R^i(z)$. Modular invariance is ensured by demanding that $\Gamma_{22,6}$ is even and self-dual, and by applying the usual GSO projection to the fermion sector [47]. The internal supercurrent of this theory is given by

$$G_{\text{int}}(z) = i\psi_{\text{R}}(z) \cdot \partial X_{\text{R}}(z)\,. \tag{2.87}$$

In order to specify the orbifold group, one defines the action of a generator of Z_N on the internal fields X_{L}^I, X_{R}^i and ψ_{R}^i:

$$\begin{aligned} X_{\text{L}} &\mapsto \theta_{\text{L}} X_{\text{L}} + 2\pi v_{\text{L}}\,, \\ X_{\text{R}} &\mapsto \theta_{\text{R}} X_{\text{R}} - 2\pi v_{\text{R}}\,, \\ \psi_{\text{R}} &\mapsto \theta_{\text{R}} \psi_{\text{R}}\,, \end{aligned} \tag{2.88}$$

where θ_{L} and θ_{R} denote (orthogonal) twist matrices and v_{L} and v_{R} are shift vectors. Note, that this twisting preserves the internal supercurrent (2.87). In order for the action (2.88) to be a symmetry of the theory, $\theta = \theta_{\text{L}} \oplus \theta_{\text{R}}$ must be an automorphism of the Narain lattice $\Gamma_{22,6}$. Then, the action on the vertex operators

$$V_k(z,\bar{z}) = \,:\!e^{ik_{\text{L}}\cdot X_{\text{L}}(\bar{z}) + ik_{\text{R}}\cdot X_{\text{R}}(z)}\!:\varepsilon(k,\hat{p})\,, \tag{2.89}$$

is given by

$$V_k \mapsto \eta_\theta(k)^{-1} e^{2\pi i (v_{\text{L}}\cdot k_{\text{L}} - v_{\text{R}}\cdot k_{\text{R}})} V_{\theta^{-1}k}\,. \tag{2.90}$$

Again, $\varepsilon(s,t)$ denotes a cocycle, and $\hat{p} = (\hat{p}_{\text{L}}, \hat{p}_{\text{R}})$ denotes the zero-mode operators of the left-mover and right-mover currents $i\bar{\partial}X_{\text{L}}$ and $i\partial X_{\text{R}}$, respectively. Moreover, $\eta_\theta(k)$ is a phase factor that must satisfy

$$\varepsilon(\theta^{-1}k, \theta^{-1}k') = \varepsilon(k,k')\,\eta_\theta(k)\,\eta_\theta(k')\,\eta_\theta(k+k')^{-1}\,. \tag{2.91}$$

This phase factor ensures that the orbifold action is an automorphism of the vertex operator algebra (that is, the transformed quantities have identical OPEs as the original ones), and is also important because it can cause that the order of G_{orb} doubles when acting on the vertex operators. The calculation of $\eta_\theta(k)$ for a convenient choice of the cocycle is described in [48].

As in the heterotic string the number of left-mover and right-mover bosons is different, the above is an intrinsically asymmetric construction. One nevertheless calls such an orbifolding *symmetric* if $\theta_{\text{L}} \cong \theta_{\text{R}} \oplus \theta_{16}$ for some 16-dimensional twist θ_{16}, and asymmetric otherwise. Here, we do not care about the difference and just call above general construction "asymmetric orbifolding".

For the following discussion it is convenient to bosonize the fermions ψ_{R}^i of the original Narain theory. Then, one obtains a covariant lattice theory with $\Gamma_{22,14} = \Gamma_{22,6} \oplus \overline{E_8^{\text{st}}}$. Since our orbifold group is Abelian, it is always possible to choose the chiral bosons H^3, H^4 and H^5 of the bosonization such that θ_{R} acts as a shift,

$$H^\alpha \mapsto H^\alpha - 2\pi v_\psi^\alpha\,. \tag{2.92}$$

Note, that here we adopt the notation of Section 2.1 and let H^1 and H^2 denote the bosonization of the ψ^μ. As these are untouched by the orbifolding, we assume that the components v_ψ^1 and v_ψ^2 vanish. The other v_ψ^α are then determined from the twist vector t^α, which appears in the eigenvalues $e^{\pm 2\pi i t^\alpha}$ of θ_R (here, we assume $\det(\theta_\mathrm{R}) = 1$) as $v_\psi^\alpha = t^\alpha$. However, there is a small ambiguity in this prescription which we will resolve in Subsection 2.2.2. Under the action (2.92), the vertex operators $:e^{i\lambda_\alpha H^\alpha}:$ obtain a phase $e^{-2\pi i \lambda_\alpha v_\psi^\alpha}$.

The twist-shift correspondence. The Narain theory that is the starting point of above construction is always equivalent to a covariant lattice theory, but the orbifold theory need not be so in general. For example, any twist θ_L that breaks the rank of the gauge group will necessarily result in a theory that is not describable by free bosons on a lattice. However, we now discuss a rather general class of asymmetric orbifolds which are equivalent to a covariant lattice theory [33–35]. Henceforth, we assume that θ_L is the identity (v_L may still be non-zero, as the shift action does not project out the $i\bar{\partial}X_\mathrm{L}$). Yet there remains the problem that θ_R might reduce the rank of the right-mover Kac-Moody algebra $\mathfrak{g}_\mathrm{R}$. However, for certain combinations of Narain lattices and twists θ_R it is possible perform a "rebosonization" of the X_R^i, i.e. a change of basis, that allows one to trade θ_R for a shift v_R [34]. Then, the complete orbifold action is given by a shift, which maps the original Narain theory on another covariant lattice theory.

Let us in the following assume that the Narain lattice decomposes as

$$\Gamma_{22,6} \supset (\Gamma_{22})_\mathrm{L} \oplus \overline{(\Gamma_6)_\mathrm{R}}\,, \tag{2.93}$$

where $(\Gamma_6)_\mathrm{R}$ shall be a root lattice of a rank 6 semi-simple Lie algebra of ADE type. Note that $\Gamma_{22,14}$ is then of the form (2.85), with

$$(\Gamma_{14})_\mathrm{R} = (\Gamma_6)_\mathrm{R} \oplus E_8^\mathrm{st}\,. \tag{2.94}$$

In this setup, it is possible to express the $i\partial X_\mathrm{R}$ in terms of exponentials of new chiral boson fields $X_\mathrm{R}'^i$:

$$i\partial X_\mathrm{R}^i = \sum_{r^2=2} \alpha(r) :e^{ir\cdot X_\mathrm{R}'}: \varepsilon(r, \hat{p}_\mathrm{R}')\,. \tag{2.95}$$

Here, the sum is over a set of vectors $r \in (\Gamma_6)_\mathrm{R}$ of norm 2, and the $\alpha(r)$ are some suitable coefficients that must be chosen such that the r.h.s. of (2.95) has the same OPEs as $i\partial X_\mathrm{R}^i$. Then, for special choices of the θ_R, the twist action on the X_R^i from (2.88) is equivalent to a shift action on the $X_\mathrm{R}'^i$ as

$$X_\mathrm{R}' \mapsto X_\mathrm{R}' - 2\pi v_\mathrm{R}\,. \tag{2.96}$$

The θ_R for which this twist-shift correspondence is applicable turn out to be exactly those that belong to the Weyl group of the simply-laced root system of $(\Gamma_6)_\mathrm{R}$.

Note, that the change of basis (2.95) can also be understood as an automorphism of the underlying vertex operator algebra. In particular, it gives an automorphism of the zero-mode Lie algebra $\mathfrak{g}^0_{\mathrm{R}}$. For Weyl twists, i.e. inner automorphisms of the corresponding Lie group G_{R}, this automorphism can be used to conjugate the original action (2.90) into the maximal torus of G_{R}, hence giving a shift action. For outer automorphisms there does not exist such a conjugation. There is also another way to see that an outer automorphism twist θ_{R} is not viable in this framework: such a twist acts as a symmetry of the Dynkin diagram corresponding to $(\Gamma_6)_{\mathrm{R}}$, and it nontrivially permutes the conjugacy classes of $(\Gamma_6)_{\mathrm{R}}$. Then, θ_{R} will not be a symmetry of the Narain lattice, unless we also twist the left-movers in a similar way.

Now, one might ask why we discussed the original twist action (2.88) at all, if we could just have started with a pure shift action action on the $\mathcal{N} = 4$ covariant lattice theory. The answer lies in the requirement of world-sheet supersymmetry: an arbitrary shift vector that acts on the nine chiral bosons X^i_{R} and $H^{3,4,5}$ would project out the supercurrent. However, we saw that the construction (2.88) retains the supercurrent, and this fact is not spoiled by bosonizing the ψ^i_{R} and by the base change (2.95). In other words, above construction provides us with a relation between v_{R} and v_ψ that ensures world-sheet supersymmetry in the orbifold theory. We also observe that in terms of the X'_{R} and H^α, the supercurrent takes the form (2.80), which we have shown to be necessary for chiral covariant lattice theories.

The orbifold spectrum. Let us now discuss the orbifold theory that is obtained from the complete shift action. The shift vector is written compactly as $v = (v_{\mathrm{L}}, v_{\mathrm{R}}, v_\psi, 0)$, where the zero entry corresponds to the D^{st}_5 part that we do not touch at all. Also, if the twist-orbifolding defines a modular invariant theory, one can always choose v such that $v^2 \in 2\mathbb{Z}$ (cf. [28, Appendix A.4]). Let in the following N denote the smallest natural number such that $Nv \in \Gamma_{22,14}$ (we assume $N > 1$ as the orbifolding would be trivial otherwise).

In order to calculate the untwisted spectrum, one has to project on G_{orb} invariant states (or vertex operators). The untwisted sector of the shift-orbifold theory is then represented by the sublattice $\Gamma^{\mathrm{u}}_{22,14}$ of vectors $x \in \Gamma_{22,14}$ with vanishing orbifold phase, i.e. $x \cdot v \in \mathbb{Z}$. This lattice always has a (self-glue free) decomposition

$$\Gamma^{\mathrm{u}}_{22,14} \supset (\Gamma^{\mathrm{u}}_{22})_{\mathrm{L}} \oplus \overline{(\Gamma^{\mathrm{u}}_{14})}_{\mathrm{R}} \,. \tag{2.97}$$

Moreover, $|\Gamma_{22,14}/\Gamma^{\mathrm{u}}_{22,14}| = N$, so $\Gamma^{\mathrm{u}}_{22,14}$ is not self-dual. However, the inclusion of the twisted sectors provides additional glue vectors of the form

$$nv + \Gamma^{\mathrm{u}}_{22,14} \,, \qquad n \in \{1, \dots, N-1\} \,. \tag{2.98}$$

These cosets complete $\Gamma^{\mathrm{u}}_{22,14}$ to an even self-dual lattice $\Gamma'_{22,14} \supset \Gamma^{\mathrm{u}}_{22,14}$. Then, in particular $(\Gamma'_{14})_{\mathrm{R}} \supseteq (\Gamma^{\mathrm{u}}_{14})_{\mathrm{R}}$, where the equality is achieved for appropriate choices of v_{L}.

2.2.2 Twists for Chiral Models

To obtain a chiral spectrum, v_ψ must break $E_8^{\rm st}$ to $E_6^{\rm st}$ ($\mathcal{N}=1$) or $D_5^{\rm st}$ ($\mathcal{N}=0$). If one considers $E_8^{\rm st}$ decomposed as

$$E_8^{\rm st} \supset D_1 \oplus D_1 \oplus D_1 \oplus D_5^{\rm st}\,, \tag{2.99}$$

one sees that all conjugacy classes of the form

$$\begin{aligned}&(v,0,0,v)\,,\\&(0,v,0,v)\,,\\&(0,0,v,v)\,,\end{aligned} \tag{2.100}$$

must be projected out, as otherwise $D_5^{\rm st}$ would be enhanced at least to some $D_{5+k}^{\rm st}$. For this to happen, it is necessary that

$$v_\psi^\alpha \notin \mathbb{Z}\,. \tag{2.101}$$

This is equivalent to demanding that $\theta_{\rm R}$ is non-degenerate, i.e. $\det(\theta_{\rm R}-\mathbb{1})\neq 0$. The decomposition (2.99) also contains the eight conjugacy classes:

$$(s,s,s,s),\ (c,c,c,c),\ \underline{(s,s,c,c)}\,, \tag{2.102}$$

the underline denoting all possible permutations. The massless states are given by the momenta $\lambda=(\lambda_2,\lambda_3,\lambda_4,\lambda_5)$ of the form

$$(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2})\,, \tag{2.103}$$

with an even number of minus signs (λ_2 corresponds to H^2 which appears in the bosonization of ψ^2 and ψ^3, equation (2.7)). In order to attain $\mathcal{N}=1$, it is necessary that

$$v_\psi^3 \pm v_\psi^4 \pm v_\psi^5 \in 2\mathbb{Z} \tag{2.104}$$

or

$$v_\psi^3 \pm v_\psi^4 \pm v_\psi^5 \in \mathbb{Z} \text{ and at least one } v_\psi^\alpha \in \frac{\mathbb{Z}}{2} \tag{2.105}$$

for any sign combination: one sees that this causes a cancellation of the phase $\lambda_\alpha v_\psi^\alpha$ for exactly one CPT-conjugate pair of momenta in (2.103), leading to a single gravitino in the spectrum.

Let us now discuss an ambiguity in the choice of the v_ψ^α. First note that the components of the twist vector t^α are only determined modulo integers and signs from the eigenvalues of $\theta_{\rm R}$. While one can show that any sign combination of the v_ψ^α gives the same theory, adding arbitrary integers to them can result in different theories. However, adding an *even* vector to $(v_\psi^3, v_\psi^4, v_\psi^5)$ produces equivalent theories because such a vector belongs to the D_3 factor in the decomposition $E_8^{\rm st}\supset D_3\oplus D_5^{\rm st}$ and thus has integral inner product with all vectors in $E_8^{\rm st}$. Thus, for a given $\theta_{\rm R}$ there are in principle two different choices for v_ψ, which act equally on the NS sector, but differently on the R sector. Now, if a shift vector v_ψ satisfies (2.105), then adding an odd vector to v_ψ will destroy this condition, *except* when any of the v_ψ^α is half-integral: in that case, adding an odd vector is reverted by a sign change. Hence, depending on the twist θ_R there are three possibilities:

1. The twist always breaks supersymmetry completely. For these, neither (2.104) nor (2.105) can be satisfied.
2. The twist always retains $\mathcal{N} = 1$ supersymmetry. In this case, (2.105) is satisfied.
3. The twist allows two inequivalent shifts: one which destroys supersymmetry completely, and one which retains $\mathcal{N} = 1$. In this case, none of the v_ψ^α is half-integral but it is possible to satisfy (2.104) by adding an odd vector to v_ψ.

Finally, let us note that it possible to restore non-chirality and further supersymmetries from twisted sectors. This usually happens when the order of the action of G_orb is different on left-movers and right-movers. It is however possible to remove these cases by imposing additional conditions on the shifts v_L.

Non-degenerate Weyl twists. In order to construct a chiral orbifold theory we require a non-degenerate Weyl twist. Non-degenerate elements of Weyl groups were classified by Carter [49] using certain generalized Dynkin diagrams that are known as Carter diagrams. In particular, for any semi-simple Lie-algebra there is always a Carter diagram which corresponds to the Coxeter element, and it is identical to the respective Dynkin diagram. Also, any Carter diagram of a simply-laced subalgebra of the same rank is allowed. A method to calculate the twist vector t^α and the shift vector v_R from a Carter diagram is given in [34]. In Table 2.1 we give a list of all degenerate Weyl twists and corresponding root lattices $(\Gamma_6)_\mathrm{R}$.

Table 2.1: Possible twists $(\Gamma_6)_{\mathrm{R}}$ and root lattices leading to chiral asymmetric orbifold models which are equivalent to a covariant lattice. Note that the twists corresponding to the Carter diagrams A_3^2 and $A_1^2 D_4(a_1)$ are conjugate to each other within D_6. The attainable number $\mathcal{N}$ of supersymmetries is also given.

Order	t^α	Carter Diagram	$\mathcal{N}$	$(\Gamma_6)_{\mathrm{R}}$
Z_2	$(1,1,1)/2$	A_1^6	0	$A_1^6, A_1^2 D_4, D_6$
Z_3	$(1,1,1)/3$	A_2^3	0,1	A_2^3, E_6
Z_4	$(1,1,2)/4$	A_3^2	1	A_3^2, D_6
	$(1,1,2)/4$	$A_1^2 D_4(a_1)$	1	$A_1^2 D_4, D_6$
	$(1,2,2)/4$	$A_1^3 A_3$	0	$A_1^3 A_3, A_1 D_5$
Z_6	$(1,1,2)/6$	$E_6(a_2)$	0,1	E_6
	$(1,1,3)/6$	$D_6(a_2)$	0	D_6
	$(1,2,3)/6$	$A_1 A_5$	1	$A_1 A_5, E_6$
	$(1,2,3)/6$	$A_2 D_4$	1	$A_2 D_4$
	$(1,3,3)/6$	$A_1^2 D_4$	0	$A_1^2 D_4, D_6$
	$(2,2,3)/6$	$A_1^2 A_2^2$	0	$A_1^2 A_2^2$
	$(2,3,3)/6$	$A_1^4 A_2$	0	$A_1^4 A_2, A_2 D_4$
Z_7	$(1,2,3)/7$	A_6	0,1	A_6
Z_8	$(1,2,3)/8$	$D_6(a_1)$	0,1	D_6
	$(1,3,4)/8$	$A_1 D_5$	1	$A_1 D_5$
Z_9	$(1,2,4)/9$	$E_6(a_1)$	0	E_6
Z_{10}	$(1,3,5)/10$	D_6	0	D_6
	$(2,4,5)/10$	$A_1^2 A_4$	0	$A_1^2 A_4$
Z_{12}	$(1,4,5)/12$	E_6	0,1	E_6
	$(2,3,6)/12$	$A_1 D_5(a_1)$	0	$A_1 D_5$
	$(3,3,4)/12$	$A_2 D_4(a_1)$	0	$A_2 D_4$
	$(3,4,6)/12$	$A_1 A_2 A_3$	0	$A_1 A_2 A_3$
Z_{15}	$(3,5,6)/15$	$A_2 A_4$	0	$A_2 A_4$

Chapter 3

Classification of Chiral Models

In the previous chapter we introduced the heterotic covariant lattice formalism and saw that it can be used to construct chiral four-dimensional string models. However, we are lacking a systematic and exhaustive understanding of this particular corner of the four-dimensional string landscape. The purpose of this chapter is to fill this gap by providing an overall classification of chiral covariant lattice models.

Let us start by giving an overview on the classification process. In Subsection 2.1.5 we concluded that, in a chiral model, the covariant lattice $\Gamma_{22,14}$ must decompose as in equation (2.85) into a left-mover lattice $(\Gamma_{22})_{\mathrm{L}}$ and a right-mover lattice $(\Gamma_{14})_{\mathrm{R}}$. The covariant lattice also obeys several consistency conditions from modular invariance and world-sheet supersymmetry. However, while modular invariance involves $\Gamma_{22,14}$ as a whole, the requirement of world-sheet supersymmetry directly affects only the right-mover lattice. In particular we saw that $(\Gamma_{14})_{\mathrm{R}}$ must contain a sublattice Γ_{14} which is constructed from a nine-dimensional lattice Ξ using the constraint vectors. On this "supercurrent lattice" Ξ there must exist a solution to equations (2.81)–(2.83). Our first step will be the classification of these supercurrent lattices, which will be treated in Section 3.1.

Once these lattices are classified, it is possible to construct all right-mover lattices that may appear in a chiral model. Modular invariance then greatly restricts the choice of the left-mover lattice $(\Gamma_{22})_{\mathrm{L}}$, and one is naturally led to the conclusion that the possible $(\Gamma_{22})_{\mathrm{L}}$ are organized in what is known as lattice genera. The well-established theory of lattice genera then allows us to calculate a lower bound on the number of left-mover lattices, and also provides us exact methods for their classification. This all is subject of Section 3.2.

Finally, one has to keep in mind the possibility that for a pair of lattices $(\Gamma_{22})_{\mathrm{L}}$ and $(\Gamma_{14})_{\mathrm{R}}$ there may be several *inequivalent* embeddings of the form (2.85). This depends on the existence of certain automorphisms of these lattices. We do not treat this issue generally, but we consider special cases in Chapter 4, where we perform explicit model building.

3.1 The Supercurrent Lattices

Let us begin with the classification of the nine-dimensional supercurrent lattices that were introduced in Section 2.1. For a chiral model, Ξ must satisfy the following properties:

1. *Basic.* Ξ is positive definite, integral, generated by its norm 3 vectors, and contains no norm 1 vectors.

2. *Supersymmetry.* There exists a solution $A(s)$ to equations (2.81)–(2.83) on Ξ.

We call a positive-definite lattice Ξ that obeys above two properties *admissible*. An admissible lattice can further be reduced to certain fundamental building blocks, that we now describe.

First, note that an orthogonal sum of admissible lattices is again admissible. One also verifies that the converse is true: if an orthogonal sum of several factors is admissible, then so is each factor. This is because any norm 3 vector v_3 that were not contained in one of the orthogonal factors would be a sum of lattice vectors v_1 and v_2, where v_1 has norm 1 and v_2 is of norm 2. However, by the "basic" properties such vectors v_1 do not exist and hence equations (2.81)–(2.83) must have a solution independently for each factor. Another point is that we do not need to care about an admissible lattice which contains an admissible sublattice of the same dimension. These facts motivate the definition of the following properties:

1. *Primitivity.* Ξ is not isomorphic to an orthogonal sum $\Xi_1 \oplus \ldots \oplus \Xi_k$, $k > 1$.

2. *Elementarity.* Ξ is admissible and does not contain a strictly smaller admissible sublattice $\Xi' \subset \Xi$ of the same dimension.

Then, any admissible lattice Ξ can be built from primitive elementary building blocks by orthogonal composition and gluing. In the following we classify these building blocks by first constructing a set of candidate lattices and then solving equations (2.81)–(2.83).

3.1.1 Construction of Candidate Lattices

Let us start by classifying all lattices satisfying above "basic" and "primitivity" properties. This is done by induction over $n = \dim(\Xi)$ up to $n = 9$.

At this point we need to discuss a rather important subtlety. First, recall that any lattice Λ has a basis $\{b_1, \ldots, b_n\}$ with basis vectors $b_i \in \Lambda$ such that each $x \in \Lambda$ is represented by a unique integer linear combination of the b_i. Then, n denotes the dimension of Λ. When we say that a lattice Λ is spanned (or generated) by a finite set $\{a_1, \ldots, a_N\}$ of vectors $a_i \in \Lambda$, we mean that any $x \in \Lambda$ can be written as a, not necessarily unique, integer linear combination of the a_i. Now, one might hope that, for any such generating set, it is possible to choose a basis where all basis vectors b_i belong to that generating set. While this is certainly true for vectors spaces, it is not so for lattices (consider, for example, the lattice $\mathbb{Z}$ and the generating set $\{2, 3\}$). A counterexample that comes close to our situation has been found in [50]. The lattice constructed there is generated by its vectors of minimal length, but does not possess a basis solely out of these minimal vectors. After this discovery such lattices have received some interest [51]: they were shown not to exist for $n \leq 9$, while for $n = 10$ a concrete example of such a lattice was found.

Although here we are only interested in the case $n \leq 9$, our lattices are allowed to contain vectors of norm 2, so the theorem of [51] is not applicable. Thus, among the lattices that satisfy our basic properties there might exist "pathological" lattices which do not possess a basis of norm 3 vectors. However, suppose Ξ' be such a pathological case. Then, there must exist a sublattice $\Xi \subset \Xi'$ that is not pathological. Hence, we can at the moment restrict our classification to lattices which do have a basis of norm 3 vectors and care about the pathological cases afterwards. Now, the idea is to enumerate these lattices by constructing all possible Gram matrices, in a way similar to the "lamination" process introduced in [52, 53]. For a lattice with basis $\{b_1, \ldots, b_n\}$, the Gram matrix is defined by

$$G_{ij} = b_i \cdot b_j \,. \tag{3.1}$$

It determines the lattice up to $O(n)$ rotations in the ambient space, but is not a basis independent quantity. In our case we assume the existence of a basis of norm 3 vectors, so we only need to consider Gram matrices where the diagonal elements are $G_{ii} = 3$. Now, suppose G' be such a Gram matrix of a $(n+1)$-dimensional lattice which satisfies our basic properties. The lattice corresponding to the restricted Gram matrix $G = (G'_{ij})_{i,j\leq n}$ then must also fulfill these properties. Now, we can write G' as

$$G' = \begin{pmatrix} G & v \\ v^T & 3 \end{pmatrix} , \tag{3.2}$$

where v is a column vector. From positive definiteness it follows that $\det(G') > 0$. This translates into the following condition:

$$v^T G^{-1} v < 3 \,. \tag{3.3}$$

Since the v_i are integers and G is positive definite, there are only finitely many possible v satisfying equation (3.3).

The naive algorithm then goes as follows. Suppose we have a set $\mathscr{B}_n$ that contains a Gram matrix for each non-pathological lattice of dimension n with the basic properties. Set $\mathscr{B}_{n+1} = \{\}$. Then, for each pair (G, v) with $G \in \mathscr{B}_n$ and v satisfying equation (3.3) do the following:

1. Construct the matrix G' as in (3.2). It is necessarily positive definite.

2. If the lattice corresponding to G' contains norm 1 vectors, continue to the next pair (G, v).

3. Otherwise, check whether $\mathscr{B}_{n+1}$ already contains a Gram matrix that is equivalent to G' by a change of basis. If not, replace $\mathscr{B}_{n+1}$ by $\mathscr{B}_{n+1} \cup \{G'\}$.

After completion, $\mathscr{B}_{n+1}$ contains a Gram matrix for each non-pathological lattice of dimension $n+1$ satisfying our basic properties. The algorithm is initialized with $\mathscr{B}_1$, which contains the only possible Gram matrix in one dimension, and then repeatedly applied until we obtain $\mathscr{B}_9$. Since we start with only finitely many lattices, we only produce finitely many new lattices in each step and hence $\mathscr{B}_n$ is finite for all n.

In this form, the algorithm also produces non-primitive lattices. However, the non-primitive cases are excluded if we consider only non-vanishing v in each step. This can be seen as follows. Suppose that a lattice Ξ with Gram matrix $G \in \mathscr{B}_i$ be an orthogonal sum two lattices Ξ_1 and Ξ_2. Then G is necessarily of the form $G_1 \oplus G_2$ up to permutations of the basis vectors b_i. Here, G_1 and G_2 denote Gram matrices of Ξ_1 and Ξ_2, respectively. This is because any potential basis vector of norm 3 in Ξ must belong to either Ξ_1 or Ξ_2. Then one sees that, in order to obtain G, we have at some stage in the naive algorithm chosen a vanishing v. Thus, by leaving out the case $v = 0$ one restricts to primitive lattices. One also sees that this prescription does not accidentally remove some primitive lattices, because for them one can always find a Gram matrix for which the vectors v are non-zero at any stage.

Finally we have to check for the existence of pathological cases. This is done as follows: for each lattice we found (including the non-primitive ones obtained by considering orthogonal sums), one constructs all overlattices Ξ' that are obtained by including additional norm 3 vectors. If we encounter a primitive lattice that was not yet obtained, we must add it to our results.

The actual computation showed that for $n \leq 9$ no pathological case exists. Also, in Table 3.1 the number of primitive basic lattices that were obtained for each dimension is listed. At this point it should also be noted that above naive algorithm is very slow for large n, and the actual implementation includes several optimizations. One such optimization arises if one also computes the automorphism groups of the lattices corresponding to the Gram matrices G. The automorphism group naturally acts on the vectors v, and we can compute its orbits. Then we only need to consider one representative vector v in each orbit.

3.1.2 Solutions to the Supercurrent Equations

At this stage we possess a list of all lattices with the basic and primitivity properties. The next step is to solve the conditions imposed by world-sheet supersymmetry. First, it is practical to consider only equation (2.81), which is a system of linear equations in the $|A(s)|^2$. It turns out that in total only 59 lattices possess a solution, so the number of candidate lattices is drastically reduced (cf. Table 3.1). For these candidates we then consider, in a somewhat case by case manner, the full set of equations (2.81)–(2.83). This is done using the following strategies:

- By means of the $U(1)^9$ symmetry discussed in Subsection 2.1.5, we may fix the phases of some $A(s)$. Moreover, if $U(1)^9$ is extended to some non-Abelian symmetry, we may use this symmetry to set some $A(s)$ to zero. This procedure radically reduces

Table 3.1: The number of lattices with the respective properties: *basic* (B), *supersymmetry* (S), *primitivity* (P), *elementarity* (E).

$\dim(\Xi)$	PB	PB + (2.81)	PBS	PE
1	1	1	1	1
2	2	0	0	0
3	7	1	1	1
4	28	0	0	0
5	136	1	1	1
6	911	6	6	3
7	8665	2	2	2
8	131316	8	7	4
9	3345309	40	36	14
Total	3486375	59	54	26

the complexity of the problem in most cases.

- Sometimes, systems of polynomial equations are easier to solve if one first computes a Gröbner basis of the corresponding ideal. Especially, if the computed Gröbner basis is trivial then the system has no solutions.

With these methods it was possible to rule out the existence of a solution for 5 out of the 59 candidate lattices. For further 26 candidates an explicit solution was found, albeit with some trial and error. Then, it was proven that these lattices are elementary and that all remaining candidates can be reproduced by gluing together orthogonal sums thereof.

Thus, for $\dim(\Xi) \leq 9$, there exist 54 primitive admissible lattices of which 26 are primitive elementary. By orthogonally combining them one obtains in $\dim(\Xi) = 9$ a total of 63 admissible lattices, and 32 of them are elementary. A summary of these results is shown in Table 3.1.

In the rest of this work we identify a primitive elementary lattice by its dimension, and, in the cases $6 \leq \dim(\Xi) \leq 9$, also by an additional uppercase Latin subscript which is assigned in alphabetical order. Moreover, we use a shorthand notation where e.g. $3^1 6_A^1$ denotes the orthogonal sum of the primitive elementary lattices 3 and 6_A. Gram matrices for the 26 primitive elementary lattices are provided in Table B.1, together with some further information.

3.2 Right-Mover and Left-Mover Lattices

We have now classified the possible supercurrent lattices Ξ that may appear in a chiral covariant lattice model. From these we can construct all possible right-mover lattices $(\Gamma_{14})_{\mathrm{R}}$ and discuss the associated left-mover lattices that are compatible with modular invariance.

Before we are able to do this, we must briefly study some facts about discriminant forms and lattice genera.

3.2.1 Discriminant Forms and Lattice Genera

Let us begin by introducing the concept of discriminant forms [54]. Let Λ be an integral lattice with Gram matrix G, and let $\det(\Lambda) = |\det(G)|$ denote its discriminant. Since Λ is integral, it must be contained in the dual lattice Λ^*, and hence we can construct the quotient group Λ^*/Λ. This quotient is a product of (finite) cyclic groups whose orders are given by the elementary divisors of Λ. These are in turn defined to be the elementary divisors obtained from the Smith normal form of G.

The important point is now that the bilinear form on Λ naturally induces a bilinear form $B_\Lambda(\cdot,\cdot) : \Lambda^*/\Lambda \times \Lambda^*/\Lambda \mapsto \mathbb{Q}/\mathbb{Z}$, given by

$$B(v + \Lambda, w + \Lambda) = v \cdot w \tag{3.4}$$

for $v, w \in \Lambda^*$. This is indeed well-defined because $(v + x) \cdot (w + y) - v \cdot w \in \mathbb{Z}$, for any $x, y \in \Lambda$. For even lattices Λ we also define a quadratic form $Q_\Lambda : \Lambda^*/\Lambda \mapsto \mathbb{Q}/2\mathbb{Z}$,

$$Q_\Lambda(v + \Lambda) = v^2 \,, \tag{3.5}$$

with $v \in \Lambda^*$. Again, this is well-defined because $(v + x)^2 - v^2 \in 2\mathbb{Z}$ for all $x \in \Lambda$.

The quotient Λ^*/Λ together with the bilinear form B_Λ is called the discriminant bilinear form, $\mathrm{disc}_-(\Lambda)$ of Λ. Similarly, for even lattices we also define the discriminant quadratic form $\mathrm{disc}_+(\Lambda)$ as the tuple $(\Lambda^*/\Lambda, Q_\Lambda)$. Moreover one defines morphisms for these structures to be group morphisms $\phi : \Lambda_1^*/\Lambda_1 \mapsto \Lambda_2^*/\Lambda_2$ which preserve the respective bilinear or quadratic form, i.e.

$$B_{\Lambda_2}(\phi(\cdot), \phi(\cdot)) = B_{\Lambda_1}(\cdot,\cdot) \,, \tag{3.6}$$

in the case of disc_-, and further

$$Q_{\Lambda_2} \circ \phi = Q_{\Lambda_1} \,. \tag{3.7}$$

in the case of disc_+. A fortiori, a morphism of a discriminant quadratic form $\mathrm{disc}_+(\Lambda)$ also gives us a morphism of the corresponding discriminant bilinear form $\mathrm{disc}_-(\Lambda)$.

Now, there is a well known theorem [38], proven in Appendix A.2:

Theorem 1. *For any integral self-dual lattice Λ with a self-glue free decomposition of the form $\Lambda \supseteq \Lambda_1 \oplus \overline{\Lambda}_2$, where $\dim(\Lambda) = \dim(\Lambda_1) + \dim(\Lambda_2)$, one obtains an isomorphism*

$$\mathrm{disc}_-(\Lambda_1) \cong \mathrm{disc}_-(\Lambda_2) \,, \tag{3.8}$$

which at the same time is an isomorphism

$$\mathrm{disc}_+(\Lambda_1) \cong \mathrm{disc}_+(\Lambda_2) \,. \tag{3.9}$$

if Λ is even. Also the "converse" statement is true: given two integral lattices Λ_1, Λ_2 and an isomorphism $\phi : \mathrm{disc}_-(\Lambda_1) \mapsto \mathrm{disc}_-(\Lambda_2)$, then there exists a self-dual lattice Λ with a self-glue free decomposition $\Lambda \supseteq \Lambda_1 \oplus \overline{\Lambda}_2$. If furthermore Λ_1 and Λ_2 are even and ϕ is also an isomorphism of the corresponding discriminant quadratic forms, then also Λ is even.

In the remainder of this work we also adopt the following convention. When we call something a discriminant form $\mathrm{disc}(\Lambda)$ of a lattice Λ (without the attribute "bilinear" or "quadratic"), we mean $\mathrm{disc}_+(\Lambda)$ in the case where Λ is even, and $\mathrm{disc}_-(\Lambda)$ in the case where Λ is odd.

Lattice genera. Now, we want to introduce the concept of lattice genera. Let Λ_1 and Λ_2 denote two integral lattices with Gram matrices G_1 and G_2. Then one defines an equivalence relation "$\equiv$" as follows: we say that $\Lambda_1 \equiv \Lambda_2$ if for every prime number p there exists an invertible p-adic integral matrix U_p such that

$$U_p G_1 U_p^T = G_2 \,, \tag{3.10}$$

and if further Λ_1 and Λ_2 have the same signature. The corresponding equivalence classes are called genera. An alternative characterization of the genus is due to Nikulin [54]: two lattices Λ_1 and Λ_2 lie in the same genus $\mathcal{G}$ if and only if they have identical signatures, they are either both even or odd, and their discriminant forms are isomorphic. Note that the isomorphism already implies $p_1 - q_1 = p_2 - q_2 = 0 \bmod 8$ for the respective signatures (p_1, q_1) and (p_2, q_2). In particular, two lattices in the same genus have identical elementary divisors.

There is a classic result that states that a genus $\mathcal{G}$ contains only finitely many lattices, and the predominant method for the enumeration of all lattices in a genus is known as Kneser's neighborhood method [55]. It is related to the "shift vector method" in [28, Appendix A.4], which is in turn related to certain shift-orbifold constructions. Also, in some cases the "replacement" lattice engineering method (cf. [28, Appendix A.4]) turns out to be useful (we will apply it in Subsection 3.2.4).

The mass formula. Another relevant tool in the study of lattice genera is the Smith-Minkowski-Siegel mass formula [39, 56]. The mass of a genus is defined as

$$m(\mathcal{G}) = \sum_{\Lambda \in \mathcal{G}} \frac{1}{|\mathrm{Aut}(\Lambda)|} \,. \tag{3.11}$$

Here, $\mathrm{Aut}(\Lambda)$ denotes the automorphism group (point group) of Λ (definiteness of the lattices is assumed). Note, that the definition of the mass depends on *all* lattices in $\mathcal{G}$. The mass formula then provides another way of computing the mass which only requires explicit knowledge of a *single* $\Lambda \in \mathcal{G}$. This computation is rather complicated (the technicalities are found in [39]) and we will not go into the details here.

An important application of the mass formula is the computation of a lower bound on $|\mathcal{G}|$: from $|\mathrm{Aut}(\Lambda)| \geq 2$ one obtains

$$|\mathcal{G}| \geq 2m(\mathcal{G}) \,. \tag{3.12}$$

However, this bound is rather crude in many cases. The mass formula also allows to verify whether an explicit enumeration of lattices in a genus is exhaustive.

Consecuences for chiral models. Using the theorem stated earlier, we now conclude from the self-duality of $\Gamma_{22,14}$ that the lattices $(\Gamma_{22})_{\mathrm{L}}$ and $(\Gamma_{14})_{\mathrm{R}}$ appearing in the decomposition (2.85) must have isomorphic discriminant forms. Hence, by the theorem of Nikulin [54], the right-mover lattice completely determines the genus $\mathcal{G}_{\mathrm{L}}$ of $(\Gamma_{22})_{\mathrm{L}}$. Moreover, $(\Gamma_{22})_{\mathrm{L}}$ can be replaced by any other lattice from $\mathcal{G}_{\mathrm{L}}$ without destroying self-duality, so set of possible left-mover lattices that can be paired with some specific $(\Gamma_{14})_{\mathrm{R}}$ is given precisely by the corresponding genus $\mathcal{G}_{\mathrm{L}}$. Not surprisingly, it is also possible to exchange $(\Gamma_{14})_{\mathrm{R}}$ with a different lattice from the same genus, provided that it also obeys the constraints from world-sheet supersymmetry.

It thus possible to use the Smith-Minkowski-Siegel mass formula to give a lower bound on the number of models — or to use one of the earlier mentioned methods and attempt their exact classification.

3.2.2 The Lattice Inclusion Graph

We now use our classification of supercurrent lattices and the theory of lattice genera to obtain an overall insight into the space of covariant lattice models. First, for each of the 32 elementary supercurrent lattices that we classified in Section 3.1, one constructs a right-mover lattice $(\Gamma_{14})_{\mathrm{R}} = \Gamma_{14}$ from the constraint vectors $(0, s, v)$. These lattices are minimal, in the sense that there is no solution to the supersymmetry constraints (2.81)–(2.83) for any strictly smaller sublattice. However, any even overlattice

$$(\Gamma_{14})'_{\mathrm{R}} \supseteq (\Gamma_{14})_{\mathrm{R}} \tag{3.13}$$

clearly inherits the solution $A(s)$ from $(\Gamma_{14})_{\mathrm{R}}$ (it may violate the chiralness constraint from Subsection 2.1.5, though), and only finitely many such overlattices can exist. The explicit construction of all these overlattices produced a total of 414 right-mover lattices.

From these lattices it is possible to construct a directed graph $\mathscr{G}$ in which each lattice is represented by a node, and two lattices A and B are connected by an arrow, $A \longrightarrow B$, if $A \supset B$ and the index $|A/B|$ is prime. It turns out that the graph splits into nine disjoint connected components $\mathscr{G}_1$ to $\mathscr{G}_9$. Consequently, for most right-mover lattices, more than one of the 32 elementary supercurrents can be chosen, and, as pointed out in Subsection 2.1.5, it is possible that these choices are related by a symmetry transformation. In Figures 3.1–3.3 we display the connected components $\mathscr{G}_1$ to $\mathscr{G}_5$ and $\mathscr{G}_9$, but only the subgraphs thereof consisting of nodes with spacetime supersymmetry. Table 3.2 lists the number of lattices in each connected component, separately for the different levels of spacetime supersymmetry. It turns out that among all 414 lattices, only 99 comply with the chiralness condition introduced in Subsection 2.1.5, and merely 19 lead to $\mathcal{N} = 1$ spacetime supersymmetry.

In the following, we discuss in detail some interesting parts of the lattice inclusion graph $\mathscr{G}$ together with the corresponding left-mover genera $\mathcal{G}_{\mathrm{L}}$.

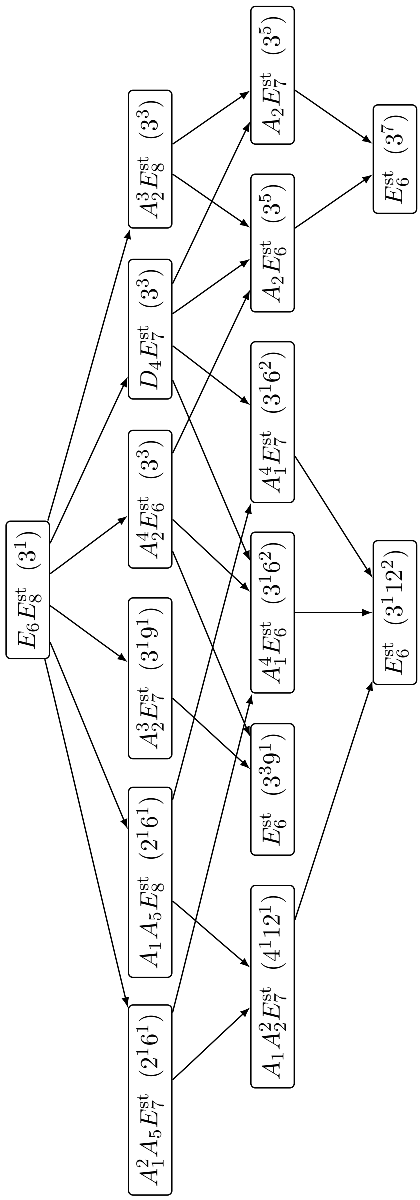

Figure 3.1: Lattices $(\Gamma_{14})_{\mathrm{R}}$ in $\mathscr{G}_1$ with spacetime supersymmetry. The node label indicates the root system of norm 2 vectors and the elementary divisors of $(\Gamma_{14})_{\mathrm{R}}$.

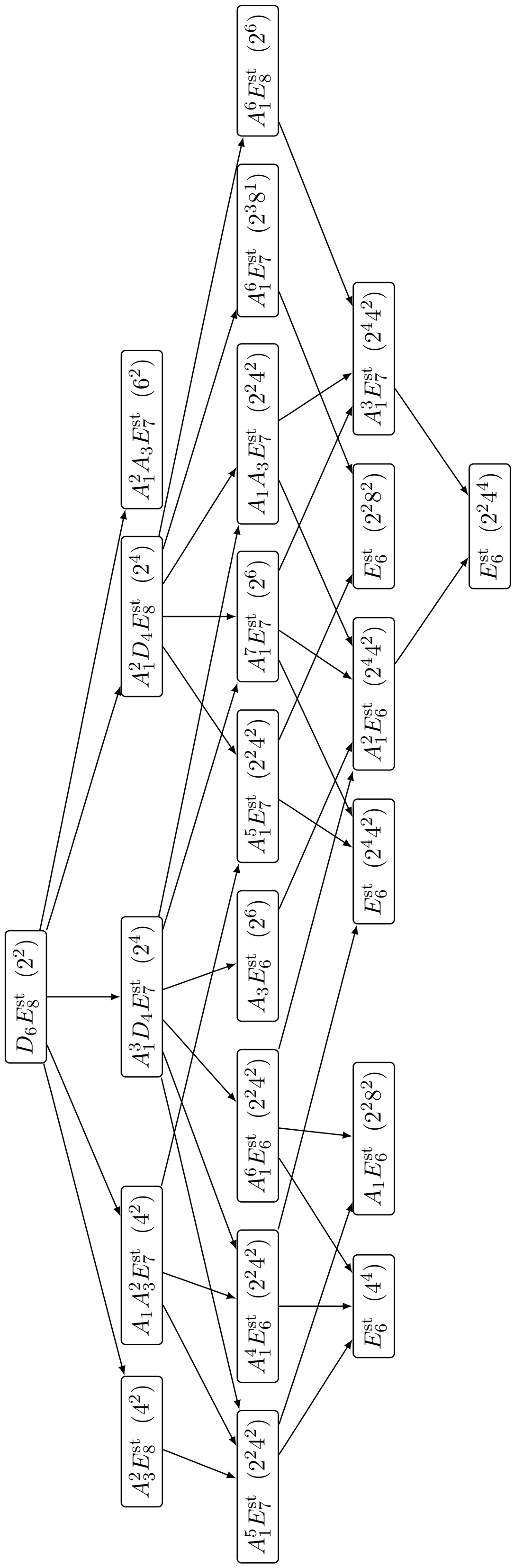

Figure 3.2: Lattices $(\Gamma_{14})_{\mathrm{R}}$ in $\mathscr{G}_2$ with spacetime supersymmetry. The node label indicates the root system of norm 2 vectors and the elementary divisors of $(\Gamma_{14})_{\mathrm{R}}$.

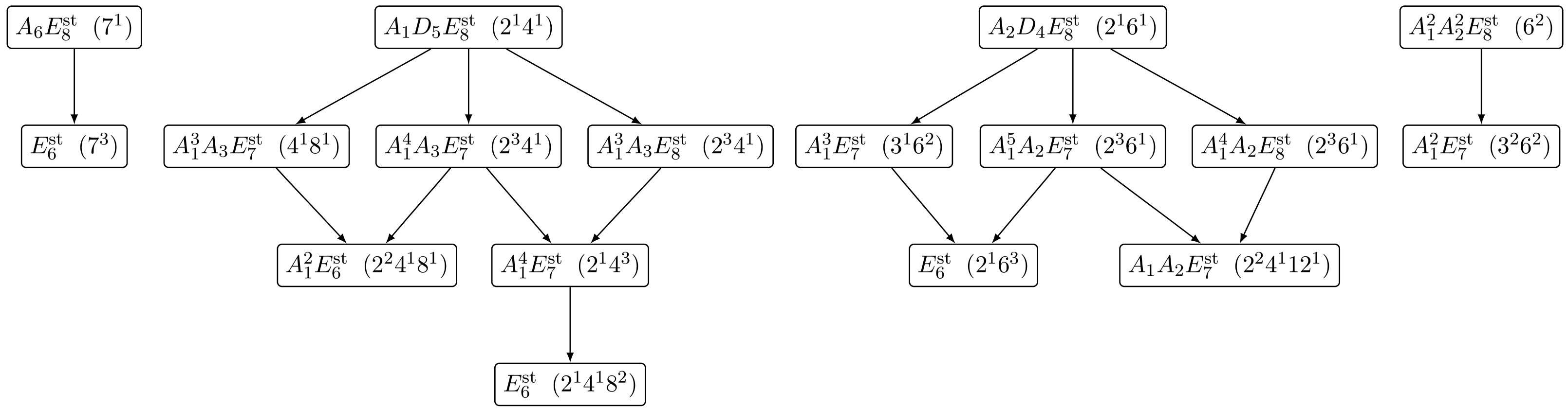

Figure 3.3: Lattices $(\Gamma_{14})_{\mathrm{R}}$ in $\mathscr{G}_3$, $\mathscr{G}_4$, $\mathscr{G}_5$ and $\mathscr{G}_9$ (from left to right) with spacetime supersymmetry. The node label indicates the root system of norm 2 vectors and the elementary divisors of $(\Gamma_{14})_{\mathrm{R}}$.

Table 3.2: Statistics of the lattice inclusion graph $\mathscr{G}$ in total, and also separately for each connected component $\mathscr{G}_i$. Denoted are the total number of lattices, as well as the number of lattices with certain spacetime sublattice ($k > 0$).

	D_5^{st}	E_6^{st}	E_7^{st}	E_8^{st}	D_{5+k}^{st}	Total
$\mathscr{G}_1$	19	6	6	3	36	70
$\mathscr{G}_2$	35	9	9	4	115	172
$\mathscr{G}_3$	1	1	0	1	1	4
$\mathscr{G}_4$	8	2	3	2	45	60
$\mathscr{G}_5$	10	1	3	2	41	57
$\mathscr{G}_6$	2	0	0	1	5	8
$\mathscr{G}_7$	1	0	0	1	6	8
$\mathscr{G}_8$	1	0	0	1	10	12
$\mathscr{G}_9$	3	0	1	1	18	23
Total $\mathscr{G}$	80	19	22	16	277	414

The bottom nodes. Our graph $\mathscr{G}$ contains certain nodes B for which there do not exist any other nodes $B' \longleftarrow B$. We call them "bottom nodes", and they clearly correspond to those 32 lattices $(\Gamma_{14})_\text{R}$ constructed from the elementary supercurrent lattices Ξ using the constraint vectors. Using the Smith-Minkowski-Siegel mass formula, it is possible to calculate a lower bound on the respective $|\mathcal{G}_\text{L}|$ (note that in order to apply this formula, we explicitly need a representative of each $\mathcal{G}_\text{L}$, which can however be given by $(\Gamma_{14})_\text{R} \oplus E_8$). The bottom nodes together with these bounds are listed in Table 3.3.

One recognizes that from the bottom nodes alone one obtains a total of at least $O(10^{23})$ models. Of course, these models are not guaranteed to be chiral. They are also not supersymmetric, and supposedly many of them contain tachyons. In any case though these high numbers rule out an explicit enumeration and evaluation of all these models.

The top nodes. We call a node $T \in \mathscr{G}$ a "top node" if there do not exist other nodes $T' \longrightarrow T$. Remarkably, each connected component contains, among others, also a top node representing a $\mathcal{N} = 4$ theory. These theories are Narain-compactifications of the ten-dimensional theory where $(\Gamma_{14})_\text{R}$ is of the form $(\Gamma_6)_\text{R} \oplus E_8^{\text{st}}$ and $(\Gamma_6)_\text{R}$ is the root lattice of a rank 6 semi-simple Lie algebra of ADE type. Recall that these right-mover lattices appeared in the twist-shift correspondence of asymmetric orbifolds in Subsection 2.2.1.

It also turns out that, separately for each connected component of $\mathscr{G}$, the respective top nodes belong to the same genus. For each of these "top node genera" we performed an exact classification using the "replacement" method described in [28, Appendix A.4] (in some cases, for practicability reasons a generalized method which involves also odd lattices was used). This classification is described in detail in Subsection 3.2.4. In Table 3.4, the top nodes and the respective $|\mathcal{G}_\text{L}|$ are listed.

Table 3.3: The bottom nodes of $\mathscr{G}$, i.e. the lattices $(\Gamma_{14})_{\mathrm{R}}$ generated from the 32 elementary lattices Ξ. Here, $\Delta_2^{\perp}$ denotes the root system of norm 2 vectors orthogonal to D_5^{st}. Also, some information on the corresponding genera $\mathcal{G}_{\mathrm{L}}$ is provided.

	Genus		Lattice $(\Gamma_{14})_{\mathrm{R}}$	
Component	$\lvert\mathcal{G}_{\mathrm{L}}\rvert$ Bound	Divisors	Ξ	$\Delta_2^{\perp}$
$\mathscr{G}_1$	$3.1\cdot10^{17}$	$3^7 6^2$	1^9	–
	$1.3\cdot10^{9}$	$2^4 4^1 12^1$	$1^1 8_A^1$	A_1^2
	$4.2\cdot10^{14}$	$3^1 6^4$	$1^1 8_D^1$	–
	$3.4\cdot10^{12}$	$3^1 18^2$	9_D^1	A_1
	$1.0\cdot10^{14}$	$2^1 6^1 12^2$	9_J^1	–
	$2.9\cdot10^{11}$	$3^2 6^1 18^1$	9_K^1	–
$\mathscr{G}_2$	$5.8\cdot10^{17}$	$6^2 12^2$	$1^2 7_B^1$	–
	$3.8\cdot10^{17}$	$2^6 12^2$	$1^1 3^1 5^1$	–
	$2.1\cdot10^{15}$	$4^2 12^2$	$1^1 8_B^1$	–
	$1.8\cdot10^{15}$	$2^2 4^6$	3^3	–
	$1.2\cdot10^{12}$	$2^4 4^4$	$3^1 6_A^1$	–
	$2.1\cdot10^{11}$	$2^4 8^2$	9_C^1	A_1^2
	$5.2\cdot10^{9}$	$2^4 8^2$	9_E^1	–
	$4.4\cdot10^{9}$	$2^2 4^4$	9_F^1	–
	$4.2\cdot10^{11}$	$4^3 16^1$	9_G^1	–
	$3.6\cdot10^{14}$	$2^2 20^2$	9_L^1	–
	$1.7\cdot10^{15}$	$2^2 4^2 8^2$	9_M^1	–
$\mathscr{G}_3$	$4.1\cdot10^{13}$	$7^1 14^2$	9_N^1	–
$\mathscr{G}_4$	$6.1\cdot10^{18}$	$2^1 4^1 24^2$	$1^1 8_C^1$	–
	$8.4\cdot10^{17}$	$2^1 4^3 8^2$	$3^1 6_B^1$	–
	$1.9\cdot10^{7}$	$2^4 4^1 8^1$	9_A^1	A_1^2
			9_B^1	A_1^2
	$5.1\cdot10^{13}$	$2^3 4^1 8^2$	9_H^1	–
	$3.0\cdot10^{14}$	$4^1 8^3$	9_I^1	–
$\mathscr{G}_5$	$1.2\cdot10^{18}$	$2^1 6^5$	$1^4 5^1$	–
	$8.2\cdot10^{21}$	$2^2 4^1 12^3$	$1^3 3^2$	–
	$6.0\cdot10^{17}$	$2^3 6^1 12^2$	$1^3 6_A^1$	–
$\mathscr{G}_6$	$6.5\cdot10^{22}$	$15^1 30^2$	$1^3 6_C^1$	–
$\mathscr{G}_7$	$2.6\cdot10^{20}$	$2^1 10^1 20^2$	$3^1 6_C^1$	–
$\mathscr{G}_8$	$5.2\cdot10^{22}$	$2^1 12^1 24^2$	$1^3 6_B^1$	–
$\mathscr{G}_9$	$2.7\cdot10^{23}$	$3^2 6^2 12^2$	$1^6 3^1$	–
	$5.6\cdot10^{11}$	$2^6 6^2$	$1^2 7_A^1$	–

Table 3.4: The top nodes in $\mathscr{G}$. The corresponding left-mover genera $\mathcal{G}_\mathrm{L}$ were enumerated completely. Here, Γ^st denotes the spacetime sublattice, and $\Delta_2^\perp$ is the root system of norm 2 vectors orthogonal to Γ^st.

	Genus		Lattice $(\Gamma_{14})_\mathrm{R}$	
Component	$\lvert\mathcal{G}_\mathrm{L}\rvert$	Divisors	$\Delta_2^\perp$	Γ^st
$\mathscr{G}_1$	31	3^1	E_6	E_8^st
			–	D_{13}^st
$\mathscr{G}_2$	68	2^2	D_6	E_8^st
			A_1^2	D_{12}^st
			–	D_{14}^st
$\mathscr{G}_3$	153	7^1	A_6	E_8^st
$\mathscr{G}_4$	326	$2^1 4^1$	$A_1 D_5$	E_8^st
			$A_1 A_3$	D_{10}^st
			A_1	D_{13}^st
$\mathscr{G}_5$	382	$2^1 6^1$	$A_2 D_4$	E_8^st
			A_1^3	D_{10}^st
			A_2	D_{12}^st
$\mathscr{G}_6$	1163	15^1	$A_2 A_4$	E_8^st
$\mathscr{G}_7$	4043	$2^1 10^1$	$A_1^2 A_4$	E_8^st
			A_4	D_{10}^st
$\mathscr{G}_8$	9346	$2^1 12^1$	$A_1 A_2 A_3$	E_8^st
			$A_1 A_2$	D_{11}^st
$\mathscr{G}_9$	19832	6^2	$A_1^2 A_2^2$	E_8^st
			A_1^2	D_{10}^st
			A_2^2	D_{10}^st
			–	D_{12}^st

Nodes with $\mathcal{N} = 1$ spacetime supersymmetry. Finally, let us consider the 19 right-mover lattices in $\mathscr{G}$ that lead to $\mathcal{N} = 1$ supersymmetry. These lattices are listed in Table 3.5, together with information on their associated left-mover genera.

For the lattice $A_2^4 E_6^{\text{st}}$ (3^3) which is contained in $\mathscr{G}_1$ (cf. Figure 3.1), a complete classification of the corresponding genus $\mathcal{G}_{\text{L}}$ was performed (this is described in detail in Subsection 3.2.5). It was found that $|\mathcal{G}_{\text{L}}| = 2030$, so one can construct 2030 models from this right-mover lattice. These models are discussed in Section 4.1. Interestingly, the right-mover lattices $A_2^4 E_6^{\text{st}}$ (3^3) and $A_2^3 E_8^{\text{st}}$ (3^3) belong to the same genus. Hence, the 2030 left-mover lattices we classified also appear in Narain-compactified $\mathcal{N} = 4$ models.

Using the same method, we also computed an upper bound of $|\mathcal{G}_{\text{L}}| \leq 5033195$ on the size of the left-mover genus corresponding to the right-mover lattice $A_1^4 E_6^{\text{st}}$ $(3^1 6^2)$. The lower bound from the mass formula is given as $|\mathcal{G}_{\text{L}}| \geq 1504$. An explicit model belonging to this genus is considered in Section 4.2.

For the other lattices in Table 3.5, only a lower bound on the respective $|\mathcal{G}_{\text{L}}|$ was calculated using the mass formula (in the case of the lattice with elementary divisors 2^6 it was lower than one and therefore meaningless). The bottom line is that, even in the $\mathcal{N} = 1$ case, we expect at least $O(10^{10})$ models. However one must keep in mind the crudeness of the lower bound. Out of curiosity, we can quantify this crudeness in the case of the right-mover lattice $A_2^4 E_6^{\text{st}}$ (3^3) where we enumerated $\mathcal{G}_{\text{L}}$ exactly: there, the lower bound calculated using the mass formula is of $O(10^{-3})$, thus the deviation is of $O(10^6)$.

3.2.3 Relationship With Asymmetric Orbifolds

In Section 2.2 we discussed certain twist-orbifolds of Narain compactified heterotic strings that could be expressed as a shift-orbifold of a covariant lattice $\Gamma_{22,14}$. We saw that the shift-orbifolding produces a new covariant lattice $\Gamma'_{22,14}$ which contains a sublattice $\Gamma^{\text{u}}_{22,14}$ describing the untwisted sector. Now, that we have a complete classifiaction of right-mover lattices for chiral models, it is interesting to investigate what right-mover lattices $(\Gamma^{\text{u}}_{14})_{\text{R}}$ actually arise due to this mechanism. Let us restrict to those $(\Gamma_6)_{\text{R}}$ and twists θ_{R} that lead to $\mathcal{N} = 1$ spacetime supersymmetry (cf. the summary in Table 3.6). Then, the obtained right-mover lattices $(\Gamma^{\text{u}}_{14})_{\text{R}}$ must be among those 19 right-mover lattices in Table 3.5. An explicit calculation verified that this in fact happens (in Table 3.5 it is also denoted which type of orbifold corresponds to which right-mover lattice). Interestingly, in several cases it happens that different types of twist-orbifold lead to the same right-mover lattice. One such case is given by the Z_6^{I} and Z_6^{II} orbifolds constructed from $(\Gamma_6)_{\text{R}} = E_6$, as both lead to the $A_1^4 E_6^{\text{st}}$ $(3^1 6^2)$ right-mover lattice shown in Figure 3.1. This indicates the possibility that some models could be obtained from either twist-orbifold construction.

Let us finally treat the question whether all elementary supercurrent lattices can be obtained from the Z_N twist-orbifold construction with Weyl twists that we introduced in Section 2.2 (or a possible generalization thereof that also includes e.g. the $Z_N \times Z_M$ case). In [34], a simple condition is provided that allows to check whether a given admissible supercurrent lattice Ξ can be obtained from the twist-orbifold construction: there must exist three orthonormal vectors e^I in Ξ^*, i.e. $e^I \cdot e^J = \delta^{IJ}$, so that for each norm 3 vector

Table 3.5: Right-mover lattices $(\Gamma_{14})_{\mathrm{R}}$ with $\mathcal{N}=1$. Here, $\Delta_2^{\perp}$ denotes the root system of norm 2 vectors orthogonal to E_6^{st}. Also, some information on the corresponding genera $\mathcal{G}_{\mathrm{L}}$ is provided. Where applicable, the corresponding Z_N twist-orbifolds are shown.

	Genus		Lattice $(\Gamma_{14})_{\mathrm{R}}$	
Component	Divisors	$\lvert\mathcal{G}_{\mathrm{L}}\rvert$	$\Delta_2^{\perp}$	$(\Gamma_6)_{\mathrm{R}}/Z_N$ Orbifolds
$\mathscr{G}_1$	3^3	2030	A_2^4	E_6/Z_3
	$3^1 6^2$	$1504 \leq \lvert\mathcal{G}_{\mathrm{L}}\rvert \leq 5033195$	A_1^4	$E_6/Z_6^{\mathrm{I}}, E_6/Z_6^{\mathrm{II}}$
	3^5	$> 6.9 \cdot 10^3$	A_2	A_2^3/Z_3
	$3^3 9^1$	$> 2.7 \cdot 10^5$	–	
	$3^1 12^2$	$> 1.5 \cdot 10^9$	–	$E_6/Z_{12}^{\mathrm{I}}, A_1A_5/Z_6^{\mathrm{II}}$
	3^7	$> 4.1 \cdot 10^8$	–	
$\mathscr{G}_2$	2^6	–	A_3	
	$2^2 4^2$	> 3	A_1^4	
	$2^2 4^2$	> 6	A_1^6	D_6/Z_4
	$2^4 4^2$	$> 1.3 \cdot 10^5$	A_1^2	$A_1^2 D_4/Z_4$
	$2^4 4^2$	$> 1.3 \cdot 10^4$	–	
	$2^2 8^2$	$> 4.8 \cdot 10^6$	A_1	$A_3^2/Z_4, D_6/Z_8^{\mathrm{I}}$
	$2^2 8^2$	$> 8.0 \cdot 10^5$	–	
	4^4	$> 8.0 \cdot 10^5$	–	
	$2^2 4^4$	$> 1.7 \cdot 10^9$	–	
$\mathscr{G}_3$	7^3	$> 4.0 \cdot 10^7$	–	A_6/Z_7
$\mathscr{G}_4$	$2^2 4^1 8^1$	$> 4.4 \cdot 10^3$	A_1^2	
	$2^1 4^1 8^2$	$> 2.3 \cdot 10^9$	–	$A_1 D_5/Z_8^{\mathrm{II}}$
$\mathscr{G}_5$	$2^1 6^3$	$> 5.2 \cdot 10^7$	–	$A_2 D_4/Z_6^{\mathrm{II}}$

Table 3.6: All different types of twist-orbifolds that were found in [34] to lead to $\mathcal{N}=1$ covariant lattice theories. For the v_R, the notation of [34] is adopted.

Component	$(\Gamma_6)_\mathrm{R}$	Type	Carter Diagram	v_ψ	v_R
$\mathscr{G}_1$	E_6	Z_3	A_2^3	$(1,1,-2)/3$	$(1,1,0,1,1,1)/3$
		Z_6^I	$E_6(a_2)$	$(1,1,-2)/6$	$(1,0,1,0,1,0)/6$
		Z_6^II	A_1A_5	$(1,2,-3)/6$	$(1,1,1,1,1,3)/6$
		Z_{12}^I	E_6	$(1,4,-5)/12$	$(1,1,1,1,1,1)/12$
	A_1A_5	Z_6^II	A_1A_5	$(1,2,-3)/6$	$(3,1,1,1,1,1)/6$
	A_2^3	Z_3	A_2^3	$(1,1,-2)/3$	$(1,1,1,1,1,1)/3$
$\mathscr{G}_2$	D_6	Z_4	$A_1^2D_4(a_1)$ or A_3^2	$(1,1,-2)/4$	$(2,0,1,0,1,1)/4$
		Z_8^I	$D_6(a_1)$	$(1,2,-3)/8$	$(1,1,0,1,1,1)/8$
	A_3^2	Z_4	A_3^2	$(1,1,-2)/4$	$(1,1,1,1,1,1)/4$
	$A_1^2D_4$	Z_4	$A_1^2D_4(a_1)$	$(1,1,-2)/4$	$(2,2,1,0,1,1)/4$
$\mathscr{G}_3$	A_6	Z_7	A_6	$(1,2,-3)/7$	$(1,1,1,1,1,1)/7$
$\mathscr{G}_4$	A_1D_5	Z_8^II	A_1D_5	$(1,3,-4)/8$	$(4,1,1,1,1,1)/8$
$\mathscr{G}_5$	A_2D_4	Z_6^II	A_2D_4	$(1,2,-3)/6$	$(2,2,1,1,1,1)/6$

$t\in\Xi$ there is exactly one e^I such that $t\cdot e^I=\pm1$.

We explicitly checked this condition for all 32 elementary supercurrent lattices that resulted from our classification. It turned out that, except for cases $1^27^1_A$, $1^18^1_A$ and 9^1_A, it was possible to satisfy the condition. Moreover, for these exceptional cases the corresponding $(\Gamma_{14})_\mathrm{R}$ built from the constraint vectors are not contained in a lattice of the form $(\Gamma_6)_\mathrm{R}\oplus E_8^\mathrm{st}$ (note, that this would be required if above condition were true). Hence, it is impossible for these supercurrents to appear in a $\mathcal{N}=1$ model.

3.2.4 Classification of Top Node Genera

In this section we classify the left-mover genera associated with the top nodes of our lattice inclusion graph $\mathscr{G}$. While these genera themselves do not lead to chiral models, they are important because they can serve as a starting point for the classification of all the other left-mover genera. The main method we use in this section is related to the "replacement" technique of [28, Appendix A.4].

We have already observed that for each connected component $\mathscr{G}_i$, the top nodes belong to a single genus. Hence, there are nine genera $\mathcal{G}_\mathrm{L}$ to classify. Furthermore, each of these genera contains a lattice of the form $\Gamma_6\oplus E_8$, where Γ_6 is the root lattice of a semi-simple rank 6 lie algebra of ADE type. Since E_8 is self-dual, this means that the discriminant

Table 3.7: Pairs of even lattices Γ_6 and Γ_2. Where appropriate, Γ_2 is represented by a Gram matrix.

Component	Divisors	Γ_6	Γ_2
$\mathscr{G}_1$	3^1	E_6	A_2
$\mathscr{G}_2$	2^2	D_6	A_1^2
$\mathscr{G}_3$	7^1	A_6	$\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$
$\mathscr{G}_4$	$2^1 4^1$	$A_1 D_5$	$A_1 \mathbb{Z}_{(4)}$
$\mathscr{G}_5$	$2^1 6^1$	$A_2 D_4$	$\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$
$\mathscr{G}_6$	15^1	$A_2 A_4$	$\begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}$
$\mathscr{G}_7$	$2^1 10^1$	$A_1^2 A_4$	$\begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix}$
$\mathscr{G}_8$	$2^1 12^1$	$A_1 A_2 A_3$	$A_1 \mathbb{Z}_{(12)}$
$\mathscr{G}_9$	6^2	$A_1^2 A_2^2$	$\mathbb{Z}_{(6)}^2$

forms of the corresponding left-mover lattices $(\Gamma_{22})_{\mathrm{L}}$ must be isomorphic to that of Γ_6:

$$\mathrm{disc}\left((\Gamma_{22})_{\mathrm{L}}\right) \cong \mathrm{disc}(\Gamma_6)\,. \tag{3.14}$$

Moreover, one shows that for each genus, there exists an even Euclidean lattice Γ_2 such that $E_8 \supset \Gamma_6 \oplus \Gamma_2$ is a self-glue free decomposition (these lattices Γ_2 are listed in Table 3.7). Hence,

$$\mathrm{disc}(\Gamma_6) \cong \mathrm{disc}(\overline{\Gamma}_2)\,. \tag{3.15}$$

By combining above isomorphisms we see that for any $(\Gamma_{22})_{\mathrm{L}} \in \mathcal{G}_{\mathrm{L}}$ there must exist an Euclidean even self-dual lattice Γ_{24} which has a self-glue free decomposition

$$\Gamma_{24} \supset (\Gamma_{22})_{\mathrm{L}} \oplus \Gamma_2\,. \tag{3.16}$$

The even self-dual lattices in 24 dimensions form a genus $\mathcal{G}_{24}^+$, for which a classification is available due to Niemeier [57] (it turns out that $|\mathcal{G}_{24}^+| = 24$). Then, by reversing above argument one notices that it is possible to classify the $\mathcal{G}_{\mathrm{L}}$ in question by computing the orthogonal complement of Γ_2 in Γ_{24} for all inequivalent embeddings of the respective Γ_2 into a $\Gamma_{24} \in \mathcal{G}_{24}^+$.

Note, that the case $\mathscr{G}_1$ where $\Gamma_6 = E_6$ has already been classified by hand using this method [24], and we explicitly list the lattices in this left-mover genus in Table 3.8. Although above technique works in principle for all the $\mathscr{G}_i$, the available computational resources were sufficient only for the cases $\mathscr{G}_1$ to $\mathscr{G}_5$. This is because these cases are in principle controlled by enumerating only norm 2 and norm 4 vectors in Γ_{24}, which turned out to be viable for all $\Gamma_{24} \in \mathcal{G}^+_{24}$. The other cases involve the enumeration of vectors of larger norm, which happened to be computationally too expensive. Nonetheless, it was possible to classify the remaining genera by similar methods which make use of odd lattices. In the following we describe these methods.

Method for A-type lattices. Let us begin by describing an technique that is in principle applicable in cases where Γ_6 consists only of factors of A-type. First, note that for any $n \in \mathbb{N}^+$ there exists a (self-glue free) decomposition of the form

$$\mathbb{Z}^{n+1} \supset A_n \oplus \mathbb{Z}_{(n+1)} \,, \tag{3.17}$$

and hence

$$\mathrm{disc}_-(A_n) \cong \mathrm{disc}_-\left(\overline{\mathbb{Z}_{(n+1)}}\right) \,. \tag{3.18}$$

With this in mind, let us now consider the genus $\mathcal{G}_\mathrm{L}$ associated with the top nodes of $\mathscr{G}_6$ (i.e. the case $\Gamma_6 = A_2A_4$). Using the embeddings $\mathbb{Z}^3 \supset A_2 \oplus \mathbb{Z}_{(3)}$ and $\mathbb{Z}^5 \supset A_4 \oplus \mathbb{Z}_{(5)}$ we then obtain

$$\mathrm{disc}_-(\Gamma_6) \cong \mathrm{disc}_-\left(\overline{\mathbb{Z}}_{(3)}\overline{\mathbb{Z}}_{(5)}\right) \,, \tag{3.19}$$

one concludes that for each $(\Gamma_{22})_\mathrm{L} \in \mathcal{G}_\mathrm{L}$ there must exist an odd self-dual lattice Γ^-_{24} which decomposes as

$$\Gamma^-_{24} \supset (\Gamma_{22})_\mathrm{L} \oplus \mathbb{Z}_{(3)}\mathbb{Z}_{(5)} \,. \tag{3.20}$$

The genus $\mathcal{G}^-_{24}$ of odd and self-dual lattices in 24 dimensions was classified in [39, 58], with the result that $|\mathcal{G}^-_{24}| = 273$. Then, by determining the orthogonal complement of $\mathbb{Z}_{(3)}\mathbb{Z}_{(5)}$ in Γ^-_{24}, for all inequivalent embeddings of $\mathbb{Z}_{(3)}\mathbb{Z}_{(5)}$ in a $\Gamma^-_{24} \in \mathcal{G}^-_{24}$, it was possible to classify $\mathcal{G}_\mathrm{L}$. Besides the even lattices $(\Gamma_{22})_\mathrm{L}$ that we are interested in, this procedure also produces odd lattices $(\Gamma_{22})_\mathrm{L}$ which we must discard.

Method for lattices with D-type factors. Let us now introduce a further alternative method which is applicable in the cases where Γ_6 contains a factor of D-type. First, note that the vector conjugacy class (v) of a D_n lattice has integral norm (in this context we use $D_1 = \mathbb{Z}_{(4)}$, $D_2 = A_1^2$ and $D_3 = A_3$), and that $\mathbb{Z}^n = D_n \cup (v)$. We can thus map a D_n factor contained in Γ_6 to $\mathbb{Z}^n$, giving us an odd lattice $\Gamma^-_6 \supset \Gamma_6$. Since

$$\mathrm{disc}_-(\Gamma_6) \cong \mathrm{disc}_-\left((\Gamma_{22})_\mathrm{L}\right) \cong \mathrm{disc}_-(\overline{\Gamma}_2) \,, \tag{3.21}$$

we can perform an analogous mapping also for $(\Gamma_{22})_\mathrm{L}$ and Γ_2, resulting in some odd lattices Γ^-_{22} and Γ^-_2. The lattices Γ^-_6 and Γ^-_2 are listed in Table 3.9. Then, by using a generalization of a lemma proven in Appendix A.2 (Lemma 1) that also covers odd lattices, one obtains

$$\mathrm{disc}_-(\Gamma^-_6) \cong \mathrm{disc}_-(\Gamma^-_{22}) \cong \mathrm{disc}_-(\overline{\Gamma}^-_2) \,. \tag{3.22}$$

Table 3.8: The genus of 22-dimensional even Euclidean lattices Γ of determinant 3, given by providing a sublattice Γ_0 and glue vectors.

No.	Γ_0	Γ/Γ_0	Generators of Γ/Γ_0
1	$D_{21}\mathbb{Z}_{(12)}$	4^1	$(s,1/4)$
2	$D_{13}E_8\mathbb{Z}_{(12)}$	4^1	$(s,0,1/4)$
3	$D_{16}E_6$	2^1	$(s,0)$
4	$E_6E_8^2$	–	–
5	$A_{21}\mathbb{Z}_{(66)}$	22^1	$(5,1/22)$
6	$D_9D_{12}\mathbb{Z}_{(12)}$	2^14^1	$(v,s,0)$, $(s,v,1/4)$
7	$A_{14}E_7\mathbb{Z}_{(10)}$	10^1	$(3,1,1/10)$
8	A_5A_{17}	6^1	$(3,3)$
9	$D_7E_7^2\mathbb{Z}_{(12)}$	2^14^1	$(v,1,1,0)$, $(s,1,0,1/4)$
10	$A_5D_{10}E_7$	2^2	$(3,s,0)$, $(0,c,1)$
11	$A_{12}D_9\mathbb{Z}_{(156)}$	52^1	$(2,s,1/52)$
12	$A_{15}D_6\mathbb{Z}_{(12)}$	2^18^1	$(0,v,1/2)$, $(2,s,1/4)$
13	$D_5D_8^2\mathbb{Z}_{(12)}$	2^24^1	$(v,s,v,0)$, $(v,v,s,0)$,$(s,v,v,1/4)$
14	$A_9A_{12}\mathbb{Z}_{(390)}$	130^1	$(1,5,1/130)$
15	$A_8D_7E_6\mathbb{Z}_{(36)}$	36^1	$(1,s,1,1/36)$
16	$A_{11}D_4E_6\mathbb{Z}_{(12)}$	2^112^1	$(0,v,0,1/2)$, $(1,s,1,1/4)$
17	$A_2^2A_{11}D_7$	12^1	$(1,1,1,s)$
18	$A_2^2E_6^3$	3^2	$(1,1,1,2,0)$, $(1,1,2,0,1)$
19	$A_6A_9D_6\mathbb{Z}_{(210)}$	2^170^1	$(0,5,v,1/2)$, $(1,2,s,1/70)$
20	$A_3A_9^2\mathbb{Z}_{(12)}$	2^120^1	$(0,5,5,1/2)$, $(1,2,9,1/4)$
21	$A_3D_6^3\mathbb{Z}_{(12)}$	2^34^1	$(2,v,v,v,0)$, $(2,c,s,v,1/2)$, $(2,s,v,c,1/2)$, $(3,v,s,0,1/4)$
22	$A_5A_8^2\mathbb{Z}_{(18)}$	3^118^1	$(0,3,6,0)$, $(1,4,1,1/18)$
23	$A_4A_7D_5^2\mathbb{Z}_{(120)}$	4^140^1	$(0,0,c,c,1/4)$, $(1,1,s,v,1/40)$
24	$A_1^2A_7^2D_5\mathbb{Z}_{(12)}$	$2^14^18^1$	$(1,1,0,0,0,1/2)$, $(1,0,2,0,c,1/4)$, $(1,1,1,7,s,0)$
25	$A_3A_6^3\mathbb{Z}_{(84)}$	7^128^1	$(0,0,2,6,1/7)$, $(1,6,2,1,1/28)$
26	$A_2A_5^3D_4\mathbb{Z}_{(6)}$	2^16^2	$(0,3,0,0,s,1/2)$, $(1,2,3,4,v,1/6)$, $(1,4,2,3,c,1/6)$
27	$A_5^4\mathbb{Z}_{(4)}\mathbb{Z}_{(12)}$	$2^16^112^1$	$(3,0,0,3,1/2,0)$, $(0,1,1,4,1/2,0)$, $(1,0,1,2,1/4,1/4)$
28	$D_4^5\mathbb{Z}_{(4)}\mathbb{Z}_{(12)}$	2^54^1	$(c,s,v,v,s,0,1/2)$, $(c,0,c,v,v,1/2,1/2)$, $(v,v,c,0,c,1/2,1/2)$, $(c,v,v,c,0,1/2,1/2)$, $(0,c,v,v,c,1/2,1/2)$, $(s,s,s,s,s,1/4,1/4)$
29	$A_1A_4^5\mathbb{Z}_{(30)}$	5^210^1	$(0,0,2,3,3,2,1/5)$, $(0,2,0,2,3,3,1/5)$, $(1,4,1,0,1,4,1/10)$
30	$A_3^7\mathbb{Z}_{(12)}$	4^4	$(0,1,1,2,0,0,1,1/4)$, $(1,0,1,1,2,0,0,1/4)$, $(0,1,0,1,1,2,0,1/4)$, $(0,0,1,0,1,1,2,1/4)$
31	A_2^{11}	3^5	$(1,1,0,0,1,0,2,0,2,0,2)$, $(1,1,2,1,0,0,0,2,2,0,0)$, $(1,0,0,0,0,1,2,2,2,1,0)$, $(0,1,2,0,1,0,2,2,0,1,0)$, $(1,0,2,1,0,0,2,0,0,1,2)$

Table 3.9: Pairs of odd lattices Γ_6^- and Γ_2^-.

Component	Γ_6^-	Γ_2^-
$\mathscr{G}_2$	$\mathbb{Z}^6$	$\mathbb{Z}^2$
$\mathscr{G}_4$	$A_1\mathbb{Z}^5$	$A_1\mathbb{Z}$
$\mathscr{G}_5$	$A_2\mathbb{Z}^4$	$\mathbb{Z}_{(3)}\mathbb{Z}$
$\mathscr{G}_7$	$A_4\mathbb{Z}^2$	$\mathbb{Z}_{(5)}\mathbb{Z}$
$\mathscr{G}_8$	$A_1A_2\mathbb{Z}^3$	$A_1\mathbb{Z}_{(3)}$
$\mathscr{G}_9$	$A_2^2\mathbb{Z}^2$	$\mathbb{Z}_{(3)}^2$

Thus, there must exist an odd self-dual lattice $\Gamma_{24}^- \in \mathcal{G}_{24}^-$ with self-glue free decomposition

$$\Gamma_{24}^- \supset \Gamma_{22}^- \oplus \Gamma_2^- \,. \tag{3.23}$$

We can then determine the genus $\mathcal{G}_\mathrm{L}^-$ of the lattices Γ_{22}^- by determining the inequivalent embeddings of Γ_2^- in an odd self-dual lattice $\Gamma_{24}^- \in \mathcal{G}_{24}^-$ (in cases where Γ_2^- contains $\mathbb{Z}$ factors the problem can be reduced to lower-dimensional odd self-dual lattices). Finally for each $\Gamma_{22}^- \in \mathcal{G}_\mathrm{L}^-$ we can determine the corresponding $(\Gamma_{22})_\mathrm{L}$ and thus the genus $\mathcal{G}_\mathrm{L}$ by computing the sublattice of even vectors. Note that the correspondence $\mathcal{G}_\mathrm{L}^- \leftrightarrow \mathcal{G}_\mathrm{L}$ is one-to-one, except for the case $\mathscr{G}_5$ which involves D_4. In that case the correspondence is many-to-one due to the triality of D_4.

Note that this method is similar to to the one described at the beginning of this subsection. However, the lattices Γ_2^- are controlled by lower-norm vectors than their even counterparts Γ_2 which made the problem solvable on a computer. The resulting numbers for $|\mathcal{G}_\mathrm{L}|$ are then listed in Table 3.4.

3.2.5 The Subgenus Method

We now introduce a method that is suitable for the enumeration of the remaining left-mover genera, but begin with a rather generic discussion. Let Λ_0 be an even lattice that belongs to a genus $\mathcal{G}_0$. Furthermore, assume that there is a vector $w \in \Lambda_0^*$ with $w \notin \Lambda_0$ that satisfies $w^2 \in 2\mathbb{Z}$, so that we can construct the even overlattice

$$\Lambda = \langle \Lambda_0 \cup \{w\} \rangle \,, \tag{3.24}$$

belonging to some genus $\mathcal{G}$. Clearly, Λ/Λ_0 is a cyclic group. Let in the following denote $N = |\Lambda/\Lambda_0|$. Then, by a theorem proven in Appendix A.2 (Theorem 3), there must exist another vector $v \in \Lambda_0^*$ satisfying

$$\begin{aligned} &Nv \in \Lambda^* \,, \\ &kv \notin \Lambda^* \text{ for } k \in \{1, \ldots, N-1\} \,, \end{aligned} \tag{3.25}$$

so that $\Lambda_0 = I_v(\Lambda)$. Here, we define $I_v(\Lambda)$ for a lattice Λ and a vector v in the $\mathbb{R}$-span of Λ as

$$I_v(\Lambda) = \{x \in \Lambda \mid x \cdot v \in \mathbb{Z}\}\,. \tag{3.26}$$

This suggests a method for the classification of $\mathcal{G}_0$. Suppose we have already completely classified the genus $\mathcal{G}$. Then, for each $\Lambda \in \mathcal{G}$ we compute the vectors satisfying (3.25) and the corresponding lattices $I_v(\Lambda)$ as in (3.26). Of course, there are infinitely many such vectors v, but two vectors v and v' satisfying (3.25) produce same $I_v(\Lambda)$ if

$$v' - \theta v \in \Lambda^* \tag{3.27}$$

for some automorphism θ of Λ. Hence, it is sufficient to consider only one representative v of each orbit of the natural action of $\mathrm{Aut}(\Lambda)$ on the set $\Lambda^*/(N\Lambda^*)$ (clearly, there are only finitely many orbits). Nevertheless, we may still obtain several isomorphic copies of a lattice $I_v(\Lambda)$ for different $\Lambda \in \mathcal{G}$ and v, and we must remove the duplicates at this stage. Now, let $\mathcal{S}$ denote the (finite and duplicate-free) set of all $I_v(\Lambda)$ obtained this way. We claim that this set always is a (disjoint) union of complete *subgenera* $\mathcal{G}_i$ of $\mathcal{G}$, the original $\mathcal{G}_0$ being among them (a subgenus of a genus $\mathcal{G}$ shall be the genus of a sublattice of a lattice in $\mathcal{G}$). We now sketch a proof of this claim. What we basically have to show is that, if some lattice $\Lambda_0 \in \mathcal{G}_i$ belongs to $\mathcal{S}$, then any other lattice $\Lambda_0' \in \mathcal{G}_i$ also belongs to $\mathcal{S}$. Clearly, for such a lattice Λ_0' there must exist a discriminant form isomorphism

$$\phi : \mathrm{disc}(\Lambda_0) \mapsto \mathrm{disc}(\Lambda_0')\,, \tag{3.28}$$

because Λ_0 and Λ_0' belong to the same genus. Also, we know that there must be a vector $w \in \Lambda_0^*$ satisfying $\Lambda = \langle \Lambda_0 \cup \{w\}\rangle \in \mathcal{G}$. Then, we can choose a representative $w' \in \phi(w + \Lambda_0)$ and construct another even lattice

$$\Lambda' = \langle \Lambda_0' \cup \{w'\}\rangle\,. \tag{3.29}$$

Clearly, $|\Lambda'/\Lambda_0'| = N$. Also, by Theorem 3 there exists a v' with the same properties as v in (3.25), but with Λ replaced by Λ', so that $\Lambda_0' = I_{v'}(\Lambda')$. Now from a lemma proven in Appendix A.2 (Lemma 1), one observes that Λ and Λ' have isomorphic discriminant forms. Thus, Λ' also belongs to $\mathcal{G}$. But then, as we constructed $\mathcal{S}$ from all lattices in $\mathcal{G}$, also $\Lambda_0' \in \mathcal{S}$, which proves our claim.

Above method does in general also produce subgenera $\mathcal{G}_i$ of $\mathcal{G}$ other than the desired $\mathcal{G}_0$. A simple way to deal with this is of course to remove the unwanted genera afterwards. It is also possible to impose additional conditions on v which allow to single out $\mathcal{G}_0$. However, the explicit form of such conditions depends on the concrete genus $\mathcal{G}$ and the subgenus of it we wish to classify.

All left-mover lattices $(\Gamma_{22})_\mathrm{L}$ belong to some genus $\mathcal{G}_\mathrm{L}$ that is a subgenus of a top node genus. Using above method for prime subgenera (i.e. N being a prime number) we can classify the left-mover genera corresponding to the right-mover lattices directly below the top nodes. By repeating this procedure in the obvious way it is in principle possible to classify all left-mover genera (in practice though this is not computationally viable, as the genera become very large).

Note, that the way we compute $I_v(\Lambda)$ in (3.26) clearly resembles the calculation of untwisted sectors in the shift-orbifold construction studied in Section 2.2 (in the following we also refer to the vectors v as shift vectors).

The E_6/Z_3 left-mover genera. The right-mover lattice $(\Gamma_{14})'_{\text{R}} = A_2^4 E_6^{\text{st}}\ (3^3)$ in Figure 3.1 is a sublattice of the top node $(\Gamma_{14})_{\text{R}} = E_6 E_8^{\text{st}}\ (3^1)$ whose associated left-mover genus $\mathcal{G}_{\text{L}}$ was classified in Subsection 3.2.4 (the 31 lattices in $\mathcal{G}_{\text{L}}$ are displayed in Table 3.8). Using the subgenus method, a classification of the left-mover genus $\mathcal{G}'_{\text{L}}$ corresponding to $(\Gamma_{14})'_{\text{R}}$ was performed. Note that $N = 3$, and in fact this right-mover lattice appears in Z_3 orbifolds of from Narain theories with $(\Gamma_6)_{\text{R}} = E_6$.

First, we calculate all inequivalent shift vectors v satisfying

$$3v \in (\Gamma_{22})^*_{\text{L}}\,, \tag{3.30}$$
$$v \notin (\Gamma_{22})^*_{\text{L}}\,, \tag{3.31}$$

for each of the 31 lattices in $(\Gamma_{22})_{\text{L}} \in \mathcal{G}_{\text{L}}$ as described above. In order to single out the left-mover genus associated with $(\Gamma_{14})'_{\text{R}}$, we impose the additional constraints:

$$3v \in (\Gamma_{22})_{\text{L}}\,, \tag{3.32}$$
$$3v^2 \in 2\mathbb{Z}\,. \tag{3.33}$$

These also originate from a modular invariance condition in the asymmetric Z_3 orbifold construction [25]. By computing orbits under the respective $\text{Aut}((\Gamma_{22})_{\text{L}})$ it turned out that in total there are 5317 inequivalent shift vectors v, so that one obtains 5317 lattices $(\Gamma_{22})'_{\text{L}} \in \mathcal{G}'_{\text{L}}$ by computing the respective $I_v((\Gamma_{22})_{\text{L}})$. After removing duplicates, 2030 lattices remained and thus $|\mathcal{G}'_{\text{L}}| = 2030$.

The E_6/Z_6 left-mover genera. From Figure 3.1 we see that the right-mover lattice $A_2^4 E_6^{\text{st}}\ (3^3)$ has a sublattice $A_1^4 E_6^{\text{st}}\ (3^1 6^2)$ of index $N = 2$. We are interested in its left-mover genus $\mathcal{G}''_{\text{L}}$, which corresponds to Z_6 orbifold models from Narain theories with $(\Gamma_6)_{\text{R}} = E_6$. Then, similarly to above case, we calculate inequivalent shift vectors of order 2,

$$2v \in (\Gamma_{22})'^*_{\text{L}}\,, \tag{3.34}$$
$$v \notin (\Gamma_{22})'^*_{\text{L}}\,, \tag{3.35}$$

for each of the 2030 lattices in $(\Gamma_{22})'_{\text{L}} \in \mathcal{G}'_{\text{L}}$. It turned out that there are in total 10069090 such shift vectors, and that the corresponding lattices as calculated using (3.26) split into precisely two genera. Among these lattices, 5033195 belong to the genus $\mathcal{G}''_{\text{L}}$. However, due to the computational complexity it was not possible to remove duplicates, thus the number 5033195 may only be interpreted as an upper bound on $|\mathcal{G}''_{\text{L}}|$.

Chapter 4

Model Building

In the previous chapter, we investigated the general structure of the space of chiral covariant lattice models by classifying the right-mover lattices $(\Gamma_{14})_{\mathrm{R}}$. We observed that only 19 right-mover lattices lead to models with $\mathcal{N} = 1$ spacetime supersymmetry, and that some of them arise in certain Z_N orbifold compactifications. In this chapter, we discuss explicit models belonging to two of these 19 classes of covariant lattice models, one class corresponding to Z_3 orbifolds and the other to Z_6 orbifolds.

4.1 A Class of Asymmetric Z_3 Orbifolds

In this section, we consider covariant lattice model building with the right-mover lattice $A_2^4 E_6^{\mathrm{st}}$ (3^3), which corresponds to asymmetric Z_3 orbifolds built from a Narain lattice containing an $\overline{E_6}$ factor. Some of these models were already considered previously in the asymmetric orbifold framework [25]. We start by giving an explicit construction of the right-mover lattice and study its phenomenological implications. Then, we search for (semi-)realistic models, and discuss one particular model in detail.

4.1.1 Massless Right-Mover Spectrum and Supercurrent

The right-mover lattice $(\Gamma_{14})_{\mathrm{R}} = A_2^4 E_6^{\mathrm{st}}$ (3^3) is defined by the following union of conjugacy classes of $A_2^4 E_6^{\mathrm{st}}$:

$$(\Gamma_{14})_{\mathrm{R}} = (0,0,0,0,0) \cup (1,1,1,1,1) \cup (2,2,2,2,2)\,. \tag{4.1}$$

One checks, using $Q_{A_2}(1) = Q_{A_2}(2) = 2/3$ and $Q_{E_6}(1) = Q_{E_6}(2) = 4/3$, that by above definition $(\Gamma_{14})_{\mathrm{R}}$ contains only even norm vectors. The quotient group, $(\Gamma_{14})^*_{\mathrm{R}}/(\Gamma_{14})_{\mathrm{R}}$ is of Z_3^3 structure, and is generated by the following unions of conjugacy classes:

$$\begin{aligned}
\chi_1 =\ & (1,0,0,0,1) \cup (2,1,1,1,2) \cup (0,2,2,2,0)\,,\\
\chi_2 =\ & (0,1,0,0,1) \cup (1,2,1,1,2) \cup (2,0,2,2,0)\,,\\
\chi_3 =\ & (0,0,1,0,1) \cup (1,1,2,1,2) \cup (2,2,0,2,0)\,.
\end{aligned} \tag{4.2}$$

In the following, we identify a coset in $(\Gamma_{14})^*_{\mathrm{R}}/(\Gamma_{14})_{\mathrm{R}}$, by an integer triple (Q_1, Q_2, Q_3) that denotes a linear combination of the generators χ_i. The "sector charges" Q_i are defined only modulo 3, and we choose them to lie in the range $\{0, 1, 2\}$. Note, that the sector charges of a state can also be determined from its gauge representation, which depends on the left-mover lattice.

Massless right-mover spectrum. Following the prescription in Subsection 2.1.4, we now determine the massless right-mover spectrum. Let us begin with the $(0, 0, 0)$-sector which corresponds to $(\Gamma_{14})_{\mathrm{R}}$. This sector generically gives us a spacetime $\mathcal{N} = 1$ vector multiplet from the **78** of E_6^{st}. The other states with $h_{\mathrm{R}} = 1$ form an adjoint representation of A_2^4, but they are unphysical due to the truncation condition (2.17)–(2.18).

Chiral multiplets arise from the **27** representation of E_6^{st}, i.e. from its (1) conjugacy class ($\overline{\mathbf{27}}$ contains the CPT conjugates). Note that such a conjugacy class is contained in each sector, but only some give rise to massless states: the norm in the A_2^4 part must be 2/3 to satisfy the masslessness condition. This happens precisely for the conjugacy classes

$$
\begin{aligned}
&(1, 0, 0, 0, 1)\,, \\
&(0, 1, 0, 0, 1)\,, \\
&(0, 0, 1, 0, 1)\,, \\
&(0, 0, 0, 1, 1)\,,
\end{aligned}
\tag{4.3}
$$

which belong to sector $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(2, 2, 2)$, respectively. Each of these gives three massless right-mover states forming a triplet **3** of the corresponding $SU(3)$. In the effective theory they are represented by chiral $\mathcal{N} = 1$ superfields Φ_{sk}, which are denoted in Table 4.1.

Interpretation in the orbifold setup. When considering $(\Gamma_{14})_{\mathrm{R}}$ in the Z_3 orbifold setup, one can distinguish between untwisted and twisted sectors. It turns out that the untwisted sector corresponds to one of above four sectors, and the other three originate from twisted sectors. Furthermore, each sector is threefold degenerate. For the untwisted sector, this degeneracy is determined from orbifold phases that are assigned to the momenta of the chiral bosons H^α. The degeneracy of the twisted sectors can be calculated from the formula [19]

$$
D = \frac{\prod_{i, t^i \notin \mathbb{Z}} (2 \sin(\pi t^i))}{\sqrt{\det(I_\theta(\Gamma_{22,6}))}}\,.
\tag{4.4}
$$

Here, θ denotes the twist $\theta_{\mathrm{L}} \oplus \theta_{\mathrm{R}}$ and $0 \leq t^i < 1$ the corresponding twist vector. Moreover, $I_\theta(\Gamma_{22,6})$ denotes the sublattice of the Narain lattice $\Gamma_{22,6}$ that is left invariant under the twist. In our construction θ_{L} is the identity and θ_{R} has twist vector $(1, 1, 2)/3$. Hence, the invariant sublattice is given by $(\Gamma_{22})_{\mathrm{L}}$ which must have $\det((\Gamma_{22})_{\mathrm{L}}) = 3$ because also $\det(E_6) = 3$. From this one calculates the degeneracy of the θ_{R} and the (CPT-conjugate)

Table 4.1: Massless chiral superfields from the $A_2^4 E_6^{\rm st}(3^3)$ right-mover lattice. The index k transforms as the fundamental triplet of the corresponding $SU(3)$.

Superfield	$SU(3)^4$	Q_1	Q_2	Q_3
Φ_{1k}	$(\mathbf{3},1,1,1)$	1	0	0
Φ_{2k}	$(1,\mathbf{3},1,1)$	0	1	0
Φ_{3k}	$(1,1,\mathbf{3},1)$	0	0	1
Φ_{4k}	$(1,1,1,\mathbf{3})$	2	2	2

$\theta_{\rm R}^{-1}$ twisted sector as $D = 3$, in accordance with our result from the covariant lattice description.

Note, that the right-mover lattice $A_2^4 E_6^{\rm st}$ (3^3) has a symmetry that permutes the four sectors. In fact, any of above four sectors serves equally well as an untwisted sector in the orbifold description. This is related to the fact that the same model may be obtained from different (Narain) left-mover lattices $(\Gamma_{22})_{\rm L}$.

The supercurrent. Let us now fix a realization of the supercurrent $G_{\rm int}(z)$. First, we determine all norm 3 vectors s that can possibly appear in the constraint vectors $(0,s,v) \in (\Gamma_{14})_{\rm R}$ of a supercurrent. For this we decompose the root lattice $E_6^{\rm st}$ as

$$E_6^{\rm st} \supset D_5^{\rm st} \oplus \mathbb{Z}_{(12)} \,. \tag{4.5}$$

The conjugacy classes of $E_6^{\rm st}$ then branch as

$$\begin{aligned}(0) &\to (0,0) \cup (s,1/4) \cup (v,1/2) \cup (c,3/4)\,,\\ (1) &\to (0,2/3) \cup (s,11/12) \cup (v,1/6) \cup (c,5/12)\,,\\ (2) &\to (0,1/3) \cup (s,7/12) \cup (v,5/6) \cup (c,1/12)\,,\end{aligned} \tag{4.6}$$

and the corresponding representations as in (2.78) under $E_6 \to SO(10) \times U(1)_R$. We also let Λ_9 denote the orhogonal complement of $D_5^{\rm st}$ in $(\Gamma_{14})_{\rm R}$. It is given by

$$\Lambda_9 = (0,0,0,0,0) \cup (1,1,1,1,2/3) \cup (2,2,2,2,1/3) \tag{4.7}$$

in conjugacy classes of $A_2^4 \oplus \mathbb{Z}_{(12)}$, as can be deduced from the definition (4.1) and the branchings (4.6). The norm 3 vectors s then must lie in the dual lattice Λ_9^*, and in particular belong to a conjugacy classes which is glued to the (v) of $D_5^{\rm st}$. Using (4.6) one finds that the these are:

$$(0,0,0,0,1/2) \cup (1,1,1,1,1/6) \cup (2,2,2,2,5/6)\,, \tag{4.8}$$

in terms of $A_2^4 \oplus \mathbb{Z}_{(12)}$. Then, we obtain two norm 3 vectors from the conjugacy class $(0,0,0,0,1/2)$. In the conjugacy class $(1,1,1,1,1/6)$ there are another 81 such vectors forming the $(\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},1/6)$ representation of $SU(3)^4 \times U(1)_R$. Among them we choose the following nine:

$$\begin{aligned} s_1 &= (w_1, w_1, w_1, w_1, 1/6)\,, \\ s_2 &= (w_2, w_2, w_2, w_1, 1/6)\,, \\ s_3 &= (w_3, w_3, w_3, w_1, 1/6)\,, \\ s_4 &= (w_1, w_2, w_3, w_2, 1/6)\,, \\ s_5 &= (w_2, w_3, w_1, w_2, 1/6)\,, \\ s_6 &= (w_3, w_1, w_2, w_2, 1/6)\,, \\ s_7 &= (w_1, w_3, w_2, w_3, 1/6)\,, \\ s_8 &= (w_2, w_1, w_3, w_3, 1/6)\,, \\ s_9 &= (w_3, w_2, w_1, w_3, 1/6)\,, \end{aligned} \tag{4.9}$$

where w_1, w_2 and w_3 denote the weights of the triplet of $SU(3)$:

$$w_1 = \frac{2\alpha_1}{3} + \frac{\alpha_2}{3}\,, \tag{4.10}$$

$$w_2 = -\frac{\alpha_1}{3} + \frac{\alpha_2}{3}\,, \tag{4.11}$$

$$w_3 = -\frac{\alpha_1}{3} - \frac{2\alpha_2}{3}\,, \tag{4.12}$$

α_1 and α_2 being simple roots. The vectors s_i fulfill $s_i \cdot s_j = 3\delta_{ij}$. Since the conjugacy class $(2,2,2,2,5/6)$ contains the vectors $-s_i$, it is possible construct the 1^9 supercurrent as

$$G_{\text{int}}(z) = \sum_{k=1}^{9} \frac{1}{\sqrt{3}} \left(:e^{is_k \cdot X_{\text{R}}(z)}: \varepsilon(s_k, \hat{p}_{\text{R}}) + :e^{-is_k \cdot X_{\text{R}}(z)}: \varepsilon(-s_k, \hat{p}_{\text{R}}) \right) . \tag{4.13}$$

Henceforth, we let Ξ denote the lattice spanned by the s_i. Note that we do not fix an explicit form of the cocycle $\varepsilon(s,t)$. This is not necessary because in this particular case the supercurrent can be taken as (4.13) for any choice of the cocycle. Note that for this right-mover lattice also the $1^1 8^1_D$, 9^1_D, 9^1_J and 9^1_K elementary supercurrents are possible.

4.1.2 Discrete Symmetries and the Superpotential

The right-mover spectrum that we determined formally comes in representations of the right-mover zero-mode Lie algebra $A_2^4 \times U(1)_R$ (the $U(1)_R$ coming from the lattice decomposition $E_6^{\text{st}} \to D_5^{\text{st}} \oplus \mathbb{Z}_{(12)}$). To this Lie algebra corresponds a symmetry group

$$G_{\text{R}} = SU(3)^4 \times U(1)_R / Z_3\,, \tag{4.14}$$

where the origin of the Z_3 becomes clear later. Although G_{R} is a symmetry of the CFT correlation functions, it is not a symmetry of the effective four-dimensional theory. This is due to insertions of the picture-changing operator $\mathscr{P}_{+1} = :G(z)e^{i\phi(z)}:$, which is not a singlet under G_{R}. However, in Subsection 2.1.4 we argued that the (necessarily finite) stabilizer subgroup $H_{\text{R}} \subset G_{\text{R}}$ which acts trivially on the supercurrent is a symmetry of the effective theory. Among these symmetries one clearly finds the Z_3^3 from the conservation of the sector charges Q_i: they are conserved (modulo 3) in any interaction because the picture-changing operator belongs to the $(0,0,0)$ sector. The representation of the remaining symmetries in $H_{\rm R}$ depends on the explicit form of the supercurrent, which we in the following assume to be that of equation (4.13).

Let us now determine the subgroup $H_{\rm R}$ of $SU(3)^4 \times U(1)_R/Z_3$ that stabilizes $G_{\rm int}(z)$. Although the supercurrent transforms inside the representation

$$(\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},1/6) \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}},\overline{\mathbf{3}},\overline{\mathbf{3}},-1/6)\,, \tag{4.15}$$

it is sufficient to consider only the $(\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},1/6)$ part due to the Hermiticity property (2.84) that the supercurrent (4.13) obeys. Henceforth this representation is denoted by a tensor c_{ijkl}, and from (4.9) we infer that the special subspace inside this representation corresponding to the supercurrent can be given by

$$c_{1111} = c_{2221} = c_{3331} = c_{\underline{123}2} = c_{\underline{132}3} = 1\,, \tag{4.16}$$

and zero for all other combinations of indices (the underline denotes all cyclic permutations). Let us now denote an element of $SU(3)^4 \times U(1)_R$ by a tuple

$$(U_1, U_2, U_3, U_4, \zeta)\,, \tag{4.17}$$

where $U_s \in SU(3)$ and $\zeta \in U(1)_R$. The supercurrent then transforms as

$$c_{ijkl} \mapsto c'_{ijkl} = \zeta^2 (U_1)_i^m (U_2)_j^n (U_3)_k^o (U_4)_l^p c_{mnop}\,, \tag{4.18}$$

where we used the fact that the $U(1)_R$ charges are quantized as multiples of $1/12$. The component fields of the chiral superfields Φ_{si},

$$\Phi_{si} = \phi_{si} + \theta \cdot \psi_{si} + \theta^2 F_{si}\,, \tag{4.19}$$

and the superspace variable θ transform as

$$\phi_{si} \mapsto \phi'_{si} = \zeta^2 (U_s)_i^k \phi_{sk}\,, \tag{4.20}$$

$$\psi_{si} \mapsto \psi'_{si} = \zeta^{-1} (U_s)_i^k \psi_{sk}\,, \tag{4.21}$$

$$F_{si} \mapsto F'_{si} = \zeta^{-4} (U_s)_i^k F_{sk}\,, \tag{4.22}$$

$$\theta \mapsto \theta' = \zeta^3 \theta\,. \tag{4.23}$$

The first thing that we immediately recognize is that the transformation

$$(\omega\mathbb{1}, \omega\mathbb{1}, \omega\mathbb{1}, \omega\mathbb{1}, \omega)\,, \tag{4.24}$$

where $\omega = e^{2\pi i/3}$, acts trivially on the supercurrent and the fields (also in the massive sector). This transformation generates a Z_3 central subgroup N of $SU(3)^4 \times U(1)_R$, and its origin can be traced back to the fact that

$$\Lambda_9/(A_2^4 \oplus \mathbb{Z}_{(12)}) = Z_3\,. \tag{4.25}$$

Due to this trivially acting Z_3 subgroup, the actual symmetry group $G_{\rm R}$ is given not as $SU(3)^4 \times U(1)_R$, but rather as the factor group (4.14). In the following, let $\tilde{H}_{\rm R}$ denote the subgroup of $SU(3)^4 \times U(1)_R$ given by those tuples (4.17) for which $c'_{ijkl} = c_{ijkl}$. The group of discrete symmetries $H_{\rm R}$ is then given by $\tilde{H}_{\rm R}/N$.

Let us begin by determining the subgroup of $\tilde{H}_{\rm R}$ that lies in the Cartan subgroup $U(1)^9$ of $SU(3)^4 \times U(1)_R$ generated by the right-mover momentum operator $\hat{p}_{\rm R}$. We denote an element of this group by $e^{2\pi i \xi \cdot \hat{p}_R}$. Now, in order for such a transformation to stabilize the supercurrent, ξ must belong to Ξ^*. Furthermore, since the internal right-mover quantum numbers $k_{\rm R}$ (the eigenvalues of $\hat{p}_{\rm R}$) of a physical state must lie in the dual lattice Λ_9^*, ξ is only determined modulo vectors in Λ_9. Hence, we obtain a symmetry of the form Ξ^*/Λ_9. This quotient has, as one shows preferably by a computer calculation, the structure $Z_3^6 \times Z_2$. The Z_2 factor corresponds to the symmetry that assigns each spacetime fermion a minus sign, and corresponds to the superspace transformation $\theta \mapsto -\theta$. It is given by a vector $\xi_\theta \in \Xi^*$ of the form

$$\xi_\theta = (0, 0, 0, 0, 1/2)\,, \tag{4.26}$$

in components of $A_2^4 \oplus \mathbb{Z}_{(12)}$. The remaining Z_3^6 is generated by the following independent vectors in Ξ^*:

$$\xi_1 = (-w_1, 0, 0, w_1, 0)\,, \tag{4.27}$$
$$\xi_2 = (0, -w_1, 0, w_1, 0)\,, \tag{4.28}$$
$$\xi_3 = (0, 0, -w_1, w_1, 0)\,, \tag{4.29}$$
$$\xi_4 = \tfrac{1}{3}(\alpha_1, \alpha_1, \alpha_1, 0, 0)\,, \tag{4.30}$$
$$\xi_5 = \tfrac{1}{3}(0, -\alpha_1, \alpha_1, -\alpha_2, 0)\,, \tag{4.31}$$
$$\xi_R = \tfrac{1}{3}(w_1, w_1, w_1, w_1, 5/3)\,. \tag{4.32}$$

One checks that above vectors ξ satisfy $\xi \cdot k_{\rm R} \in \mathbb{Z}$ for $k_{\rm R} \in \Xi$. Note, that we recover the sector charges (Q_1, Q_2, Q_3) as $3(\xi_1 \cdot k_{\rm R}, \xi_2 \cdot k_{\rm R}, \xi_3 \cdot k_{\rm R})$ for the states in our spectrum. We also define $Q_4 = 3\xi_4 \cdot k_{\rm R}$ and $Q_5 = 3\xi_5 \cdot k_{\rm R}$. These charges are different for the component fields in the $SU(3)$ triplets. Finally, one checks using the branching rules (2.78) that $R = 3\xi_R \cdot k_{\rm R}$ gives different values for bosonic and fermionic states in the same chiral multiplet:

$$R(\text{fermion}) = R(\text{boson}) + 1 \mod 3\,. \tag{4.33}$$

Hence it is a discrete Z_3 R-symmetry. Since the R-charge of a chiral multiplet is defined to be that of its bosonic state, the discrete R-charge of the superpotential must add up to 1 mod 3. The discrete charges for the massless chiral multiplets are listed in Table 4.2.

Let us now rewrite above $Z_3^6 \times Z_2$ symmetry in the language of the tuples (4.17). The Z_2 symmetry associated with ξ_θ is then represented by

$$(\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, -1)\,. \tag{4.34}$$

Table 4.2: Discrete Z_3 charges of the massless $\mathcal{N} = 1$ chiral superfields from the $A_2^4 E_6^{\text{st}}(3^3)$ right-mover lattice. The discrete R-charge of the superspace variable θ is defined as 2 (mod 3).

Superfield	Q_1	Q_2	Q_3	Q_4	Q_5	R
Φ_{11}	1	0	0	1	0	1
Φ_{12}	1	0	0	2	0	0
Φ_{13}	1	0	0	0	0	0
Φ_{21}	0	1	0	1	2	1
Φ_{22}	0	1	0	2	1	0
Φ_{23}	0	1	0	0	0	0
Φ_{31}	0	0	1	1	1	1
Φ_{32}	0	0	1	2	2	0
Φ_{33}	0	0	1	0	0	0
Φ_{41}	2	2	2	0	0	1
Φ_{42}	2	2	2	0	2	0
Φ_{43}	2	2	2	0	1	0

The symmetries corresponding to above discrete charges are given by

$$Q_1 : \quad (\omega\mathbb{1}, \mathbb{1}, \mathbb{1}, \omega^2\mathbb{1}, 1)\,, \tag{4.35}$$
$$Q_2 : \quad (\mathbb{1}, \omega\mathbb{1}, \mathbb{1}, \omega^2\mathbb{1}, 1)\,, \tag{4.36}$$
$$Q_3 : \quad (\mathbb{1}, \mathbb{1}, \omega\mathbb{1}, \omega^2\mathbb{1}, 1)\,, \tag{4.37}$$
$$Q_4 : \quad (A, A, A, \mathbb{1}, 1)\,, \tag{4.38}$$
$$Q_5 : \quad (\mathbb{1}, A^2, A, \omega A^2, 1)\,, \tag{4.39}$$
$$R : \quad (B, B, B, B, \epsilon\omega)\,, \tag{4.40}$$

and one may verify again that they stabilize c_{ijkl}. Here we use $\epsilon = e^{4i\pi/9}$ and the $SU(3)$ matrices

$$A = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon\omega^2 & 0 \\ 0 & 0 & \epsilon\omega^2 \end{pmatrix}. \tag{4.41}$$

Let us now discuss the possibility of symmetries other than the above. First, note that the projection of all tuples $(U_1, U_2, U_3, U_4, \zeta) \in \tilde{H}_{\text{R}}$ on the i-th $SU(3)$ must give a finite subgroup of $SU(3)$ which we call G_i in the following. From (4.35)–(4.40) we see that, regardless of which component we choose, G_i contains the $Z_3 \times Z_9$ subgroup generated by A and B. The possible finite subgroups of $SU(3)$ are well studied [40, 59, 60], and the existence of a $Z_3 \times Z_9$ subgroup essentially restricts possible extensions of the G_i to the groups $C_{9,3}^{(1)}$ and $D_{9,3}^{(1)}$ of [61], and to the exceptional group $\Sigma(216 \times 3)$ which has received some interest as flavor group in the past [62]. These groups are interrelated as

$$C_{9,3}^{(1)} \subset D_{9,3}^{(1)} \subset \Sigma(216 \times 3)\,, \tag{4.42}$$

and one can choose them to be generated them as follows:

$$C_{9,3}^{(1)}: \quad A, B, C_1C_2\,, \tag{4.43}$$

$$D_{9,3}^{(1)}: \quad A, B, C_1, C_2\,, \tag{4.44}$$

$$\Sigma(216 \times 3): \quad A, B, C_2, D\,. \tag{4.45}$$

Here we additionally use the $SU(3)$ matrices

$$C_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad D = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{4.46}$$

and $C_3 = C_1C_2C_1$. Note that any two of the C_i generate a lift of the Weyl group S_3 of $SU(3)$ into $SU(3)$, and that $D^2 = C_1$. The next step is to check that the elements

$$(C_2, C_2, C_2, C_1, 1)\,, \tag{4.47}$$

$$(C_1, C_2, C_3, C_2, 1)\,, \tag{4.48}$$

$$(D, D, D, D, 1)\,, \tag{4.49}$$

of $SU(3)^4 \times U(1)_R$ stabilize c_{ijkl}. For the first three this can easily be seen because they only act as permutations on the vectors s_i. The last one is better verified on a computer. As a consequence, one observes that the G_i are given by $\Sigma(216 \times 3)$, so $\tilde{H}_{\mathrm{R}}$ must be a subgroup of $\Sigma(216 \times 3)^4$. In fact one can check that the group generated by the tuples (4.34), (4.35)–(4.40) and (4.47)–(4.49) already span the whole group $\tilde{H}_{\mathrm{R}}$, which turns out to be of order $|\tilde{H}_{\mathrm{R}}| = 314928$. Hence, the discrete symmetry group $H_{\mathrm{R}} = \tilde{H}_{\mathrm{R}}/Z_3$ has order $|H_{\mathrm{R}}| = 104976$.

The superpotential. Let us now construct the most general effective superpotential, up to fourth order, that is consistent with the symmetries we found. Obviously, the three-point coupling in the superpotential does not require picture changing, and hence it must be an $SU(3)^4$ invariant. The only such invariant that we can write is of the form

$$\mathscr{W}_3 = \sum_{s=1}^{4} \varepsilon^{ijk} \Phi_{si} \otimes \Phi_{sj} \otimes \Phi_{sk}\,, \tag{4.50}$$

where we use the notation "$\otimes$" to indicate that we must also produce gauge-singlets, whose form depends on the choice of the left-mover lattice. From invariance under the discrete symmetries, the four-point term boils down to the somewhat expected form

$$\mathscr{W}_4 = c^{ijkl} \Phi_{1i} \otimes \Phi_{2j} \otimes \Phi_{3k} \otimes \Phi_{4l}\,, \tag{4.51}$$

where $c^{ijkl} = c^*_{ijkl}$. The next higher terms of the superpotential are six-point couplings, by invariance under the sector charges.

4.1.3 Search for Three-Generation Models

In order to construct a complete model, we have to provide a left-mover lattice $(\Gamma_{22})_{\mathrm{L}}$ that belongs the genus $\mathcal{G}_{\mathrm{L}}$ associated with the right-mover lattice $A_2^4 E_6^{\mathrm{st}}$ (3^3). In Subsection 3.2.5 we described the classification of this genus, the result being $|\mathcal{G}_{\mathrm{L}}| = 2030$. Thus, we can construct 2030 models from this right-mover lattice. Two models belonging to this genus were already described in [25].

Let us now perform a systematic search for standard-model like models. The first thing we do is to restrict to models that contain the standard model gauge group $G_{\mathrm{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$ (without extending to a larger group), and three generations (or families) of chiral matter. As most of the 2030 models contain several $U(1)$ factors, it will be convenient to define the family number in terms of the non-Abelian factors only:

$$N_{\mathrm{F}} = \#(\mathbf{3}, \mathbf{2}) - \#(\overline{\mathbf{3}}, \mathbf{2}), \tag{4.52}$$

Furthermore, a model may contain several $SU(3)$ or $SU(2)$ factors, and we may obtain different family numbers for different assignments. Note that due to anomaly cancellation also

$$2N_{\mathrm{F}} = \#(\overline{\mathbf{3}}, 1) - \#(\mathbf{3}, 1), \tag{4.53}$$

as the only representations allowed at the massless level are singlets and doublets in the case of $SU(2)$, as well as singlets and triplets in the case of $SU(3)$. Note that since the chiral fields in these models always come in a threefold degeneracy, the family number must necessarily be a multiple of three. An explicit calculation showed that among the 2030 spectra, 499 allowed for an assignment of $SU(3)_C \times SU(2)_L$ with $|N_{\mathrm{F}}| = 3$, the total number of such assignments being 1507 (here, we do not exclude possible equivalent assignments, for which a permutation of the corresponding G_{SM} factors is induced by a lattice automorphism).

All of these assignments contain exotic fields (in particular colored fields) which one must decouple at some high scale. However, there is a general difficulty to decouple such fields at three-point level of the superpotential, which we now explain. Let us consider in the following a singlet s uncharged under G_{SM} (including hypercharge), and two fields d and $\overline{d}$ oppositely charged under G_{SM}. For these fields there shall exist a superpotential term of the form $s\overline{d}d$, which reads

$$\varepsilon^{ijk} s_i \overline{d}_j d_k \,, \tag{4.54}$$

where we explicitly write out the $SU(3)$ triplet indices of the right-mover G_{R} representation. Now, assuming s obtains a vacuum expectation value (VEV) the mass matrix of the fields d_i and $\overline{d}_j$ is completely antisymmetric. Such a matrix always has a zero eigenvalue and hence one pair of chiral fields remains massless. Also, the remaining two vector-like fields would obtain the same mass. Then, in order to decouple the exotic fields we must also study the four-point terms. Note that this argument also applies for a Yukawa term such as $H_u Q u^c$, resulting in a light up quark but equally heavy charm and top quarks.

Hence, it will be difficult to obtain realistic Yukawa terms, and the inclusion of higher terms of the effective superpotential is indispensable.

Let us now consider a concrete $|N_{\mathrm{F}}| = 3$ model by choosing a specific left-mover lattice $(\Gamma_{22})_{\mathrm{L}}$. In order to describe this lattice, we define a sublattice Γ_{22} in terms of root lattices and specify a set of conjugacy classes. The sublattice is as follows:

$$\Gamma_{22} = A_1^5 A_2 A_3 A_4 \mathbb{Z}_{(4)} \mathbb{Z}_{(6)}^2 \mathbb{Z}_{(10)} \mathbb{Z}_{(12)} \mathbb{Z}_{(24)} \mathbb{Z}_{(60)} \mathbb{Z}_{(120)} \tag{4.55}$$

Note, that Γ_{22} has elementary divisors $2^7 6^1 12^2 60^2 120^2$. The cosets in $(\Gamma_{22})_{\mathrm{L}}/\Gamma_{22}$ are generated by the following conjugacy classes:

$$\begin{aligned}
&(0,0,0,0,1,0,0,0;\tfrac{1}{2},0,0,\tfrac{1}{2},0,0,0,0)\,,\\
&(1,0,0,0,0,0,v,0;0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,0)\,,\\
&(0,0,0,1,0,0,0,0;\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0)\,,\\
&(1,1,1,0,0,0,v,0;0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})\,,\\
&(1,0,0,1,1,0,c,0;\tfrac{3}{4},0,0,0,0,0,0,\tfrac{3}{4})\,,\\
&(1,0,1,0,0,2,0,2;0,\tfrac{1}{6},\tfrac{2}{3},0,\tfrac{1}{12},\tfrac{7}{12},\tfrac{7}{20},\tfrac{3}{20})\,,\\
&(0,0,0,0,0,2,v,2;\tfrac{1}{4},\tfrac{5}{6},\tfrac{1}{3},\tfrac{7}{10},\tfrac{3}{4},\tfrac{17}{24},\tfrac{3}{20},\tfrac{61}{120})\,,
\end{aligned} \tag{4.56}$$

where we choose to separate the semi-simple part and its orthogonal complement by a semicolon. The group $(\Gamma_{22})_{\mathrm{L}}/\Gamma_{22}$ then is of the structure $Z_2^4 Z_4 Z_{60} Z_{120}$.

We now have to glue $(\Gamma_{22})_{\mathrm{L}}$ and $(\Gamma_{14})_{\mathrm{R}}$ together to an even self-dual lattice $\Gamma_{22,14}$. For the right-mover lattice we consider here, there is only one inequivalent way to perform this gluing. To see this, first note that in $(\Gamma_{14})^*_{\mathrm{R}}/(\Gamma_{14})_{\mathrm{R}}$ there are only 8 cosets of even norm, which correspond to the four sectors in Table 4.1 and their CPT conjugates. Furthermore, the automorphism group of $\mathrm{disc}((\Gamma_{14})_{\mathrm{R}})$ is of structure $O(3,\mathbb{F}_3) \cong Z_2 \times S_4$, where the S_4 acts by permuting the four sectors, and the Z_2 as CPT conjugation (here, $\mathbb{F}_3$ denotes the finite field of three elements). However, all these discriminant form automorphisms are induced by lattice automorphisms of $(\Gamma_{14})_{\mathrm{R}}$ (note that there is an S_4 group which permutes the four A_2 factors), so all embeddings in an even self-dual lattice $\Gamma_{22,14}$ are equivalent. Hence, there is no need to explicitly write down the glue vectors.

Now, separately for each sector, we read off the massless left-mover spectrum by solving equation (2.66). First, we recognize that this model has gauge group

$$SU(2)^5 \times SU(3) \times SU(4) \times SU(5) \times U(1)^8\,, \tag{4.57}$$

(divided by some central subgroup). The chiral multiplets are displayed in Table 4.3, where we understand that the right-mover structure of a sector is as in Table 4.1. In Table 4.3, we have already chosen an assignment of the standard model gauge group G_{SM}, for which $SU(2)_L$ corresponds to the fifth A_1 factor in (4.55). The hypercharge is given as the following linear combination of the q_i:

$$Y = \frac{q_1}{4} + \frac{q_4}{5} + \frac{q_5}{6} - \frac{q_6}{24} - \frac{q_8}{120}\,. \tag{4.58}$$

This model also possesses an anomalous $U(1)_A$, as is apparent by computing the traces:

$$\frac{1}{36}\mathrm{Tr}(q_i) = (-3, 1, 0, -1, 5, -1, 1, -9)\,. \tag{4.59}$$

One can show that above assignment of the hypercharge is non-anomalous, and that it further ensures that all exotic matter is vector-like under G_{SM}.

Next, we write down the gauge-invariant terms of the superpotential (the right-mover structure is as in (4.50) and (4.51)).

$$\begin{aligned}\mathscr{W}_3 = {} & LE^c H_d + QU^c H_u + n_5 d_9 D^c \\ & + s_1 n_1 n_2 + s_2 d_1 d_2 + s_3 n_3 n_4 + s_4 d_8 d_8 \\ & + s_5 d_6 d_7 + s_6 d_{10} d_{11} + s_7 d_{12} d_{13} + s_8 d_{16} d_{17}\,.\end{aligned} \tag{4.60}$$

$$\begin{aligned}\mathscr{W}_4 = {} & n_1 n_3 n_5 n_7 + n_1 n_4 d_{13} d_{17} + d_1 d_5 n_5 U^c + d_2 d_7 d_{13} U^c \\ & + d_2 d_7 d_{17} D^c + n_4 n_6 L H_u + n_5 d_8 d_{16} L + d_6 D^c H_u H_d \\ & + s_1 d_8 d_{12} H_u + s_2 d_8 d_{11} H_u + s_3 D^c Q H_d + s_4 n_1 n_6 n_7 \\ & + s_4 d_1 d_{10} U^c + s_4 d_1 d_{15} D^c + s_4 d_9 E^c U^c + s_5 d_3 n_5 d_{14} \\ & + s_5 d_{12} d_{17} E^c + s_6 n_2 n_4 n_7 + s_6 d_2 d_6 d_{16} + s_6 d_2 d_8 Q \\ & + s_7 d_7 Q L + s_8 d_1 d_7 d_{11} + s_8 d_8 d_{13} H_d + s_1 s_3 d_{12} d_{16} \\ & + s_1 s_6 d_5 d_{15} + s_1 s_8 d_7 d_9 + s_2 s_3 d_{11} d_{16} + s_2 s_5 d_{10} d_{17} \\ & + s_2 s_5 d_{13} d_{15} + s_2 s_7 d_5 d_{17} + s_4 s_7 d_3 d_{14} + s_5 s_8 n_2 n_6 \\ & + s_6 s_8 d_3 d_4 + s_7 s_8 n_2 n_3\,.\end{aligned} \tag{4.61}$$

We observe that there are Yukawa terms for the charged leptons and the up-type quarks. However, from the preceding discussion we know that this results in $m_\mu \approx m_\tau$ and $m_c \approx m_t$, so this model does not appear very realistic. By giving the field s_3 a VEV we obtain further Yukawa terms for the down-type quarks. Let us now discuss the decoupling of exotic matter fields d_i. Except for d_1, d_2, d_6 and d_8, we obtain mass terms for these fields from $\mathscr{W}_4$ by giving VEVs to the s_i fields. The remaining four fields obtain mass terms from $\mathscr{W}_3$, but due to the antisymmetric mass matrix only two of the three flavors decouple. Thus, in order to completely decouple the exotic fields it will be necessary to obtain mass terms from $\mathscr{W}_6$ or higher superpotential terms.

4.2 A Class of Asymmetric Z_6 Orbifolds

The Z_3 models that we discussed in the last section had some obvious phenomenological deficiences: it appeared difficult to decouple exotic matter fields, and the Yukawa sector is rather unrealistic. Both problems originate from the fact that the chiral fields in these models always appear in formal triplets of some right-mover A_2 algebra, and that the three-point coupling must be an $SU(3)$ invariant. In this section, we consider covariant lattice models that correspond to Z_6 asymmetric orbifolds built from a Narain lattice with

Table 4.3: Chiral left-mover spectrum of a E_6/Z_3 asymmetric orbifold/covariant lattice model. Here, $G_{\text{hidden}} = SU(2)^4 \times SU(4) \times SU(5)$ denotes the hidden non-Abelian gauge group and the q_i are the $U(1)^8$ charges. The hypercharge is given from the linear combination (4.58).

Sector	Field	No.	G_{SM}	G_{hidden}	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
Φ_{1k}	s_1	1	$(1,1)_0$	$(1,1,\mathbf{2},1,1,1)$	0	−1	−1	0	−1	−4	5	0
	s_2	2	$(1,1)_0$	$(\mathbf{2},1,1,\mathbf{2},1,1)$	0	−1	−1	1	−1	2	−1	−6
	n_1	3	$(1,1)_0$	$(1,1,1,1,\overline{\mathbf{4}},1)$	1	1	1	−1	0	2	−4	−4
	n_2	4	$(1,1)_0$	$(1,1,\mathbf{2},1,\mathbf{4},1)$	−1	0	0	1	1	2	−1	4
	d_1	5	$(\mathbf{3},1)_{1/6}$	$(1,1,1,\mathbf{2},1,1)$	0	1	1	0	1	0	5	0
	d_2	6	$(\overline{\mathbf{3}},1)_{-1/6}$	$(\mathbf{2},1,1,1,1,1)$	0	0	0	−1	0	−2	−4	6
	d_3	7	$(1,1)_{-1/5}$	$(1,\mathbf{2},1,1,1,\overline{\mathbf{5}})$	−1	0	0	−1	1	−1	−1	−5
	L	8	$(1,\mathbf{2})_{-1/2}$	$(1,1,1,1,1,1)$	0	−2	1	−2	0	2	2	2
	E^c	9	$(1,1)_1$	$(1,1,1,1,1,1)$	1	1	−2	1	3	−1	−1	−1
	H_d	10	$(1,\mathbf{2})_{-1/2}$	$(1,1,1,1,1,1)$	−1	1	1	1	−3	−1	−1	−1
Φ_{2k}	s_3	11	$(1,1)_0$	$(1,1,1,1,1,1)$	1	1	0	−1	1	3	1	11
	s_4	12	$(1,1)_0$	$(1,1,1,1,1,1)$	0	−2	0	2	−2	0	−2	8
	s_5	13	$(1,1)_0$	$(1,\mathbf{2},1,1,1,1)$	1	0	2	−1	−1	−3	1	1
	n_3	14	$(1,1)_0$	$(1,1,1,1,\overline{\mathbf{4}},1)$	−1	−1	1	1	0	0	4	−6
	n_4	15	$(1,1)_0$	$(1,1,1,1,\mathbf{4},1)$	0	0	−1	0	−1	−3	−5	−5
	d_4	16	$(1,1)_{1/5}$	$(1,1,1,1,1,\mathbf{5})$	−1	1	0	1	1	−3	1	5
	d_5	17	$(1,1)_{1/2}$	$(1,1,1,\mathbf{2},1,1)$	0	−1	1	0	3	0	−5	0
	d_6	18	$(\mathbf{3},1)_{-1/3}$	$(1,1,1,1,1,1)$	−1	−1	−2	−1	1	1	1	1
	d_7	19	$(\overline{\mathbf{3}},1)_{1/3}$	$(1,\mathbf{2},1,1,1,1)$	0	1	0	2	0	2	−2	−2
	d_8	20	$(1,\mathbf{2})_0$	$(\mathbf{2},1,\mathbf{2},1,1,1)$	0	1	0	−1	1	0	1	−4
Φ_{3k}	s_6	21	$(1,1)_0$	$(1,1,\mathbf{2},1,1,1)$	1	0	1	1	−2	3	4	−1
	s_7	22	$(1,1)_0$	$(1,\mathbf{2},1,1,1,1)$	1	2	0	−1	−1	−3	1	1
	n_5	23	$(1,1)_0$	$(1,1,1,1,1,1)$	0	0	−2	2	−2	0	−2	8
	n_6	24	$(1,1)_0$	$(1,1,1,1,\overline{\mathbf{4}},1)$	−1	1	−1	1	0	0	4	−6
	d_9	25	$(\mathbf{3},1)_{-1/3}$	$(1,1,1,1,1,1)$	−1	1	2	−1	1	1	1	1
	d_{10}	26	$(1,1)_{1/2}$	$(1,1,1,\mathbf{2},1,1)$	0	1	−1	0	3	0	−5	0
	d_{11}	27	$(1,1)_{-1/2}$	$(1,1,\mathbf{2},\mathbf{2},1,1)$	−1	−1	0	−1	−1	−3	1	1
	d_{12}	28	$(1,1)_{-1/2}$	$(\mathbf{2},1,1,1,1,1)$	−1	−1	0	0	−1	3	−5	−5
	d_{13}	29	$(1,1)_{1/2}$	$(\mathbf{2},\mathbf{2},1,1,1,1)$	0	−1	0	1	2	0	4	4
	D^c	30	$(\overline{\mathbf{3}},1)_{1/3}$	$(1,1,1,1,1,1)$	1	−1	0	−1	1	−1	1	−9
Φ_{4k}	s_8	31	$(1,1)_0$	$(1,\mathbf{2},\mathbf{2},1,1,1)$	1	−1	−1	−1	0	1	−4	1
	n_7	32	$(1,1)_0$	$(1,1,1,1,\mathbf{6},1)$	0	0	0	−2	2	−2	2	2
	d_{14}	33	$(1,1)_{1/5}$	$(1,1,1,1,1,\mathbf{5})$	0	0	0	0	2	4	2	−4
	d_{15}	34	$(1,1)_{-1/2}$	$(1,1,1,\mathbf{2},1,1)$	−1	2	−1	−1	0	1	−4	1
	d_{16}	35	$(1,1)_{1/2}$	$(\mathbf{2},1,\mathbf{2},1,1,1)$	0	1	1	1	1	−2	−1	−6
	d_{17}	36	$(1,1)_{-1/2}$	$(\mathbf{2},\mathbf{2},1,1,1,1)$	−1	0	0	0	−1	1	5	5
	Q	37	$(\mathbf{3},\mathbf{2})_{1/6}$	$(1,1,1,1,1,1)$	−1	−1	−1	1	1	−1	−1	−1
	U^c	38	$(\overline{\mathbf{3}},1)_{-2/3}$	$(1,1,1,1,1,1)$	0	0	0	−2	−2	0	2	−8
	H_u	39	$(1,\mathbf{2})_{1/2}$	$(1,1,1,1,1,1)$	1	1	1	1	1	1	−1	9

a right-mover E_6 factor. The relevant right-mover lattice is the lattice $A_1^4 E_6^{\rm st}$ $(3^1 6^2)$ in Figure 3.1. We immediately see that the difficulties of the Z_3 models cannot be present in the same form for these Z_6 models, as there are no A_2 factors in the right-mover lattice. Further note that although we do not have a strict classification of the genus $\mathcal{G}_{\rm L}$ as in the Z_3 case, we obtained an exhaustive but not duplicate-free list of 5033195 left-mover lattices in $\mathcal{G}_{\rm L}$ by the method described in Subsection 3.2.5.

4.2.1 Massless Right-Mover Spectrum

Let us first describe $(\Gamma_{14})_{\rm R} = A_1^4 E_6^{\rm st}$ $(3^1 6^2)$ right-mover lattice. For this we define a sublattice $\Gamma_{14} \subset (\Gamma_{14})_{\rm R}$:

$$\Gamma_{14} = A_1^4 Z_{(6)}^4 E_6^{\rm st}\,, \tag{4.62}$$

which has elementary divisors $2^3 6^5$. Then, $(\Gamma_{14})_{\rm R}/\Gamma_{14} \cong Z_2^3 \times Z_3$ is generated by the following conjugacy classes:

$$\begin{aligned} &(1,1,0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,0)\,,\\ &(0,1,1,0,0,\tfrac{1}{2},\tfrac{1}{2},0,0)\,,\\ &(0,0,1,1,0,0,\tfrac{1}{2},\tfrac{1}{2},0)\,,\\ &(0,0,0,0,\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3},1)\,. \end{aligned} \tag{4.63}$$

Similarly, $(\Gamma_{14})_{\rm R}^*/(\Gamma_{14})_{\rm R}$ is generated by the following unions of conjugacy classes:

$$\begin{aligned} \chi_1 &= (0,0,0,0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,0) \cup \ldots\,,\\ \chi_2 &= (1,0,0,0,\tfrac{5}{6},0,\tfrac{1}{3},\tfrac{1}{3},0) \cup \ldots\,,\\ \chi_3 &= (1,0,0,0,0,\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},0) \cup \ldots\,, \end{aligned} \tag{4.64}$$

where the ellipsis represents the other conjugacy classes which are obtained by adding those from (4.63). In the following, we identify a coset in the quotient group $(\Gamma_{14})_{\rm R}^*/(\Gamma_{14})_{\rm R}$, by an integer triple (Q_1, Q_2, Q_3) that denotes a linear combination of the generators χ_i. Note that $3\chi_1 = 6\chi_2 = 6\chi_3 = (\Gamma_{14})_{\rm R}$, and that the quotient group has the structure $Z_3 \times Z_6^2$. The sector charge Q_1 is then defined only modulo 3, whereas Q_2 and Q_3 are defined modulo 6.

The massless right-mover spectrum is then calculated similarly as in Section 4.1. Here, we do not carry out the calculation in detail, but only show the result in Table 4.4. Note, that for this right-mover lattice both the $1^1 8^1_D$ and the 9^1_J elementary supercurrents are possible.

The superpotential. Here, we do not determine the discrete symmetry group $H_{\rm R}$ as we did for the E_6/Z_3 case in Section 4.1. Thus, we only write down the three-point superpotential, as it is controlled by $SU(2)^4 \times U(1)^4$ conservation:

$$\mathscr{W}_3 = \sum_{s=1}^{4} U \otimes R_s \otimes S_s + \sum_{s=1}^{4} \varepsilon^{ij} S_s \otimes T_{si} \otimes T_{sj} \tag{4.65}$$

Table 4.4: Massless chiral superfields from the $A_1^4 E_6^{\mathrm{st}}(3^1 6^2)$ right-mover lattice. The index k transforms as the doublet of the corresponding $SU(2)$. The c_i denote the $U(1)^4$ charges.

Superfield	$SU(2)^4$	c_1	c_2	c_3	c_4	Q_1	Q_2	Q_3
U	$(1,1,1,1)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	3	3
R_1	$(1,1,1,1)$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	3	1
R_2	$(1,1,1,1)$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	5	3
R_3	$(1,1,1,1)$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	1	1	5
R_4	$(1,1,1,1)$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	2	3	3
S_1	$(1,1,1,1)$	$-\frac{1}{3}$	0	0	0	0	0	2
S_2	$(1,1,1,1)$	0	$-\frac{1}{3}$	0	0	0	4	0
S_3	$(1,1,1,1)$	0	0	$-\frac{1}{3}$	0	2	2	4
S_4	$(1,1,1,1)$	0	0	0	$-\frac{1}{3}$	1	0	0
T_{1k}	$(\mathbf{2},1,1,1)$	$\frac{1}{6}$	0	0	0	0	3	2
T_{2k}	$(1,\mathbf{2},1,1)$	0	$\frac{1}{6}$	0	0	0	1	0
T_{3k}	$(1,1,\mathbf{2},1)$	0	0	$\frac{1}{6}$	0	2	5	4
T_{4k}	$(1,1,1,\mathbf{2})$	0	0	0	$\frac{1}{6}$	1	3	0

4.2.2 A Three-Generation Model

Let us now construct a three-generation model based on this right-mover lattice by specifying a left-mover lattice $(\Gamma_{22})_{\mathrm{L}}$. For this, we define the following sublattice:

$$\Gamma_{22} = A_1^5 A_2 \mathbb{Z}_{(4)}^8 \mathbb{Z}_{(6)} \mathbb{Z}_{(8)} \mathbb{Z}_{(12)}^4 \mathbb{Z}_{(24)} \,, \tag{4.66}$$

which has elementary divisors $2^6 4^7 12^5 24^2$. Then, $(\Gamma_{22})_{\mathrm{L}}/\Gamma_{22} \cong Z_2^2 Z_4^5 Z_{12} Z_{24}$ is generated by the conjugacy classes

$$\begin{aligned}
&(1,1,1,1,1,0;0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0)\,,\\
&(0,1,0,1,0,0;\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},0,0,0,\tfrac{1}{2})\,,\\
&(0,1,0,1,0,0;\tfrac{1}{2},\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{2},\tfrac{1}{4},0,\tfrac{1}{2},0,0,0,0,\tfrac{1}{2})\,,\\
&(0,0,0,0,1,0;\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{4},\tfrac{3}{4},\tfrac{3}{4},\tfrac{1}{2},0,\tfrac{1}{4},\tfrac{3}{4},0,\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{4})\,,\\
&(1,0,0,0,1,0;\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{2},0,\tfrac{3}{4},0,\tfrac{3}{4},0,\tfrac{1}{2},\tfrac{3}{4},0,\tfrac{3}{4})\,,\\
&(1,0,1,0,1,0;\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{4},0,0,0,\tfrac{1}{2},0,0,\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})\,,\\
&(1,0,1,0,1,0;\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{3}{4},\tfrac{3}{4},0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2},0,\tfrac{1}{4})\,,\\
&(0,0,1,1,0,0;\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{4},0,\tfrac{1}{4},\tfrac{5}{6},\tfrac{1}{4},\tfrac{1}{3},\tfrac{1}{6},\tfrac{5}{12},\tfrac{11}{12},\tfrac{1}{2})\,,\\
&(1,0,1,0,1,1;0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{4},\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{6},\tfrac{5}{8},\tfrac{11}{12},\tfrac{1}{12},\tfrac{3}{4},\tfrac{2}{3},\tfrac{5}{24})\,.
\end{aligned} \tag{4.67}$$

Here, for better readability, we separate the semi-simple part from its orthogonal complement by a semi-colon.

We now have to glue $(\Gamma_{22})_{\rm L}$ and $(\Gamma_{14})_{\rm R}$ together to an even self-dual lattice $\Gamma_{22,14}$. Unlike the E_6/Z_3 case, there is the possibility of having several inequivalent embeddings of a chosen $(\Gamma_{22})_{\rm L} \oplus \overline{(\Gamma_{14})_{\rm R}}$ into an even self-dual lattice. In fact one can show that depending on the left-mover lattice, there are either one or two inequivalent embeddings. In the following we discuss only a single embedding. However, we will not give an explicit form of the glue vectors and instead just show the spectrum. This is sufficient, because the spectrum uniquely determines the embedding.

Let us now calculate the massless left-mover spectrum by solving equation (2.66). First, we observe that there is a gauge group

$$SU(2)^5 \times SU(3) \times U(1)^{15}\,, \tag{4.68}$$

(divided by a central subgroup). The chiral multiplets are given as in Table 4.5. There, we already assigned the fifth A_1 factor to $SU(2)_L$ and the hypercharge is given by the linear combination

$$Y = \frac{q_1}{2} - \frac{q_3}{4} - \frac{q_4}{4} - \frac{q_6}{4} - \frac{q_7}{4} + \frac{q_9}{3} + \frac{q_{10}}{4} - \frac{q_{11}}{6} - \frac{q_{12}}{12} + \frac{q_{13}}{12} - \frac{q_{15}}{6}\,, \tag{4.69}$$

which one can show to be non-anomalous. This model has an anomalous $U(1)_A$, which one shows by calculating the traces:

$$\frac{1}{12}\mathrm{Tr}(q_i) = (2,3,1,2,-1,-1,0,-5,-2,4,6,7,5,1,-2)\,. \tag{4.70}$$

This model has three generations of standard model matter and only vector-like exotics. In Table 4.5, the fields s_i denote singlets of $SU(3)_C \times SU(2)_L$, whereas the L_i are lepton or Higgs doublets. The Q_i denote quark doublets, and U_i^c and D_i^c are up-type and down-type quark singlets. Note, that in this model all exotics are color singlets. Besides the lepton doublets, there are five additional pairs of $(1,\mathbf{2})_{1/2}$ and $(1,\mathbf{2})_{-1/2}$, which may serve as Higgs doublets or as exotic matter. Furthermore, Q_2 and Q_3 come as a doublet of a right-mover $SU(2)$, and (U_2^c, U_3^c) form a doublet of an extra $SU(2)$ gauge flavor symmetry. Here, we do not write down the entire three-point superpotential, but merely note that the only term involving quark fields is

$$\mathscr{W}_3^{\rm Quarks} = L_1 Q_1 U_1^c\,. \tag{4.71}$$

It is therefore natural to identify L_1 as a H_u Higgs doublet and interpret $\mathscr{W}_3^{\rm Quarks}$ as a top quark Yukawa term. Thus, at three-point level only the top quark obtains a mass, indicating that all other quark masses are suppressed.

Finally, note that there is another interesting possibility to take the hypercharge in this model by replacing Y in (4.69) with $Y' = Y + I_2$, where I_2 denotes the isospin of the second $SU(2)$ factor in $G_{\rm hidden}$. Using this hypercharge assignment all exotic fields obtain integer electric charge.

Table 4.5: Chiral left-mover spectrum of a E_6/Z_6 asymmetric orbifold/covariant lattice model. Here, $G_{\text{hidden}} = SU(2)^4$ denotes the hidden non-Abelian gauge group and the q_i denote the $U(1)^{15}$ charges. The hypercharge is given from the linear combination (4.69).

Sector	Field	No.	G_{SM}	G_{hidden}	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}
	s_1	1	$(1,1)_{-1}$	$(1,1,1,1)$	−1	0	0	0	1	1	1	−1	1	0	2	1	1	1	0
	s_2	2	$(1,1)_{-1}$	$(1,1,1,1)$	−1	1	−1	1	0	0	−1	0	−1	0	1	2	−1	−1	0
	s_3	3	$(1,1)_{1}$	$(1,1,1,1)$	1	−1	1	−1	0	0	1	0	1	0	−1	−2	1	1	0
	s_4	4	$(1,1)_{1}$	$(1,1,1,1)$	1	0	0	0	−1	−1	−1	1	−1	0	−2	−1	−1	−1	0
	s_5	5	$(1,1)_{0}$	$(1,1,1,\mathbf{2})$	0	0	−1	0	0	0	0	1	1	−1	−1	1	1	1	3
U	s_6	6	$(1,1)_{0}$	$(1,1,1,\mathbf{2})$	0	0	1	0	0	0	0	−1	−1	1	1	−1	−1	−1	−3
	s_7	7	$(1,1)_{0}$	$(1,1,\mathbf{2},1)$	0	−1	0	1	1	0	0	1	0	−1	0	0	0	0	−3
	s_8	8	$(1,1)_{0}$	$(1,1,\mathbf{2},1)$	0	1	0	−1	−1	0	0	−1	0	1	0	0	0	0	3
	s_9	9	$(1,1)_{-3/2}$	$(1,\mathbf{2},1,1)$	0	−1	0	0	0	1	0	0	−1	−1	1	−1	−1	−1	3
	s_{10}	10	$(1,1)_{3/2}$	$(1,\mathbf{2},1,1)$	0	1	0	0	0	−1	0	0	1	1	−1	1	1	1	−3
	L_1	11	$(1,\mathbf{2})_{1/2}$	$(1,1,1,1)$	0	0	0	−1	1	0	0	0	1	0	−1	1	−2	−2	0
	L_2	12	$(1,\mathbf{2})_{-1/2}$	$(1,1,1,1)$	0	0	0	1	−1	0	0	0	−1	0	1	−1	2	2	0
	s_{11}	13	$(1,1)_{0}$	$(1,1,1,1)$	−1	0	0	−1	−2	0	0	0	1	1	1	0	0	−1	1
	s_{12}	14	$(1,1)_{0}$	$(1,1,1,1)$	−1	0	0	−1	2	0	0	0	1	1	1	0	0	−1	1
	s_{13}	15	$(1,1)_{1}$	$(1,1,1,1)$	−1	0	0	0	0	−1	−1	−1	1	0	−2	0	0	2	−2
	s_{14}	16	$(1,1)_{-1}$	$(1,1,1,1)$	−1	1	1	0	0	0	1	0	−1	0	0	−2	−2	0	−2
R_1	s_{15}	17	$(1,1)_{1}$	$(1,1,1,1)$	1	−1	−1	1	0	−1	0	−1	1	1	1	0	0	−1	1
	s_{16}	18	$(1,1)_{0}$	$(1,1,1,1)$	1	0	0	0	1	0	0	0	−1	1	0	1	−2	3	1
	s_{17}	19	$(1,1)_{0}$	$(1,1,1,1)$	1	1	1	1	0	1	0	1	1	1	1	0	0	−1	1
	s_{18}	20	$(1,1)_{0}$	$(1,1,1,\mathbf{2})$	0	0	1	−1	0	1	−1	0	−1	0	0	1	1	0	−2
	s_{19}	21	$(1,1)_{0}$	$(1,1,\mathbf{2},1)$	0	1	0	0	0	0	0	−1	0	−1	−1	−1	2	−2	1

Table 4.5: Chiral left-mover spectrum of a E_6/Z_6 asymmetric orbifold/covariant lattice model (cont'd).

Sector	Field	No.	G_{SM}	G_{hidden}	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}
R_1	s_{20}	22	$(1,1)_{1/2}$	$(1,\mathbf{2},1,1)$	0	0	−1	−1	0	−1	1	0	−1	0	0	1	1	0	−2
	L_3	23	$(1,\mathbf{2})_{1/2}$	$(1,1,1,1)$	0	0	0	0	0	−1	−1	1	−1	1	0	−2	1	0	1
R_2	s_{21}	24	$(1,1)_0$	$(1,1,\mathbf{2},1)$	0	0	−1	0	−1	0	1	−1	0	0	−1	0	−2	1	0
	s_{22}	25	$(1,1)_{1/2}$	$(1,\mathbf{2},1,1)$	0	0	1	0	0	−1	−1	0	1	0	1	1	2	−1	0
	s_{23}	26	$(1,1)_0$	$(1,1,1,1)$	−1	−1	−1	0	0	0	−1	0	−1	2	0	2	0	0	0
	s_{24}	27	$(1,1)_0$	$(1,1,1,1)$	−1	1	−1	1	0	0	−1	0	1	0	1	−2	−1	−1	0
	s_{25}	28	$(1,1)_1$	$(1,1,1,1)$	1	0	0	0	−1	0	0	0	−1	1	0	−1	0	−3	−3
	s_{26}	29	$(1,1)_1$	$(1,1,1,1)$	1	0	0	0	1	1	−1	−1	1	0	−2	1	−1	−1	0
	s_{27}	30	$(1,1)_{-1}$	$(1,1,1,1)$	1	1	1	0	0	0	1	0	−1	−2	0	2	0	0	0
	s_{28}	31	$(1,1)_0$	$(1,1,1,\mathbf{2})$	0	0	−1	0	0	1	1	0	1	0	1	1	2	−1	0
	s_{29}	32	$(1,1)_1$	$(1,1,\mathbf{2},1)$	0	1	0	0	1	0	0	1	0	1	−1	0	1	1	−3
	s_{30}	33	$(1,1)_{-1/2}$	$(1,\mathbf{2},1,1)$	0	1	0	−1	0	1	0	0	−1	1	0	−1	0	0	3
	s_{31}	34	$(1,1)_{-1}$	$(\mathbf{2},1,1,1)$	−1	0	1	0	0	1	0	0	1	0	1	1	−1	2	0
	s_{32}	35	$(1,1)_0$	$(\mathbf{2},1,1,\mathbf{2})$	0	0	0	0	1	−1	0	−1	−1	0	0	−1	0	0	0
	U_1^c	36	$(\overline{\mathbf{3}},1)_{-2/3}$	$(1,1,1,1)$	−1	0	0	0	−1	0	0	0	−1	−1	0	−1	2	1	−1
R_3	s_{33}	37	$(1,1)_{-1}$	$(1,1,1,1)$	−1	1	−1	0	1	1	0	−1	−1	−1	0	1	0	1	−1
	s_{34}	38	$(1,1)_{-1}$	$(1,1,1,1)$	−1	−1	−1	0	0	0	1	0	−1	0	0	−2	0	−2	2
	s_{35}	39	$(1,1)_{-1}$	$(1,1,1,1)$	−1	−1	1	0	1	−1	0	1	−1	−1	0	1	0	1	−1
	s_{36}	40	$(1,1)_0$	$(1,1,1,1)$	−1	0	0	0	0	1	−1	1	1	0	−2	0	2	0	2
	s_{37}	41	$(1,1)_1$	$(1,1,1,1)$	1	0	0	−1	0	0	0	0	−1	−1	−3	−2	0	1	−1
	L_4	42	$(1,\mathbf{2})_{-1/2}$	$(1,1,1,1)$	0	0	0	0	−1	1	1	1	−1	1	0	1	0	1	−1

Table 4.5: Chiral left-mover spectrum of a E_6/Z_6 asymmetric orbifold/covariant lattice model (cont'd).

Sector	Field	No.	G_{SM}	G_{hidden}	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}
R_3	D_1^c	43	$(\overline{\mathbf{3}},1)_{1/3}$	$(1,1,1,1)$	1	0	0	0	1	0	0	0	−1	1	0	1	2	−1	1
R_4	s_{38}	44	$(1,1)_{1/2}$	$(1,\mathbf{2},1,1)$	0	1	0	0	−1	1	0	0	1	−1	−1	0	0	0	−3
	s_{39}	45	$(1,1)_0$	$(1,1,1,1)$	1	1	−1	1	0	0	1	0	−1	0	1	−2	1	1	0
	s_{40}	46	$(1,1)_0$	$(1,1,1,1)$	−1	0	0	−1	0	0	0	0	−1	1	1	−2	−2	1	−3
	s_{41}	47	$(1,1)_{-1}$	$(1,1,1,1)$	−1	0	0	0	0	−1	1	−1	1	−2	2	0	0	0	0
	s_{42}	48	$(1,1)_{-1}$	$(1,1,1,1)$	−1	0	0	0	1	1	1	−1	−1	0	−2	1	1	1	0
	s_{43}	49	$(1,1)_1$	$(1,1,1,1)$	1	0	0	0	0	1	−1	1	1	2	2	0	0	0	0
	s_{44}	50	$(1,1)_1$	$(1,1,\mathbf{2},1)$	0	0	−1	−1	0	0	−1	−1	0	0	0	−1	2	−1	0
	s_{45}	51	$(1,1)_{-1}$	$(\mathbf{2},1,1,1)$	−1	1	0	0	0	0	0	1	−1	0	1	1	1	−2	0
S_1	s_{46}	52	$(1,1)_0$	$(1,1,1,1)$	0	0	0	−1	−1	0	0	0	0	2	1	1	−2	2	2
	s_{47}	53	$(1,1)_0$	$(1,1,1,1)$	0	0	0	1	0	−1	1	−1	0	1	1	−2	−2	−1	−1
	s_{48}	54	$(1,1)_1$	$(1,1,1,1)$	0	1	1	1	0	0	−1	0	2	1	−1	0	0	1	−1
	s_{49}	55	$(1,1)_0$	$(1,1,\mathbf{2},1)$	1	1	0	0	1	0	0	−1	−1	0	−1	0	0	1	2
	s_{50}	56	$(1,1)_{1/2}$	$(1,\mathbf{2},1,1)$	1	1	0	0	0	0	1	1	0	1	1	1	1	−1	−1
	s_{51}	57	$(1,1)_1$	$(1,1,1,\mathbf{2})$	1	−1	0	0	0	0	−1	−1	0	1	1	1	1	−1	−1
	s_{52}	58	$(1,1)_{-1}$	$(1,1,1,1)$	0	0	0	1	−1	1	1	−1	0	−1	1	1	1	2	−1
	L_5	59	$(1,\mathbf{2})_{1/2}$	$(1,1,1,1)$	1	0	0	0	−1	0	0	0	0	−1	−2	1	−2	−1	−1
	U_2^c, U_3^c	60	$(\overline{\mathbf{3}},1)_{-2/3}$	$(\mathbf{2},1,1,1)$	0	0	1	0	0	−1	0	0	0	−1	1	1	0	0	1
S_2	s_{53}	61	$(1,1)_{-1}$	$(1,1,1,1)$	0	−2	0	1	0	0	0	0	0	−2	1	0	−1	−1	0
	s_{54}	62	$(1,1)_0$	$(1,1,1,1)$	0	0	0	0	0	−1	−1	1	0	−1	−2	0	−1	2	3
	s_{55}	63	$(1,1)_{-1}$	$(1,1,1,1)$	0	0	0	0	0	0	0	0	−2	0	0	4	0	0	0

Table 4.5: Chiral left-mover spectrum of a E_6/Z_6 asymmetric orbifold/covariant lattice model (cont'd).

Sector	Field	No.	G_{SM}	G_{hidden}	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}
S_2	s_{56}	64	$(1,1)_{-1}$	$(1,1,1,\mathbf{2})$	−1	0	1	0	1	0	0	−1	0	0	1	0	−1	2	0
	s_{57}	65	$(1,1)_{1/2}$	$(1,\mathbf{2},\mathbf{2},1)$	0	0	0	0	−1	−1	0	−1	1	0	0	1	0	0	0
	s_{58}	66	$(1,1)_0$	$(\mathbf{2},1,1,1)$	0	0	−1	0	1	0	1	−1	0	0	1	0	2	−1	0
	s_{59}	67	$(1,1)_0$	$(\mathbf{2},1,1,1)$	0	0	1	0	−1	1	0	0	0	1	1	0	−1	−1	−3
	s_{60}	68	$(1,1)_1$	$(1,1,1,1)$	0	1	−1	0	0	0	1	0	0	1	−2	0	−1	2	−3
	s_{61}	69	$(1,1)_0$	$(1,1,1,1)$	0	1	1	−1	0	0	−1	0	0	1	1	0	2	−1	3
	s_{62}	70	$(1,1)_{-1}$	$(1,1,1,1)$	0	2	0	1	0	0	0	0	0	−2	1	0	−1	−1	0
	L_6	71	$(1,\mathbf{2})_{-1/2}$	$(1,1,1,1)$	−1	0	0	−1	0	−1	1	1	0	0	1	0	−1	−1	0
	D_2^c	72	$(\overline{\mathbf{3}},1)_{1/3}$	$(1,1,1,1)$	0	0	0	0	0	1	−1	−1	0	−1	−2	0	1	0	−1
	Q_1	73	$(\mathbf{3},\mathbf{2})_{1/6}$	$(1,1,1,1)$	1	0	0	1	0	0	0	0	0	1	1	0	0	1	1
S_3	s_{63}	74	$(1,1)_0$	$(1,1,1,1)$	0	0	0	−1	−1	0	0	0	0	−2	1	1	2	−2	−2
	s_{64}	75	$(1,1)_0$	$(1,1,1,1)$	0	0	0	1	0	−1	−1	−1	0	−1	1	−2	2	1	1
	s_{65}	76	$(1,1)_{-1/2}$	$(1,\mathbf{2},1,1)$	1	0	1	0	0	1	−1	0	0	−1	1	1	−1	1	1
	s_{66}	77	$(1,1)_0$	$(\mathbf{2},1,1,1)$	0	−1	0	1	0	0	0	−1	0	1	−2	1	−1	1	1
	s_{67}	78	$(1,1)_0$	$(\mathbf{2},1,1,1)$	0	1	0	0	1	0	0	1	0	1	1	−2	−1	1	1
	s_{68}	79	$(1,1)_0$	$(1,1,1,1)$	0	0	0	1	−1	−1	−1	1	0	1	1	1	−1	−2	1
	s_{69}	80	$(1,1)_0$	$(1,1,1,1)$	0	−1	1	0	0	0	1	0	2	1	2	0	1	0	1
	s_{70}	81	$(1,1)_0$	$(1,1,1,\mathbf{2})$	1	0	−1	0	0	−1	1	0	0	−1	1	1	−1	1	1
	s_{71}	82	$(1,1)_1$	$(1,1,\mathbf{2},1)$	1	0	−1	1	0	1	0	0	1	0	0	−1	0	−1	−2
S_4	s_{72}	83	$(1,1)_0$	$(1,1,1,1)$	0	0	0	0	0	0	0	0	2	0	4	0	0	0	0
	s_{73}	84	$(1,1)_0$	$(1,1,1,1)$	0	0	0	0	1	2	0	0	0	2	0	1	1	1	0

Table 4.5: Chiral left-mover spectrum of a E_6/Z_6 asymmetric orbifold/covariant lattice model (cont'd).

Sector	Field	No.	$G_{\rm SM}$	$G_{\rm hidden}$	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}
S_4	s_{74}	85	$(1,1)_1$	$(1,1,1,1)$	0	0	0	0	1	−2	0	0	0	2	0	1	1	1	0
	s_{75}	86	$(1,1)_{-1}$	$(1,1,1,1)$	0	1	1	0	−1	0	1	0	0	1	0	1	−2	1	3
	s_{76}	87	$(1,1)_0$	$(1,1,1,\mathbf{2})$	1	0	1	1	0	0	0	−1	0	0	0	1	1	−2	0
	s_{77}	88	$(1,1)_0$	$(\mathbf{2},1,1,1)$	0	0	−1	1	0	0	−1	−1	0	0	0	1	−2	1	0
	L_7	89	$(1,\mathbf{2})_{-1/2}$	$(1,1,1,1)$	−1	−1	1	0	−1	0	−1	0	0	0	0	1	1	1	0
	D_3^c	90	$(\overline{\mathbf{3}},1)_{1/3}$	$(1,1,1,1)$	0	−1	−1	0	−1	0	1	0	0	1	0	1	0	−1	−1
T_{1k}	s_{78}	91	$(1,1)_0$	$(1,1,\mathbf{2},1)$	0	1	1	1	0	0	−1	0	−1	1	−1	0	0	1	−1
	s_{79}	92	$(1,1)_{-1/2}$	$(1,\mathbf{2},1,1)$	0	0	0	1	1	0	1	0	0	1	1	1	−2	−1	−1
	s_{80}	93	$(1,1)_0$	$(1,1,1,1)$	−1	0	−1	0	0	−1	0	0	0	2	1	1	1	2	2
	s_{81}	94	$(1,1)_1$	$(1,1,1,1)$	1	1	0	0	−1	−1	1	0	0	1	1	−2	1	−1	−1
	s_{82}	95	$(1,1)_1$	$(1,1,1,1)$	1	1	0	0	1	0	0	−1	2	0	−1	0	0	1	2
	s_{83}	96	$(1,1)_0$	$(1,1,1,1)$	−1	0	1	−1	0	0	1	−1	0	1	−2	1	1	−1	−1
	s_{84}	97	$(1,1)_1$	$(1,1,1,1)$	1	−1	0	0	0	0	0	−1	0	0	1	1	1	2	−4
	s_{85}	98	$(1,1)_{-1}$	$(1,1,1,\mathbf{2})$	0	0	0	1	−1	1	0	−1	0	0	1	1	1	−1	2
	L_8, L_9	99	$(1,\mathbf{2})_{1/2}$	$(1,1,1,1)$	0	0	−1	1	0	−1	0	0	0	−1	−2	1	1	−1	−1
	L_{10}, L_{11}	100	$(1,\mathbf{2})_{-1/2}$	$(1,1,1,1)$	0	0	1	0	1	1	0	0	0	−1	1	−2	1	−1	−1
T_{2k}	s_{86}	101	$(1,1)_0$	$(1,1,1,1)$	1	−1	0	0	−1	0	0	1	0	−2	1	0	2	−1	0
	s_{87}	102	$(1,1)_1$	$(1,1,1,\mathbf{2})$	0	1	−1	0	0	0	0	0	0	2	−2	0	−1	−1	0
	s_{88}	103	$(1,1)_0$	$(1,1,1,\mathbf{2})$	0	1	1	−1	0	0	0	0	0	0	1	0	2	2	0
	s_{89}	104	$(1,1)_0$	$(1,1,\mathbf{2},1)$	0	0	0	0	0	0	0	0	1	0	0	4	0	0	0
	s_{90}	105	$(1,1)_{-1/2}$	$(1,\mathbf{2},1,1)$	0	0	0	0	−1	−1	0	−1	−2	0	0	1	0	0	0

Table 4.5: Chiral left-mover spectrum of a E_6/Z_6 asymmetric orbifold/covariant lattice model (cont'd).

Sector	Field	No.	G_{SM}	G_{hidden}	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}
T_{2k}	s_{91}	106	$(1,1)_0$	$(\mathbf{2},1,1,1)$	−1	0	0	1	0	0	0	0	0	1	1	0	2	−1	−3
	s_{92}	107	$(1,1)_0$	$(\mathbf{2},1,1,1)$	1	0	0	−1	0	1	1	−1	0	0	1	0	−1	−1	0
	s_{93}	108	$(1,1)_{-1}$	$(1,1,1,1)$	−1	0	1	0	1	0	−1	−1	0	1	1	0	−1	−1	3
	s_{94}	109	$(1,1)_1$	$(1,1,1,1)$	1	1	0	0	1	−1	−1	0	0	−1	1	0	−1	−1	−3
T_{3k}	s_{95}	110	$(1,1)_{-1/2}$	$(1,\mathbf{2},1,1)$	0	0	0	1	1	0	−1	0	0	−1	1	1	2	1	1
	s_{96}	111	$(1,1)_1$	$(\mathbf{2},1,1,1)$	1	0	0	0	−1	0	0	0	0	1	−2	1	2	1	1
	s_{97}	112	$(1,1)_{-1}$	$(1,1,1,1)$	−1	−1	0	0	0	0	0	−1	0	−2	1	1	−1	−2	−2
	s_{98}	113	$(1,1)_0$	$(1,1,1,1)$	−1	1	0	−1	0	−1	−1	0	0	−1	−2	1	−1	1	1
	s_{99}	114	$(1,1)_0$	$(1,1,1,1)$	1	0	1	0	−1	0	−1	−1	0	−1	1	−2	−1	1	1
	s_{100}	115	$(1,1)_0$	$(1,1,1,\mathbf{2})$	0	0	0	1	−1	−1	0	1	0	0	1	1	−1	1	−2
	s_{101}	116	$(1,1)_{-1}$	$(1,1,\mathbf{2},1)$	0	−1	1	0	0	0	1	0	−1	1	2	0	1	0	1
	s_{102}	117	$(1,1)_0$	$(1,1,1,1)$	1	0	−1	0	0	−1	0	0	0	0	1	1	−1	−2	4
	s_{103}	118	$(1,1)_0$	$(1,1,1,1)$	1	0	−1	1	0	1	0	0	−2	0	0	−1	0	−1	−2
	Q_2, Q_3	119	$(\mathbf{3},\mathbf{2})_{1/6}$	$(1,1,1,1)$	0	1	0	0	0	0	0	−1	0	1	1	1	0	0	−1
T_{4k}	s_{104}	120	$(1,1)_{-1}$	$(1,1,\mathbf{2},1)$	0	0	0	0	0	0	0	0	−1	0	4	0	0	0	0
	s_{105}	121	$(1,1)_{1/2}$	$(1,\mathbf{2},1,1)$	0	0	0	0	0	1	0	−1	0	2	0	−2	1	1	0
	s_{106}	122	$(1,1)_{-1}$	$(1,1,1,1)$	−1	1	0	1	0	−1	1	0	0	1	0	1	1	1	3
	s_{107}	123	$(1,1)_1$	$(1,1,1,1)$	1	0	1	−1	0	−1	0	0	0	2	0	1	−2	1	0
	s_{108}	124	$(1,1)_0$	$(1,1,1,1)$	1	0	1	1	0	0	1	−1	0	−1	0	1	1	1	−3
	s_{109}	125	$(1,1)_1$	$(\mathbf{2},1,1,1)$	1	1	−1	0	−1	0	−1	0	0	0	0	1	1	1	0
	L_{12}, L_{13}	126	$(1,\mathbf{2})_{-1/2}$	$(1,1,1,1)$	0	−1	0	−1	0	0	0	−1	0	−1	0	1	1	1	3

Chapter 5

Conclusions and Outlook

In this work, chiral four-dimensional covariant lattice models were revisited and a classification of all possible right-mover lattices was performed. Also, it was found that once a right-mover lattice is fixed, modular invariance requires that the set of possible left-mover lattices forms a genus. The result is that there are in total 99 right-mover lattices which may lead to chiral models, and only 19 of them lead to $\mathcal{N} = 1$ spacetime supersymmetry. Then, using the Smith-Minkowski-Siegel mass formula, a lower bound of $O(10^{10})$ models with $\mathcal{N} = 1$ supersymmetry was calculated. Furthermore, some of the relevant genera were enumerated completely. In particular, for two classes of covariant lattice models that correspond to Z_3 and Z_6 asymmetric orbifolds we performed an exhaustive enumeration of models, which resulted in exactly 2030 models in the Z_3 case, and in $O(10^7)$ models (including duplicates) for the Z_6 case. We also considered some generic phenomenological properties of these models, such as discrete flavor and R-symmetries, and also gave spectra of three-generation models. Finally, we studied how the equivalence between certain covariant lattice and twist-orbifold models fits into our picture, and found that there exist some covariant lattices which cannot be obtained as a twist-orbifold theory.

As in the case of the genus corresponding to E_6/Z_3 orbifold models that we enumerated exactly, some smaller genera (e.g. the D_6/Z_4 case) may be studied exactly in the future. However, an exact evaluation of the larger genera does not seem to be practicable, both from the viewpoint of computation time and required memory. This became already manifest in our enumeration of the left-mover lattices corresponding to asymmetric E_6/Z_6 orbifolds, where it was not feasible to eliminate duplicates. For even larger genera, one might resort to other methods. For example, a randomized search that just produces a large number of models is perfectly viable, as long as one does not mind obtaining duplicate models. Another approach would be to impose more phenomenological constraints. Then, it might be possible to circumvent the lower bounds that we calculated.

Furthermore, there is no need to restrict on supersymmetric models. In fact, the non-supersymmetric case seems even richer, as for the number of these models we obtained a lower bound of $O(10^{23})$. However, some of these theories inevitably contain tachyons. Nevertheless, four-dimensional non-supersymmetric models from string theory have received some interest recently, and in some specific symmetric orbifold setup a quite large fraction of tachyon-free models was obtained [63].

There is another remark on the supercurrent lattices that we found. Here, we used a rather brute force approach to classify them. However, it would be interesting to have a more fundamental and geometrical understanding of these lattices, maybe in a way similar to how we understand root systems in terms of simple roots. One could also ask which of the admissible supercurrent lattices allow for an additional world-sheet supercurrent that completes the $N = 2$ super-Virasoro algebra. Clearly, such a supercurrent must be allowed for the maximal admissible lattice obtained from a right-mover lattice with $\mathcal{N} = 1$ spacetime supersymmetry. This is because spacetime supersymmetry always implies at least $N = 2$ supersymmetry on the world-sheet. An example for this would be the 9_N supercurrent lattice that appears in the A_6/Z_7 orbifold. Also, the one-dimensional supercurrent lattice allows for an additional world-sheet supersymmetry because it corresponds to a $N = 2$ minimal model.

The lattice theories discussed in this work only cover CFTs with Kac-Moody algebras of level one, so a generalization that takes into account also higher levels would be desirable (note that a generalization of the theory of lattice genera to general CFTs was attempted in [64]). Nevertheless, our results may be useful in the construction of some sort of hybrid models. For example, one could combine our primitive elementary lattices with $N = 2$ minimal models to fill up the required central charge $c_{\mathrm{R}} = 9$.

Note on the Computational Methods

Most of the computations for this work were performed using the computer algebra system GAP [65]. The calculation of lattice automorphism groups and isomorphisms between lattices was a crucial part for which a modified version of the algorithm described in [66] was implemented. The computation of the Smith-Minkowski-Siegel mass formula relied on the built-in method `conway_mass()` of the computer algebra system SAGE [67]. Gröbner bases were calculated using Singular [68] and the GAP package "singular". The calculation of orbits in Subsection 3.2.5 was carried out using a highly specialized C++ implementation.

Appendix A

Lattices

A.1 Basic Definitions

Lattices are crucial for most of this work, and here we wish to formalize some aspects of them. A lattice Λ shall be defined as a discrete subset of points in a real Euclidean or Lorentzian inner product space that is closed under addition and subtraction. As such, we also regard it as an Abelian group or as a free $\mathbb{Z}$-module that is endowed with a bilinear form $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \mapsto \mathbb{R}$. For two vectors $x, y \in \Lambda$ we often also write $x \cdot y$ for $\langle x, y \rangle$ and x^2 for $\langle x, x \rangle$. Any lattice Λ has a basis $\{b_1, \ldots, b_n\}$ with basis vectors $b_i \in \Lambda$, so that each $x \in \Lambda$ can be written as a unique linear combination of the b_i. The number n of basis vectors is called the dimension $\dim(\Lambda)$ of Λ. Moreover, for such a basis one defines the Gram matrix G as

$$G_{ij} = \langle b_i, b_j \rangle . \tag{A.1}$$

A subset $\Lambda' \subseteq \Lambda$ ($\Lambda' \subset \Lambda$) that is closed under addition and subtraction is called a sublattice (strict sublattice) of Λ. Let Λ be a lattice with Gram matrix G, and let $\det(\Lambda) = |\det(G)|$ denote its discriminant. Then, for any sublattice $\Lambda' \subseteq \Lambda$ one can construct the quotient group Λ/Λ'. Its order, $|\Lambda/\Lambda'|$, is called the index of Λ' in Λ and is given by

$$|\Lambda/\Lambda'| = \sqrt{\det(\Lambda')/\det(\Lambda)} . \tag{A.2}$$

Furthermore, we define the orthogonal sum $\Lambda_1 \oplus \Lambda_2$ of two lattices Λ_1 and Λ_2 as the Cartesian product $\Lambda_1 \times \Lambda_2$ endowed with the bilinear form

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle , \tag{A.3}$$

for all $(x_1, x_2), (y_1, y_2) \in \Lambda_1 \times \Lambda_2$.

We also call a map $\phi : \Lambda_1 \mapsto \Lambda_2$ between two lattices Λ_1 and Λ_2 a lattice homomorphism, if it is a homomorphism of the corresponding Abelian groups (and a fortiori also of the $\mathbb{Z}$-modules), and if it preserves the bilinear form, i.e.

$$\langle \phi(x), \phi(y) \rangle_{\Lambda_2} = \langle x, y \rangle_{\Lambda_1} , \quad \text{for all } x, y \in \Lambda_1 . \tag{A.4}$$

Such a map is called an isomorphism if it is bijective, and and automorphism if it is bijective and further $\Lambda_1 = \Lambda_2$.

The dual lattice Λ^* of a lattice is defined as the set of vectors in the $\mathbb{R}$-span of Λ that has integer inner product with all vectors in Λ. Taking the dual lattice is an involution, i.e. $\Lambda^{**} = \Lambda$. For a lattice Λ, let $\overline{\Lambda}$ denote the lattice that is identical to Λ except that the inner product is amended by a minus sign.

A special role is played by integral lattices. A lattice Λ is called integral if $\langle x, y\rangle \in \mathbb{Z}$ for all $x, y \in \Lambda$. Furthermore, an integral lattice is called even if $x^2 \in 2\mathbb{Z}$ for all $x \in \Lambda$. Integral lattices that are not even are called odd. For an integral lattice Λ, one defines the quotient group Λ^*/Λ, whose order is given by $|\Lambda^*/\Lambda| = \det(\Lambda)$. A lattice for which $\Lambda = \Lambda^*$ is called self-dual. Furthermore, we define the elementary divisors of a lattice to be the elementary divisors obtained from the Smith normal form of the Gram matrix G. For these divisors we often use the abbreviated notation $d_1^{n_1} \cdots d_r^{n_r}$, which also gives a factorization of $\det(\Lambda)$, but do not explicitly write down the number of appearances of the unit 1, as it can be deduced from the dimension of the lattice.

Discriminant forms. Let in the following Λ be an integral lattice. The bilinear form on Λ naturally induces a bilinear form on $B_\Lambda(\cdot,\cdot) : \Lambda^*/\Lambda \times \Lambda^*/\Lambda \mapsto \mathbb{Q}/\mathbb{Z}$, given by

$$B(v + \Lambda, w + \Lambda) = v \cdot w \tag{A.5}$$

for $v, w \in \Lambda^*$. This is indeed well-defined because

$$\langle x + \Lambda, y + \Lambda\rangle - \langle x, y\rangle = \langle x + y, \Lambda\rangle + \langle \Lambda, \Lambda\rangle \in \mathbb{Z}\,, \tag{A.6}$$

for any $x, y \in \Lambda$. For Λ even we also define a quadratic form $Q_\Lambda : \Lambda^*/\Lambda \mapsto \mathbb{Q}/2\mathbb{Z}$,

$$Q_\Lambda(x + \Lambda) = \langle x, x\rangle\,, \tag{A.7}$$

with $x \in \Lambda^*$. Again, this is well-defined because $(v + x)^2 - v^2 \in 2\mathbb{Z}$ for all $x \in \Lambda$.

The quotient Λ^*/Λ together with the bilinear form B_Λ is called the discriminant bilinear form, $\mathrm{disc}_-(\Lambda)$ of Λ. Similarly, for even lattices we also define the discriminant quadratic form $\mathrm{disc}_+(\Lambda)$ as the tuple $(\Lambda^*/\Lambda, Q_\Lambda)$. Moreover one defines morphisms for these structures as group morphisms $\phi : \Lambda_1^*/\Lambda_1 \mapsto \Lambda_2^*/\Lambda_2$ which preserve the respective bilinear or quadratic form, i.e.

$$B_{\Lambda_2}(\phi(\cdot), \phi(\cdot)) = B_{\Lambda_1}(\cdot,\cdot)\,, \tag{A.8}$$

in the case of disc_-, and further

$$Q_{\Lambda_2} \circ \phi = Q_{\Lambda_1}\,. \tag{A.9}$$

in the case of disc_+. A fortiori, a morphism of a discriminant quadratic form $\mathrm{disc}_+(\Lambda)$ also gives us a morphism of the corresponding discriminant bilinear form $\mathrm{disc}_-(\Lambda)$. When we just write "discriminant form" without the attribute "bilinear" or "quadratic", we mean disc_+ in the case of even lattices and disc_- in the case of odd lattices. Furthermore, we say that two lattices are in the same genus if simultaneously their signatures are equal, they are either both even or odd, and their discriminant forms are isomorphic.

A.2 Lattice Gluing

In this section, we formalize some aspects of lattice gluing. Most of the following is used implicitly in the main text. A decomposition $\Lambda \supset \Lambda_1 \oplus \Lambda_2$ of an integral lattice Λ shall be called self-glue free if $\Lambda/(\Lambda_1 \oplus \Lambda_2)$ does not contain cosets of the form (η, Λ_2) and (Λ_1, ξ), with $\eta \in \Lambda_1^*/\Lambda_1$ and $\xi \in \Lambda_2^*/\Lambda_2$ where neither η nor ξ is the zero coset. Now, there is a theorem about self-glue free decompositions of self-dual lattices [39]:

Theorem 1. *For any integral self-dual lattice Λ with a self-glue free decomposition of the form $\Lambda \supseteq \Lambda_1 \oplus \overline{\Lambda}_2$, where $\dim(\Lambda) = \dim(\Lambda_1) + \dim(\Lambda_2)$, one obtains an isomorphism*

$$\mathrm{disc}_-(\Lambda_1) \cong \mathrm{disc}_-(\Lambda_2)\,, \tag{A.10}$$

which at the same time is an isomorphism

$$\mathrm{disc}_+(\Lambda_1) \cong \mathrm{disc}_+(\Lambda_2)\,. \tag{A.11}$$

if Λ is even. Also the "converse" statement is true: given two integral lattices Λ_1, Λ_2 and an isomorphism $\phi : \mathrm{disc}_-(\Lambda_1) \mapsto \mathrm{disc}_-(\Lambda_2)$, then there exists a self-dual lattice Λ with a self-glue free decomposition $\Lambda \supseteq \Lambda_1 \oplus \overline{\Lambda}_2$. If furthermore Λ_1 and Λ_2 are even and ϕ is also an isomorphism of the corresponding discriminant quadratic forms, then also Λ is even.

Proof. First, one realizes that each coset in $\Lambda/(\Lambda_1 \oplus \overline{\Lambda}_2)$ can be uniquely represented by a pair (ξ_1, ξ_2) of cosets $\xi_1 \in \Lambda_1^*/\Lambda_1$ and $\xi_2 \in \Lambda_2^*/\Lambda_2$. Since we are considering a self-glue free decomposition, each coset in Λ_1^*/Λ_1 and Λ_2^*/Λ_2 can appear *at most once* in such a pair (assuming otherwise immediately produces a contradiction). This gives us the following inequalities:

$$\left|\frac{\Lambda}{\Lambda_1 \oplus \overline{\Lambda}_2}\right| \leq \left|\frac{\Lambda_i^*}{\Lambda_i}\right|\,, \qquad i \in \{1, 2\}\,. \tag{A.12}$$

Also, since Λ is self-dual, each $\xi_1 \in \Lambda_1^*/\Lambda_1$ and $\xi_2 \in \Lambda_2^*/\Lambda_2$ must appear *at least once* in a pair (ξ_1, ξ_2). This can be seen by deriving

$$\left|\frac{\Lambda}{\Lambda_1 \oplus \overline{\Lambda}_2}\right|^2 = \frac{\det(\Lambda_1 \oplus \overline{\Lambda}_2)}{\det(\Lambda)} = \left|\frac{\Lambda_1^*}{\Lambda_1}\right|\left|\frac{\Lambda_2^*}{\Lambda_2}\right|\,, \tag{A.13}$$

using $\det(\Lambda) = 1$. Hence, in (A.12) equalities must hold and the set of pairs (ξ_1, ξ_2) defines a bijective map $\phi : \Lambda_1^*/\Lambda_1 \mapsto \Lambda_2^*/\Lambda_2$ which preserves the group structure. Thus, we have established the following group isomorphies:

$$\frac{\Lambda}{\Lambda_1 \oplus \overline{\Lambda}_2} \cong \frac{\Lambda_1^*}{\Lambda_1} \cong \frac{\Lambda_2^*}{\Lambda_2} \tag{A.14}$$

Moreover, $B_{\Lambda_1}(\xi_1, \xi_1') - B_{\Lambda_2}(\phi(\xi_1), \phi(\xi_1')) = 0$, and if Λ is even it follows further that $Q_{\Lambda_1}(\xi_1) - Q_{\Lambda_2}(\phi(\xi_1)) = 0$. Here, the minus sign originates from the fact that Λ_2 appears with negated inner product in the decomposition of Λ. The "converse" is shown by giving an explicit construction:

$$\Lambda = \bigcup_{\xi \in \Lambda_1^*/\Lambda_1} \xi \times \phi(\xi)\,, \tag{A.15}$$

This lattice clearly has the desired properties. Here, the Cartesian product $\xi \times \phi(\xi)$ is to be interpreted as an element of $(\Lambda_1 \oplus \overline{\Lambda}_2)^*/(\Lambda_1 \oplus \overline{\Lambda}_2)$. □

Let Λ be a lattice and Λ_0 be a sublattice of Λ. For some coset $\xi \in \Lambda/\Lambda_0$ let $|\xi|$ denote the order of that coset, i.e. the smallest natural number n such that $n\xi = \Lambda_0$. Furthermore, $\langle\xi\rangle$ shall be the linear span of all vectors in that coset, i.e. $\langle\xi\rangle = \bigcup_{k=1}^{|\xi|} k\xi$.

Lemma 1. *Let Λ_1 and Λ_2 denote two even lattices and $\phi : \mathrm{disc}(\Lambda_1) \mapsto \mathrm{disc}(\Lambda_2)$ be an isomorphism. Moreover, let $\xi \in \Lambda_1^*/\Lambda_1$ be a non-zero coset which contains only even vectors. Then, it follows that $\mathrm{disc}(\langle\xi\rangle) \cong \mathrm{disc}(\langle\phi(\xi)\rangle)$.*

Proof. First, note that $\langle\xi\rangle$ and $\langle\phi(\xi)\rangle$ are again even lattices. Let us now define a homomorphism $\psi : \langle\xi\rangle^*/\langle\xi\rangle \mapsto \langle\phi(\xi)\rangle^*/\langle\phi(\xi)\rangle$:

$$\psi(x + \langle\xi\rangle) = \phi(x + \Lambda_1)|_{\mathrm{rep}} + \langle\phi(\xi)\rangle\ , \tag{A.16}$$

and a homomorphism $\tilde\psi : \langle\phi(\xi)\rangle^*/\langle\phi(\xi)\rangle \mapsto \langle\xi\rangle^*/\langle\xi\rangle$:

$$\tilde\psi(y + \langle\phi(\xi)\rangle) = \phi^{-1}(y + \Lambda_2)|_{\mathrm{rep}} + \langle\xi\rangle\ , \tag{A.17}$$

where $x \in \langle\xi\rangle^*$, $y \in \langle\phi(\xi)\rangle^*$, and $|_{\mathrm{rep}}$ means taking any representative. One checks that they are well defined. Now one shows

$$\tilde\psi \circ \psi\,(x + \langle\xi\rangle) = \phi^{-1}(\phi(x + \Lambda_1)|_{\mathrm{rep}} + \Lambda_2)|_{\mathrm{rep}} + \langle\xi\rangle = x + \langle\xi\rangle\ . \tag{A.18}$$

It follows that $\tilde\psi = \psi^{-1}$ so ψ is an isomorphism. Now it remains to prove that the isomorphism fulfills $Q_{\langle\phi(\xi)\rangle} \circ \psi = Q_{\langle\xi\rangle}$. However, we see for some $x \in \langle\xi\rangle^*$ that

$$\begin{aligned} Q_{\langle\phi(\xi)\rangle}(\psi(x + \langle\xi\rangle)) &= Q_{\langle\phi(\xi)\rangle}(\phi(x + \Lambda_1)|_{\mathrm{rep}} + \langle\phi(\xi)\rangle) \\ &= \langle\phi(x + \Lambda_1)|_{\mathrm{rep}},\ \phi(x + \Lambda_1)|_{\mathrm{rep}}\rangle_{\Lambda_2} \\ &= Q_{\Lambda_2}(\phi(x + \Lambda_1)) \\ &= Q_{\Lambda_1}(x + \Lambda_1) \\ &= \langle x, x\rangle_{\Lambda_1} \\ &= Q_{\langle\xi\rangle}(x + \langle\xi\rangle)\,, \end{aligned} \tag{A.19}$$

equalities being understood modulo 2. So, we established that ψ is an isomorphism of discriminant forms. □

Theorem 2 (Regluing Theorem)**.** *Let $\Lambda \supset \Lambda_1 \oplus \overline{\Lambda}_2$ be a self-glue free decomposition of an even self-dual lattice Λ, and ξ be a coset in Λ_1^*/Λ_1 with $Q_{\Lambda_1}(\xi) = 0$ not containing the null vector. Then, there exists an even self-dual lattice Λ' with self-glue free decomposition $\Lambda' \supset \langle\xi\rangle \oplus \overline{\langle\phi(\xi)\rangle}$, where $\phi : \mathrm{disc}(\Lambda_1) \mapsto \mathrm{disc}(\Lambda_2)$ is an isomorphism.*

Proof. The existence of ϕ is ensured by Theorem 1. Then, from Lemma 1 it follows that $\mathrm{disc}(\langle\xi\rangle) \cong \mathrm{disc}(\langle\phi(\xi)\rangle)$. Again, from Theorem 1 follows the existence of Λ'. □

Lemma 2. *Let Λ be an even lattice and $\xi \in \Lambda^*/\Lambda$ be a coset of only even vectors that does not contain the null vector. Then, there exists another coset $\mu \in \Lambda^*/\langle\xi\rangle^*$ so that $\Lambda = \{x \in \langle\xi\rangle \mid \langle v, x\rangle \in \mathbb{Z}\}$ for any given $v \in \mu$.*

Proof. Let us define a bilinear form $B(\cdot,\cdot) : \Lambda^*/\langle\xi\rangle^* \times \langle\xi\rangle/\Lambda \mapsto \mathbb{Z}/|\xi|\mathbb{Z}$:

$$B(x + \langle\xi\rangle^*, y + \Lambda) = |\xi|\langle x, y\rangle \ , \tag{A.20}$$

with $x \in \Lambda^*$ and $y \in \langle\xi\rangle$. Note, that both $\Lambda^*/\langle\xi\rangle^*$ and $\langle\xi\rangle/\Lambda$ are cyclic and of the same order, as seen from the following argument. Define a homomorphism $\psi : \Lambda^*/\langle\xi\rangle^* \mapsto \langle\xi\rangle/\Lambda$:

$$\psi(\eta) = B(\eta, \xi)\xi \ , \tag{A.21}$$

This is injective because $\eta \in \ker\psi \implies B(\eta,\xi) = 0 \pmod{|\xi|}$ which is only possible for $\eta = \langle\xi\rangle^*$. However, since $|\Lambda^*/\langle\xi\rangle^*| = |\xi| = |\langle\xi\rangle/\Lambda|$ it follows that ψ is also surjective, so it is an isomorphism. Now, choose $\mu = \psi^{-1}(\xi)$ so from $\psi(\mu) = \xi$ one obtains $B(\mu,\xi) = 1 \pmod{|\xi|}$. It follows that

$$B(\mu,\nu) = 0 \pmod{|\xi|} \iff \nu = \Lambda \, ,$$

and thus $\Lambda = \{x \in \langle\xi\rangle \mid \langle v, x\rangle \in \mathbb{Z}\}$, for any $v \in \mu$. □

Theorem 3. *Let Λ be an even lattice and Λ_0 be a sublattice of Λ so that Λ/Λ_0 is nontrivial and cyclic. Further, let N denote the order of Λ/Λ_0. Then, there exists a vector v fulfilling*

(a) $Nv \in \Lambda^*$

(b) $kv \notin \Lambda^*$ *for* $k \in \{1, \ldots, N-1\}$

so that $\Lambda_0 = \{x \in \Lambda \mid \langle v, x\rangle \in \mathbb{Z}\}$, and an integer $n \neq 1$ dividing the exponent of Λ_0^*/Λ_0 such that

(a) $N \mid n$

(b) $nv \in \Lambda$

(c) $n\langle v, v\rangle \in \mathbb{Z}$ *and* $n\langle v, v\rangle \in 2\mathbb{Z}$ *if* $2 \nmid n$

Proof. Let ξ generate Λ/Λ_0, so we can write $\Lambda = \langle\xi\rangle$. From Lemma 2 we then obtain a $\mu \in \Lambda_0^*/\Lambda^*$ so that $\Lambda_0 = \{x \in \Lambda \mid \langle v, x\rangle \in \mathbb{Z}\}$ for any fixed $v \in \mu$. Since $N\mu = \Lambda^*$ we have $Nv \in \Lambda^*$. Furthermore, because $v \notin \Lambda_0$ but $v \in \Lambda_0^*$, there is some smallest $n \neq 1$ dividing the exponent of Λ_0^*/Λ_0 so that $nv \in \Lambda_0$. Then we have $n\langle v, v\rangle \in \mathbb{Z}$ and $\langle nv, nv\rangle \in 2\mathbb{Z}$, so it follows that actually $n\langle v, v\rangle \in 2\mathbb{Z}$ if $2 \nmid n$. Clearly, N must divide n because $\Lambda \subset \Lambda^*$. □

A.3 The Lorentzian Lattices $D_{n,m}$

Let us define the lattice $D_{n,m}$ as the set of vectors $a \in \mathbb{Z}^{n+m}$ satisfying $a^2 = a \cdot a \in 2\mathbb{Z}$. The bilinear form is given by

$$a \cdot a' = \sum_{i=1}^{n} a_i a'_i - \sum_{i=n+1}^{n+m} a_i a'_i \,, \tag{A.22}$$

and has signature (n, m). For $m = 0$ one also denotes these lattices by D_n, and they are isomorphic to the root lattices of $SO(2n)$.

One then checks that the dual lattice of $D^*_{n,m}$ is given by the union of four cosets (0), (v), (s) and (c), which are defined as

$$(0) = D_{n,m} \,, \tag{A.23}$$

$$(v) = (1, 0, \ldots, 0) + D_{n,m} \,, \tag{A.24}$$

$$(s) = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}) + D_{n,m} \,, \tag{A.25}$$

$$(c) = (-\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}) + D_{n,m} \,. \tag{A.26}$$

These are also called root (0), vector (v), spinor (s), and cospinor (c) conjugacy classes due to their connection to Lie theory. Vectors in these conjugacy classes have the following norms:

$$x^2 \in 2\mathbb{Z} \,, \text{ for } x \in (0) \,, \tag{A.27}$$

$$x^2 \in \tfrac{n-m}{4} + 2\mathbb{Z} \,, \text{ for } x \in (s) \,, \tag{A.28}$$

$$x^2 \in 1 + 2\mathbb{Z} \,, \text{ for } x \in (v) \,, \tag{A.29}$$

$$x^2 \in \tfrac{n-m}{4} + 2\mathbb{Z} \,, \text{ for } x \in (c) \,. \tag{A.30}$$

Furthermore, note that these cosets form a group, $D^*_{n,m}/D_{n,m}$, which is isomorphic to $Z_2 \times Z_2$ in the case where $n + m$ is even, and to Z_4 otherwise.

A lattice $D_{n,m}$ can be decomposed into an orthogonal sum of sublattices,

$$D_{n,m} \supset D_{n_1,m_1} \oplus D_{n_2,m_2} \,, \tag{A.31}$$

where $n = n_1 + n_2$ and $m = m_1 + m_2$. The conjugacy classes then decompose as follows:

$$(0) \longrightarrow (0,0) \cup (v,v) \,, \tag{A.32}$$

$$(s) \longrightarrow (s,s) \cup (c,c) \,, \tag{A.33}$$

$$(v) \longrightarrow (v,0) \cup (0,v) \,, \tag{A.34}$$

$$(c) \longrightarrow (s,c) \cup (c,s) \,. \tag{A.35}$$

Note that the pairs on the r.h.s. in above equations are to be interpreted as a Cartesian product of the corresponding conjugacy classes of the sublattices.

From a lattice $D_{n,m}$ we can always construct the self-dual lattice $(0) \cup (v) = \mathbb{Z}^{n,m}$. In the cases where $n - m \in 4\mathbb{Z}$ also the lattice $E_{n,m} = (0) \cup (s)$ (or equivalently $(0) \cup (c)$) is self-dual. Furthermore, this lattice is even if $n - m \in 8\mathbb{Z}$. Also, due to triality E_4 is isomorphic to $\mathbb{Z}^4$.

Appendix B

Supplementary Material

Table B.1: Gram matrices for the primitive elementary supercurrent lattices. Also shown are the elementary divisors, the root system Δ_2 of norm 2 vectors and the number n_3 of norm 3 vectors.

Ξ	Gram Matrix	Divisors	Δ_2	n_3
1	$\begin{pmatrix} 3 \end{pmatrix}$	3^1	–	2
3	$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$	4^2	–	8
5	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & 1 \\ -1 & -1 & 3 & 1 & -1 \\ -1 & -1 & 1 & 3 & -1 \\ -1 & 1 & -1 & -1 & 3 \end{pmatrix}$	$2^3 6^1$	–	20
6_A	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & 1 \\ -1 & -1 & 3 & 1 & 1 & -1 \\ -1 & -1 & 1 & 3 & 1 & -1 \\ -1 & -1 & 1 & 1 & 3 & -1 \\ -1 & 1 & -1 & -1 & -1 & 3 \end{pmatrix}$	$2^4 4^1$	–	32
6_B	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & 1 & 1 \\ -1 & -1 & 3 & 1 & -1 & 1 \\ -1 & -1 & 1 & 3 & 0 & 0 \\ -1 & 1 & -1 & 0 & 3 & 0 \\ -1 & 1 & 1 & 0 & 0 & 3 \end{pmatrix}$	$2^1 8^2$	–	20
6_C	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & -1 & 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ -1 & 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 0 & -1 & 0 & 3 \end{pmatrix}$	5^3	–	20
7_A	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 3 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 3 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 3 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 3 \end{pmatrix}$	2^6	–	56
7_B	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 3 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 3 & 0 & -1 \\ -1 & 1 & -1 & -1 & 0 & 3 & 1 \\ -1 & 1 & -1 & 0 & -1 & 1 & 3 \end{pmatrix}$	12^2	–	32
8_A	$\begin{pmatrix} 3 & -2 & -2 & -1 & -1 & -1 & -1 & -1 \\ -2 & 3 & 1 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 3 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 3 & 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 & 3 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 & 3 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 & 1 & 3 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & -1 & 3 \end{pmatrix}$	$2^4 4^1$	A_1^2	80

Table B.1: The primitive elementary supercurrent lattices (cont'd).

Ξ	Gram Matrix	Divisors	Δ_2	n_3
8_B	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 3 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 3 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 1 & 3 & 0 & -1 \\ -1 & 1 & -1 & -1 & -1 & 0 & 3 & 1 \\ -1 & 1 & -1 & 0 & 0 & -1 & 1 & 3 \end{pmatrix}$	$4^2 12^1$	–	44
8_C	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 3 & 1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 3 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 3 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 3 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$	$2^1 8^1 24^1$	–	32
8_D	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 3 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 3 & 1 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 3 & 1 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & -1 & 3 \end{pmatrix}$	$3^2 6^2$	–	32
9_A	$\begin{pmatrix} 3 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 3 & 1 & 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 & 3 & 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 & 3 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 0 \\ -1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 3 \end{pmatrix}$	$2^4 8^1$	A_1^2	88
9_B	$\begin{pmatrix} 3 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -2 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -2 & 1 & 3 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 3 & 1 & 1 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 3 & 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 3 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & -1 & -1 & -1 & -1 & 3 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$	$2^2 4^1 8^1$	A_1^2	84
9_C	$\begin{pmatrix} 3 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & 0 \\ -2 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & 1 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 & 1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 3 & 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 3 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 & 3 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 \end{pmatrix}$	$2^2 8^2$	A_1^2	60

Table B.1: The primitive elementary supercurrent lattices (cont'd).

Ξ	Gram Matrix	Divisors	Δ_2	n_3
9_D	$\begin{pmatrix} 3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 3 & 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 3 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 3 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 3 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 3 \end{pmatrix}$	$3^1 9^2$	A_1	54
9_E	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 3 & 1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 3 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 1 & 3 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ -1 & 1 & -1 & -1 & -1 & -1 & 0 & 3 & 1 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$	$2^2 8^2$	–	64
9_F	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 3 & 1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 3 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 1 & 3 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 3 & 0 & -1 \\ -1 & 1 & -1 & -1 & -1 & 0 & 0 & 3 & 1 \\ -1 & 1 & -1 & 0 & 0 & -1 & -1 & 1 & 3 \end{pmatrix}$	4^4	–	56
9_G	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 3 & 1 & 1 & 1 & -1 & -1 & 0 \\ -1 & -1 & 1 & 3 & 1 & 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 3 & 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 3 & 0 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & 0 & 3 & 1 & 1 \\ -1 & 1 & -1 & 0 & 0 & -1 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 3 \end{pmatrix}$	$4^2 16^1$	–	56
9_H	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 3 & 1 & 1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 3 & 1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 1 & 3 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & -1 & 0 & 3 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 1 & 3 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$	$2^1 4^1 8^2$	–	44
9_I	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 3 & 1 & 1 & 1 & -1 & -1 & 0 \\ -1 & -1 & 1 & 3 & 1 & 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 3 & 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 3 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & -1 & 0 & 3 & 1 & 1 \\ -1 & 1 & -1 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 3 \end{pmatrix}$	8^3	–	44

Table B.1: The primitive elementary supercurrent lattices (cont'd).

Ξ	Gram Matrix	Divisors	Δ_2	n_3
9_J	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 & -1 & -1 & 0 \\ -1 & -1 & 1 & 3 & 1 & 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 3 & 1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 3 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 0 & 3 & 1 & 1 \\ -1 & 1 & -1 & 0 & -1 & 0 & 1 & 3 & 1 \\ -1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 3 \end{pmatrix}$	$3^1 12^2$	–	44
9_K	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 3 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 3 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 & -1 & 3 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & -1 & 1 & 3 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & 3 \end{pmatrix}$	$3^3 9^1$	–	56
9_L	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 3 & 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 3 & 1 & -1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 3 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 3 \end{pmatrix}$	20^2	–	44
9_M	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 3 & 1 & 1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 3 & 1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 3 & 0 & 0 & -1 & 1 \\ -1 & 1 & -1 & 0 & 0 & 3 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 3 \end{pmatrix}$	$4^2 8^2$	–	32
9_N	$\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 1 & 3 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 3 & 0 & -1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & -1 & 0 & 3 & 1 & -1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 3 & 0 \\ -1 & 0 & 1 & 1 & 1 & 1 & -1 & 0 & 3 \end{pmatrix}$	7^3	–	44

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